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Monetary policy rules: An approach based on the theory of chaos control

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ABSTRACT

This article explores the relationship between Taylor rules for monetary policy and those derived from chaos control methods. A similar structure of both rule types would theoretically support the stabilizing role of the Taylor rule for the control of inflation, which until now has been based on an empirical framework. This link between monetary policy and chaos control rules is illustrated using the OGY method of chaos control, resulting in a control rule that is applied to a monetary model that presents chaotic solutions and becomes stable at an objective equilibrium point with a stable inflation rate.

1. Introduction

Monetary policy rules are used by central banks to conduct monetary policy within a strategy that sets the objectives to be achieved in terms of inflation, unemployment, or economic growth. The economic authorities use rules as a guide for driving their policy instruments according to the evolution of the deviation of the objective variables from their targets or desired levels. The rules provide a clear specification of the actions that central banks must follow in order to achieve their macroeconomic objectives, thereby reducing the uncertainty caused by the discretionary actions of the economic authorities and ensuring a greater certainty for economic agents as to the conduct of monetary policy.

The Taylor rule [1] is one of the most studied and debated rules in monetary policy literature since it has successfully described the stabilizing policy followed by many countries to control inflation. In fact, the Taylor rule has acquired a normative significance for the establishment of interest rates and has been adopted as a reference by an increasing number of monetary authorities. However, the rule lacks a precise theoretical foundation since it was developed empirically from the follow-up to the evolution of the US federal funds rate between 1984 and 1992 [2].

Although there is considerable literature in which Chaos Theory is applied to Economics for the theoretical construction of chaotic models [3], it is less frequent to find contributions showing that chaotic behavior in economics models can be controlled. Chaos control theory studies the design of intervention rules to eliminate chaotic behavior. This control theory of chaotic systems can provide a rigorous foundation for Taylor's monetary policy rules. In fact, there is a great similarity between the structure and specification of these monetary policy rules and the control rules arising from the chaos control theory to stabilize chaotic evolutions. In this paper we aim to link the study of chaotic economic dynamic systems with chaos control and monetary policy rules.

In other contributions [4,5] the Taylor's rules have been used as the feedback factor that can give rise to chaotic behavior in economic models. Our approach here is radically different. We propose that these types of monetary policy rules are indeed

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effective in stabilizing and eliminating chaotic behavior. We propose that the power as stabilizers of the economy of these political rules arises precisely from their relationship and foundation in chaos control theory. We also suggest that the link between Taylor's policy rules and those for controlling chaos could be explaining the paradox of chaos in Economics [6]. This paradox states that it is relatively simple to build highly plausible and economically credible theoretical models that give rise to chaotic behavior; but in turn, it is very complicated to find strong evidence of chaotic behavior in real economic time series, why? The successful use of monetary policy rules would be stabilizing the economy, and eliminating, consequently, the chaotic behavior from economic time series. Therefore, it is the policy rules themselves that prevent the appearance of chaotic behavior, thus making it impossible to detect chaotic behaviors in economic time series.

The article is organized as follow. The second section addresses different aspects of the monetary policy rules, especially the Taylor rule. The third section links the control theory to the monetary policy rules. The fourth section describes the OGY method for the control of chaos. The fifth section presents a dynamic model for a monetary economy, and the sixth section establishes the control rule of this model using the OGY method to stabilize the inflation and unemployment rates at the desired values. Finally, seventh section presents the main conclusions about the relationship between chaos control and monetary policy rules.

2. Monetary policy rules: The Taylor rule

Since the pioneering works of the nineteenth century [7,8], or the later ones [9–11], many contributions have already advocated the application of policy or action rules in the field of Economics. In fact, in the current strategy of inflation targeting, most of the economies have abandoned discretionary monetary policy and instead have based their monetary policy decisions on the flexible rules that automatically govern the actions and decisions of the monetary authorities.

According to Taylor [12], a monetary policy rule is a contingent plan that specifies, as clearly as possible, the circumstances under which a central bank must modify monetary policy instruments. McCallum [13] defines a policy rule as a formula that specifies the adjustments that must be made to a policy instrument to maintain a target variable near its specified goal. A rule can then be considered the strategy followed by the monetary authorities to keep their target variables stable within specified values.

There are two priority objectives of the short-term economic policy [14]. On the one hand, the objective of inflation, that is, to maintain the increase in prices of goods and services at minimum rates compatible with the proper functioning of the economy. On the other hand, the objective of full employment, or reduction of the percentage of unemployed workers, which can also be expressed in terms of potential production, understood as the level of production of goods and services that guarantees full employment for all workers and the whole labor force in a country. Based on these two objectives, a monetary policy rule can be defined as that criterion behind the decision of the monetary authorities aimed to reduce the deviations of the observed inflation with respect to the target inflation and the deviations of the level of production of goods and services with respect to the level of potential production or full employment (output gap).

In general, there are two types of economic policy rules [15]. First, there are the fixed rules (without feedback or open-loop), which specify, at the beginning of the planning period, the current and future values that the monetary policy must follow, regardless of the disturbances or changes that may affect the economy [16]. An example of a fixed rule is *Friedman's rule*, which consists in a constant growth rate of the money supply compatible with the inflation target. A growth rate that must remain fixed over time regardless of the perturbations or disruptions that might affect the evolution of the economy [17]. Secondly, the flexible rules (contingent rules, conditional rules, closed-loop or with feedback) that allow monetary policy to take into account new information in order to adapt and react to any eventual perturbation or disturbances that might occur in the economy.

It is precisely in this context of flexible rules that the *Taylor rule* arises [18], formulated as a flexible monetary policy rule that uses feedback between the short-term interest rate (the policy instrument) and the value assumed over time by the objective variables of the monetary policy (inflation and unemployment or production). Thus, the Taylor rule establishes a reaction function for the short-term interest rate established by the monetary authority in its open market operations. More specifically, this rule proposes that central banks adjust the short-term interest rate according to the deviation suffered by the inflation rate and the level of production of their target values:

$$i_t = r^* + \alpha(\pi_t - \pi^*) + \beta\left(Y_t - Y^*\right) \tag{1}$$

where: i_t is the short-term nominal interest rate, r^* is the real equilibrium interest rate, π^* is the inflation target or objective established by the central bank, π_t is the current or effective inflation rate in the instant t, $\left(Y_t - Y^*\right)$ is the output or production gap (deviation from actual production Y_t with regard to its potential level Y^*), and α and β are positive parameters that measure the magnitude of the response of the monetary policy instrument i_t to the inflation $(\pi_t - \pi^*)$ and production gap $\left(Y_t - Y^*\right)$.

This formulation of the rules of political action recalls in some way those already used in the late 1950s in economic policy decision models with the first mathematical formulation of the decision problem by Henri Theil [19]. These decision models established the economic policy as the solution of a problem that minimize the deviations of the objectives variables (inflation, economic growth, unemployment…) with respect to the values considered optimal from a social point of view.

The Taylor rule must contain two important elements to achieve economic stabilization [20]. On the one hand, the so-called Taylor principle that establishes $\alpha > 1$ in order to achieve inflation stability [21,22] and as a necessary condition for the uniqueness and stability of the equilibrium point [23]; and on the other hand, $\beta > 0$, or principle of "leans against the wind", cooling the economy when production exceeds its potential level and stimulating it when it is below this level. In short, the Taylor rule recommends that if production and inflation exceed the objective, the monetary authority should raise the short-term nominal interest rate. This will increase the real interest rate and consequently reduce the aggregate demand for goods and services, which

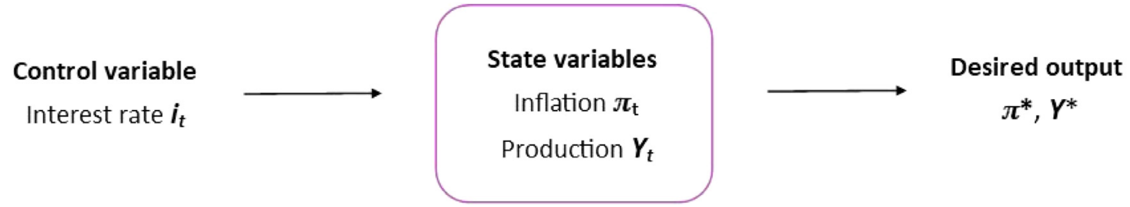

Fig. 1. The overall control scheme of an inflation-production system. The state variables describe the state of the economy, and the control variables are modified directly by policymakers to lead the state variables to behave as planned by the policy makers.

will ultimately reduce production levels and inflationary pressures. On the contrary, when the product and the inflation rate are below objectives, the Taylor rule states that monetary authorities should reduce the nominal interest rate. Through a transfer mechanism like the previous one, this would reduce the real interest rate and thereby stimulate the aggregate demand for goods and services, increasing production and inflation. Finally, if the inflation rate is equal to its objective and the output gap is equal to zero, then the short-term nominal interest rate must be equal to the real equilibrium interest rate r^*, which will be precisely that which is compatible with the equilibrium between inflation and the production of goods and services [24].

Monetary policy of most developed economies, at least since the mid-1960s, is well represented empirically by the Taylor rule [18]. In next sections we place this normative monetary rule in the general context of the dynamic system control rules. More specifically, we want to show that the Taylor rule can be the result of the application of chaos control mechanisms on monetarist models of inflation and unemployment dynamics, thus giving this Taylor rule a foundation in terms of a Control Theory for chaotic dynamic systems.

3. Control rules and macroeconomic stabilization

As already mentioned, there are two main objectives of the short-term or stabilizing economic policy: inflation and potential output (or full employment). A monetary policy rule can be specified as the stabilization of the inflation π_t and production Y_t targets at their predetermined values π^* and Y^* through adjustments or modifications in the monetary control i_t instrument or short-term nominal interest rate[1] (Fig. 1).

