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PHYSICAL REVIEW D VOLUME 10, NUMBER 2 15 JULY 1974

# Classical electrodynamics of a nonlinear Dirac field with anomalous magnetic moment

Antonio F. Rañada and Manuel F. Rañada
*Departamento de Física Teórica, Universidad de Zaragoza, Zaragoza, Spain*

Mario Soler
*Division de Física Teórica, Junta de Energía Nuclear, Madrid 3, Spain*

Luis Vázquez
*Departamento de Física Teórica, Universidad de Zaragoza, Zaragoza, Spain*
(Received 28 January 1974)

The classical electromagnetic interactions of a nonlinear spinor field are studied in perturbation theory. When Pauli terms are included, the model describes with reasonable accuracy (within the assumed approximations) such properties of the nucleons as spin, charge, magnetic moment, and the proton mass. With no other information one can calculate the proton-neutron mass difference, which comes out of the wrong sign and of the same size as in quantum electrodynamics.

## I. INTRODUCTION

The purpose of this paper is to explore the classical electrodynamics of a nonlinear spinor field as a possible model of elementary particles.

Since the work by Rosen[1] the interaction of electromagnetism with other classical fields has been studied by many authors. These attempts have not been in general very successful, one of the reasons probably being the lack of satisfactory solutions for the "free" (noninteracting) classical fields. The absence of free solutions invalidates the use of perturbation methods, since the free zero-order states are a necessary first step for the perturbative procedure.

It has been shown, however,[2] that the classical theory of a spinor field with a positive $(\bar{\psi}\psi)^2$ self-interaction provides a satisfactory model for a free particle. The presence of fourth-order terms in the Lagrangian as a dynamical consequence of spin was first discussed by Weyl,[3] and it was shown along the same lines[4] that in a certain simple model of the universe one is led to precisely the same theory with a $(\bar{\psi}\psi)^2$ self-interaction through only the assumption of generalized covariance for the spinor field.

Once solutions of the free spinor equations are available, it is quite natural to study the coupling of the free spinor and electromagnetic fields. This was done in Ref. 5 for the particular case in which, of all four electromagnetic potentials $A_\mu$, only $A_0 \neq 0$—when the particle is at rest. This model happens to be spherically separable and soluble (numerically). Assuming that the suppression of the vector potentials $A_k$ ($k=1,2,3$) would not introduce a drastic change in the solutions, it was shown that these might roughly describe the physical nucleons. Moreover, the model was proposed as an interesting test case in which exact and perturbative solutions might be compared.

A classical perturbation theory for the model proposed in Ref. 5 was developed in Ref. 6. The numerical results were in complete accordance with those previously obtained in treating the exact solutions.

It seems, therefore, that there are grounds to trust perturbation theory in those cases in which an exact solution cannot be obtained. Such is indeed the general case when the vector electromagnetic potentials $A_k$ are not zero. We now address ourselves to this problem. Since we will try to compare the properties of this model with the physical nucleons, we will assume the existence of Pauli terms, so that anomalous magnetic moments can be included.

We present in this paper numerical results which have been obtained under two different approximations. One is the above-mentioned perturbative approach, and another is the multipole expansion. It turns out that, at any order of $e^2$, only a finite number of multipoles are nonvanishing. This allows the exact calculation of any quantity at a certain order. For example, in the case of the ground state of our model only the first partial wave contributes to the energy at order $e^2$.

In Sec. II we derive the equations for the model and explain the approximations which are made. Section III contains the numerical results; they show how a remarkable picture of some of the outstanding physical features of the nucleon can be obtained within this simple model. The significance of these results is discussed in Sec. IV, where we compare the assumptions which have been made to set up the model with those generally taken to derive similar results.

## II. GENERAL DESCRIPTION OF THE MODEL

Our model is based on the Lagrangian

$$\mathcal{L} = \mathcal{L}_D + \mathcal{L}_{\rm EM} + \mathcal{L}_I , \tag{1}$$

where

$$\mathcal{L}_D = \tfrac{1}{2}i[\bar{\psi}\gamma^\mu\partial_\mu\psi - (\partial_\mu\bar{\psi})\gamma^\mu\psi] - m\bar{\psi}\psi + \lambda(\bar{\psi}\psi)^2 , \tag{2}$$

$$\mathcal{L}_{\rm EM} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} , \tag{3}$$

$$\mathcal{L}_I = -e\delta\bar{\psi}\gamma^\mu\psi A_\mu - k\,\frac{e}{4m}\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu} . \tag{4}$$

Our notation will be

$$g_{\nu\nu} = (1,-1,-1,-1) ,$$

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$$

($\sigma^k$ the Pauli matrices),

$$A^\mu = (A^0, \vec{A}), \quad F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu ,$$

$$\sigma^{\mu\nu} = \frac{1}{2i}[\gamma^\mu, \gamma^\nu] .$$

$e$ is the electromagnetic coupling constant, $\delta$ is a parameter taking the values 0, 1 in order to represent both charged and neutral solutions, and $k$ is the coupling constant for the Pauli term. When $k=0$, we have the case of minimal coupling.

It was proved in Ref. 2 that the Lagrangian $\mathcal{L}_D$ provides a satisfactory theory for spinorial particles with rest mass. In Ref. 6 the case $\delta=1$, $k=0$ was studied under the simplifying assumption $A^k=0$, i.e., neglecting the magnetic effects. In the present paper these effects are also considered.

