UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS MATEMÁTICAS

TESIS DOCTORAL

Weak Hopf algebras, Matrix Product Operators and the classification of quantum phases of matter.

Álgebras de Hopf débiles, operadores producto de matrices y la clasificación de las fases cuánticas de la materia.

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Tesis doctoral

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Memoria para la obtención del
Doctorado en Investigación Matemática
presentada por

Alberto Pascual Ruiz de Alarcón Torregrosa

Directores
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Octubre, 2022

Acknowledgements

First of all, I am deeply grateful to my supervisors, David Pérez García and András Molnár for their unceasing support and invaluable insights. I would also like to express my special gratitude to José Garre Rubio for his time and patience.

I am very fortunate to have been part of the mathQI group at the Complutense University of Madrid and the Institute of Mathematical Sciences (ICMAT) - CSIC. I have met many colleagues there, whom I would like to thank for their company, thank you Aleksander K., Patricia C., Ángela C., Abderramán A., Carlos P., Sofyan I., Alejandro P., David T., Antonio H., Giannicola S., Angelo L., Carlos G. and Ignacio V.

I would like to give special thanks also to Norbert Schuch for his hospitality during my research stay at the University of Vienna, as well as to all the colleagues there from the Quantum Information and Quantum Many-Body Physics group.

I would like to acknowledge the financial support that I received from the Spanish Ministry of Science and Innovation through the grant Formación de Personal Investigador - Severo Ochoa at ICMAT-CSIC.

Finally, this thesis would not have been possible without the support of my family, especially my mother. Thank you from the bottom of my heart.

Contents

Resumen

La comprensión de la estructura de entrelazamiento de los sistemas cuánticos de muchos cuerpos es fuerza motriz de crecientes esfuerzos teóricos en las últimas décadas. Esto ha conducido, en particular, al estudio de las redes tensoriales, un paradigma revolucionario que surgió de la colaboración entre la teoría de la información cuántica y la teoría de la materia condensada. Estos modelos, fácilmente descritos en términos de tensores locales contraídos a lo largo de una estructura gráfica subyacente, resultan sorprendentemente poderosos para describir sistemas cuánticos de muchos cuerpos en términos de sus grados de libertad de entrelazamiento, dilucidar las propiedades esenciales de las fases de la materia cuántica y caracterizar cómo las diferentes simetrías son codificadas, mediante el análisis de sus tensores constituyentes.

Esta tesis, titulada "Álgebras de Hopf débiles, operadores producto de matrices y la clasificación de las fases cuánticas de la materia", está dedicada al estudio matemático de las redes tensoriales unidimensionales que surgen a partir de una amplia variedad de estructuras algebraicas. Originalmente, dos modelos centrales de estados y operadores de redes tensoriales unidimensionales, conocidos como *estados producto de matrices* y *operadores producto de matrices* (abreviadamente, MPS y MPO por sus siglas en inglés), respectivamente, modelan una amplia gama de sistemas. Entre ellas se encuentran los estados fundamentales de los Hamiltonianos locales unidimensionales con *gap* (la diferencia de energía entre su estado fundamental y su primer estado excitado), los estados mixtos en sistemas cuánticos unidimensionales, los autómatas celulares cuánticos, etc. Aquí establecemos rigurosamente cómo las representaciones de estructuras algebraicas bien conocidas, como las coálgebras, las bialgébras débiles o las álgebras de Hopf débiles, dan lugar a redes tensoriales que exhiben propiedades exóticas, y proporcionamos un diccionario entre las propiedades puramente algebraicas y las de los tensores locales. Esto permite transferir resultados muy interesantes entre dichas dos configuraciones.

Las redes tensoriales pueden utilizarse para representar una gran clase de estados topológicos y proporcionan herramientas muy prometedoras para el estudio analítico y numérico de las fases topológicas. Por ejemplo, un escenario en el que los MPOs son particularmente útiles es en el caso representar álgebras de simetrías no triviales: las

fronteras de las fases de los sistemas cuánticos bidimensionales protegidas por simetrías u ordenadas topológicamente exhiben simetrías en forma de MPOs. En esta tesis definimos las álgebras *pulling-through*, que aíslan las hipótesis mínimas necesarias para definir las redes tensoriales bidimensionales con orden topológico a partir de las álgebras de MPOs, y demostramos que las álgebras de Hopf débiles co-semisimples y co-pivotales son álgebras *pulling-through*. Además, demostramos su potencia mostrando que pueden utilizarse para construir los modelos dobles cuánticos de Kitaev para álgebras de Hopf, del mismo modo que las simetrías de MPOs obtenidas de las categorías de fusión pueden utilizarse para construir modelos de redes de cuerdas de Levin-Wen, y para describir todas sus características topológicas.

Finalmente, el puente entre las dos materias nos ha proporcionado las herramientas para estudiar las fases en una amplia gama de sistemas cuánticos abiertos unidimensionales. Motivados por la exitosa clasificación de las fases de los estados fundamentales de los Hamiltonianos locales unidimensionales con *gap*, aquí consideraremos que dos estados mixtos están en la misma fase si ambos pueden transformarse en el otro mediante un circuito de longitud finita de canales cuánticos locales. Para empezar a entender el diagrama de fases emergente, en esta tesis nos restringimos al estudio de los MPOs que son operadores de densidad y en ausencia de escalas de longitud, es decir, en aquellos que son puntos fijos de renormalización. Estos estados surgen, por ejemplo, como fronteras de estados bidimensionales con orden topológico. Para ello, construimos primero familias de tales estados basadas en álgebras de Hopf débiles, proporcionando canales cuánticos explícitos de reescalado que definen el proceso de renormalización. Finalmente, demostramos que un subconjunto de tales estados, más concretamente aquellos construidos a partir de álgebras de Hopf, se encuentran de hecho en la fase trivial.

Dada la generalidad de las técnicas y resultados de esta tesis, esperamos que resulten útiles en el futuro próximo para el estudio de las fases topológicas de la materia, tanto en una como en dos dimensiones.

Abstract

Understanding the entanglement structure of quantum many-body systems is a driving force behind increasing theoretical efforts in recent decades. This has led, in particular, to the study of tensor networks, a revolutionary paradigm that emerged from the interplay between quantum information theory and condensed matter theory. These models, easily described in terms of local tensors contracted along an underlying graph structure, are surprisingly powerful for describing interacting quantum many-body systems in terms of their entanglement degrees of freedom, elucidating the essential properties of phases of quantum matter and characterizing how different symmetries are encoded by the analysis of their constituent tensors.

This thesis, entitled "Weak Hopf algebras, Matrix Product Operators and the classification of quantum phases of matter", is devoted to the mathematical study of aspects pertaining to one-dimensional tensor networks arising from a wide variety of algebraic structures. Originally, two central models of one-dimensional tensor network states and operators, known as *matrix product states* (MPS) and *matrix product operators* (MPO), respectively, modeled a wide range of situations. They can describe gapped ground states of one-dimensional local Hamiltonians, mixed states in one-dimensional quantum systems, quantum cellular automata, etc. Here, we rigorously establish how representations of well-known algebraic structures, such as coalgebras, weak bialgebras and weak Hopf algebras, give rise to tensor networks exhibiting exotic properties, and provide a dictionary between different properties of the algebra and that of the corresponding tensors. This allows very interesting results to be transferred between these two configurations.

Tensor networks can be used to represent a large class of topological states and they provide promising tools for the analytical and numerical study of topological phases. For instance, one scenario in which MPOs are particularly useful is to represent algebras of non-trivial symmetries: The boundary of both symmetry protected and topologically ordered phases in two-dimensional quantum systems exhibit symmetries described by MPOs. In this dissertation we define pulling-through algebras, which isolate the minimal assumptions needed to define topologically ordered two-dimensional tensor networks from algebras of MPOs, and prove that co-semisimple co-pivotal weak Hopf algebras provide examples for pulling-trough algebras. In addition, we demonstrate their

power by showing that they can be used to construct Kitaev's quantum double models for Hopf algebras, in the same way as MPO symmetries obtained from fusion categories can be used to construct Levin-Wen string-net models, and to describe all their topological features.

Finally, the bridge between the two subjects has provided us with the tools to study phases in a wide range of one-dimensional open quantum systems. Encouraged by the successful classification of phases of gapped ground states of one-dimensional local Hamiltonians, two mixed states are regarded here to be in the same phase if both states can be transformed into the other by means of a shallow circuit of local quantum channels. In order to start understanding the emergent phase diagram, in this dissertation we restrict ourselves to the study of *matrix product density operators* in the absence of length scales, i.e., which are renormalization fixed points. These states arise, for example, as boundaries of two-dimensional topologically ordered states. To this end, we first construct families of such states based on weak Hopf algebras by providing explicit local rescaling quantum channels for the renormalization procedure of these states. Finally, we show that a subset of such states, more specifically those built using Hopf algebras, are in fact in the trivial phase.

Due to the generality of the techniques and results obtained in this thesis, we expect them to be useful in the near future for the study of toplogical phases of matter, both in one and two dimensions.

CHAPTER 1

Introduction

Since the birth of quantum physics, in the course of the first quarter of the twentieth century, predicting the properties of quantum systems composed of several particles from the foundational principles of quantum mechanics has been one of the driving forces of theoretical physics. This includes a wide diversity of problems of fundamental importance in quantum chemistry, condensed matter physics, materials science, etc., known under the name of *the quantum many-body problem*. However, the lack of techniques to solve this range of problems in a closed analytic form has led the theorists to the use of approximate methods such as mean field theory, perturbative expansions in a small parameter or effective descriptions obtained from symmetries or renormalization arguments.

1. Tensor networks

A particularly fruitful approach to the quantum many-body problem is based on the analysis of simple models that are easy to understand, either *toy models* or *trial wave functions*. *Tensor networks*, the main objects of this dissertation, constitute a class of such ansatzes that has emerged in past decades from the interplay of quantum information theory and condensed matter theory. They allow one to express certain high-dimensional tensors with a large number of indices efficiently as the contraction of simple tensors. To construct such states, elementary tensors are arranged on a graph and their auxiliary indices, known as *virtual indices*, are contracted with those of the adjacent tensors, as prescribed by the edges of the graph. The resulting object describes hence a multi-dimensional tensor whose indices are given by the remaining uncontracted indices of the original network, known as *physical indices*. This construction allows one to express certain vectors with a few number of parameters, typically scaling linearly in the number of particles, while retaining highly non-trivial properties between the different indices.

These ansatzes have been discovered in different fields independently and have found applications in a multitude of areas throughout data science [**116**, **97**, **28**, **29**, **79**, **88**, **64**, **51**, **5**, **22**, **53**] and physics [**110**, **115**, **89**, **35**, **101**, **128**, **100**, **123**, **111**, **8**, **102**, **65**, **117**, **99**]. Notably, their suitability for approximating low-energy states of local Hamiltonians can be rigorously proven in a broad range of settings

[63, 62, 33], providing a formal justification of their success in the study of such systems. Moreover, the fact that one-dimensional tensor networks allow to represent physical symmetries locally [104] (i.e. as certain operators acting on the auxiliary degrees of freedom) has enabled a comprehensive classification of unconventional phases protected by symmetries [26, 113, 33].

1.1. Matrix product states. An exceptionally successful instance of this paradigm is starred by one-dimensional tensor networks known as *matrix product states* (MPS) [124]. Their origins date back to the 40s with the study of the 2D classical Ising model by H. A. Kramers and G. H. Wannier [75] and later, in the late 60s, in early works of R. Baxter [12]. In the late 80s, MPS were discovered as powerful analytical tools to study the properties of quantum states. First, they appeared as expressions of the seminal one-dimensional model by I. Affleck, T. Kennedy, E. H. Lieb and H. Tasaki, *the AKLT model* [1]. This gave rise to a broader class of states known as *finitely correlated states* [44, 45, 46, 47, 48, 49], which can now be thought of as *injective* MPS. Also, shortly after S. R. White developed the algorithm known as the *density matrix renormalization group* (DMRG) [124] in the context of quantum spin chain simulations, it was found out that DMRG can be reformulated as a variational algorithm over the set of MPS. Its success was well understood when M. B. Hastings established the existence of accurate MPS approximations to ground states of gapped 1D systems [62]. Additionally, MPS originated within the framework of quantum repeaters from the point of view of entanglement theory [119, 121]. A *fundamental theorem* clarified their structure [103] and this led to the classification of all symmetry protected topological phases for one-dimensional quantum spin systems [26, 113].

In this thesis we first present an alternative view on the construction of MPS: In Chapter 3 we show that they naturally appear as "representations" of *coalgebras*. More concretely, coalgebra elements are encoded in the boundary condition of some MPS. Reciprocally, we show that any family of translationally invariant MPS induces a coalgebra and thus MPS and representations of coalgebras are equivalent.

1.2. Matrix product operators. The first step forward in the generalization of one-dimensional tensor networks is naturally led by *matrix product operators* (MPOs). These operators act on multi-dimensional tensors such that themselves can be expressed as one-dimensional tensor networks preserving the underlying one-dimensional locality structure [95]. For example, MPOs that are density operators constitute the mixed state analogue of pure MPS and are usually known as *matrix product density operators* (MPDOs). In particular, they arise in simulations of quantum many-body systems at finite temperature

or in the presence of dissipation [120] and provide powerful numerical methods [58, 95]. Another class of MPOs that appear in the dynamics of quantum many-body systems are *matrix product unitaries* (MPUs), as they generate unitary operators [32, 107, 54]. MPUs can describe evolutions with a local structure (but not necessarily locally generated) such as in driven quantum systems, i.e. Floquet physics.

1.3. Projected entangled pair states. Higher-dimensional generalizations of the one-dimensional MPS are commonly known under the name of *projected-entangled pair states* (PEPS). They provide a natural generalization in the sense that they parametrize states exhibiting an area law. Just as in MPS, increasing the bond dimension, i.e. the number of variational parameters in the ansatz, grows the family of states described by them [118]. As expected, PEPS form a much richer family of states than MPS. For example, they are able to represent critical systems and systems with topological quantum order [122, 114, 108]. However, there is still a wide variety of open problems and important fundamental limitations in understanding the properties of the states PEPS can efficiently describe. It is conjectured that all ground states of gapped local Hamiltonians in higher dimensions can be represented faithfully as PEPS, and although there are strong indications for this fact [90, 61], this has not been proven. In addition, a full proof of the area law for 2D local gapped Hamiltonians does not exist yet but it has been supported by a large amount of numerical and analytical evidence [39, 2, 3, 87]. The most general form of the fundamental theorem of MPS on which much of the previous results above are built upon does not have an immediate generalization to 2D. Furthermore, it was shown in [109] that determining whether a PEPS has a certain symmetry is undecidable, i.e. there is no algorithm which can decide whether the family of states generated by that tensor will be symmetric, given a PEPS tensor, and hence it is actually impossible to fully characterize all the ways in which symmetries can be realized in PEPS. Hence, one concentrates on the cases for which the symmetries on the virtual level can be described in terms of tensor products of local operators or MPOs [105, 125, 91, 92]

1.4. Algebras of matrix product operators. The use of MPOs in many-body physics extends deeply into the study of PEPS: Boundaries of such systems and, in particular, the correlations across those boundaries, i.e., the entangement spectrum, play a key role in understanding the physics of strongly correlated quantum matter [81, 76]. The boundary of such a system has a natural 1D structure, and it can be made explicit by cutting a PEPS description of the bulk system at the boundary [30]. The density operator which carries the entanglement spectrum can then be described by an MPDO at the boundary.

On the other hand, just as physical symmetries in MPS can be represented as acting at the auxiliary indices, symmetries in PEPS can be understood as MPOs acting at the 1D boundary [**27, 125, 92**]. These MPOs naturally form a representation of the symmetry group; indeed, the classification of MPO representations of groups enabled the classification of symmetry protected phases in 2D. MPO symmetries of the boundary MPDO, however, do not necessarily originate from physical symmetries of the corresponding PEPS. They can also appear as symmetries purely of the entanglement degrees of freedom. These entanglement symmetries are precisely the ones underlying topological order in 2D, as we explain below in Section 2. The study of these symmetries allows one to comprehensively understand topologically ordered systems, encompassing their ground space structure as well as anyons and their braiding [**112, 20, 33**].

In many of these cases, MPOs naturally form algebraic structures, such as group representations in the case of symmetries or evolutions. A particularly strong structure arises in the case where the MPO describes the entanglement symmetries which appear in topologically ordered systems. As both products and linear combinations of symmetries are again symmetries, the MPOs which appear in topologically ordered systems naturally form MPO algebras. Given the widespread use of MPS and MPOs, the importance of understanding topologically ordered systems, and the key role played by MPO algebras in this context, it is thus highly desirable to formalize the representation theory of MPO algebras and their underlying algebraic structure, and to understand the way in which additional conditions imposed on the corresponding algebraic structures are reflected in properties of their MPO representation, and vice versa [**20**].

In Chapter 4 of this dissertation we show that MPOs with boundary conditions which form a closed family under composition are, in fact, representations of *pre-bialgebras*, i.e. coalgebras which are also algebras and satisfy certain (minimal) compatibility conditions between their algebra and coalgebra structure. In this context, when the pre-bialgebras are cosemisimple, the irreducible blocks of the MPO tensor give rise to a notion of *sectors* and their fusion, which links to topological order, as we will explain below. We then study in Chapter 5 the effect of introducing the additional conditions which make a pre-bialgebra to be a *weak bialgebra*, a *weak Hopf algebra* or a *Hopf algebra*. We show how those conditions give rise to additional properties of the sectors of the MPO and their fusion, giving them the structure of a monoidal category in the case of a weak bialgebra and multi-fusion category in the case of a weak Hopf algebra. In particular, we show how to construct a *vacuum* sector, as well as a special element whose representation is a projector used later to construct the topological models. Finally, we restrict our attention to more specific structures, such as *pivotal* and

spherical weak Hopf algebras, *C*-weak Hopf algebras* and *C*-Hopf algebras*.

2. Connections to topological order

For a long time, it was believed that all phases of matter were described by Landau's symmetry-breaking theory, and the transitions between those phases were realized by the change of symmetry-breaking orders. However, since the discovery of the fractional quantum Hall effect, the existence of a new type of order beyond Landau's theory, now known as *topological order*, was realized.

2.1. The arena of topologically ordered phases. Topologically ordered phases are phases which exhibit order that cannot be detected by any local order parameter. Instead, they are characterized by a global order in their quantum correlations or entanglement. Characteristic to these systems are their degenerate ground states, which are locally indistinguishable and whose number depends on the topology of the surface on which the system is defined (both incompatible with local order parameters) as well as the presence of excitations with non-trivial statistics in the system, termed *anyons*.

In a seminal work, A. Kitaev [72] proposed a Hamiltonian model for a spin system with the aforementioned properties, the *toric code model*, as well as its generalization to finite groups, the *quantum double models*. This highly entangled topological state supports emergent excitations that turn out to be *Abelian anyons*. Also, there is a direct relation between this type of model and quantum error correction: The toric code is an example of a *stabilizer code* [55, 21].

An alternative construction of toy models exhibiting topological order was provided by the *string-net models* of M. A. Levin and X.-G. Wen [80], which are conjectured to provide a complete characterization of non-chiral topological theories in two dimensions. While string-net models can be understood as generalizations of the toric code model, they are in fact motivated by topological quantum field theories. Hence, they use a category theoretical language (the construction utilizes unitary fusion categories with some additional restrictions, the tetrahedral symmetry of the F-symbols), as opposed to the algebraic approach used for the quantum double models.