We can represent, then, the problem of monetary policy regulation as that which seeks the stabilization of inflation and production in its target values:

$$\lim_{t\to\infty} \pi_t = \pi^* \tag{2}$$

$$\lim_{t\to\infty} Y_t = Y^* \tag{3}$$

In the context of the control of chaotic dynamic systems, applying a stabilization policy means leading the economy towards a desired equilibrium values π^*, Y^*. As stated by Barnett and He [25] the concept of "economic stabilization policy" implicitly assumes that the dynamics of the economy is unstable in the absence of the application of economic policy. In general, stabilization policy means the actions of policymakers to mitigate or eliminate unwanted fluctuations in the economy, which might be caused by factors that are exogenous or endogenous to the system.

These stabilization actions are also designed as automatic rules of action, especially in the case of monetary policy, such as flexible Taylor type rules of monetary policy. The stabilization capacity of these monetary policy rules, from a control theory perspective, emerge from the fact that they are negative feedback rules (Fig. 2): the policy maker will look for the controlled variable x_t to take the desired value x_t^* by adjusting the control variable i_t; this variable should be reduced (increased) if the objective variable x_t is below (above) its target value x_t^* in the previous period.

As mentioned earlier, the Taylor rule has been implemented in many countries under a direct inflation targeting regime due to its ability to stabilize inflation and production around equilibrium objectives. The Taylor rule in Eq. (1), employs feedback between the objective variables (π_t, Y_t) and the control instrument (i_t) and can be represented according to Fig. 3:

The design of the specific policy rule to be used by the monetary authorities requires the choice of values for the parameters that moderate the feedback between the objectives and the control instrument. That is, the solution to the problem of control or stabilization of π_t and Y_t at its desired values π^* and Y^* will imply finding a specific value for the feedback parameters *K1* and *K2* that leads to achieving the inflation and employment or production objectives determined. This control objective will involve the design of a closed loop control law of the form:

$$i_t = -K_1\pi_t - K_2 Y_t \tag{4}$$

We will use the OGY method proposed by Ott, Grebogi, and Yorke [26] in next sections to obtain the values of parameters *K1* and *K2*, that ultimately determine the intensity and direction of the monetary policy intervention and decision rule to be followed by policy makers.

[1] Alternatively, and as seen below, the amount of money could be considered as a monetary policy instrument, since both instruments are related through the money market.

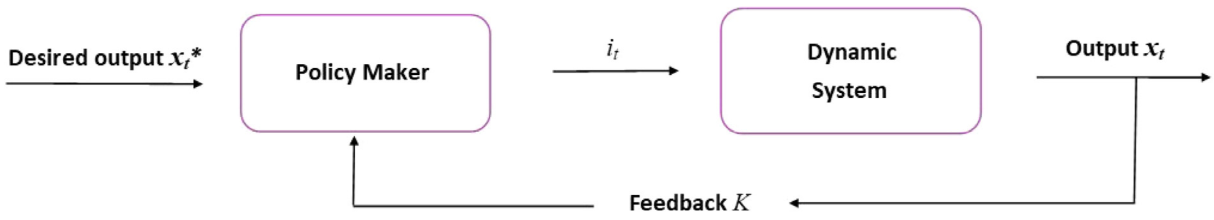

Fig. 2. Negative feedback system in a monetary policy rule. The policy maker will seek that the controlled variable x_t take the desired value x_t^* by adjusting the control variable i_t.

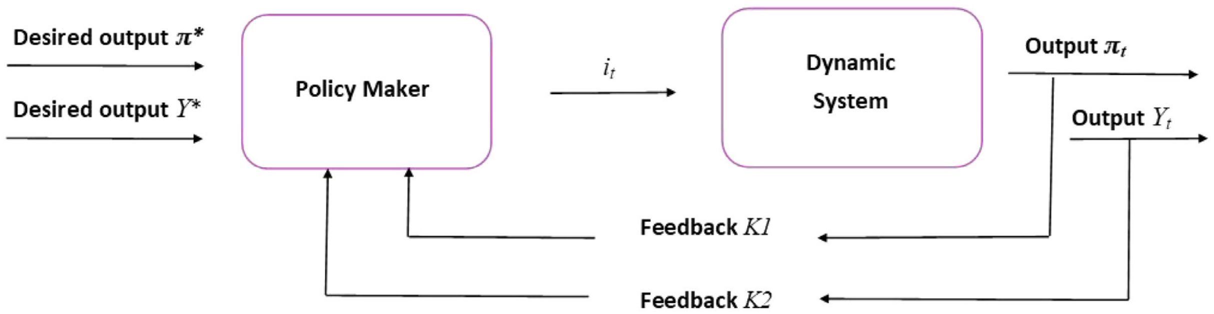

Fig. 3. Negative feedback system in the Taylor rule, interest rate as an instrument to control or stabilizing inflation and output.

4. Chaos control using the OGY method

The OGY method employs local linear feedback to stabilize the unstable periodic orbits of a chaotic discrete system [27]. The type of equilibrium in which a chaotic dynamic system is stabilized can be modified through the application of the control law derived from this method. The basis of the method is to make slight variations in the values of the system parameters to be controlled to change the dynamics of the system, that is, eliminating the irregularity of chaotic solutions by stabilizing irregular behaviors (when they are unwanted). This requires removing the system from the strange attractor and drive it to some periodic equilibria (fixed points or limit cycles) and keeping the system stabilized in those periodic equilibria.

Below is a summary of the well-known OGY method and the result of its control equations. Be the discrete dynamic system:

$$\boldsymbol{x}(t+1) = \boldsymbol{F}(\boldsymbol{x}(t), \boldsymbol{\mu}(t)) \tag{5}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the vector of state variables, $\boldsymbol{\mu}\epsilon\mathbb{R}^p$ are the parameters that can be exogenously controlled, and $\boldsymbol{F}$ is a smooth function. The state of the system depends on its previous state and the parameter vector value $\boldsymbol{\mu}$. Suppose without loss of generality, that a single control parameter is available, that is, $p = 1$ and that the objective is targeted as an unstable fixed point $\boldsymbol{x}^*$. The aim of OGY method is the design of a rule to control $\boldsymbol{x}$ through variation of the $\boldsymbol{\mu}$ parameter when it is quite close to its nominal value $\boldsymbol{\mu}^{\boldsymbol{0}}$ and whose variation will be restricted within a range $\delta \ll 1$:

$$\left|\mu - \mu^0\right| < \delta$$

The OGY procedure starts from linearly approximating equation (5) around the unstable fixed point $(\boldsymbol{x}^*, \mu^0)$:

$$\boldsymbol{x}(t+1) - x^* = \boldsymbol{A}\left[\boldsymbol{x}(t) - \boldsymbol{x}^*\right] + \boldsymbol{B}(\mu(t) - \mu^{\boldsymbol{0}}) \tag{6}$$

where $\boldsymbol{A}$ is the Jacobian matrix $n \times n$ and $\boldsymbol{B}$ is a $1 \times n$-dimensional vector of derivatives with respect to the μ parameter, and $\boldsymbol{A}$ and $\boldsymbol{B}$ are evaluated at the fixed point $(\boldsymbol{x}^*, \mu^0)$:

$$\boldsymbol{A} = D_x F(x, \mu)\big|_{x=x^*, \mu=\mu^0}$$
$$\boldsymbol{B} = D_p F(x, \mu)\Big|_{x=x^*, \mu=\mu^0}$$

The control is entered into the system assuming that the control parameter $\boldsymbol{\mu}$ is a linear function of the variable $\boldsymbol{x}(t)$:

$$\mu - \mu^0 = -\boldsymbol{K}^{\boldsymbol{T}}\left[\boldsymbol{x}(t) - \boldsymbol{x}^*\right] \tag{7}$$

The expression (7) is the **control law**, where $\boldsymbol{K}^{\boldsymbol{T}}$ is a $1 \times n$-dimensional matrix called the *gains matrix*, which reflects the sensitivity of the intervention or control of the system with respect to its deviations from the fixed point. Solving the system control problem mentioned in rule (7) consists mainly of finding this gains matrix $\boldsymbol{K}^{\boldsymbol{T}}$ that makes the fixed point (x^*, μ^0) stable.

Policy rules, such as the Taylor rule, are very similar to those of chaos control. Note that this structure of the control law (7) is the same structure as that specified by the Taylor law. In fact, if we remember the specification of the rule (1), the monetary policy feedback parameters α and β would form the gains matrix ($K1$ and $K2$) of the Taylor rule (1):

$$i_t - r^* = \alpha(\pi_t - \pi^*) + \beta\left(Y_t - Y^*\right)$$

As mentioned above, the basic idea of the control of chaotic systems, expression (7), is that by applying small linear disturbances on the system parameter (μ), the stabilization of a desired stable fixed point ($\boldsymbol{x}^*$) is achieved. This task is also behind the Taylor rule: economic authorities act by adjusting their monetary policy instrument, the nominal short-term interest rate (i_t), in order to stabilize inflation and income at the desired fixed point (π^*, Y^*).

