An overview of generalised Kac-Moody algebras on compact real manifolds

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Rutwig Campoamor-Stursberg[1*], Marc de Montigny[2†], Michel Rausch de Traubenberg[3‡]

[1] Instituto de Matemática Interdisciplinar and Dpto. Geometría y Topología, UCM,E-28040 Madrid, Spain

[2] Faculté Saint-Jean, University of Alberta, 8406 91 Street, Edmonton, Alberta T6B 0M9, Canada

[3] Université de Strasbourg, CNRS, IPHC UMR7178, F-67037 Strasbourg Cedex, France

[*] Email: rutwig@ucm.es

[†] Email: mdemonti@ualberta.ca

[‡] Email: Michel.Rausch@iphc.cnrs.fr

Abstract

A generalised notion of Kac-Moody algebra is defined using smooth maps from a compact real manifold $\mathcal{M}$ to a finite-dimensional Lie group, by means of complete orthonormal bases for a Hermitian inner product on the manifold and a Fourier expansion. The Peter–Weyl theorem for the case of manifolds related to compact Lie groups and coset spaces is discussed, and appropriate Hilbert bases for the space $L^2(\mathcal{M})$ of square-integrable functions are constructed. It is shown that such bases are characterised by the representation theory of the compact Lie group, from which a complete set of labelling operator is obtained. The existence of central extensions of generalised Kac-Moody algebras is analysed using a duality property of Hermitian operators on the manifold, and the corresponding root systems are constructed. Several applications of physically relevant compact groups and coset spaces are discussed.

1 Introduction

Kac-Moody algebras have been used in theoretical physics from the beginning 1980s onwards in various different contexts, such as string theory, the study of critical phenomena in two-dimensional statistical systems, Yang–Mills theory as well as in applications to exact solvable models (see *e.g.* [1] and references therein). Besides the axiomatic construction, Kac-Moody algebras (or more precisely, affine Lie algebras) can be obtained from affine extensions of the loop algebra of smooth maps from the unit circle $\mathbb{S}^1$ into a simple Lie group [2, 3, 4, 5, 6, 7].

Another infinite-dimensional Lie algebra widely encountered in two-dimensional Conformal Field Theory, as well as in string theory, is the Virasoro algebra, the central extension of the Witt algebra, that is, the centrally extended Lie algebra of polynomial vector fields on the circle $\mathbb{S}^1$ [8] (and *e.g.* [9] and references therein). Various types of generalisations of Kac-Moody algebras respectively affine Lie algebras [1] have been proposed in the literature, such as the so-called quasi-simple Lie algebras in Ref. [10], the generalised Kac-Moody algebras in Ref. [11], the Borcherds algebra [12], as well as related structures like the Monster algebra [13, 14] and the Monstrous Moonshine [15, 16, 17].

As a matter of fact, Kac-Moody and Virasoro algebras are deeply related to the one-dimensional compact manifold $\mathbb{S}^1$. In this context, it is natural to expect that physical theories with more than two dimensions involve richer structures. It is with such possibilities in mind that we discuss hereafter a generalisation of the notion of Kac-Moody algebras associated to compact manifolds $\mathcal{M}$ of dimension higher than one. In particular, we shall restrict ourselves to certain type of manifolds, namely compact Lie groups $\mathcal{M} = G_c$ or coset spaces $\mathcal{M} = G_c/H$, where $H \subset G_c$ is a closed subgroup. The reason for these choices lies on the fact that the harmonic functions on the corresponding manifold $\mathcal{M}$ can be classified in terms of the representation theory of the Lie group G_c. The algebras described in this paper do not belong to the general classification of Kac-Moody algebras given by Kac [3]; rather, they represent generalisations of affine Lie algebras which, as we will observe below, admit roots but not simple roots and thus no Cartan matrices (except, as we will prove in Section 4, when the number of central charges, or 'order of centrality', is equal to one). Moreover, unlike the usual Kac-Moody algebras, we can construct all the generators of our generalised algebras.

The notion of generalised Kac-Moody algebras is motivated by various phenomena in higher-dimensional physics, and possess the salient feature of being fully specified by harmonic expansions on $\mathcal{M}$. These algebras are potentially of use in the Kaluza-Klein theory (see *e.g.* [18, 19, 20], with the latter reference being motivated in the supergravity context), where the space-time takes the form $K = \mathbb{R}^{1,3} \times \mathcal{M}$. Symmetries in K, in particular the Nœther theorem, lead naturally to such generalised Kac-Moody algebras. Similarly, this type of structure emerges naturally through the consideration of current algebras [21, 22]. For instance, the authors of reference [23] analysed the symmetries corresponding to the massive states in the Fourier expansion for a Kaluza-Klein compactification in five dimensions, with these symmetries involving Kac-Moody and Virasoro algebras without central extensions. These authors suggested that similar infinite-dimensional symmetries should also appear in more complicated higher-dimensional theories with non-Abelian symmetry of the extra dimensions. The relevant point is that this type of algebras admits central extensions. These central extensions can be introduced in two different but related ways, either by introducing two-cocycles in their Lie brackets or, more physically, adding Schwinger terms [24] to the current algebra.

The structure of the paper is the following: In Section 2 we define generalised Kac-Moody algebras by means of the set of smooth maps from a compact real manifold to a real or complex finite-dimensional Lie group, in terms of a complete orthonormal basis for

[1] In the following (unless otherwise stated) we will always refer to Kac-Moody algebras instead of affine Lie algebras in order to be coherent with the physical literature.

the Hermitian scalar product on the manifold with a Fourier expansion. In the following, we shall restrict our discussion to manifolds related to compact Lie groups, mainly due to technical reasons. Although there is no doubt that the case of non-compact Lie groups is full of interest, with potential applications to non-Euclidean spaces and general manifolds, their study require techniques somewhat different from those used in this work. The main difference between compact and non-compact Lie groups is that unitary representations of the former are finite dimensional, whilst those of the latter are infinite dimensional. In addition, non-compact Lie groups exhibit irreducible unitary representations occurring outside the space of square integrable functions on the group, implying that a more general positive measure on the space of irreducible unitary representations must be defined, thus leading to more general integral formulae as the Plancherel formula instead of the Peter–Weyl theorem of the compact case [25]. Another technical difficulty resides in the division into discrete and continuous series, specifically in the context of the normalisation problem for discrete and the continuous spectra. For these reasons, in this paper we shall only offer a glimpse of the (rather different) constructions based on non-compact groups, the general analysis of which would be beyond the scope of our work. In Section 3, we discuss the Fourier expansion on manifolds taken as Lie groups and coset spaces, and discuss the Peter-Weyl theorem in this context. The corresponding Hilbert basis $\mathcal{B}$ of $L^2(\mathcal{M})$ is appropriately identified. As elements of $\mathcal{B}$ are characterised by the representation theory of G_c, we consider the labelling problem and identify a minimal set of operators, beyond the usual Casimir operators and Cartan subalgebra of G_c, to identify unambiguously all elements of $\mathcal{B}$. With these considerations, in Section 4, we construct the generalised Kac-Moody algebras for the case where the underlying manifold is a compact Lie group and the coset space is a factor space of a compact Lie group by a closed subgroup. It is shown that these algebras admit central extensions related by some kind of duality to certain Hermitian operators of $\mathcal{M}$. The root system of the centrally extended algebra is identified and some elements of the representation theory are given, at least for the simplest case, corresponding to the $n-$dimensional tori $\mathcal{M} = \mathbb{T}^n$. In Section 5, some applications of the construction are presented in detail. Finally, in Section 6, some conclusions are drawn and potential generalisations of the approach discussed.

2 Algebras associated to compact manifolds

In the following we shall assume that $\mathcal{M}$ is a compact real manifold. Let $L^2(\mathcal{M})$ denote the space of square integrable functions on $\mathcal{M}$ and $\mathrm{d}\mu(\mathcal{M})$ the integration measure on $\mathcal{M}$. If $\mathcal{B} = \{\rho_I(m), I \in \mathcal{I}\}$ is a complete orthonormal basis for the Hermitian scalar product on $\mathcal{M}$, with $\mathcal{I}$ a countable set, the identity

$$(\rho_I, \rho_J) = \int_{\mathcal{M}} \mathrm{d}\mu(\mathcal{M}) \overline{\rho^I(m)} \rho_J(m) = \delta^I_J \ , \quad m \in \mathcal{M} \ ,$$

is satisfied. As a consequence, a function $\Phi \in L^2(\mathcal{M})$ can be described in terms of the basis $\mathcal{B}$ as

$$\Phi(m) \;\; = \;\; \sum_{I \in \mathcal{I}} \Phi^I \rho_I(m) \equiv \Phi^I \rho_I(m) \ ,$$

where

$$\Phi^I = (\rho_I, \Phi)$$

correspond to the expansion coefficients. An important question concerns the problem whether, given two elements $\rho_I, \rho_J \in \mathcal{B}$, the product still belongs to the space $L^2(\mathcal{M})$. In this work, this will be the case, as we assume that all functions are bounded, *i.e.*, $|\rho_I| < M_I$ for some M_I, so that $\rho_I\rho_J \in L^2(\mathcal{M})$. This enables us to consider the Fourier expansion of product of elements of $\mathcal{B}$

$$\rho_I(m)\rho_J(m) = c_{IJ}{}^K \rho_K(m) \ , \tag{1}$$

with $c_{IJ}{}^K \in \mathbb{C}$. In general, it is difficult to derive precise formulae for the coefficients $c_{IJ}{}^K$, but we shall discuss some examples where they can be explicitly computed, at least partially.

In this paper we will restrict our analysis to manifolds associated to compact Lie groups G_c. Therefore, the functions ρ_I of the orthonormal basis $\mathcal{B}$ can be organised using the representation theory of G_c, *i.e.*, each ρ_I belongs to a given representation of G_c. As G_c is compact, the functions ρ_I are automatically bounded. In addition, the product (1) can be evaluated using representations of G_c and the corresponding Clebsch-Gordan coefficients. It is important to observe that for ρ (resp. ρ') belonging to a representation $\mathcal{D}$ (resp. $\mathcal{D}'$), the fact that ρ and ρ' are commuting functions implies that the product $\rho\rho'$ belongs to $\mathcal{S}(\mathcal{D}\otimes\mathcal{D}')$, where $\mathcal{S}$ denotes the symmetric tensor product of $\mathcal{D}$ and $\mathcal{D}'$.

Consider now a simple complex or real finite-dimensional Lie group G with Lie algebra $\mathfrak{g}$. We denote its basis elements by T_a with $a = 1, \cdots, \dim \mathfrak{g}$. The Lie bracket is given by

$$[T_a, T_b] = \mathrm{i} f_{ab}{}^c T_c \ .$$

It is well known that a Kac-Moody algebra can be associated to the Lie algebra $\mathfrak{g}$ *via* the set of smooth maps from the circle $\mathbb{S}^1$ to G [2, 3, 4, 6, 7]. Similarly, the notion of generalised Kac-Moody algebra associated to the manifold $\mathcal{M}$, denoted by $\mathfrak{g}(\mathcal{M})$, can be defined by using the set of smooth maps from $\mathcal{M}$ to G as described hereafter [10, 11, 26]. Let $G(\mathcal{M})$ denote the group of smooth maps from $\mathcal{M}$ to G and let $g \in G$, so that

$$g = e^{\mathrm{i}\,\theta^a T_a} \tag{2}$$

holds if G is compact. In these conditions, any element of G can be represented by the exponential of an appropriate element of $\mathfrak{g}$, while for the non-compact case, we have to replace it by a finite product of exponentials. The element in $G(\mathcal{M})$ associated to (2) is given by

$$\hat{g}(m) = e^{\mathrm{i}\,\theta^a(m) T_a} \ ,$$

where now $\theta^a(m)$ are square-integrable functions of $\mathcal{M}$. In a neighbourhood of the identity the following approximation holds

$$\hat{g}(m) \sim 1 + \mathrm{i}\,\theta^a(m) T_a = 1 + \mathrm{i}\,\theta^{aI} \rho_I(m) T_a \ ,$$

where $\rho_I(m) \in \mathcal{B}$. In particular, the set of functions from $\mathcal{M} \to G$ leads, at the infinitesimal level, to the Lie algebra $\mathfrak{g}(\mathcal{M})$ with basis

$$\mathfrak{g}(\mathcal{M}) = \Big\{ T_{aI}(m) = T_a \rho_I(m), a = 1, \cdots, \dim \mathfrak{g}, I \in \mathcal{I} \Big\} ,$$

and Lie brackets

$$[T_{aI}, T_{bJ}] = \mathrm{i}\ f_{ab}{}^c c_{IJ}{}^K T_{cK} \ . \tag{3}$$

If $\mathfrak{g}$ is a real Lie algebra, then the generalised Kac-Moody algebra, denoted by $\mathfrak{g}(\mathcal{M})$, will be real, because the manifold $\mathcal{M}$ is real. Clearly $\mathfrak{g}(\mathcal{M})$ constitutes a generalisation of the usual notion of Kac-Moody algebras, but restricted hereafter to the context of compact manifolds $\mathcal{M}$. The algebra (3) can be further enlarged introducing central charges and additional operators. Actually, the possible central extensions of (3) were fully classified in [6]. It is worthy to be mentioned that the construction can be naturally adapted to Lie supergroups and Lie superalgebras, resulting in the notion of generalised super-Kac-Moody algebras [27, 28, 29, 30].

An alternative physical motivation for considering generalised Kac-Moody algebras is related to Kaluza-Klein theories [18, 19, 20] (and references therein) and current algebras [21, 22]. Indeed if we consider a $(4+n)$-dimensional compactified space-time of the form

$$K = \mathbb{R}^{1,3} \times \mathcal{M} \ ,$$

where $\mathbb{R}^{1,3}$ is the four-dimensional space-time and $\mathcal{M}$ a compact $n-$dimensional real manifold, it follows from the Nœther theorem that the conserved charges can be expressed in terms of the fields belonging to the $(4+n)-$dimensional space-time. If we denote by T_a the conserved charge associated to a Lie algebra $\mathfrak{g}$, and by y^A $(A = 1 \cdots, n)$ the coordinates on $\mathcal{M}$, then integration over the space part of $\mathbb{R}^{1,3}$ but not over the internal space $\mathcal{M}$ and the equal-time commutation relations lead to the current algebra

$$\big[T_a(y), T_{a'}(y')\big] = \mathrm{i}\ f_{aa'}{}^b T_b(y) \delta^n(y-y') \ , \tag{4}$$

where the $\delta-$distribution is defined in Appendix A, equation (76). Now consider G_c, a compact Lie group and $H \subset G_c$. Let us introduce a Hilbert basis of $L^2(\mathcal{M})$ as above, and set

$$T_a(y) = T_{aI}\ \bar{\rho}^I(y) \ ,$$

then upon integration by $\int \mathrm{d}^n y \int \mathrm{d}^n y'$ (see Appendix A) where $\mathcal{M} = G_c$ or $\mathcal{M} = G_c/H$, gives rise to

$$\big[T_{aI}, T_{a'I}\big] = \mathrm{i}\ f_{aa'}{}^b c_{II'}{}^J T_{bJ} \ . \tag{5}$$

We thus obtain a generalised Kac-Moody algebra as defined in (3). If we add a Schwinger term to (4), we can define possible central extensions in close analogy with the Pressley–Segal analysis of central extensions of generalised Kac-Moody algebras [6, 24]. Now, considering the Lie algebra of vector fields on $\mathcal{M}$ generated by $L_{AI} = -\mathrm{i}\ \rho_I \partial_A$ (where $\partial_A = \frac{\partial}{\partial y^A}$), the algebra (5) extends to

$$\begin{aligned}
\big[T_{aI}, T_{bJ}\big] &= \mathrm{i}\ f_{ab}{}^c c_{IJ}{}^K T_{cK}\ , \\
\big[L_{AI}, L_{BJ}\big] &= -\mathrm{i}\Big((\partial_A \rho_J) L_{BI} - (\partial_B \rho_I) L_{AJ}\Big)\ , \\
\big[L_{AI}, T_{aJ}\big] &= \rho_I \partial_A \rho_J T_{aJ} = d_{AI,J}{}^K T_{aK}\ ,
\end{aligned} \qquad (6)$$

where the summation over repeated indices is implicit and $\rho_I \partial_A \rho_J = d_{AI,J}{}^K \rho_K$.

The authors of Ref. [23] analysed the symmetries induced by the massive modes appearing in the Fourier expansion for a five dimensional compactified space-time $\mathbb{R}^{1,3} \times \mathbb{S}^1$ by using an algebra of the type (6); however in the context of centreless (usual) Kac-Moody and Virasoro algebras. It was further mentioned that these results can potentially be extrapolated to higher dimensional space-times.

3 Fourier expansion on compact manifolds

In this section we briefly discuss the Fourier expansion on compact manifolds. Indeed, the usual Fourier analysis on the circle $\mathbb{S}^1$ can be extended to compact manifolds. Specifically, we consider two types of manifolds: compact Lie groups and coset spaces of compact Lie groups. For the two situations, we obtain the basic functions appearing in the Fourier analysis by group theoretical arguments.

3.1 The compact manifold as a Lie group

Let G_c be a simple real compact Lie group and $\hat{\mathcal{R}} = \{\mathcal{R}_k, k \in \hat{G}_c\}$ be the set of all irreducible unitary representations of G_c, and $\hat{G}_c$ the set of labels of such representations.[2] As G_c is compact, each of such representations is finite-dimensional; we denote the corresponding dimension of $\mathcal{R}_k$ by d_k. For a matrix representation $D_{(k)}(g) \in \mathcal{R}_k$, $g \in G_c$, we denote the matrix elements by $D_{(k)}{}^i{}_j(g)$. As G_c is a group, for two matrix representations $D_{(k)}(g), D_{(k)}(g') \in \mathcal{R}_k$, the matrix product $D_{(k)}(g'') = D_{(k)}(g'g) = D_{(k)}(g)D_{(k)}(g')$ is again a representation and therefore belongs to $\mathcal{R}_k$. In other words, each column (resp. each line) of the matrix elements $D_{(k)}{}^i{}_j(g), j = 1 \cdots, d_k$ (resp. $D_{(k)}{}^i{}_j(g), i = 1 \cdots, d_k$) is a G_c-representation.

Let $\mathrm{d}\mu(G_c)$ be the Haar measure of G_c and consider the space of square integrable functions, $L^2(G_c)$, defined on the manifold G_c and normalised as

$$\int_{G_c} \mathrm{d}\mu(G_c) = 1\ .$$

This allows us to state the following theorem.

[2]The notation $\hat{G}_c$ used here for the set of labels should not be confused with $\hat{G}$, which is often used in the literature to denote the affine algebras.

Theorem 3.1 (Peter-Weyl [31]) *Let $\hat{\mathcal{R}} = \{\mathcal{R}_k, k \in \hat{G}_c\}$ be the set of all unitary irreducible representations of G_c, and $D_{(k)}(g) \in \mathcal{R}_k$, for $g \in G_c$. Then the set of functions on G_c,*

$$\psi_{(k)}{}^i{}_j(g) = \sqrt{d_k} D_{(k)}{}^i{}_j(g), \quad k \in \hat{G}_c \ , i, j = 1, \cdots, d_k, g \in G_c, \tag{7}$$

forms a complete Hilbert basis of $L^2(G_c)$ with inner product

$$(\psi_{(k)}{}^i{}_j, \psi'_{(k')}{}^{i'}{}_{j'}) = \int_{G_c} \mathrm{d}\mu(G_c) \overline{\psi}^{(k)}{}_i{}^j(g) \psi_{(k')}{}^{i'}{}_{j'}(g) = \delta^k_{k'} \delta^{i'}_i \delta^j_{j'} \ .$$

For any function Φ in $L^2(G_c)$, we have

$$\Phi(g) = \sum_{k \in \hat{G}_c} \sum_{i,j=1}^{d_k} \phi^k{}_i{}^j \psi_{(k)}{}^i{}_j(g) \equiv \phi^k{}_i{}^j \psi_{(k)}{}^i{}_j(g) \ ,$$

where the coefficients $\phi^k{}_i{}^j$ are given by

$$\phi^k{}_i{}^j = \int_{G_c} \mathrm{d}\mu(G_c) \overline{\psi}^{(k)}{}_i{}^j(g) \Phi(g) \ .$$

As the representation is unitary, it follows that

$$\overline{\psi}^{(k)}{}_i{}^j(g) = \psi_{(k)}{}^i{}_j(g^{-1}) \ .$$

In the following, we describe left-coset manifolds $\mathcal{M} = G_c/H$ for compact groups G_c. As mentioned earlier, we do not discuss manifolds based on non-compact groups in detail. Albeit formally feasible, as we shall illustrate with some examples, the construction involves various subtleties that go beyond the scope of this paper. Essentially, instead of using the Peter-Weyl theorem for the particular compact groups discussed here, the construction in the non-compact case requires the more general Plancherel theorem (see, e.g. [25, 32, 33]), compounded by the infinite-dimensional unitary representations of non-compact groups. Instead of the direct sums used in the context of square-integrable functions, the corresponding expression in the non-compact case uses the Plancherel integral.

3.2 Compact manifolds as a coset space G_c/H

In this section, we consider the manifold to be a left coset $\mathcal{M} = G_c/H$ with respect to a closed subgroup H of G_c. Denote by $\mathfrak{h}$ the Lie algebra associated to H. In general, G_c/H, which is the set of equivalence classes

$$g_1 \sim g_2 \quad \textit{iff} \quad \exists\, h \in H \ \text{ such that } \ g_2 = g_1 h \ ,$$

does not form a group unless H is normal in G_c. The elements of the coset space G_c/H are denoted by $[g]$ and we have $[g_1] = [g_2]$ *iff* $\exists\, h \in H$ such that $g_2 = g_1 h$. Thus if $r \in [r]$ then

$$[gr] = [r'] \quad \Leftrightarrow \quad gr = r'h \quad \Leftrightarrow \quad r' = grh^{-1} \ ,$$

which defines the left action of G_c on G_c/H.

At the Lie algebra level, we write the generators of $\mathfrak{g}_c$, namely T_a (with $a = 1, \cdots, \dim \mathfrak{g}_c$), as follows: U_i with $i = 1, \cdots, \dim \mathfrak{h}$, and V_p with $p = 1, \cdots, \dim \mathfrak{g}_c - \dim \mathfrak{h}$. The elements V_p belong to the space $\mathfrak{g}_c/\mathfrak{h}$, which is not generally a Lie algebra. The commutations relations take the form

$$\begin{array}{ll} (a) & [U_j, U_k] = \mathrm{i}\; g_{jk}{}^{\ell} U_\ell \ , \\ (b) & [U_j, V_p] = \mathrm{i}\; (R_j)_p{}^q V_q \ , \\ (c) & [V_p, V_q] = \mathrm{i}\; g_{pq}{}^{j} U_j + \mathrm{i}\; g_{pq}{}^{r} V_r \ . \end{array}$$

The relations (a) are trivially satisfied as $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}_c$, whereas the relations (b) imply that $\mathfrak{g}_c/\mathfrak{h}$ is a representation of $\mathfrak{g}_c$. We remark that if $g_{pq}{}^r = 0$ holds in (c), then the manifold G_c/H is said to be a symmetric space.

Now let us extend the harmonic expansion of Theorem 3.1 to a coset space G_c/H seen as a manifold [18]. As the elements of G_c/H are equivalence classes, we consider now a function $\Phi \in L^2(G_c/H)$, with components Φ^i, which belongs to a certain representation $\mathcal{R}_H$ of H and, for a given $h \in H$, we denote by $D^i{}_j(h)$ its matrix elements in $\mathcal{R}_H$. Thus, we have

$$\Phi^i(gh) = D^i{}_j(h)\Phi^j(g) \ , \quad h \in H, \ g \in G_c/H \ .$$

In order to apply the harmonic expansion to a coset manifold G_c/H, we do not need to consider all the representations of G_c. Let us consider the representation $\mathcal{R}_{(k)} \in \mathcal{R}, k \in \hat{G}_c$ with matrix representative $D_{(k)}(g)$ for $g \in G_c$, such that

$$D_{(k)}(gh) = D_{(k)}(g)D(h) \ .$$

This relation is possible if and only if, in the embedding $H \subset G_c$, we have

$$\mathcal{R}_{(k)} = m_k \mathcal{R}_H \oplus \cdots \ .$$

In other words, $\mathcal{R}_H$ is contained m_k times (with $m_k > 0$) in $\mathcal{R}_{(k)}$, so that m_k is the multiplicity of $\mathcal{R}_H$ in the decomposition $H \subset G_c$. We denote by $\hat{\mathcal{R}}|_{\mathcal{R}_H}$ the set of representations of G_c satisfying this property, while $\hat{G}_c|_{\mathcal{R}_H}$ denotes the set of corresponding labels.

