A CHARACTERIZATION OF THE NOWHERE DIFFERENTIABLE FUNCTIONS IN THE GENERALIZED TAKAGI CLASS

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ABSTRACT. In this paper, we explore the non-differentiability behavior of the functions belonging to the Generalized Takagi class, a recent generalization of the Takagi function. It is derived by replacing the dyadic numbers used in the definition of classical Takagi function with an arbitrary countable and dense subset of $[0,1]$. Subject to specific conditions on the decomposition of such a set, we demonstrate that a function within this class is nowhere differentiable if and only if the sequence of weights does not belong to c_0.

1. INTRODUCTION

After the discovery of Weierstrass's example, continuous nowhere differentiable functions have captivated the attention of a large number of mathematicians. Proof thereof is the myriad of research into such functions and their properties that can be found in the existing literature (see [16], [4] or [21]).

The Takagi function is probably the simplest example of a continuous nowhere differentiable function (see [20]). It is usually defined by

$$T(x) = \sum_{n=0}^{\infty} g_n(x), \quad x \in [0,1], \tag{1.1}$$

where $g_n(x)$ denotes the distance from x to the set $D_n = \{k2^{-n} : k = 0, \ldots, 2^{-n}\}$. The Takagi function exhibits remarkable properties from different perspectives (see [3] for instance). For this reason, it has been thoroughly studied over the years, leading to numerous generalizations aiming to extend the intrinsic properties of the Takagi function to a broader family of functions. The surveys [1] or [18], as well as the thesis [19], contain a lot of information about the Takagi function and its generalizations.

One of the most famous generalizations of the Takagi function is the so-called Takagi class. This family of functions was introduced by M. Hata and M. Yamaguti in 1984 (see [15]). It is formed by all the functions $T_w : [0,1] \to \mathbb{R}$ defined as

$$T_w(x) = \sum_{n=0}^{\infty} w_n g_n(x)$$

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where $w = (w_n)_n$ is a sequence of weights satisfying that $(2^{-n}w_n)_n \in \ell_1$. They proved that the Takagi class is a closed subspace of $C[0,1]$ endowed with the supremum-norm, and it is isomorphic to ℓ_1.

In 1987, N. Kôno conducted a deep study of the differentiability properties of the functions in the Takagi class. More specifically, he proved that T_w is nowhere differentiable if and only if $w \notin c_0$; T_w is nondifferentiable almost everywhere but it is differentiable at an uncountable null set and the range of the derivative is all of $\mathbb{R}$, if and only if $w \in c_0 \setminus \ell_2$; and finally T_w is absolutely continuous, and consequently differentiable almost everywhere, if and only if $w \in \ell_2$.

Recently, the first two authors have introduced a new generalization of the Takagi function (see [6]): let D be a dense and countable subset of $[0,1]$ with $0,1 \in D$. A decomposition of the set D is an increasing sequence of finite sets $\mathcal{D} = (D_n)_n$ fulfilling the following properties:

(1) $0,1 \in D_0$.
(2) $D = \cup_{n=0}^{\infty} D_n$.
(3) There exist $\rho \in (0,1]$ and a non-increasing sequence of positive numbers $(\alpha_n)_n \in \ell_1$ such that

$$\rho\alpha_n \leq |x-y| \leq \alpha_n$$

for every $x, y \in D_n$ such that $(x,y) \cap D_n = \emptyset$.

It is worth mentioning that Proposition 1.5 of [6] yields that such a decomposition always exists, no matter the set D we are considering. For every decomposition $\mathcal{D} = (D_n)_n$ of the set D, the Generalized Takagi function $T_{\mathcal{D}} : [0,1] \to \mathbb{R}$ is defined by

$$T_{\mathcal{D}}(x) = \sum_{n=0}^{\infty} g_n(x)$$

where $g_n(x)$ denotes the distance from x to the set D_n. As a consequence of their results, J. Ferrera and J. Gómez Gil obtained that $T_{\mathcal{D}}$ is nowhere differentiable.

Following the same reasoning as M. Hata and M. Yamaguti, the first two authors also introduced the so-called Generalized Takagi class (see [7]). For a given decomposition $\mathcal{D} = (D_n)_n$ of the set D, the Generalized Takagi class is composed by all the functions $T_{\mathcal{D},w} : [0,1] \to \mathbb{R}$ defined by

$$T_{\mathcal{D},w}(x) = \sum_{n=0}^{\infty} w_n g_n(x)$$

where $w = (w_n)_n$ is a sequence of weights satisfying that $(w_n\alpha_n)_n \in \ell_1$. Under some extra but mild conditions on the decomposition, they proved that the Generalized Takagi Class is a closed subspace of $C[0,1]$ isomorphic to ℓ_1.

The Generalized Takagi class includes also the following interesting case: let $\mathbf{r} = (r_n)_n$ be a strictly increasing sequence of non-negative integers such that $r_0 = 1$ and r_n divides r_{n+1} for every n. We consider the decomposition $\mathcal{D} = (D_n)$ of the set $D = \cup D_n$ where

$$D_n = \left\{ \frac{k}{r_n} : k = 0, \ldots, r_n \right\}$$

for every n. We take $\alpha_n = r_n^{-1}$ and $\rho = 1$. The corresponding class receives the name of Generalized Takagi-Van der Waerden class, and it was originally introduced in [7]. The approximate differentiability of the function belonging to such a class was recently studied in [11]. We must mention that for $w = \mathbf{1}$ and $r_n = r^n$ where $r \geq 2$ is a fixed integer, we obtain the so-called Takagi-Van der Waerden function. The set of points where the Takagi-Van der Waerden function has an infinite derivative was characterized in [9].

In their paper, the first two authors studied the differentiability properties of the functions in the Generalized Takagi class when the sequence of weights w belongs to c_0. More precisely, they obtained a theorem similar to that of N. Kôno, provided that the decomposition fulfills certain additional requirements.

The aim of this paper is to provide sufficient conditions (if any) on the decomposition that guarantee that the functions of the Generalized Takagi Class are nowhere differentiable whenever $w \notin c_0$. In a sense, the results obtained in this manuscript together with those appearing in [7], allow us to provide a quite general version of Kôno's theorem for the functions belonging to the Generalized Takagi Class. Among other results, we obtain Theorem 1 below. We introduce some notation that will allow us to explain this result more precisely.