The control procedure will consist of activating the law (7) when the trajectory $\boldsymbol{x}(t)$ approaches a small neighborhood of the chosen periodic orbit $\boldsymbol{x}^*$, that is, when $\left[\boldsymbol{x}(t) - \boldsymbol{x}^*\right] \ll 1$. On the contrary, when $\boldsymbol{x}(t)$ is not sufficiently close to its objective $\boldsymbol{x}^*$, the system trajectory will evolve according to the nominal value of the parameter μ^0. Therefore, the following control law is introduced into the dynamic system to stabilize the chaotic trajectory in the chosen periodic orbit:

$$\begin{cases} \mu = \mu^0 & , if \left|x(t) - \boldsymbol{x}^*\right| > \delta_0 \\ \mu = \mu^0 - \boldsymbol{K}^T\left[x(t) - \boldsymbol{x}^*\right] & , if \left|x(t) - \boldsymbol{x}^*\right| \leq \delta_0 \end{cases}$$

Assuming that we are in the neighborhood of the fixed point and that the control law is, therefore, activated, and by replacing (7) with (6), the local linear approximation of the system would be:

$$\boldsymbol{x}(t+1) - \boldsymbol{x}^* = \left(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}^T\right)\left[\boldsymbol{x}(t) - \boldsymbol{x}^*\right] \tag{8}$$

The fixed point chosen will be stable at (8) if the matrix ($\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}^T$) is asymptotically stable, that is, if all the eigenvalues of the matrix ($\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}^T$) have a smaller modulus than the unit. The control of system (5), therefore, requires finding the gain matrix $\boldsymbol{K}^T$ in the control rule in (7), which implies making the fixed point (x^*, μ^0) stable in (8).[2]

5. A chaotic system for the dynamics of inflation and unemployment in a monetary economy

Next section shows how the chaos control techniques described above can correct undesired economic fluctuations. These chaos control methodologies, such as the OGY, have already been used in economics to propose intervention rules for exchange rate and financial markets [30,31]. In this section we show an illustrative example where the monetary policy rules can be used to stabilize the evolution of the inflation and unemployment. We will not consider in our example any problem related to the design, articulation, or implementation of monetary policies (such as the problem of uncertainty or the internal and external policy lags), or any derived from the lack of credibility and reputation of the monetary authorities. This simplification is justified in a clear presentation of what we understand is the true contribution of this article, that is, to show how the application of chaos control techniques in the design of monetary policy intervention rules enables the stabilization of dynamic equilibrium, which would be unstable in the absence of control by policy makers. Therefore, in our example, without loss of generality, we will use a simple monetary hyperinflation model proposed by Soliman [32] that represents the dynamics between inflation and unemployment, and that allows the presence of chaotic behaviors for certain values of the parameters.

The dynamic system must be non-linear for the presence of chaotic behaviors. The monetary model of Soliman incorporates non-linearity through the Phillips curve, which shows the inverse and non-linear relationship between inflation and unemployment. This type of non-linear relationship has been widely discussed in economic literature, and theoretically justified by the possibility of decreasing marginal returns by reducing unemployment [33], although there is no consensus on the functional form it should take [34]. The hyperinflation model presented by Soliman also incorporates a money market and its relationship with the aggregate demand for goods and services, income, and employment through a demand for money for transaction purposes:

$$\begin{aligned} &\pi_t = f\left(u_t\right) + a\pi_t^e, \quad 0 \leq a \leq 1, \quad \frac{df}{du} < 0 \\ &\pi_{t+1}^e = \pi_t^e + c\left(\pi_t - \pi_t^e\right), \quad 0 \leq c \leq 1 \\ &u_{t+1} = -b\left(m - \pi_t\right) + u_t, \quad b > 0 \\ &f\left(u\right) = \alpha_1 + \frac{\alpha_2}{u} + \frac{\alpha_3}{u^2} \end{aligned} \tag{9}$$

where π_t and π_t^e are the current and future inflation rates expected in the period t; and u_t is the level of unemployment in the period t. The non-linear relationship between inflation and unemployment (the Phillips curve) is determined by the function $f\left(u_t\right)$. The parameter a represents the degree to which inflationary expectations are incorporated into current inflation. The second equation represents the mechanism used to form inflationary expectations; in this case, adaptive expectations, where the parameter c shows the degree of correction of errors in the prediction of current inflation. The third equation incorporates the money market where the demand for money depends on the level of income, so that the growth rate of the money supply, m, directly influences production and employment through changes induced in the demand for goods and services. More specifically, this equation assumes that an increase in the real amount of money ($m - \pi_t$) increases the level of production and consequently reduces unemployment rates, where b represents the elasticity of the variation in unemployment ($(u_{t+1} - u_t)$) with respect to the growth of the real money supply.

[2] The *pole location* technique can be used to calculate the gain matrix so that the fixed point is stable at (8) [28,29].

In this model, the endogenous variables are inflation and unemployment rates, and the money growth rate m is considered an exogenous factor in economic dynamics, that is determined or regulated by the monetary authorities through their monetary policy. In this way, following the tradition of the monetarist models, we will use the money supply as an instrument to control the entire system through the monetary policy. Note that unlike the traditional formulation of the Taylor rule (1), in this case, the control instrument is not the nominal interest rate but the money supply[3] and the policy objective is not established in terms of production but in terms of employment, so the Taylor rule or monetary authorities' action (1) should now be established as:

$$m_t = \overline{m} - k_1 \left(\pi_t^e - \pi^*\right) + k_2(u_t - u^*) \tag{10}$$

being m_t the growth of the money at time t, which is considered as the policy or control instrument of the Central Bank. On the one hand, π^* and u^* are the objectives of expected inflation and unemployment that are to be achieved and stabilized through the intervention of the monetary authorities. On the other, $\overline{m}$ will be the growth rate of the fixed money supply compatible with the inflation and unemployment objectives, that is, monetary growth in the absence of intervention, when the control rule is not activated since there are no deviations in either the inflation or the unemployment objectives.

We will now examine how the model works and in the next section we analyze how the Taylor rule (10) can be reached by applying the OGY control method when the model is in chaotic regime. We can reduce the model (9) as a 2-dimensional dynamic system, with two state variables π^e and u:

$$\begin{aligned} \boldsymbol{\pi}^e_{t+1} &= \boldsymbol{G}\left(\boldsymbol{\pi}^e_t, \boldsymbol{u}_t\right) = \boldsymbol{c}\,\boldsymbol{f}\left(\boldsymbol{u}_t\right) + (\mathbf{1} - \boldsymbol{c}\,(\mathbf{1} - \boldsymbol{a}))\boldsymbol{\pi}^e{}_t \\ \boldsymbol{u}_{t+1} &= \boldsymbol{H}\left(\boldsymbol{\pi}^e_t, \boldsymbol{u}_t\right) = -\boldsymbol{bm} + \boldsymbol{bf}\left(\boldsymbol{u}_t\right) + \boldsymbol{b}\ \boldsymbol{a}\boldsymbol{\pi}^e{}_t + \boldsymbol{u}_t \end{aligned} \tag{11}$$

with

$$f(u) = \alpha_1 + \frac{\alpha_2}{u} + \frac{\alpha_3}{u^2}$$

Let us start by analyzing the steady-state equilibrium points of the system (11). These balances are defined precisely as the stationary states of the system, their fixed points that, once reached, keep the state variables steady, that is, $\pi^e_{t+1} = \pi^e_t$ and $u_{t+1} = u_t$. This fixed-point will correspond, then, to constant levels of inflation and unemployment in (11):

$$\begin{aligned} \pi^e_E &= m \\ f(u) &= m(1-a) \end{aligned} \tag{12}$$

or:

$$u_E = \frac{-\alpha_2 \pm \sqrt{\alpha_2^2 - 4(\alpha_1 - m(1-a))\alpha_3}}{2(\alpha_1 + (a-1)\,m)}$$

This steady state will depend on the structural parameters of the model, including the growth rate of the money supply m, which will be directly regulated by the Central Bank. In fact, according to (12) in stationary equilibrium, inflation rates will be determined directly by the growth rate of the money supply ($\pi^e_E = m$). Monetary policy will also have a nonlinear influence in the equilibrium unemployment rate (u_E) (Fig. 4).

We can analyze now the stability of this steady state and the global response of the model to variations in monetary policy. That is, we can observe the solution of the model (11) when only the control parameter m varies, keeping the other parameters of the model fixed. To perform this exercise of comparative dynamics, we will establish $f(u)$ at values $\alpha_1 = -1.14, \alpha_2 = 5.53, \alpha_3 = 3.68$, those estimated by Lipsey [33] for the relationship between inflation and unemployment. The rest of the parameters will be fixed, without loss of generality, at $a = 0.1, b = 0.1$ and $c = 0.1$

The asymptotic behavior analysis of the system can be shown through the *bifurcation diagrams* (Figs. 5 and 6). These diagrams represent the values (π^e, u) to which the system converges in the long term for the different values of the growth rate of the money supply m. As can be seen, by starting with initial values of $m = 11$, the system converges to the fixed point (12) $(\pi^e_E, u_E) = (11, 0.8797)$.[4] From there, the value of this fixed-point changes as m increases, according to (12), that is, inflation will increase, and the unemployment rate will decrease with m (Fig. 4).

A characteristic property of the non-linear dynamic system (11), which the bifurcation diagram also shows, is that as the money creation, m, increases the equilibrium point $\left(\pi^e_E, u_E\right)$ not only changes in value, but also loses its stability. Initially, the fixed point, although changing value with m, remains stable (the system converges to a single point for each value of m). However, from a certain critical value of m, a first bifurcation occurs, that is, the fixed point (π^e_E, u_E) loses its stability, and the system, instead of

[3] Both control variables, however, are related through the money market and the demand for real balances depending on interest rates, so we can actually consider that the choice of money supply or interest rates as an intervention variable does not affect the design of a monetary policy rule. When the monetary policy instrument is the interest rate, the money supply is determined endogenously in the money market to guarantee the balance between demand and money supply precisely for the interest rate established by the policy rules. When the money supply is the intervention variable, the interest rates are those that will be determined endogenously in the model to reach the balance between money supply and demand. In this simple example, the demand for money has not been made on the basis of interest rates, so we cannot use them as instruments of monetary policy control since they are not determined in the money market. For the relationship between Monetary Rules based on interest rates or money supply see [14] chapters 5, 9 and 10.

[4] Note that in fact in (12) two fixed points are defined, but only one of them is asymptotically stable, and that is why initially only that point is represented for each m on the bifurcation diagram. The stability of the fixed points (12) will be determined by the eigenvalues of the Jacobian matrix associated to the system evaluated at the fixed points. For stability to exist, both eigenvalues must be, in module, smaller than the unit.