The harmonic expansion takes the form

$$\Phi^i(g) = \sum_{k \in \hat{G}_c|_{\mathcal{R}_H}} \sum_{n=1}^{m_k} \sum_{j=1}^{d_k} \sqrt{\frac{d_k}{d_D}} \Phi^{(k)jn} D_{(k)}{}^{i}{}_{j,n}(g) \ ,$$

where d_D is the dimension of the representation $\mathcal{R}_H$. Let $g = rh$, with r a representative of the equivalence class $[r] \in G_c/H$. From the identities

$$\begin{aligned}
\int \mathrm{d}\mu(G_c)\sqrt{\frac{d_k}{d_D}} D_{(k)}{}^{j}{}_{i,n}(g^{-1})\Phi^i(g) &= \int \mathrm{d}\mu(G_c)\sqrt{\frac{d_k}{d_D}} D_{(k)}{}^{j}{}_{i,n}(h^{-1}r^{-1})\Phi^i(rh) \\
&= \int \mathrm{d}\mu(G_c)\sqrt{\frac{d_k}{d_D}} D_{(k)}{}^{j}{}_{\ell',n}(r^{-1}) D^{\ell'}{}_i(h^{-1}) D^i{}_\ell(h)\Phi^\ell(r) \\
&= \int \mathrm{d}\mu(H) D^{\ell'}{}_i(h^{-1}) D^i{}_\ell(h) \\
&\quad \times \int \mathrm{d}\mu(G_c/H)\sqrt{\frac{d_k}{d_D}} D_{(k)}{}^{j}{}_{\ell'}(r^{-1})\Phi^\ell(r) \\
&= \int \mathrm{d}\mu(G_c/H)\sqrt{\frac{d_k}{d_D}} D_{(k)}{}^{j}{}_{\ell',n}(r^{-1})\Phi^\ell(r) \ ,
\end{aligned}$$

we conclude that the coefficients of the expansion are given by

$$\Phi^{(k)jn} = \int \mathrm{d}\mu(G_c/H)\sqrt{\frac{d_k}{d_D}} D_{(k)}{}^{j}{}_{i,n}(r^{-1})\Phi^i(r) \ .$$

Here, we shall be interested only in functions Φ in the trivial representation of H, so that the expansion simplifies to

$$\Phi(g) = \sum_{k\in \hat{G}_c|_{\mathcal{R}_0}} \sum_{n=1}^{m_k} \sum_{j=1}^{d_k} \sqrt{\frac{d_k}{d_D}} \Phi^{(k)jn} D_{(k)}{}^{i_0}{}_{j,n}(g) \ ,$$

with $\mathcal{R}_0$ and $D_{(k)}{}^{i_0}{}_{j,n}(g)$ the trivial representation of H. In all examples that will be presented in Section 5, we have $m_k = 1$.

Unitary representations of compact Lie algebras are classified either by their Dynkin labels or their Young tableaux which correspond to tensors of a certain type, notably for the classical series $\mathfrak{a}_n = \mathfrak{su}(n+1), \mathfrak{b}_n = \mathfrak{so}(2n+1), \mathfrak{c}_n = \mathfrak{usp}(2n), \mathfrak{d}_n = \mathfrak{so}(2n)$. The representations which contain the scalar representation can then be deduced from either the Dynkin label or from the Young tableau. For instance, the only representations that lead to a scalar representation for the embedding $SO(n-1) \subset SO(n)$ are the representations with dominant weight $|n, 0, \cdots, 0\rangle$ corresponding to n-th order symmetric traceless tensors, *i.e.*, the n^{th}-order symmetric power of the fundamental representation $[1, 0^{n-1}]$.

3.3 Labeling functions in the Peter-Weyl theorem

The functions appearing in harmonic analysis on $\mathcal{M} = G_c$ or $\mathcal{M} = G_c/H$ are associated to all the finite-dimensional unitary representations of G_c. It is however well known that, within a given representation, a weight vector is generally not uniquely defined by its eigenvalues with respect to a given Cartan subalgebra. The purpose of this section is to identify a minimal set of operators to characterise unambiguously all weight vectors in an arbitrary representation. In essence, the main properties of the labelling problem for semisimple Lie algebras are deduced from a theorem due to Racah [34], which in modern terminology can be stated as follows:

Proposition 3.2 *Let $\mathfrak{g}$ be a simple (compact) Lie algebra of rank ℓ. Then the following conditions hold:*

1. *$\mathfrak{g}$ admits ℓ independent primitive Casimir operators $\{C_{d_1}, \cdots, C_{d_\ell}\}$.*[3]

2. *Each C_{d_k} can be represented as a homogeneous polynomial of degree d_k in the generators.*

3. *The degrees d_k of the invariants satisfy the following numerical identity:*

$$\sum_{k=1}^{\ell} d_k = \frac{\dim \mathfrak{g} + \ell}{2}.$$

4. *Any irreducible representation $\mathcal{D}$ of $\mathfrak{g}$ is completely determined by $\frac{\dim \mathfrak{g}+\ell}{2}$ labels, from which*

   (a) *ℓ labels characterise the representation $\mathcal{D}$ as eigenvalues of the Casimir operators $\{C_{d_1}, \cdots, C_{d_\ell}\}$.*

   (b) *A number of $\dfrac{\dim \mathfrak{g} - \ell}{2}$ internal labels are required to separate the states within the multiplet $\mathcal{D}$.*

Although $\mathcal{D}$ can be distinguished from other non-equivalent representations by means of the eigenvalues of the Casimir operators or, alternatively, the highest weight with respect to a given Cartan subalgebra $\mathfrak{h}$, the choice of internal labels is far from being unique, and usually depends on a specific chain of proper subalgebras

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}$$

such that, in each step, the Casimir operators of the subalgebra are used to separate states [36, 37]. We give in Appendix B some details on the construction of internal labels beyond the Cartan subalgebra.

In fact, when $\mathcal{M} = G_c$, we understand the action of the group G_c on a matrix M in a representation $\mathcal{D}$ as a right action and left action, *i.e.*, considering $g \in G_c$ and its corresponding matrix $D(g)$ in the representation $\mathcal{D}$ we have

$$M \to M' = D(g) M D(g)^t \ .$$

This means that the matrix elements of M are labeled by three types of indices associated to their corresponding operators:

1. ℓ labels which specify the representation: they can be the eigenvalues of the Casimir operators or the eigenvalues of the Cartan generators on the highest weight;

2. $\frac{\dim \mathfrak{g}-\ell}{2}$ labels which characterise the lines and are associated to internal labels and Cartan subalgebra for the right action.

[3] By primitive Casimir operators we mean those of minimal degree in the generators [35, 34].

3. $\frac{\dim \mathfrak{g} - \ell}{2}$ labels which characterise the columns and are associated to internal labels and Cartan subalgebra for the left action.

Therefore $\dim \mathfrak{g}$ operators are needed to label unambiguously all states when $\mathcal{M} = G_c$, whilst for $\mathcal{M} = G_c/H$ the number of operators needed is ℓ to label all representations and $\frac{\dim \mathfrak{g} - \ell}{2}$ internal labels.[4]

4 Generalised Kac-Moody algebras

In this section we build explicitly the generalised Kac-Moody algebra associated to the manifold $\mathcal{M}$, where $\mathcal{M}$ is either a compact Lie group G_c or a coset space G_c/H with respect to a closed subgroup $H \subset G_c$. This construction proceeds in several steps.

4.1 Construction of the algebra

Let $\mathfrak{g}$ be a simple real (or complex) Lie algebra with basis $\{T_a, a = 1, \cdots, \dim \mathfrak{g}\}$. The Lie brackets take the form

$$\big[T_a, T_b\big] = \mathrm{i}\, f_{ab}{}^c T_c \ .$$

Further, denote the Killing form by

$$\Big\langle T_a, T_b \Big\rangle_0 = g_{ab} \equiv \ \mathrm{Tr}\Big(\mathrm{ad}(T_a)\mathrm{ad}(T_b)\Big) \ .$$

Let $\mathcal{M}$ be a compact $n = (p+q)-$dimensional manifold of volume V, isomorphic to either G_c or G_c/H, and suppose that we have a parameterisation $y^A = (\varphi^i, u^r) = (\varphi^1, \cdots, \varphi^p, u^1, \cdots, u^q)$ such that

$$\int_{\mathcal{M}} \mathrm{d}\mu(\mathcal{M}) = \frac{1}{V} \int_{\mathcal{M}} \mathrm{d}^p\varphi \ d^q u = 1 \ .$$

Consider the set of square integrable functions on $\mathcal{M}$ which are periodic in all $\varphi-$directions, but not in the $u-$directions. As done in Sections 3.1 and 3.2, we introduce a Hilbert basis of $L^2(\mathcal{M})$ identified with a minimal set of labels (see Section 3.3)

$$\mathcal{B} = \Big\{ \rho_I(\varphi, u) \ , \quad I \in \mathcal{I} \Big\} \ ,$$

where $\mathcal{I}$ denotes the set of all labels needed to identify the states unambiguously. Let $\mathfrak{g}(\mathcal{M})$ be the set of smooth maps from $\mathcal{M}$ into $\mathfrak{g}$:

$$\mathfrak{g}(\mathcal{M}) = \Big\{ T_{aI} = T_a \rho_I(\varphi, u) \ , a = 1, \ldots, \dim \mathfrak{g} \ , I \in \mathcal{I} \Big\} \ .$$

In this case (see Appendix A), the Lie brackets take the form

$$\big[T_{aI}, T_{bJ}\big] = \mathrm{i}\, f_{ab}{}^c c_{IJ}{}^K T_{cK} \ . \tag{8}$$

[4] Actually the full set of labels is only necessary for the generic case. For representations of G_c exhibiting some kind of symmetry, the number of internal labels needed is usually smaller.

The precise form of the coefficients ${c_{IJ}}^K$ defined in (1) is irrelevant at this stage. We shall present several explicit examples in Section 5. Finally, the Killing form in $\mathfrak{g}(\mathcal{M})$ is given by

$$\Big\langle X, Y \Big\rangle_1 = \int_{\mathcal{M}} \mathrm{d}\mu(\mathcal{M}) \Big\langle X, Y \Big\rangle_0 \ , \tag{9}$$

for $X, Y \in \mathfrak{g}(\mathcal{M})$. It follows (see Appendix A) that

$$\rho_I(\varphi, u) = \eta_{IJ} \overline{\rho}^J(\varphi, u) \ ,$$

so that

$$\Big\langle T_{aI}, T_{bJ} \Big\rangle_1 = g_{ab} \eta_{IJ} \ .$$

Central extensions of the generalised algebra. One natural question is whether the algebra determined by (8) admits central extensions. This problem was completely solved by Pressley and Segal in [6] (see Proposition 4.28 therein). Given a one-chain C (*i.e.*, a closed one-dimensional piecewise smooth curve), the central extension is given by the two-cocycle

$$\omega_C(X, Y) = \oint_C \big\langle X, \mathrm{d}Y \big\rangle_0 \ , \tag{10}$$

where $\mathrm{d}Y = \partial_A Y \ \mathrm{d}y^A = \partial_i Y \ \mathrm{d}\varphi^i + \partial_s Y \ \mathrm{d}u^s$ is the exterior derivative of Y. We observe that the two-cocycle ω_C is non-trivial even if the manifold $\mathcal{M}$ has a trivial fundamental group. In fact this result extends also for non-compact differentiable manifolds. Furthermore, as pointed out by Pressley and Segal, this result is somewhat disappointing, as central extensions are characterised by maps from $C \to \mathcal{M}$, where C is one-dimensional. Stated differently, there is no central extension built up from maps $\mathcal{N} \to \mathcal{M}$ when $\dim \mathcal{N} > 1$. The two-cocycle can be written in alternative form. Indeed we have [38]

$$\omega_C(X, Y) = \int_{\mathcal{M}} \big\langle X, \mathrm{d}Y \big\rangle_0 \wedge \gamma \ , \tag{11}$$

where γ is a closed $(n-1)-$current (a distribution) associated to C. In particular if

$$\gamma = \sum_{A=1}^{n} (-1)^A \gamma_A \ \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^{A-1} \wedge \mathrm{d}y^{A+1} \wedge \cdots \wedge \mathrm{d}y^n,$$

then

$$\mathrm{d}\gamma = \sum_{A=1}^{n} \partial_A \gamma_A \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n = 0 \ .$$

Using the topological properties of the manifolds $\mathbb{S}^2$ and $\mathbb{S}^1 \times \mathbb{S}^1$, the authors of Ref. [11] classified all possible central extensions for the case in which γ is defined only by functions and not distributions. For our purposes, in order to have some contact with the current algebra (13), hereafter, we will consider n specific $(n-1)-$forms :

$$\gamma_{(A)} = (-1)^A k_A \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^{A-1} \wedge \mathrm{d}y^{A+1} \wedge \cdots \mathrm{d}y^n \ , \ A = 1, \cdots, n$$

where $k_A \in \mathbb{R}$. Thus

$$\begin{aligned}\omega_{(A)}(T_{aI}, T_{bJ}) &= k_A g_{ab} \int_{\mathcal{M}} \mathrm{d}\mu(\mathcal{M})\ \rho_I(\varphi, u)\partial_A \rho_J(\varphi, u) \\ &= k_A g_{ab} d_{AIJ} \ .\end{aligned}$$

The brackets of the centrally extended algebra $\mathfrak{g}(\mathcal{M})$ take the form

$$\big[T_{aI}, T_{bJ}\big] = \mathrm{i}\ f_{ab}{}^c c_{IJ}{}^K T_{cK} + g_{ab} \sum_{A=1}^{n} k_A d_{AIJ} \ . \tag{12}$$

The algebra constructed by this procedure is closely related to a current algebra with Schwinger terms

$$\big[T_a(y), T_{a'}(y')\big] = \mathrm{i}\ f_{aa'}{}^b T_b(y)\delta^n(y-y') - \mathrm{i} \sum_{A=1}^{n} k_A \partial_A \delta^n(y-y') \ . \tag{13}$$

Indeed, upon integration by $\int \mathrm{d}^n y \int \mathrm{d}^n y'$ (see Appendix A, in particular, equation (76)) equation (13) leads to equation (12). In Ref. [26], Bars constructed centrally extended extensions of the generalised Kac-Moody algebras $\mathfrak{g}(\mathbb{S}^2)$ and $\mathfrak{g}(\mathbb{S}^1 \times \mathbb{S}^1)$ by using a current algebra approach.

As already mentioned, a generalisation of Kac-Moody algebras to the case of non-compact manifolds can be considered, hence the natural question whether these algebras admit central extensions arises. This question was briefly studied in [30], where it was shown that (10) still defines a two-cocycle as $\mathcal{C}$ has no boundary, but that the cocycle reformulation (11) must be treated with care because the manifold $\mathcal{M}$ is non-compact and divergence problems for the integrals may appear. A generic ansatz to circumvent this technical difficulty has not yet been found.

Derivations of the generalised algebra. The last step in the construction of the generalised Kac-Moody algebra associated to the manifold $\mathcal{M}$ is to introduce the derivations ∂_A. However, due to the specific parametrisation of $\mathcal{M}$, the variables φ and u have different periodicity properties. The former are periodic whereas the latter are not. This in particular means that the operators $d_j = -\mathrm{i}\partial_{\varphi^j}$ associated to the variables φ^j are Hermitian whilst, due to the boundary term in the integration by parts, the operators $d_s = -\mathrm{i}\partial_{u^s}$ associated to the variables u^s are not Hermitian. However, as we shall see, additional Hermitian operators beyond $d_j, j = 1, \cdots, p$ can be considered. The existence of these additional operators follows from the relation between the manifold $\mathcal{M}$ and the Lie group G_c. We can thus identify a maximal set of commuting Hermitian operators. Of course, the operators $d_j, j = 1, \cdots, p$ are commuting Hermitian operators. As just mentioned, since the manifolds that we consider are of the form G_c or G_c/H, there exists a largest Lie algebra $\mathfrak{g}_m$ such that $\mathfrak{g}_c \subseteq \mathfrak{g}_m$, with $\mathfrak{g}_c$ the Lie algebra of G_c (see examples below), such that the basic functions ρ_I belong to some unitary irreducible representation of $\mathfrak{g}_m$. Furthermore the generators of the Lie algebra $\mathfrak{g}_m$ can be realised as differential Hermitian operators acting on $\mathcal{M}$. Thus, among those generators we can extract the generators of the Cartan subalgebra $H_1, \cdots, H_k$, where k is the rank of $\mathfrak{g}_m$. We express these operators as

$$H_j = -\mathrm{i} f_j^A(y)\partial_A \ .$$

The Hermicity condition translates into

$$\partial_A f_j^A(y) = 0 \ \text{ and } \ f_j^r| = 0 \ , \ r = 1, \cdots, p, \tag{14}$$

where $f_j^r| = 0$ means that the boundary term associated to all $u-$directions vanishes. Now we identify among the generators $d_1, \cdots, d_p, H_1, \cdots, H_k$ the maximal set of commuting operators that we denote $D_1, \cdots, D_r$. These generators are easily seen to adopt the form

$$D_j = -\mathrm{i} f_j^A(y)\partial_A \ , \ j = 1, \cdots, r$$

and satisfy (14). Naturally, the functions ρ_I are eigenfunctions of H_j and we note

$$H_j(\rho_I(y)) = I(j)\rho_I(y) \ ,$$

with $I(j)$ the corresponding eigenvalue.

It is worthy to be observed that there exists some kind of duality between the Hermitian operators D_j and central extensions. Indeed, one can easily show that the $(n-1)-$forms

$$\gamma_j = k_j \sum_{A=1}^{n} (-1)^A f_j^A(y) \, \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^{A-1} \wedge \mathrm{d}y^{A+1} \wedge \cdots \wedge \mathrm{d}y^n \ , \ j = 1, \ldots, r \tag{15}$$

$k_j \in \mathbb{R}$ are closed because of the condition (14), and the corresponding two-cocycles are given by

$$\omega_k(T_{aI}, T_{bJ}) = k_k J(k) g_{ab} \eta_{IJ} \ . \tag{16}$$

The generalised Kac-Moody algebra is thus generated by

1. T_{aI} which belong to $\mathfrak{g}(\mathcal{M})$;
2. the Hermitian operators $D_1, \cdots, D_r$;
3. the central charges $k_1, \cdots, k_r$ associated to the Hermitian operators.

The non-vanishing brackets of the generalised Kac-Moody algebra associated to $\mathcal{M}$ have the form

$$\begin{aligned} \big[T_{aI}, T_{bJ}\big] &= \mathrm{i} \ f_{ab}{}^c c_{IJ}{}^K T_{cK} + g_{ab}\eta_{IJ} \sum_{j=1}^{r} k_j I(j) \ , \\ \big[D_j, T_{aI}\big] &= I(j) T_{aI} \ , \end{aligned} \tag{17}$$

where $I(j)$ is the eigenvalue of D_j. The authors of Ref. [10] defined generalised Kac-Moody algebras associated to the torus $\mathbb{T}^n$, which coincides with our construction for $G_c = U(1)^n$, and showed that these algebras correspond to specific examples of what they called 'quasi-simple Lie algebras'. It should be observed that all their operators d_A are Hermitian, hence they did not encounter the problem mentioned above.

4.2 Root system of generalised Kac-Moody algebras

The purpose of this section is to identify a root structure for the generalised algebras defined by equation (17). We begin with the roots of the finite-dimensional simple Lie algebra $\mathfrak{g}$. Suppose that $\mathfrak{g}$ is of rank ℓ. Let $H^i, i = 1, \cdots, \ell$, be the generators of the Cartan subalgebra of $\mathfrak{g}$ and let Σ be the root system of $\mathfrak{g}$. We consider the corresponding operators $E_\alpha, \alpha \in \Sigma$, in the usual Cartan-Weyl basis. If we introduce

$$\hat{\mathfrak{g}}(\mathcal{M}) = \mathrm{Span}\Big\{T_{aI}, D_j, k_j, a = 1, \cdots, \dim \mathfrak{g}, I \in \mathcal{I}, j = 1, \cdots, r\Big\} , \tag{18}$$

then we observe from the algebra (17) that, in addition to the elements of the Cartan subalgebra of $\mathfrak{g}$, the operators D_j and k_j commute with each other for $j = 1, \ldots, r$. Thus the Cartan subalgebra of $\hat{\mathfrak{g}}(\mathcal{M})$ is then generated by H^i, D_j and k_j, where $i = 1, \ldots, \ell$, $j = 1, \cdots, r$, and the Cartan-Weyl basis takes the form H^i_I and $E_{\alpha I}$, where the non-vanishing brackets read

$$\begin{aligned}
\big[H^i_I, H^{i'}_{I'}\big] &= \eta_{II'} h^{ii'} \sum_{p=1}^{r} I'(k) k_p \ , \\
\big[H^i_I, E_{\alpha J}\big] &= c_{IJ}{}^K \alpha^i \, E_{\alpha K} \ , \\
\big[E_{\alpha_I}, E_{\beta_J}\big] &= \begin{cases} \mathcal{N}_{\alpha,\beta} \, c_{IJ}{}^K \, E_{\alpha+\beta K} \ , & \alpha+\beta \in \Sigma, \\ c_{IJ}{}^K \, \alpha \cdot H_K + \eta_{IJ} \sum\limits_{p=1}^{r} J(k) k_p \ , & \alpha + \beta = 0, \\ 0, & \begin{cases} \alpha+\beta \neq 0 \ , \\ \alpha+\beta \notin \Sigma, \end{cases} \end{cases} \\
\big[D_i, E_{\alpha J}\big] &= J(i) E_{\alpha_J} \ , \\
\big[D_i, H^j_J\big] &= J(i) H^i_J \ ,
\end{aligned} \tag{19}$$

where

$$h^{ij} = \big\langle H^i, H^j \big\rangle_0 \ ,$$

with the Killing form $\langle \cdot, \cdot \rangle_0$ defined at the beginning of Section 4, and the operators associated to roots of $\mathfrak{g}$ are normalised as

$$\big\langle E_\alpha, E_\beta \big\rangle_0 = \delta_{\alpha,-\beta} \ .$$

Proceeding along the same lines as for usual Kac-Moody algebras (see *e.g.* [7], p. 343-344), we have for the Killing form of $\hat{\mathfrak{g}}(\mathcal{M})$

$$\begin{aligned}
\Big\langle T_{aI}, T_{bJ} \Big\rangle &= \eta_{IJ} g_{ab} \ , \\
\Big\langle D_j, T_{aI} \Big\rangle &= \Big\langle k_j, T_{aI} \Big\rangle = 0 \ , \\
\Big\langle k_i, k_j \Big\rangle &= \Big\langle D_i, D_j \Big\rangle = 0 \ , \\
\Big\langle D_i, k_j \Big\rangle &= \delta^i_j \ .
\end{aligned} \tag{20}$$

The root spaces are given by

$$\begin{aligned}
\mathfrak{g}_{(\alpha,n_1,\cdots,n_r)} &= \Big\{E_{\alpha I} \text{ with } I(1)=n_1,\cdots,I(r)=n_r\Big\}\ ,\alpha\in\Sigma,n_1,\cdots,n_r\in\mathbb{Z}\ ,\\
\mathfrak{g}_{(0,n_1,\cdots,n_r)} &= \Big\{H^i_I \text{ with } I(1)=n_1,\cdots,I(r)=n_r\Big\}\ ,n_1,\cdots,n_r\in\mathbb{Z}\ . \qquad (21)
\end{aligned}$$

Unlike the usual Kac-Moody algebras, the root spaces associated to roots are infinite dimensional and we have

$$\begin{aligned}
\Big[\mathfrak{g}_{(0,\mathbf{n})},\mathfrak{g}_{(\alpha,\mathbf{m})}\Big] &\subset \mathfrak{g}_{(\alpha,\mathbf{m}+\mathbf{n})},\\
\Big[\mathfrak{g}_{(\alpha,\mathbf{m})},\mathfrak{g}_{(\beta,\mathbf{n})}\Big] &\subset \mathfrak{g}_{(\alpha+\beta,\mathbf{m}+\mathbf{n})},\quad \alpha+\beta\in\Sigma
\end{aligned}$$

with $\mathbf{n}=(n_1,\cdots,n_r)$. Introduce also $\mathbf{0}=(0,\cdots,0)$. It is important to observe that the Lie bracket between two elements involves not only the root structure, but also the representation theory of G_c, in the form of the Clebsch-Gordan coefficients ${c_{IJ}}^K$ (see (19)).