For every n, we denote by $\mathcal{F}_n$ the family of all connected components of $[0,1]\setminus D_n$, and by $\widetilde{D}_n$ the set of midpoints of all the intervals belonging to $\mathcal{F}_n$. Furthermore, we denote

$$\widetilde{D} = \bigcup_{n=1}^{\infty} \widetilde{D}_n \quad \text{and} \quad \mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n.$$

The requirements on the decomposition that appear in the following result are illuminated by those properties that arise when considering the Takagi-Van der Waerden class. In such a case, we have $\widetilde{D}_n \subset D_{n+1}$ for every n provided that r is even, and we have $\widetilde{D}_n \subset \widetilde{D}_{n+1}$ for every n provided that r is odd.

Theorem 1. *Assume that the decomposition $\mathcal{D} = (D_n)_n$ satisfies that for each $n \geq 0$ one of the following situations arises:*

(1) $\widetilde{D}_n \subset D_{n+1}$, or

(2) $I \cap D_{n+1} \neq \emptyset$ for every $I \in \mathcal{F}_n$ and $\widetilde{D}_n \subset \widetilde{D}_{n+1}$.

Then, $T_{\mathcal{D},w}$ is nowhere differentiable if and only if $w \notin c_0$.

Throughout this paper, we will provide examples to demonstrate that requirements on the decomposition, such as those mentioned in the previous theorem, are necessary for the function $T_{\mathcal{D},w}$ to be nowhere differentiable. Contrary to this, we will show that if the sequence $w = (w_n)_n$ is non-negative, the function $T_{\mathcal{D},w}$ is nowhere differentiable without any conditions on the decomposition.

We conclude this introduction with a lemma that will be frequently used. While it may be well-known, we provide a proof for the sake of completeness.

Lemma 2. *Let $I \subset \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ a function, and $x \in I$.*

(1) Let $a_n < x < b_n$ for every n. Assume that $\lim_n a_n = \lim_n b_n = x$. If f is derivable at x, then

$$\lim_n \frac{f(b_n) - f(a_n)}{b_n - a_n} = f'(x).$$

(2) Let $x < u_n < v_n$ for every n. Assume that $\lim_n v_n = x$, and $\limsup_n \frac{u_n - x}{v_n - u_n} < +\infty$. If f is right derivable at x, then

$$\lim_n \frac{f(v_n) - f(u_n)}{v_n - u_n} = f'^{+}(x).$$

(3) Let $u_n < v_n < x$ for every n. Assume that $\lim_n u_n = x$, and $\limsup_n \frac{x-v_n}{v_n-u_n} < +\infty$. If f is left derivable at x, then

$$\lim_n \frac{f(v_n)-f(u_n)}{v_n-u_n} = f'^{-}(x).$$

Proof. Let us prove (1). For every n we denote

$$r_n = \frac{f(b_n)-f(x)}{b_n-x} \quad \text{and} \quad s_n = \frac{f(a_n)-f(x)}{a_n-x}.$$

We have $\lim_n r_n = \lim_n s_n = f'(x)$. Since

$$\begin{aligned} &f(b_n)-f(a_n) = f(b_n)-f(x)+f(x)-f(a_n) \\ &= r_n(b_n-x)-s_n(a_n-x) = r_n(b_n-a_n)+(s_n-r_n)(x-a_n) \end{aligned}$$

we obtain

$$\frac{f(b_n)-f(a_n)}{b_n-a_n} = r_n + (s_n-r_n)\frac{x-a_n}{b_n-a_n}.$$

Taking limits we get the result.

Let us prove (2). For every n we denote

$$r_n = \frac{f(v_n)-f(x)}{v_n-x} \quad \text{and} \quad s_n = \frac{f(u_n)-f(x)}{u_n-x}.$$

We have $\lim_n r_n = \lim_n s_n = f'^{+}(x)$. Since

$$\begin{aligned} &f(v_n)-f(u_n) = f(v_n)-f(x)+f(x)-f(u_n) \\ &= r_n(v_n-x)-s_n(u_n-x) = r_n(v_n-u_n)+(s_n-r_n)(x-u_n) \end{aligned}$$

we obtain

$$\frac{f(v_n)-f(u_n)}{v_n-u_n} = r_n + (r_n-s_n)\frac{u_n-x}{v_n-u_n}.$$

Taking limits we get the result.

The proof of (3) is similar. □

2. Lateral derivatives on the set D

In 1936, E. G. Begle and W. L. Ayres proved that the Takagi function has cusps pointing downwards at every dyadic point of the unit interval (see [2]). Although not explicitly stated, this result was extended to the Takagi-Van der Waerden function case by J. Ferrera and J. Gómez Gil (see [6]). These results are the motivation to look for different conditions on the decomposition that guarantee that $T_{\mathcal{D},w}$ does not have finite lateral derivatives at $x \in D$ provided that $w \notin c_0$. As we will see in Example 9 below, the decomposition can behave in a manner that the corresponding generalized Takagi function is differentiable at every point of D_1.

If $x \in D$, then there exists $n_0 \in \mathbb{N}$ such that $x \in D_{n_0}$ but $x \notin D_{n_0-1}$. We are assuming that $x \in (0,1)$ since the study of $T'^{+}_{\mathcal{D},w}(0)$ and $T'^{-}_{\mathcal{D},w}(1)$ is similar. Thus, the function

$$G_{n_0-1} = \sum_{k=0}^{n_0-1} w_k g_k$$

has lateral derivatives at x given by

$$G'^{+}_{n_0-1}(x) = \sum_{k=0}^{n_0-1} w_k g'^{+}_k(x) \quad \text{and} \quad G'^{-}_{n_0-1}(x) = \sum_{k=0}^{n_0-1} w_k g'^{-}_k(x).$$

Therefore, in order to study the existence of lateral derivatives of $T_{\mathcal{D},w}$ we may assume that $x \in D_1$ and $w_0 = 0$.

Proposition 3. *Assume that the decomposition $\mathcal{D} = (D_n)_n$ satisfies that for each $n \geq 0$ one of the following situations arises:*

(1) $\widetilde{D}_n \subset D_{n+1}$, or
(2) $I \cap D_{n+1} \neq \emptyset$ for every $I \in \mathcal{F}_n$ and $\widetilde{D}_n \subset \widetilde{D}_{n+1}$.

Then, $T_{\mathcal{D},w}$ has not finite lateral derivatives at $x \in D$ provided that $w \notin c_0$.