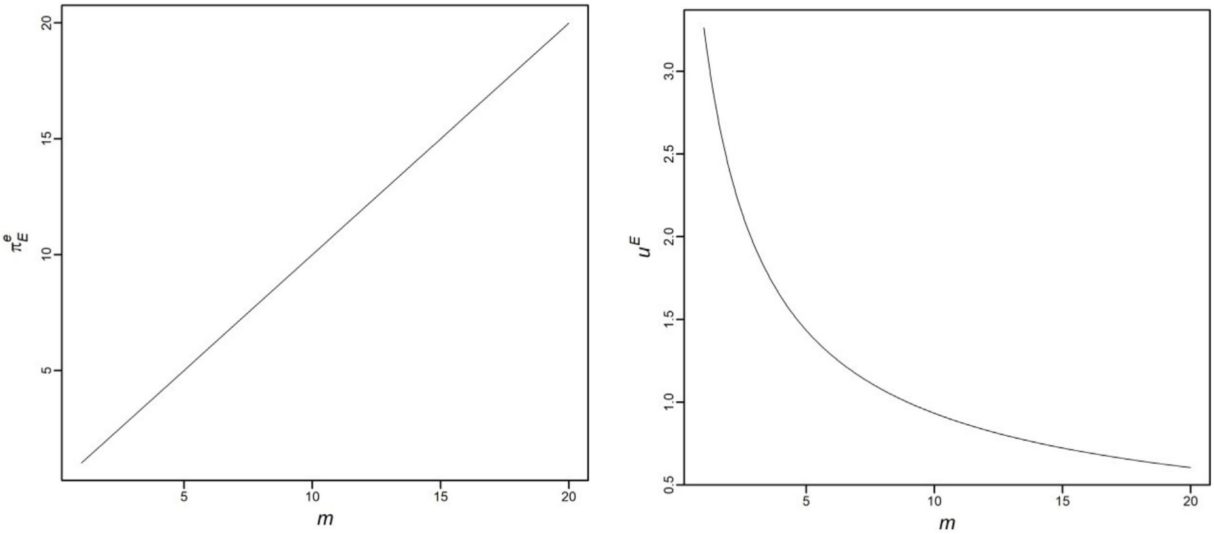

Fig. 4. Effect of m, growth rate of the money supply, on the steady-state (π_E^e, u_E), with $1 < m < 20$ and $\alpha_1 = -1.14, \alpha_2 = 5.53, \alpha_3 = 3.68,\ a = 0.1, b = 0.1, c = 0.1$.

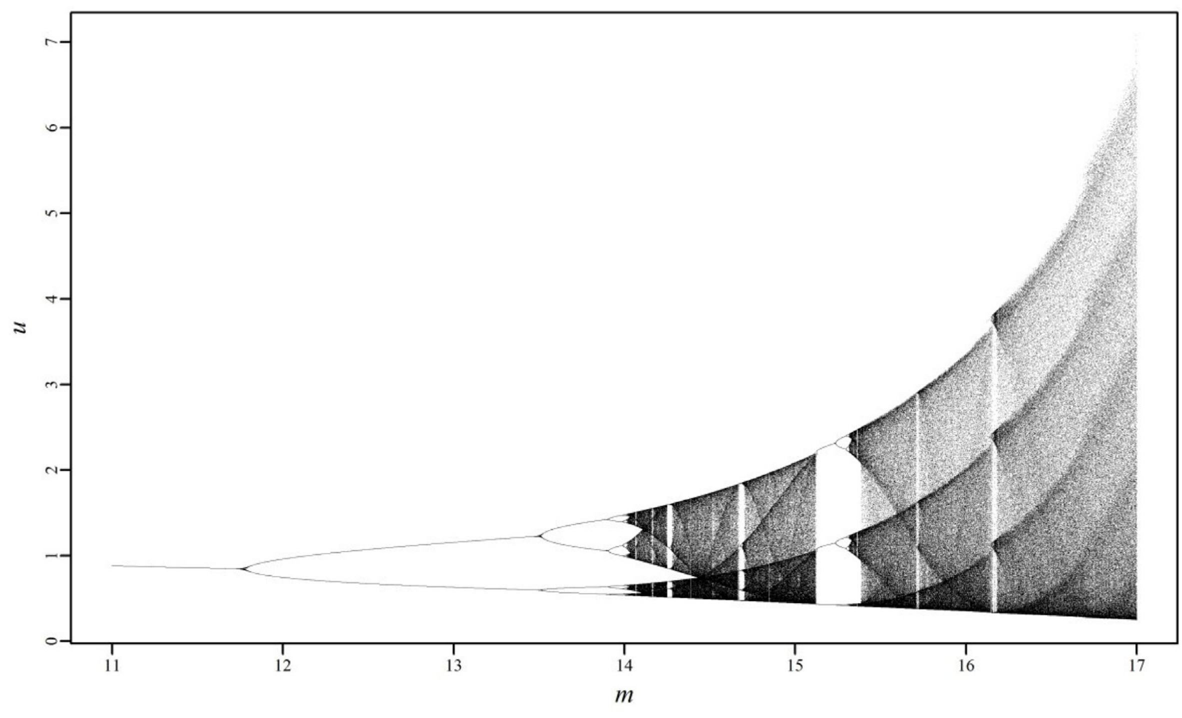

Fig. 5. Bifurcation diagram of the unemployment rate u_E under (11), with $11 < m < 17$ and $\alpha_1 = -1.14, \alpha_2 = 5.53, \alpha_3 = 3.68,\ a = 0.1, b = 0.1, c = 0.1, \pi(0) = 0.5, \pi^e(0) = 11, u(0) = 0.8797$.

ending in a single point, converges towards two points or two states that it will alternate between sequentially. The new *dynamic equilibrium* of the system will no longer be a steady state, but a period 2 *limit cycle*. As m continues increasing the period 2 cycle loses its stability, bifurcating the system solution towards a period 4 limit cycle, and then to a 16, 32... period cycle. This cascade of bifurcations continues to increase until, from a new critical value of m^c, the system converges to an aperiodic limit cycle formed by a bounded but infinite set of points or states (the shaded areas of the bifurcation diagram). The system has then converged to a *strange aperiodic attractor*, to a *chaotic dynamic equilibrium*.

Chaotic long-term equilibrium behaviors are characterized by high sensitivity to initial conditions, that is, although the system converges to a bounded region within the phase space, the evolution within this attractor is highly unstable. This instability can be

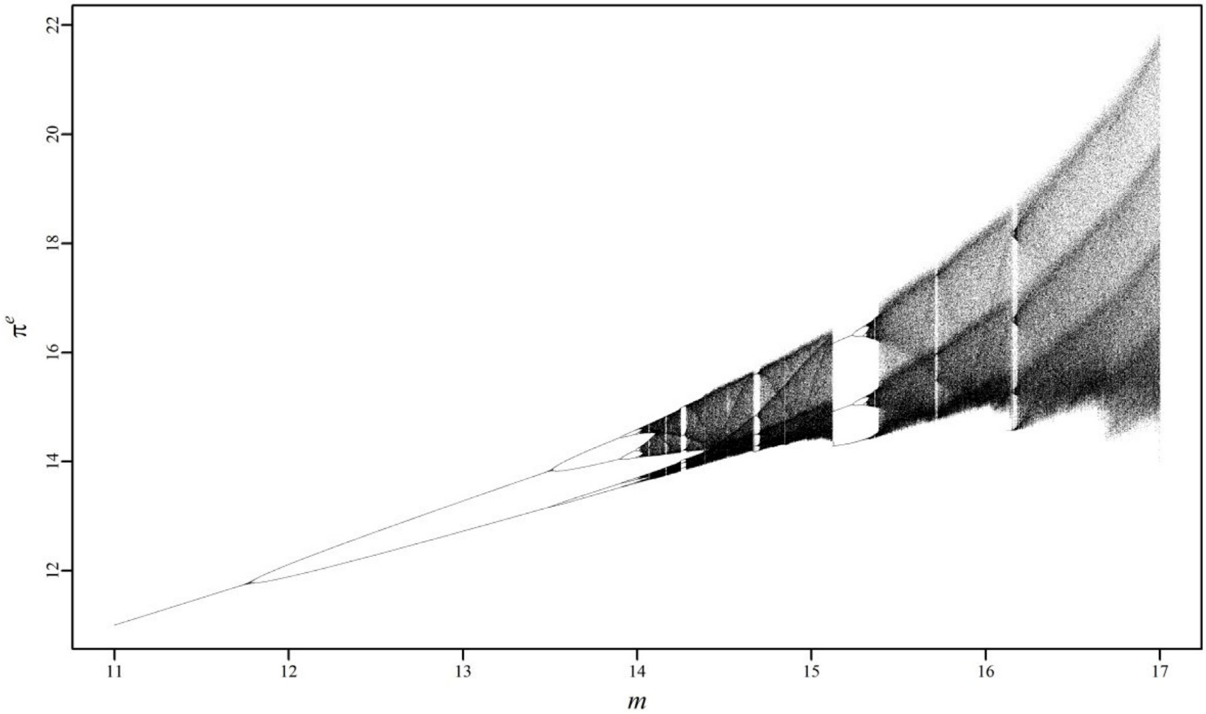

Fig. 6. Bifurcation diagram of the inflation expectation variable π^e under (11), with $11 < m < 17$ and $\alpha_1 = -1.14, \alpha_2 = 5.53, \alpha_3 = 3.68,\ a = 0.1, b = 0.1, c = 0.1, \pi(0) = 0.5, \pi^e(0) = 11, u(0) = 0.8797$.

measured by the Lyapunov exponents that measure the average separation rate of initially nearby orbits. For systems that converge to regular or periodic dynamic equilibria (fixed points, finite period limit cycles, or quasi-periodic orbits), the exponents of Lyapunov are negative, indicating convergence and stability of the equilibria. On the contrary, the existence of a positive Lyapunov exponent implies local instability, separation, and sensitivity to the initial conditions, that is, chaotic behaviors. Fig. 7 complements the bifurcation diagram showing the maximum Lyapunov exponent of the system (11) for each value of m. As we can see, the exponents are initially negative, indicating the stability of the equilibria. In the neighborhood of each bifurcation, the exponents reach a local maximum, but always with negative values. However, from the critical value of m^c, the exponents become positive, indicating that they have entered the chaotic region. Note that there are specific values of m within this chaotic region for which negative exponents are reached, which are the well-known regular windows in which the system converges again to periodic cycles.