Over the years, generalizations of both classes of models have been devised. A. Kitaev already noted in his original work [72] that the same construction also works for C*-Hopf algebras. This has later been worked out in detail [7, 19], and further generalized to C*-weak Hopf algebras [23]. For string-net models, it has been shown that the requirement of tetrahedral symmetry can be dropped [59], and thus string-net models can be built from arbitrary unitary fusion categories;

further, the construction has been generalized to build on bimodule categories instead of fusion categories [85].

2.2. Topological order and tensor networks. Both Kitaev models and string-net models admit tensor network descriptions. Such a description has been developed first for Kitaev's toric code model [122] and later generalized to Kitaev models based on finite groups [112] and Hopf algebras [19], and separately for string-net models [18, 57]. A characteristic feature of the PEPS description both of Kitaev models based on finite groups and of string-net models is that the tensors which define the state possess symmetries which act solely on the auxiliary degrees of freedom of the tensor. This symmetry is intimately tied to the topological features of the system, as it allows to explain both its ground space degeneracy and the presence of anyonic excitations [112, 108, 20]. Breaking the symmetry, even slightly, leads to an immediate breakdown of topological order in 2D [25] (but not in 3D [126, 36]). A crucial property of these symmetries is their size-independence: They are given either by tensor powers of a local symmetry generator (for the double models) or, more generally, by homogenous MPOs (for string-net models), such that every region in the PEPS possesses the same MPO symmetries. The key ingredient relating the PEPS and the MPOs is known as a *pulling-through condition*, introduced in [108]. The specific properties of the underlying topological phase can then be inferred by studying the algebraic properties of the corresponding MPO algebra. Remarkably, it is even possible to build topological models in this very phase from nothing but the MPO symmetry itself: the PEPS tensor is then constructed from the MPO by placing it on a fixed-size ring with suitable boundaries [112, 20, 33].

Despite the success of MPO symmetries in understanding, characterizing, and simulating topological order in the phases of the Kitaev double models of finite groups and of the string-net models [112, 6, 84, 38], the picture is unfortunately not complete. First off, for the Kitaev model based on Hopf algebras the known tensor network constructions [19] do not evidently display any such symmetries; and moving to an even broader setting, for the Kitaev models constructed from weak Hopf algebras not even a tensor network description is known – which, in turn, should be possible to construct once the underlying MPO symmetries have been identified. This lack of knowledge is even more surprising given that multi-fusion categories are precisely the representation categories of weak Hopf algebras [41], and thus, they exhibit the same type of topological order as the corresponding string-net models. A key reason why, despite this connection, an understanding of the MPO symmetries underlying Kitaev models for (weak) Hopf algebras is missing is the fact that the MPO symmetries for string-net models are constructed in a category theoretical language. To understand the

MPO symmetries relevant for describing Kitaev models, however, an algebraic language is clearly more natural.

2.3. String-net models and MPO algebras. On a more abstract level, the relation between the MPO symmetries of the string-net models and the MPO representations of semisimple weak Hopf algebra is as follows. The MPOs constructed in this work are representations of semisimple weak Hopf algebras and representations of semisimple weak Hopf algebras are known to be exactly multi-fusion categories [**41**]. Vice versa, every multi-fusion category arises as the category of representations of some weak Hopf algebra [**41**]. The MPO symmetries of the string-net model based on any given fusion category will thus form a representation of the corresponding weak Hopf algebra. In fact, that weak Hopf algebra can be constructed from the MPO symmetries themselves, when closed with arbitrary boundary conditions. It is important to note that while (ordinary) string-net models are built using a single fusion category, general representations of weak Hopf algebras involve two different fusion categories: one is the representation category of the weak Hopf algebra, the other is the category (or a subcategory) of the representations of the dual weak Hopf algebra. The correspondence between fusion categories, MPOs, and weak Hopf algebras then suggests that string-net models based on bimodule categories [**85**] and PEPS constructed from weak Hopf algebras are actually the same.

In Chapter 5 of this thesis we describe how pivotal and spherical weak Hopf algebras and C*-weak Hopf algebras possess a *pulling-through structure*, pointed out in Section 2.2, satisfying a sequence of increasingly strong conditions. This motivates the definition of *pulling-through algebras* in Chapter 6, whose MPO representation possesses such a pulling-through structure which satisfies the corresponding conditions. In Chapter 7, we construct PEPS based on pulling-through algebras, show that their symmetry structure is scale invariant, which is central for follow-up work, and demonstrate that in the case of C*-Hopf algebras the resulting models are equivalent to the class of generalized Kitaev models.

3. Phases of matter in 1D open systems

Finally, we address the analysis and classification of exotic topological phases of quantum matter in 1D open systems, which is at the very core of condensed-matter physics.

A definition of such a phase which tries to capture the global properties in the closed regime, motivated by quantum information, is the following: Two ground states are in the same phase if there exists a short-depth geometrically local quantum circuit mapping one ground

state into the other [24, 17]. A phase is simply an element in the quotient set defined by that equivalence relation. Using Hastings-Wen's quasi-adiabatic evolution [60] and Lieb-Robinson bounds [82] one can prove that this property is implied by the more standard definition of phase based on the existence of a gapped path of Hamiltonians connecting both systems [4].

The main advantage of the definition based on quantum circuits is that it focuses on states rather than on Hamiltonians, which is crucial to extend it to more general setups, like the one we are addressing here: open quantum systems. However, this approach poses an additional problem: one has to identify the relevant class of states to classify. For closed quantum systems this relevant class is precisely the set of ground states of gapped short-range Hamiltonians. As commented above, quantum information theory provided us with a characterization of this set: ground states of short-ranged gapped Hamiltonians fulfill an area law for the entanglement between neighbouring regions, which implies that they are well approximated by tensor network states, in particular by MPS and PEPS [62, 3, 33].

A natural approach to classify phases is to first restrict the classification to "simple" states that nevertheless are representatives for each phase. Since topological properties are global, these representatives are taken to be insensitive to real space renormalization steps (being those a finite depth circuit), that is, they are renormalization fixed points (RFP). In 2D, for instance, the string-net models of Levin and Wen [80] are believed to provide a complete set of renormalization fixed points for non-chiral 2D topological phases.

The restriction to RFPs has two important benefits. On the one hand, RFPs in gapped phases have zero correlation length and thus they are *exactly* MPS and PEPS [33]; no approximation is needed. On the other hand, it is easier to identify the key global invariants and thus identify the different phases of RFP states.

These two points have been the crucial insights to successfully complete the classification of 1D phases with symmetries, the so-called symmetry protected topological (SPT) phases, already introduced above in Section 1.1. Let us illustrate that this is the case by recalling the steps that led to the classification of 1D SPT phases. The first step was to prove that any MPS can be transformed into an RFP MPS in the same phase [113]. This restricts the classification problem to just RFP MPS. The second step was to identify the invariants of the phases using the set of RFP MPS. These invariants are a set of quantities which, on the one hand, are robust against short depth circuits and, on the other, are sufficient to identify each phase uniquely. For SPT phases with unique ground state, the invariants are the different equivalent classes of the second cohomology group of the symmetry group [26, 113]. For SPT phases with symmetry breaking and therefore degenerate ground

states, the invariants are the different induced representations of the non-symmetry broken subgroup together with its second cohomology group [113]. The third step was to prove that any two RFP MPS that share the same invariants can be mapped into each other with a short depth quantum circuit. On top of that, a final and important step has been recently made: the breakthrough results of Y. Ogata [98] show that one can even extend these arguments beyond the framework of MPS to cover all gapped short-range Hamiltonians.

All the previous results stand for closed quantum systems, where the object of interest is the ground state of a Hamiltonian. However, the question of classifying phases is far from being answered for open quantum systems, even in one dimension. Since isolation is never practically achieved, the characterization of those systems play a fundamental role in real applications.

In this dissertation, we take the first steps towards the classification of open quantum systems in 1D. A main difference between open and closed quantum systems is that evolutions in closed quantum systems (either Hamiltonian evolution or quantum circuits) are *reversible*, whereas this is no longer true in open quantum systems evolved under a Linblad master equation. For instance, if one starts in a topologically ordered state, like the toric code [72], one cannot find a short depth quantum circuit mapping it into a product state. Short depth quantum circuits cannot create *or destroy* global correlations. However, local depolarizing noise can convert the toric code (and indeed any topologically ordered state, no matter how complex) into a product state in a short amount of time [34]. Destroying global correlations is therefore *easy* in the open quantum systems regime. Constructing global correlations is, on the other hand, still hard. In fact, local fast dissipative evolutions cannot create global correlations [73]. This shows that in the open quantum setting, phases should not be thought of as classes of an equivalence relation, but rather as a partial order given by the existence of a local fast dissipative evolution mapping one state into another one. This partial order can also be understood as the complexity present in the different topological phases. This proposal, due to [34], is the one we are taking here. Concretely, we will say that a mixed state ρ_1 is more complex than another one ρ_2 if there is a short-depth (geometrically local) circuit of quantum channels, i.e. completely positive trace-preserving linear maps, mapping ρ_1 into ρ_2 .

There are several subtleties to make this definition formal. First of all, ρ_1 and ρ_2 should be well defined for all system size n. Second, one should ask only for getting sufficiently close to ρ_2, allowing for both polylog(n) depth and polylog(n) locality in the gates of the circuit. Finally, one could take either a discrete point of view, as here, or a continuous one, asking for a rapid mixing quasi-local Linbladian evolution that approximates ρ_2 starting from ρ_1. Since in this thesis we are

working only with RFP states, we will not need any of those subtleties here and we refer to [34] for a detailed analysis of those.

We notice that there are other definitions of phases in the open quantum system setting, like the works of Diehl et al. for Gaussian mixed states [37, 9, 10] and for quasi thermal states [56], where the authors generalize the notion of phases via gapped paths of Hamiltonians or via local unitary transformations respectively. We refer also to [34] for a detailed discussion about why the definition we are taking here seems more appropriate.

Encouraged by the successful classification of pure states sketched above, we will focus on RFP that are *gapped* mixed states, that is, mixed states which fulfill an area law for the mutual information. This is motivated by two facts. On the one hand, it is known that Gibbs states of short-range Hamiltonians fulfill an area law for the mutual information [127]. On the other hand, it is known that fixed points of rapid mixing dissipative evolutions also fulfill an area law for the mutual information [16].

This restriction naturally leads us to the set of RFP mixed states with an MPDO representation. The structure of RFP MPDOs has been studied in detail in [31] where, up to minor technical conditions, the following is shown: (i) An MPDO is an RFP if there exist two quantum channels $\mathfrak{T}$ and $\mathfrak{S}$ that implement the local coarse graining and the local fine graining respectively, for which the given MPDO is a fixed point. (ii) The RFP condition for MPDOs is characterized operationally by the absence of length scales in the system; in particular by having zero correlation length and saturation of the area law. (iii) The existence of such $\mathfrak{T}$ and $\mathfrak{S}$ maps is equivalent to the fact that from the MPDO an MPO algebra can be constructed.

This result brings the classification of 1D mixed states into the realm of the understanding and classification of MPO algebras. As explained above in Section 1.4, MPO algebras are precisely the mathematical objects behind the classification of RFP 2D topologically ordered pure states in terms of PEPS. Indeed, RFP MPDOs are expected to contain the set of boundary states associated to RFP 2D non-chiral topologically ordered systems [31].

In Chapter 8 we recall the definition of RFP MPDOs given in [31] and provide the construction of a family of RFP MPDOs arising from any given biconnected C*-weak Hopf algebra. In particular, we provide explicit constructions of the local coarse-graining and local fine-graining quantum channels $\mathfrak{T}$ and $\mathfrak{S}$ commented before. In Chapter 9 we describe the previous RFP MPDOs as the boundary states of topological 2D PEPS. In Chapter 10 we prove that the previous families of RFP MPDOs are in the trivial phase in the C*-HA case, in the sense that they can be obtained via a finite-depth and bounded-range circuit of quantum channels acting on the maximally mixed state. Moreover,

we show that this result can be extended to the trivial sector of any biconnected C*-weak Hopf algebra.

Finally, Chapter 11 covers the conclusions, as well as the questions that have been left open and our ideas for further work.

4. Contributions

Two pre-prints have been produced during the development of this dissertation.

The first one, [**93**], entitled *Matrix Product Operator Algebras I: Representations of Weak Hopf Algebras and Projected Entangled Pair States*, covers Chapters 3 to 7 and Appendices A and B, and has been the result of joint work with András Molnár, José Garre Rubio, Norbert Schuch, José Ignacio Cirac and David Pérez García. As a summary, in this paper we have introduced the correspondence between coalgebras and MPS, as well as a "dictionary" between the different properties that manifest more specific structures, such as pre-bialgebras, weak bialgebras, weak Hopf algebras and Hopf algebras, in terms of tensor networks or, more specifically, Matrix Product Operators. In addition, we define the notion of pulling-through algebra, which abstracts the minimal requirements for defining topologically ordered 2D tensor networks from MPO algebras, and demonstrate the usefulness of this framework by constructing Kitaev's quantum double models from Hopf algebras solely from an MPO representation of the Hopf algebra.

The second one, [**106**], under the title *Matrix Product Operator Algebras II: Phases of Matter for 1D Mixed States*, covers the results presented in Chapters 8 to 10 and Appendices C to G, and has been the result of the joint work with András Molnár, José Garre Rubio and David Pérez García. The bridge established in the first paper between weak Hopf algebras and Matrix Product Operators has provided us, among other things, with the tools to rigorously define and study, in this second paper, the phases of a wide family of open quantum systems (via shallow circuits of quantum channels) in the absence of length scales, i.e. described by density operators that are renormalization fixed points.

All results in Chapters 3 to 10, as well as in Appendices A to G, constitute original content, unless otherwise stated. Finally, let us remark that some results included in this thesis, for which we now give proofs based on tensor networks, are well-known results in the literature of weak Hopf algebras, while other results are completely new. We explicitely specify whenever we are recovering a known result about weak Hopf algebras.

CHAPTER 2

Preliminaries

In this chapter we recall elementary notions from linear algebra, the axiomatic framework of quantum mechanics and the graphical notation for tensor networks. From now on, we assume that all vector spaces in this dissertation are finite dimensional and their ground field is the field of complex numbers $\mathbb{C}$.

An associative *algebra* is a vector space $\mathcal{A}$ endowed with an associative linear map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, called *multiplication*, denoted by juxtaposition, and it is said to be *unital* if there exists an element $1 \in \mathcal{A}$, called *unit*, satisfying $1x = x1 = x$ for all elements $x \in \mathcal{A}$. A (unital) *C*-algebra* is an associative (unital) algebra $\mathcal{A}$ with an anti-linear involutive algebra anti-homomorphism $(\,\cdot\,)^* : \mathcal{A} \to \mathcal{A}$, $x \mapsto x^*$, called $*$-*operation*, and a compatible Banach space structure. In this context, positive elements of $\mathcal{A}$ are elements of the form $x = y^*y$ for some element $y \in \mathcal{A}$. As usual, the multiplication, the unit element and the $*$-operation of two C*-algebras $\mathcal{A}$ and $\mathcal{B}$ are implicitly extended to their tensor product space $\mathcal{A} \otimes \mathcal{B}$ componentwise.

For any two vector spaces V and W, we denote by $\mathfrak{B}(V, W)$ the associative algebra of $\mathbb{C}$-linear maps from V to W. In addition, let $\mathfrak{B}(V) := \mathfrak{B}(V, V)$ denote the associative unital algebra of $\mathbb{C}$-linear endomorphisms on V, where the unit element is the *identity map*, denoted Id_V or simply Id. If V is a Hilbert space, i.e. it can be equipped with an inner product $\langle\,\cdot\,|\,\cdot\,\rangle : V \times V \to \mathbb{C}$ (as usually prescribed in physics, it is linear in the second argument and anti-linear in the first argument), then $\mathfrak{B}(V)$ is a unital C*-algebra. In this setting, for every operator $X \in \mathfrak{B}(V)$ there exists a unique operator $X^\dagger \in \mathfrak{B}(V)$ such that $\langle w|X(v)\rangle = \langle X^\dagger(w)|v\rangle$ for all $v, w \in V$ (and this is the $*$-operation). It is said that an operator $X \in \mathfrak{B}(V)$ is *normal* if $XX^\dagger = X^\dagger X$, *Hermitian* or *self-adjoint* if $X = X^\dagger$ and *positive semidefinite* if it is a positive element of the C*-algebra, i.e. $X = Y^\dagger Y$ for some operator $Y \in \mathfrak{B}(V)$. Finally, we denote by $V^* := \mathfrak{B}(V, \mathbb{C})$ the dual vector space of V.

Finally, it is common in quantum physics to denote vectors and linear functionals using the *bra-ket notation*, which consists in denoting them between a vertical bar and an angle bracket. For instance, a vector $v \in V$ would be usually denoted in the form $|v\rangle$, called *ket*. On the other hand, since every linear functional $f \in V^*$ can be expressed in the form $f = \langle w|\,\cdot\,\rangle$ for some $w \in V$, it is common to denote it simply by $\langle w|$, called *bra*.

1. The postulates of quantum mechanics

Since its origin, quantum mechanics has become a general theoretical framework for the description of almost every physical system at the microscopic level. This framework consists of a mathematical core formulated as postulates (or axioms) and becomes a physical theory by adding a set of correspondence rules that tell us which mathematical objects we have to use in different physical situations. Unlike classical physical theories, these rules do not turn out to be very intuitive. In this section we will briefly describe the most basic principles.

First of all, let us note that in quantum mechanics it is useful to consider physical experiments divided into two procedures: *preparation* and *measurement*. Although this may seem irrelevant, this fact emphasizes one of the basic differences between classical and quantum physics, since in the former it is not necessary to talk about measurements per se.

In addition, we may think that for a physical theory to be "correct", we can ask it to predict the *exact* result of any given measurement given all the information about the setup or, in other words, the initial conditions of the system. However, in the quantum world this is not possible and we can only predict the *probabilities* of the results of our experiments. Therefore, in this framework one does not predict individual events in general. Moreover, by virtue of Bell's theorem, there is good reason to believe that this is not because the theory is incomplete, but is intrinsic to nature.

1.1. Description of the state of a system. Since many different preparations may give rise to the same probability distributions, it is reasonable to introduce the concept of *state*, that specifies the effect of a preparation regardless of how it was carried out. Mathematically, associated with any physical system, its possible states are collected in a complex Hilbert space $\mathcal{H}$, known as the *state space* of the system. The system is described then by a *normalized* vector $\psi \in \mathcal{H}$, called *quantum pure state*.

In this setting, given any two quantum pure states $\psi, \phi \in \mathcal{H}$, any coherent superposition of the two states $\lambda\psi + \mu\phi \in \mathcal{H}$ where $\lambda, \mu \in \mathbb{C}$ are two complex numbers such that the resulting state above is again a normalized vector is also a quantum pure state. This obvious property for vectors is known as the *superposition principle of quantum states*, and it is a fundamental feature that is not present in classical mechanics.

Example 2.1. The simplest example, $\mathcal{H} := \mathbb{C}^2$, is known as a *qubit system*, the quantum analog of a bit. Contrasting to a bit, which has only two possible values, a qubit could be in any kind of superposition of the two basis states.