The field equations are

$$\begin{gathered}\partial_\mu F^{\nu\mu} = e\bar{\psi}\gamma^\nu\psi - e\,\frac{k}{2m}\partial_\mu(\bar{\psi}\sigma^{\nu\mu}\psi) , \\ i\gamma^\mu\partial_\mu\psi - m\psi + 2\lambda(\bar{\psi}\psi)\psi - e\gamma^\mu A_\mu\psi - e\,\frac{k}{4m}\sigma^{\mu\nu}F_{\mu\nu}\psi = 0 .\end{gathered} \tag{5}$$

In contrast to the simplified models considered in Refs. 2 and 5, these general equations admit no stationary solutions which are separable in spherical coordinates when the system is at rest at the origin.

In general, if no separable solutions can be found, the method to follow is to expand the functions in a multipole series and write equations for the coefficients of each partial wave. These equations, however, are nonlinear and difficult to solve.

In our particular case we have found separable solutions for those equations (5) which would correspond to $e=0$. This means that the lack of separable solutions is related in this model to the existence of electromagnetism. Therefore, if an expansion in powers of the coupling constant $e$ is introduced for each of the functions in (5), the lowest order in $e$ (corresponding to $e=0$) should still be separable, and the nonseparable multipole expansions should be considered as electromagnetic corrections.

As we said in the Introduction, the reliability of perturbation theory has been established in Ref. 6 for the special case in which $A_0 \neq 0$, $A_k = 0$ ($k=1,2,3$). There is no reason for doubting that it holds in the general case when $A_k \neq 0$, to be considered presently.

Our approach to the problem will essentially consist in writing down a multipole expansion for each order of perturbation in the coupling constant $e$. However, since the natural dimensionless quantity in this theory is not $e$ but $\epsilon = e^2/2\lambda m^2$, the relevant orders of perturbation will also be in $\epsilon^n$, and before we come to the multipole expansions we must express the fields in the form

$$\psi = e^{-i\omega t}(\psi_0 + \epsilon\psi_1 + \cdots),$$
$$A_\mu = \frac{2\lambda m^2}{e}(\epsilon A_\mu^1 + \cdots). \tag{6}$$

As was shown in Ref. 6, the free parameter $\omega$, which is supposed to adjust itself to the minimum energy, should also be expanded in the form

$$\omega = \omega_0 + \epsilon\omega_1 + \cdots .$$

We will not go into the details concerning this expansion again here. Let it only be recalled that $\omega_1$ can be easily determined and that $\omega_0$ is known from the free case.

Substitution of these expansions in the general equations (5) gives in the successive orders 0, 1, 2 in $\epsilon$

$$i\gamma^k \partial_k \psi_0 + \omega_0 \gamma^0 \psi_0 - m\psi_0 + 2\lambda(\bar\psi_0\psi_0)\psi_0 = 0, \tag{7}$$

$$\Box A_1^\mu - \delta\bar\psi_0\gamma^\mu\psi_0 + \frac{k}{2m}\partial_\nu(\bar\psi_0\sigma^{\mu\nu}\psi_0) = 0, \tag{8}$$

$$i\gamma^k \partial_k \psi_1 + \omega_0\gamma^0\psi_1 + \omega_1\gamma^0\psi_0 - m\psi_1 + 2\lambda(\bar\psi_0\psi_0\psi_1 + \bar\psi_0\psi_1\psi_0 + \bar\psi_1\psi_0\psi_0) - \delta\gamma^\mu A_\mu^1\psi_0 - \frac{k}{4m}\sigma^{\mu\nu}F_{\mu\nu}^1\psi_0 = 0. \tag{9}$$

One way to obtain the equations of motion for each partial wave would be, of course, to substitute in these equations the multipole expansions of each field. This procedure is, however, extremely tedious. It is far simpler in practice to use a method advocated by Finkelstein *et al.*[7] in a similar problem, adapting it to the special case of perturbation theory.

We first substitute the expansion (6) in the total Lagrangian density, and split it in the form

$$\mathcal{L} = \mathcal{L}_0(\bar\psi_0, \psi_0) + \epsilon\mathcal{L}_1(A_\mu^1) + \epsilon^2\mathcal{L}_2(\bar\psi_1, \psi_1) + \cdots .$$

It is understood that the expression for the term $\mathcal{L}_n$ includes, besides the highest-order fields [$\psi_{n/2}$, $\bar\psi_{n/2}$ if $n$ is even, or $A_{(n+1)/2}$ if $n$ is odd] and their first derivatives, also the lower-order fields. The variation of any particular term $\mathcal{L}_n$ in the expansion of $\mathcal{L}$ will provide the equations for the fields of highest order contained in it, while all the other (lower-order) fields are supposed already determined by the variation of $\mathcal{L}_0, \mathcal{L}_1, \ldots,$ up to $\mathcal{L}_{n-1}$. It can be easily checked that one thus obtains the same equations (7), (8), and (9) already derived, and in general the equations for any order.

In case the fields appearing in a certain $\mathcal{L}_n$ do not admit solutions which can be separated in spherical coordinates, one can write for these a multipole expansion. But instead of substituting this expansion in the equations of motion, the alternative method, which happens to be quite simpler, is to substitute this expansion in the Lagrangian and integrate it over the sphere. The resulting Lagrangian density can be varied with respect to the remaining radial functions.