To define the set of positive roots, we use the lexicographic order:

$$(\alpha,0,\cdots,0,n_1,\cdots,n_r)>0 \ \text{ if}\begin{cases} \text{either} & \begin{cases}\exists\, k\in\{1,\cdots,r\} \ \text{ s.t.}\\ n_r=\cdots=n_{k+1}=0 \ \text{ and } \ n_k>0\end{cases}\\ \text{or} & n_r=\cdots=n_1=0 \ \text{ and } \ \alpha>0\ .\end{cases} \qquad (22)$$

By (20), we can endow the weight space with a scalar product. Indeed

$$(\alpha,c_1,\cdots,c_r,n_1,\cdots,n_r)\cdot(\alpha',c'_1,\cdots,c'_r,n'_1,\cdots,n'_r)=\alpha\cdot\alpha'+\sum_{j=1}^r\Big(n_jc'_j+n'_Jc_j\Big)\ .$$

We further observe that, as happens for usual Kac-Moody algebras [3], we have two types of roots. The set of roots $(\alpha,\mathbf{0},\mathbf{n})$ of $\mathfrak{g}_{(\alpha,\mathbf{n})}$ with $\alpha\in\Sigma,\mathbf{n}\in\mathbb{Z}^r$ satisfy

$$(\alpha,\mathbf{0},\mathbf{n})\cdot(\alpha,\mathbf{0},\mathbf{n})=\alpha\cdot\alpha>0\ ,$$

and are called *real roots*, whilst the set $(0,\mathbf{0},\mathbf{n})$ of $\mathfrak{g}_{(0,\mathbf{n})}$ with $\mathbf{n}\in\mathbb{Z}^r$ and satisfying

$$(0,\mathbf{0},\mathbf{n})\cdot(0,\mathbf{0},\mathbf{n}')=0\ ,$$

is called the set of *imaginary roots*.

Recall that r denotes the number of central charges (see (18)), that we called the order of centrality. We now show that unless $r=1$, we cannot find a system of simple roots for $\hat{\mathfrak{g}}$. To this extent, introduce $\alpha_i, i=1,\cdots,\ell$ the simple roots of $\mathfrak{g}$. If $r=1$, and we denote by ψ the highest root of $\mathfrak{g}$, it is easy to see that

$$\hat\alpha_i=(\alpha_i,0,0)\ ,\quad i=1,\cdots,\ell\ ,\quad \hat\alpha_{\ell+1}=(-\psi,0,1) \qquad (23)$$

is a system of simple roots of $\hat{\mathfrak{g}}$. Now, if we suppose that $r=2$, as the positive roots are given by (i) $(\alpha,0,0,0,0)$ with $\alpha>0$, or (ii) $(\alpha,0,0,n_1,0)$ with $\alpha\in\Sigma,n_1>0$, or (iii) $(\alpha,0,0,n_1,n_2),\alpha\in\Sigma,n_1\in\mathbb{Z},n_2>0$ and since the roots $(\alpha,0,0,n_1,0)$ are neither bounded from below nor from above because $n_1 \in \mathbb{Z}$, we cannot define a simple root of the form $(-\psi, 0, 0, -n_{\max}, 1)$, where $n_{\max}$ corresponds to the highest possible value of n_1 (or $-n_{\max}$ the lowest possible value of n_1). This means that for $r \geq 2$ we cannot construct a system of simple roots. In other words, the only generalised Kac-Moody algebras that admit simple roots are (obviously) the usual Kac-Moody algebras, but also the Kac-Moody algebras associated to $SU(2)/U(1)$ studied in Section 5.2. In the latter case, the Dynkin diagram of $\hat{\mathfrak{g}}(SU(2)/U(1))$ is analogous to the Dynkin diagram of the corresponding usual Kac-Moody algebra $\hat{\mathfrak{g}}(U(1))$, but is dressed by the representation theory of $SO(3)$. Indeed, in this case the root space is infinite dimensional (see (21)).

We have seen that for a generalised Kac-Moody algebra of centrality order $r > 1$, the set of imaginary roots is r-dimensional. We may then wonder whether the algebra associated to the manifold $\mathcal{M}$ has some relationship with a degenerate Kac-Moody algebra with Cartan matrix of co-rank r. In fact, the algebra associated to the manifold $\mathcal{M}$ does not belong to the general classification of Kac-Moody algebras as given by Kac in [3]. Indeed, Kac-Moody algebras are defined by a (symmetrisable) Cartan matrix and thus admit a Chevalley-Serre presentation. Differently, the algebras considered in this paper represent generalisations of affine Lie algebras. In particular, we have seen that for centrality orders strictly higher than one, there does not exist a system of simple roots, and hence no Cartan matrix or Chevalley-Serre basis exist. Moreover, we can construct all the generators of our generalised algebra, whilst this is not the case for the Kac-Moody algebras (different from affine Lie algebras). Observe moreover that for a centrality order $r = 1$ corresponding to the algebra $\hat{\mathfrak{g}}(U(2)/U(1))$, we have a system of simple roots and a Chevalley-Serre presentation of the algebra. Even if the Dynkin diagram of $\hat{\mathfrak{g}}(U(2)/U(1))$ coincides with the Dynkin diagram of $\hat{\mathfrak{g}}(U(1))$, the former is dressed by the representation theory of $SO(3)$.

To finish this section, we briefly show that the generalised Kac-Moody algebras constructed so far share some properties with the so-called Lorentzian Kac-Moody algebras [39, 40]. Lorentzian Kac-Moody algebras appear in $M-$theory or in eleven-dimensional supergravity compactifed on tori. Such algebras are defined by a Cartan matrix (or a Dynkin diagram) subjected to some constraints.

To this extent, introduce the root-lattice of $\hat{\mathfrak{g}}(\mathcal{M})$. We begin with the root lattice of $\mathfrak{g}$

$$\Lambda_R(\mathfrak{g}) = \Big\{ \sum_{i=1}^{\ell} n^i \alpha_i \ , \ n^i \in \mathbb{Z} \Big\} \ ,$$

with $\alpha_1, \cdots, \alpha_\ell$ the simple roots of $\mathfrak{g}$, supposed of rank ℓ. We then introduce the two-dimensional Lorentzian even self-dual lattice [39, 40]

$$\Pi^{1,1} = \Big\{ (m, n) \ , \ m, n \in \mathbb{Z} \ \Big\} \ ,$$

endowed with the Lorentzian scalar product

$$(m, n) \cdot (m', n') = mn' + nm' \ .$$

Let $(e = (0, 1), \bar{e} = (1, 0))$ satisfying $e \cdot \bar{e} = 1, e \cdot e = \bar{e} \cdot \bar{e} = 0$ be a basis of $\Pi^{1,1}$. Now assume that $\mathcal{M} = \mathbb{T}^r$ and introduce $r-$copies of $\Pi^{1,1}$, and the corresponding basis $(e_i, \bar{e}_i), i = 1, \cdots, r$.

Therefore we have (see (21))

$$\Lambda_R(\hat{\mathfrak{g}}(\mathbb{T}^r)) \subset \Lambda_R(\mathfrak{g}) \oplus \underbrace{\Pi^{1,1} \oplus \cdots \oplus \Pi^{1,1}}_{r-\text{times}} ,$$

i.e., the root lattice of $\hat{\mathfrak{g}}(\mathbb{T}^r)$ is a sublattice of $\Lambda_R(\mathfrak{g}) \oplus \Pi^{1,1} \oplus \cdots \oplus \Pi^{1,1}$. More precisely, $\alpha \in \Lambda_R(\hat{\mathfrak{g}}(\mathcal{M}))$ if $\alpha \cdot e_i = 0, i = 1, \cdots, r$. Thus the root system, as well as all generators of $\hat{\mathfrak{g}}(\mathbb{T}^r)$, are known whereas, as we have seen previously, for $r > 1$ it is not possible to identify a system of simple roots. As a consequence, a Chevalley-Serre basis is not available.

In the same manner, some types of Lorentzian Kac-Moody algebras can be obtained from any semisimple Lie algebra $\mathfrak{g}$. For instance, the so-called 'very extended Lie algebra' $\mathfrak{g}^{+++}$ is a rank $\ell + 3$ Lorentzian Lie algebra associated to the semisimple Lie algebra $\mathfrak{g}$ (of rank ℓ), where the simple roots are constructed from the simple roots of $\mathfrak{g}$ and two copies of $\Pi^{1,1}$ [39, 40]. Therefore,

$$\Lambda_R(\mathfrak{g}^{+++}) \subset \Lambda_R(\mathfrak{g}) \oplus \Pi^{1,1} \oplus \Pi^{1,1} .$$

The simple roots of the very extended Lie algebra $\mathfrak{g}^{+++}$ are known. This means that one can introduce a Chevalley-Serre basis for these algebras. However, in this case explicit formulae for all generators of $\mathfrak{g}^{+++}$ are not available.

Thus even if the root lattices of $\hat{\mathfrak{g}}(\mathbb{T}^2)$ and $\mathfrak{g}^{+++}$ are both sub-lattices of $\Lambda_R(\mathfrak{g}) \oplus \Pi^{1,1} \oplus \Pi^{1,1}$ those two Lie algebras have different properties.

4.3 Representations of generalised Kac-Moody algebras

In this paragraph we outline some relevant points concerning the representations of generalised Kac-Moody algebras, with special emphasis on the existence of central charges in connection with the unitarity of representations. In the following, we consider the quasi-simple Lie algebras as introduced in [10], and assume that $\mathfrak{g}$ is a compact real Lie algebra. In the Cartan-Weyl basis, the algebra is generated by

$$\hat{\mathfrak{g}}(U(1)^r) = \left\{ H^i_{\mathbf{m}}, E_{\alpha,\mathbf{m}} \ , \ \alpha \in \Sigma, \ \mathbf{m} \in \mathbb{Z}^n, \ d_i, k_i, i = 1, \cdots, r \right\} ,$$

and the Lie brackets are given by ($\mathbf{m} = (m_1, \cdots, m_r)$)

$$\begin{aligned}
\left[H^i_{\mathbf{m}}, H^{i'}_{\mathbf{m}'}\right] &= \delta_{\mathbf{m}+\mathbf{m}'} h^{ii'} \sum_{i=1}^r m_i k_i \ , \\
\left[H^i_{\mathbf{m}}, E_{\alpha\mathbf{n}}\right] &= \alpha^i \ E_{\alpha\mathbf{m}+\mathbf{n}} \ , \\
\left[E_{\alpha\mathbf{m}}, E_{\beta\mathbf{n}}\right] &= \begin{cases} \mathcal{N}_{\alpha,\beta} \ \ E_{\alpha+\beta\mathbf{m}+\mathbf{n}} \ , & \alpha+\beta \in \Sigma, \\ \alpha \cdot H_{\mathbf{m}+\mathbf{n}} + \delta_{\mathbf{m}+\mathbf{m}'} \sum\limits_{i=1}^r m_i k_i, & \alpha+\beta = 0, \\ 0, & \begin{cases} \alpha+\beta \neq 0 \ , \\ \alpha+\beta \notin \Sigma, \end{cases} \end{cases} \\
\left[d_i, E_{\alpha\mathbf{m}}\right] &= m_i E_{\alpha\mathbf{m}} \ , \\
\left[d_i, H^j_{\mathbf{m}}\right] &= m_i H^i_{\mathbf{m}} \ .
\end{aligned} \tag{24}$$

In the following, in order to increase the readability of some long formulae, we sometimes use the convention that $\delta_{a+b} = \delta^0_{a+b} = \delta^{-b}_a$.

As before, we consider the set $\alpha_i, i = 1, \cdots, \ell$ of simple roots and ψ the highest root of $\mathfrak{g}$. Consider also the fundamental weights $\mu^i, i = 1, \cdots, \ell$ of $\mathfrak{g}$ satisfying

$$2\mu^i \cdot \frac{\alpha_j}{\alpha_j \cdot \alpha_j} = \delta^i_j \ .$$

We also suppose to be given a representation of $\hat{\mathfrak{g}}(U(1)^r)$ with highest weight

$$|\hat{\mu}_0\rangle = |\mu_0, \mathbf{c}, \mathbf{m}\rangle \ , \quad \text{where} \ \ \mu_0 = p_i \mu^i \ , \quad (p_1, \cdots, p_r) \in \mathbb{N}^r$$

such that the relations

$$\begin{array}{rcl} H^i \, |\hat{\mu}_0\rangle &=& \mu^i_0 \, |\hat{\mu}_0\rangle \ , \\ k_i \, |\hat{\mu}_0\rangle &=& c_i \, |\hat{\mu}_0\rangle \ , \\ d_i \, |\hat{\mu}_0\rangle &=& m_i \, |\hat{\mu}_0\rangle \, , \end{array}$$

and

$$\begin{array}{rcl} E_{\alpha \mathbf{m}} \, |\hat{\mu}_0\rangle &=& 0 \ , \quad (\alpha, \mathbf{0}, \mathbf{m}) > 0 \ , \\ H^i_{\mathbf{m}} \, |\hat{\mu}_0\rangle &=& 0 \ , \quad \mathbf{m} > 0 \ . \end{array}$$

are satisfied. For any positive real root $\hat{\alpha} = (-\alpha, \mathbf{0}, \mathbf{m})$, the generators

$$X^{\pm}_{\alpha, \mathbf{m}} = \sqrt{\frac{2}{\alpha \cdot \alpha}} E_{\mp\alpha, \pm \mathbf{m}} \ , \quad h_\alpha = \frac{2}{\alpha \cdot \alpha}\Big(- \alpha_i H^i_{\mathbf{0}} + \sum_{i=1}^r m_i k_i \Big) \ , \tag{25}$$

span an $\mathfrak{su}(2)-$subalgebra. The unitarity condition implies the constraints

$$\begin{array}{c} \frac{2}{\alpha \cdot \alpha}\Big(- \alpha \cdot \mu_0 + \sum\limits_{i=1}^r c_i m_i \Big) \in \mathbb{Z} \ , \\ \sum\limits_{i=1}^r c_i m_i \geq \alpha \cdot \mu_0 \ , \end{array} \tag{26}$$

with the latter identity being a consequence of the relation

$$\begin{array}{rcl} ||X^-_{\alpha, \mathbf{m}} \, |\hat{\mu}\rangle||^2 = \langle \hat{\mu} \mid X^+_{\alpha, \mathbf{m}} X^-_{\alpha, \mathbf{m}} |\hat{\mu}\rangle &=& \langle \hat{\mu} \mid \big[X^+_{\alpha, \mathbf{m}}, X^-_{\alpha, \mathbf{m}}\big] \ |\hat{\mu}\rangle = \langle \hat{\mu} \mid h_\alpha \mid \hat{\mu}\rangle \\ &=& \dfrac{2}{\alpha \cdot \alpha}\Big(- \alpha \cdot \mu_0 + \sum\limits_{i=1}^r c_i m_i \Big) \geq 0 \ . \end{array}$$

If $\alpha > 0$ then $\alpha \cdot \mu_0 > 0$, whereas for $\alpha < 0$ we obtain $\alpha \cdot \mu_0 < 0$. We thus suppose that $\alpha > 0$. In this case the second relation in (34) is very strong. Indeed if $(-\alpha, \mathbf{0}, \mathbf{m}) > 0$, this means that $\mathbf{m} = (m_1, \cdots, m_{k-1}, m_k, 0, \cdots, 0)$ with $m_k > 0$ and $m_1, \cdots, m_{k-1} \in \mathbb{Z}$. The second condition of (34), which must be satisfied for any $k = 1, \cdots, r$, is equivalent to impose that only one central charge is non-vanishing. Therefore, without loss of generality we can suppose $c_r = c \neq 0$ and $c_i = 0, i = 1, \cdots, r-1$.

Next, prior to analyse unitary representations, we observe that, since $c_1 = \cdots = c_{r-1} = 0$, the algebra $\hat{\mathfrak{g}}(U(1)^r)$ does not admit highest weight unitary representations. For this purpose, consider now $\hat{\alpha} = (-\alpha, \mathbf{0}, m_1, \ldots, m_k, 0, \ldots 0) > 0$ with $1 \leq k \leq r$, $m_k > 0$ and $(m_1, \ldots, m_{k-1}) \in \mathbb{Z}^{k-1}$. The operators (with $\mathbf{m} = (m_1, \cdots, m_k, 0, \cdots, 0)$)

$$Y^{\pm}_{\hat{\alpha}} = \sqrt{\frac{2}{\alpha \cdot \alpha}} E_{\mp\alpha, \pm\mathbf{m}} \ , \quad h'_\alpha = \frac{2}{\alpha \cdot \alpha}\Big(- \alpha_i H^i_{\mathbf{0}} \Big) \ , \tag{27}$$

also generate an $\mathfrak{su}(2)-$subalgebra. As before, the condition $||Y^{\pm}_{\hat{\alpha}} |\hat{\mu}\rangle||^2 \geq 0$ holds, from which we deduce that

$$\sum_{p=1}^{k} m_i c_i \geq \alpha \cdot \mu \ . \tag{28}$$

However, as $c_1 = \cdots = c_{r-1} = 0$, this cannot be satisfied. This contradiction arises from the fact that the central charges vanish for $k_1, \ldots, k_{r-1}$, and is in direct agreement with the well known result of classical Kac-Moody algebras that states that the unique highest weight unitary representation for $k = 0$ is the trivial one (the adjoint representation, for which the central charge vanishes, is not a highest weight representation).

With these preliminaries, we will construct unitary representations in two steps. As there is only one non-vanishing central charge k_r, in the first step we consider the usual Kac-Moody algebra

$$\tilde{\mathfrak{g}} = \mathfrak{g}\left(U(1)\right) = \{T_{am}, k_r, d_r, \alpha \in \Sigma, m \in \mathbb{Z}\} \, . \tag{29}$$

Representations of the latter are well known and correspond to the unitary representations of $\tilde{\mathfrak{g}}$ with highest weight $|\tilde{\mu}\rangle = |\mu, c_r, m_r\rangle$. In this case, with ψ being the highest root of $\mathfrak{g}$, we obtain:

$$\frac{\psi}{\psi \cdot \psi} = \sum_{i=1}^{\ell} q^i \frac{\alpha_i}{\alpha_i \cdot \alpha_i} \ , \quad q^i \in \mathbb{N}, \tag{30}$$

and the second condition of (34) translates to

$$x \geq p_i q^i \ , \tag{31}$$

where $x = 2\frac{c}{\psi \cdot \psi}$ is the level of the representation. We now recall some known results on unitary representations of $\tilde{\mathfrak{g}}$ (see *e.g.* [7]).

The simple roots of $\tilde{\mathfrak{g}}$ are given by

$$\hat{\alpha}_0 = (-\psi, 0, 1) \ , \quad \hat{\alpha}_i = (\alpha_i, 0, 0) \ , i = 1, \cdots, \ell$$

where the second entry corresponds to the eigenvalue of the non-vanishing central charge and the third one to the eigenvalue of the corresponding Hermitean operator, say k_r and d_r respectively. Introduce also

$$\hat{\Sigma} = \Big\{ \hat{\alpha} = (\alpha, 0, n) \ , \quad (0, 0, n) \ , \quad \alpha \in \Sigma, n \in \mathbb{Z} \Big\} \ .$$

The fundamental weights are defined by

$$\hat{\mu}^0 = (0, \frac{1}{2}q^0\psi\cdot\psi, 0) \ \text{ with } \ q^0 = 1 \ , \ \hat{\mu}^i = (\mu^i, \frac{1}{2}q^i\psi\cdot\psi, 0) \ , \ i = 1, \cdots, \ell \ ,$$

with the $q^i, i = 1, \cdots, r$ defined by (30). We obviously have

$$2\hat{\mu}^i \cdot \frac{\hat{\alpha}_j}{\hat{\alpha}_j\cdot\hat{\alpha}_j} = \delta^i_j \ .$$

A highest weight is then specified by

$$\hat{\mu}_0 = p_i\hat{\mu}^i, \quad \text{with } p_i \in \mathbb{N} \ , i = 0, \cdots, \ell \ ,$$

and the level of the representation is then given by

$$x = \sum_{i=0}^{\ell} p_i q^i \geq \sum_{i=1}^{\ell} p_i q^i \ .$$

We denote the corresponding representation space as $\mathcal{D}_{\hat{\mu}_0}$.

In a second step, let $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}}\left(U(1)^{r-1}\right)$, which corresponds to the set of smooth maps from $U(1)^{r-1}$ into $\widetilde{\mathfrak{g}}$. We deduce that (see (29))

$$\begin{aligned} \hat{\mathfrak{g}}(U(1)^r) &= \widetilde{\mathfrak{g}}(U(1)^{r-1}) && (32) \\ &= \Big\{ T_{am_r} e^{\mathrm{i}\sum\limits_{k=1}^{n-1} m_k\varphi_k}, \ \ \mathbf{m} \in \mathbb{Z}^{r-1} \ , \ d_r \ , k_r \ \text{ and } \ d_j = -\mathrm{i}\partial_j \ , \ \ j = 1, \cdots, n-1 \Big\} \ . \end{aligned}$$

As seen in (34), the central charges associated to $d_j, j = 1, \cdots, r-1$ vanish. Consider now

$$\mathcal{R} = \Big\{ |\mathbf{m}\rangle, \mathbf{m} \in \mathbb{Z}^{r-1} \Big\} \ , \tag{33}$$

the set of all unitary representations of $U(1)^{r-1}$. We have

$$d_j|\mathbf{m}\rangle = m_j|\mathbf{m}\rangle \ .$$

Using the harmonic expansion on $U(1)^{r-1}$

$$\langle(\varphi_1, \cdots, \varphi_{r-1})|\mathbf{m}\rangle = \mathbf{e}^{\mathrm{i}(\mathrm{m}_1\varphi_1 + \ldots \mathrm{m}_{\mathrm{r}-1}\varphi_{\mathrm{r}-1})} \ ,$$

unitary representations of $\hat{\mathfrak{g}}$ are given by the tensor product

$$\widehat{\mathcal{D}_{\tilde{\mu}}} = \mathcal{D}_{\tilde{\mu}} \otimes \mathcal{R} \tag{34}$$

and correspond to a harmonic expansion of the unitary representation $\mathcal{D}_{\tilde{\mu}}$ of $\tilde{\mathfrak{g}}$ on the manifold $U(1)^{r-1}$.

A unitary representation of $\hat{\mathfrak{g}}(U(1)^r)$ follows directly from unitary representations of $\widetilde{\mathfrak{g}}$. As the space (33) is neither bounded from above nor bounded from below, unitary representations of $\hat{\mathfrak{g}}(U(1)^r)$ are not highest weight representations. As we have just seen, unitarity of representations implies only *one* non-vanishing central charge. This result seems to be contradictory at a first sight, in particular when considering the tensor product of two-representations for two different non-vanishing central charges. As an illustration, consider for instance the generalised Kac-Moody algebra associated to the manifold $\mathcal{M} = \mathbb{T}^2$. Denote with $\mathbf{k} = (k_1, k_2)$ the central charges $(k_1, k_2 \neq 0)$ of $\mathfrak{g}_{12} = \hat{\mathfrak{g}}(U(1)^2)$, without any unitarity constraints. From the previous result, two cases can be considered if one wants unitary highest weight representations: (1) $\mathbf{k} = (k_1, 0)$ or (2) $\mathbf{k} = (0, k_2)$. These two choices lead to two possible isomorphic (*but different*) algebras that we denote $\mathfrak{g}_1$ respectively $\mathfrak{g}_2$ (see Eq.[32]). We can now consider unitary representations of the first or the second algebra. Let $\mathcal{D}_1 \otimes \mathcal{R}_2$ (resp. $\mathcal{R}_1 \otimes \mathcal{D}_2$) be a unitary representation of $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$) with the notations of (34). While $\Big(\mathcal{D}_1 \otimes \mathcal{R}_2\Big) \otimes \Big(\mathcal{R}_1 \otimes \mathcal{D}_2\Big)$ is certainly a representation of the algebra $\mathfrak{g}_1 \times \mathfrak{g}_2$, *it is not* a representation of the algebra $\mathfrak{g}_{12}$, which has *two* non-vanishing central charges. This shows that the contradiction is only apparent, with no conflict emerging from the construction.