6. Chaos control in the inflation and unemployment model

In the previous section, it has been shown (through bifurcation diagram) that, *ceteris-paribus*, from a certain critical value of the growth rate of the money supply, the steady-state equilibrium point (π^e_E, u_E) loses its stability, leading to qualitative changes in system dynamics (11). We have shown how an excessive money growth can be the cause hyperinflation and also the origin of instability and irregular and chaotic economic dynamics. Traditionally, short-term economic policy sets its inflation and unemployment targets in terms of fixed points. The inflation target is usually set at values considered to be the minimum compatible with the proper functioning of the economy (for example the European Central Bank sets it at 2%), and the unemployment rate at the minimum compatible with the frictional unemployment rate of full employment. In our case we will consider, without loss of generality, that the monetary policy targets have been set at the fixed point (π^e_E, u_E) given by (12).

In this section we illustrate, by applying the OGY control method how we can redirect chaotic dynamics back to the target fixed point (π^e_E, u_E), which even when it has lost its stability, it is still an (unstable) steady-state equilibrium. And all this without the need for making major modifications to the control parameter m.

Consider, for example, that the control parameter is $\overline{m} = 16$. In this case, the system (11) is in the chaotic region (Figs. 5, 6, and 7), the evolution of the system in the phase space is trapped in a strange attractor (Fig. 8), and the trajectories of both inflation and unemployment rates are chaotic (Fig. 9). Inside that strange attractor, infinite unstable periodic orbits coexist arbitrarily close to the chaotic solution in the strange attractor. One of these periodic orbits is the fixed point or steady-state $\left(\pi^e_E, u_E\right)$ (12) now unstable and, therefore, which the chaotic system will never reach on its own. The objective of chaos control is precisely to redirect the dynamic system towards one of this unstable fixed point by making minor modifications on control parameter, the money growth rate m. If the control is successful, and once the targeted fixed point has been reached, the system will remain stationary in it unless

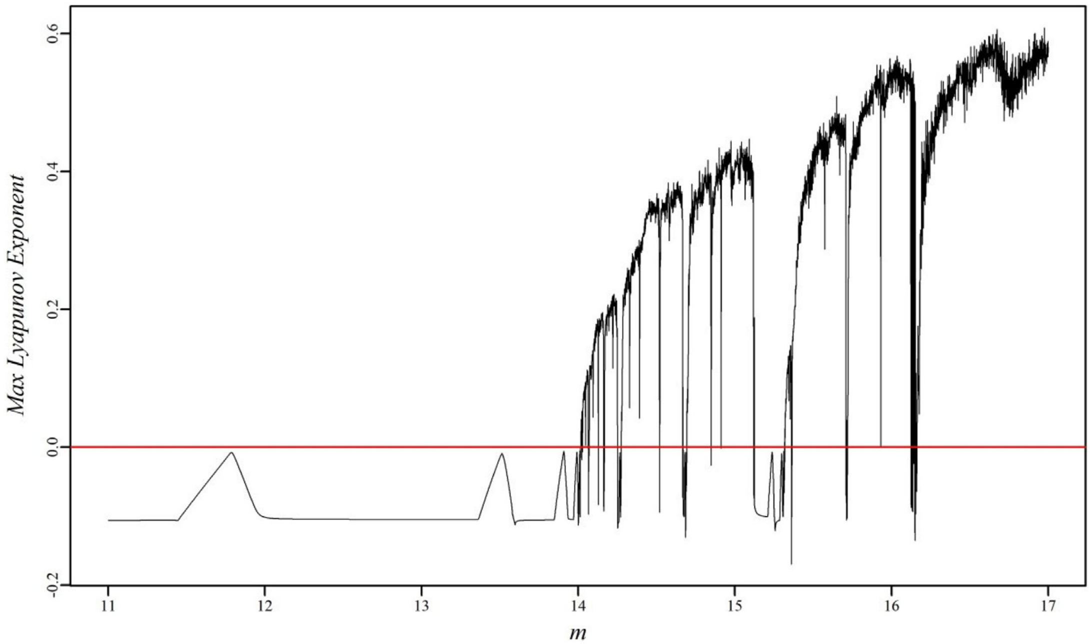

Fig. 7. Maximum Lyapunov exponent of the system (11) for different values of m $11 < m < 17$; with $\alpha_1 = -1.14, \alpha_2 = 5.53, \alpha_3 = 3.68, a = 0.1, b = 0.1, c = 0.1, .\pi(0) = 0.5, \pi^e(0) = 11, u(0) = 0.8797$.

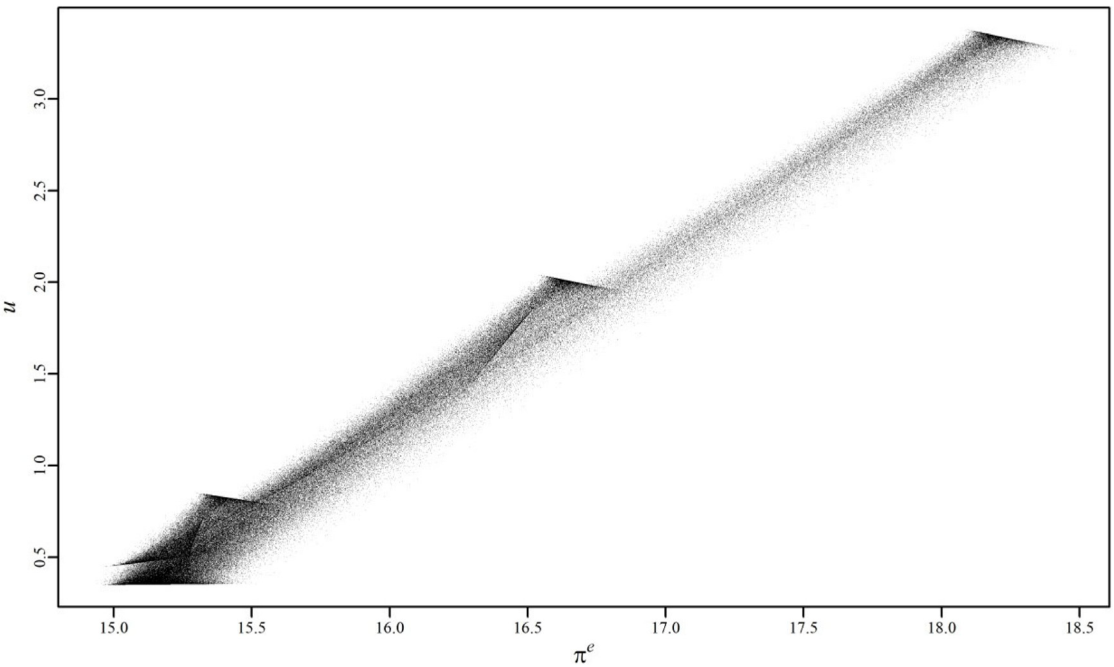

Fig. 8. Strange attractor in the phase space (u, π^e) *of* (11) with $m = 16$ without applying a control rule, the trajectories of inflation and unemployment are chaotic and trapped in a strange attractor.

it suffers some additional disturbance that will revert the system again to the chaotic behavior. It will then be necessary to apply the control again to re-redirect the system to a steady state.

Consider, to continue with our example, that policy makers wish to stabilize and target as an economic policy objective the steady state defined in (12), which with $\overline{m} = 16$ has become unstable. To apply OGY control method on the system it will be necessary to make a linear approximation of the system (11) around the control parameter m values in chaotic regime $\overline{m} = 16$ and the fixed

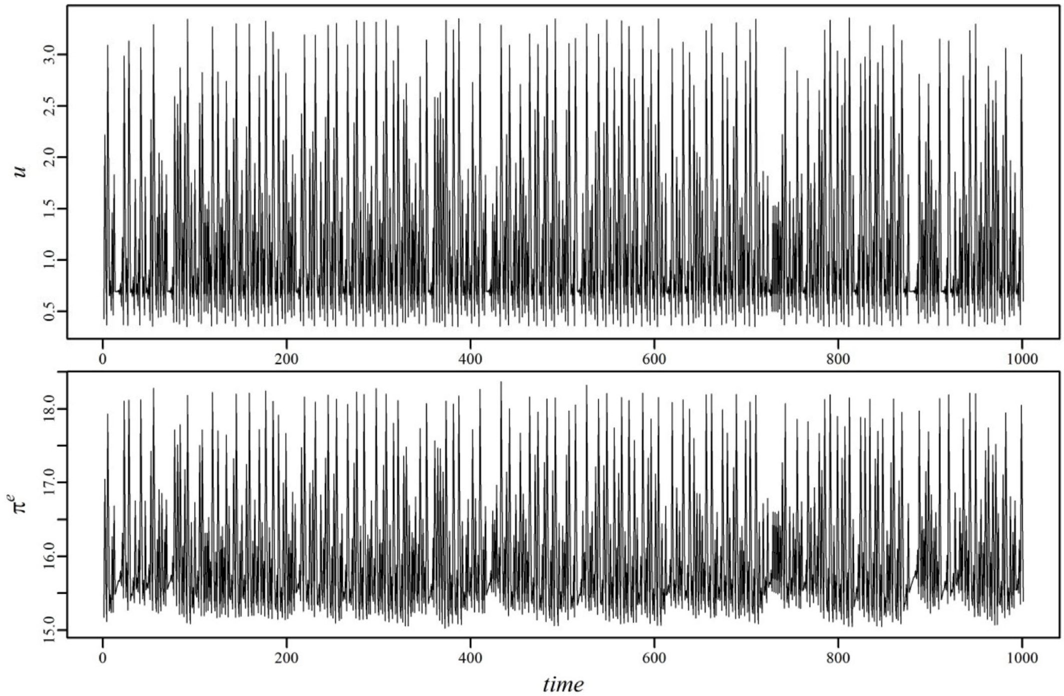

Fig. 9. Chaotic trajectories of inflation π^e and unemployment u with $m = 16$ and without the application of any control rule.