For this particular kind of solution, the three first $\mathcal{L}_n$ in $\mathcal{L}$ are

$$\mathcal{L}_0 = \tfrac{1}{2}i[\bar\psi_0\gamma^k\partial_k\psi_0 - (\partial_k\bar\psi_0)\gamma^k\psi_0] + \omega_0\bar\psi_0\gamma^0\psi_0 - (m - \lambda\bar\psi_0\psi_0)\bar\psi_0\psi_0, \tag{10}$$

$$\begin{aligned}\mathcal{L}_1 = {}&\tfrac{1}{2}i[\bar\psi_1\gamma^k\partial_k\psi_0 + \bar\psi_0\gamma^k\partial_k\psi_1 - (\partial_k\bar\psi_1)\gamma^k\psi_0 - (\partial_k\bar\psi_0)\gamma^k\psi_1]\\ &+ \omega_1\bar\psi_0\gamma^0\psi_0 + \omega_0(\bar\psi_0\gamma^0\psi_1 + \bar\psi_1\gamma^0\psi_0) - (m - 2\lambda\bar\psi_0\psi_0)(\bar\psi_0\psi_1 + \bar\psi_1\psi_0) - \delta\bar\psi_0\gamma^\mu\psi_0A_\mu^1 - \tfrac{1}{4}F_{\mu\nu}^1F_1^{\mu\nu} - \frac{k}{4m}\bar\psi_0\sigma^{\mu\nu}\psi_0F_{\mu\nu}^1,\end{aligned} \tag{11}$$

$$\begin{aligned}\mathcal{L}_2 = {}&\tfrac{1}{2}i[\bar\psi_1\gamma^k\partial_k\psi_1 - (\partial_k\bar\psi_1)\gamma^k\psi_1] + \omega_0\bar\psi_1\gamma^0\psi_1 - m\bar\psi_1\psi_1 + \omega_1(\bar\psi_0\gamma^0\psi_1 + \bar\psi_1\gamma^0\psi_0) + \lambda[2(\bar\psi_1\psi_1)\bar\psi_0\psi_0 + (\bar\psi_0\psi_1 + \bar\psi_1\psi_0)^2]\\ &- \delta(\bar\psi_1\gamma^\mu\psi_0 + \bar\psi_0\gamma^\mu\psi_1)A_\mu^1 - \frac{k}{4m}(\bar\psi_1\sigma^{\mu\nu}\psi_0 + \bar\psi_0\sigma^{\mu\nu}\psi_1)F_{\mu\nu}^1 .\end{aligned} \tag{12}$$

Equations (7) are obtained from $\mathcal{L}_0$. It was shown in Ref. 2, where we refer for more details, that they admit a separable solution of the form

$$\psi_0 = \begin{bmatrix} g(r)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ if(r)\begin{pmatrix} \cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix} \end{bmatrix}$$

corresponding to an $S_{1/2}$ state.

From $\psi_0$ we obtain (8), which can be completely separated with $A^1_\mu$ taking the simple form

$$A^1_\mu = (A(r),\ V(r)\sin\theta\sin\phi,\ -V(r)\sin\theta\cos\phi,\ 0). \quad (13)$$

Substitution in (8) gives

$$\frac{d^2A}{dr^2} + \frac{2}{r}\frac{dA}{dr} = -\delta(f_0^{\,2} + g_0^{\,2}) + \frac{k}{m}\left[(f_0g_0)' + \frac{2f_0g_0}{r}\right], \quad (14)$$

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} - \frac{2}{r^2}V = -2\delta f_0 g_0 + \frac{k}{m}\left[g_0g_0' + f_0f_0' + \frac{f_0^{\,2}}{r}\right]. \quad (15)$$

The functions $\bar{\psi}_1$, $\psi_1$ appearing in $\mathcal{L}_2$ no longer admit separable solutions of the corresponding equations (9). We therefore use our method, i.e., substitute in $\mathcal{L}_2$ the exact expression for $\psi_0$, $\bar{\psi}_0$, $A^1_\mu$ and an expansion of $\psi_1$ in spinorial spherical harmonics:

$$\psi^1 = \sum_j \psi^1_{j+} + \sum_j \psi^1_{j-}, \quad (16)$$

where

$$\psi^1_{j+} = \begin{pmatrix} g_{2j}\,\mathcal{Y}^{1/2}_{jl_A} \\ if_{2j}\,\mathcal{Y}^{1/2}_{jl_B} \end{pmatrix}, \quad l_A \text{ even} \quad (17)$$

$$\psi^1_{j-} = \begin{pmatrix} h_{2j}\,\mathcal{Y}^{1/2}_{jl_A} \\ ik_{2j}\,\mathcal{Y}^{1/2}_{jl_B} \end{pmatrix}, \quad l_A \text{ odd}. \quad (18)$$

Once it is integrated over the angles, the resulting expression (which is fairly simple because of the orthonormal property of the angular functions) is varied with respect to the radial functions $g_i$, $f_i$, $k_i$, and $h_i$.

The system of equations thereby obtained is, of course, linear, and each partial wave would be uncoupled with the rest if the nonlinear $\lambda(\bar{\psi}\psi)^2$ term were absent. More precisely, the system decouples in an infinity of two-dimensional subsystems each one containing the two waves with the same values of $l_A$, coupled to $\psi_0$.

Moreover, only the $S_{1/2}$ and $D_{3/2}$ waves are coupled to $\psi_0$ by the electromagnetic interaction. It follows that only three waves ($S_{1/2}, D_{3/2}, D_{5/2}$) are nonvanishing; the rest are zero.

The equations for these three waves, which can be easily obtained by the method explained above, are the following:

$$f_1' + \frac{2}{r}f_1 + [m - \omega_0 + 2\lambda(f_0^{\,2} - 3g_0^{\,2})]g_1 + 4\lambda f_0g_0f_1 - (\omega_1 - \delta A)g_1 - \tfrac{2}{3}\delta f_0 V - \frac{k}{2m}\left(\tfrac{4}{3}g_0\frac{V}{r} + \tfrac{2}{3}g_0V' - f_0A'\right) = 0,$$

$$g_1' + [m + \omega_0 + 2\lambda(3f_0^{\,2} - g_0^{\,2})]f_1 - 4\lambda f_0g_0g_1 + (\omega_1 - \delta A)f_0 + \tfrac{2}{3}\delta g_0 V - \frac{k}{2m}(-\tfrac{2}{3}f_0V' + g_0A') = 0, \quad (19)$$