Proof. We may assume without loss of generality that $x \in D_1$ and $w_0 = 0$. For every $n \in \mathbb{N}$ we let $y_n \in D_n$ satisfying that $(x, y_n) \in \mathcal{F}_n$. Observe that conditions (1) and (2) imply that $y_{n+1} < y_n$ for every n. It is immediate to see that

$$\frac{T_{\mathcal{D},w}(y_n) - T_{\mathcal{D},w}(x)}{y_n - x} = \sum_{k=1}^{n-1} w_k \frac{g_k(y_n)}{y_n - x} = \sum_{k=1}^{n-1} w_k$$

since $g_k(y_n) = y_n - x$ for every $k < n$. Indeed, if $g_k(y_n) = y_k - y_n$ for some $k < n$ then the midpoint of (x, y_k) belongs to (x, y_n) and consequently, we are necessarily under the second alternative. This implies that $y_{k+1} \in (x, y_n)$ which is a contradiction. Hence, if $T'^{+}_{\mathcal{D},w}(x)$ exists then the series $\sum_{k=1}^{\infty} w_k$ converges which implies $w \in c_0$. The proof for the left-hand derivative is similar. □

From now on, we will strive to obtain different kinds of conditions on the decomposition. For a given point $x \in D_1$, we will consider $y_n \in D_n$ such that $(x, y_n) \in \mathcal{F}_n$. Throughout the subsequent results, we will assume that the sequence $(y_n)_n$ is strictly decreasing. This requirement on the decomposition is very mild. For instance, it holds when we have that $I \cap D_{n+1} \neq \emptyset$ for every $I \in \mathcal{F}_n$ and for every $n \in \mathbb{N}$. It means that between two points of D_n there is at least one point belonging to D_{n+1}.

Lemma 4. *Assume that the decomposition satisfies $\alpha_{n+1} \leq \frac{\rho}{1-\rho}\alpha_n$ for every n. If $x \in D_1$, $w_0 = 0$, and $y_{n+1} < y_n$ for every n, then*

$$\frac{T_{\mathcal{D},w}(y_n) - T_{\mathcal{D},w}(x)}{y_n - x} = \sum_{k=1}^{n-2} w_k + \delta_{n-1} w_{n-1}$$

where

$$\delta_{n-1} = \begin{cases} \frac{y_{n-1} - y_n}{y_n - x} & \text{if } g_{n-1}(y_n) = y_{n-1} - y_n \\ 1 & \text{if } g_{n-1}(y_n) = y_n - x \end{cases}$$

Furthermore, we have $\delta_{n-1} \in [\rho, 1)$ provided that $g_{n-1}(y_n) = y_{n-1} - y_n$.

Proof. We have that

$$\frac{T_{\mathcal{D},w}(y_n) - T_{\mathcal{D},w}(x)}{y_n - x} = \sum_{k=1}^{n-1} w_k \frac{g_k(y_n)}{y_n - x}.$$

If $g_{n-1}(y_n) = y_n - x$ then $g_k(y_n) = y_n - x$ for every $k < n$, and hence

$$\sum_{k=1}^{n-1} w_k \frac{g_k(y_n)}{y_n - x} = \sum_{k=1}^{n-1} w_k$$

so the claim holds with $\delta_{n-1} = 1$.

Otherwise we have $g_{n-1}(y_n) = y_{n-1} - y_n$. We observe that if $g_{n-2}(y_n) = y_{n-2} - y_n$ then

$$\alpha_n \geq y_n - x > y_{n-2} - y_n = y_{n-2} - y_{n-1} + y_{n-1} - y_n \geq \rho(\alpha_{n-1} + \alpha_n)$$

which is impossible. Therefore, we obtain that $g_k(y_n) = y_n - x$ for every $k \leq n-2$ and

$$\delta_{n-1} = \frac{y_{n-1} - y_n}{y_n - x} \geq \frac{\rho\alpha_n}{\alpha_n} = \rho.$$

□

Lemma 5. *Assume that* $\alpha_{n+1} \leq \frac{\rho}{1-\rho}\alpha_n$ *for every* n*. Let* $x \in D_1$*,* $w_0 = 0$*, and* $y_{n+1} < y_n$ *for every* n*. If* $\delta_{n-1} < 1$ *and* $\delta_n < 1$ *then*

$$\frac{T_{\mathcal{D},w}(y_n) - T_{\mathcal{D},w}(y_{n+1})}{y_n - y_{n+1}} = \sum_{k=1}^{n-2} w_k + \frac{y_{n-1} - y_n - (y_{n+1} - x)}{y_n - y_{n+1}} w_{n-1} - w_n.$$

Proof. First, we have that $\delta_{n-1} < 1$ implies $g_{n-1}(y_n) = y_{n-1} - y_n$, whereas $\delta_n < 1$ implies $g_n(y_{n+1}) = y_n - y_{n+1}$. Now, we observe that if $g_{n-1}(y_{n+1}) = y_{n-1} - y_{n+1}$ then

$$\alpha_{n+1} \geq y_{n+1} - x > y_{n-1} - y_{n+1} = (y_{n-1} - y_n) + (y_n - y_{n+1}) \geq \rho(\alpha_n + \alpha_{n+1})$$

which contradicts our assumption on the decomposition. Therefore, $g_{n-1}(y_{n+1}) = y_{n+1} - x$ necessarily. Proceeding in a similar way we get that $g_{n-2}(y_n) = y_n - x$, so we deduce that $g_k(y_n) = y_n - x$ and $g_k(y_{n+1}) = y_{n+1} - x$ for every $k \leq n-2$. □

Proposition 6. *Let* $w \notin c_0$*. Assume that* $\rho > 1/2$*. If* $x \in D_1$*,* $w_0 = 0$*, and* $y_{n+1} < y_n$ *for every* n*, then* $T_{\mathcal{D},w}$ *is not right-hand differentiable at* x*.*

Proof. First, we observe that $\rho > 1/2$ implies $\alpha_{n+1} \leq \frac{\rho}{1-\rho}\alpha_n$ for every n. For the sake of contradiction, we suppose that $T_{\mathcal{D},w}$ is right-hand differentiable at x. Hence,

$$T'^{+}_{\mathcal{D},w}(x) = \lim_{n\to\infty} \frac{T_{\mathcal{D},w}(y_n) - T_{\mathcal{D},w}(x)}{y_n - x} := \lim_{n\to\infty} \Delta_n$$

and by Lemma 4 we obtain

$$0 = \lim_{n\to\infty} (\Delta_{n+1} - \Delta_n) = \lim_{n\to\infty} ((1 - \delta_{n-1})w_{n-1} + \delta_n w_n) := \lim_{n\to\infty} \eta_n.$$