This framework can be further extended to allow us to work when the precise state of the system is not known, but some statistical knowledge about it is available. More specifically, we would like to describe a quantum system that can be in a quantum pure state $\psi_i \in \mathcal{H}$ with probability $p_i \in [0,1]$, $i = 1, \ldots, k$. For this purpose, we will extend the concept of state as follows: a *state* is a positive semidefinite automorphism with trace one, $\rho \in \mathfrak{B}(\mathcal{H})$, or more precisely, a *density operator* or *quantum mixed state*. A vital feature of the set of density operators is that it is convex: given a set of density operators ρ_i and probabilities p_i, $i = 1, \ldots, k$, the "mixture" $\rho = \sum_i p_i \rho_i$ is again a density operator. In this setting, quantum pure states are rank-one projectors and they form the extreme points of the set of density operators.

1.2. Composite systems. Contrary to the case of the classical joint probability of two systems A and B, in the quantum world we deal with a vector space (while an experiment in the classical world would be restricted to the use of a certain basis) and hence composite quantum systems are described by tensor products: Consider two state spaces described by Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, then, the state space of the *composite system* is the tensor product Hilbert space $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$.

The simplest quantum pure states of a composite system are *product states*, i.e., tensor products of individual states $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$. On the other hand, states that do not factorize are said to be *entangled* or to exhibit *correlations*. This is commonly written in a shorter form by introducing the concept of *reduced density operator*, defined by the action of the mapping $\mathfrak{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathfrak{B}(\mathcal{H}_A)$, $\rho \mapsto \rho_A := (\mathrm{Id} \otimes \mathrm{Tr})(\rho)$. This map is called the *partial trace* (here, performed in the second subsystem, B) and the resulting quantum mixed state ρ_A is usually known as a *marginal* of ρ, since it encodes precisely the information that can be extracted from ρ by acting only on A.

1.3. Time evolution of a system. In contrast to other quantities, in quantum mechanics time is treated in a "classical" way, in the sense that it will be parameterized by real numbers. The time evolution of a quantum pure state $\psi(t) \in \mathcal{H}$ is governed by *Schrödinger's equation*

$$i\hbar \frac{d\psi(t)}{dt} = H(\psi(t)),$$

where $H \in \mathfrak{B}(\mathcal{H})$ is a self-adjoint operator called *Hamiltonian*. Hence the evolution is unitary: There is a semigroup $\mathbb{R} \to \mathfrak{B}(\mathcal{H})$, $t \mapsto U_t$, of unitary operators such that $\psi(t) = U_t(\psi(0))$ for all $t \in \mathbb{R}$.

However, there are many factors one has to take into account. For instance, for a many-body system of n particles, not all unitaries can be realized in terms of a Hamiltonian as $U = e^{-iH(t-t_0)}$, since in such context one usually considers Hamiltonians involving only few-body

interactions, i.e. $H = \sum_i h_i$, where each h_i acts non-trivially on only a small number of nearby particles.

Another issue for time evolution of a quantum system is due to *decoherence*. As already discussed above, the marginal ρ_{S} of the system comes from a larger system $\mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{E}}$, composed of both the *system* and its environment, whose evolution is unitary. However, when one only has access to the system but not the environment, the dynamics of the system alone is in general non-unitary.

The question then becomes what the form of the general dynamics $\mathcal{E} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ of the system could be. First, linearity is an inherent quantum mechanical requirement. Second, $\mathcal{E}$ has to map density operators onto density operators. Therefore, since every element in $\mathfrak{B}(\mathcal{H})$ is a linear combination of density operators we obtain by virtue of linearity that $\mathrm{Tr}(\mathcal{E}(X)) = \mathrm{Tr}(X)$ for all $X \in \mathfrak{B}(\mathcal{H})$, that is, $\mathcal{E}$ is *trace-preserving*. Besides, positivity alone is, however, not sufficient: Consider $\mathcal{H}$ as a part of a bipartite system $\mathcal{H} \otimes \mathcal{K}$, so that the evolution of a larger system is described by $\mathcal{E} \otimes \mathrm{Id}_{\mathcal{K}}$; that is, the additional system merely plays the role of a spectator as the evolution of this part is trivial. Yet, $\mathcal{E} \otimes \mathrm{Id}_{\mathcal{K}}$ should again be a positive map, which is a stronger requirement, known under the name of *complete positivity* of $\mathcal{E}$. Notably, the set of completely positive trace preserving linear maps, called for simplicity *quantum channels*, precisely coincides with the set of maps obtained by the following procedure: initialize the environment in a pure state, evolve the joint system-enviroment by a unitary evolution on the Hilbert space $\mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{E}}$ and finally implement the partial trace on the environment.

1.4. Description of physical quantities. Physical quantities are represented by self-adjoint operators $O \in \mathfrak{B}(\mathcal{H})$, known as *observables*. The *average value* or the *expectation value* of the observable $O \in \mathfrak{B}(\mathcal{H})$ in the system described by the quantum mixed state $\rho \in \mathfrak{B}(\mathcal{H})$ is $\mathrm{Tr}(\rho O)$, and it coincides with the expectation value $\langle \psi | O(\psi) \rangle$ for a quantum pure state $\psi \in \mathcal{H}$.

1.5. Tensor networks states. *Tensor network states* are a particular family of quantum states representing many-body systems defined on graphs or lattices. They are generated by a set of tensors which are assigned to each vertex and are contracted according to the geometry of the graph. More concretely, let us consider a directed pseudograph Γ, i.e. such that multiple edges are allowed between vertices, with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. Intuitively, one can think of a graph as (a part of) a system of spins or particles, where vertices represent them and edges indicate the interactions between them. Every edge $e = (p, q) \in \mathcal{E}$ consists of two distinct elements from $\mathcal{V}$, called endpoints of e; p is said to be the *tail* of e and q is said to be the *head* of e. For any vertex p, we denote by $e_t(p) \subset \mathcal{E}$ (resp. $e_h(p)$) the set of edges whose

tail (resp. head) is p. To every edge $e \in \mathcal{E}$ we associate a complex vector space W_e of dimension $D_e := \dim(W_e)$. In turn, to every vertex $p \in \mathcal{V}$ we associate a tensor $A_p \in (\otimes_{e \in e_h(p)} W_e^*) \otimes (\otimes_{e \in e_t(p)} W_e) \otimes \mathcal{H}_p$, where $\mathcal{H}_p$ is a complex Hilbert space of dimension $d_p := \dim(\mathcal{H}_p)$ describing the physical degrees of freedom at p. In order to obtain a physical state we build first the global fiducial state $\otimes_{p \in \mathcal{V}} A_p$ by taking tensor products and then contract together the virtual indices corresponding to the same edge. The result is a vector $|\psi\rangle \in \otimes_{p \in \mathcal{V}} \mathcal{H}_p$, where individual physical states are now correlated according to the graph and constitutes a tensor network state of a quantum many-body system.

2. Graphical notation for tensor networks

In this section we introduce the graphical notation of tensor calculus that we use throughout the dissertation. This graphical notation is especially useful to visualize equations that involve the contraction of many higher rank tensors (i.e. tensors with more than two indices) and it is standard in the field of tensor networks. In this thesis, however, we face extra challenges as we use the graphical language parallel to an algebraic one, and thus we have to modify the usual graphical language in order to be able to translate between the two languages.

From a computational point of view, tensors are just multi-dimensional arrays. In the usual graphical notation of tensor calculus, one denotes tensors by dots (and various shapes) with lines connected to them. The number of lines connected to the dot (or other shape) is the rank of the tensor, and each line corresponds to one of the vector spaces in the tensor product. For example, the following diagrams represent a scalar, a vector and a matrix, respectively:

$s =$, $v =$, $A =$  .

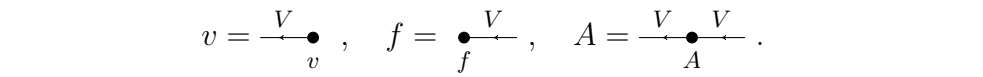
Tensor contraction is denoted by joining the lines corresponding to the contracted indices of the two tensors. For example the scalar product of two vectors, a matrix acting on a vector and the product of two matrices are denoted by the following diagrams, respectively:

$\sum_i w_i v_i =$  , $\sum_j A_{ij} v_j =$  , $\sum_j A_{ij} B_{jk} =$  .

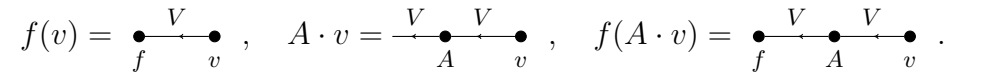
Here we have made the implicit assumption that the first index is the left line, the second one is the right one. For higher rank tensors and more complicated contraction schemes one has to keep track of which index belongs to which line (e.g. fixing a convention such as in the previous figure). In the following we outline a notation that allows us to distinguish the different indices of the tensor without requiring them to be always at the same position. This notation is thus suitable to depict more complicated tensor constructions such as the definition of a PEPS.

To introduce our modification of the graphical notation, let us first formalize what tensors and tensor contractions are. Rank-n tensors are elements of the tensor product $V_1 \otimes \cdots \otimes V_n$ for some finite dimensional vector spaces $V_1 \dots V_n$. One can naturally take tensor products of tensors: for example, if V_1, V_2 and V_3 are vector spaces and $r = v_1 \otimes v_2 \in V_1 \otimes V_2$ and $s = v_3 \otimes f \in V_3 \otimes V_2^*$, where V_2^* denotes the space of linear functionals on V_2, then their tensor product $t = r \otimes s$ is $t = v_1 \otimes v_2 \otimes v_3 \otimes f$. Tensor contraction (without introducing a scalar product) is then the following operation: if amongst the n components of the tensor product both a vector space V and its dual V^* appears, then one can form a rank-$(n-2)$ tensor by acting with the linear functional in V^* on the vector in V. For example, the tensor $t = v_1 \otimes v_2 \otimes v_3 \otimes f$ defined above is an element of the space $V_1 \otimes V_2 \otimes V_3 \otimes V_2^*$, and thus one can contract its second and fourth components to obtain a rank-two tensor $\mathcal{C}_{24}(t) = f(v_2) \cdot v_1 \otimes v_3 \in V_1 \otimes V_3$.

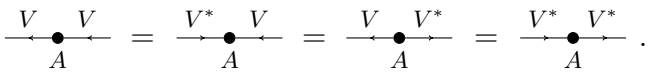
As we have seen, to make sense of tensor contraction without a scalar product, it is important to differentiate between vector spaces and linear functionals. We will denote indices corresponding to vectors by outgoing arrows, while indices corresponding to linear functionals by incoming arrows. Moreover, to distinguish between the different indices of the tensor, we will label the lines with vector spaces. For outgoing arrow, the label is the corresponding tensor component. For incoming arrow, the label is the vector space the given tensor component is the dual of. For example, a vector $v \in V$, a linear functional $f \in V^*$ and a rank-two tensor $A \in V \otimes V^*$ are denoted, respectively, by



Tensor contraction is still denoted by joining lines. Note, however, that now only lines with the same label can be joined that also point in the same direction. For example,



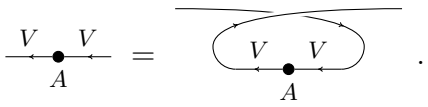
The vector space V is canonically isomorphic to V^{**}, and thus one can equally think of V as the space of linear functionals on V^*. Correspondingly, in our graphical notation every arrow can be reversed by changing the label from V to V^*. For example, there are four different ways to depict a rank-two tensor $A \in V \otimes V^*$:



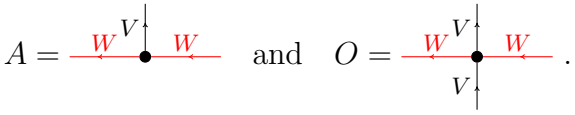
The first depiction of A suggests to interpret it as a linear map $\hat{A} : V \to V$, while the last equation suggests to interpret it as a linear map $V^* \to V^*$; this linear map is $\hat{A}^T$. Such rank-two tensors are sometimes

labeled by the linear map $\hat{A}$; when this is the case, we will try to be consistent and label the vector spaces by V and not by V^*.

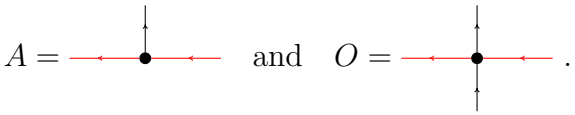
As we will depict tensor networks in two dimensions, sometimes it will be convenient to rotate tensors. This means that vectors don't always point to the left, and thus we actually need the arrows to distinguish between the two indices of a rank-two tensor. Such a rotation is, for example, the following:



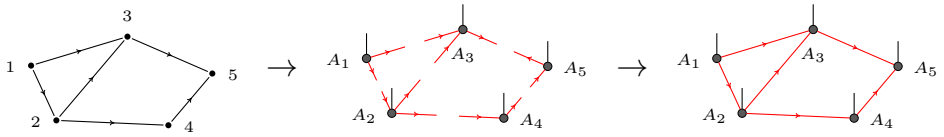
In the rest of the paper we will often deal with rank-three and rank-four tensors of the form $A \in V \otimes W \otimes W^*$ and $O \in V \otimes V^* \otimes W \otimes W^*$. These tensors are denoted by



For better visual distinction, we have introduced colors: the edges labeled W are colored red and the edges labeled V are colored black. In fact, in cases where the vector spaces V and W are fixed and we only need to distinguish between V, V^*, W and W^*, we will drop the labels and keep the colors only, denoting A and O by



Finally, let us exemplify the construction procedure of a tensor network state described in Section 1.5, now using graphical notation:



We start from a directed pseudograph $\Gamma = (\mathcal{V}, \mathcal{E})$ as first represented, and a family $\{A_p : p \in \mathcal{V}\}$ of individual tensors, each of them associated to a vertex of the graph. The global fiducial state, $\otimes_{p \in \mathcal{V}} A_p$, i.e. the tensor product of all tensors, is simply depicted by placing them next to each other. Finally, contracting the corresponding indices is illustrated by joining the corresponding lines. Let us stress that, as commented above, physical indices are represented by black lines, in contrast to virtual indices, represented by red lines.

CHAPTER 3

Coalgebras and matrix product states

In this chapter we define coalgebras and matrix product states (MPS) and show that the latter can be thought of as representations of coalgebras. We also show that this correspondence holds the other way around as well: given an MPS tensor, one can construct a coalgebra such that the MPS forms a representation of the constructed coalgebra. This observation thus makes MPS and coalgebras completely equivalent. We elaborate on a special case: when the coalgebra is cosemisimple, the corresponding MPS tensor is a sum of injective tensors (see Definition 3.5), and vice versa, given an MPS that is a sum of injective tensors, the constructed coalgebra is cosemisimple. We also define the notion of cocentral and non-degenerate coalgebra elements and show how these properties are reflected in the MPS representation.

We start by defining coalgebras.

Definition 3.1. The triple $(\mathcal{C}, \Delta, \varepsilon)$ is a *coalgebra* if $\mathcal{C}$ is a finite dimensional vector space over $\mathbb{C}$, $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ is a linear map called *comultiplication* such that it is *coassociative*:

$$(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta,$$

and $\varepsilon \in \mathcal{C}^*$ is a linear functional called *counit*, such that

$$(\varepsilon \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \varepsilon) \circ \Delta = \mathrm{Id},$$

where we have identified $\mathbb{C} \otimes \mathcal{C} \cong \mathcal{C} \otimes \mathbb{C} \cong \mathcal{C}$.

Coalgebras emerge as the dual of algebras: Given a finite dimensional algebra $\mathcal{A}$ with multiplication $\mu_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and unit 1, we can define a comultiplication on $\mathcal{A}^*$ by defining

$$\Delta_{\mathcal{A}^*} : \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*, \quad \Delta_{\mathcal{A}^*} := \mu_{\mathcal{A}}^T,$$

that is,

$$\Delta_{\mathcal{A}^*}(f)(x \otimes y) := f(xy),$$

where $x, y \in \mathcal{A}$ and $f \in \mathcal{A}^*$: Coassociativity of $\Delta_{\mathcal{A}^*} = \mu_{\mathcal{A}}^T$ is equivalent to the associativity of $\mu_{\mathcal{A}}$, and the map given by $f \mapsto f(1)$ defines the counit of $\mathcal{A}^*$. Vice versa, if $\mathcal{C}$ is a coalgebra with comultiplication $\Delta_{\mathcal{C}}$ and counit ε, then we can naturally give $\mathcal{C}^*$ an algebra structure by defining the multiplication via $\mu_{\mathcal{C}^*} := \Delta_{\mathcal{C}}^T$, i.e. by

$$(fg)(x) := (f \otimes g) \circ \Delta_{\mathcal{C}}(x),$$

where $f, g \in \mathcal{C}^*$ and $x \in \mathcal{C}$; the unit of $\mathcal{C}^*$ is then ε.

Coassociativity of the comultiplication allows us to write $\Delta^2(x)$ instead of $(\Delta \otimes \mathrm{Id}) \circ \Delta$, and $\Delta^n(x)$ for n repeated application of Δ on x. In the following we will use Sweedler's notation of the coproduct and write

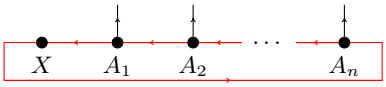
$$\Delta^n(x) =: x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(n+1)}.$$

We will show below that this shorthand notation actually hides a more complicated sum that has a special structure called matrix product state.

Definition 3.2. Let $\{V_1, \ldots, V_n\}$ and $\{W_0, W_1, \ldots, W_n\}$ be two collections of finite dimensional vector spaces over $\mathbb{C}$. A *matrix product state* (MPS) is given by tensors $A_i \in V_i \otimes W_{i-1} \otimes W_i^*$, $i = 1, \ldots, n$, $A_i = \sum_k |k\rangle \otimes A_i^k$ and a matrix $X \in W_n \otimes W_0^*$; the state generated by the MPS is given by

$$|\psi\rangle = \sum_k \mathrm{Tr}(X A_1^{k_1} \cdots A_n^{k_n}) |k_1 \ldots k_n\rangle$$

or, in graphical notation,

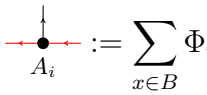
$|\psi\rangle =$ 

.

We say that the MPS is *translation invariant* (TI) with *open boundary condition* if $V_1 = \cdots = V_n$, $W_0 = \cdots = W_n$ and $A_1 = \cdots = A_n$.

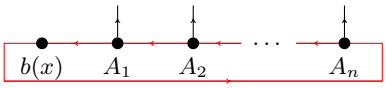
Let us now show that in any coalgebra $\mathcal{C}$ the repeated coproduct of an element $x \in \mathcal{C}$, $\Delta^{n-1}(x)$, can be represented as an MPS on n sites.

Theorem 3.3. *Let $\mathcal{C}$ be a coalgebra, V_i be finite dimensional vector spaces over $\mathbb{C}$ and $\Phi_i : \mathcal{C} \to V_i$ linear maps, $i = 1, \ldots, n$. Let W be a vector space and $\Psi : \mathcal{C}^* \to \mathfrak{B}(W)$ be an injective representation of the algebra $\mathcal{C}^*$. Let $A_i \in V_i \otimes \mathfrak{B}(W)$ be defined by*



$:= \sum_{x \in B} \Phi_i(x) \otimes \Psi(\delta_x),$

where B is a basis of $\mathcal{C}$, and δ_x denotes the dual basis elements (i.e. $\delta_x \in \mathcal{C}^*$ and $\delta_x(y) = \delta_{x,y}$ for any $x, y \in B$). Then for all $x \in \mathcal{C}$ there exists a matrix $b(x) \in \mathfrak{B}(W)$ such that for all $n > 0$,

$(\Phi_1 \otimes \cdots \otimes \Phi_n) \circ \Delta^{n-1}(x) =$ 

.