Now we can extend part of the results to the general case, *i.e.*, when $\mathcal{M} = G_c$ or G_c/H. Let α be a root of the compact Lie algebra $\mathfrak{g}$. Next, introduce

$$X^+_{\alpha,\mathbf{m}} \in \mathfrak{g}_{(-\alpha,\mathbf{m})} \ , \quad X^-_{\alpha,\mathbf{m}} \in \mathfrak{g}_{(\alpha,-\mathbf{m})} \ , \quad h_\alpha \in \mathfrak{g}_{(0,\mathbf{0})} \ ,$$

with $\mathbf{m} = (m_1, \cdots, m_r) > 0$. Note that in this case, with the notations of equation (19), for $X^+_{\alpha,\mathbf{m}} \in \mathfrak{g}_{(-\alpha,\mathbf{m})}$, $X^-_{\alpha,\mathbf{m}} \in \mathfrak{g}_{(\alpha,-\mathbf{m})}$ we have $\mathbf{m}(k) = m_k$ and $-\mathbf{m}(k) = -m_k$ respectively. Furthermore, the coefficient $\eta_{\mathbf{mn}}$ appearing in the bracket $\big[X^+_{\alpha,\mathbf{m}}, X^-_{\alpha,\mathbf{m}}\big]$ simplifies in this case to $\eta_{\mathbf{m}}$, which is a sign (see the examples given in the next section). Assume further that the operators $X^\pm_{\alpha,\mathbf{m}}, h_\alpha - \frac{2}{\alpha\cdot\alpha}\eta_{\mathbf{m}} \sum\limits_{i=1}^{r} m_i k_i$ are chosen in such way that they generate an $\mathfrak{su}(2)-$subalgebra:

$$\big[h_\alpha, X^\pm_{\alpha,\mathbf{m}}\big] = \pm X^\pm_{\alpha,\mathbf{m}} \ , \quad \big[X^+_{\alpha,\mathbf{m}}, X^-_{\alpha,\mathbf{m}}\big] = h_\alpha - \frac{2}{\alpha\cdot\alpha}\eta_{\mathbf{m}} \sum_{i=1}^{r} m_i k_i \ . \tag{35}$$

Then, in analogy to the previous discussion of the unitarity of representations, it follows that all central charges except one must be equal to zero. It is important to observe that, in absence of symmetries between the generators D_i, we can have r different possibilities given by (eventually reordering the eigenvalues of the operators D_i to define positive roots, see equation (22))

$$\mathbf{c} = (0, \cdots, 0, c_p, 0, \cdots, 0) \ \text{ or } \ \mathbf{k} = (0, \cdots, 0, k_p, 0, \cdots, 0) \ \ p \in \{1, \cdots, r\} \ .$$

In this situation, it remains to identify the precise form of the operators in (35) and to establish a condition analogous to (31) to characterise unitary representations. Independently of these conditions, unitarity leads to only one non-vanishing central charge. Consequently, there are no obstructions to introduce a system of simple roots, as seen in Section 4.2. The resulting Dynkin diagram of $\hat{\mathfrak{g}}(\mathcal{M})$ is analogous to the Dynkin diagram of the corresponding usual Kac-Moody algebra $\hat{\mathfrak{g}}(U(1))$, but dressed with the representation theory of G_c.

Let us emphasise again that, given the Lie algebra $\hat{\mathfrak{g}}(\mathcal{M})$, all central charges except one must be equal to zero in order to guarantee the unitarity of a representation. This result can be compared to that of the previous section, where we proved that only when the order of centrality, *i.e.*, the number of non-vanishing central charges is equal to one, the algebra admits a system of simple roots.

5 Explicit construction in low rank

In this section, we illustrate the procedure previously developed with some physically relevant examples in which we shall identify the coefficients ${c_{IJ}}^K$ together with the Hermitian operators D_i and their corresponding central extensions k_i. The brackets will always be given by equation (17). We begin naturally with the Lie group $SU(2)$, then we turn to the coset spaces $SU(2)/U(1)$, $SO(4)/SO(3)$, $SU(3)/SU(2)$ and $G_2/SU(3)$. For elementary definitions see *e.g.* [41].

5.1 Real Lie group $SU(2)$

The group $SU(2)$ is defined as the set of special unitary 2×2 complex matrices:

$$SU(2)=\Big\{U\in\mathcal{M}_2(\mathbb{C})\ \text{ such that }\ U^\dagger U=1,\ \det U=1\Big\}\ ,$$

that can be written in the form

$$U=\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix}\ ,\quad \alpha,\beta\in\mathbb{C}\ ,\quad |\alpha|^2+|\beta|^2=1\ ,$$

showing that the isomorphism of manifolds $SU(2)\cong\mathbb{S}^3$ holds. Indeed, setting (see [41])

$$\left\{\begin{array}{rcl}\alpha&=&\cos\theta e^{i\varphi_1}\ ,\\ \beta&=&\sin\theta e^{i\varphi_2}\ ,\end{array}\right.\quad 0\le\theta\le\frac{\pi}{2}\ ,\quad 0\le\varphi_1<2\pi\ ,\quad 0\le\varphi_2<2\pi \tag{36}$$

we obtain a parameterisation of the sphere $\mathbb{S}^3$. In the language of Appendix A, this leads to the parameterisation of $\mathbb{S}^3$ given by

$$0\le\varphi_1,\varphi_2\le 2\pi\ ,0\le u=\frac{1}{2}\sin^2\theta\le\frac{1}{2}\ . \tag{37}$$

This parameterisation is a bijection on a dense subset of $SU(2)$, namely when $\theta\ne\big\{0,\frac{\pi}{2}\big\}$. We observe that this parameterisation is not a homeomorphism from $[0,\frac{\pi}{2}]\times[0,2\pi)\times[0,2\pi)$ onto $\mathbb{S}^3$, as the interval is non-compact. The scalar product on $SU(2)$ is given by (see [41])

$$(f,g)=\frac{1}{2\pi^2}\int\limits_0^{\frac{\pi}{2}}\sin\theta\cos\theta\mathrm{d}\theta\int\limits_0^{2\pi}\mathrm{d}\varphi_1\int\limits_0^{2\pi}\mathrm{d}\varphi_2\ \overline{f(\theta,\varphi_1,\varphi_2)}\ g(\theta,\varphi_1,\varphi_2)\ .$$

It is easy to observe that the functions

$$\psi_{a,b}(\theta,\varphi_1,\varphi_2)=\sqrt{\frac{(a+b+1)!}{a!\ b!}}\alpha^a(-\overline{\beta})^b\ ,a,b\in\mathbb{N}\ ,$$

satisfy the relations

$$(\psi_{a,b}, \psi_{a',b'}) = \delta^a_{a'}\delta^b_{b'} \ . \tag{38}$$

Furthermore, we can see from equation (38) that the functions $\alpha, \beta \in \mathbb{S}^3$ defined in equation (36) enable us to obtain all the matrix elements introduced in Section 3.1 in a ready manner. To this extent, we introduce a differential realisation of the generators of the Lie algebra $\mathfrak{su}(2)$ in the above parameterisation of $SU(2)$:

$$\begin{aligned}
J_+ &= J_1 + \mathrm{i}J_2 = \frac{1}{2}e^{\mathrm{i}(\varphi_1-\varphi_2)}\Big(-\mathrm{i}\tan\theta\frac{\partial}{\partial\varphi_1} - \mathrm{i}\cot\theta\frac{\partial}{\partial\varphi_2} + \frac{\partial}{\partial\theta}\Big) \ , \\
J_- &= J_1 - \mathrm{i}J_2 = \frac{1}{2}e^{-\mathrm{i}(\varphi_1-\varphi_2)}\Big(-\mathrm{i}\tan\theta\frac{\partial}{\partial\varphi_1} - \mathrm{i}\cot\theta\frac{\partial}{\partial\varphi_2} - \frac{\partial}{\partial\theta}\Big) \ , \\
J_3 &= -\frac{\mathrm{i}}{2}\big(\frac{\partial}{\partial\varphi_1} - \frac{\partial}{\partial\varphi_2}\big) \ ,
\end{aligned} \tag{39}$$

with Lie brackets

$$\big[J_3, J_\pm\big] = \pm J_\pm \ , \quad \big[J_+, J_-\big] = 2J_3 \ .$$

This differential realisation acts on the rows of matrices and thus corresponds to a right action. Similarly we can define a left action acting on the columns. The generators are the same as in (39), except that we have to replace φ_2 by $-\varphi_2$. We do not give the form of the generators except for the last one

$$J'_3 = -\frac{\mathrm{i}}{2}\left(\frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2}\right) \ , \tag{40}$$

but it can be explicitly checked that right and left actions commute.

The space of irreducible unitary representations is given by $\widehat{\mathcal{R}} = \Big\{\mathcal{R}_\ell, \ \ell \in \frac{1}{2}\mathbb{N} \ \Big\}$ with the representation $\mathcal{R}_\ell$ of dimension $d_\ell = 2\ell + 1$. Thus, for each ℓ, we have to identify $2\ell+1$ equivalent representations associated to the right action. The key observation for this identification is given by the two complex-conjugate two-dimensional spinor representations defined as

$$\begin{aligned}
\mathcal{D}_{\frac{1}{2},\frac{1}{2}} &= \Big\{\Phi_{\frac{1}{2},\frac{1}{2},\frac{1}{2}} = \sqrt{2}\alpha \ , \Phi_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}} = \sqrt{2}\beta\Big\} \ , \\
\mathcal{D}_{-\frac{1}{2},\frac{1}{2}} &= \Big\{\Phi_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}} = -\sqrt{2}\overline{\beta} \ , \Phi_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} = \sqrt{2}\overline{\alpha}\Big\} \ ,
\end{aligned}$$

such that $\mathcal{D}_{-\frac{1}{2},\frac{1}{2}} = \overline{\mathcal{D}}_{\frac{1}{2},\frac{1}{2}} \cong \mathcal{D}_{\frac{1}{2},\frac{1}{2}}$, as expected. In the notation above, the first index corresponds to the eigenvalue of the Cartan generator of the left action or J'_3 (see (40)), the last index to the eigenvalue of the Cartan generator of the right action or J_3 (see (39)), whereas the second index corresponds to the eigenvalue of the Casimir operator of the spinor representation. This identification is in accordance with Section 3.3 and the labelling problem.

This can be extended easily to an arbitrary representation. Indeed, for any $\ell \in \frac{1}{2}\mathbb{N}$, define the $2\ell+1$ equivalent representation spaces corresponding to the right action $\mathcal{D}_{m',\ell}$,

$-\ell \leq m' \leq \ell$. Each space admits the highest weight vector

$$\Phi_{m',\ell,\ell} = \sqrt{\frac{(2\ell+1)!}{(\ell+m')!(\ell-m')!}}\alpha^{\ell+m'}(-\overline{\beta})^{\ell-m'} \tag{41}$$
$$= (-1)^{\ell-m'}\sqrt{\frac{(2\ell+1)!}{(\ell+m')!(\ell-m')!}}e^{\mathrm{i}(\ell+m')\varphi_1-\mathrm{i}(\ell-m')\varphi_2}\cos^{\ell+m'}(\theta)\sin^{\ell-m'}(\theta)\ .$$

In order to obtain the remaining vectors of the representation space $\mathcal{D}_{m',\ell}$ we use the relation

$$L_{\pm}^{k}\Big(e^{\mathrm{i}m_1\varphi_1-\mathrm{i}m_2\varphi_2}F(\theta)\Big) = \frac{(\mp)^k}{2^k}e^{\mathrm{i}(m_1\pm k)\varphi_1-\mathrm{i}(m_2\pm k)\varphi_2}\sin^{k\pm m_2}\theta\cos^{\pm m_1}\theta$$
$$\frac{\mathrm{d}^k}{\mathrm{d}(\cos\theta)^k}\Big[\sin^{\mp m_2}\theta\ \cos^{\mp m_1}\theta\ F(\theta)\Big]\ ,$$

which can be proved by induction. It follows that

$$\Phi_{m',\ell,m} = \sqrt{\frac{(\ell+m)!}{(2\ell)!(\ell-m)!}}\Big(J_-\Big)^{\ell-m}\Bigg[(-1)^{\ell-m'}\sqrt{\frac{(2\ell+1)!}{(\ell+m')!(\ell-m')!}}$$
$$\times e^{\mathrm{i}(\ell+m')\varphi_1-\mathrm{i}(\ell-m')\varphi_2}\cos^{\ell+m}\theta\sin^{\ell-m'}\theta\Bigg]$$
$$= (-1)^{\ell-m'}\frac{1}{2^{\ell-m}}\sqrt{\frac{2\ell+1}{(\ell+m')!(\ell-m')!}\frac{(\ell+m)!}{(\ell-m)!}}\ e^{\mathrm{i}(m+m')\varphi_1-\mathrm{i}(m-m')\varphi_2}$$
$$\times\sin^{-m+m'}\theta\ \cos^{-\ell-m'}\theta\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)^{\ell-m}}\Big[\sin^{2\ell-2m'}\theta\cos^{2\ell+2m'}\theta\Big]\ .$$

When ℓ is an integer number and $m' \geq 0$, the formula simplifies. If we define $u = \cos 2\theta$ and the polynomials

$$P_{m',\ell,m}(u) = (-1)^{\ell-m'}\frac{1}{2^\ell}\sqrt{\frac{1}{(\ell+m')!(\ell-m')!}}\frac{\mathrm{d}^{\ell-m}}{\mathrm{d}u^{\ell-m}}\Big[(1-u^2)^{\ell-m'}(1+u)^{2m'}\Big],$$

we have

$$\Phi_{m',l,m} = \sqrt{\frac{(\ell+m)!}{(\ell-m)!}}e^{\mathrm{i}(m+m')\varphi_1-\mathrm{i}(m-m')\varphi_2}\sin^{-m+m'}2\theta(1+\cos 2\theta)^{-m'}P_{m',\ell,m}(u)\ .$$

In particular

$$\Phi_{m',\ell,0} = e^{\mathrm{i}m'(\varphi_1-\varphi_2)}\sin^{m'}2\theta(1+\cos 2\theta)^{-m'}P_{m',\ell,0}(u)\ .$$

There are analogous formulæ for $m' \leq 0$.

We conclude that the Φ-functions are orthonormal

$$(\Phi_{m_1',\ell_1,m_1},\Phi_{m_2',\ell_2,m_2}) = \delta_{m_2'}^{m_1'}\delta_{\ell_2}^{\ell_1}\delta_{m_2}^{m_1}\ ,$$

and constitute an orthonormal Hilbert basis which is well adapted to the Peter-Weyl theorem applied to $SU(2)$:

$$\mathcal{B} = \left\{ \Phi_{m',\ell,m},\ \ell \in \frac{1}{2}\mathbb{N}, -\ell \leq m, m' \leq \ell \right\} . \tag{42}$$

From the highest weight (41) we have the conjugacy property

$$\bar{\Phi}^{m',\ell,m}(\theta,\varphi_1,\varphi_2) = (-1)^{m-m'}\Phi_{-m',\ell,-m}(\theta,\varphi_1,\varphi_2) . \tag{43}$$

In order to define the Lie algebra $\hat{\mathfrak{g}}(SU(2))$, we introduce $T_{a,m',\ell,m} = T_a\Phi_{m',\ell,m}(\theta,\varphi_1,\varphi_2)$, the two Hermitian operators J'_3, J_3 and their associated central charges k, k' (see (15) and (16)) . The Lie brackets take then the form (see (17))

$$\begin{aligned} \left[T_{a_1,m'_1,\ell_1,m_1}, T_{a_2,m'_2,\ell_2,m_2}\right] &= \mathrm{i} f_{a_1a_2}{}^{a_3} c_{m'_1,\ell_1,m_1,m'_2,\ell_2,m_2}{}^{m'_3,\ell_3,m_3} T_{a_3,m'_3,\ell_3,m_3} \\ &\quad + (-1)^{m_1-m'_1} g_{a_1a_2}\delta_{\ell_1,\ell_2}\delta_{m_1+m_2}\delta_{m'_1+m'_2,}(km_2 + k'm'_2) , \\ \left[J'_3, T_{a,m',\ell_1,m}\right] &= m' T_{a,m',\ell_1,m} , \\ \left[J_3, T_{a,m',\ell_1,m}\right] &= m T_{a,m',\ell_1,m} . \end{aligned} \tag{44}$$

We now proceed with the evaluation of the $c_{IJ}{}^K$ coefficients. By using well-known results from the coupling theory of angular momenta (see *e.g.* [42, 43]) we obtain

$$\begin{aligned} &\Phi_{m'_1,\ell_1,m_1}(\theta,\varphi_1,\varphi_2)\Phi_{m'_2,\ell_2,m_2}(\theta,\varphi_1,\varphi_2) \\ &= \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \lambda(m'_1,m'_2,\ell,\ell_1,\ell_2)\left(\begin{smallmatrix}\ell_1 & \ell_2 \\ m_1 & m_2\end{smallmatrix}\middle|\begin{smallmatrix}\ell \\ m_1+m_2\end{smallmatrix}\right)\Phi_{m'_1+m'_2,\ell,m_1+m_2}(\theta,\varphi_1,\varphi_2), \end{aligned} \tag{45}$$

where $\left(\begin{smallmatrix}\ell_1 & \ell_2 \\ m_1 & m_2\end{smallmatrix}\middle|\begin{smallmatrix}\ell \\ m_1+m_2\end{smallmatrix}\right)$ are the Clebsch-Gordan coefficients associated to the right action. In this expansion, we consider only the allowed values of $m'_1+m'_2$, such that $-\ell \leq m'_1+m'_2 \leq \ell$. There is no generic closed expression to compute the coefficients $\lambda(m'_1,m'_2,\ell,\ell_1,\ell_2)$, although they can be computed recursively. For instance, for the highest value of $m'_1,m'_2,\ell,\ell_1,\ell_2$ we obtain :

$$\lambda(m'_1,m'_2,\ell_1+\ell_2,\ell_1,\ell_2) = \sqrt{\frac{(2\ell_1+1)!(2\ell_2+1)!}{(2(\ell_1+\ell_2)+1)!}\frac{(\ell_1+\ell_2+m'_1+m'_2)!(\ell_1+\ell_2-(m'_1+m'_2))!}{(\ell_1+m'_1)!(\ell_1-m'_1)!(\ell_2+m'_2)!(\ell_2-m'_2)!}} .$$

5.2 Coset space $SU(2)/U(1)$

Assume that $Q \in U(1) \subset SU(2)$ is given by

$$Q = e^{2\mathrm{i}\,\theta J'_3} = \begin{pmatrix} e^{\mathrm{i}\theta} & 0 \\ 0 & e^{-\mathrm{i}\theta}\end{pmatrix} .$$

This means that α,β have a $U(1)-$charge equal to 1 and $\overline{\alpha},\overline{\beta}$ have a $U(1)-$charge -1. More generally, the functions $\Phi_{m',\ell,m}(\theta,\varphi_1,\varphi_2)$ have a charge $2m'$. From Section 3.2, we just need to consider functions that are neutral, $i.e$, $\Phi_{0,\ell,m}$. This is possible if ℓ is an integer number. For such functions, we have

$$\begin{aligned}\Phi_{0,\ell,m} &= \frac{(-1)^\ell}{2^\ell \ell!}\sqrt{(2\ell+1)\frac{(\ell+m)!}{(\ell-m)!}}\, e^{\mathrm{i}m(\varphi_1-\varphi_2)} \sin^{-m} 2\theta \frac{\mathrm{d}^{\ell-m}}{\mathrm{d}(\cos 2\theta)^{\ell-m}}\Big((1-\cos^2 2\theta)^\ell\Big) \\ &= Y_{\ell m}(2\theta, \varphi_1-\varphi_2)\ ,\end{aligned}$$

where $Y_{\ell m}$ are the usual spherical harmonics on the sphere $\mathbb{S}^2$. (Note, however, the unconventional normalisation factor for the Y-functions.) If we perform the change of coordinates

$$0 \le \psi = 2\theta \le \pi\ ,\quad 0 \le \varphi = \varphi_1 - \varphi_2 < 2\pi\ ,\quad 0 \le \widetilde{\varphi} = \varphi_1 + \varphi_2 < 4\pi\ ,$$

then the points $m = (\psi, \varphi, \widetilde{\varphi} = \text{cons.})$ parameterise points on the manifold $\mathbb{S}^2 \cong SU(2)/U(1) \subset \mathbb{S}^3 \cong SU(2)$. With this parameterisation, for the level surfaces $\varphi = \text{cons.}$ we have, on the one hand

$$\begin{aligned}L_\pm &= e^{\pm \mathrm{i}\varphi}\Big(\mathrm{i}\cot\psi\frac{\partial}{\partial\varphi} \pm \frac{\partial}{\partial\psi} - \mathrm{i}\frac{1}{\sin\psi}\frac{\partial}{\partial\widetilde{\varphi}}\Big) \\ &= \Big|_{\widetilde{\varphi}=\text{cons.}}\ e^{\pm \mathrm{i}\varphi}\Big(\mathrm{i}\cot\psi\frac{\partial}{\partial\varphi} \pm \frac{\partial}{\partial\psi}\Big)\ , \\ L_3 &= -\mathrm{i}\frac{\partial}{\partial\varphi}\ ,\end{aligned}$$

as well as the relation

$$\begin{aligned}(f,g) &= \frac{1}{2\pi^2}\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\ \mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\varphi_1\int_0^{2\pi}\mathrm{d}\varphi_2\ \overline{f(\theta,\varphi_1,\varphi_2)}\ g(\theta,\varphi_1,\varphi_2) \\ &= \Big|_{\widetilde{\varphi}=\text{cons.}}\ \frac{1}{4\pi}\int_0^{\pi}\sin\psi\ \mathrm{d}\psi\int_0^{2\pi}\mathrm{d}\varphi\overline{f(\psi,\varphi,\text{cte})}\ g(\psi,\varphi,\text{cons.})\ ,\end{aligned}$$

thus reducing to the generators of $\mathfrak{so}(3)$ (resp. to the Hilbert scalar product on $\mathbb{S}^2$). In the language of Appendix A, $\mathbb{S}^2$ is parameterised by

$$0 \le \varphi \le 2\pi\ ,\quad -1 \le u = \cos\psi \le 1\ .$$

It follows that the adapted Hilbert basis for $SU(2)/U(1)$ is given by the usual spherical harmonics,

$$\mathcal{B} = \Big\{Y_{\ell m}, \ell \in \mathbb{N},\ -\ell \le m \le \ell\Big\}\ ,$$

with the conjugacy relation

$$\bar{Y}^{\ell m}(\psi,\varphi) = (-1)^m Y_{\ell,-m}(\psi,\varphi)\ .$$

In order to define the Lie algebra $\hat{\mathfrak{g}}\big(SU(2)/U(1)\big)$ we introduce $T_{a,\ell,m} = T_a Y_{\ell m}(\psi,\varphi)$, the Hermitian operator L_3 and its associated central charge k (see (15) and (16)). The Lie brackets take the form (see (17))

$$\begin{aligned}
\big[T_{a_1,\ell_1,m_1}, T_{a_2,\ell_2,m_2}\big] &= \mathrm{i} f_{a_1 a_2}{}^{a_3} c_{\ell_1,m_1,\ell_2,m_2}{}^{\ell_3,m_3} T_{a_3,\ell_3,m_3} \\
&\quad + (-1)^{m_1} k m_2 \, g_{a_1 a_2} \delta_{\ell_1,\ell_2} \delta_{m_1+m_2} \ , \\
\big[J_3, T_{a,\ell,m}\big] &= m T_{a,\ell,m} \ .
\end{aligned} \tag{46}$$

For the spherical harmonics, it is well known that

$$Y_{\ell_1 m_1} Y_{\ell_2 m_2} = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{2\ell+1}} \begin{pmatrix} \ell_1 & \ell_2 & \big| \ell \\ 0 & 0 & \big| 0 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \big| \ell \\ m_1 & m_2 & \big| m_1+m_2 \end{pmatrix} Y_{\ell m_1+m_2} \ , \tag{47}$$

which leads to the $c_{IJ}{}^K$ coefficients in (46).