$$f_3' - \frac{1}{r}f_3 + [m - \omega_0 + 2\lambda(f_0^{\,2} - g_0^{\,2})]g_3 + \tfrac{8}{5}\lambda g_0(f_0f_3 - g_0g_3) + \tfrac{8}{5}\lambda g_0(f_0f_5 - g_0g_5) + \tfrac{2}{3}\delta f_0V - \frac{k}{3m}\left(-g_0V' + g_0\frac{V}{r}\right) = 0,$$

$$g_3' + \frac{3}{r}g_3 + [m + \omega_0 + 2\lambda(f_0^{\,2} - g_0^{\,2})]f_3 + \tfrac{8}{5}\lambda f_0(f_0f_3 - g_0g_3) + \tfrac{8}{5}\lambda f_0(f_0f_5 - g_0g_5) - \tfrac{2}{3}\delta g_0V - \frac{k}{3m}\left(f_0V' + 3f_0\frac{V}{r}\right) = 0, \quad (20)$$

$$f_5' + \frac{4}{r}f_5 + [m - \omega_0 + 2\lambda(f_0^{\,2} - g_0^{\,2})]g_5 + \tfrac{12}{5}\lambda g_0(f_0f_3 - g_0g_3) + \tfrac{12}{5}\lambda g_0(f_0f_5 - g_0g_5) = 0,$$

$$g_5' - \frac{2}{r}g_5 + [m + \omega_0 + 2\lambda(f_0^{\,2} - g_0^{\,2})]f_5 + \tfrac{12}{5}\lambda f_0(f_0f_3 - g_0g_3) + \tfrac{12}{5}\lambda f_0(f_0f_5 - g_0g_5) = 0. \quad (21)$$

As stated above, the equation for the $D_{5/2}$ wave does not contain electromagnetic terms. However, the wave does not vanish because of the source terms.

It is easy to show that only the lowest wave $S_{1/2}$ contributes to the energy at order $\epsilon$. This is because of the orthogonality properties of the angular functions. In fact, $\psi_1$ appears in the expression of the energy at order $\epsilon$ [see Eq. (28)] only in the integrals

$$\int (\psi_0^\dagger\psi_1 + \psi_1^\dagger\psi_0)\,d^3x,$$

$$\int (\bar{\psi}_0\psi_0)(\bar{\psi}_0\psi_1 + \bar{\psi}_1\psi_0)\,d^3x,$$

to which $D_{3/2}$ and $D_{5/2}$ do not contribute. On the other hand, $S_{1/2}$ is independent of the other two. For these reasons we include in our calculations only the first wave. Though this amounts to neglecting some first-order terms in the value of the spinors, the energy of the ground state will be exact at order $\epsilon$.

We will now write down the uncoupled equations for the wave $\psi^1_{1/2+}$ which have been used in the numerical calculations after the following convenient changes in functions and variables are performed:

$$
\begin{aligned}
&(g_0, f_0, g_1, f_1) = \left(\frac{m}{2\lambda}\right)^{1/2} (G_0, F_0, G_1, F_1),\\
&(A, V) = m(\mathfrak{A}, \mathfrak{V}),\\
&\rho = mr,\\
&\left(\frac{\omega_0}{m}, \frac{\omega_1}{m}\right) = (\Lambda_0, \Lambda_1).
\end{aligned}
\tag{22}
$$

With this notation the equations which determine $G_0, F_0$ and $\mathfrak{A}, \mathfrak{V}$ are

$$
\begin{aligned}
&F_0' + \frac{2}{\rho} F_0 + (1 - \Lambda_0 + F_0^2 - G_0^2) G_0 = 0,\\
&G_0' + (1 + \Lambda_0 + F_0^2 - G_0^2) F_0 = 0,
\end{aligned}
\tag{23}
$$

$$
\mathfrak{A}'' + \frac{2}{\rho}\mathfrak{A}' + \delta(G_0^2 + F_0^2) - k\left(G_0' F_0 + G_0 F_0' + \frac{2}{\rho} G_0 F_0\right) = 0,
\tag{24}
$$

$$
\mathfrak{V}'' + \frac{2}{\rho}\mathfrak{V}' - \frac{2}{\rho^2}\mathfrak{V} + 2\delta G_0 F_0 - k\left(G_0 G_0' + F_0 F_0' + \frac{2}{\rho} F_0^2\right) = 0.
$$

The equations for $F_1$ and $G_1$ are

$$
\begin{aligned}
&F_1' + \frac{2}{\rho} F_1 + (1 - \Lambda_0 + F_0^2 - 3G_0^2) G_1 - \Lambda_1 G_0 + 2 G_0 F_0 F_1\\
&\quad + \delta(G_0 \mathfrak{A} - \tfrac{2}{3} F_0 \mathfrak{V}) - k\left(\tfrac{2}{3} G_0 \frac{\mathfrak{V}}{\rho} + \tfrac{1}{3} G_0 \mathfrak{V}' - \tfrac{1}{2} F_0 \mathfrak{A}'\right) = 0,\\
&G_1' + [(1 + \Lambda_0) + 3F_0^2 - G_0^2] F_1 + \Lambda_1 F_0 - 2 G_0 F_0 G_1\\
&\quad - \delta(F_0 \mathfrak{A} - \tfrac{2}{3} G_0 \mathfrak{V}) + k(\tfrac{1}{3} F_0 \mathfrak{V}' - \tfrac{1}{2} G_0 \mathfrak{A}') = 0.
\end{aligned}
\tag{25}
$$

We will be interested in three types of solutions, namely the "normal" type, with $\delta \neq 0$, $k = 0$, the "proton" type, $\delta \neq 0$, $k \neq 0$, and the "neutron" type, $\delta = 0$, $k \neq 0$.