Let us remark that the tensor A_i is independent of the concrete choice of the basis B of $\mathcal{C}$, as the expression $\sum_{x \in B} x \otimes \delta_x$ is independent of B. Let us also remark that there might be many different choices for the matrix $b(x)$ satisfying the required equation.

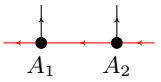
PROOF. Notice that

$$\sum_{x,y} f(x)g(y)\delta_x\delta_y = fg = \sum_{z\in B} fg(z)\delta_z = \sum_{z\in B} f(z_{(1)})g(z_{(2)})\delta_z,$$

where in both equations we have used that any linear functional $f \in \mathcal{C}^*$ can be decomposed in the form $f = \sum_{x\in B} f(x)\delta_x$. As this is true for all $f, g \in \mathcal{C}^*$, we conclude that

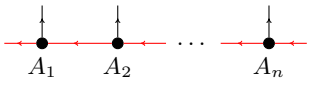
$$\sum_{x,y\in B} x \otimes y \otimes \delta_x\delta_y = \sum_{z\in B} z_{(1)} \otimes z_{(2)} \otimes \delta_z = \sum_{z\in B} \Delta(z) \otimes \delta_z.$$

This means that



$$= \sum_{x,y\in B} \Phi_1(x) \otimes \Phi_2(y) \otimes \Psi(\delta_x\delta_y)$$
$$= \sum_{z\in B} (\Phi_1 \otimes \Phi_2) \circ \Delta(z) \otimes \Psi(\delta_z).$$

A similar equation holds for any number of consecutive tensors, i.e.

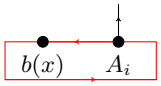


$$= \sum_{x\in B} (\Phi_1 \otimes \cdots \otimes \Phi_n) \circ \Delta^{n-1}(x) \otimes \Psi(\delta_x).$$

As Ψ is an injective representation of $\mathcal{C}^*$, there exists a matrix $b(x)$ for all $x \in \mathcal{C}$ such that

$$f(x) = \mathrm{Tr}\left(b(x)\Psi(f)\right)$$

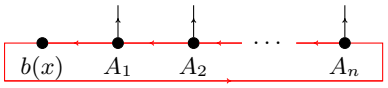
holds for all $f \in \mathcal{C}^*$. Note that this equation does not uniquely determine $b(x)$; there may be many different choices of $b(x)$ satisfying this equation. Using any such choice of $b(x)$, the following holds:



$$= \sum_{y\in B} \mathrm{Tr}\left(b(x)\Psi(\delta_y)\right) \Phi_i(y) = \sum_{y\in B} \delta_y(x)\Phi_i(y) = \Phi_i(x).$$

Finally, combining this statement with the previous one results in the desired equation

$$(\Phi_1 \otimes \cdots \otimes \Phi_n) \circ \Delta^{n-1}(x) =$$

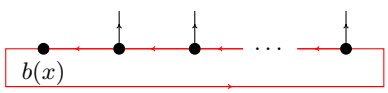


.

□

From now on, unless otherwise specified, we only consider translation invariant representations of coalgebras. This restriction is not essential and most statements obviously generalize to non-TI representations, but it eases the notation. For example, as all tensors are the same, we will drop the label A denoting the MPS tensor and simply write

$$(\Phi \otimes \cdots \otimes \Phi) \circ \Delta^{n-1}(x) =$$



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Remark 3.4. Let us now show that the construction of Theorem 3.3 can be reversed in the translation invariant case: Given a translation invariant MPS with open boundary condition, one can construct a coalgebra $\mathcal{C}$ such that the MPS becomes a representing MPS of $\mathcal{C}$. Indeed, let $A \in V \otimes W \otimes W^*$ be an MPS tensor and let us write

$$A = \sum_i |i\rangle \otimes A^i$$

where $i = 1, \dots, d$ with $d = \dim(V)$ and $A^i \in \mathfrak{B}(W)$. Let us define the algebra $\mathcal{A}$ as

$$\mathcal{A} = \mathrm{Span} \bigcup_{n \in \mathbb{N}} \left\{ A^{i_1} A^{i_2} \cdots A^{i_n} : (i_1, \dots, i_n) \in \{1, \dots, d\}^{\times n} \right\},$$

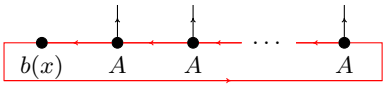
and let $\mathcal{C} = \mathcal{A}^*$. Let us fix now a basis B of $\mathcal{C}$; elements of this basis are denoted by $x, y, \dots$ This basis then also fixes a basis (the dual basis) on $\mathcal{C}^* = \mathcal{A}$. Elements of this basis are denoted by $\delta_x, \delta_y, \dots$ By definition the matrices A^i are elements of $\mathcal{A}$, and thus one can expand them in this (dual) basis. One can thus write $A = \sum_{x \in B} |v_x\rangle \otimes \delta_x$ for some vectors $|v_x\rangle \in V$. This tensor then can be interpreted as a map $\Phi : \mathcal{C} \to V$ by

$$\Phi(x) = \sum_{y \in B} \delta_y(x) |v_y\rangle.$$

Note that, by the definition of the dual basis, $\Phi(x) = |v_x\rangle$ for any $x \in B$. This implies that the following equation also holds:

$$A = \sum_{x \in B} \Phi(x) \otimes \Psi(\delta_x),$$

with $\Psi = \mathrm{Id}$. Finally, as any element $x \in \mathcal{C}$ is a linear functional on $\mathcal{A}$ ($\mathcal{C} = \mathcal{A}^*$), one can find a matrix $b(x)$ such that $x(m) = \mathrm{Tr}(b(x)m)$ for all matrices $m \in \mathcal{A}$. With this, we have obtained that for any $x \in \mathcal{C}$,

$\Phi^{\otimes n} \circ \Delta^{n-1}(x) =$ 

,

i.e. the MPS defined above forms a representation of $\mathcal{C}$ with the properties listed in Theorem 3.3.

1. Cosemisimplicity and injectivity of the representing MPS

In this section we introduce cosemisimple coalgebras as well as injective MPS and examine the connection between these two properties: the MPS representation of a cosemisimple coalgebra decomposes into a sum of injective MPS, and conversely, given an MPS that decomposes into a direct sum of injective MPS, the corresponding coalgebra is cosemisimple.

Let $\mathcal{C}$ be a coalgebra (finite dimensional, over $\mathbb{C}$). As we have seen above, $\mathcal{C}^*$ has a natural algebra structure. In the following we will

talk about representations of this algebra $\mathcal{C}^*$, as the MPS construction above uses the representations of that algebra. Recall that two representations $\Psi_1 : \mathcal{C}^* \to \mathfrak{B}(W_1)$ and $\Psi_2 : \mathcal{C}^* \to \mathfrak{B}(W_2)$ are called equivalent if there is an invertible linear map $Z : W_1 \to W_2$ such that $\Psi_1(f) = Z^{-1}\Psi_2(f)Z$ for all $f \in \mathcal{C}^*$. In particular, the dimension of the two equivalent representations coincide, $\dim(W_1) = \dim(W_2)$. The set of irreducible representation equivalence classes, also called *sectors*, of the algebra $\mathcal{C}^*$ is denoted by $\mathrm{Irr}(\mathcal{C}^*)$, and the elements of this set will be denoted by small Roman letters $a, b, c, \ldots$ The dimension of (all) irreducible representations from the sector a will be denoted by D_a. For convenience, let us fix a concrete representation Ψ_a on vector space W_a from each sector a. Recall that by the density theorem [40], $\Psi_a(\mathcal{C}^*) = \mathfrak{B}(W_a) \simeq \mathcal{M}_{D_a}$ for all irreducible representations Ψ_a, where $\mathcal{M}_{D_a}$ denotes the set of $D_a \times D_a$ matrices over $\mathbb{C}$; in fact $\Phi(\mathcal{C}^*) = \bigoplus_{a \in I \subseteq \mathrm{Irr}(\mathcal{C}^*)} \mathcal{M}_{D_a} \otimes \mathrm{Id}_{m_a}$ for all representations of the form $\Phi(x) = \bigoplus_{a \in I \subseteq \mathrm{Irr}(\mathcal{C}^*)} \Phi_a(x) \otimes \mathrm{Id}_{m_a}$.

We say that the coalgebra $\mathcal{C}$ is *cosemisimple* if the algebra $\mathcal{C}^*$ is semisimple, i.e. if $\mathcal{C}^* \simeq \bigoplus_{a \in \mathrm{Irr}(\mathcal{C}^*)} \mathcal{M}_{D_a}$. In particular, $\mathcal{C}^*$ has finitely many sectors. If $\mathcal{C}^*$ is semisimple, then any representation Ψ of it, up to a basis transformation, is of the form

$$\Psi(x) \simeq \bigoplus_{a \in I \subseteq \mathrm{Irr}(\mathcal{C}^*)} \Psi_a(x) \otimes \mathrm{Id}_{m_a},$$

where a runs over a subset I of sectors of $\mathcal{C}^*$ and Ψ_a are the previously fixed representatives from the class a, and the numbers m_a denote the multiplicity of the irreducible representation Ψ_a in the decomposition of Ψ. The representation Ψ is injective if and only if all sectors are present in this decomposition[1], i.e. if $I = \mathrm{Irr}(\mathcal{C}^*)$. As Ψ is a direct sum of irreducible representations, the density theorem applies and thus $\Psi(\mathcal{C}^*) = \bigoplus_{a \in I \subseteq \mathrm{Irr}(\mathcal{C}^*)} \mathcal{M}_{D_a} \otimes \mathrm{Id}_{m_a}$.

The MPS tensor A in Theorem 3.3 is constructed using a representation Ψ of $\mathcal{C}^*$ that is assumed to be injective. Therefore, if $\mathcal{C}$ is cosemisimple, A decomposes as

$$A \simeq \sum_{x \in B} \Phi(x) \otimes \bigoplus_{a \in \mathrm{Irr}(\mathcal{C}^*)} \Psi_a(\delta_x) \otimes \mathrm{Id}_{m_a}.$$

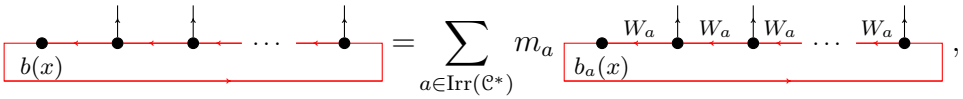
As the defining property of $b(x)$ is that $\mathrm{Tr}\,(b(x)\Psi(f)) = f(x)$, the matrix $b(x)$ can also be chosen w.l.o.g. in this form, i.e. such that in the same basis as Ψ, it reads

$$b(x) \simeq \bigoplus_{a \in \mathrm{Irr}(\mathcal{C}^*)} b_a(x) \otimes \mathrm{Id}_{m_a}.$$

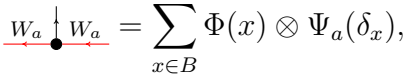
[1]Note that even if $\mathcal{C}^*$ arises as the group algebra $\mathbb{C}G$ of some finite group G, we are talking about the representation of the group algebra, and not the representation of the group; that is, we require the map $\Phi : \mathbb{C}G \to \mathfrak{B}(V)$ to be injective, not the map $\Phi|_G : G \to \mathfrak{B}(V)$.

If $b(x)$ is in this form, then it is uniquely defined by the equation $\mathrm{Tr}\,(b(x)\Psi(f)) = f(x)$ and the map $x \mapsto b(x)$ is linear and a bijection between $\mathcal{C}$ and $\bigoplus_{a\in \mathrm{Irr}(\mathcal{C}^*)} \mathcal{M}_{D_a}$.

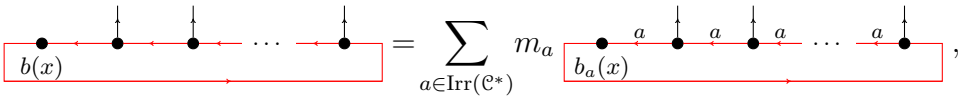
We have thus obtained that the MPS representing a cosemisimple coalgebra decomposes into a sum of MPS with smaller bond dimension,



where

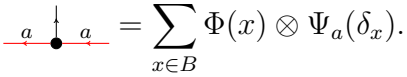


for the previously fixed irreducible representation representatives Ψ_a in the sector a. In the following, we will choose the representation Ψ such that it contains exactly one irreducible representation from each sector (i.e. such that $m_a = 1$). The previously fixed irreducible representation representatives Ψ_a determine the vector spaces W_a for each a, and thus from now on, we do not display the labels W_a in graphical representation, only the label a. Therefore we will write



with

(1)



The MPS tensors in Equation 1, provided that Φ has certain properties, are special:

Definition 3.5. An MPS tensor $A \in V \otimes \mathfrak{B}(W)$, $A = \sum_i |i\rangle \otimes A^i$ is *normal* if there is an $n \in \mathbb{N}$ such that

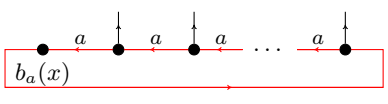
$$\mathrm{Span}\{A^{i_1} \cdots A^{i_n} : (i_1, \ldots, i_n) \in \{1, \ldots, \dim(V)\}^{\times n}\} = \mathfrak{B}(W).$$

The tensor is called *injective* if it is normal with $n = 1$.

It is immediate to see that the MPS tensors defined in Equation 1 are injective if and only if Φ is injective and they are normal if and only if there is an n such that $\Phi^{\otimes n} \circ \Delta^{n-1}$ is injective. We will call such a linear map *normal*.

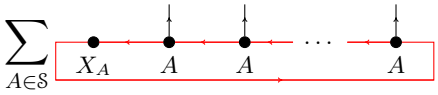
We have thus obtained that if $\mathcal{C}$ is a cosemisimple coalgebra and Φ is a $\mathcal{C} \to V$ linear map such that $\Phi^{\otimes n} \circ \Delta^{n-1}$ is injective for some n, then the coproduct of an element x has a special MPS representation of the form

$$\Phi^{\otimes n} \circ \Delta^{n-1}(x) = \sum_{a\in \mathrm{Irr}(\mathcal{C}^*)}$$



where each MPS tensor is normal. This statement now can be reversed: let us consider a set $\mathcal{S}$ of injective MPS tensors $A \in V \otimes \mathfrak{B}(W_A)$ for each

$A \in \mathcal{S}$ such that no two of them are related to each other with a basis transformation. One then can construct a *cosemisimple* coalgebra $\mathcal{C}$, a map $\Phi : \mathcal{C} \to V$ and a bijection $B : \bigoplus_{A\in\mathcal{S}} \mathfrak{B}(W_A) \to \mathcal{C}$, the inverse of the map $x \mapsto b(x)$, such that



$= \Phi^{\otimes n} \circ \Delta^{n-1} \circ B(\oplus_{A\in\mathcal{S}} X_A).$

2. Cocentral and non-degenerate elements

In this section we define the co-center of a coalgebra and show that in a cosemisimple coalgebra the set of cocentral elements have special MPS representations: They are translation invariant MPS with periodic boundary condition.

Let $\mathcal{C}$ be a coalgebra with comultiplication $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$. Then the map $\Delta_{\text{op}} : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ defined by swapping the two components of the tensor product in $\Delta(x)$,

$$\Delta_{\text{op}}(x) := x_{(2)} \otimes x_{(1)},$$

is also a comultiplication. Using the *opposite comultiplication*, we define:

Definition 3.6. An element $x \in \mathcal{C}$ is called *cocentral* or *trace-like* if it satisfies $\Delta_{\text{op}}(x) = \Delta(x)$.

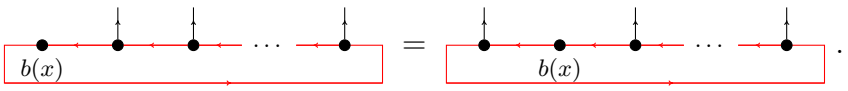
Due to the definition of the multiplication in $\mathcal{C}^*$, if $x \in \mathcal{C}$ is cocentral,

$$\begin{aligned}(fg)(x) &= (f \otimes g) \circ \Delta(x) = (g \otimes f) \circ \Delta_{\text{op}}(x) \\ &= (g \otimes f) \circ \Delta(x) = (gf)(x)\end{aligned}$$

for all $f, g \in \mathcal{C}^*$. This thus means that the map $x : f \mapsto f(x)$ is a trace-like (cyclic) linear functional on $\mathcal{C}^*$, i.e. the set of cocentral elements of $\mathcal{C}$ is exactly the set of trace-like linear functionals of $\mathcal{C}^*$. Due to their cyclicity, repeated coproducts of these elements are translation invariant:

$$\begin{aligned}x_{(n)} \otimes x_{(1)} \otimes \cdots \otimes x_{(n-1)} &= (\text{Id} \otimes \Delta^{n-2}) \circ \Delta_{\text{op}}(x) \\ &= (\text{Id} \otimes \Delta^{n-2}) \circ \Delta(x) = \Delta^{n-1}(x) = x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(n)}.\end{aligned}$$

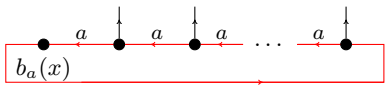
The corresponding MPS representation is also translation invariant, i.e. $b(x)$ is such that



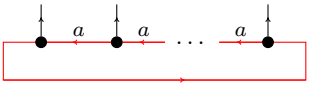
Let us assume now that $\mathcal{C}$ is cosemisimple. In this case, cocentral elements of $\mathcal{C}$ have very simple MPS representations: If $x \in \mathcal{C}$ is cocentral

and Φ is normal, then

$$\Phi^{\otimes n} \circ \Delta^{n-1}(x) = \sum_{a \in \mathrm{Irr}(\mathcal{C}^*)}$$



$$= \sum_{a \in \mathrm{Irr}(\mathcal{C}^*)} \lambda_a$$



;

indeed, the boundary matrix $b(x) = \oplus_a b_a(x)$ is defined by

$$\mathrm{Tr}(b(x)\Psi(f)) = f(x)$$

for all $x \in \mathcal{C}$, and as x is trace-like,

$$\mathrm{Tr}(b_a(x)\Psi_a(f)\Psi_a(g)) = \mathrm{Tr}(\Psi_a(f)b_a(x)\Psi_a(g))$$

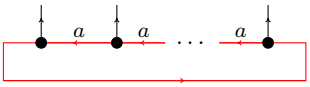
for all $f, g \in \mathcal{C}^*$ and $a \in \mathrm{Irr}(\mathcal{C}^*)$. As Ψ_a is an irreducible representation, both $\Psi_a(f)$ and $\Psi_a(g)$ span the full matrix algebra and thus $b_a(x)$ is necessarily proportional to the identity.

Let τ_a denote the character of the sector a. As τ_a is a linear functional on $\mathcal{C}^*$, it can also be viewed as an element of the coalgebra $\mathcal{C}$. By definition, $\tau_a(f) = \mathrm{Tr}(\Psi_a(f))$, and thus it can be written as

$$\tau_a(f) = \mathrm{Tr}(b(\tau_a)\Psi(f))$$

with $b_b(\tau_a) = \delta_{ab}\mathrm{Id}_a$. That is, the MPS representation of τ_a is

$$\Phi^{\otimes n} \circ \Delta^{n-1}(\tau_a) =$$



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Another set of special elements in the coalgebra are those that have full-rank coproduct.

Definition 3.7. An element $x \in \mathcal{C}$ is called *non-degenerate* if its coproduct $\Delta(x)$ is full rank.