Potential applications can *e.g.* be conceived in Supergravity [44, 45], using the space $SL(2,\mathbb{R})/U(1)$ related to the non-compact group $SL(2,\mathbb{R})$, the unitary representations of which are known [46], and correspond to the discrete series (either lower or upper bounded) and the continuous series (principal and supplementary). Discrete (respectively continuous) series are characterised by a discrete (respectively continuous) spectrum of the Casimir operator of $\mathfrak{sl}(2,\mathbb{R})$, from which we conclude that the discrete series are normalisable, whereas the continuous series are not. They can be found, for instance, in Section 2 of Ref. [32], and can be related to Eqs. (5.26) and (5.27) of Ref. [41]; (the former corresponding to the discrete series, while the latter, within the continuous series, requires to distinguish between "bosons" (n even in [32], p. 197) and "fermions" (n odd in [32]), with the correspondence: $2\mu = E_0 - \frac{1}{2} + s$ and $2\lambda = -E_0 - \frac{1}{2} + s$ (μ, λ from Ref. [41]) with $E_0 = 0$ for bosons and $E_0 = \frac{1}{2}$ for fermions and with $s \in \mathrm{i}\mathbb{R}$ (resp. $s \in \mathbb{R}$) for the principal (respectively complementary) series. In the context of harmonic functions, the discussion of non-compact groups rapidly shows to be considerably intricate [47]. Indeed, as the number s for the continuous series (see [47]) is either real or purely imaginary, the expressions in Section 4.3 of this reference [47] are meaningless for the continuous series, whereas for discrete series, convergence occurs for spin greater than 1/2, whereas here, $s = 0,\ 1/2$. Harmonic analysis of homogeneous spaces G/H, with G a non-compact Lie group and H a compact closed subgroup, of the same type as $SL(2,\mathbb{R})/U(1)$, has been considered to some extent in [33], Chapter 15.

5.3 Coset space $SO(4)/SO(3)$

The manifold $SO(4)/SO(3)$ is well-known to be isomorphic to the three-sphere $\mathbb{S}^3$. Its interest in our context is that it gives rise to an equivalent realisation of the algebra (44), but with a different Hilbert basis. This construction can furthermore be extended to the coset spaces $SO(n)/SO(n-1) \cong \mathbb{S}^{n-1}$ for values $n > 4$.

The sphere $\mathbb{S}^3$ can be parameterised by

$$\left.\begin{aligned}
x_1 &= \sin\psi \sin\theta \cos\varphi \ , \\
x_2 &= \sin\psi \sin\theta \sin\varphi \ , \\
x_3 &= \sin\psi \cos\theta \ , \\
x_4 &= \cos\psi \ ,
\end{aligned}\right\} \begin{aligned} &0 \le \psi \le \pi \ , \\ &0 \le \theta \le \pi \ , \\ &0 \le \varphi < 2\pi \ , \end{aligned} \tag{48}$$

and endowed with a scalar product defined by

$$(f,g) = \frac{1}{2\pi^2}\int\limits_0^\pi \sin^2\psi \mathrm{d}\psi \int\limits_0^\pi \sin\theta \mathrm{d}\theta \int\limits_0^{2\pi} \mathrm{d}\varphi \overline{f(\theta,\varphi,\psi)}\, g(\theta,\varphi,\psi)\ .$$

Again, using the terminology of Appendix A, the sphere $\mathbb{S}^3$ is parameterised by

$$0 \le \varphi \le 2\pi\ ,\quad -1 \le u_1 = \cos\theta \le 1\ , 0 \le u_2 = \frac{1}{2}\psi - \frac{1}{2}\cos\psi\sin\psi \le \frac{\pi}{2}\ .$$

Representations of $SO(4)$ are characterised by their Dynkin labels or by a Young tableau associated to a tensor with a certain symmetry. Among tensors, only traceless n^{th}-order symmetric tensors admit a scalar representation with respect to the embedding $SO(3) \subset SO(4)$. Let us denote by $\mathcal{D}_{\frac{n}{2},\frac{n}{2}}, n \in \mathbb{N}$ the representation corresponding to the set of traceless n^{th}-order tensors, which is of dimension $(n+1)^2$. If $D^i_{(n)j}$ are the corresponding matrix elements, from the result of Section 3.2 and because $\mathcal{D}_{\frac{n}{2},\frac{n}{2}}$ contains the scalar representation in the decomposition through the embedding $SO(3) \subset SO(4)$, for any Fourier expansion on $\mathbb{S}^3$ we have to consider the indices $i = i_0$, where $D^{i_0}_{(n)j}$ are in the scalar representation of $SO(3)$.

These matrix elements can be easily obtained. To that purpose, we introduce the generators of the Lie algebra $\mathfrak{so}(4)$ in the usual $\{N_0, N_\pm, N'_0, N'_\pm\}$ basis:

$$\begin{array}{rclrcl} \big[N_0, N_\pm\big] &=& \pm N_\pm, & \big[N'_0, N'_\pm\big] &=& \pm N'_\pm, \\ \big[N_+, N_-\big] &=& 2N_0, & \big[N'_+, N'_-\big] &=& 2N'_0, \end{array} \qquad \big[N_a, N'_b\big] = 0 \ .$$

From the expression of the generators of the Lie algebra defined on $\mathbb{S}^3$ and the expansion on the sphere, we obtain:

$$\begin{aligned} N_0 &= -\frac{\mathrm{i}}{2}\bigg[\frac{\partial}{\partial\varphi} + \cot\psi\sin\theta\frac{\partial}{\partial\theta} - \cos\theta\frac{\partial}{\partial\psi}\bigg]\ , \\ N_+ &= \frac{1}{2}e^{\mathrm{i}\varphi}\bigg[\Big(\mathrm{i}\cot\theta - \frac{\cot\psi}{\sin\theta}\Big)\frac{\partial}{\partial\varphi} + \Big(1 + \mathrm{i}\cot\psi\cos\theta\Big)\frac{\partial}{\partial\theta} + \mathrm{i}\sin\theta\frac{\partial}{\partial\psi}\bigg]\ , \\ N_- &= \frac{1}{2}e^{-\mathrm{i}\varphi}\bigg[\Big(\mathrm{i}\cot\theta + \frac{\cot\psi}{\sin\theta}\Big)\frac{\partial}{\partial\varphi} + \Big(-1 + \mathrm{i}\cot\psi\cos\theta\Big)\frac{\partial}{\partial\theta} + \mathrm{i}\sin\theta\frac{\partial}{\partial\psi}\bigg]\ , \end{aligned}$$

$$\begin{aligned} N'_0 &= -\frac{\mathrm{i}}{2}\bigg[\frac{\partial}{\partial\varphi} - \cot\psi\sin\theta\frac{\partial}{\partial\theta} + \cos\theta\frac{\partial}{\partial\psi}\bigg]\ , \\ N'_+ &= \frac{1}{2}e^{\mathrm{i}\varphi}\bigg[\Big(\mathrm{i}\cot\theta + \frac{\cot\psi}{\sin\theta}\Big)\frac{\partial}{\partial\varphi} + \Big(1 - \mathrm{i}\cot\psi\cos\theta\Big)\frac{\partial}{\partial\theta} - \mathrm{i}\sin\theta\frac{\partial}{\partial\psi}\bigg]\ , \\ N'_- &= \frac{1}{2}e^{\mathrm{i}\varphi}\bigg[\Big(i\cot\theta - \frac{\cot\psi}{\sin\theta}\Big)\frac{\partial}{\partial\varphi} - \Big(1 + \mathrm{i}\cot\psi\cos\theta\Big)\frac{\partial}{\partial\theta} - \mathrm{i}\sin\theta\frac{\partial}{\partial\psi}\bigg]\ . \end{aligned}$$

We can construct spherical harmonics, since we have

$$
\begin{aligned}
\Phi_{\frac{1}{2}\frac{1}{2};\frac{1}{2},\frac{1}{2}}(\theta,\varphi,\psi) &= \sqrt{2}e^{\mathrm{i}\varphi}\sin\theta\sin\psi = \sqrt{2}\big(x_1+\mathrm{i}x_2\big) \ , \\
\Phi_{\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2}}(\theta,\varphi,\psi) &= \sqrt{2}(-\cos\theta\sin\psi+\mathrm{i}\cos\psi) = \sqrt{2}(-x_3+\mathrm{i}x_4) \ , \\
\Phi_{\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}}(\theta,\varphi,\psi) &= -\sqrt{2}(\cos\theta\sin\psi+\mathrm{i}\cos\psi) = -\sqrt{2}(x_3+\mathrm{i}x_4) \ , \\
\Phi_{\frac{1}{2},-\frac{1}{2};\frac{1}{2},-\frac{1}{2}}(\theta,\varphi,\psi) &= -\sqrt{2}e^{-\mathrm{i}\varphi}\sin\theta\sin\psi = -\sqrt{2}\big(x_1-\mathrm{i}x_2\big) \ ,
\end{aligned}
$$

denoting this representation by $\mathcal{D}_{\frac{1}{2},\frac{1}{2}}$, and whose highest weight is given by $\Phi_{\frac{1}{2}\frac{1}{2};\frac{1}{2},\frac{1}{2}}$. According to this prescription, the highest weight of the representation $\mathcal{D}_{\frac{n}{2},\frac{n}{2}}$ is given by

$$
\Phi_{\frac{n}{2},\frac{n}{2};\frac{n}{2},\frac{n}{2}}(\theta,\varphi,\psi) = \sqrt{n+1}e^{\mathrm{i}n\varphi}\sin^n\theta\sin^n\psi \ .
$$

The remaining vectors $\Phi_{\frac{n}{2},m_1;\frac{n}{2},m_2}(\theta,\varphi,\psi)$ are explicitly obtained by the action of the operators N_-, N'_-, where m_1, m_2 indicate the eigenvalues of N_0, N'_0. Moreover, we have the conjugacy properties

$$
\overline{\Phi}^{\frac{n}{2},m_1;\frac{n}{2},m_2}(\theta,\varphi,\psi) = (-1)^{m_1+m_2}\Phi_{\frac{n}{2},-m_1;\frac{n}{2},-m_2}(\theta,\varphi,\psi) \ .
$$

Since the first and third indices are redundant, we set

$$
\Phi_{\frac{n}{2},m_1;\frac{n}{2},m_2} \to \Phi_{n,m_1,m_2} \ .
$$

The representation space $\mathcal{D}_{\frac{n}{2},\frac{n}{2}}$ can also be obtained in another way. Introduce $y_i = rx_i$ (see equation (48)) in spherical coordinates, as well as the Laplacian ∇^2 in this system of coordinates (to be defined below, in equation (54)) and the space of polynomials of degree n, $\mathbb{R}_n[y_1,y_2,y_3,y_4]$, so that we have

$$
\mathcal{D}_{\frac{n}{2},\frac{n}{2}} = \Big\{P(y_1,y_2,y_3,y_4)\big|_{r=1} \ \text{ where } \ P\in\mathbb{R}_n[y_1,y_2,y_3,y_4] \ \text{ s.t. } \ \nabla^2P(y_1,y_2,y_3,y_4)=0\Big\} \ . \tag{49}
$$

In equation (49), the Laplacian constraint simply projects on traceless polynomials. Then the Hilbert basis on $\mathbb{S}^3$ relevant for our purpose is defined by

$$
\mathcal{B} = \Big\{\Phi_{n;m_1,m_2}, \ \ n\in\mathbb{N}, -\frac{n}{2}\le m_1,m_2\le\frac{n}{2}\Big\} \ . \tag{50}
$$

We further have the orthonormality relations

$$
\Big(\Phi_{n,m_1,m_2}, \Phi_{n',m'_1,m'_2}\Big) = \delta^n_{n'}\delta^{m_1}_{m'_1}\delta^{m_2}_{m'_2} \ .
$$

In the notations above for Φ_{n,m_1,m_2}, we have taken the following conventions: n labels the representation space, here $\mathcal{D}_{\frac{n}{2},\frac{n}{2}}$, and m_1 (resp. m_2) labels the eigenvalue of N_0 (resp. N'_0). From the results of Section 3.3, we would expect to need four labels to classify all representations of $\mathfrak{so}(4)$. However, as only certain types of representations appear in the Fourier expansion, it will turn out that three labels are sufficient. In order to define the Lie algebra $\hat{\mathfrak{g}}\big(SO(4)/SO(3)\big)$, we introduce $T_{a,n,m_1,m_2} = T_a\Phi_{n,m_1,m_2}(\theta,\varphi,\psi)$ the Hermitian operator

N_0, N_0'. The associated central charges k_0 and k_0' are more involved and are obtained using the two-forms dual to N_0, N_0' given by

$$
\begin{aligned}
\gamma_0 &= -\frac{\mathrm{i}}{2}k_0\Big[\mathrm{d}\theta\wedge\mathrm{d}\psi - \cot\psi\sin\theta\mathrm{d}\varphi\wedge\mathrm{d}\psi - \cos\theta\mathrm{d}\varphi\wedge\mathrm{d}\theta\Big]\sin^2\psi\sin\theta\ , \\
\gamma_0' &= -\frac{\mathrm{i}}{2}k_0'\Big[\mathrm{d}\theta\wedge\mathrm{d}\psi + \cot\psi\sin\theta\mathrm{d}\varphi\wedge\mathrm{d}\psi + \cos\theta\mathrm{d}\varphi\wedge\mathrm{d}\theta\Big]\sin^2\psi\sin\theta\ .
\end{aligned}
$$

The Lie brackets take the form (see (17))

$$
\begin{aligned}
\big[T_{a,n,m_1,m_2}, T_{a',n',m_1',m_2'}\big] &= \mathrm{i}f_{aa'}{}^{a''}c_{n,m_1,m_2,n',m_1',m_2'}{}^{n'',m_1'',m_2''}T_{a'',n'',m_1'',m_2''}, \\
&\quad +(-1)^{m_1+m_2}g_{a'a'}\delta_{nn'}\delta_{m_1+m_1'}\delta_{m_1+m_2'}(k_0m_1'+k_0'm_2')\ , \qquad (51)\\
\big[N_0, T_{a,n,m_1,m_2}\big] &= m_1T_{a,n,m_1,m_2}\ , \\
\big[N_0', T_{a,n,m_1,m_2}\big] &= m_2T_{a,n,m_1,m_2}\ .
\end{aligned}
$$

Since the two Hilbert bases (42) and (50) are admissible bases for the space $L^2(\mathbb{S}^3)$, the algebra (51) is isomorphic to the algebra (44). The former is presented using the representation theory of $\mathbb{S}^3 \cong SU(2)$, whilst the latter is described in terms of the representation theory of $\mathbb{S}^3 = SO(4)/SU(3)$.

The coefficients $c_{IJ}{}^K$ in (51) can be obtained from the relation

$$
\begin{aligned}
\Phi_{n,m_1,m_2}(\theta,\varphi,\psi)\Phi_{n',m_1',m'}(\theta,\varphi,\psi) &= \sum_{N=|n-n'|}^{n+n'}\lambda(N,n,n')\left(\begin{smallmatrix} n \\ m_1,m_2\end{smallmatrix}\ \begin{smallmatrix} n' \\ m_1',m_2'\end{smallmatrix}\Big|\begin{smallmatrix} N \\ m_1+m_1',m_2+m_2'\end{smallmatrix}\right) \\
&\qquad\times\Phi_{N,m_1+m_1',m_2+m_2'}(\theta,\varphi,\psi), \qquad (52)
\end{aligned}
$$

where $\left(\begin{smallmatrix} n \\ m_1,m_2\end{smallmatrix}\ \begin{smallmatrix} n' \\ m_1',m_2'\end{smallmatrix}\Big|\begin{smallmatrix} N \\ m_1+m_1',m_2+m_2'\end{smallmatrix}\right)$ are the Clebsch-Gordan coefficients of the decomposition

$$
\mathcal{D}_{\frac{n}{2},\frac{n}{2}}\otimes\mathcal{D}_{\frac{n'}{2},\frac{n'}{2}} = \bigoplus_{N=|n-n'|}^{n+n'}\mathcal{D}_{\frac{N}{2},\frac{N}{2}}\ ,
$$

and $\lambda(N,n,n')$ are coefficients which can be computed recursively.

There is a third presentation of the algebra given in (51). The two Casimir operators of the $\mathfrak{so}(4)$ algebra are given by

$$
Q = N_0^2 + \frac{1}{2}(N_+N_- + N_-N_+)\ ,\quad Q' = N_0'^2 + \frac{1}{2}(N_+'N_-' + N_-'N_+')\ ,
$$

or by

$$
C = 2(Q+Q')\ ,\quad C' = 2(Q-Q')\ .
$$

For the representation $\mathcal{D}_{\frac{n}{2},\frac{n}{2}}$, we have

$$
C = n(n+2)\ ,\quad C' = 0\ . \qquad (53)
$$

However, the Laplacian in spherical coordinates (r,ϕ,θ,ψ) takes the form

$$\nabla^2 = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j) = \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} - \frac{C}{r^2}\ , \tag{54}$$

with the metric given by

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\big(\mathrm{d}\psi^2 + \sin^2\psi\ \mathrm{d}\theta^2 + \sin^2\psi\sin^2\theta\ \mathrm{d}\phi^2\big)\ . \tag{55}$$

Hence, the condition (53) for functions of the sphere $\mathbb{S}^3$ in the representation $\mathcal{D}_{\frac{n}{2},\frac{n}{2}}$ is equivalent to impose the constraint

$$\nabla^2 Y(\theta,\phi,\psi) = -\frac{n(n+2)}{r^2}Y(\theta,\phi,\psi)\ . \tag{56}$$

This definition is obviously equivalent to equation (49), as

$$\nabla^2\big(r^n Y(\theta,\phi,\psi)\big) = 0\ .$$

If we set

$$Y(\theta,\phi,\psi) \equiv Y_{n\ell m}(\theta,\phi,\psi) = H_{n\ell}(\psi)Y_{\ell m}(\theta,\phi)\ ,\quad n\in\mathbb{N}\ ,\quad 0\leq\ell\leq n\ ,\quad -\ell\leq m\leq\ell\ ,$$

then the condition (56) is equivalent to the second-order linear homogeneous ordinary differential equation

$$\frac{\mathrm{d}^2H_{n\ell}(\psi)}{\mathrm{d}\psi^2} + 2\cot\psi\frac{\mathrm{d}H_{n\ell}(\psi)}{\mathrm{d}\psi} + \Big(n(n+2) - \frac{\ell(\ell+1)}{\sin^2\psi}\Big)H_{n\ell}(\psi) = 0\ . \tag{57}$$

Indeed, we can express the functions $H_{n\ell}(\psi)$ as follows

$$H_{n\ell}(\psi) = N_{n\ell}\sin^\ell\psi\ C^{1+\ell}_{n-\ell}(\cos\psi)\ ,$$

where $C^{1+\ell}_{n-\ell}$ are the Gegenbauer polynomials defined by (see *e.g.* [48, 49])

$$\frac{1}{(1+x^2-2x\cos\psi)^\alpha} = \sum_{n=0}^{\infty}C^\alpha_n(\cos\psi)x^n\ ,\quad \text{for}\quad |x|<1\ ,$$

and the normalisation coefficients are given by

$$N_{n\ell} = (-1)^\ell(2\ell)!\sqrt{\frac{(n+1)(n-\ell)!}{(n+\ell+1)!}}\ .$$

The harmonic functions $Y_{n\ell m}$ are characterised by the eigenvalues of the Casimir operators in the embedding chain $SO(4)\supset SO(3)\supset SO(2)$, with n being associated to $SO(4)$, ℓ to $SO(3)$ and m to $SO(2)$. This is another illustration of the missing label problem studied in Section 3.3. The functions $Y_{n\ell m}$ in (56) can be expressed in terms of the functions Φ_{n,m_1,m_2}, by means of the decomposition

$$\mathcal{D}_{\frac{n}{2},\frac{n}{2}} = \bigoplus_{k=0}^{n}\mathcal{D}_n\ ,$$

valid for subduced representations in the embedding $SO(3) \subset SO(4)$. These functions can be considered as an alternative Hilbert basis on the 3-sphere, with

$$(Y_{n\ell m}, Y_{n'\ell' m'}) = \delta^n_{n'}\delta^\ell_{\ell'}\delta^m_{m'} \ ,$$

and constitute the standard basis used in the harmonic analysis on $\mathbb{S}^3$.

It is possible to define the Lie algebra $\hat{\mathfrak{g}}(SO(4)/SO(3))$ by means of the generators $T_{a,n,\ell,m} = T_a Y_{n\ell m}$, $N_0 + N'_0, N_0 - N'_0$ and $k_0 + k'_0, k_0 - k'_0$. However, the functions $Y_{n\ell m}$ are not simultaneous eigenfunctions of the operators N_0 and N'_0, but of their sum $N_0 + N'_0$. This follows at once from the identities

$$Y_{11\pm1} = \Phi_{1,\frac{1}{2},\pm\frac{1}{2}} \ , Y_{110} = \frac{1}{\sqrt{2}}\Big(\Phi_{1,-\frac{1}{2},\frac{1}{2}} + \Phi_{1,\frac{1}{2},-\frac{1}{2}}\Big) \ , \quad Y_{100} = \frac{\mathrm{i}}{\sqrt{2}}\Big(\Phi_{1,-\frac{1}{2},\frac{1}{2}} - \Phi_{1,\frac{1}{2},-\frac{1}{2}}\Big) \ .$$

Thus, in this new basis, the commutation relations are more involved. In particular, the fact that the Ys are not eigenfunctions of $N_0 - N'_0$ in the second line of (51) implies that the coefficient of $k_0 - k'_0$ together with the commutator with $N_0 - N'_0$ have to be consequently modified.

The construction described in this section can be further extended to the coset spaces $SO(n)/SO(n-1) \cong \mathbb{S}^{n-1}$. As a matter of fact, the only representations of $SO(n)$ that contain a scalar representation with respect to the embedding $SO(n-1) \subset SO(n)$ are the traceless symmetric n^{th}-order tensors. These representations are obtained from the symmetric powers of the fundamental representation $[1, 0^{n-1}]$ after extracting the trace, and correspond to representations with Dynkin labels $[n, 0, \cdots, 0]$, in the notation of [50].

Similarly to what happened with the results of Section 5.2, the construction above could be potentially of interest for applications to either Supergravity or Einstein-Maxwell scalar theories in $4 + 1$ dimensions, with a non-compact Riemannian version $SO(3, 1)/SO(3)$ of our manifold $SO(4)/SO(3)$, that can be seen as the special case for $n = 3$ of the sequence $SO(n, 1)/SO(n)$ of symmetric real manifolds in supergravity theories, which yield non-symmetric homogeneous special Kähler (respectively special quaternionic Kähler) manifolds [44, 45]. Unitary representations of $SL(2, \mathbb{C})$ were originally obtained by Gel'fand [51], while the unitary representations of $SO(1, n)$ were studied in [52].

5.4 Coset space $SU(3)/SU(2)$

In order to construct the generalised Kac-Moody algebra associated to the coset space $SU(3)/SU(2)$, we must first conveniently parameterise the manifold $SU(3)$. To this extent, we proceed in four steps:

1. If w_1, w_2, w_3 are three orthonormal vectors of $\mathbb{C}^3$, *i.e.*, satisfying $w_i^\dagger w_j = \delta^i_j$, the matrix $M_1 = \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix}$ subjected to the additional constraint $\det M_1 = 1$, is such that $M_1^\dagger M_1 = 1$, and therefore belongs to $SU(3)$. We choose the matrix M_1 as

$$M_1 = \begin{pmatrix} \cos\theta e^{\mathrm{i}\omega_1} & \sin\theta\cos\xi e^{\mathrm{i}\omega_2} & \sin\theta\sin\xi e^{-\mathrm{i}(\omega_1+\omega_2)} \\ -\sin\theta e^{\mathrm{i}\omega_1} & \cos\theta\cos\xi e^{\mathrm{i}\omega_2} & \cos\theta\sin\xi e^{-\mathrm{i}(\omega_1+\omega_2)} \\ 0 & -\sin\xi e^{\mathrm{i}\omega_2} & \cos\xi e^{-\mathrm{i}(\omega_1+\omega_2)} \end{pmatrix} \ ,$$

with $0 \le \omega_1, \omega_2 < 2\pi, 0 \le \xi, \theta \le \frac{\pi}{2}$.