For all three cases our main interest is in evaluating the energy at rest and the magnetic moment. Other relevant quantities will be the total charge and spin. In the classical theory of fields all these quantities must be calculated as volume integrals of the corresponding densities. The total electric charge is, in lowest order of the dimensionless constant $\epsilon$,

$$
q = \int_V j_0 \, dV = \frac{2\pi}{\lambda m^2} \delta e \int_0^\infty (G_0^2 + F_0^2) \rho^2 d\rho.
\tag{26}
$$

In the case $\delta = 0$, corresponding to the neutron model, Eq. (26) gives, of course, zero electric charge. There is, however, a conserved current as a consequence of gauge invariance of the first kind. This invariance provides a conserved baryonic charge or normalization which in lowest order is common to the "normal," proton, and neutron models. As we will immediately see, the normalization must be

$$
\frac{2\pi}{\lambda m^2} \int_0^\infty (G_0^2 + F_0^2) \rho^2 d\rho = \hbar
\tag{27}
$$

in order to have the experimentally known spin $S_3 = \frac{1}{2}\hbar$.

The definition of spin comes from the spin vector

$$
S_k = \tfrac{1}{2} \epsilon_{ijk} J^{ij},
$$

where

$$
J^{ij} = \int d^3 r (x^i T^{j0} - x^j T^{i0}).
$$

Using the energy-momentum tensor

$$
\begin{aligned}
T^{\alpha\beta} = {}& F^{\nu\alpha} F^{\beta}{}_{\nu} + \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\alpha\beta} + \tfrac{1}{4} i [\bar\psi \gamma^\alpha \partial^\beta \psi + \bar\psi \gamma^\beta \partial^\alpha \psi - (\partial^\alpha \bar\psi) \gamma^\beta \psi - (\partial^\beta \bar\psi) \gamma^\alpha \psi]\\
& - \delta \frac{e}{2} (\bar\psi \gamma^\alpha \psi A^\beta + \bar\psi \gamma^\beta \psi A^\alpha) - k \frac{e}{4m} (\bar\psi \sigma^{\nu\alpha} \psi F_\nu{}^\beta + \bar\psi \sigma^{\nu\beta} \psi F_\nu{}^\alpha) + \lambda (\bar\psi\psi)^2 g^{\alpha\beta},
\end{aligned}
\tag{28}
$$

we find in our case, always in lowest order of $\epsilon$,

$$
\begin{aligned}
&S_1 = S_2 = 0,\\
&S_3 = \frac{\pi}{\lambda m^2} \int_0^\infty (G_0^2 + F_0^2) \rho^2 d\rho,
\end{aligned}
$$

so, in fact, the normalization (27) gives the correct spin. As a consequence of (27) we also see that the total electric charge is $q = e\hbar$.

We need not assume in our present classical treatment that $\hbar$ is a new physical constant, i.e., a quantum of action as is postulated in quantum mechanics. All we really do is adjust the charge and spin of the model to the physical nucleon values.

Once the normalization is fixed the two relevant quantities in which we are interested are the magnetic moments in lowest order and the energy including first-order corrections.

The magnetic moment is given by

$$\mathfrak{M} = \tfrac{1}{2}\int_V \vec{r}\times\vec{j}\, d^3r\,,$$

where $\vec{j}$, the current density, is

$$j^k = e\delta\bar{\psi}\gamma^k\psi + \frac{ke}{2m}\partial_\mu(\bar{\psi}\sigma^{\mu k}\psi)\,.$$

Using our definitions (22) we get in lowest order

$$\begin{aligned}\mathfrak{M} &= \frac{\pi}{3}\frac{e}{\lambda m^3}\left[4\delta\int_0^\infty F_0G_0\rho^3d\rho + k\int_0^\infty (F_0^2+3G_0^2)\rho^2d\rho\right]\\ &= \frac{1}{\lambda m^2}\frac{2\pi}{3}\frac{m_p}{m}\int_0^\infty [4\delta F_0G_0\rho + k(F_0^2+3G_0^2)]\,\rho^2 d\rho\mu_N\,,\end{aligned} \tag{29}$$

where $m_p$ is the proton mass, and $\mu_N$ is a nuclear magneton.

Before we deal with the rest energy, for which, as we just mentioned, first-order corrections will be included, it must be stressed that we leave out for the moment the important question of higher-order corrections to the electric and baryonic charges and to the spin. It should be kept in mind that the situation is quite similar in this respect for the physical charge and for the spin. The twofold aspect of charge, as a coupling constant $e$, and as the integral of the fourth component of the electric current, which we have called $q$, is well known. Less known is the close analogy which exists classically between the coupling constant $e$ appearing in the definition of the generalized derivative $\partial_\mu \to \partial_\mu - ieA_\mu$ and the "coupling constant" which must be used to define the generalized covariant derivative for a spinor field.[8,9] As a consequence, one can also define spin as a coupling constant ($\tfrac{1}{2}$ in our case) or as the integral of the fourth component of a conserved current.

In this case the question of higher-order corrections to the total spin is on the same footing as that of higher-order corrections to the total electric charge. We intend to analyze these questions in a future publication. For the phenomenological purposes of this paper we will see that first-order normalization is sufficient to fit the data with very good accuracy.