For every n we have

$$|w_n| \leq \frac{1}{\delta_n}|\eta_n| + \frac{1 - \delta_{n-1}}{\delta_n}|w_{n-1}| \leq \frac{1}{\rho}|\eta_n| + \frac{1-\rho}{\rho}|w_{n-1}|. \tag{2.1}$$

There is $n_0 \in \mathbb{N}$ such that $|\eta_n| \leq \varepsilon\,\rho$ for every $n \geq n_0$, and from (2.1) we get

$$|w_{n_0+k}| \leq \varepsilon \sum_{j=0}^{k} \left(\frac{1-\rho}{\rho}\right)^j + |w_{n_0-1}| \left(\frac{1-\rho}{\rho}\right)^{k+1}$$

for every $k \geq 0$. The previous inequality together with the fact that $\rho > 1/2$ gives that $w \in c_0$, which is a contradiction. □

Proposition 7. *Let* $w \notin c_0$*. Assume that* $\alpha_{n+1} \leq \rho\alpha_n$ *for every* n*. If* $x \in D_1$*,* $w_0 = 0$*, and* $y_{n+1} < y_n$ *for every* n*, then* $T_{\mathcal{D},w}$ *is not right-hand differentiable at* x*.*

Proof. First, we observe that $\alpha_{n+1} \le \rho\alpha_n$ implies $\alpha_{n+1} \le \frac{\rho}{1-\rho}\alpha_n$ for every n. Proceeding via contradiction, let us suppose that $T_{\mathcal{D},w}$ is right-hand differentiable at x. For every n we denote

$$\Delta_n := \frac{T_{\mathcal{D},w}(y_n) - T_{\mathcal{D},w}(x)}{y_n - x} \quad \text{and} \quad \Gamma_n := \frac{T_{\mathcal{D},w}(y_n) - T_{\mathcal{D},w}(y_{n+1})}{y_n - y_{n+1}},$$

so we have

$$T'^{+}_{\mathcal{D},w}(x) = \lim_{n\to\infty} \Delta_n = \lim_{n\to\infty} \Gamma_n,$$

where the second equality follows from Lemma 2. As before, we have

$$0 = \lim_{n\to\infty} (\Delta_{n+1} - \Delta_n) = \lim_{n\to\infty} \big((1-\delta_{n-1})w_{n-1} + \delta_n w_n\big)$$

by Lemma 4, and we let

$$\eta_n := \delta_n w_n + (1-\delta_{n-1})w_{n-1}.$$

If $\delta_{n-1} = 1$ then

$$|w_n| = \frac{1}{\delta_n}|\eta_n| \le \frac{1}{\rho}|\eta_n|.$$

If $\delta_n = 1$ then

$$|w_n| \le |\eta_n| + (1-\delta_{n-1})|w_{n-1}| \le |\eta_n| + (1-\rho)|w_{n-1}|.$$

From now on, we restrict to the situation when $\delta_{n-1} < 1$ and $\delta_n < 1$. It follows that $g_{n-1}(y_n) = y_{n-1} - y_n$ and $g_n(y_{n+1}) = y_n - y_{n+1}$. Recall that

$$\delta_{n-1} = \frac{y_{n-1} - y_n}{y_n - x} \quad \text{and} \quad \delta_n = \frac{y_n - y_{n+1}}{y_{n+1} - x}.$$

If we denote $\lambda_n := \Delta_n - \Gamma_n$, we have that $\lim_n \lambda_n = 0$, and

$$\begin{aligned}
\lambda_n &= w_{n-1}\left(\delta_{n-1} - \frac{y_{n-1} - y_n - (y_{n+1} - x)}{y_n - y_{n+1}}\right) + w_n \\
&= w_{n-1}\delta_{n-1}\left(1 - \frac{y_n - x}{y_n - y_{n+1}}\right) + \frac{w_{n-1}}{\delta_n} + w_n = \frac{w_{n-1}}{\delta_n}(1-\delta_{n-1}) + w_n
\end{aligned}$$

by Lemma 4 and Lemma 5.

If $\delta_{n-1} \ge 1 - \frac{\rho}{2}$ then

$$0 < \frac{1-\delta_{n-1}}{\delta_n} \le \frac{\rho}{2\delta_n} \le \frac{1}{2},$$

and therefore $|w_n| \le |\lambda_n| + \frac{1}{2}|w_{n-1}|$.

Otherwise, we have $\delta_{n-1} \le 1 - \frac{\rho}{2}$. Since $\alpha_{n+1} \le \rho\alpha_n$ we get

$$\frac{y_{n-1} - y_n - (y_{n+1} - x)}{y_n - y_{n+1}} \ge 0,$$

and consequently, we obtain that $|w_n| \le |\lambda_n| + \delta_{n-1}|w_{n-1}| \le |\lambda_n| + (1 - \frac{\rho}{2})|w_{n-1}|$.

Finally, joining all the previous inequalities we conclude that

$$|w_n| \le \max\left\{|\lambda_n| + \frac{1}{\rho}|\eta_n|\right\} + \left(1 - \frac{\rho}{2}\right)|w_{n-1}|$$

which implies that $w \in c_0$. □

It is immediately observed that similar results hold for the left-hand derivative, so we deduce the following theorem:

Theorem 8. *Let $w \notin c_0$. Assume that the decomposition satisfies that $D_{n+1} \cap I \neq \emptyset$ for every $I \in \mathcal{F}_n$, and that either $\rho > 1/2$, or $\alpha_{n+1} \leq \rho\alpha_n$ for every n. If $x \in D$, then $T_{\mathcal{D},w}$ has not finite lateral derivatives at x.*

It is worth noting that for a given countable and dense set D of the unit interval, Proposition 1.5 of [6] mentioned above, states that it is always possible to construct a decomposition of such a set satisfying that $\alpha_{n+1} \leq \frac{1}{4}\alpha_n$ for every n.

In a sense, this was the motivation to pursue conditions on the decomposition like those appearing in Theorem 8.

The following example illustrates that a requirement as $D_{n+1} \cap I \neq \emptyset$ for every $I \in \mathcal{F}_n$ is needed in order to ensure that $T_{\mathcal{D},w}$ has not finite lateral derivatives at a point belonging to the set D.