2. We introduce the diagonal matrix

$$U_1 = \begin{pmatrix} e^{\mathrm{i}\lambda} & 0 & 0 \\ 0 & e^{\mathrm{i}\lambda} & 0 \\ 0 & 0 & e^{-2\mathrm{i}\lambda} \end{pmatrix} \ , \ 0 \le \lambda < 2\pi$$

   which belongs to $U(1) \subset SU(3)$.

3. We further consider the matrix

$$U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\overline{\beta} & \overline{\alpha} \end{pmatrix} \ ,$$

   with

$$\begin{cases} \alpha = \cos\rho e^{\mathrm{i}\psi_1} \ , & 0 \le \rho \le \frac{\pi}{2} \ , \\ \beta = \sin\rho e^{\mathrm{i}\psi_2} \ , & 0 \le \psi_1, \psi_2 < 2\pi \ . \end{cases}$$

   This implies that $U_2 \in SU(2) \subset SU(3)$.

4. Finally, we define the matrix

$$U \ \equiv \ U_2 U_1 M_1 \in SU(3) \ ,$$

   which parameterises a point on the manifold $SU(3)$.

If we define $\lambda + \omega_1 = \varphi_3$, $\lambda + \omega_2 = \varphi_1$ and $\lambda - \omega_1 - \omega_2 = \varphi_2$, then we have the following identities in the first row of the matrix U,

$$\begin{aligned} U_{11} &= \cos\theta e^{\mathrm{i}\varphi_3} \ , \\ U_{12} &= \sin\theta \cos\xi e^{\mathrm{i}\varphi_1} \ , \\ U_{13} &= \sin\theta \sin\xi e^{\mathrm{i}\varphi_2} \ . \end{aligned}$$

Similarly, we have

$$\begin{aligned} U_{21} &= -\alpha \sin\theta e^{\mathrm{i}\varphi_3} \ , \\ U_{22} &= \alpha \cos\theta \cos\xi e^{\mathrm{i}\varphi_1} - \beta \sin\xi e^{-\mathrm{i}(\varphi_2+\varphi_3)} \ , \\ U_{23} &= \alpha \cos\theta \sin\xi e^{\mathrm{i}\varphi_2} + \beta \cos\xi e^{-\mathrm{i}(\varphi_1+\varphi_3)} \ , \end{aligned}$$

for the second row of U, and for the third row,

$$\begin{aligned} U_{31} &= \overline{\beta} \sin\theta e^{\mathrm{i}\varphi_3} \ , \\ U_{32} &= -\overline{\beta} \cos\theta \cos\xi e^{\mathrm{i}\varphi_1} - \overline{\alpha} \sin\xi e^{-\mathrm{i}(\varphi_2+\varphi_3)} \ , \\ U_{33} &= -\overline{\beta} \cos\theta \sin\xi e^{\mathrm{i}\varphi_2} + \overline{\alpha} \cos\xi e^{-\mathrm{i}(\varphi_1+\varphi_3)} \ . \end{aligned}$$

The three rows $\{U_{i1}, U_{i2}, U_{i3}\}, i = 1, 2, 3$ span the fundamental three-dimensional representation $[1, 0]$ of $SU(3)$, whereas $\{\overline{U}_{i1}, \overline{U}_{i2}, \overline{U}_{i3}\}, i = 1, 2, 3$ span the anti-fundamental three-dimensional representation $[0, 1]$ of $SU(3)$. It is well known that any representation of $SU(3)$

can be obtained from appropriate tensor products of the fundamental and anti-fundamental representations, from which we conclude that the Hilbert basis for the manifold $SU(3)$ can be deduced from the U_{ij} and $\overline{U}_{ij}$ functions.

From the Peter-Weyl theorem, for a given representation $\mathcal{D}$ of dimension d, we obtain, with the functions U_{ij} and $\overline{U}_{ij}$, d copies of the representation $\mathcal{D}$. If we denote by $\mathcal{D}_{n,m}$ the representation of highest weight $n\mu_1 + m\mu_2$ (see below for the notations), then it is straightforward to obtain the correct number of copies of the representations $\mathcal{D}_{n,0}$ and $\mathcal{D}_{0,n}$, say $(n+2)!/(2!n!)$. For general representations $\mathcal{D}_{n,m}$, however, it is more involved. For instance, we have eight copies of the adjoint representation $\mathcal{D}_{1,1}$, rather than nine, as it would seem at a first glance. Indeed, the highest weight of the adjoint representations are $\overline{U}_{11}U_{12}$, $\overline{U}_{11}U_{22}$, $\overline{U}_{11}U_{32}$, $\overline{U}_{21}U_{12}$, $\overline{U}_{21}U_{22}$, $\overline{U}_{21}U_{32}$, $\overline{U}_{31}U_{12}$, $\overline{U}_{31}U_{22}$, $\overline{U}_{31}U_{32}$, but because of the relation $\overline{U}_{11}U_{12} + \overline{U}_{21}U_{12} + \overline{U}_{31}U_{32} = 0$, there are actually eight independent copies of the adjoint representation. Thus, having constructed all representations with the correct multiplicity, an appropriate Kac-Moody algebra $\hat{\mathfrak{g}}(SU(3))$ can be defined.

It is worth noticing that the embedding of Kostant's principal subalgebra $\mathfrak{su}(2)_P$, isomorphic to $\mathfrak{so}(3)$, into the Lie algebra $\mathfrak{su}(3)$, is actually maximal, constituting the unique symmetric embedding of Kostant's $\mathfrak{su}(2)_P$ into any simple, compact Lie algebra [53].

However, we are mainly interested in the Fourier expansion on the coset space

$$SU(3)/SU(2) \cong \mathbb{S}^5 \ .$$

This isomorphism is related to the possible description of the corresponding tangent space as the 5-dimensional irreducible representation space of either $\mathfrak{so}(5) \simeq \mathfrak{usp}(4)$ or $\mathfrak{su}(2)$. Since this representation of $\mathfrak{su}(2)$ is self-conjugate, this implies, by Theorem 1.5 of [54], the maximal embedding $\mathfrak{su}(2) \subset \mathfrak{usp}(4)$, so that the subduced representation remains irreducible. It should be mentioned in this respect that the action of $SU(2)$ on its 5-dimensional irreducible representation is an example of a θ-group [55, 56, 57], a remarkable class of linear groups of transformations related to symmetric, (pseudo-)Riemannian coset spaces that has shown to be of current interest in several physical applications. According to the results of Section 3.2, the only functions that must be considered are

$$\left\{ \begin{array}{lcll} z_3 = U_{11} &=& \cos\theta e^{\mathrm{i}\varphi_3} \ , & 0 \le \theta \le \frac{\pi}{2} \ , \\ z_2 = U_{13} &=& \sin\theta \sin\xi e^{\mathrm{i}\varphi_2} \ , & 0 \le \xi \le \frac{\pi}{2} \ , \\ z_1 = U_{12} &=& \sin\theta \cos\xi e^{\mathrm{i}\varphi_1} \ , & 0 \le \varphi_i < 2\pi \ . \end{array} \right.$$

They parameterise the five-sphere as a consequence of $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$. In the language of Appendix A, the parameterisation of $\mathbb{S}^5$ is given by [58]

$$0 \le \varphi_1, \varphi_2, \varphi_3 < 2\pi \ , \quad 0 \le u_1 = \frac{1}{2}\sin^2\xi \le \frac{1}{2} \ \ , \ 0 \le u_2 = \frac{1}{4}\sin^4\theta \le \frac{1}{4} \ .$$

The scalar product on the five-sphere is taken as

$$\begin{aligned} (f,g) &= \frac{1}{\pi^3}\int\limits_0^{\frac{\pi}{2}} \sin^3\theta\cos\theta \mathrm{d}\theta \int\limits_0^{\frac{\pi}{2}} \sin\xi\cos\xi \mathrm{d}\xi \int\limits_0^{2\pi} \mathrm{d}\varphi_1 \int\limits_0^{2\pi} \mathrm{d}\varphi_2 \int\limits_0^{2\pi} \mathrm{d}\varphi_3 \\ & \times \overline{f(\theta,\xi,\varphi_1,\varphi_2,\varphi_3)}\, g(\theta,\xi,\varphi_1,\varphi_2,\varphi_3) \ , \end{aligned}$$

while the generators of the Lie algebra $\mathfrak{su}(3)$ are

$$\begin{aligned}
E_1^+ &= \frac{1}{2}e^{\mathrm{i}(\varphi_1-\varphi_2)}\Big(\frac{\partial}{\partial\xi} - \mathrm{i}\tan\xi\frac{\partial}{\partial\varphi_1} - \mathrm{i}\cot\xi\frac{\partial}{\partial\varphi_2}\Big) ,\\
E_2^+ &= \frac{1}{2}e^{\mathrm{i}(\varphi_2-\varphi_3)}\Big(- \sin\xi\frac{\partial}{\partial\theta} - \cot\theta\cos\xi\frac{\partial}{\partial\xi} - \mathrm{i}\frac{\cot\theta}{\sin\xi}\frac{\partial}{\partial\varphi_2} - \mathrm{i}\tan\theta\sin\xi\frac{\partial}{\partial\varphi_3}\Big) ,\\
E_3^+ &= \frac{1}{2}e^{\mathrm{i}(\varphi_1-\varphi_3)}\Big(- \cos\xi\frac{\partial}{\partial\theta} + \cot\theta\sin\xi\frac{\partial}{\partial\xi} - \mathrm{i}\frac{\cot\theta}{\cos\xi}\frac{\partial}{\partial\varphi_1} - \mathrm{i}\tan\theta\cos\xi\frac{\partial}{\partial\varphi_3}\Big) ,\\
E_1^- &= \frac{1}{2}e^{\mathrm{i}(-\varphi_1+\varphi_2)}\Big(- \frac{\partial}{\partial\xi} - \mathrm{i}\tan\xi\frac{\partial}{\partial\varphi_1} - \mathrm{i}\cot\xi\frac{\partial}{\partial\varphi_2}\Big) ,\\
E_2^- &= \frac{1}{2}e^{\mathrm{i}(-\varphi_2+\varphi_3)}\Big(\sin\xi\frac{\partial}{\partial\theta} + \cot\theta\cos\xi\frac{\partial}{\partial\xi} - \mathrm{i}\frac{\cot\theta}{\sin\xi}\frac{\partial}{\partial\varphi_2} - \mathrm{i}\tan\theta\sin\xi\frac{\partial}{\partial\varphi_3}\Big) ,\\
E_3^- &= \frac{1}{2}e^{\mathrm{i}(-\varphi_1+\varphi_3)}\Big(\cos\xi\frac{\partial}{\partial\theta} - \cot\theta\sin\xi\frac{\partial}{\partial\xi} - \mathrm{i}\frac{\cot\theta}{\cos\xi}\frac{\partial}{\partial\varphi_1} - \mathrm{i}\tan\theta\cos\xi\frac{\partial}{\partial\varphi_3}\Big) ,\\
h_1 &= -\mathrm{i}\Big(\frac{\partial}{\partial\varphi_1} - \frac{\partial}{\partial\varphi_2}\Big) ,\\
h_2 &= -\mathrm{i}\Big(\frac{\partial}{\partial\varphi_2} - \frac{\partial}{\partial\varphi_3}\Big) .
\end{aligned} \tag{58}$$

Here $(E_i^\pm, h_i), i = 1, 2$, are the generators associated with the two simple roots $\alpha_i, i = 1, 2$ of the complexification of $\mathfrak{su}(3)$, and $E_3^\pm = \pm[E_1^\pm, E_2^\pm]$. We denote the fundamental weights by μ_i with $i = 1, 2$.

A direct computation with equation (58) shows that the functions

$$\psi^n_{n_1,n_2,n_3}(\theta,\xi,\varphi) = \sqrt{\frac{(n+2)!}{2n_1!n_2!n_3!}}\sin^{n_1+n_2}\theta\cos^{n_3}\theta\cos^{n_1}\xi\sin^{n_2}\xi\; e^{\mathrm{i}(n_1\varphi_1+n_2\varphi_2+n_3\varphi_3)} ,\quad n_1+n_2+n_3 = n$$

span the representation with highest weight $\big|n\mu_1\big\rangle$, while the functions $\overline{\psi}_n^{n_1,n_2,n_3}$ span the representation with highest weight $\big|n\mu_2\big\rangle$. These highest weights are explicitly given by

$$\begin{aligned}
\big\langle\theta,\xi,\varphi_1,\varphi_2,\varphi_3\big|n\mu_1\big\rangle &= \psi^n_{n,0,0}(\theta,\xi,\varphi_1,\varphi_2,\varphi_3) = \sqrt{\frac{(n+2)!}{2n!}}\sin^n\theta\cos^n\xi e^{\mathrm{i}n\varphi_1} ,\\
\big\langle\theta,\xi,\varphi_1,\varphi_2,\varphi_3\big|n\mu_2\big\rangle &= \overline{\psi}_n^{0,0,n}(\theta,\xi,\varphi_1,\varphi_2,\varphi_3) = \sqrt{\frac{(n+2)!}{2n!}}\cos\theta^n e^{-\mathrm{i}n\varphi_3} .
\end{aligned}$$

Furthermore,

$$(\psi^n_{n_1,n_2,n_3}, \psi^m_{m_1,m_2,m_3}) = \delta^m_n\delta^{n_1}_{m_1}\delta^{n_2}_{m_2}\delta^{n_3}_{m_3} ,\quad (\overline{\psi}_n^{n_1,n_2,n_3}, \psi^m_{m_1,m_2,m_3}) = 0 .$$

The representations of highest weight $\big|n\mu_1 + m\mu_2\big\rangle$, denoted by $\mathcal{D}_{n,m}$, are obtained from the highest weight

$$\psi^{n,m}_{n,0,0;0,0,m} = \sqrt{\frac{(n+m+2)!}{2n!m!}}\sin^n\theta\cos^n\xi\cos^m\theta e^{\mathrm{i}(n\varphi_1-m\varphi_2)} ,$$

which enables us to obtain the full representation from the operators E_i^-, where $i = 1, 2$. The functions $\psi^{n,m}_{n_1,n_2,n_3;m_1,m_2,m_3}$ that span the representation $\mathcal{D}_{n,m}$ (which can alternatively be obtained by combining $\psi^n_{n_1,n_2,n_3}$ with $\overline{\psi}_m^{m_1,m_2,m_3}$ and substracting the trace). We now identify the minimal set of indices to characterise any function. From Section 3.3, each representation is characterised by two numbers. We denote by $\mathcal{D}_{n,m}$ the representation of highest weight $n\mu_1 + m\mu_2$. Then three internal labels are needed to distinguished elements inside each representation space. The inner states are distinguished by the eigenvalues of h_1, h_2 as well as the value of the Casimir operator of the $\mathfrak{su}(2)$-subalgebra generated by $E_1^\pm, h_1$ that we take equal to

$$Q = \frac{1}{4}h_1^2 + \frac{1}{2}(E_1^+ E_1^- + E_1^- E_1^+) \ .$$

Then for $n, m \in \mathbb{N}$, the representation space reduces to

$$\mathcal{D}_{n,m} = \Big\{\psi_{n,m,n_1,n_2,\ell}, \quad n_1, n_2 \ \text{ s.t. } \ n_1\mu_1 + n_2\mu_2 \text{ is a weight, } 0 \le \ell \le \frac{1}{2}(n+m)\Big\}$$

and the Hilbert basis of $\mathbb{S}^5$ adapted to our construction is defined by

$$\mathcal{B} = \Big\{\psi_{n,m,n_1,n_2,\ell}, n, m \in \mathbb{N}, n_1, n_2, \quad \text{s.t. } \ n_1\mu_1 + n_2\mu_2 \text{ is a weight, } 0 \le \ell \le \frac{1}{2}(n+m)\Big\} \ . \quad (59)$$

In the notation above, n, m correspond to the representation $D_{n,m}$, n_1, n_2 are the eigenvalues of h_1, h_2 and ℓ is the eigenvalue of the additional internal label Q. In order to obtain the conjugacy relation, we observe that we have the following normalisation for the fundamental and anti-fundamental representations

$$\mathcal{D}_{1,0} = \big\{\sqrt{3}z_1, \sqrt{3}z_2, \sqrt{3}z_3\big\} \ , \quad \mathcal{D}_{1,0} = \overline{\mathcal{D}}_{1,0} = \big\{\sqrt{3}\bar{z}_3, -\sqrt{3}\bar{z}_2, \sqrt{3}\bar{z}_3\big\} \ , \quad (60)$$

and thus

$$\overline{\psi}^{n,m,n_1,n_2,\ell}(\theta, \xi, \varphi_1, \varphi_2, \varphi_3) = (-1)^{\frac{1}{3}(n-m-n_1+n_2)}\psi_{m,n,-n_1,-n_2,\ell}(\theta, \xi, \varphi_1, \varphi_2, \varphi_3) \ .$$

Then we define the Lie algebra $\hat{\mathfrak{g}}\big(SU(3)/SU(2)\big)$ by introducing $T_{a,n,m,n_1,n_2,\ell} = T_a\psi_{n,m,n_1,n_2,\ell}$. We identify the maximal set of commuting operators by observing that relations (58) can be extended to define a differential realisation of the Lie algebra $\mathfrak{so}(6) \supset \mathfrak{su}(3)$. Within this differential realisation, all the representations $D_{n,m}$ are rearranged into representations of $\mathfrak{so}(6)$ corresponding to symmetric traceless tensors. The two constructions based on $\mathbb{S}^5 = SU(3)/SU(2)$ or on $\mathbb{S}^5 = SO(6)/SO(5)$ lead to isomorphic algebras in straight analogy with Sections 5.1 and 5.3. We will not use the representations of $\mathfrak{so}(6)$ to build the generalised Kac-Moody algebra.

The Hermitian operators are taken to be h_1, h_2 and

$$h = -\mathrm{i}\left(\frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2} + \frac{\partial}{\partial\varphi_3}\right) , \quad (61)$$

the latter being associated to the Cartan subalgebra of $\mathfrak{so}(6)$, and we have

$$h\psi_{n,m,n_1,n_2,\ell} = (n-m)\psi_{n,m,n_1,n_2,\ell} \ .$$

We finally introduce the associated central charge k_1, k_2, k (see (15) and (16)). The Lie brackets take the form (see (17))

$$\begin{aligned}
\big[T_{a,n,m,n_1,n_2,\ell}, T_{a',n',m',n'_1,n'_2,\ell'}\big] &= \mathrm{i} f_{aa'}{}^{a''} c_{n,m,n_1,n_2,\ell,n',m',n'_1,n'_2,\ell'}{}^{n'',m'',n''_1,n''_2,\ell''} T_{a'',n'',m'',n''_1,n''_2,\ell''} \\
&\quad +(-1)^{\frac{1}{3}(n-m-n_1+n_2)}(k_1 n_2 + k_2 n'_1 + k(n'-m')) \times \\
&\qquad g_{aa'}\delta_{nm'}\delta_{mn'}\delta_{n_1+n'_2}\delta_{n_2+n'_1}\delta_{\ell\ell'} \ , \\
\big[h_1, T_{a,n,m,n_1,n_2,\ell}\big] &= n_1 T_{a,n,m,n_1,n_2,\ell} \ , \\
\big[h_2, T_{a,n,m,n_1,n_2,\ell}\big] &= n_2 T_{a,n,m,n_1,n_2,\ell} \ , \\
\big[h, T_{a,n,m,n_1,n_2,\ell}\big] &= (n-m) T_{a,n,m,n_1,n_2,\ell} \ ,
\end{aligned} \tag{62}$$

Observe that, because of the identity $\overline{\mathcal{D}}_{n,m} = \mathcal{D}_{m,n}$, the structure constants involve the normalisation term $\delta_{mn'}\delta_{nm'}\delta_{n_1+n'_2}\delta_{n_2+n'_1}$, and not $\delta_{mm'}\delta_{nn'}\delta_{n_1+n'_1}\delta_{n_2+n'_2}$ as may be naively expected. Such a subtlety is only encountered when the Lie algebra $\mathfrak{g}$ admits complex (*i.e.*, neither real nor pseudo-real) representations, and is not present for the remaining examples studied in this article.

The $c_{IJ}{}^K$ coefficients are obtained from

$$\begin{aligned}
\psi_{n,m,n_1,n_2,\ell}\psi_{n',m',n'_1,n'_2,\ell'} &= \sum \lambda(N,M,n,n',m,m',\ell,\ell',L) \left(\begin{smallmatrix} n & m & \ell & & n' & m' & \ell' \\ n_1 & n_2 & & & n'_1 & n'_2 & \end{smallmatrix} \middle| \begin{smallmatrix} N & M & L \\ n_1+n'_1 & n_2+n'_2 & \end{smallmatrix} \right) \\
&\qquad \times \psi_{N,M,n_1+n'_1,n_2+n'_2,L} .
\end{aligned}$$

Here $\left(\begin{smallmatrix} n & m & \ell & & n' & m' & \ell' \\ n_1 & n_2 & & & n'_1 & n'_2 & \end{smallmatrix} \middle| \begin{smallmatrix} N & M & L \\ n_1+n'_1 & n_2+n'_2 & \end{smallmatrix} \right)$ are the Clebsch-Gordan coefficients of the decomposition

$$\mathcal{D}_{n,m} \otimes \mathcal{D}_{n',m'} = \bigoplus_{N,M} D_{N,M} \ .$$

As before, the coefficients $\lambda(N,M,n,n',m,m',\ell,\ell',L)$ can be computed recursively.

This construction can be extended naturally, along the same lines, to the generic coset space $SU(n+1)/SU(n) \cong \mathbb{S}^{2n+1}$. The only representations that contain the scalar representations with respect to the embedding $SU(n-1) \subset SU(n)$ are $\mathcal{D}_{n,0,\cdots,0,m}$ (in the notations of [41]) and correspond to traceless tensor products of the fundamental and the anti-fundamental representations. It should be observed that the generalised Kac-Moody algebra that we obtain from $SU(n-1) \subset SU(n)$ is isomorphic to the construction from the coset space $SO(2n+2)/SO(2n+1) \cong \mathbb{S}^{2n+1}$.