Equivalently, $x \in \mathcal{C}$ is non-degenerate if and only if

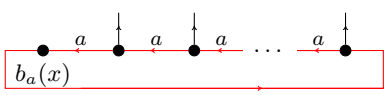
$$(\mathrm{Id} \otimes \mathcal{C}^*) \circ \Delta(x) = (\mathcal{C}^* \otimes \mathrm{Id}) \circ \Delta(x) = \mathcal{C},$$

or, with other words, if for all $y \in \mathcal{C}$ there exist linear functionals $f, g \in \mathcal{C}^*$ such that

$$y = (\mathrm{Id} \otimes f) \circ \Delta(x) = (g \otimes \mathrm{Id}) \circ \Delta(x).$$

Remark 3.8. Let us now show that if $\mathcal{C}$ is cosemisimple, then x is non-degenerate if and only if its MPS representation is of the form

$$\Phi^{\otimes n} \circ \Delta^{n-1}(x) = \sum_{a \in \mathrm{Irr}(\mathcal{C}^*)}$$



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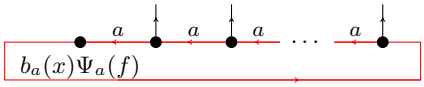
where all $b_a(x)$ are invertible. To prove this, note first that for any linear functional $f \in \mathcal{C}^*$ the element $y = (f \otimes \mathrm{Id}) \circ \Delta(x)$ is described

by the MPS

$$\Phi^{\otimes n}\circ\Delta^{n-1}(y)=\sum_{a\in\mathrm{Irr}(\mathcal{C}^*)}$$ 

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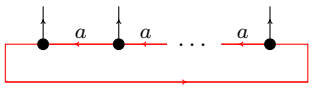
i.e. the boundary describing $y=(f\otimes\mathrm{Id})\circ\Delta(x)$ is given by $b_a(y)=b_a(x)\Psi_a(f)$ in every sector $a\in\mathrm{Irr}(\mathcal{C}^*)$. Similarly, the boundary describing $y=(\mathrm{Id}\otimes g)\circ\Delta(x)$ is given by $b_a(y)=\Psi_a(g)b_a(x)$. Therefore x is non-degenerate if and only if for all $y\in\mathcal{C}$ there are $f,g\in\mathcal{C}^*$ such that

$$b_a(y)=\Psi_a(g)b_a(x)=b_a(x)\Psi_a(f).$$

As $\Psi_a(f)$ can be any matrix, this is equivalent with the invertibility of $b_a(x)$.

As a particular case of the previous statement, let us consider a cocentral coalgebra element $x=\sum_a\lambda_a\tau_a$, where $\tau_a\in\mathcal{C}$ are the irreducible representation characters of $\mathcal{C}^*$. Then, x is non-degenerate if and only if $\lambda_a\neq 0$ for all $a\in\mathrm{Irr}(\mathcal{C}^*)$. For example, the cocentral element $\Theta=\sum_a\tau_a$ with MPS representation

$$\Phi^{\otimes n}\circ\Delta^{n-1}(\Theta)=\sum_{a\in\mathrm{Irr}(\mathcal{C}^*)}$$ 

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is a non-degenerate cocentral element. Using this element Θ, one can interpret $b(x)$ (or, more precisely, $\Psi^{-1}(b(x))$) as the linear functional that satisfies $(\Psi^{-1}(b(x))\otimes\mathrm{Id})\circ\Delta(\Theta)=x$.

CHAPTER 4

Pre-bialgebras and their matrix product operator representations

In this chapter we define pre-bialgebras as well as matrix product operators (MPO). We show that the MPS representations of coalgebras from the previous chapter naturally generalize to pre-bialgebras providing an MPO representation for them. We further investigate this MPO representation for cosemisimple pre-bialgebras.

Definition 4.1. $\mathcal{A}$ is a *pre-bialgebra* if it is both an algebra and a coalgebra such that the comultiplication Δ is multiplicative, i.e.

$$\Delta(xy) = \Delta(x)\Delta(y)$$

for all $x, y \in \mathcal{A}$

Recall that multiplication in $\mathcal{A} \otimes \mathcal{A}$ is taken componentwise, i.e.

$$(x \otimes y)(z \otimes w) := xz \otimes yw.$$

If $\mathcal{A}$ is a pre-bialgebra with multiplication $\mu_{\mathcal{A}}$ and comultiplication $\Delta_{\mathcal{A}}$, then $\mathcal{A}^*$ is also a pre-bialgebra with multiplication $\mu_{\mathcal{A}^*} := \Delta_{\mathcal{A}}^T$ and comultiplication $\Delta_{\mathcal{A}^*} := \mu_{\mathcal{A}}^T$. It is clear from context whether we refer to the comultiplication of $\mathcal{A}$ or that of $\mathcal{A}^*$, and thus in the following we drop the subscripts $\mathcal{A}$ and $\mathcal{A}^*$, and simply write Δ for both $\Delta_{\mathcal{A}}$ and $\Delta_{\mathcal{A}^*}$. That is, the multiplication and comultiplication in $\mathcal{A}^*$ are such that for all $f, g \in \mathcal{A}^*$ and $x, y \in \mathcal{A}$,

$$(fg)(x) := (f \otimes g) \circ \Delta(x) \quad \text{and} \quad \Delta(f)(x \otimes y) := f(xy).$$

Furthermore, the unit of $\mathcal{A}^*$ is the counit of $\mathcal{A}$ and the counit of $\mathcal{A}^*$ is the unit of $\mathcal{A}$.

In the following, we will talk about *representations* of the pre-bialgebra $\mathcal{A}$. These representations should be understood as representations of the algebraic structure of $\mathcal{A}$, i.e. disregarding the coalgebra structure. The extra structure given by the comultiplication allows us to define the tensor product of representations. Let $\Phi_1 : \mathcal{A} \to \mathfrak{B}(V_1)$ and $\Phi_2 : \mathcal{A} \to \mathfrak{B}(V_2)$ be two representations of $\mathcal{A}$. Then, as the comultiplication Δ is multiplicative,

$$(\Phi_1 \otimes \Phi_2) \circ \Delta : \mathcal{A} \to \mathfrak{B}(V_1 \otimes V_2)$$

is also multiplicative. This map is not a representation on $V_1 \otimes V_2$, however, since $(\Phi_1 \otimes \Phi_2) \circ \Delta(1)$ is not the identity, unless $\Delta(1) = 1 \otimes 1$. The element $(\Phi_1 \otimes \Phi_2) \circ \Delta(1)$ is a projector and is absorbed by any

element from $(\Phi_1 \otimes \Phi_2) \circ \Delta(\mathcal{A})$, as $a1 = 1a = a$ for all $a \in \mathcal{A}$. This means that all operators in $(\Phi_1 \otimes \Phi_2) \circ \Delta(\mathcal{A})$ can be restricted to the range of $(\Phi_1 \otimes \Phi_2) \circ \Delta(1)$. Let $V_1 \boxtimes V_2$ denote this subspace of $V_1 \otimes V_2$. By definition, $(\Phi_1 \otimes \Phi_2) \circ \Delta(1)$ restricted to $V_1 \boxtimes V_2$, $(\Phi_1 \otimes \Phi_2) \circ \Delta(1)|_{V_1 \boxtimes V_2}$, is the identity, and thus the map defined by

$$(\Phi_1 \boxtimes \Phi_2)(x) := (\Phi_1 \otimes \Phi_2) \circ \Delta(x)|_{V_1 \boxtimes V_2}$$

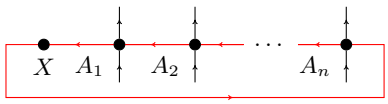
is a representation of $\mathcal{A}$. This representation is then called the tensor product of the representations Φ_1 and Φ_2. Using associativity of the comultiplication, one can define the n-fold tensor product of representations for any $n > 2$ integer as well: given $\Phi_i : \mathcal{A} \to \mathfrak{B}(V_i)$ representation for $i = 1 \dots n$, the tensor product representation $\Phi_1 \boxtimes \cdots \boxtimes \Phi_n$ is given by the restriction of $(\Phi_1 \otimes \cdots \otimes \Phi_n) \circ \Delta^{n-1}(x)$ onto the range of $(\Phi_1 \otimes \cdots \otimes \Phi_n) \circ \Delta^{n-1}(1)$ in the vector space $V_1 \otimes \cdots \otimes V_n$.

Just as in the previous chapter, using the coalgebra structure of $\mathcal{A}$, one can form MPS representations of $\mathcal{A}$. As $\mathcal{A}$ has an algebra structure as well, it is natural to choose the linear map Φ used at the construction of the MPS to be a representation of the algebra. The resulting MPS is then interpreted as an operator, and in fact, this structure is called a matrix product operator.

Definition 4.2. Let $\{V_1, \dots, V_n\}$ and $\{W_0, W_1, \dots, W_n\}$ be two collections of finite dimensional vector spaces over $\mathbb{C}$. A *matrix product operator* (MPO) is given by tensors $A_i \in V_i \otimes V_i^* \otimes W_{i-1} \otimes W_i^*$. $i = 1, \dots, n$, $A_i = \sum_{k,l} |k\rangle\langle l| \otimes A_i^{kl}$ and a matrix $X \in W_n \otimes W_0^*$; the operator generated by the MPO is given by

$$O = \sum_{k,l} \mathrm{Tr}(X A_1^{k_1 l_1} \cdots A_n^{k_n l_n}) |k_1 \dots k_n\rangle\langle l_1 \dots l_n|$$

or, in graphical notation,

$O =$ 

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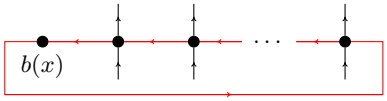
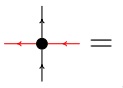
Fixing a representation $\Phi : \mathcal{A} \to \mathfrak{B}(V)$ of $\mathcal{A}$ and an injective representation $\Psi : \mathcal{A}^* \to \mathfrak{B}(W)$ of $\mathcal{A}^*$, one can repeat the procedure described in the previous chapter to form MPOs representing the coalgebra structure of $\mathcal{A}$:

$\Phi^{\otimes n} \circ \Delta^{n-1}(x) =$ 

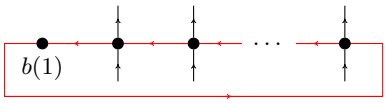
with

 $= \sum_{x \in B} \Phi(x) \otimes \Psi(\delta_x),$

where B is a basis of $\mathcal{A}$, and Φ is a representation of $\mathcal{A}$, and the matrices $b(x)$ are such that $\mathrm{Tr}\,(b(x)\Psi(f)) = f(x)$. As described above, the map

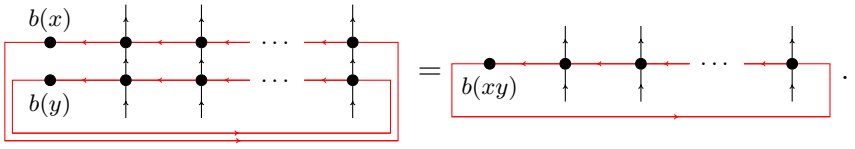
$\Phi^{\otimes n} \circ \Delta^{n-1}$, in general, is not a representation of $\mathcal{A}$ as 1 is not mapped to the identity operator on the space $V^{\otimes n}$:

$$\Phi^{\otimes n} \circ \Delta^{n-1}(1) = \quad \neq \mathrm{Id}_V^{\otimes n}.$$

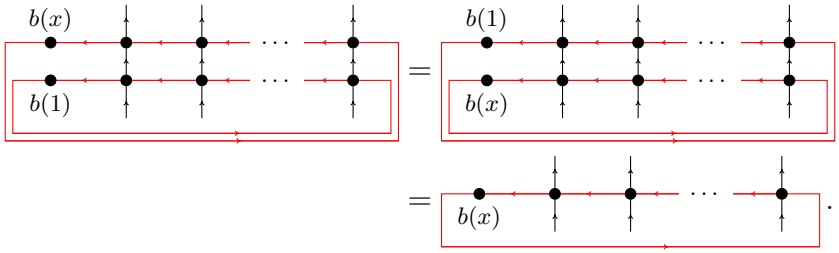


These MPOs, nevertheless, are multiplicative as $\Phi^{\otimes n} \circ \Delta^{n-1}$ is multiplicative:

(2)

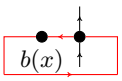


In particular, these MPOs are invariant under multiplying with $\Phi^{\otimes n} \circ \Delta^{n-1}(1)$ from either side,



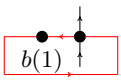
One can thus restrict $V^{\otimes n}$ to the range of $\Phi^{\otimes n} \circ \Delta^{n-1}(1)$ (notice that this is a projector as $1 \cdot 1 = 1$), and on this space the MPOs form a representation of $\mathcal{A}$. Note that this restriction is only necessary for $n > 1$; for $n = 1$ the representing MPOs are simply $x \mapsto \Phi(x)$:

$$\Phi(x) = \quad .$$



In particular, on a single site, unlike for $n > 1$ sites, the unit is represented by the identity operator:

$$\mathrm{Id} = \Phi(1) = \quad .$$



1. Cosemisimplicity

Let $\mathcal{A}$ be a pre-bialgebra, then $\mathcal{A}^*$ is a pre-bialgebra as well. As such, one can form tensor products of its representations, i.e. if $\Psi_1 : \mathcal{A}^* \to \mathfrak{B}(W_1)$ and $\Psi_2 : \mathcal{A}^* \to \mathfrak{B}(W_2)$ are representations of $\mathcal{A}^*$, then $(\Psi_1 \otimes \Psi_2) \circ \Delta_{\mathcal{A}^*}$ restricted to the range of $(\Psi_1 \otimes \Psi_2) \circ \Delta_{\mathcal{A}^*}(\varepsilon)$ is a representation[1] of $\mathcal{A}^*$ denoted by $\Psi_1 \boxtimes \Psi_2$.

Just as for coalgebras, we say that a pre-bialgebra $\mathcal{A}$ is cosemisimple if the algebra $\mathcal{A}^*$ is semisimple, i.e. if every representation of it decomposes into a direct sum of irreducible representations. In particular, let us fix irreducible representations $\Psi_a : \mathcal{A}^* \to \mathfrak{B}(W_a)$ for each

[1]Remember that the unit of $\mathcal{A}^*$ is ε.

sector $a \in \mathrm{Irr}(\mathcal{A}^*)$; then given two of these irreducible representations, Ψ_a and Ψ_b, their tensor product $\Psi_a \boxtimes \Psi_b$ decomposes into a direct sum of irreducible representations as follows:

$$\Psi_a \boxtimes \Psi_b \simeq \bigoplus_{c \in \mathrm{Irr}(\mathcal{A}^*)} \Psi_c \otimes \mathrm{Id}_{N_{ab}^c},$$

i.e. in the decomposition of $\Psi_a \boxtimes \Psi_b$ the irreducible representation Ψ_c appears N_{ab}^c times. These N_{ab}^c non-negative integers are called *fusion multiplicities*. Note that N_{ab}^c might be 0; in that case the irreducible representation Ψ_c does not appear in the decomposition. The above equation holds up to a basis transformation, i.e. for all $a, b, c \in \mathrm{Irr}(\mathcal{A}^*)$ there are invertible operators $Z_{ab} : \bigoplus_c W_c \otimes \mathbb{C}^{N_{ab}^c} \to W_a \boxtimes W_b$ such that for all $f \in \mathcal{A}^*$

$$(\Psi_a \boxtimes \Psi_b)(f) = Z_{ab} \left(\bigoplus_{c \in \mathrm{Irr}(\mathcal{A}^*)} \Psi_c(f) \otimes \mathrm{Id}_{N_{ab}^c} \right) (Z_{ab})^{-1}, \tag{3}$$

or equivalently, for all $a, b, c \in \mathrm{Irr}(\mathcal{A}^*)$ there are N_{ab}^c operators $Z_{ab}^{c\mu} : V_c \to V_a \boxtimes V_b$ and $Y_{ab}^{c\mu} : V_a \boxtimes V_b \to V_c$ (here $\mu = 1, \ldots, N_{ab}^c$ are integers) such that

$$(\Psi_a \boxtimes \Psi_b)(f) = \sum_{c \in \mathrm{Irr}(\mathcal{A}^*)} \sum_{\mu=1}^{N_{ab}^c} Z_{ab}^{c\mu} \Psi_c(f) Y_{ab}^{c\mu} \quad \text{and} \quad Y_{ab}^{c\mu} Z_{ab}^{d\nu} = \delta_{cd} \delta_{\mu\nu} \mathrm{Id}_{V_c}.$$

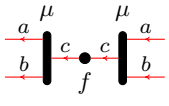
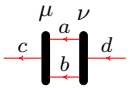
These operators then can be extended[2] to map from (and to) $W_a \otimes W_b$ instead of $W_a \boxtimes W_b$, i.e. there are linear maps $V_{ab}^{c\mu} : W_c \to W_a \otimes W_b$ and $W_{ab}^{c\mu} : W_a \otimes W_b \to W_c$, called *fusion tensors*, such that

$$(\Psi_a \otimes \Psi_b) \circ \Delta(f) = \sum_{c} \sum_{\mu=1}^{N_{ab}^c} V_{ab}^{c\mu} \Psi_c(f) W_{ab}^{c\mu}, \; W_{ab}^{c\mu} V_{ab}^{d\nu} = \delta_{cd} \delta_{\mu\nu} \mathrm{Id}_{W_c}, \tag{4}$$

for all $a, b, c \in \mathrm{Irr}(\mathcal{A}^*)$ and $f \in \mathcal{A}^*$. The fusion tensors are rank-four tensors; the index μ, however, plays a very different role than the rest of its indices. This allows us to think of the fusion tensors as a set for rank-three tensors instead, and write

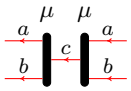
$V_{ab}^{c\mu} =$  and $W_{ab}^{c\mu} =$  .

Using this notation, the graphical representation of Equation 4 is the following:

$(\Psi_a \otimes \Psi_b) \circ \Delta(f) =$  and  $= \delta_{cd} \delta_{\mu\nu} \mathrm{Id}_c.$ (5)

[2] Note that as ε is a projector, $V_a \otimes V_b = \mathrm{Ker}((\Psi_a \boxtimes \Psi_b)(\varepsilon)) \oplus \mathrm{Im}(\Psi_a \boxtimes \Psi_b)(\varepsilon)$. The extension of the operators $Y_{ab}^{c\mu}$ is such that they act as zero on the space $\mathrm{Ker}((\Psi_a \boxtimes \Psi_b)(\varepsilon))$.