Example 9. Let $w \notin c_0$ be defined as $w_0 = 0$, $w_{3n-2} = 1$, $w_{3n-1} = -1$ for every n, whereas $w_3 = -2^{-1}$ and $w_{3n} = 2^{-n}$ for every $n > 1$. Let D_1 be a finite subset of $[0, 1]$ such that $0, 1 \in D_1$. For every n we define $D_{3n-2} = D_{3n-1}$, and $D_{3(n+1)-2} = D_{3n} = D_{3n-1} \cup \widetilde{D}_{3n-1}$. We consider the countable and dense set $D = \cup_{n=1}^{\infty} D_n$ and its decomposition given by $\mathcal{D} = (D_n)_n$. It satisfies $\widetilde{D}_n \subset D_{n+1} \cup \widetilde{D}_{n+1}$ for every n. We have that $T'_{\mathcal{D},w}(x) = 0$ for every $x \in D_1$.

Proof. First, we observe that

$$T_{\mathcal{D},w}(z) = -\frac{1}{2} g_3(z) + \sum_{k=2}^{\infty} \frac{1}{2^k} g_{3k}(z)$$

for every $z \in [0, 1]$. From Proposition 2.15 in [7] we obtain

$$T'^{+}_{\mathcal{D},w}(x) = -\frac{1}{2} + \sum_{k=2}^{\infty} \frac{1}{2^k} = 0 \quad \text{and} \quad T'^{-}_{\mathcal{D},w}(x) = \frac{1}{2} + \sum_{k=2}^{\infty} \left(-\frac{1}{2^k}\right) = 0,$$

which gives us the result. □

3. Nowhere differentiability

We begin this section with an example that shows the necessity of a condition such as the existence of $\rho \in (0, 1]$ satisfying that $\rho\alpha_n \leq |x - y|$ for every $x, y \in D_n$ with $(x, y) \cap D_n = \emptyset$, in order to obtain positive results. For the sake of simplicity, in such an example we define the generalized Takagi function on the interval $[-1, 1]$ instead of $[0, 1]$.

Example 10. Let us consider the sets $D_0^+ = \{1\}$ and $D_1^+ = \left\{\frac{2}{3}, 1\right\}$. For every $n \in \mathbb{N}$ we define the sets

$$D_{2n}^+ = \left\{\frac{k}{2^n} \in (0, 1] : k \in \mathbb{Z}\right\} \cup D_{2n-1}^+$$

and

$$D_{2n+1}^+ = \left\{\frac{k}{2^n} - \frac{1}{3^{n+1}} \in (0, 1] : k \in \mathbb{Z}\right\} \cup D_{2n}^+.$$

For all $n \geq 0$ we also define $D_n^- = \{-x : x \in D_n^+\}$ and we consider the set $D_n = D_n^+ \cup D_n^-$. Let $w \notin c_0$ be defined as $w_{2n} = 1$ and $w_{2n+1} = -1$ for every n. Then, $T_{\mathcal{D},w}$ is differentiable at 0 and $T'_{\mathcal{D},w}(0) = 0$.

Proof. We may rewrite the function $T_{\mathcal{D},w} : [-1,1] \to \mathbb{R}$ as

$$T_{\mathcal{D},w}(x) = \sum_{n=0}^{\infty} H_n(x)$$

where $H_n(x) = w_{2n}g_{2n}(x) + w_{2n+1}g_{2n+1}(x)$. It is immediate that, for every n, $|H_n(x)| \leq 3^{-(n+1)}$ for all $x \in [-1,1]$ and

$$H_n(x) = \frac{1}{3^{n+1}} \qquad \text{for all} \qquad x \in [-\frac{1}{2^n} + \frac{1}{3^{n+1}}, \frac{1}{2^n} - \frac{1}{3^{n+1}}].$$

If $0 < |h| < \frac{1}{2}$ there exists an integer n such that $2^{-(n+1)} \leq |h| < 2^{-n}$ and as $H_k(0) = H_k(h)$ for all $0 \leq k \leq n-1$ we obtain

$$\left|\frac{T_{\mathcal{D},w}(h) - T_{\mathcal{D},w}(0)}{h}\right| = \frac{1}{|h|}\left|\sum_{k=n}^{\infty} H_k(h) - \sum_{k=n}^{\infty}\frac{1}{3^{k+1}}\right| \leq 2^{n+2}\sum_{k=n}^{\infty}\frac{1}{3^{k+1}} = 2\cdot\left(\frac{2}{3}\right)^n.$$

Letting h to zero and therefore, n to infinity, we obtain that $T'_{\mathcal{D},w}(0) = 0$. □

Remark 11. *In the previous example we have that* $0 \in \widetilde{D}$. *Furthermore, if we modify such an example by defining* $D_0^+ = \{0,1\}$, *we also have that* $T_{\mathcal{D},w}$ *is differentiable at* $0 \in D$.

For a point $x \notin D$ we let $a_n, b_n \in D_n$ such that $a_n < x < b_n$ and $(a_n,b_n) \cap D_n = \emptyset$. We denote by c_n the midpoint of the interval (a_n, b_n).

The problem in the general situation is that we cannot control the position of the midpoints c_k with respect to (a_n, b_n) for $k < n$. This forces us to require additional restrictions on the decomposition. However, they are quite mild and allow us to generalize Kono's theorem for the Takagi Class (see [17]).

Theorem 12. *Assume that the decomposition* $\mathcal{D} = (D_n)_n$ *satisfies that for each* $n \geq 0$ *one of the following situations arises:*

(1) $\widetilde{D}_n \subset D_{n+1}$, *or*

(2) $I \cap D_{n+1} \neq \emptyset$ *for every* $I \in \mathcal{F}_n$ *and* $\widetilde{D}_n \subset \widetilde{D}_{n+1}$.

If $w \notin c_0$, *then* $T_{\mathcal{D},w}$ *is nowhere differentiable.*

Proof. By Proposition 3 it is enough to prove that $T'_{\mathcal{D},w}(x)$ does not exist whenever $x \notin D$. For a given point $x \in \widetilde{D}$ we may assume without loss of generality that $x \in \widetilde{D}_1$ and consequently, $x \in \widetilde{D}_n$ for every n since it would belong to D otherwise. If this is the case, then x is the midpoint of (a_n, b_n) for every n, which implies that $a_n < a_{n+1} < x < b_{n+1} < b_n$ for every n since $I \cap D_{n+1} \neq \emptyset$ for every $I \in \mathcal{F}_n$. From

$$\frac{T_{\mathcal{D},w}(b_{n+1}) - T_{\mathcal{D},w}(b_n)}{b_{n+1} - b_n} = \sum_{k=1}^{n} w_k \frac{g_k(b_{n+1}) - g_k(b_n)}{b_{n+1} - b_n} = -\sum_{k=1}^{n} w_k$$

we deduce that if $T'_{\mathcal{D},w}(x)$ exists, then

$$-\lim_{n\to\infty}\sum_{k=1}^{n} w_k = T'_{\mathcal{D},w}(x)$$

by Lemma 2 since

$$\frac{b_{n+1} - x}{b_n - b_{n+1}} \leq \frac{\alpha_{n+1}}{\rho\alpha_{n+1}} = \frac{1}{\rho}.$$

This implies that the series $\sum_{k=1}^{\infty} w_k$ converges, which is impossible.