5.5 Coset space $G_2/SU(3)$

Prior to the construction of parameterisations of the manifolds G_2 and $G_2/SU(3)$, we briefly recall some fundamental properties of the exceptional Lie algebra $\mathfrak{g}_2$ (see *e.g.* [59]). The Cartan matrix of $\mathfrak{g}_2$ is given by

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

and its simple roots and fundamental weights are given respectively by

$$\begin{array}{lclclcl} \alpha_1 &=& \big|2,-3\big\rangle \ , & \alpha_2 &=& \big|-1,2\big\rangle \ , \\ \mu_1 &=& \big|1,0\big\rangle \ , & \mu_2 &=& \big|0,1\big\rangle \ . \end{array}$$

The representation of highest weight μ_2 is seven-dimensional and real, explicitly:

$$\begin{array}{c} \big\langle z\big|1,-2\big\rangle = z_{[1,-2]} \ , \quad \big\langle z\big|-1,1\big\rangle = z_{[-1,1]} \ , \quad \big\langle z\big|0,1\big\rangle = z_{[0,1]} \ , \\ \big\langle z\big|-1,2\big\rangle = \overline{z}_{[-1,2]} \ , \quad \big\langle z\big|1,-1\big\rangle = \overline{z}_{[1,-1]} \ , \quad \big\langle z\big|0,-1\big\rangle = \overline{z}_{[0,-1]} \ , \\ \big\langle z\big|0,0\big\rangle = x_0 \ . \end{array}$$

Because the representation is real, x_0 is a real number, while $z_{[1,-2]}, z_{[-1,1]}, z_{[0,1]}$ are complex and conjugate to $\overline{z}_{[-1,2]}, \overline{z}_{[1,-1]}, \overline{z}_{[0,-1]}$, respectively. A differential realisation of the algebra deduced from this representation is given by (see Chapter 10 in [60])

$$\begin{array}{rcl} E_{\alpha_1[2,-3]} &=& z_{[1,-2]}\partial_{[-1,1]} - \overline{z}_{[1,-1]}\overline{\partial}_{[-1,2]} \ , \\ E_{-\alpha_1[-2,3]} &=& z_{[-1,1]}\partial_{[1,-2]} - \overline{z}_{[-1,2]}\overline{\partial}_{[1,-1]} \ , \\ E_{\alpha_2[-1,2]} &=& z_{[0,1]}\overline{\partial}_{[1,-1]} - z_{[-1,1]}\overline{\partial}_{[0,-1]} + \sqrt{2}\Big(x_0\partial_{[1,-2]} - \overline{z}_{[-1,2]}\partial_0\Big) \ , \\ E_{-\alpha_2[1,-2]} &=& \overline{z}_{[0,-1]}\partial_{[-1,1]} - \overline{z}_{[1,-1]}\partial_{[0,1]} + \sqrt{2}\Big(x_0\overline{\partial}_{[-1,2]} - z_{[1,-2]}\partial_0\Big) \ , \\ E_{\alpha_1+\alpha_2[1,-1]} &=& z_{[1,-2]}\overline{\partial}_{[0,-1]} - z_{[0,1]}\overline{\partial}_{[-1,2]} + \sqrt{2}\Big(x_0\partial_{[-1,1]} - \overline{z}_{[1,-1]}\partial_0\Big) \ , \\ E_{-\alpha_1-\alpha_2[-1,1]} &=& \overline{z}_{[-1,2]}\partial_{[0,1]} - \overline{z}_{[0,-1]}\partial_{[1,-2]} + \sqrt{2}\Big(x_0\overline{\partial}_{[1,-1]} - z_{[-1,1]}\partial_0\Big) \ , \\ E_{\alpha_1+2\alpha_2[0,1]} &=& \overline{z}_{[1,-1]}\partial_{[1,-2]} - \overline{z}_{[-1,2]}\partial_{[-1,1]} + \sqrt{2}\Big(x_0\overline{\partial}_{[0,-1]} - z_{[0,1]}\partial_0\Big) \ , \\ E_{-\alpha_1-2\alpha_2[0,1]} &=& z_{[-1,1]}\overline{\partial}_{[-1,2]} - z_{[1,-2]}\overline{\partial}_{[1,-1]} + \sqrt{2}\Big(x_0\partial_{[0,1]} - \overline{z}_{[0,-1]}\partial_0\Big) \ , \\ E_{\alpha_1+3\alpha_2[-1,3]} &=& z_{[0,1]}\partial_{[1,-2]} - \overline{z}_{[-1,2]}\overline{\partial}_{[0,-1]} \ , \\ E_{-\alpha_1-3\alpha_2[1,-3]} &=& z_{[1,-2]}\partial_{[0,1]} - \overline{z}_{[0,-1]}\overline{\partial}_{[-1,2]} \ , \\ E_{2\alpha_1+3\alpha_2[1,0]} &=& z_{[0,1]}\partial_{[-1,1]} - \overline{z}_{[1,-1]}\overline{\partial}_{[0,-1]} \ , \\ E_{-2\alpha_1-3\alpha_2[1,0]} &=& z_{[-1,1]}\partial_{[0,1]} - \overline{z}_{[0,-1]}\overline{\partial}_{[1,-1]} \ , \\ h_1 &=& z_{[1,-2]}\partial_{[1,-2]} - z_{[-1,1]}\partial_{[-1,1]} - \overline{z}_{[-1,2]}\overline{\partial}_{[-1,2]} + \overline{z}_{[1,-1]}\overline{\partial}_{[1,-1]} \ , \\ h_2 &=& -2z_{[1,-2]}\partial_{[1,-2]} + z_{[-1,1]}\partial_{[-1,1]} + z_{[0,1]}\partial_{[0,1]} \\ && +2\overline{z}_{[-1,2]}\overline{\partial}_{[-1,2]} - \overline{z}_{[1,-1]}\overline{\partial}_{[1,-1]} - \overline{z}_{[0,-1]}\overline{\partial}_{[0,-1]} \ . \end{array} \tag{63}$$

Due to the embedding $\mathfrak{g}_2 \subset \mathfrak{so}(7)$, the quadratic form

$$q(z) = x_0^2 + 2z_{[1,-2]}\overline{z}_{[-1,2]} + 2z_{[-1,1]}\overline{z}_{[1,-1]} + 2z_{[0,1]}\overline{z}_{[0,-1]} \ ,$$

is preserved *i.e.*, for any element ξ in the realisation (63) of $\mathfrak{g}_2$, we have $\xi(q(z)) = 0$. As a consequence, the representations with highest weight $n\mu_2$ can be described in terms of

n^{th}-order polynomials in the variables z. In order to factor out those terms proportional to $q(z)$, we establish that

$$\mathcal{D}_{0,n} = \{P \in \mathbb{R}_n[x_0, z_{[1,-2]}, z_{[-1,1]}, z_{[0,1]}, \overline{z}_{[-1,2]}, \overline{z}_{[1,-1]}, \overline{z}_{[0,-1]}] \text{ s.t. } \nabla^2(P) = 0\} \ , \tag{64}$$

where

$$\nabla^2 = \partial_0^2 + 2\partial_{[1,-2]}\overline{\partial}_{[-1,2]} + 2\partial_{[-1,1}\overline{\partial}_{[1,-1]} + 2\partial_{[0,1]}\overline{\partial}_{[0,-1]} \ ,$$

and $\mathbb{R}_n[x_0, z_{[1,-2]}, z_{[-1,1]}, z_{[0,1]}, \overline{z}_{[-1,2]}, \overline{z}_{[1,-1]}, \overline{z}_{[0,-1]}]$ denotes the space of n^{th}-order polynomials. For the embedding $\mathfrak{su}(3) \subset \mathfrak{g}_2$, the branching rule for the adjoint representation (with highest weight μ_1) of $\mathfrak{g}_2$ is given by

$$\underset{\sim}{\mathbf{14}} = \underset{\sim}{\mathbf{8}} \oplus \underset{\sim}{\mathbf{6}} \ .$$

The adjoint representation is always real, whereas the two *complex conjugate* fundamental and anti-fundamental representations regroup into a six-dimensional *real* representation:

$$\underset{\sim}{\mathbf{3}} \oplus \overline{\underset{\sim}{\mathbf{3}}} = \underset{\sim}{\mathbf{6}} \ . \tag{65}$$

The embedding $\mathfrak{su}(3) \subset \mathfrak{g}_2$ can be explicitly described as follows. Let

$$\begin{aligned} \beta_1 &= 2\alpha_1 + 3\alpha_2 \ , \\ \beta_2 &= -\alpha_1 \ , \end{aligned}$$

be the two simple roots of $\mathfrak{su}(3)$, so that the generators of the subalgebra $\mathfrak{su}(3) \subset \mathfrak{g}_2$ are expressed as

$$\begin{array}{lcllcl} E_{\beta_1} &=& E_{2\alpha_1+3\alpha_2} \ , & E_{-\beta_1} &=& E_{-2\alpha_1-3\alpha_2} \ , \\ E_{\beta_2} &=& E_{-\alpha_1} \ , & E_{-\beta_2} &=& E_{\alpha_1} \ , \\ E_{\beta_1+\beta_2} &=& E_{\alpha_1+3\alpha_2} \ , & E_{-\beta_1-\beta_2} &=& E_{-\alpha_1-3\alpha_2} \\ H_1 &=& 2h_1 + h_2 \ , & H_2 &=& -h_1 \ . \end{array}$$

By introducing also the fundamental weight,

$$\begin{aligned} \mu_1 &= \frac{2}{3}\beta_1 + \frac{1}{3}\beta_2 = \alpha_1 + 2\alpha_2 \ , \\ \mu_2 &= \frac{1}{3}\beta_1 + \frac{2}{3}\beta_2 = \alpha_2 \ , \end{aligned}$$

the generators of the coset $\mathfrak{g}_2/\mathfrak{su}(3)$ read as

$$\begin{array}{lcllcl} & \underset{\sim}{\mathbf{3}} & & & \overline{\underset{\sim}{\mathbf{3}}} & \\ \hline E_{\mu_1} &=& E_{\alpha_1+2\alpha_2} \ , & E_{\mu_2} &=& E_{\alpha_2} \ , \\ E_{\mu_1-\beta_1} &=& E_{-\alpha_1-\alpha_2} \ , & E_{\mu_2-\beta_2} &=& E_{\alpha_1+\alpha_2} \ , \\ E_{\mu_1-\beta_1-\beta_2} &=& E_{-\alpha_2} \ , & E_{\mu_2-\beta_2-\beta_1} &=& E_{-\alpha_1-2\alpha_2} \ . \end{array}$$

The next step in the construction is to derive a matrix representation. This can be easily done by means of the vectors

$$\begin{aligned} Z &= (z_{[-1,2]}, z_{[-1,1]}, z_{[0,1]}, \overline{z}_{[1,-2]}, \overline{z}_{[1,-1]}, \overline{z}_{[0,-1]}, x_0), \\ \partial Z &= (\partial_{[1,-2]}, \partial_{[1,-1]}, \partial_{[0,-1]}, \overline{\partial}_{[-1,2]}, \overline{\partial}_{[-1,1]}, \overline{\partial}_{[0,1]}, \partial_0)^t \ , \end{aligned}$$

such that, to any (first-order) differential operator D of $\mathfrak{g}_2$, we can associate the matrix M defined by

$$D = ZM\partial_Z \ .$$

The matrices M_γ associated to the roots γ of $\mathfrak{g}_2$, as well as those h_1, h_2, corresponding to the Cartan subalgebra, can be constructed in a straightforward manner *via* this prescription. For the subalgebra $\mathfrak{su}(3)$, these matrices reduce to

$$T_i = \begin{pmatrix} \lambda_i & 0 & 0 \\ 0 & -\overline{\lambda}_i & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \quad i = 1, \ldots, 8 \ ,$$

where λ_i are the 3×3 Gell-Mann matrices. For the coset $\mathfrak{g}_2/\mathfrak{su}(3)$ we have

$$x^i U_i = \begin{pmatrix} 0 & 0 & 0 & 0 & x^5 + \mathrm{i}x^6 & x^3 + \mathrm{i}x^4 & \sqrt{2}(x^1 - \mathrm{i}x^2) \\ 0 & 0 & 0 & -x^5 - \mathrm{i}x^6 & 0 & -x^1 - \mathrm{i}x^2 & \sqrt{2}(x^3 - \mathrm{i}x^4) \\ 0 & 0 & 0 & -x^3 - \mathrm{i}x^4 & x^1 + \mathrm{i}x^2 & 0 & -\sqrt{2}(x^5 - \mathrm{i}x^6) \\ 0 & -x^5 + \mathrm{i}x^6 & -x^3 + \mathrm{i}x^4 & 0 & 0 & 0 & -\sqrt{2}(x^1 + \mathrm{i}x^2) \\ x^5 - \mathrm{i}x^6 & 0 & x^1 - \mathrm{i}x^2 & 0 & 0 & 0 & -\sqrt{2}(x^3 + \mathrm{i}x^4) \\ x^3 - \mathrm{i}x^4 & -x^1 + \mathrm{i}x^2 & 0 & 0 & 0 & 0 & \sqrt{2}(x^5 + \mathrm{i}x^6) \\ \sqrt{2}(x^1 + \mathrm{i}x^2) & \sqrt{2}(x^3 + \mathrm{i}x^4) & -\sqrt{2}(x^5 + \mathrm{i}x^6) & -\sqrt{2}(x^1 - \mathrm{i}x^2) & -\sqrt{2}(x^3 - \mathrm{i}x^4) & \sqrt{2}(x^5 - \mathrm{i}x^6) & 0 \end{pmatrix} ,$$

with the matrices U_i defined by

$$\begin{array}{lllllll} U_1 & = & M_{\alpha_2} - M_{-\alpha_2} \ , & U_2 & = & \mathrm{i}(M_{\alpha_2} + M_{-\alpha_2}) \ , \\ U_3 & = & M_{\alpha_1+\alpha_2} - M_{-\alpha_1-\alpha_2} \ , & U_4 & = & \mathrm{i}(M_{\alpha_1+\alpha_2} + M_{-\alpha_1-\alpha_2}) \ , \\ U_5 & = & M_{\alpha_1+2\alpha_2} - M_{-\alpha_1-2\alpha_2} \ , & U_6 & = & \mathrm{i}(M_{\alpha_1+2\alpha_2} + M_{-\alpha_1-2\alpha_2}) \ . \end{array}$$

It should be observed that these matrices are not well adapted, because the representation is real. With equation (65), we consider the real basis

$$X = \begin{pmatrix} x_{[-1,2]} = \frac{1}{\sqrt{2}}(z_{[-1,2]} + \overline{z}_{[1,-2]}) \\ x_{[-1,1]} = \frac{1}{\sqrt{2}}(z_{[-1,1]} + \overline{z}_{[1,-1]}) \\ x_{[0,1]} = \frac{1}{\sqrt{2}}(z_{[0,1]} + \overline{z}_{[0,-1]}) \\ y_{[-1,2]} = -\frac{\mathrm{i}}{\sqrt{2}}(z_{[-1,2]} - \overline{z}_{[1,-2]}) \\ y_{[-1,1]} = -\frac{\mathrm{i}}{\sqrt{2}}(z_{[-1,1]} - \overline{z}_{[1,-1]}) \\ y_{[0,1]} = -\frac{\mathrm{i}}{\sqrt{2}}(z_{[0,1]} - \overline{z}_{[0,-1]}) \\ x_0 \end{pmatrix} \ . \tag{66}$$

Over this basis, the generators of the $\mathfrak{su}(3)$-subalgebra take the form

$$S_i = \begin{pmatrix} \frac{1}{2}(\lambda_i - \overline{\lambda}_i) & \frac{\mathrm{i}}{2}(\lambda_i + \overline{\lambda}_i) & 0 \\ -\frac{\mathrm{i}}{2}(\lambda_i + \overline{\lambda}_i) & \frac{1}{2}(\lambda_i - \overline{\lambda}_i) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

while for the coset $\mathfrak{g}_2/\mathfrak{su}(3)$ the generators V_i are given by

$$\mathrm{i}x^j V_j = \begin{pmatrix} 0 & -x^6 & -x^4 & 0 & -x^5 & -x^3 & 2x^2 \\ x^6 & 0 & x^2 & -x^5 & 0 & x^1 & 2x^4 \\ x^4 & -x^2 & 0 & -x^3 & -x^1 & 0 & -2x^6 \\ 0 & x^5 & x^3 & 0 & -x^6 & -x^4 & 2x^1 \\ x^5 & 0 & x^1 & x^6 & 0 & -x^2 & -2x^3 \\ x^3 & -x^1 & 0 & x^4 & x^2 & 0 & 2x^5 \\ -2x^2 & -2x^4 & 2x^6 & -2x^1 & 2x^3 & -2x^5 & 0 \end{pmatrix}.$$

Given an appropriate real basis, we can construct a parameterisation of the manifold G_2. From the coset space structure, we rewrite a matrix for G_2 in the form

$$G = M_3 M_2$$

where M_3 is a matrix that parameterises $SU(3)$, and M_2 a matrix that parameterises the factor space $G_2/SU(3)$. The matrix M_3 can be directly obtained from Section 5.4:

$$M_3 = \begin{pmatrix} \frac{1}{2}(U + \overline{U}) & \frac{\mathrm{i}}{2}(U - \overline{U}) & 0 \\ -\frac{\mathrm{i}}{2}(U - \overline{U}) & \frac{1}{2}(U + \overline{U}) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{67}$$

The matrix M_2 is constructed as follows. Take

$$U_1(\theta) = e^{\mathrm{i}\theta V_1}$$

and consider the specific point on $G_2/SU(3)$ given by $U_1(\pi)$. Next, introduce the matrix $P = RR_1R_2R_3R_4R_5$, with the $R-$matrices being appropriate rotations: R angle $-\varphi$ in the plane $(x_{[-1,2]}, x_{[-1,1]})$, R_1 angle $-\theta_1$ in the plane $(x_{[-1,1]}, x_{[0,1]})$, R_2 angle $-\theta_2$ in the plane $(x_{[0,1]}, y_{[-1,,2]})$, R_3 angle $-\theta_3$ in the plane $(y_{[-1,,2]}, y_{[-1,1]})$, R_4 angle $-\theta_4$ in the plane $(y_{[-1,1]}, y_{[0,1]})$, and R_5 angle $-1/2\theta_5$ in the plane $(y_{[0,1]}, x_0)$ with the notations of (66) and define M_2 by

$$M_2 = P^{-1}U_1(\pi)P\ .$$

The parameterisation of the manifold is thus given by

$$G = M_3P^{-1}U_1(\pi)P\ . \tag{68}$$

At a first glance, the matrix M_2 seems not to be very illuminating. Fortunately, however, as follows from Section 3.2, only the last column of the matrix M_2 will be relevant for the harmonic analysis on $G_2/SU(3)$. It reduces to a very simple expression:

$$\begin{aligned}
M_{2,71} &= \sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\sin\theta_5\sin\varphi \equiv -\frac{\mathrm{i}}{2}(z_1-\bar z_1)\ , \\
M_{2,72} &= \sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\sin\theta_5\cos\varphi \equiv \frac{1}{2}(z_1+\bar z_1)\ , \\
M_{2,73} &= \cos\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\sin\theta_5 \equiv \frac{1}{2}(z_2+\bar z_2)\ \ , \\
M_{2,74} &= \cos\theta_2\sin\theta_3\sin\theta_4\sin\theta_5 \equiv -\frac{\mathrm{i}}{2}(z_2-\bar z_2)\ , \\
M_{2,75} &= \cos\theta_3\sin\theta_4\sin\theta_5 \equiv \frac{1}{2}(z_3+\bar z_3)\ , \\
M_{2,76} &= \cos\theta_4\sin\theta_5 \equiv -\frac{\mathrm{i}}{2}(z_3-\bar z_3)\ , \\
M_{2,77} &= \cos\theta_5 \equiv x_0\ ,
\end{aligned} \tag{69}$$

which is a parameterisation of the sphere $\mathbb{S}^6 \cong G_2/SU(3)$ with $0\le\varphi<2\pi$, $0\le\theta_i\le\pi$, $i=1,\cdots,5$, that corresponds to the usual spherical coordinates.

Comparing with the approach of Appendix A, the sphere $\mathbb{S}^6$ is parameterised by

$$\begin{aligned}
0\le\varphi\le 2\pi\ ,\quad -1\le u_1 &= \cos\theta_1\le 1\ \ , \\
0\le u_2 &= \frac{1}{2}(\theta_2-\cos\theta_2\sin\theta_2)\le\frac{\pi}{2} \\
-\frac{2}{3}\le u_3 &= -\frac{1}{3}(\sin^2\theta_3\cos\theta_3+2\cos\theta_3)\le\frac{2}{3}\ , \\
0\le u_4 &= \frac{1}{8}((3\theta_4-3\sin\theta_4\cos\theta_4-2\sin^3\theta_4\cos\theta_4)\le\frac{3}{8}\pi. \\
-\frac{8}{15}\le u_5 &= -\frac{8}{15}\cos\theta_5-\frac{4}{15}\sin\theta_5^2\cos\theta_5-\frac{1}{5}\sin\theta_5^4\cos\theta_5\le\frac{8}{15}\ .
\end{aligned}$$

If we define the scalar product on $\mathbb{S}^6$ by

$$\begin{aligned}
(f,g) \;=\; \frac{15}{16\pi^3}\int_0^\pi \mathrm{d}\theta_5\sin^5\theta_5\int_0^\pi \mathrm{d}\theta_4\sin^4\theta_4\int_0^\pi \mathrm{d}\theta_3\sin^3\theta_3\int_0^\pi \mathrm{d}\theta_2\sin^2\theta_2\int_0^\pi \mathrm{d}\theta_1\sin\theta_1\int_0^{2\pi}\mathrm{d}\varphi \\
\overline{f(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\varphi)}\ g(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\varphi),
\end{aligned}$$

without loss of generality (*e.g.* after having conjugated the matrix M_2 by an appropriate permutation) we can introduce the harmonic functions

$$\begin{aligned}
\Phi_{1;1,-2} &= \sqrt{\frac{7}{4}}\big(M_{2,72}+\mathrm{i}M_{2,71}\big) = -\sqrt{\frac{7}{2}}z_{[-1,2]} \\
&= \sqrt{\frac{7}{2}}e^{\mathrm{i}\varphi}\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\sin\theta_5\ , \\
\Phi_{1;-1,2} &= \sqrt{\frac{7}{4}}\big(M_{2,72}-\mathrm{i}M_{2,71}\big) = \sqrt{\frac{7}{2}}\overline{z}_{[1,-2]}
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{7}{2}}e^{-\mathrm{i}\varphi}\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\sin\theta_5 \quad , \\
\Phi_{1;-1,1} &= \sqrt{\frac{7}{4}}\big(M_{2,73}+\mathrm{i}M_{2,74}\big) = -\sqrt{\frac{7}{2}}z_{[-1,1]} \\
&= \sqrt{\frac{7}{2}}\sin\theta_3\sin\theta_4\sin\theta_5\Big(\cos\theta_1\sin\theta_2+\mathrm{i}\cos\theta_2\Big) \ , \\
\Phi_{1;1,-1} &= \sqrt{\frac{7}{4}}\big(M_{2,73}-\mathrm{i}M_{2,74}\big) = -\sqrt{\frac{7}{2}}\overline{z}_{[1,-1]} \\
&= \sqrt{\frac{7}{2}}\sin\theta_3\sin\theta_4\sin\theta_5\Big(\cos\theta_1\sin\theta_2-\mathrm{i}\cos\theta_2\Big) \ , \\
\Phi_{1;0,1} &= \sqrt{\frac{7}{4}}\big(M_{2,75}+\mathrm{i}M_{2,76}\big) = \sqrt{\frac{7}{2}}z_{[0,1]} \\
&= \sqrt{\frac{7}{2}}\sin\theta_5\big(\cos\theta_3\sin\theta_4+\mathrm{i}\cos\theta_4\big) \ , \\
\Phi_{1;0,-1} &= \sqrt{\frac{7}{4}}\big(M_{2,75}-\mathrm{i}M_{2,76}\big) = -\sqrt{\frac{7}{2}}\overline{z}_{[0,-1]} \\
&= \sqrt{\frac{7}{2}}\sin\theta_5\big(\cos\theta_3\sin\theta_4-\mathrm{i}\cos\theta_4\big) \ , \\
\Phi_{1;0,0} &= \sqrt{7}M_{2,77} = \sqrt{7}x_0 = \sqrt{7}\cos\theta_5,
\end{aligned} \tag{70}
$$

which are orthonormal with respect to the scalar product on $\mathbb{S}^6$. The precise signs are obtained from (63). These functions parameterise the representation $\mathcal{D}_{0,1}$. The highest weight of the representation $\mathcal{D}_{0,n}$ (see equation (64)) is therefore given by

$$
\Phi^{\alpha}_{n,0,n} = \sqrt{\frac{1}{60}\frac{1}{4^n}\frac{(2n+5)!}{n!(n+2)!}}z^n_{[0,1]} = \sqrt{\frac{1}{60}\frac{1}{4^n}\frac{(2n+5)!}{n!(n+2)!}}\sin\theta_5^n\big(\cos\theta_3\sin\theta_4+\mathrm{i}\cos\theta_4\big)^n \ ,
$$

and $\mathcal{D}_{0,n}$ is constructed by the action of the operators given in equation (63). Only at the very end, we substitute equation (70) into the n^{th}-order polynomials of $\mathcal{D}_{0,n}$, given in equation (64), to obtain the corresponding harmonic functions.

In contrast to the previous cases, for G_2 we have a degeneracy problem. According to Proposition 3.2, we need 6 internal labels to separate states within an irreducible representation of G_2, the Casimir operators of G_2 being used to characterise the representation. Considering the reduction chain

$$
G_2 \supset SU(3) \supset SU(2) \supset U(1), \tag{71}
$$

provides us with five internal labels, namely the Casimir operators of $SU(3)$ and $SU(2)$, as well as the generators of the Cartan subalgebra. It is thus necessary to consider an additional label. This operator can be constructed by the method of elementary multiplets (see [61]) observing that the adjoint representation of G_2 decomposes as the direct sum of an octet T and two conjugate triplets $V, \overline{V}$ of $SU(3)$. The simplest labelling problem resulting from this method is a cubic operator $TV\overline{V}$ in the generators of G_2, such that in each monomial one generator belongs to the octet and each of the triples, respectively. This operator is Hermitian and commutes with the elements of $SU(3)$ [62].