If we include first-order corrections, from the energy-momentum tensor (28) we get for the energy $E=\int_V T_{00}dV$, $E=E_0+\epsilon E_1$, where

$$E_0 = \frac{2\pi}{\lambda m}\int_0^\infty \rho^2 d\rho[\Lambda_0(F_0^2+G_0^2)+\tfrac{1}{2}(F_0^2-G_0^2)^2]\,, \tag{30}$$

$$\begin{aligned}E_1 = \frac{2\pi}{\lambda m}\Big\{&\int_0^\infty \rho^2 d\rho[\Lambda_1(F_0^2+G_0^2)+2\Lambda_0(F_0F_1+G_0G_1)+2(F_0^2-G_0^2)(F_0F_1-G_0G_1)]\\ &+\int_0^\infty[\tfrac{1}{2}\mathfrak{A}'^2\rho^2+\mathfrak{V}^2+\tfrac{1}{3}\mathfrak{V}'^2\rho^2+\tfrac{2}{3}\mathfrak{V}\mathfrak{V}'\rho-\delta\mathfrak{A}(F_0^2+G_0^2)\rho^2-kF_0G_0\mathfrak{A}'\rho^2]\,d\rho\Big\}\,.\end{aligned} \tag{31}$$

Up to this approximation the energy is not changed if we include first-order corrections in the normalization (27), i.e., if we write

$$\hbar = \frac{2\pi}{\lambda m^2}(I_0+2\epsilon I_1)\,,$$

where

$$I_0 = \int_0^\infty (F_0^2+G_0^2)\rho^2 d\rho\,,$$

$$I_1 = \int_0^\infty (F_0F_1+G_0G_1)\rho^2 d\rho$$

($I_1$ is evaluated at $\Lambda_0$ corresponding to the minimum energy). This normalization provides a second-order equation for the quantity $\lambda m^2$, since $\epsilon = e^2/2\lambda m^2$. For the relevant solution, up to order $e^2$ the old $\epsilon$ would go over to the new value $\epsilon - 2\epsilon^2(I_1/I_0)$ with the new normalization. As a consequence of this change in $\epsilon$, the energy $E=E_0+\epsilon E_1$ would only get a correction of second order, which we can neglect.

We therefore see that the mass difference between the proton and neutron models can be established in first order using the same baryonic norm (in zero order) for both models.

## III. NUMERICAL RESULTS

In this section we present the numerical solutions which have been obtained to the equations, and the fit to experimental quantities.

At this stage of the theory, which is rather unrefined, we have not judged it necessary to keep an exact control of numerical precision. Our quoted calculations are, at any rate, always supposed reliable to within a 1% error, and are in general much better.

For Eq. (23) we find the system with a minimum energy at $\Lambda_0 = 0.936$. We obtain for this value $E_0 = (2\pi/\lambda m)3.7548$. For comparison, for the values $\Lambda_0 = 0.934$ and $\Lambda_0 = 0.938$ one gets respectively $E_0 = (2\pi/\lambda m)3.7554$ and $E_0 = (2\pi/\lambda m)3.7555$.

Once $\Lambda_0$ is fixed, and correspondingly the functions $G_0$ and $F_0$, one solves Eq. (24) with boundary values such that

$$\mathcal{a} \sim \frac{1}{\rho}$$

and

$$\mathcal{V} \sim \frac{1}{\rho^2} \quad \text{for } \rho \to \infty \,.$$

In practice, what is done is to choose arbitrary values of $\mathcal{a}(0)$ and $\mathcal{V}'(0)$, which gives $\mathcal{a} \to C$ and $\mathcal{V} \sim D\rho$ for $\rho \to \infty$. From these, the "good" physical initial conditions are $\mathcal{a}(0) - C$ and $\mathcal{V}'(0) - D$, besides $\mathcal{V}(0) = 0$.

The next equations to be solved are (25), which contain the as yet undetermined constant $\Lambda_1$. It has been proved in Ref. 6 that the energy in first order is independent of $\Lambda_1$. The wave functions $G_1$ and $F_1$ depend of course on $\Lambda_1$, which may be easily determined as we will see later, if their shape is needed.

It may be easily seen that the energy in first order, $E_0 + \epsilon E_1$, can be written with $E_1$ in the form

$$E_1 = \frac{2\pi}{\lambda m}(A\delta^2 + B\delta k + Ck^2 + D\Lambda_1)\,.$$

$D$ should be exactly zero if we have chosen $\Lambda_0$ corresponding exactly to the minimum energy. In fact we get the following numerical constants for $\Lambda_0 = 0.936$:

$$A = 11.922\,, \quad B = -6.102\,,$$
$$C = -0.1075\,, \quad D = 0.00226\,.$$

For comparison we also give the values corresponding to $\Lambda_0 = 0.934$,

$$A = 12.388\,, \quad B = -6.189\,,$$
$$C = -0.0935\,, \quad D = -0.665\,,$$

and to $\Lambda_0 = 0.938$,

$$A = 11.436\,, \quad B = -6.014\,,$$
$$C = -0.1216\,, \quad D = 0.763\,.$$

We now start fitting the model to the physical constants of the baryons.

For the dimensionless constant $\epsilon = e^2/2\lambda m^2$ we use the normalization

$$\frac{2\pi}{\lambda m^2}\int_0^\infty (F_0^{\,2} + G_0^{\,2})\rho^2 d\rho = \hbar$$

corresponding to $\Lambda_0 = 0.936$. Since the value of the integral is 3.6576, and $q = e\hbar$, we find $\epsilon = 0.00199$. We will take $\epsilon = 0.002$.

Once $\epsilon$ is fixed, we still have to adjust the parameters $m$ and $\lambda$ in order to describe the proton and neutron. Before this, however, it is interesting to consider the case of a particle with a proton mass and without anomalous magnetic moment. $\lambda$ and $m$ can be deduced, once $\lambda m^2$ is known, if we adjust the total rest energy $E = E_0 + \epsilon E_1$ to the proton mass 938.2 MeV through

$$E = \frac{2\pi}{\lambda m}\, 3.7786 \text{ MeV}\,. \tag{32}$$

We get $m = 908.2$ MeV$\hbar^{-1}$, $\lambda = 2.786 \times 10^{-5}\hbar$ MeV$^{-2}$.