From now on, we assume that $x \notin \widetilde{D} \cup D$. For every n, we have that if $c_k \notin (a_n, b_n)$ for every $k < n$ then

$$\frac{T_{\mathcal{D},w}(b_n) - T_{\mathcal{D},w}(a_n)}{b_n - a_n} = \sum_{k=1}^{n-1} w_k \frac{g_k(b_n) - g_k(a_n)}{b_n - a_n} = \sum_{k=1}^{n-1} w_k g_k'(x).$$

Alternatively, if $c_k \in (a_n, b_n)$ for some $k < n$ then $c_k = c_{k+1} = \cdots = c_n$, since we are necessarily under the second alternative. It also implies that

$$a_k < a_{k+1} < \cdots < a_n < b_n < \cdots < b_{k+1} < b_k.$$

If $c_n < x$ then

$$\frac{T_{\mathcal{D},w}(b_n) - T_{\mathcal{D},w}(b_{n-1})}{b_n - b_{n-1}} = \sum_{k=1}^{n-1} w_k \frac{g_k(b_n) - g_k(b_{n-1})}{b_n - b_{n-1}} = \sum_{k=1}^{n-1} w_k g_k'(x)$$

and

$$\frac{b_n - x}{b_{n-1} - b_n} \leq \frac{\alpha_n}{\rho\alpha_n} = \frac{1}{\rho}.$$

If $x < c_n$ then

$$\frac{T_{\mathcal{D},w}(a_n) - T_{\mathcal{D},w}(a_{n-1})}{a_n - a_{n-1}} = \sum_{k=1}^{n-1} w_k \frac{g_k(a_n) - g_k(a_{n-1})}{a_n - a_{n-1}} = \sum_{k=1}^{n-1} w_k g_k'(x)$$

and

$$\frac{x - a_{n-1}}{a_n - a_{n-1}} = \frac{x - a_n}{a_n - a_{n-1}} + 1 \leq \frac{\alpha_n}{\rho\alpha_n} + 1 = \frac{1}{\rho} + 1.$$

Therefore, if $T'_{\mathcal{D},w}(x)$ exists then taking $u_n = b_n$ or a_n, and $v_n = a_n, b_{n-1}$ or a_{n-1} accordingly, and invoking Lemma 2 we obtain

$$T'_{\mathcal{D},w}(x) = \lim_n \frac{T_{\mathcal{D},w}(u_n) - T_{\mathcal{D},w}(v_n)}{u_n - v_n} = \sum_{k=1}^{\infty} w_k g_k'(x)$$

which is a contradiction. □

From now on, we will concentrate our efforts on proving Proposition 16 below. It will allow us to obtain the converse result of Theorem 12.

Let $x \notin D \cup \widetilde{D}$. If $0 < |h| < \rho\alpha_1$, then there exists $n(h) \in \mathbb{N}$ such that

$$\rho\alpha_{n(h)+1} \leq |h| < \rho\alpha_{n(h)}.$$

Furthermore, we denote

$$N_2(h) = \{n < n(h) : a_n < x < c_n < x + h \leq b_n\}.$$

The following result was proven by the first two authors (see Lemma 2.2 in [7]):

Lemma 13. *Let $w \in c_0$ and $x \notin D \cup \widetilde{D}$. If the series $\sum_{n=1}^{\infty} w_n g_n'(x)$ converges, then $T_{\mathcal{D},w}$ is differentiable at x with $T'_{\mathcal{D},w}(x) = \sum_{n=1}^{\infty} w_n g_n'(x)$ if and only if*

$$\lim_{h\to 0} \sum_{n\in N_2(h)} \left(\frac{g_n(x+h) - g_n(x)}{h} - g_n'(x) \right) w_n = 0.$$

Proposition 14. *Let $w \in c_0$. Assume that the decomposition $\mathcal{D} = (D_n)_n$ satisfies that for each $n \geq 0$ one of the following situations arises:*

(1) $\widetilde{D}_n \subset D_{n+1}$, or

(2) $I \cap D_{n+1} \neq \emptyset$ for every $I \in \mathcal{F}_n$ and $\widetilde{D}_n \subset \widetilde{D}_{n+1}$.

If $x \notin D \cup \widetilde{D}$ and the series $\sum_{n=1}^{\infty} w_n g_n'(x)$ converges, then $T_{\mathcal{D},w}$ is differentiable at x with $T'_{\mathcal{D},w}(x) = \sum_{n=1}^{\infty} w_n g_n'(x)$.

Proof. We suppose that $h > 0$ and the set $N_2(h)$ is non-empty. The case when $h < 0$ is similar. Let $n \in N_2(h)$ be. We are under one of the following scenarios:

If (1) holds, that is $\widetilde{D}_n \subset D_{n+1}$, then $c_n \in D_{n+1}$ and hence, for every $n+1 \leq k \leq n(h) - 1$ we have $x < b_k \leq c_n < x + h$, so we obtain that $k \notin N_2(h)$.

If (2) holds, that is $I \cap D_{n+1} \neq \emptyset$ for every $I \in \mathcal{F}_n$ and $\widetilde{D}_n \subset \widetilde{D}_{n+1}$, then $c_{n+1} \leq c_n$. If $n < n(h) - 1$ and $c_{n+1} < c_n$ then $c_{n+1} < b_{n+1} < c_n < x + h$, so we have $n + 1 \notin N_2(h)$.