It is worthy to be observed that the sphere can be obtained in two different ways, either as the coset space $\mathbb{S}^6 = G_2/SU(3)$, or alternatively as $\mathbb{S}^6 = SO(7)/SO(6)$. On the other hand, as the subduced representations $\mathcal{D}_{n,0,0}$ of $SO(7)$ are isomorphic to the representation representation $\mathcal{D}_{0,n}$ of G_2 (see *e.g.* [63]), it follows that the harmonic functions on $G_2/SU(3)$ are the same as the harmonic functions on $SO(7)/SO(6)$, except that the former are labeled with the quantum numbers of G_2, whereas the latter are labeled by the quantum numbers of $SO(7)$. This, in particular, implies that we have the Lie algebra isomorphism

$$\hat{\mathfrak{g}}\big(G_2/SU(3)\big) \cong \hat{\mathfrak{g}}\big(SO(7)/SO(6)\big) \ .$$

This enables us to construct harmonic functions on the sphere $\mathbb{S}^6$ using two alternative ways, either using the representation theory of $\mathfrak{g}_2$ or the representation theory of $\mathfrak{so}(7)$. In the second case, we can extend the differential realisation of $\mathfrak{g}_2$ given in (63) to a differential realisation of $\mathfrak{so}(7)$. Moreover all representations $\mathcal{D}_{0,n}$ turn out to be representations of $\mathfrak{so}(7)$ corresponding to symmetric traceless tensors.

The differential realisation of $\mathfrak{so}(7)$ is given on page 407 in [41]. It is not useful to reproduce the expression for all the generators of $\mathfrak{so}(7)$, but only for the Cartan subalgebra:

$$\begin{aligned} h_1 &= -\mathrm{i}\frac{\partial}{\partial\varphi} \ , \\ h_2 &= -\mathrm{i}\Big(\cot\theta_2\sin\theta_1\frac{\partial}{\partial\theta_1} - \cos\theta_1\frac{\partial}{\partial\theta_2}\Big) \ , \\ h_3 &= -\mathrm{i}\Big(\cot\theta_4\sin\theta_3\frac{\partial}{\partial\theta_3} - \cos\theta_3\frac{\partial}{\partial\theta_4}\Big) \ . \end{aligned} \tag{72}$$

According to Proposition 3.2, we need nine internal labels. Actually, as we merely consider symmetric traceless tensors in fact, only 6 labels are required [64]. The reduction chain [5]

$$SO(7) \supset SO(5) \supset SO(3) \ ,$$

provides three additional operators, namely the two Casimir operators of $SO(5)$ and the Casimir operator of $SO(3)$. Thus with the generators of the Cartan subalgebra we have identified six labels. We introduce the vector representation (with the notations of (69))

$$\begin{aligned} \mathcal{D}_{1,0,0} = \Big\{ &\Psi_{1,1,0,0} = \sqrt{\frac{7}{2}}z_1, \Psi_{1,-1,0,0} = -\sqrt{\frac{7}{2}}\bar{z}_1, \Psi_{1,0,1,0} = \sqrt{\frac{7}{2}}z_2, \Psi_{1,0,-1,0} = \sqrt{\frac{7}{2}}\bar{z}_2, \\ &\Psi_{1,0,0,1} = \sqrt{\frac{7}{2}}z_3, \Psi_{1,0,0,-1} = -\sqrt{\frac{7}{2}}z_3, \Psi_{1,0,0,0} = \sqrt{7}x_0 \Big\} \ , \end{aligned} \tag{73}$$

[5] As in Section 5.3, the set of Gegenbauer polynomials obtained from the reduction chain $SO(7) \supset SO(6) \supset SO(5) \supset SO(4) \supset SO(3) \supset SO(2)$ of $\mathbb{S}^6$ does not constitute an adapted set of harmonic functions in our case. This will further hold for all $n-$spheres.

obtained explicitly (signs included) from the differential realisation of $\mathfrak{so}(7)$. The first label is associated to the vector representation $\mathcal{D}_{1,0,0}$, whilst the last three indices correspond to the eigenvalues of the Cartan subalgebra. In a similar manner, with the highest weight vector of the representation $\mathcal{D}_{n,0,0}, n \in \mathbb{N}$ being given by

$$\Psi_{n,n,0,0} = \sqrt{\frac{1}{60}\frac{1}{4^n}\frac{(2n+5)!}{n!(n+2)!}} z_1^n = \sqrt{\frac{1}{60}\frac{1}{4^n}\frac{(2n+5)!}{n!(n+2)!}} e^{\mathrm{i}n\varphi} \sin^n\theta_1 \sin^n\theta_2 \sin^n\theta_3 \sin^n\theta_4 \sin^n\theta_5 \ ,$$

the representation $\mathcal{D}_{n,0,0}$ can be easily obtained. The labels introduced previously enables us to determine an adapted Hilbert basis:

$$\mathcal{B} = \Big\{ \Psi_{n,m_1,m_2,m_3,\ell_1,\ell_2,\ell_3} \ , \ n \in \mathbb{N} \Big\} \ . \tag{74}$$

In this notation, the first index corresponds to the representation $\mathcal{D}_{n,0,0}$, the three last indices to the eigenvalues of the Cartan subalgebra of $\mathfrak{so}(7)$ and the remaining indices to the additional internal labels. From (73) we have the conjugacy relation

$$\bar{\Psi}^{n,m_1,m_2,m_3,\ell_1,\ell_2,\ell_3} = (-1)^{n_1+n_2} \Psi_{n,-m_1,-m_2,-m_3,\ell_1,\ell_2,\ell_3} \ .$$

The generators of $\hat{\mathfrak{g}}(SO(7)/SO(6))$ are then given by $T_{a,n,m_1,m_2,m_3,\ell_1,\ell_2,\ell_3} = T_a \Psi_{n,m_1,m_2,m_3,\ell_1,\ell_2,\ell_3}$, the Hermitian operators (72). The corresponding $5-$forms are thus

$$\begin{aligned}
\gamma_1 &= -\mathrm{i}k_1 \ \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \wedge \mathrm{d}\theta_3 \wedge \mathrm{d}\theta_4 \wedge \mathrm{d}\theta_5 \sin^5\theta_5 \sin^4\theta_4 \sin^3\theta_3 \sin^2\theta_2 \sin\theta_1 \ , \\
\gamma_2 &= -\mathrm{i}k_2 \Big(-\cot\theta_2 \sin\theta_1 \mathrm{d}\varphi \wedge \mathrm{d}\theta_2 \wedge \mathrm{d}\theta_3 \wedge \mathrm{d}\theta_4 \wedge \mathrm{d}\theta_5 - \cos\theta_1 \mathrm{d}\varphi \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_3 \wedge \mathrm{d}\theta_4 \wedge \mathrm{d}\theta_5 \Big) \\
&\qquad\qquad \times \sin^5\theta_5 \sin^4\theta_4 \sin^3\theta_3 \sin^2\theta_2 \sin\theta_1 \\
\gamma_3 &= -\mathrm{i}k_3 \Big(-\cot\theta_4 \sin\theta_3 \mathrm{d}\varphi \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \wedge \mathrm{d}\theta_4 \wedge \mathrm{d}\theta_5 - \cos\theta_3 \mathrm{d}\varphi \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \wedge \mathrm{d}\theta_3 \wedge \mathrm{d}\theta_5 \Big) \\
&\qquad\qquad \times \sin^5\theta_5 \sin^4\theta_4 \sin^3\theta_3 \sin^2\theta_2 \sin\theta_1
\end{aligned}$$

and the associated central charges are noted k_1, k_2, k_3. The Lie brackets take the form (see (17)

$$\begin{gathered}
\Big[T_{a,n,m_1,m_2,m_3,\ell_1,\ell_2,\ell_3}, T_{a',n',m'_1,m'_2,m'_3,\ell'_1,\ell'_2,\ell'_3} \Big] = \\
\mathrm{i} f_{aa'}{}^{a''} c_{a,n,m_1,m_2,m_3,\ell_1,\ell_2,\ell_3,a',n',m'_1,m'_2,m'_3,\ell'_1,\ell'_2,\ell'_3}{}^{a'',n'',m''_1,m''_2,m''_3,\ell''_1,\ell''_2,\ell''_3} T_{a'',n'',m''_1,m''_2,m''_3,\ell''_1,\ell''_2,\ell''_3} \\
+(-1)^{m_1+n_2}(k_1 n'_1 + k_2 n'_2 + k_3 n'_3) g_{aa'} \delta_{n,n'} \delta_{\ell_1,\ell'_1} \delta_{\ell_2,\ell'_2} \delta_{\ell_3,\ell'_3} \delta_{m_1+m'_1} \delta_{m_2+m'_2} \delta_{m_3+m'_3} \ , \\
\Big[h_1, T_{a,n,m,n_1,n_2,\ell} \Big] = n_1 T_{a,n,m,n_1,n_2,\ell} \ , \\
\Big[h_2, T_{a,n,m,n_1,n_2,\ell} \Big] = n_2 T_{a,n,m,n_1,n_2,\ell} \ , \\
\Big[h_3, T_{a,n,m,n_1,n_2,\ell} \Big] = n_3 T_{a,n,m,n_1,n_2,\ell} \ ,
\end{gathered} \tag{75}$$

The $c_{IJ}{}^K$ coefficients can be obtained either from the Clebsch-Gordan coefficients in the decomposition

$$\mathcal{D}_{0,n} \otimes \mathcal{D}_{0,n'} = \bigoplus_{0,N} D_{0,N} \ ,$$

associated to G_2, or using the decomposition

$$\mathcal{D}_{n,0,0} \otimes \mathcal{D}_{n',0,0} = \bigoplus_{N,0,0} D_{N,0,0} \ ,$$

corresponding to $SO(7)$.

A question that arises naturally in this context is whether for the non-compact real form $G_{2(2)}$ of the exceptional algebra G_2, a consistent construction can be obtained for $G_{2(2)}/SL(3,\mathbb{R})$, which can be seen as a non-compact pseudo-Riemannian version of the non-symmetric coset $G_2/SU(3)$. This case is of great physical relevance due to its relation with the super-Ehlers embedding of minimal supergravity without matter coupling in $4+1$ space-time dimensions (see [65]).

6 Concluding remarks

We have considered a notion of generalised Kac-Moody algebras based on the set of smooth maps from an $n-$dimensional compact manifold $\mathcal{M}$ (associated to a compact Lie group G_c) to a real or complex Lie group G, and studied the conditions that ensure that such generalisations admit central extensions, that have been denoted $\hat{\mathfrak{g}}(\mathcal{M})$. From this point of view, it turns out that the harmonic analysis on the manifold $\mathcal{M}$ as well as the representation theory of G_c constitute a crucial ingredient to properly express the commutators in the generalised Kac-Moody algebra.

We also observed that the non-centrally extended algebras $\mathfrak{g}(\mathcal{M})$ can be obtained naturally from a $(4+n)$-dimensional Kaluza-Klein theory compactified on the compact manifold $\mathcal{M}$, from which it is easily deduced that to any unitary representation of $\mathfrak{g}$ there corresponds a uniquely determined unitary representation of $\mathfrak{g}(\mathcal{M})$. The converse of this assertion also holds. This correspondence suggests to try an extrapolation of the condition obtained for $\mathcal{M} = \mathbb{T}^r$ to the general case $\hat{\mathfrak{g}}(\mathcal{M})$, for the case of non-vanishing central charges. This may provide an alternative tool to inspect highest weight unitary representations.

In Section 2 we have seen that considering the set Diff$(\mathcal{M})$ of vector fields on $\mathcal{M}$, it is possible to define an algebra with a semidirect product structure Diff$(\mathcal{M}) \ltimes \mathfrak{g}(\mathcal{M})$, in analogy with the commutator structure described by equation (6). At this point, one may wonder whether the centrally extended algebra $\hat{\mathfrak{g}}(\mathcal{M})$ is compatible with Diff$(\mathcal{M})$. In this context, it turns out that the compatibility condition can be expressed in terms of the two-cocycles associated to the central extensions, leading to the constraint

$$\omega(L \cdot X, Y) + \omega(X, L \cdot Y) = 0 \ \ \forall X, Y \in \mathfrak{g}(\mathcal{M}), \forall L \in \text{Diff}(\mathcal{M}) \ ,$$

where $L \cdot X$ denotes the natural action of Diff$(\mathcal{M})$ on $\mathfrak{g}(\mathcal{M})$. Compatibility in the latter sense was discussed in [11, 30, 66]. On a different footing, and in the context of bosonic membranes, central extensions of Diff$(\mathbb{S}^1 \times \mathbb{S}^1)$ and of Diff$(\mathbb{S}^2)$ have been studied by several authors (see *e.g.* [67, 68, 69] and references therein). A question that remains currently unanswered is whether the symmetric nature of the coset manifolds has any consequences for the structural properties of the generalised Kac–Moody algebras. Albeit it seems that the answer is in the negative, as can be suspected from the examples presented, a definitive answer requires a more detailed analysis, as well as a careful comparison with other examples, possibly in higher ranks. We hope to provide more evidence in this respect in future work.

Several additional possibilities emerge from the generic approach described in this paper, such as the problem whether this notion of generalised Kac-Moody algebra can be applied and leads to useful insights in the description of extended objects, such as, for example, those arising in the framework of $M-$theory (M_2- or M_5-branes) [70]. The extension of these results to the non-compact case *via* the formalism provided by the Plancherel formula is certainly a problem worthy to be considered in detail, not only because of its geometrical significance, but also due to its current physical applications. In this context, it could be suggested that (super)membrane solutions of extended theories of (super)gravity in higher dimensions might be related to the various central extensions of the generalised Kac-Moody algebras introduced in the manuscript, like M_2- and M_5-branes are central extensions of the $N = 1$, $D = 10+1$ M-theory superalgebra. An eventual extension of the generalised Kac-Moody algebras to non-compact Lie groups or non-compact (and possibly pseudo-Riemannian) coset manifolds (*e.g.* non-Euclidean tori) could then be associated to exotic versions of the $M-$theory, such as the M^*-theory of the M'-theory [71]. However, as commented in Section 5.2, the extension of this work to manifolds involving non-compact groups present subtleties that require additional techniques to surmount the difficulties posed by the non-compacity, the details of which have not yet been solved in fully satisfactory manner, but that warrant further investigation.

As a final observation, also of physical interest, we point out that the motivation of the algebras $\mathfrak{g}(\mathcal{M})$ and $\hat{\mathfrak{g}}(\mathcal{M})$ in terms of current algebras is an aspect that deserves to be analysed more in detail, considering for instance specific fields, as it may lead to some concrete realisations of $\hat{\mathfrak{g}}(\mathcal{M})$. Work in this direction is currently in progress.

A Some identities

Let $\mathcal{M}$ be a an $n = p+q-$dimensional compact real manifold of volume V with parametrisation $y^A = (\varphi^i, u^r) = (\varphi^1, \cdots, \varphi^p, u^1, \cdots, u^q)$. Recall that there are two types of parameters. Angles $\varphi^1, \cdots, \varphi^p$ such that functions on $\mathcal{M}$ are periodic in all $\varphi-$directions, as well as parameters, $u^1, \cdots, u^p$ that do not correspond to angles and such that the functions on $\mathcal{M}$ are not periodic in all the $u-$directions. (For instance, for the sphere $\mathbb{S}^2$, the two parameters are the angle $0 \leq \varphi < 2\pi$ and the parameter $-1 \leq u = \cos\theta \leq 1$.)

From the integration measure, we can write

$$\int_{\mathcal{M}} \mathrm{d}\mu(\mathcal{M}) = \frac{1}{V} \int_{\mathcal{M}} \mathrm{d}^p\varphi \; d^q u = 1 \; ,$$

and let $\mathcal{B} = \big\{\rho_I(\varphi, u) \; , I \in \mathcal{I}\big\}$, where $\mathcal{I}$ is a countable set (see Section 3.3), be a orthonormal Hilbert basis of $L^2(\mathcal{M})$. Assume further that all functions are bounded. Since $\mathcal{B}$ is a complete orthonormal basis we have

$$\int_{\mathcal{M}} \mathrm{d}\mu(\mathcal{M}) \; \overline{\rho}^I(\varphi, u) \; \rho_J(\varphi, u) = \delta^I{}_J$$

and

$$\rho_I(\varphi,u)\overline{\rho}^I(\varphi',u') = \delta^p(\varphi-\varphi')\delta^q(u-u') \tag{76}$$

(the sum over repeated indices is implicit). Since the functions are bounded and $\mathcal{B}$ is a complete Hilbert basis we have on the one hand

$$\begin{aligned} \overline{\rho}^I(\varphi,u) &= \eta^{IJ}\rho_J(\varphi,u)\ , \\ \rho_I(\varphi,u) &= \eta_{IJ}\overline{\rho}^J(\varphi,u)\ , \end{aligned}$$

with

$$\eta^{IJ}\eta_{JK} = \delta^I_K\ ,$$

and on the other hand

$$\begin{aligned} \rho_I(\varphi,u)\rho_J(\varphi,u) &= c_{IJ}{}^K\rho_K(\varphi,u)\ , \\ \overline{\rho}^I(\varphi,u)\overline{\rho}^J(\varphi,u) &= c^{IJ}{}_K\overline{\rho}^K(\varphi,u)\ , \end{aligned} \tag{77}$$

where

$$\overline{c_{IJ}{}^K} = c^{IJ}{}_K = \eta^{IL}\eta^{JM}\eta_{KN}c_{LM}{}^N\ .$$

We now assume that $\mathcal{M}$ is either G_c of G_c/H, where G_c is a compact Lie group and $H \subset G_c$, so that the coefficients $c_{IJ}{}^K$ can be expressed by means of Clebsch-Gordan coefficients. By using the standard Hilbert basis $\left\{\big|I\big\rangle\ ,\quad I \in \mathcal{I}\right\}$ corresponding to all unitary representation of G_c, with the notations of Section 3.3, and setting $\rho_I(\varphi,u) = \big\langle\varphi,u\big|I\big\rangle$, we can extend the usual techniques of quantum mechanics for the composition of spherical harmonics with $G_c = SU(2)$ to other groups G_c. We thus obtain the relations

$$\left\{\begin{array}{rcl} \rho_I(\varphi,u)\rho_J(\varphi,u) &=& c_{IJ}{}^K\rho_K(\varphi,u)\ , \\ \rho_K(\varphi,u) &=& c^{IJ}{}_K\rho_I(\varphi,u)\rho_J(\varphi,u)\ , \end{array}\right.$$

and

$$\begin{aligned} c_{IJ}{}^K c^{IJ}{}_L &= \delta^K_L\ , \\ c_{IJ}{}^K c^{LM}{}_K &= \delta^L_I\delta^M_J\ . \end{aligned}$$

B Missing label operators

As observed, it may be convenient to describe the representations of a semisimple Lie algebra $\mathfrak{g}$ with respect to some distinguished (semisimple) subalgebra $\mathfrak{g}'$ that may correspond to an internal symmetry. The question that arises is whether in such a description the labels are sufficient to separate the degeneracies that may appear. This is known as the 'internal labelling problem' (see *e.g.* [72, 62]). The subalgebra $\mathfrak{g}'$ provides $\frac{1}{2}(\dim\mathfrak{g}'+\ell'))$ labels, where it may happen that $\mathfrak{g}'$ and $\mathfrak{g}$ have some Casimir operator in common. Therefore, subtracting the number ℓ_0 of such common functions, we still need

$$n_0 = \frac{1}{2}\left(\dim\mathfrak{g} - \ell - \dim\mathfrak{g}' - \ell'\right) + \ell_0$$

operators to separate the irreducible representations of $\mathfrak{g}'$ that appear with multiplicity greater than one in the decomposition of $\mathcal{D}$. Such operators must necessarily commute with the generators of $\mathfrak{g}'$, and are commonly called 'missing label operators' or 'subgroup scalars'. In order to prevent undesired interactions and to allow simultaneous diagonalisation, these operators are additionally required to commute with each other [61]. Among the various approaches, differential operators constitute a convenient procedure to determine internal labelling operators [34, 73, 74]: Given a Lie algebra $\mathfrak{g}$ with generators $\{X_1, \dots, X_n\}$ and commutators [6] $[X_i, X_j] = f_{ij}{}^k X_k$, the generators X_i are realised as differential operators in the space $C^\infty(\mathfrak{g}^*)$ by:

$$\widehat{X}_i = -f_{ij}{}^k x_k \frac{\partial}{\partial x_j} \ ,$$

where $\{x_1, \dots, x_n\}$ are the coordinates of a covector in a dual basis of $\{X_1, \dots, X_n\}$. The invariants of $\mathfrak{g}$ correspond to solutions of the system of partial differential equations:

$$\widehat{X}_i F = 0, \quad 1 \leq i \leq n \ ,$$

with the number $\mathcal{N}(\mathfrak{g})$ of independent solutions given by the formula

$$\mathcal{N}(\mathfrak{g}) := \dim\,\mathfrak{g} - \sup_{x_1,\dots,x_n} \operatorname{rank}\left(A(\mathfrak{g})\right) \ ,$$

where $A(\mathfrak{g}) = \left(f_{ij}{}^k x_k\right)$ corresponds to the functional matrix associated with the commutator table of $\mathfrak{g}$ over the given basis. For polynomial solutions, the standard symmetrisation map defined by

$$\Lambda\left(x_{i_1} \dots x_{i_p}\right) = \frac{1}{p!} \sum_{\sigma \in S_p} X_{\sigma(i_1)} \dots X_{\sigma(i_p)}$$

with S_r the permutation group with p elements, allows to recover the Casimir operators in their usual form as elements belonging to the centre of the enveloping algebra $U(\mathfrak{g})$ [75].

If $\mathfrak{g}' \subset \mathfrak{g}$ is an embedding of Lie algebras, it induces branching rules of representations [63]. In particular, the adjoint representation of $\mathfrak{g}$ decomposes as:

$$\operatorname{ad}(\mathfrak{g}) = \operatorname{ad}(\mathfrak{g}') \oplus R \ ,$$

where R is a (completely reducible) representation of $\mathfrak{g}'$ called the characteristic representation.[7] In order to compute the missing labels analytically, we can proceed as follows. Let $\{X_1, \dots, X_m\}$ be a basis of $\mathfrak{g}'$ and extend it to a basis $\mathfrak{B} = \{X_1, \dots, X_m, Y_1, \dots Y_{n-m}\}$ of $\mathfrak{g}$. The brackets adopt the form:

$$[X_i, X_j] = f_{ij}{}^k X_k \ , \quad [X_i, Y_p] = g_{ip}{}^q Y_q \ , \quad [Y_p, Y_q] = E_{pq}{}^k X_k + F_{pq}{}^r Y_r \ ,$$

[6] Pay attention that there is no i factor in the Lie brackets, which is more convenient to identify the missing label operators.

[7] Complete reducibility is actually ensured only if the subalgebra $\mathfrak{g}'$ is semisimple.

where $i,j,k \in \{1,\ldots,m\}$ and $p,q,r \in \{1,\ldots,n-m\}$. Now we consider those differential operators that are associated to generators of $\mathfrak{g}'$, *i.e.*, the system of partial differential equations

$$\widehat{X}_i = -f_{ij}{}^k x_k \frac{\partial}{\partial x_j} - g_{ip}{}^q y_q \frac{\partial}{\partial y_p}, \quad 1 \le i \le m \ , \tag{78}$$

where $\{x_1,\ldots,x_m,y_1,\ldots,y_{n-m}\}$ are the coordinates in a dual basis of $\mathfrak{B}$. We observe that solutions F to the system (78) such that $\dfrac{\partial F}{\partial y_p} = 0$ for all $1 \le p \le n-m$ correspond to the Casimir invariants of the subalgebra, while a genuine missing label must explicitly depend on the variables $\{y_1,\ldots,y_{n-m}\}$. Now the system (78) has exactly $n-r'$ independent solutions, where r' denotes the rank of the $m \times n$ polynomial coefficient matrix. From these solutions, $\ell + \ell' - \ell_0$ correspond to the Casimir operators of either $\mathfrak{g}$ or $\mathfrak{g}'$, so that the number of available labelling operators is given by $\chi = n - r' - \ell - \ell' + \ell_0$. It can be easily shown (see *e.g.* [73]) that $m - r' = \ell_0$, which implies that $\chi = 2n_0$, showing that there are n_0 more labels available than required. It should however be noted that among these $2n_0$ solutions, at most n_0 correspond to operators that commute with each other [76].

Once a complete set of $\dfrac{\dim \mathfrak{g} + \ell}{2}$ labelling operators has been found, they can be simultaneously diagonalised, from which an orthonormal basis of states for the representation $\mathcal{D}$ is obtained. A practical recipe for the orthonormalisation can be found *e.g.* in [76].

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