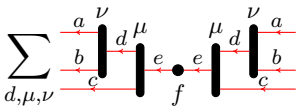
Let us stress again that the fusion tensors $V_{ab}^{c\mu}$ (and $W_{ab}^{c\mu}$) do not map to (and from) the whole space $W_a \otimes W_b$, instead only to (and from) the subspace $W_a \boxtimes W_b$. A projector onto this subspace is $(\Psi_a \otimes \Psi_b) \circ \Delta(\varepsilon) \neq \mathrm{Id}_a \otimes \mathrm{Id}_b$, that has the graphical representation

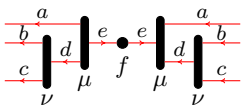
(6) $$(\Psi_a \otimes \Psi_b) \circ \Delta(\varepsilon) = \sum_{c,\mu}$$ 

The operators $V_{ab}^{c\mu}$ and $W_{ab}^{c\mu}$ are not unique, instead only their tensor product $\sum_\mu V_{ab}^{c\mu} \otimes W_{ab}^{c\mu}$ is fixed. This allows for a basis change: $\sum_\mu V_{ab}^{c\mu} \otimes W_{ab}^{c\mu} = \sum_\mu \hat{V}_{ab}^{c\mu} \otimes \hat{W}_{ab}^{c\mu}$ if and only if $\hat{V}_{ab}^{c\mu} = \sum_\nu K_{\mu\nu} V_{ab}^{c\nu}$ and $\hat{W}_{ab}^{c\mu} = \sum_\nu (K^{-1})_{\nu\mu} W_{ab}^{c\nu}$ for some $N_{ab}^c \times N_{ab}^c$ complex invertible matrix K. Note as well that the value of $\sum_\mu V_{ab}^{c\mu} \otimes W_{ab}^{c\mu}$ depends on the choices of the irreducible representation representatives Ψ_a that we have made at the beginning of this chapter; different choices will lead to a gauge transformation of $V_{ab}^{c\mu}$ and $W_{ab}^{c\mu}$, i.e. if the irreducible representation representatives are chosen to be $\hat{\Psi}_a = U_a \Psi_a U_a^{-1}$ instead of Ψ_a for each $a \in \mathrm{Irr}(\mathcal{A}^*)$ (here U_a are general invertible matrices), then the corresponding $\hat{V}_{ab}^{c\mu}$ and $\hat{W}_{ab}^{c\mu}$ are given by

$$\hat{V}_{ab}^{c\mu} = (U_a \otimes U_b) V_{ab}^{c\mu} U_c^{-1} \quad \text{and} \quad \hat{W}_{ab}^{c\mu} = U_c W_{ab}^{c\mu} (U_a^{-1} \otimes U_b^{-1}).$$

Let us now show using this graphical language how the so-called F-symbols of the representation category of $\mathcal{A}^*$ emerge. Using Equation 5, coassociativity of the comultiplication of $\mathcal{A}^*$ implies that for all $f \in \mathcal{A}^*$ and $a, b, c, e \in \mathrm{Irr}(\mathcal{A}^*)$,

$$\sum_{d,\mu,\nu}$$ 

$$= \sum_{d,\mu,\nu}$$ 

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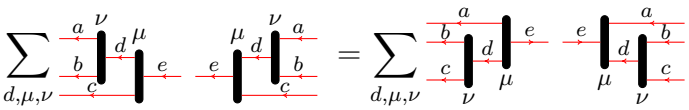
As this equation holds for all $f \in \mathcal{A}^*$, and $\Psi_e(\mathcal{A}^*) = \mathcal{M}_{D_e}$ for each irreducible representation block e, i.e. the matrix $\Psi_e(f)$ can take any value, we conclude that

(7) $$\sum_{d,\mu,\nu}$$ 

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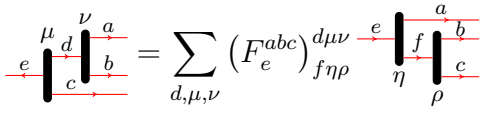
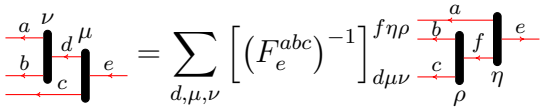
or equivalently, that there is an invertible (square) matrix F_e^{abc} of dimension $\sum_d N_{ab}^d N_{dc}^e = \sum_d N_{bc}^d N_{ad}^e$ such that



$$= \sum_{d,\mu,\nu} \left(F_e^{abc}\right)_{f\eta\rho}^{d\mu\nu}$$,

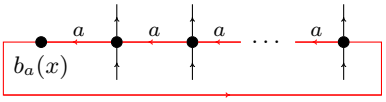


$$= \sum_{d,\mu,\nu} \left[\left(F_e^{abc}\right)^{-1}\right]_{d\mu\nu}^{f\eta\rho}$$.

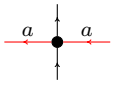
Standard arguments show that these F-symbols satisfy a consistency condition called the pentagon equation.

Let us now investigate the MPO representations of a cosemisimple pre-bialgebra $\mathcal{A}$. Just as in the MPS case, the MPOs representing $\mathcal{A}$ decompose into a sum of MPOs with smaller bond dimension corresponding to the irreducible representations of $\mathcal{A}^*$. We denote these MPOs by writing the label $a \in \mathrm{Irr}(\mathcal{A}^*)$ on the virtual indices of the tensor:

$$\Phi^{\otimes n} \circ \Delta^{n-1}(x) = \sum_{a \in \mathrm{Irr}(\mathcal{A}^*)}$$



with

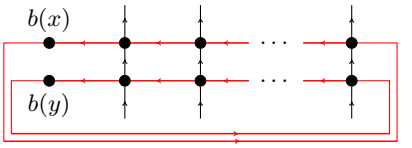


$$= \sum_{x \in B} \Phi(x) \otimes \Psi_a(\delta_x).$$

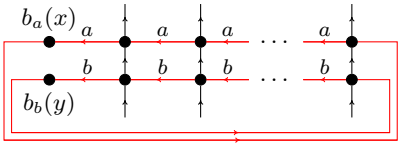
If Φ is injective, then each smaller MPO tensor is injective and if there is a positive integer n such that $\Phi^{\otimes n} \circ \Delta^{n-1}$ is injective, then they are normal. The above decomposition is thus nothing but the decomposition of the MPO into its normal components.

Let us now consider the product of two MPOs each representing an element of $\mathcal{A}$, such as in Equation 2. The l.h.s. of this equation then decomposes to injective blocks as described above. On the r.h.s. both MPOs can be decomposed into injective blocks, therefore their product can also be decomposed into a sum:

(8)

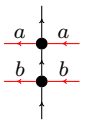


$$= \sum_{a,b \in \mathrm{Irr}(\mathcal{A}^*)}$$



.

It turns out, however, that these MPOs are still not injective, i.e. they can be further decomposed. To understand why, let us investigate the tensor describing such an MPO:



$$:= \sum_{x,y \in B} \Phi(x)\Phi(y) \otimes (\Psi_a(\delta_x) \otimes \Psi_b(\delta_y))$$

$$= \sum_{x \in B} \Phi(x) \otimes (\Psi_a \otimes \Psi_b) \circ \Delta(\delta_x),$$

where the last equality, analogous to the proof of Theorem 3.3, holds because for every $f \in \mathcal{A}^*$,

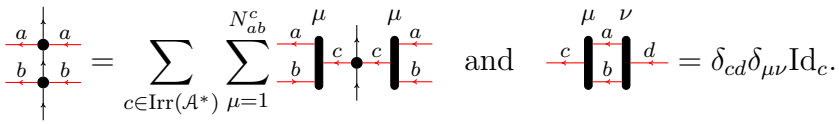
$$\sum_{x,y\in B} f(xy)\delta_x \otimes \delta_y = \sum_{x,y\in B} f_{(1)}(x)f_{(2)}(y)\delta_x \otimes \delta_y = f_{(1)} \otimes f_{(2)}$$
$$= \sum_{x\in B} f(x)\Delta(\delta_x),$$

and thus

$$\sum_{x,y\in B} xy \otimes \delta_x \otimes \delta_y = \sum_{x\in B} x \otimes \Delta(\delta_x).$$

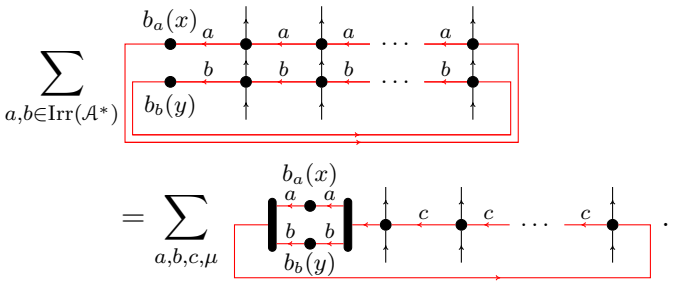
This form of the MPO tensor thus involves the tensor product of the irreducible representations Ψ_a and Ψ_b. In Equation 5 we have seen how to decompose such a representation into irreducible representations. Using that equation, the product of the MPO tensors satisfy

(9)



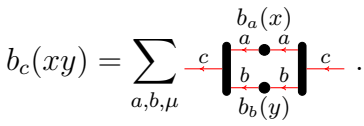
Using Equation 9 in Equation 8, we can finally decompose the r.h.s. of Equation 2 into the sum of injective MPOs:

(10)

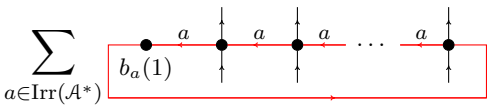


We have thus obtained that the product of two algebra elements x and y is described by the boundary

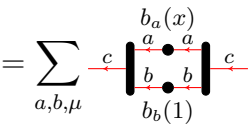
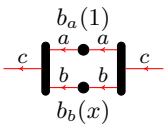
(11)



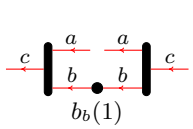
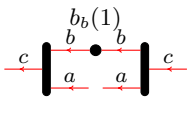
Let us note here that so far we have not investigated how the unit of the pre-bialgebra is represented. The existence of the unit in $\mathcal{A}$, in fact, imposes further restrictions on the structure of the MPOs representing pre-bialgebras. Remember that $\Phi^{\otimes n} \circ \Delta^{n-1}(1)$ is a non-trivial projector that is represented by:



The relation $1x = x1 = x$, using Equation 11, then implies that the matrices $b_a(1)$ have the following property:

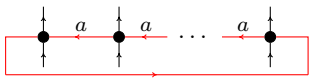
(12) $$b_c(x) = \sum_{a,b,\mu}$$  $$= \sum_{a,b,\mu}$$ 

or equivalently,

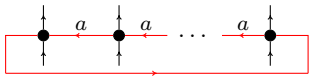
(13) $$\delta_{a,c}\mathrm{Id}_a \otimes \mathrm{Id}_a = \sum_{b,\mu}$$  $$= \sum_{b,\mu}$$ 

2. Cocentral elements and the Grothendieck ring

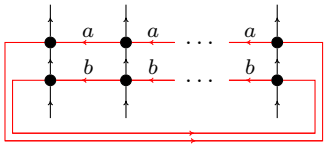
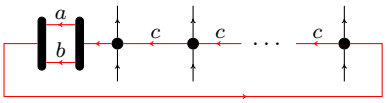
Let us consider a cosemisimple pre-bialgebra $\mathcal{A}$. Then the cocenter of $\mathcal{A}$, as we have seen in Chapter 3, consists of elements x of the form

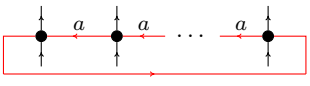
$$\Phi^{\otimes n} \circ \Delta^{n-1}(x) = \sum_{a \in \mathrm{Irr}(\mathcal{A}^*)} \lambda_a$$ 

Let $\tau_a \in \mathcal{A}$ be the character of the sector $a \in \mathrm{Irr}(\mathcal{A}^*)$, i.e. the cocentral element with MPO representation

$$\Phi^{\otimes n} \circ \Delta^{n-1}(\tau_a) =$$ 

Let us evaluate the product of two irreducible representation characters τ_a and τ_b. Using the previously derived results, it is immediate to see from their MPO representations that

 $= \sum_{c,\mu}$ 

$= \sum_c N_{ab}^c$  ,

where the first equation is Equation 10 using that $b_a(\tau_c) = \delta_{ac}\mathrm{Id}_c$ for all $a, c \in \mathrm{Irr}(\mathcal{A}^*)$, and the second is just the orthogonality relations from Equation 9 together with the fact that μ runs from 1 to N_{ab}^c. This equation then reads as

$$(\Phi^{\otimes n} \circ \Delta^{n-1}(\tau_a))(\Phi^{\otimes n} \circ \Delta^{n-1}(\tau_b)) = \sum_c N_{ab}^c (\Phi^{\otimes n} \circ \Delta^{n-1}(\tau_c)),$$

and thus it implies, as $\Phi^{\otimes n} \circ \Delta^{n-1}$ is a homomorphism and it is w.l.o.g. injective, that

(14) $$\tau_a \tau_b = \sum_c N_{ab}^c \tau_c.$$

Let us remark here that to obtain this well-known result, one does not have to consider MPO representations. Instead, the same result can be obtained directly from the decomposition of the tensor product representation into irreducible representations: let Ψ_a be an irreducible representation on a vector space V_a such that it is from the sector a (and thus its character is τ_a), and Ψ_b an irreducible representation on a vector space V_b such that it is from the sector b (and thus its character is τ_b). Then, by definition of the multiplication in $\mathcal{A}^*$,

$$\begin{aligned}(\tau_a\tau_b)(f) &= (\tau_a \otimes \tau_b) \circ \Delta(f) \\ &= (\mathrm{Tr}_{V_a} \otimes \mathrm{Tr}_{V_b}) \circ (\Psi_a \otimes \Psi_b) \circ \Delta(f) \\ &= \mathrm{Tr}((\Psi_a \otimes \Psi_b) \circ \Delta(f)).\end{aligned}$$

In this last trace the operator $(\Psi_a \otimes \Psi_b) \circ \Delta(f)$ is supported on $V_a \boxtimes V_b$ instead of the whole tensor product space $V_a \otimes V_b$, and thus restricting $(\Psi_a \otimes \Psi_b) \circ \Delta(f)$ to $V_a \boxtimes V_b$ does not change its trace:

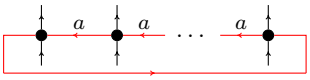
$$(\tau_a\tau_b)(f) = \mathrm{Tr}(\Psi_a \otimes \Psi_b) \circ \Delta(f) = \mathrm{Tr}(\Psi_a \boxtimes \Psi_b)(f).$$

Finally, this representation decomposes into irreducible representations (see Equation 3), and thus the trace can be evaluated:

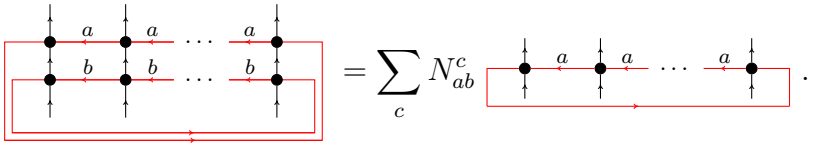
$$(\tau_a\tau_b)(f) = \mathrm{Tr}(\Psi_a \boxtimes \Psi_b)(f) = \sum_c N_{ab}^c \,\mathrm{Tr}\, \Psi_c(f) = \sum_c N_{ab}^c \tau_c(f),$$

that is equivalent to Equation 14. We have thus seen that:

Proposition 4.3. *In a finite dimensional cosemisimple pre-bialgebra $\mathcal{A}$ over $\mathbb{C}$ the irreducible representation characters of $\mathcal{A}^*$ correspond to the injective blocks in the MPO representation of $\mathcal{A}$. For $a \in \mathrm{Irr}(\mathcal{A}^*)$, the irreducible representation character $\tau_a \in \mathcal{A}$ has the following MPO representation:*

$\Phi^{\otimes n} \circ \Delta^{n-1}(\tau_a) =$ 

These MPOs form a closed ring[3] *over $\mathbb{Z}$, i.e. for all a, b, c there are non-negative integer numbers N_{ab}^c such that $\tau_a\tau_b = \sum_c N_{ab}^c \tau_c$, or graphically,*

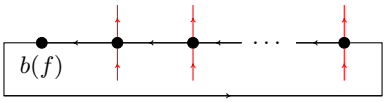


This ring is then called the Grothendieck ring of $\mathcal{A}$.

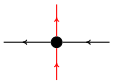
[3]Not necessarily unital; note that the unit of the algebra $1 \in \mathcal{A}$ might have a non-trivial boundary, and thus, in general, it is not in the Grothendieck ring.

3. Duality

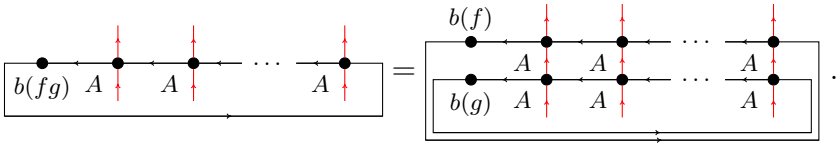
As we have seen before, the dual $\mathcal{A}^*$ of a pre-bialgebra $\mathcal{A}$ is a pre-bialgebra as well. As such, it also has MPO representations. Let us fix a representation Ψ of $\mathcal{A}^*$ on a vector space W, and a representation Φ of $\mathcal{A}^{**} = \mathcal{A}$ on a vector space V. Then the previous construction leads to the following MPO representation:

(15) $\Psi^{\otimes n} \circ \Delta^{n-1}(f) =$ 

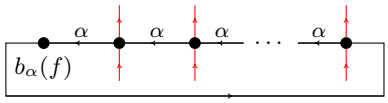
with

 $:= \sum_{x \in B} \Psi(\delta_x) \otimes \Phi(x),$

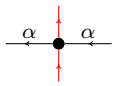
where B is a basis of $\mathcal{A}$. Notice that this MPO tensor is exactly 90 degree rotation of the MPO tensor describing the coproduct of elements in $\mathcal{A}$. As above, these MPOs form a representation of $\mathcal{A}^*$,



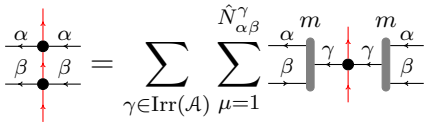
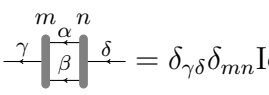
If $\mathcal{A}$ is semisimple (in general this does not follow from semisimplicity of $\mathcal{A}^*$), then this MPO representation of $\mathcal{A}^*$ decomposes into a sum of smaller bond dimensional MPOs corresponding to the characters of $\mathcal{A}$:

$\Psi^{\otimes n} \circ \Delta^{n-1}(f) = \sum_{\alpha \in \mathrm{Irr}(\mathcal{A})}$ 

with

 $:= \sum_{x \in B} \Psi(\delta_x) \otimes \Phi_\alpha(x),$

where B, as above, is a basis of $\mathcal{A}$. We will denote sectors of $\mathcal{A}$ with Greek letters, and sectors of $\mathcal{A}^*$ with Latin letters. Just as above, there are linear maps $\hat{V}^{\gamma m}_{\alpha\beta}$ and $\hat{W}^{\gamma m}_{\alpha\beta}$, $m = 1, \ldots, \hat{N}^{\gamma}_{\alpha\beta}$, such that

 $=$ $\sum_{\gamma \in \mathrm{Irr}(\mathcal{A})} \sum_{\mu=1}^{\hat{N}^{\gamma}_{\alpha\beta}}$ and  $= \delta_{\gamma\delta}\delta_{mn}\mathrm{Id}_\gamma.$

These linear maps give rise to a set of F-symbols (also satisfying the pentagon equations) that are, in general, different from the ones in the previous sections.

CHAPTER 5

Weak bialgebras and weak Hopf algebras

In this chapter we introduce weak bialgebras, weak Hopf algebras, pivotal weak Hopf algebras, spherical weak Hopf algebras and C*-weak Hopf algebras. All these structures are all special pre-bialgebras, and as such, we use their MPO representations to reason about the structure of these objects; this is possible as the MPO representations are w.l.o.g. injective. Our main result is Theorem 5.9, where we construct a special normalized integral Λ in any cosemisimple weak Hopf algebra over $\mathbb{C}$. This integral Λ is then used to prove that in a cosemisimple co-pivotal weak Hopf algebra over $\mathbb{C}$, there is a cocommutative projector with a property reminiscent to the definition of an integral that will allow us to define MPO-injective PEPS. We then further specialize these results to spherical and C*-weak Hopf algebras.