From the previous facts, we conclude that there exist $m_1, m_2 \in \mathbb{N}$ such that $m_1 \leq m_2 \leq n(h) - 1$ and $N_2(h) = \{m_1, \dots, m_2\}$. Moreover, we have $c_{m_1} = \dots = c_{m_2}$ necessarily. Hence,

$$
\begin{aligned}
&\sum_{n \in N_2(h)} \left(\frac{g_n(x+h) - g_n(x)}{h} - g_n'(x) \right) w_n \\
&= \sum_{n=m_1}^{m_2} \left(\frac{-2(x + h - c_n)}{h} \right) w_n g_n'(x) \\
&= \frac{-2(x + h - c_{m_1})}{h} \sum_{n=m_1}^{m_2} w_n g_n'(x).
\end{aligned} \tag{3.1}
$$

Since the series $\sum_{n=1}^{\infty} w_n g_n'(x)$ converges, the first item of (3.1) tends to 0 as $h \to 0$ because $m_1 \to +\infty$ whenever $h \to 0$. Therefore, we conclude that $T_{\mathcal{D},w}$ is differentiable at x with $T'_{\mathcal{D},w}(x) = \sum_{n=1}^{\infty} w_n g_n'(x)$ by Lemma 13. □

The proof of the following result is the same as that of Lemma 2.12 in [7]. Such a lemma is proved under the assumption that the decomposition satisfies $\alpha_{n+1} \leq \frac{\rho \alpha_n}{2}$ for every n. This assumption is used only to ensure that the decomposition has the following property: every half interval of $\mathcal{F}_n$ contains an interval belonging to $\mathcal{F}_{n+1}$. It is immediately observed that such a property holds under the hypotheses on the decomposition that appear in Lemma 15 below.

Lemma 15. *Assume that the decomposition $\mathcal{D} = (D_n)_n$ satisfies that for each $n \geq 0$ one of the following situations arises:*

(1) $\widetilde{D}_n \subset D_{n+1}$, or
(2) $I \cap D_{n+1} \neq \emptyset$ for every $I \in \mathcal{F}_n$ and $\widetilde{D}_n \subset \widetilde{D}_{n+1}$.

For every not eventually constant sequence $(\eta_n)_n$, with $|\eta_n| = 1$ for every n, there is $x \notin D \cup \widetilde{D}$ such that $g'_{n-1}(x) = \eta_n$ for every n.

Proposition 16. *Assume that the decomposition $\mathcal{D} = (D_n)_n$ satisfies that for each $n \geq 0$ one of the following situations arises:*

(1) $\widetilde{D}_n \subset D_{n+1}$, or
(2) $I \cap D_{n+1} \neq \emptyset$ for every $I \in \mathcal{F}_n$ and $\widetilde{D}_n \subset \widetilde{D}_{n+1}$.

If $w \in c_0 \setminus \ell_1$, then $T_{\mathcal{D},w}$ is differentiable at an uncountable set.

Proof. Since $w \in c_0 \setminus \ell_1$, it is well-known that for every $r \in \mathbb{R}$ there exists a not eventually constant sequence $(\eta_n)_n$, with $|\eta_n| = 1$ for every n, such that

$$\sum_{n=1}^{\infty} \eta_n w_{n-1} = r$$

(see page 56 of [13] for instance). There exists $x \notin D \cup \widetilde{D}$ satisfying that $g'_{n-1}(x) = \eta_n$ for every n by Lemma 15, and we obtain that

$$T'_{\mathcal{D},w}(x) = \sum_{n=0}^{\infty} w_n g'_n(x)$$

by Proposition 14. □

The next theorem together with Theorem 12 allows us to obtain Theorem 1.

Theorem 17. *Assume that the decomposition $\mathcal{D} = (D_n)_n$ satisfies that for each $n \geq 0$ one of the following situations arises:*

(1) $\widetilde{D}_n \subset D_{n+1}$, or

(2) $I \cap D_{n+1} \neq \emptyset$ for every $I \in \mathcal{F}_n$ and $\widetilde{D}_n \subset \widetilde{D}_{n+1}$.

If $T_{\mathcal{D},w}$ is nowhere differentiable, then $w \notin c_0$.

Proof. For the sake of contradiction, suppose that $w \in c_0$. If $w \in c_0 \setminus \ell_1$ then $T_{\mathcal{D},w}$ is differentiable at an uncountable set of points by Proposition 16. Furthermore, if $w \in \ell_1$ then $T_{\mathcal{D},w}$ is Lipschitz since

$$|T_{\mathcal{D},w}(x) - T_{\mathcal{D},w}(y)| \leq \sum_{k=0}^{\infty} |w_k||g_k(x) - g_k(y)| \leq |x-y|\|w\|_1$$

for all $x, y \in [0,1]$, and consequently $T_{\mathcal{D},w}$ is differentiable almost everywhere by Rademacher's theorem. This gives us the result. □

Remark 18. *Under the hypothesis on the decomposition that appear in Theorem 17, a similar argument as that used in Proposition 3.1. of [11] allows us to easily obtain that $T_{\mathcal{D},w}$ is Lipschitz if and only if $w \in \ell_1$. In such a case, the Lipschitz norm of $T_{\mathcal{D},w}$ is given by $\|w\|_1$.*

It is worth noting that if the set D satisfies that $\widetilde{D} \subset D$, then we can construct a decomposition $\mathcal{D} = (D_n)_n$ such that $\widetilde{D}_n \subset D_{n+1}$ for every n. Furthermore, such a decomposition can be constructed so that $\rho = 1/4$ and $2\alpha_{n+1} \leq \alpha_n$ for every n. This fact yields that Theorem 1 can be applied in a large number of situations. For instance, $D = \mathbb{Q}$.

4. Non-negative sequence of weights

As we saw previously, some restrictions on the decomposition are necessary in order to guarantee that $T_{\mathcal{D},w}$ is nowhere differentiable. However, the state of affairs change drastically when considering a sequence $w = (w_n)_n$ of non-negative weights. In such a case, we do not require any restriction on the decomposition. As the reader will observe, the existence of $\rho \in (0,1]$ such that $|x-y| \geq \rho\alpha_n$ for every $x, y \in D_n$ with $(x,y) \cap D_n = \emptyset$ is not even necessary for proving the results. Furthermore, we only require that $w \notin \ell_1$ in the proof of Proposition 19 below.

Throughout the subsequent results, we study the subdifferentiability behavior of the function $T_{\mathcal{D},w}$.

Recall that for a lower semicontinuous function $f : \mathbb{R} \to \mathbb{R}$ and a point $x \in \mathbb{R}$, the subdifferential of f at x, denoted by $\partial^- f(x)$, is defined as the set of all $\xi \in \mathbb{R}$ that satisfy

$$\liminf_{h \to 0} \frac{f(x+h) - f(x) - \xi h}{|h|} \geq 0.$$

The function f is said to be subdifferentiable at x whenever $\partial^- f(x) \neq \emptyset$. It is well-known that such a subdifferential can be characterized in terms of the Dini derivatives

$$D^- f(x) = \limsup_{h\to 0^-} \frac{f(x+h)-f(x)}{h} \quad \text{and} \quad d_+ f(x) = \liminf_{h\to 0^+} \frac{f(x+h)-f(x)}{h}.$$

More precisely, the function f is subdifferentiable at x if and only if

$$D^- f(x) \leq d_+ f(x) \quad \text{and} \quad [D^- f(x), d_+ f(x)] \cap \mathbb{R} \neq \emptyset. \tag{4.1}$$

In such a case, we have $\partial^- f(x) = [D^- f(x), d_+ f(x)] \cap \mathbb{R}$.