With these values and the knowledge of $\psi_0$, we get the magnetic moment from (29). The result is 1.04 nuclear magnetons, which differs by less than 1% from the normal magnetic moment corresponding to a Dirac particle of "bare" mass 908.2 MeV (1.03 magnetons).

The picture one obtains in this first model is a system which, aside from higher-order electromagnetic corrections, has the normal charge, spin, and magnetic moment, i.e., those corresponding to a Dirac particle. In other words, the nonlinearity does not affect these quantities in an appreciable way. The total rest energy is relatively more affected, since it is not $m\hbar = 908.2$ MeV, but $E = 938.2$ MeV, of which 5.96 are of electromagnetic origin, and 932.30 are "mechanic" (not electromagnetic).

We now consider the proton and neutron cases, $\delta = 1, 0,\ k \neq 0$.

The parameter $\epsilon$ has already been fixed. We determine $\lambda$ and $m$ from the observed proton mass, and $k$ from the observed magnetic moment of each particle. For the proton one gets $m = 913.7$

TABLE I. Numerical results for the "normal," proton, and neutron models.

| | Electromagnetic mass | Bare mass | Total mass | $k$ | $m$ (MeV $\hbar^{-1}$) | $\lambda$ (MeV$^{-2}$ $\hbar$) |
|---|---|---|---|---|---|---|
| "Normal" model | 5.96 MeV | 932.30 MeV | 938.26 MeV | 0 | 908.2 | $2.786 \times 10^{-5}$ |
| Proton model | 0.28 MeV | 937.98 MeV | 938.26 MeV | 1.798 | 913.7 | $2.753 \times 10^{-5}$ |
| Neutron model | −0.17 MeV | 937.98 MeV | 937.81 MeV | −1.964 | 913.7 | $2.753 \times 10^{-5}$ |

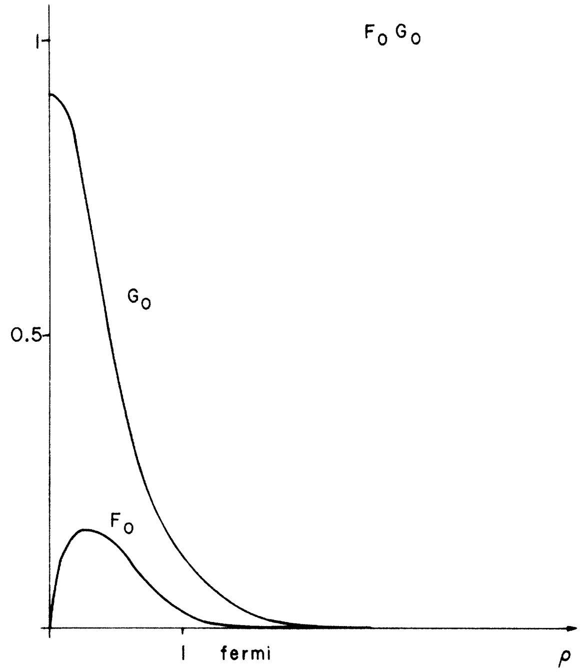


FIG. 1. Shape of the functions $F_0$, $G_0$ corresponding to the bare field $\psi_0$ at its minimum energy.

MeV $\hbar^{-1}$, $\lambda = 2.753\times 10^{-5}\hbar$ MeV$^{-2}$, and $k = 1.798$. This value of $k$ is surprisingly close to the one normally used in the linear quantum theory (1.793).

In contrast with the case considered before (a charged baryon with no anomalous magnetic moment) the contribution to the total energy coming from electromagnetic sources is here very small. It goes down to 0.28 MeV, compared with 5.96 MeV which we got before. This depression is due to the opposing effects of the electric and magnetic fields.

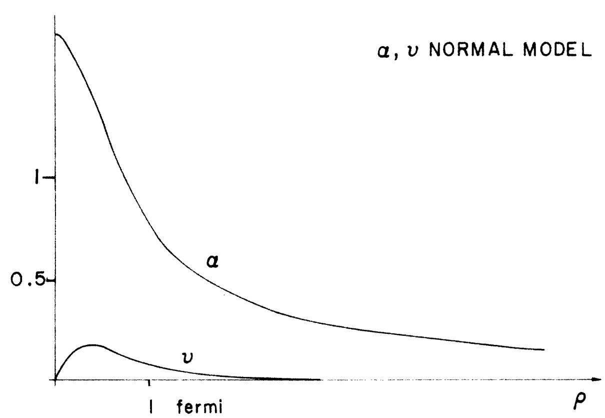


FIG. 2. Shape of the radial components of the electromagnetic potentials in first order of perturbation and lowest multipole, corresponding to $\psi_0$ as a source with minimal interaction.

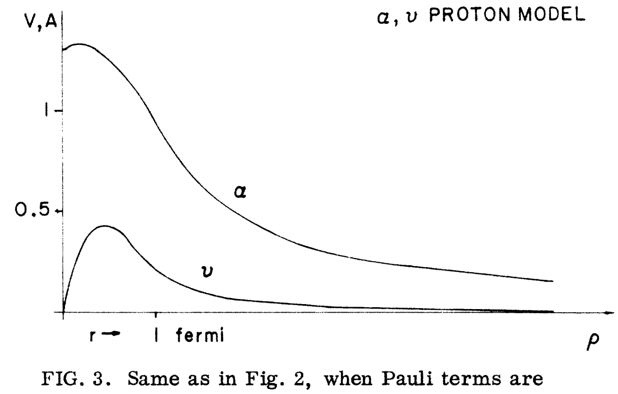


FIG. 3. Same as in Fig. 2, when Pauli terms are added besides the minimal electromagnetic coupling. The magnitude of the Pauli interaction is fixed by the proton anomalous magnetic moment.