1. Weak bialgebras

In this section we define weak bialgebras and show that the MPO representation of the unit of a cosemisimple WBA $\mathcal{A}$ has the following property: for all sectors a of $\mathcal{A}^*$, $b_a(1)$ is either 0 or rank-one. We show, moreover, that the Grothendieck ring of $\mathcal{A}$ has a unit. This unit can be written in the form

$$\tau_{\mathrm{Vac}(\mathcal{A}^*)} := \sum_{e \in \mathrm{Vac}(\mathcal{A}^*)} \tau_e,$$

where $\mathrm{Vac}(\mathcal{A}^*) \subseteq \mathrm{Irr}(\mathcal{A}^*)$ consists of the sectors a for which $b_a(1) \neq 0$.

Definition 5.1. A *weak bialgebra* (WBA) is a pre-bialgebra $\mathcal{A}$ such that the unit $1 \in \mathcal{A}$ and counit $\varepsilon \in \mathcal{A}^*$ satisfy

$$1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} = 1_{(1)} \otimes 1_{(2)} 1_{(1')} \otimes 1_{(2')} = 1_{(1)} \otimes 1_{(1')} 1_{(2)} \otimes 1_{(2')}, \tag{16}$$

$$\varepsilon_{(1)} \otimes \varepsilon_{(2)} \otimes \varepsilon_{(3)} = \varepsilon_{(1)} \otimes \varepsilon_{(2)} \varepsilon_{(1')} \otimes \varepsilon_{(2')} = \varepsilon_{(1)} \otimes \varepsilon_{(1')} \varepsilon_{(2)} \otimes \varepsilon_{(2')}. \tag{17}$$

We will refer to Equation 16 as the *unit axiom* and to Equation 17 as the *counit axiom*.

In the equations above 1 and ε appears twice in the same Sweedler notation. To distinguish between the two appearances, we added a prime to the subindices of one of them. One can also write the unit axiom, Equation 16, as

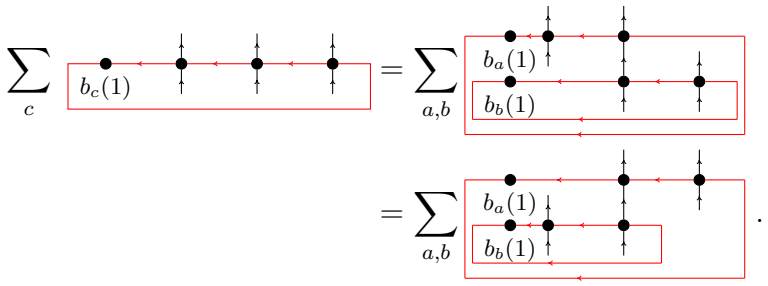
$$\Delta^2(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)),$$

while the counit axiom, Equation 17, is more convenient to think of as

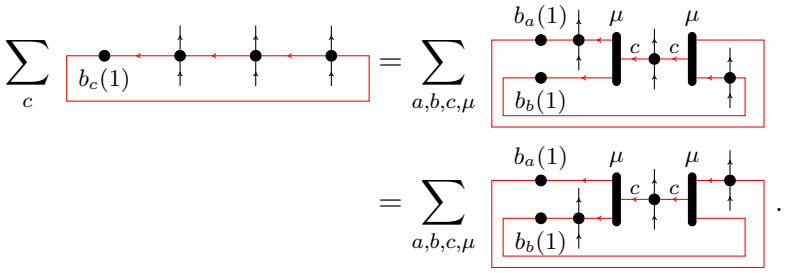
$$\varepsilon(xyz) = \varepsilon(xy_{(1)})\varepsilon(y_{(2)}z) = \varepsilon(xy_{(2)})\varepsilon(y_{(1)}z).$$

As pre-bialgebras are self-dual and the above two axioms are the dual of one another, weak bialgebras are also self-dual: if $\mathcal{A}$ is a WBA, then $\mathcal{A}^*$ is also a WBA.

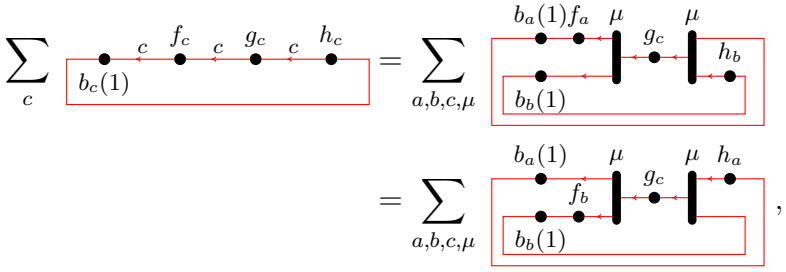
Let $\mathcal{A}$ be a cosemisimple WBA. The graphical representation of the unit axiom, Equation 16, is



Decomposing the product of the two MPO tensors using the fusion tensors (Equation 9), we arrive at



Applying linear functionals f, g and h on the three components, we arrive at the equations

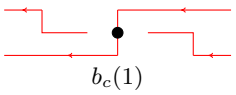
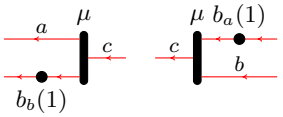


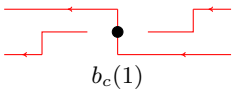
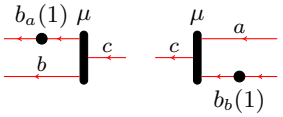
where f_c denotes the part of f supported on the sector c, i.e.

$$f_c = p_c f = f p_c,$$

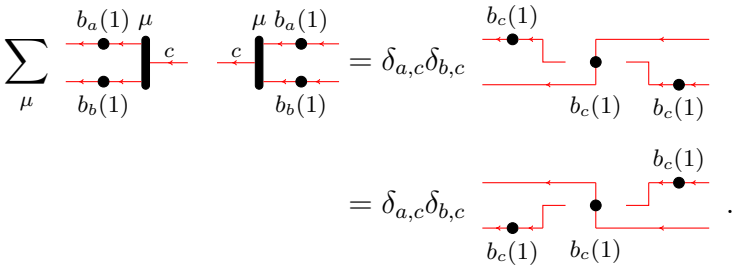
where $p_c \in Z(\mathcal{A}^*)$ is the irreducible representation projector onto the sector c. This equation is true for all f, g and h, and thus, since $\Psi_c(\mathcal{A}^*)$

is the full matrix algebra $\mathcal{M}_{D_c}$ for all sectors, we obtain

(18) $\delta_{a,c}\delta_{b,c}$  $= \sum_{\mu}$  ,

(19) $\delta_{a,c}\delta_{b,c}$  $= \sum_{\mu}$  .

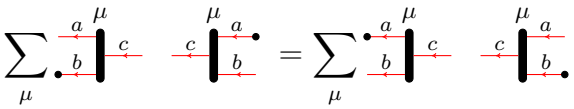
Combining these two equations, we arrive at



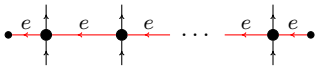
Comparing the last two expressions, we obtain that $b_c(1)$ is rank-1 for all c where it is non-zero, i.e. there is a vector  $\in V_c$ and a linear functional  $\in V_c^*$ such that

(20)  .

Let $\mathrm{Vac}(\mathcal{A}^*)$ be the set of sectors such that $b_c(1) \neq 0$. Let us now consider Equations 18 and 19 such that the sectors a and b are from this set $\mathrm{Vac}(\mathcal{A}^*)$, while c is an arbitrary sector. Using Equation 20, we obtain

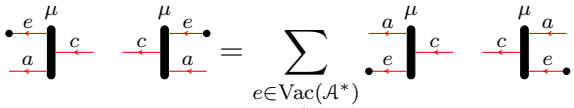
(21)  $= \delta_{a,c}\delta_{b,c}(\mathrm{Id}_c \otimes \mathrm{Id}_c)$.

In particular, $N_{ab}^c \neq 0$ for $a, b \in \mathrm{Vac}(\mathcal{A}^*)$ and $c \in \mathrm{Irr}(\mathcal{A}^*)$ if and only if $a = b = c$. As $b(1)$ is rank-one in every sector where it is non-zero, the MPO representation of $1 \in \mathcal{A}$ can be written as

(22) $\Phi^{\otimes n} \circ \Delta^{n-1}(1) = \sum_{e \in \mathrm{Vac}(\mathcal{A}^*)}$  .

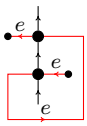
Note that the fact that $1 \in \mathcal{A}$ is the unit of the algebra implies, using Equation 13, that for all $a, c \in \mathrm{Irr}(\mathcal{A}^*)$,

(23)

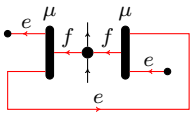
$\sum_{\substack{e \in \mathrm{Vac}(\mathcal{A}^*) \\ \mu}}$  $= \delta_{a,c}(\mathrm{Id}_c \otimes \mathrm{Id}_c)$.

Let us now consider the element $1_{(1)}1_{(2)}$. The MPS representation of this element is given by

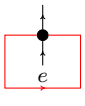
$$\Phi(1_{(1)}1_{(2)}) = \Phi(1_{(1)})\Phi(1_{(2)}) = \sum_{e\in \mathrm{Vac}(\mathcal{A}^*)}$$



$$= \sum_{e\in \mathrm{Vac}(\mathcal{A}^*),f,\mu}$$



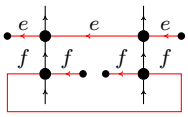
$$= \sum_{e\in \mathrm{Vac}(\mathcal{A}^*)}$$



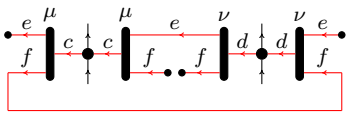
$$= \sum_e \Phi(\tau_e),$$

where in the fourth equality we have used Equation 21. As Φ is injective w.l.o.g., this implies that $1_{(1)}1_{(2)} = \sum_{e\in \mathrm{Vac}(\mathcal{A}^*)} \tau_e$. Let us denote this element of the algebra as $\tau_{\mathrm{Vac}(\mathcal{A}^*)}$. Notice now that $\Delta(1)\Delta_{\mathrm{op}}(1)$ is represented by the product

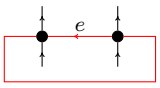
$$\sum_{e,f\in \mathrm{Vac}(\mathcal{A}^*)}$$



$$= \sum_{\substack{e,f\in \mathrm{Vac}(\mathcal{A}^*)\\ c,d,\mu,\nu}}$$



$$= \sum_{e\in \mathrm{Vac}(\mathcal{A}^*)}$$



,

where in the last equation we have used Equation 21. Again, as $\Phi\otimes\Phi$ is w.l.o.g. injective, we conclude that $\Delta(1)\Delta_{\mathrm{op}}(1) = \Delta(\tau_{\mathrm{Vac}(\mathcal{A}^*)})$. Similar calculation shows that $\Delta_{\mathrm{op}}(1)\Delta(1) = \Delta(\tau_{\mathrm{Vac}(\mathcal{A}^*)})$ as well, and thus we have proven[1] that

$$\Delta(\tau_{\mathrm{Vac}(\mathcal{A}^*)}) = \Delta(1)\Delta_{\mathrm{op}}(1) = \Delta_{\mathrm{op}}(1)\Delta(1).$$

This, in fact, implies that $\tau_{\mathrm{Vac}(\mathcal{A}^*)}$ is the unit of the Grothendieck ring of $\mathcal{A}^*$ (and of the character algebra as well), as for any co-central $\tau \in \mathcal{A}$,

$$\begin{aligned}\Delta(\tau_{\mathrm{Vac}(\mathcal{A}^*)}\tau) &= \Delta_{\mathrm{op}}(1)\Delta(1)\Delta(\tau) = \Delta_{\mathrm{op}}(1)\Delta(\tau)\\ &= \Delta_{\mathrm{op}}(1)\Delta_{\mathrm{op}}(\tau) = \Delta_{\mathrm{op}}(\tau) = \Delta(\tau),\end{aligned}$$

i.e. $\tau_{\mathrm{Vac}(\mathcal{A}^*)}\tau = \tau$, and similarly $\tau\tau_{\mathrm{Vac}(\mathcal{A}^*)} = \tau$ as well. Notice that as $\tau_{\mathrm{Vac}(\mathcal{A}^*)}$ decomposes into a sum of irreducible representation characters, given any irreducible representation character τ_a, $a \in \mathrm{Irr}(\mathcal{A}^*)$, the following equations hold:

$$\tau_a = \tau_{\mathrm{Vac}(\mathcal{A}^*)}\tau_a = \sum_{e\in \mathrm{Vac}(\mathcal{A}^*),b} N_{ea}^b \tau_b,$$

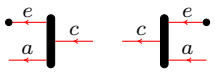
$$\tau_a = \tau_a\tau_{\mathrm{Vac}(\mathcal{A}^*)} = \sum_{e\in \mathrm{Vac}(\mathcal{A}^*),b} N_{ae}^b \tau_b.$$

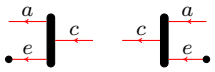
[1] The same result can be obtained by direct calculation as well [**96**]: using the unit axiom Equation 16 twice, $\Delta(\tau_{\mathrm{Vac}(\mathcal{A}^*)}) = 1_{(1)}1_{(3)} \otimes 1_{(2)}1_{(4)} = \Delta(1)(1_{(2)} \otimes 1_{(1)}1_{(3)}) = \Delta(1)\Delta_{\mathrm{op}}(1)\Delta(1)$. Finally, again due to the unit axiom, $\Delta(1)\Delta_{\mathrm{op}}(1) = \Delta_{\mathrm{op}}(1)\Delta(1)$, that leads to the desired result.

Here both $N_{ea}^b \geq 0$ and $N_{ae}^b \geq 0$ are integers, and thus, as the equations above are equivalent to $\sum_{e\in \mathrm{Vac}(\mathcal{A}^*)} N_{ea}^b = \delta_{ab}$ and $\sum_{e\in \mathrm{Vac}(\mathcal{A}^*)} N_{ae}^b = \delta_{ab}$, there are unique labels $\ell_a, r_a \in \mathrm{Vac}(\mathcal{A}^*)$ such that for all $e \in \mathrm{Vac}(\mathcal{A}^*)$ and $a, b \in \mathrm{Irr}(\mathcal{A}^*)$

$$N_{ea}^b = \begin{cases} 0 & \text{if } e \neq \ell_a, \\ \delta_{ab} & \text{if } e = \ell_a, \end{cases} \quad \text{and} \quad N_{ae}^b = \begin{cases} 0 & \text{if } e \neq r_a, \\ \delta_{ab} & \text{if } e = r_a. \end{cases} \tag{24}$$

Using this property of the fusion multiplicities N_{ab}^c, we notice that in Equation 23 in the sum over $e \in \mathrm{Vac}(\mathcal{A}^*)$ the summand is non-zero only for $e = \ell_a$ on the l.h.s. and $e = r_a$ on the r.h.s. and in these cases the sum over μ is trivial (because $N_{\ell_a a}^c = N_{a r_a}^c = \delta_{ac}$). We can thus simplify Equation 23 to

(25a)  $= \delta_{e,\ell_a}\delta_{a,c}(\mathrm{Id}_c \otimes \mathrm{Id}_c)$,

(25b)  $= \delta_{e,r_a}\delta_{a,c}(\mathrm{Id}_c \otimes \mathrm{Id}_c)$,

that hold for all $e \in \mathrm{Vac}(\mathcal{A}^*)$, and $a, c \in \mathrm{Irr}(\mathcal{A}^*)$. Let us note that Equation 25 together with Equation 24 implies Equation 21, and thus it is easy to check that the element defined by Equation 22 is indeed the unit of $\mathcal{A}$ and that it satisfies the unit axiom. We have thus seen that

Proposition 5.2. *In a finite dimensional cosemisimple pre-bialgebra $\mathcal{A}$ over $\mathbb{C}$ the unit axiom Equation 16 is equivalent to the following:*

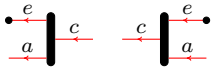
(1) *there is a set $\mathrm{Vac}(\mathcal{A}^*) \subseteq \mathrm{Irr}(\mathcal{A}^*)$ such that*

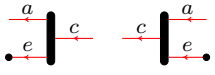
$$\tau_{\mathrm{Vac}(\mathcal{A}^*)} = \sum_{e\in \mathrm{Vac}(\mathcal{A}^*)} \tau_e$$

is the unit of the Grothendieck ring of $\mathcal{A}^*$, or equivalently, for all $a \in \mathcal{A}$ there are unique labels $\ell_a, r_a \in \mathrm{Vac}(\mathcal{A}^*)$ such that for all $e \in \mathrm{Vac}(\mathcal{A}^*)$ and $b \in \mathrm{Irr}(\mathcal{A}^*)$

$$N_{ea}^b = \begin{cases} 0 & \text{if } e \neq \ell_a, \\ \delta_{ab} & \text{if } e = \ell_a, \end{cases} \quad \text{and} \quad N_{ae}^b = \begin{cases} 0 & \text{if } e \neq r_a, \\ \delta_{ab} & \text{if } e = r_a, \end{cases}$$

(2) *and for all $e \in \mathrm{Vac}(\mathcal{A}^*)$ there are vectors*  $\in V_e$ *and linear functionals*  $\in V_e^*$ *such that for all $e \in \mathrm{Vac}(\mathcal{A}^*)$ and $a, c \in \mathrm{Irr}(\mathcal{A}^*)$,*

 $= \delta_{e,\ell_a}\delta_{a,c}(\mathrm{Id}_c \otimes \mathrm{Id}_c)$,

 $= \delta_{e,r_a}\delta_{a,c}(\mathrm{Id}_c \otimes \mathrm{Id}_c)$.

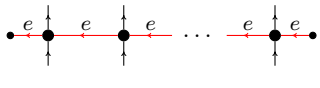
Definition 5.3. We will refer to the subset $\mathrm{Vac}(\mathcal{A}^*)$ of $\mathrm{Irr}(\mathcal{A}^*)$ as *vacuum* and the sectors in $\mathrm{Vac}(\mathcal{A}^*)$ as *vacuum sectors*. In this context, if $\mathrm{Vac}(\mathcal{A}^*)$ contains a single element, it is said that $\mathcal{A}$ is a *coconnected*

pre-bialgebra, if $\mathrm{Vac}(\mathcal{A})$ contains a single element, it is said that it is a *connected* pre-bialgebra, and it is said to be a *biconnected* pre-bialgebra if it is both a connected and coconnected pre-bialgebra.

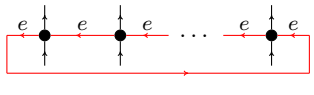
Let us now apply Equation 24 for the case that $a \in \mathrm{Vac}(\mathcal{A}^*)$. As in this case both equations in Equation 24 apply, we obtain that for all $a \in \mathrm{Irr}(\mathcal{A})$ and $e, f \in \mathrm{Vac}(\mathcal{A}^*)$,

$$N_{ef}^{a} = \begin{cases} 1 & \text{if } e = f = a, \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

Stating otherwise, we have obtained that for any $e \in \mathrm{Vac}(\mathcal{A}^*)$, $\ell_e = r_e = e$ holds. Note the difference between the MPO representation of the unit of the algebra and the element $\tau_{\mathrm{Vac}(\mathcal{A}^*)}$. The MPO representation of the unit of the algebra $1 \in \mathcal{A}$ is

$\Phi^{\otimes n} \circ \Delta^{n-1}(1) =$ 

and the MPO representation of the unit of the Grothendieck ring, $\tau_{\mathrm{Vac}(\mathcal{A}^*)}$, is

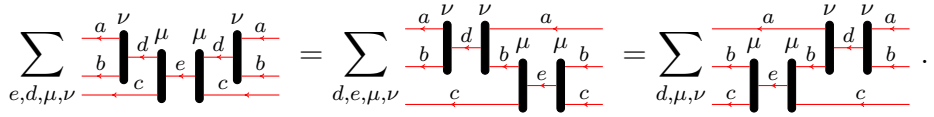
$\Phi^{\otimes n} \circ \Delta^{n-1}(\tau_{\mathrm{Vac}(\mathcal{A}^*)}) =$ 

These two MPOs coincide if and only if the sector e is one-dimensional for all $e \in \mathrm{Vac}(\mathcal{A}^*)$.