For an upper semicontinuous function $f : \mathbb{R} \to \mathbb{R}$ and a point $x \in \mathbb{R}$, the superdifferential of f at x, denoted by $\partial^+ f(x)$, can be defined by the formula

$$\partial^+ f(x) = -\partial(-f)(x). \tag{4.2}$$

It is well-known that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x \in \mathbb{R}$ if and only if both $\partial^- f(x)$ and $\partial^+ f(x)$ are non-empty. Moreover, in this case, $\partial^- f(x) = \partial^+ f(x) = \{f'(x)\}$.

Among the properties of the subdifferential of a function $f : \mathbb{R} \to \mathbb{R}$, we must mention that the set of points with non-empty subdifferential is dense in the domain of f (see Theorem 4.21 of [5]), and the cardinality of the subdifferential of f is less than or equal to one at almost every point in the domain of f (see Theorem 4.4.3 of [12]).

In 2011, P. Góra and R. J. Stern proved that the subdifferential of the Takagi function is $\mathbb{R}$ at every dyadic point; whereas, it is empty otherwise (see [14]). In this sense, the Takagi function constitutes an extreme case for the subdifferential.

We extend this kind of subdifferentiability behavior to the function $T_{\mathcal{D},w}$.

Proposition 19. *Assume that $w \notin \ell_1$ and $w_n \geq 0$ for every n. If $x \in D$, then the function $T_{\mathcal{D},w}(z) - \zeta z$ has a local minimum at x for every $\zeta \in \mathbb{R}$. In particular, we have*

$$\partial^- T_{\mathcal{D},w}(x) = \mathbb{R}.$$

Proof. If $x \in D$, there exists $n_0 \in \mathbb{N}$ such that $x \in D_{n_0}$ and $x \notin D_k$ provided that $k < n_0$. For every $\zeta \in \mathbb{R}$ we let $n \in \mathbb{N}$ be satisfying that

$$\sum_{k=n_0}^{n} w_k > \sum_{k=1}^{n_0-1} w_k + |\zeta|.$$

If $2|h| < \text{dist}\,(x, D_n \setminus \{x\})$ then we have $g_k(x+h) = |h|$ for every $n_0 \leq k \leq n$, and consequently we obtain

$$\begin{aligned} T_{\mathcal{D},w}(x+h) - T_{\mathcal{D},w}(x) - \zeta h &\geq \sum_{k=1}^{n_0-1} w_k \left(g_k(x+h) - g_k(x)\right) \\ + \sum_{k=n_0}^{n} w_k g_k(x+h) - |\zeta||h| &\geq |h| \left[\sum_{k=n_0}^{n} w_k - \left(\sum_{k=1}^{n_0-1} w_k + |\zeta|\right)\right] \geq 0, \end{aligned}$$

which gives us the result. $\square$

Corollary 20. *Assume that $w \notin \ell_1$ and $w_n \geq 0$ for every n. If $x \in D$, then $T'^{+}_{\mathcal{D},w}(x) = +\infty$ and $T'^{-}_{\mathcal{D},w}(x) = -\infty$.*

Proposition 21. *Assume that $w \notin c_0$ and $w_n \geq 0$ for every n. If $x \notin D$, then $\partial^- T_{\mathcal{D},w}(x) = \emptyset$.*

Proof. For every n we let $a_n = \max\{z \in D_n : z < x\}$ and $b_n = \min\{z \in D_n : x < z\}$. Observe that $a_n < x < b_n$. Thus,

$$d_+ T_{\mathcal{D},w}(x) \leq \liminf_n \frac{T_{\mathcal{D},w}(b_n) - T_{\mathcal{D},w}(x)}{b_n - x} \leq \liminf_n \sum_{k=1}^n w_k \frac{g_k(b_n) - g_k(x)}{b_n - x}$$
$$\leq \liminf_n \sum_{k=1}^n w_k g_k'(x),$$

since we have

$$\frac{g_k(b_n) - g_k(x)}{b_n - x} = g_k'(x)$$

provided that $g_k'(x) = -1$. We also obtain

$$D^- T_{\mathcal{D},w}(x) \geq \limsup_n \frac{T_{\mathcal{D},w}(a_n) - T_{\mathcal{D},w}(x)}{a_n - x} \geq \limsup_n \sum_{k=1}^n w_k \frac{g_k(a_n) - g_k(x)}{a_n - x}$$
$$\geq \limsup_n \sum_{k=1}^n w_k g_k'(x)$$

since

$$\frac{g_k(a_n) - g_k(x)}{a_n - x} = g_k'(x)$$

provided that $g_k'(x) = 1$.

Therefore, if $\partial^- T_{\mathcal{D},w}(x) \neq \emptyset$, by (4.1) we get

$$\limsup_n \sum_{k=1}^n w_k g_k'(x) \leq D^- T_w(x) \leq d_+ T_w(x) \leq \liminf_n \sum_{k=1}^n w_k g_k'(x),$$

and hence the series $\sum_{k=1}^\infty w_k g_k'(x)$ converges, which is impossible since $w \notin c_0$. □

Corollary 22. *Assume that $w \notin c_0$ and $w_n \geq 0$ for every n. Then, $T_{\mathcal{D},w}$ is nowhere differentiable.*

The following example illustrates that such an extreme subdifferentiability behavior may disappear when considering an arbitrary sequence of weights.

Example 23. Let us consider the function $f : [0,1] \to \mathbb{R}$ defined as $f(x) = -T(x)$ where T denotes the Takagi function (1.1). From Proposition 19, we have that if $x \in D$ then $\partial^- T(x) = \mathbb{R}$, and by (4.2) we obtain

$$\partial^+ f(x) = -\partial^-(-f)(x) = -\partial^- T(x) = \mathbb{R},$$

so we get $\partial^- f(x) = \emptyset$ since the Takagi function is nowhere differentiable. The superdifferential of the Takagi function has been recently studied in [8] and [10]. As a consequence of Theorem 4.5 in [10], we have that $\partial^+ T(1/3) = [0,1]$ and by (4.2) again, we get

$$\partial^- f(1/3) = -\partial^+ T(1/3) = [-1, 0].$$

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