point $\left(\pi^e_E, u_E\right)$:

$$\boldsymbol{Z}_{t+1} - \boldsymbol{Z}_E = \boldsymbol{A}\left(\boldsymbol{Z}_t - \boldsymbol{Z}_E\right) + \boldsymbol{B}\left(m - \overline{m}\right) \tag{13}$$

where: $\boldsymbol{Z}_t = (\boldsymbol{\pi^e}_t, \boldsymbol{u}_t)^T$, $\boldsymbol{Z}_E = (\boldsymbol{\pi^e}_E, \boldsymbol{u}_E)^T$, $\boldsymbol{A}$ is the Jacobian matrix of the system (11) and $\boldsymbol{B}$ is an n-dimensional vector of derivatives with respect to the control parameter, both evaluated at the fixed point:

$$\boldsymbol{A} = \begin{pmatrix} \frac{\partial G}{\partial \pi^e_t} & \frac{\partial G}{\partial u_t} \\ \frac{\partial H}{\partial \pi^e_t} & \frac{\partial H}{\partial u_t} \end{pmatrix}_{|u_E;\pi^e{}_E;\overline{m}} = \begin{pmatrix} 1 - c\left(1-a\right) & cf'\left(u_E\right) \\ ba & bf'\left(u_E\right) + 1 \end{pmatrix}$$

with

$$f\left(u_t\right) = \alpha_1 + \frac{\alpha_2}{u_t} + \frac{\alpha_3}{{u_t}^2} \Rightarrow f'\left(u_t\right) = \frac{-\alpha_2}{{u_t}^2} + \frac{-2\alpha_3}{{u_t}^3}$$

and

$$\boldsymbol{B} = \begin{pmatrix} \frac{\partial G}{\partial m} \\ \frac{\partial H}{\partial m} \end{pmatrix}_{|u_E;\pi^e{}_E;\overline{m}} = \begin{pmatrix} 0 \\ -b \end{pmatrix}$$

The control is entered into the system assuming that the m parameter is a linear function of the state variables (*control rule*):

$$\begin{cases} m = \overline{m} & , if \left|\boldsymbol{K}^T\left(\boldsymbol{Z}_t - \boldsymbol{Z}_E\right)\right| > \varepsilon_0 \\ m = \overline{m} - \boldsymbol{K}^T(\boldsymbol{Z}_t - \boldsymbol{Z}_E) & , if \left|\boldsymbol{K}^T\left(\boldsymbol{Z}_t - \boldsymbol{Z}_E\right)\right| \le \varepsilon_0 \end{cases} \tag{14}$$

Substituting (14) in (13) we obtain the linearized closed-loop system:

$$\boldsymbol{Z}_{t+1} - \boldsymbol{Z}_E = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}^T)\left(\boldsymbol{Z}_t - \boldsymbol{Z}_E\right) \tag{15}$$

To set the control rule (14), we need to find the value of $\boldsymbol{K}^T$ for the matrix $\left(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}^T\right)$ that let (15) be asymptotically stable and let the system (11) be stabilized at the chosen fixed point. That is, we need to find a solution where all the eigenvalues of that matrix are in module less than unity. The well know pole placement technique or pole assignment allows to solve this problem. So, to determine the vector $\boldsymbol{K}^T$, first we get the partial derivative matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, then we will have to check that the controllability

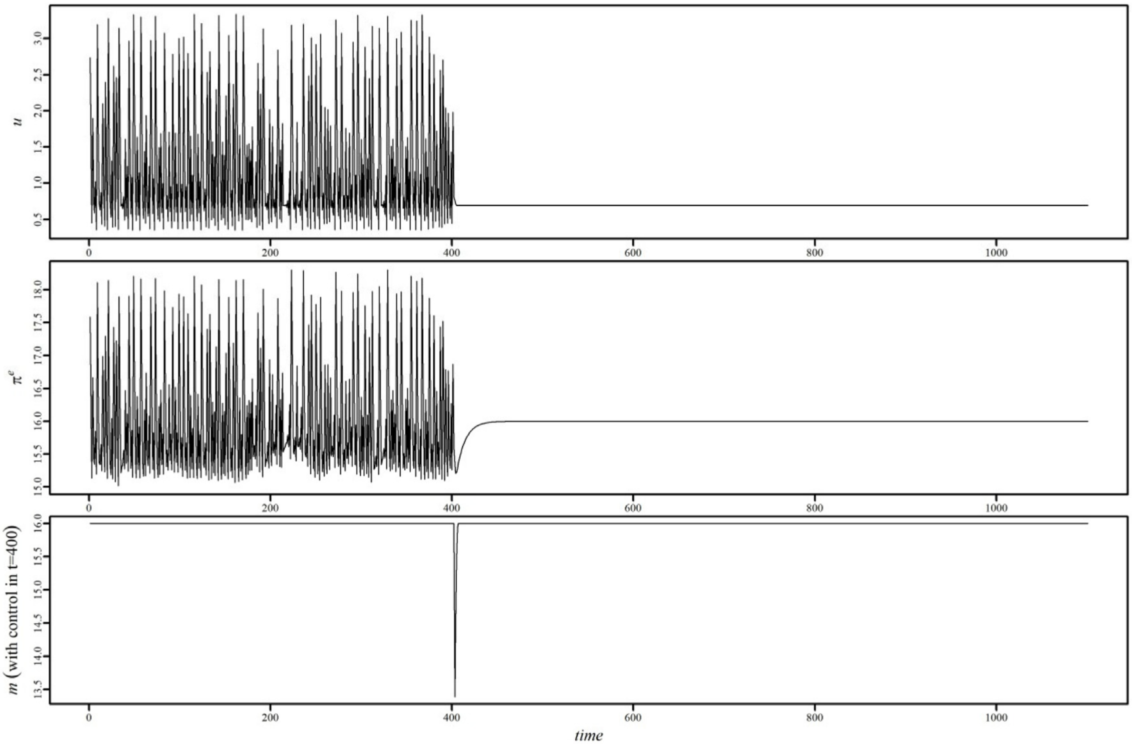

Fig. 10. Stabilization of the equilibrium fixed point when the control rule is applied. Chaotic time series of the variables u and π^e with $\overline{m} = 16$ have been stabilized after applying the control rule in $t = 40$).

conditions of the system are met and finally calculate the location of regulating poles $\{\mu_1, \mu_2\}$ or eigenvalues of $(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}^{\boldsymbol{T}})$ to be in modules smaller than unity. $\boldsymbol{A}$ proper choice will be to match these regulating poles $\{\mu_1, \mu_2\}$ with the stable eigenvalues of $\boldsymbol{A}$, and replacing the unstable ones with zero: $\mu_1 = \lambda^*_{un} = 0$ and $\mu_2 = \lambda_{st}$. With this election we make that under the control (14), the initially chaotic trajectory move toward the fixed point following its stable direction and then stay stationary on it. Considering the above, the solution to the problem of pole placement will be given by Ackermann's formula, that will provide the values wanted for the control rule, that in this case will take the following values (See Appendix):

$$\boldsymbol{K}^{\boldsymbol{T}} = \{-0.07176911, 23.1343\}$$

The monetary policy rule or control rule (10), therefore, will be:

$$m_t = 16 - 0.07176911 * (\boldsymbol{\pi^e}_{\boldsymbol{t}} - \boldsymbol{\pi}^{\boldsymbol{e}}_{\boldsymbol{E}}) + 23.1343(\boldsymbol{u}_{\boldsymbol{t}} - \boldsymbol{u}_{\boldsymbol{E}}) \quad (16)$$

With the targets $\pi^e_E = 16$ and $u_E = .0.6960655$. This control rule will only be activated when (π^e_t, u_t) are sufficiently close to the equilibrium value (π^e_E, u_E) and, therefore, the money growth rate should never be too distant from its equilibrium value $\overline{m} = 16$. Note that this monetary policy rule (16) implies that when expected inflation rises above its equilibrium $(\pi^e_t > \pi^e_E)$ or when the unemployment rate falls below the equilibrium unemployment rate $(u_t < u_E)$, monetary policy must be restrictive, reducing the growth rate of the amount of money. On the contrary, when inflation is below the level of equilibrium $(\pi^e_t < \pi^e_E)$ or when the unemployment rate is above the target unemployment rate $(u_t > u_E)$, the monetary policy should be expansive. The weights or influence of the deviations from the equilibrium values on the money growth rate are specified in (16) and, in any case, will depend on the specific parameters that define both the model (11) and the targeted equilibrium points to be stabilized (12).

We illustrate in Figs. 10 and 11 the stabilization of the equilibrium fixed point when the control rule (16) is applied in the model (11) from a certain moment in time ($t = 400$). There are two phases in that control. First, we must wait for the system (11), which is in a chaotic behavior regime, to come close enough to the target point. This prevents the monetary policy from varying beyond predetermined margins. The lower these margins of monetary policy action, the longer the waiting time. Second, once the system is in an environment close enough to the targeted state, the value of the monetary policy is determined according to (16). Thereafter, monetary policy shifts the system away from its trajectory, taking it out of the strange attractor and reorienting it toward the policy target. Once the target of the control has been achieved, which is an unstable stationary fixed point, it is not necessary to return to act the policy rule. That is, monetary policy also remains stationary at its equilibrium rate $\overline{m}$. Only when some exogenous force moves the system away from this unstable steady state, the policy rule will be put back into operation to return the system to the control target again.