With the same values of $\lambda$ and $m$, we determine $k$ for the neutron case ($\delta = 0$), so that the total magnetic moment [Eq. (29)] of the system coincides with the physical value. We obtain $k = -1.964$, also very close to $-1.913$, the number used in quantum mechanics.

The electromagnetic energy for this solution is $-0.17$ MeV, and the total energy at rest is 937.81 MeV.

The difference in energy between the proton and neutron models is $m_p - m_n = 0.45$ MeV. This result is very close to what is obtained in quantum electrodynamics when use is made of the experimentally determined form factors. In our determination we have made no use of the form factors, and have simply adjusted the model with the observable magnetic moments and the charge.

A resumé of our results is given in Table I. We also show in Figs. 1–4 the shapes of the different fields $\psi$, $A_\mu$. The generally accepted value for the nucleon radius, 0.8 F, approximately coincides with the limit zone where the nonlinear effects start to be appreciable. Outside this region the theory is very approximately linear.

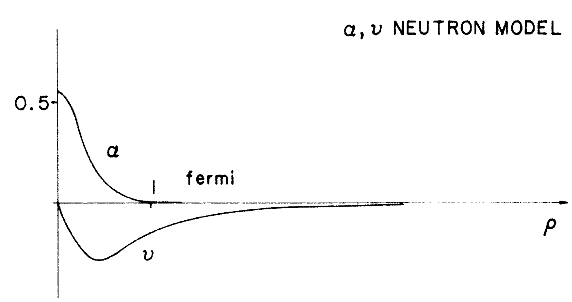


FIG. 4. Radial components for the neutron model electromagnetic potentials produced by the Pauli interaction. Their size is fixed by the neutron anomalous magnetic moment.

Some relevant features of the solutions are the comparatively smaller electrostatic fields of the proton-type solution compared with the "normal" case, and the small internal electromagnetic structure of the neutron-type solution.

In Fig. 5 we show the different energies for the three values of $\Lambda_0$: 0.934, 0.936, 0.938. The energies vary linearly with $\Lambda_1$, but for each line (corresponding to a certain value of $\Lambda_0$) only that point is significant where the line is tangent to its envelope. For the minimum ($\Lambda_0 = 0.936$) the actual values for $\Lambda_1$ in the "normal," proton, and neutron models are respectively 0.8, 0.6, 0.1. These are the values which provide the actual physical waves when inserted in Eq. (25).

## IV. CONCLUSIONS

We have obtained stable localized solutions for the system of interacting Dirac and Maxwell fields, including anomalous Pauli terms. Within the approximations which have been made, the localized objects provide a remarkably simple and accurate picture of many of the properties of the proton and neutron when the parameters appearing in the ordinary interacting Lagrangian are adjusted to the physical observables.

In particular, within the accuracy which can be expected from the approximations that have been made, we obtain the right spin, charge, rest energy, and magnetic moment, and a reasonable size of the proton. For the neutron, with no other adjustment than making the charge zero and changing the anomalous magnetic moment, we get a similar picture, except for the rest energy, which comes out incorrect. The mass difference $m_p - m_n = 0.45$ MeV coincides (within our expected accuracy) with the result obtained in quantum electrodynamics. In our case, however, we need not make use of other physical information, such as the form factors, which must be borrowed from observation in order to obtain the quantum-electrodynamical result.

Since we have not introduced pseudoscalar, vector, or other fields in the model, no effects or properties in which these fields are supposed to intervene can be expected to appear. Among these is probably the real mass difference $m_p - m_n$. On the other hand, one gets the general impression that those effects depending on the "mechanical" (self-energy) and electromagnetic properties of the nucleons are well described by the model.

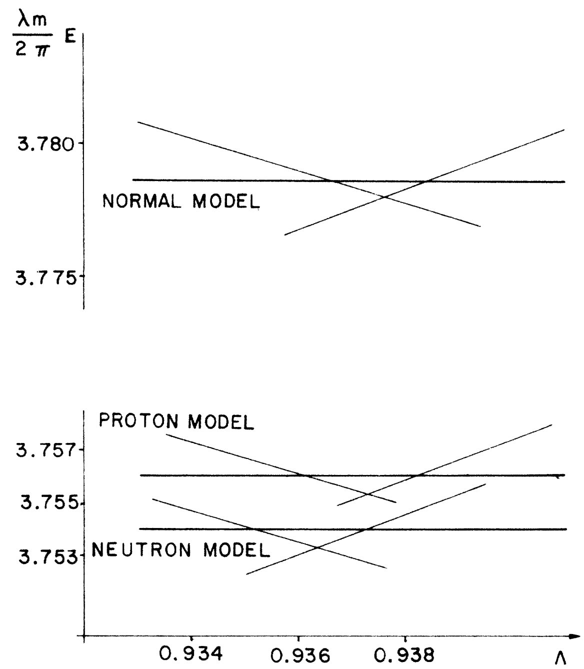


FIG. 5. Tangents to the three curves describing the dependence of the energy with $\Lambda = \Lambda_0 + \epsilon\Lambda_1$ for the "normal," proton, and neutron models. The physical particles are supposed to be described by that value of $\Lambda$ which corresponds to the minimum energy. In each case this value is fixed by the point where the horizontal tangent touches its envelope. Since $\Lambda_0$ is already known, one easily obtains $\Lambda_1 = (\Lambda - \Lambda_0)/\epsilon$.

## ACKNOWLEDGMENT

We wish to express our appreciation to Dr. B. Carreras for carefully reading our manuscript and for very useful discussions and suggestions during the completion of this paper.

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