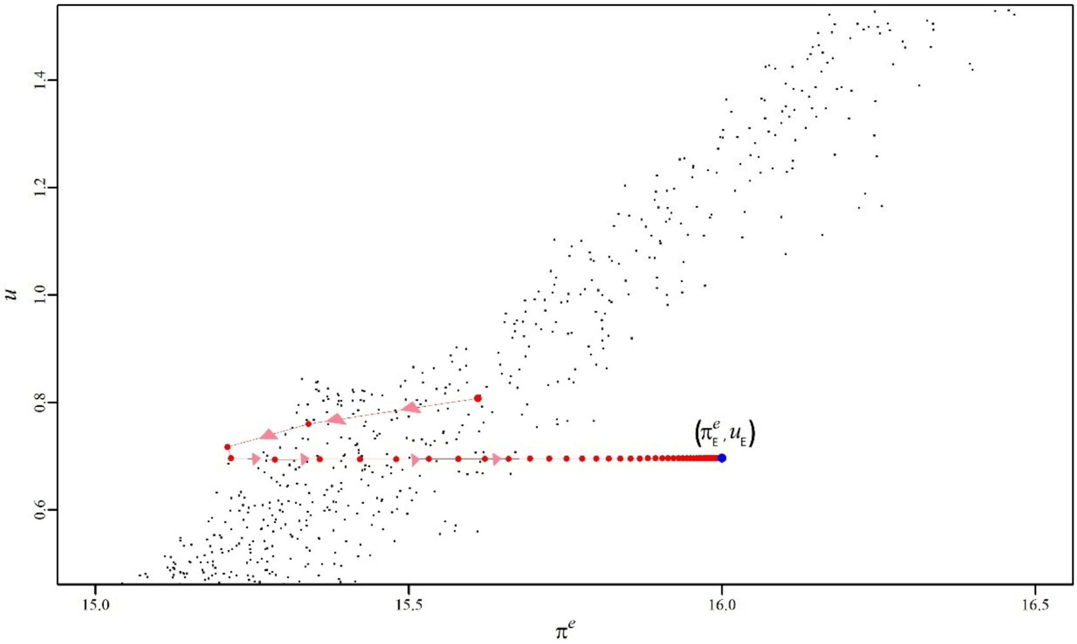

Fig. 11. Control of the strange attractor in the phase space (u and π^e) with $\overline{m} = 16$ after applying the control rule (in $t = 400$). In red, the evolution after the application of control rule towards the targeted fixed point (π^e_E, u_E).

7. Conclusions

The monetary policy implemented of most Western economies are not strictly linked to any Taylor rule. Despite this, the ability of these rules to stabilize inflation, production, or unemployment around equilibrium values has led many countries to adopt them as a normative guide for monetary policy. At least, the behavior of the monetary policy of many economies is well represented empirically by this type of Taylor monetary rule, assuming growing importance as an operational and guiding rule for monetary policy. In this sense, it can be stated that Taylor's rule has, in fact, an empirical foundation, and not a theoretical one.

We have shown how this type of monetary rule is linked to the *control theory of chaotic dynamic systems*. We have shown that the Taylor rule can be deduced and founded on the application of chaos control mechanisms on monetary models of inflation and unemployment. It is that control theory that constitutes the basis for the use of Taylor rules, and, therefore, such a control theory should be used to deepen these rule types when applied to monetary policy. In addition, these monetary rules do not arise from the specific doctrinal assumptions of the underlying economic model (neoclassical, keynesian, monetarism …), but from the chaos control theory itself, and, therefore, are applicable to any model representing economic dynamics.

The main objective of the chaos control theory is to suppress the undesired chaotic oscillations of the endogenous or state variables and make a dynamic system behave in a predetermined manner, generally reaching a steady-state (fixed point) considered as the objective of the control. The way to stabilize the system in the desired state is to use feedback between the instrument and the gap or difference between the target values and the current values of the state variables, using a control instrument to reduce the difference between the two values (targeted and effective). It is precisely this feedback mechanism that is behind the Taylor monetary policy rules, and that makes it possible to link these monetary policy rules with the chaos control theory.

Chaos control methods as the OYG applies only small changes or variations to the system's control parameter to stabilize any orbit or fixed point of equilibrium without changing the inherent properties of the dynamic system or substantially interfering in the relationships between its state variables. In economics, this means that the economic authorities will be able to achieve a specific economic policy target through slight changes in the control instrument (fine-tuning policies), without structurally altering the normal functioning of the economy.

One auxiliary but relevant result deriving from our work is that it could be precisely the use of Taylor monetary rules that might be providing an explanation for the so-called *paradox of chaos in economics*: on the one hand, there is little empirical evidence of chaotic behaviors in economic time series, but on the other, literature on economic dynamics has shown the widespread existence of chaotic behaviors in theoretical models (with non-linear feedback) that present irregular fluctuations, very similar to those observed in time series. A possible explanation for this *paradox of chaos* would be the following. If the monetary authorities are effectively applying some type of Taylor rule when it comes to directing their monetary policy and, in turn, according to the Control Theory, these rules can be considered effective for controlling and stabilizing chaotic economic dynamics; then, these rules implemented by the Central Banks are eliminating chaotic behavior and, therefore, making it impossible to detect chaos in the time series of the real economy.

In this article, we have illustrated the application of chaos control techniques in a simple model where inflation and unemployment depend on the growth rate of the money supply. On the one hand, it has been shown how the monetary policy itself, with excessive growth of the money, can cause the relationship between inflation and unemployment to become unstable, moving from a stationary equilibrium to cyclical behavior and, eventually, to an irregular, chaotic movement. And on the other hand, we have illustrated how to eliminate chaotic behavior by using a control rule on money growth, similar to the Taylor rules, and without the need to adopt an excessively aggressive monetary policy.

Chaotic dynamics offers a new perspective on economic control strategies with important results for the economic policy. Chaos can be controlled. Such control takes the form of rules or control laws that resemble the Taylor economic policy rules.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix. Obtaining the control rule for the Soliman model

The following are the steps required to carry out control of Soliman's monetary system [32]:

$$\boldsymbol{\pi}^e_{t+1} = \boldsymbol{G}\left(\boldsymbol{\pi}^e_t, \boldsymbol{u}_t\right) = \boldsymbol{c}\,\boldsymbol{f}\left(\boldsymbol{u}_t\right) + (\boldsymbol{1} - \boldsymbol{c}\,(\boldsymbol{1} - \boldsymbol{a}))\boldsymbol{\pi}^e{}_t \tag{11}$$

$$\boldsymbol{u}_{t+1} = \boldsymbol{H}\left(\boldsymbol{\pi}^e_t, \boldsymbol{u}_t\right) = -\boldsymbol{bm} + \boldsymbol{bf}\left(\boldsymbol{u}_t\right) + \boldsymbol{b}\;\boldsymbol{a}\boldsymbol{\pi}^e{}_t + \boldsymbol{u}_t$$

with

$$f(u) = \alpha_1 + \frac{\alpha_2}{u} + \frac{\alpha_3}{u^2}$$

Control requires the implementation of a rule that follows from the application of the OGY method on the system. Thus, first it is necessary to make a linear approximation of the system (11) around the control parameter m values in a chaotic regime $\overline{m} = 16$ and the fixed point $\left(\pi^e_E, u_E\right)$. This linear approximation of model (11) will be:

$$\boldsymbol{Z}_{t+1} - \boldsymbol{Z}_E = \boldsymbol{A}\left(\boldsymbol{Z}_t - \boldsymbol{Z}_E\right) + \boldsymbol{B}\left(m - \overline{m}\right) \tag{13}$$

where: $\boldsymbol{Z}_t = (\boldsymbol{\pi}^e{}_t, \boldsymbol{u}_t)^T$, $\boldsymbol{Z}_E = (\boldsymbol{\pi}^e{}_E, \boldsymbol{u}_E)^T$, $\boldsymbol{A}$ is the Jacobian matrix of the system (11) and $\boldsymbol{B}$ is an n-dimensional vector of derivatives with respect to the control parameter, both evaluated at the fixed point:

$$\boldsymbol{A} = \begin{pmatrix} \dfrac{\partial G}{\partial \pi^e_t} & \dfrac{\partial G}{\partial u_t} \\ \dfrac{\partial H}{\partial \pi^e_t} & \dfrac{\partial H}{\partial u_t} \end{pmatrix}_{\big| u_E;\pi^e{}_E;\overline{m}} = \begin{pmatrix} 1 - c\,(1-a) & c f'(u) \\ ba & bf'(u) + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - c(1-a) & c f'(u_E) \\ ba & bf'\left(u_E\right) + 1 \end{pmatrix}$$

with

$$f\left(u_t\right) = \alpha_1 + \frac{\alpha_2}{u_t} + \frac{\alpha_3}{{u_t}^2} \Rightarrow f'\left(u_t\right) = \frac{-\alpha_2}{{u_t}^2} + \frac{-2\alpha_3}{{u_t}^3}$$

and

$$\boldsymbol{B} = \begin{pmatrix} \dfrac{\partial G}{\partial m} \\ \dfrac{\partial H}{\partial m} \end{pmatrix}_{\big| u_E;\pi^e{}_E;\overline{m}} = \begin{pmatrix} 0 \\ -b \end{pmatrix}$$

The control is entered into the system assuming that the m parameter is a linear function of the state variables (*control rule*):

$$\begin{cases} m = \overline{m} & , if \left|\boldsymbol{K}^T\left(\boldsymbol{Z}_t - \boldsymbol{Z}_E\right)\right| > \varepsilon_0 \\ m = \overline{m} - \boldsymbol{K}^T(\boldsymbol{Z}_t - \boldsymbol{Z}_E) & , if \left|\boldsymbol{K}^T\left(\boldsymbol{Z}_t - \boldsymbol{Z}_E\right)\right| \le \varepsilon_0 \end{cases} \tag{14}$$

Substituting (14) in (13) we obtain the linearized closed-loop system:

$$\boldsymbol{Z}_{t+1} - \boldsymbol{Z}_E = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}^T)\left(\boldsymbol{Z}_t - \boldsymbol{Z}_E\right) \tag{15}$$

To set the control rule to [14], the value of $\boldsymbol{K}^T$ for the matrix $\left(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}^T\right)$ should be asymptotically stable, and the system stabilized at the chosen fixed point. That is, when all the eigenvalues of that matrix are in module less than the unit. The pole placement technique or pole assignment allows to solve this problem, that is to say, to determine the vector $\boldsymbol{K}^T$ in such a way that $\left(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}^T\right)$ has its eigenvalues (regulating poles $\left\{\mu_1, \dots \mu_n\right\}$) with module less than the unit and previously specified so that the fixed point

$\left(\pi_E^e, u_E\right)$ were stable in the closed-loop system. To apply this method of assignment regulating poles, we first obtain the matrices of partial derivatives matrices **A** and **B**.

$$A = \begin{pmatrix} \frac{\partial G}{\partial \pi_t^e} & \frac{\partial G}{\partial u_t} \\ \frac{\partial H}{\partial \pi_t^e} & \frac{\partial H}{\partial u_t} \end{pmatrix}_{|u_E;\pi^e{}_E;\overline{m}} = \begin{pmatrix} 0.91 & -3.323731 \\ 0.01 & -2.323731 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{\partial G}{\partial m} \\ \frac{\partial H}{\partial m} \end{pmatrix}_{|u_E;\pi^e{}_E;\overline{m}} = \begin{pmatrix} 0 \\ -0.1 \end{pmatrix}$$

We check the controllability condition of the system, that is, that the following matrix $\boldsymbol{C}_{2\times2}$ has rank 2 (its determinant is different from zero)[5]:

$$\boldsymbol{C} = (\boldsymbol{B} \vdots \boldsymbol{AB}) = \begin{pmatrix} 0 & 0.3323731 \\ -0.1 & 0.2323731 \end{pmatrix}$$

We obtain the determinant $|\boldsymbol{C}| = 0.03323731$ so the rank of the matrix C is 2, in addition,

$$\boldsymbol{C} \times \boldsymbol{C}^{-1} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

Therefore, the conditions for applying the control are satisfied, and the arbitrary assignation of poles is possible, and all eigenvalues of matrix $\boldsymbol{A}$ can be arbitrarily located. The solution to the problem of pole location is given by Ackermann's formula[6]:

$$\boldsymbol{K^T} = \left\{\boldsymbol{\alpha_2 - a_2}, \boldsymbol{\alpha_1 - a_1}\right\} \boldsymbol{T^{-1}} \tag{17}$$

where $\{a_1, a_2\}$ are the coefficients of the characteristic polynomial of $\boldsymbol{A}$[7] and $\{\alpha_1, \alpha_2\}$ are the coefficients of the desired characteristic polynomial $(\boldsymbol{A} - \boldsymbol{BK^T})$[8] and $\boldsymbol{T} = \boldsymbol{CW}$, with

$$\boldsymbol{W} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}$$

It is, therefore, a question of finding the coefficients of the characteristic polynomials $\{a_1, a_2\}$ and $\{\alpha_1, \alpha_2\}$ in order to, as we have already commented, the system [11] is stabilized at the chosen fixed point, i.e., when all the eigenvalues of $(\boldsymbol{A} - \boldsymbol{BK^T})$ are in modulus less than the unit.

We have to start looking for $\{a_1, a_2\}$. The characteristic polynomial of $\boldsymbol{A}$:

$$|\lambda \boldsymbol{I} - A| = \begin{vmatrix} \lambda - 0.91 & -3.323731 \\ 0.01 & \lambda + 2.323731 \end{vmatrix} = \lambda^2 + 1.413731\lambda - 2.081358$$

So, the coefficients $\{a_1, a_2\}$ needed to find the control vector [17] will be:

$$a_1 = 1.413731$$
$$a_2 = -2.081358$$

[5] In general, for a n dimensional system (with n state variables), the condition will be that the controllability matrix $\boldsymbol{C}_{n\times n}$has range n: $\boldsymbol{C} = \left(\boldsymbol{B} \vdots \boldsymbol{AB} \vdots \boldsymbol{A}^2\boldsymbol{B} \vdots \dots \vdots \boldsymbol{A}^{n-1}\boldsymbol{B}\right)$.

[6] In general, for an n-dimensional system, the solution to the problem of assignment pole is given by Ackermann's formula:

$$\boldsymbol{K^T} = \left\{\alpha_n - a_n, \dots, \alpha_1 - a_1\right\} \boldsymbol{T^{-1}}$$

where $\{a_1, \dots, a_n\}$ are the coefficients of the characteristic polynomial of $\boldsymbol{A}$, and $\{\alpha_1, \dots, \alpha_n\}$ are the coefficients of the desired characteristic polynomial $(\boldsymbol{A}-\boldsymbol{BK^T})$ and $\boldsymbol{T} = \boldsymbol{CW}$, with

$$\boldsymbol{W} = \begin{pmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & 1 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

[7] For a n-dimensional system: $|\lambda \boldsymbol{I} - \boldsymbol{A}| = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = \prod_{i=1}^n(\lambda - \lambda_i)$, being λ_i the eigenvalues of $\boldsymbol{A}$, the roots of its characteristic polynomial.

[8] For a n-dimensional system: $\left|\mu \boldsymbol{I} - (\boldsymbol{A} + \boldsymbol{BK^T})\right| = \mu^n + \alpha_1\mu^{n-1} + \cdots + \alpha_n = \prod_{j=1}^n(\mu - \mu_j)$, being μ_j the eigenvalues of $(\boldsymbol{A} + \boldsymbol{BK^T})$, the roots of its characteristic polynomial.

On the other hand, the roots of that characteristic polynomial[9] provide the two eigenvalues of matrix $\boldsymbol{A}$:

$$\lambda_s = 0.8996888$$
$$\lambda_u = -2.3134198$$

The chaotic trajectories of inflation and unemployment approach the unstable fixed point $\left(\pi^e_E, u_E\right)$ following the stable direction $\lambda_{s=}$0.8996888 and move away from the point following the unstable direction $\lambda_u = -2.3134198$.

With the variation of the control parameter m, the objective is for the inflation-unemployment system to follow the stable direction of the periodic orbit chosen in each iteration. As we have mentioned above, the choice of $\boldsymbol{K^T}$, given $\boldsymbol{A}$ and $\boldsymbol{B}$, is established so that the system is stable, that is to say, that the eigenvalues $\{\mu_1, \mu_2\}$ or regulating poles of the matrix $(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K^T})$ are, in the module, less than the unit.

According to Romeiras et al. (1992), there are multiple possible values for the gain matrix $\boldsymbol{K^T}$. In principle, any choice of the regulating poles $\{\mu_1, \mu_2\}$ within the unit circle is valid. However, as with the control [14], the chaotic trajectory followed by the system towards the stable direction of the equilibrium point to be stabilized is sought, an appropriate choice will be to match the regulatory poles $\{\mu_1, \mu_2\}$ with the matrix eigenvalues $\boldsymbol{A}$, thereby: $\mu_1 = \lambda_u = 0$ and $\mu_2 = \lambda_s = 0.8996888$.

With this choice, we make the chaotic trajectory approach the fixed point in each iteration following the stable direction and then remain there.

Following this criterion, a suitable choice for our case will be to match the regulating poles $\{\mu_1, \mu_2\}$ with the eigenvalues of matrix A in the following way:

$$\mu_1 = \lambda_u^* = 0$$
$$\mu_2 = \lambda_s = 0.8996888$$

Given this choice of regulating poles, we can already find the desired coefficients $\left\{\alpha_1, \alpha_2\right\}$ of the characteristic polynomial:

$$\left|\mu \boldsymbol{I} - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}^T\right| = \mu^2 + \alpha_1\mu + \alpha_2 = \left(\mu - \mu_1\right)\left(\mu - \mu_2\right) = \mu^2 - \left(\mu_1 + \mu_2\right)\mu + \mu_1\mu_2$$
$$= \mu^2 - \left(0 + \lambda_s\right)\mu + 0 \cdot \lambda_s = \mu^2 - 0.8996888\mu$$

and from the desired polynomial, we obtain the coefficients:

$$\alpha_1 = -0.8996888$$
$$\alpha_2 = 0.$$

Once we have found the coefficients $\left\{\alpha_1, \alpha_2\right\}$ and $\left\{a_1, a_2\right\}$ and had calculated the matrix $\boldsymbol{T} = \boldsymbol{C}\boldsymbol{W}$, the ***Ackermann formula*** will remain in our example as:

$$\boldsymbol{K^T} = \{-0.07176911, 23.1343\}$$

The monetary policy rule or control rule [10] therefore, will be:

$$m_t = 16 - 0.07176911 * (\boldsymbol{\pi^e}_{\boldsymbol{t}} - \boldsymbol{16}) + 23.1343(\boldsymbol{u_t} - 0.6960655)$$

in an environment of $\pi^e_E = 16$ and $u_E = 0.6960655$

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[9] Note that $a_1 = \left(-\lambda_s - \lambda_u\right)$ y $a_2 = (\lambda_s\lambda_u)$, relationship between the coefficients of the characteristic polynomial and its roots (the eigenvalues of the matrix $\boldsymbol{A}$) derived from the definition of the characteristic polynomial:

$$|\lambda \boldsymbol{I} - \boldsymbol{A}| = \lambda^2 + a_1\lambda + a_2 = \left(\lambda - \lambda_{st}\right)\left(\lambda - \lambda_{un}\right) = \lambda^2 - (\lambda_{st} + \lambda_{un})\lambda + \lambda_{st}\lambda_{un}.$$

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