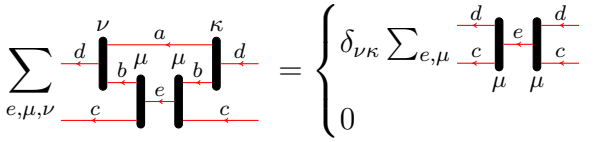
Until now, we have only used the unit axiom, and not the counit axiom: the unit $\tau_{\mathrm{Vac}(\mathcal{A}^*)} \in \mathcal{A}$ of the Grothendieck ring exists and the MPO representation of the unit $1 \in \mathcal{A}$ is rank-one in each injective block even for pre-bialgebras that satisfy the unit axiom but not the counit axiom. Let us now derive a consequence of the counit axiom that we will use later on. First, note that the counit axiom can equally be written as

$$\Delta^2(\varepsilon) = (1 \otimes \Delta(\varepsilon))(\Delta(\varepsilon) \otimes \varepsilon) = (\Delta(\varepsilon) \otimes \varepsilon)(\varepsilon \otimes \Delta(\varepsilon)),$$

and therefore the graphical representation of the counit axiom is



Using now the orthogonality relations in the first equation, we conclude that the following equation holds:



This equation, however, is not a sufficient condition for the unit of the pre-bialgebra to satisfy the counit axiom Equation 17.

2. Weak Hopf algebras

In this section we define weak Hopf algebras. In these algebras we define a set of elements called integrals, and show that a cosemisimple weak Hopf algebra over $\mathbb{C}$ has a special integral Λ that is a projector and is such that $\Lambda(fg) = \Lambda(S^2(g)f)$ holds for all $f, g \in \mathcal{A}^*$.

Definition 5.4. A *weak Hopf algebra* (WHA) is a WBA $\mathcal{A}$ together with a linear map $S \in \mathfrak{B}(\mathcal{A})$, called *antipode*, such that

$$\begin{aligned} S(x_{(1)})x_{(2)} &= 1_{(1)}\varepsilon(x1_{(2)}), \\ x_{(1)}S(x_{(2)}) &= \varepsilon(1_{(1)}x)1_{(2)}, \\ S(x_{(1)})x_{(2)}S(x_{(3)}) &= S(x). \end{aligned}$$

Given a WHA $\mathcal{A}$ with antipode $S_{\mathcal{A}}$, it is easy to check that the map $S_{\mathcal{A}^*} : \mathcal{A}^* \to \mathcal{A}^*$ defined by

$$S_{\mathcal{A}^*}(f) := f \circ S_{\mathcal{A}}, \tag{28}$$

for all $f \in \mathcal{A}^*$ (i.e. $S_{\mathcal{A}^*} = S_{\mathcal{A}}^T$), satisfies the antipode axioms as well, and thus $\mathcal{A}^*$ is a WHA too. From now on, we do not differentiate between the antipode of $\mathcal{A}$ and that of $\mathcal{A}^*$, and denote both by S. The antipode S of a WHA is an anti-homomorphism (i.e. $S(xy) = S(y)S(x)$), an anti-cohomomorphism (i.e. $\Delta \circ S = (S \otimes S) \circ \Delta_{\mathrm{op}}$) and a bijection (see [14] for a proof). In fact, an equivalent characterization of the antipode (of $\mathcal{A}^*$) is that it is a bijective anti-homomorphism of $\mathcal{A}^*$ such that for every $f \in \mathcal{A}^*$

$$S(f_{(1)})f_{(2)} \otimes f_{(3)} = \varepsilon_{(1)} \otimes f\varepsilon_{(2)}, \tag{29}$$

$$f_{(1)} \otimes f_{(2)}S(f_{(3)}) = \varepsilon_{(1)}f \otimes \varepsilon_{(2)}. \tag{30}$$

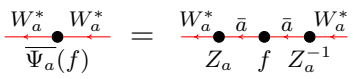
Let $\mathcal{A}$ be a cosemisimple WHA and $\Psi : \mathcal{A}^* \to \mathfrak{B}(W)$ be a representation of $\mathcal{A}^*$ on a vector space W. The antipode S of $\mathcal{A}^*$ is an anti-homomorphism, and thus the linear map $\bar{\Psi} : \mathcal{A}^* \to \mathfrak{B}(W^*)$ defined by

$$\bar{\Psi}(f) = (\Psi \circ S(f))^T$$

is a representation of $\mathcal{A}^*$ on the vector space W^*. As S is a bijection, $\bar{\Psi}$ is an irreducible representation if Ψ is an irreducible representation. Let $\mathrm{Irr}(\mathcal{A}^*)$, as above, denote the irreducible representation equivalence classes of $\mathcal{A}^*$, also called *sectors* of $\mathcal{A}^*$, and for every $a \in \mathrm{Irr}(\mathcal{A}^*)$ let us fix a representation Ψ_a on the vector space V_a from the sector a. For each irreducible representation Ψ_a the representation $\bar{\Psi}_a$ is an irreducible representation in another sector, that we denote by $\bar{a}$ (note that $\bar{a}$ might coincide with a). As $\bar{\Psi}_a$ and $\Psi_{\bar{a}}$ are in the same sector, they are related to each other by a basis transformation, i.e. there are linear maps $Z_a : W_{\bar{a}} \to W_a^*$ such that

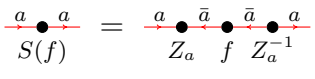
$$\overline{\Psi_a}(f) = Z_a \Psi_{\bar{a}}(f) Z_a^{-1}.$$

Let us denote this equation using the graphical notation of tensor networks. We have to be careful with the notation, as $\overline{\Psi_a}(f)$ is not one of the representations that we have fixed previously. We thus have to use a more verbose notation and display the representation itself and that it is a linear map from W_a^* to W_a^*:



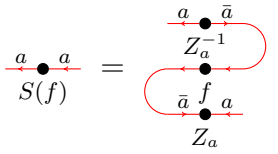
Let us now reverse the two arrows at the two ends of the figure. This changes W_a^* to W_a, and thus, as it is one of the previously fixed vector spaces, we can simply write a on the line. Similarly, $\overline{\Psi_a}(f)$ changes to $\overline{\Psi_a}(f)^T = \Psi_a \circ S(f)$, and thus, as Ψ_a is one of the previously fixed representations, the rank-two tensor on the l.h.s. of the equation can simply be labeled by $S(f)$:

(31)

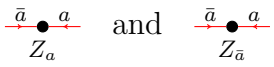


The indices of $S(f)$ are now oriented in the opposite direction as that of f. Because of this, we often bend back the indices on the r.h.s. of this equation such that we obtain

(32)

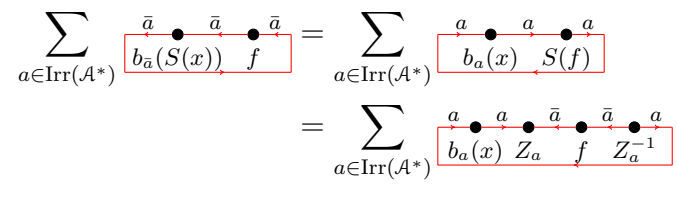


Let us warn here the reader about the subtleties of our notation. First, Z_a has two incoming indices, and thus the only way to differentiate between the two indices is via the labels a and $\bar{a}$. Note that even for sectors for which $a = \bar{a}$, this forces us to write a and $\bar{a}$ on the two legs of the tensor. Second, there are two different operators with very similar graphical representations:

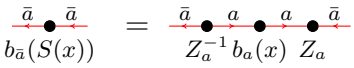


These two tensors should be read in the opposite direction: the first index of Z_a is the one labeled by $\bar{a}$ on its left, while the first index of $Z_{\bar{a}}$ is the one labeled by a on its right.

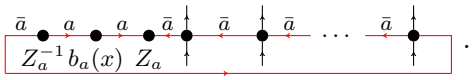
As we have mentioned above, the antipode of $\mathcal{A}$ is the transpose of the antipode of $\mathcal{A}^*$, see Equation 28. Therefore for all $f \in \mathcal{A}^*$ and $x \in \mathcal{A}$ the following holds:



where in the second equation we have used Equation 31. As this equation holds for all $f \in \mathcal{A}^*$, the boundary condition $b(S(x))$ describing $S(x)$ can be expressed as

(33)


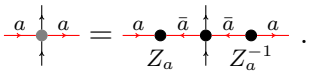
This allows us to write the repeated coproduct of $S(x)$ as

(34)
$$\Phi^{\otimes n} \circ \Delta^{n-1} \circ S(x) = \sum_a$$


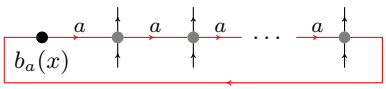
Let us consider now a new rank-4 tensor, denoted by gray dot, given by

$$:= \sum_y (\Phi \circ S)(y) \otimes \Psi_a(\delta_y) = \sum_x \Phi(x) \otimes (\Psi_a \circ S)(\delta_x),$$

where in the second equation we have used that if B is a basis with basis elements y, then the set $\{x = S(y) : y \in B\}$ forms a basis as well and its dual basis is $\{\delta_x = S^{-1}(\delta_y) : y \in B\}$. Note that we have oriented the virtual indices of this tensor in the opposite direction as in the MPO tensor given by black dots. This is because S is an anti-homomorphism; more precisely, we can relate the two tensors using Equation 31:

(35)


Using this gray tensor, one can rewrite Equation 34 as

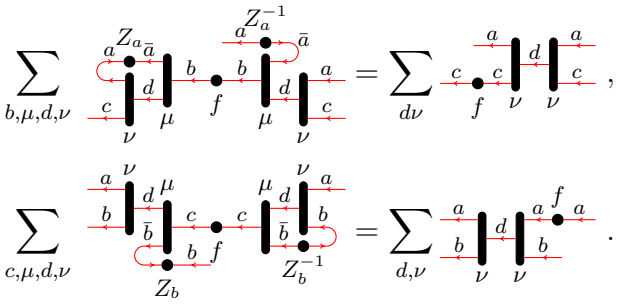
(36)
$$\Phi^{\otimes n} \circ \Delta^{n-1} \circ S(x) = \sum_a$$


This MPO, by definition, is $\Phi^{\otimes n} \circ S^{\otimes n} \circ \Delta_{\text{op}}^{n-1}(x)$ (read from left to right), as the virtual index of the MPO tensor is written in the opposite direction than as usual. As Φ is w.l.o.g. injective, we have obtained the relation

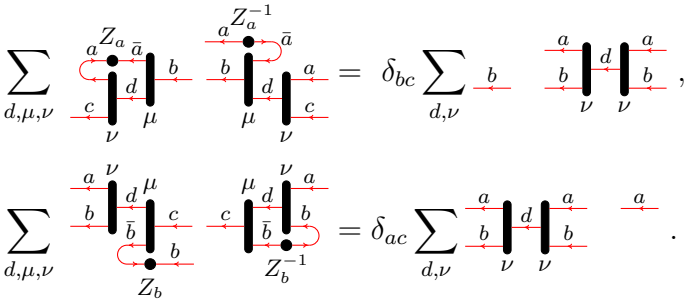
$$\Delta^{n-1} \circ S(x) = S^{\otimes n} \circ \Delta_{\text{op}}^{n-1}(x),$$

that is, in fact, a simple consequence of the fact that S is an anti-cohomomorphism (i.e. $\Delta \circ S = (S \otimes S) \circ \Delta_{\text{op}}$).

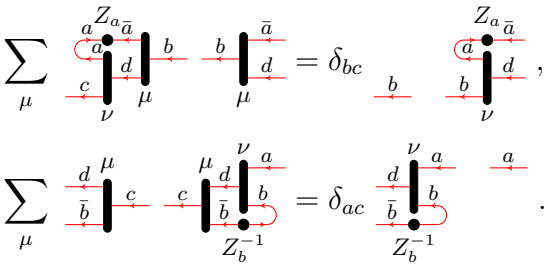
Using the tensors Z_a that we defined in Equation 31, the antipode axioms of $\mathcal{A}^*$, Equations 29 and 30, can be expressed as



Let us explain briefly how to arrive at these graphical representations of Equation 29 and Equation 30. In Equation 29, on the l.h.s. we first need to take two repeated coproduct of f: this is achieved by the two pairs of fusion tensors (see Equation 5). Then the antipode of the first component is taken (see Equation 32) and it is multiplied with the second component of the coproduct. On the r.h.s. of the same equation, we multiply f with the second component of $\Delta(\varepsilon)$ (see Equation 6). The second equation is obtained similarly. As these equations hold for any $f \in \mathcal{A}^*$, we conclude that

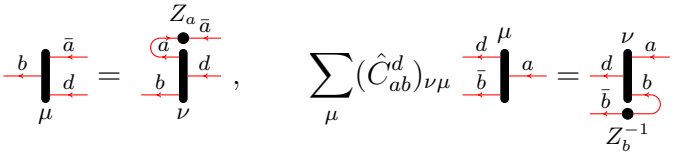


Using the orthogonality relations of the fusion tensors and multiplying the first equation by Z_a, and the second equation by Z_b^{-1}, we obtain



In particular, there are complex matrices C_{ab}^d of size $N_{ab}^d \times N_{\bar{a}d}^b$ and $\hat{C}_{ab}^d$ of size $N_{ab}^d \times N_{d\bar{b}}^a$ such that

(37)

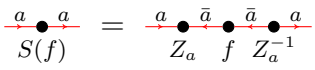


Let us now show that C_{ab}^d and $\hat{C}_{ab}^d$ are both invertible. In both equations the tensors on both sides are linearly independent, therefore $N_{ab}^d \leq N_{\bar{a}d}^b$ and $N_{ab}^d \leq N_{d\bar{b}}^a$. Applying these inequalities twice, we obtain that $N_{ab}^d \leq N_{\bar{a}d}^b \leq N_{\bar{a}\bar{b}}^d$, and that $N_{ab}^d \leq N_{d\bar{b}}^d \leq N_{a\bar{\bar{b}}}^d$. As $a \mapsto \bar{\bar{a}}$ is a permutation of $\mathrm{Irr}(\mathcal{A}^*)$, repeating these equations a few[2] times we obtain that $N_{ab}^d \leq N_{\bar{a}\bar{b}}^d \leq \cdots \leq N_{ab}^d$ and that $N_{ab}^d \leq N_{a\bar{\bar{b}}}^d \leq \cdots \leq N_{ab}^d$, i.e. in all of the inequalities above equality holds. In particular, for all $a, b, c \in \mathrm{Irr}(\mathcal{A}^*)$, the numbers N_{ab}^d possess the following symmetries:

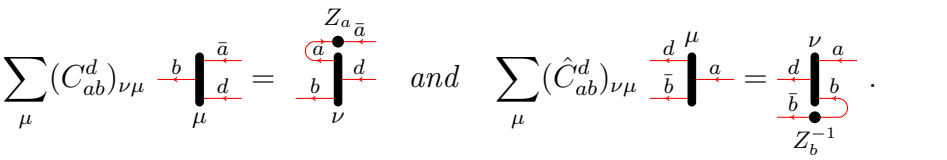
$$N_{ab}^d = N_{\bar{a}d}^b \quad \text{and} \quad N_{ab}^d = N_{d\bar{b}}^a.$$

This implies that in both equations in Equation 37 we transform N_{ab}^d linearly independent vectors to N_{ab}^d linearly independent vectors, i.e. both C_{ab}^d and $\hat{C}_{ab}^d$ are invertible. It is easy to check that the argument above can be repeated backwards, i.e. Equation 37 implies the antipode axioms. We have thus seen that

Proposition 5.5. *Let $\mathcal{A}$ be a finite dimensional cosemisimple WBA. Then the map $S : \mathcal{A}^* \to \mathcal{A}^*$ given by*



is an antipode making $\mathcal{A}^*$ a WHA if and only if there are invertible matrices C_{ab}^d of size $N_{ab}^d \times N_{\bar{a}d}^b$ and $\hat{C}_{ab}^d$ of size $N_{ab}^d \times N_{d\bar{b}}^a$ such that



As it is interesting on its own, let us restate here the symmetry properties of the fusion multiplicities N_{ab}^c together with some of the consequences of these symmetries:

Proposition 5.6. *Let $\mathcal{A}$ be a finite dimensional cosemisimple WHA, and $a, b, c \in \mathrm{Irr}(\mathcal{A}^*)$. Then*

$$N_{ab}^d = N_{\bar{a}d}^b \quad \text{and} \quad N_{ab}^d = N_{d\bar{b}}^a. \tag{38}$$

Moreover, the following equations hold:

- $\bar{\bar{a}} = a$ *for all* $a \in \mathrm{Irr}(\mathcal{A}^*)$,
- $\ell_a = r_{\bar{a}}$ *and* $r_a = \ell_{\bar{a}}$ *for all* $a \in \mathrm{Irr}(\mathcal{A}^*)$, *where* $\ell_a, r_a \in \mathrm{Vac}(\mathcal{A}^*)$ *are as in Equation 24,*
- $e = \bar{e}$ *for all* $e \in \mathrm{Vac}(\mathcal{A}^*)$.

Proof. We have already proven Equation 38 above. Let us now show that $a = \bar{\bar{a}}$ for all sectors $a \in \mathrm{Irr}(\mathcal{A}^*)$. Applying the left side of Equation 38 twice with $d = a$ and $b = r_a$, we obtain that

$$1 = N_{ar_a}^a = N_{\bar{a}a}^{r_a} = N_{\bar{\bar{a}}r_a}^a.$$

[2] the order of the permutation $a \mapsto \bar{\bar{a}}$

Using now Equation 24, this latter fusion multiplicity can only be non-zero if $a = \bar{\bar{a}}$.

Let us now show that for all $a \in \mathrm{Irr}(\mathcal{A})$, $\ell_{\bar{a}} = r_a$ holds. Using the left equation in Equation 38 with $b = r_a$ and $d = a$, we obtain that

$$1 = N_{ar_a}^{a} = N_{\bar{a}a}^{r_a} = N_{r_a\bar{a}}^{\bar{a}},$$

where the last equation is the right equation in Equation 38. As $r_a \in \mathrm{Vac}(\mathcal{A}^*)$ and the above fusion number is non-zero, Equation 24 implies that for all $a \in \mathrm{Irr}(\mathcal{A})$, $r_a = \ell_{\bar{a}}$. The equation $r_{\bar{a}} = \ell_a$ can be shown in a similar way.

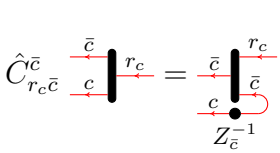
Let us finally prove that the permutation $a \mapsto \bar{a}$ leaves the set $\mathrm{Vac}(\mathcal{A}^*)$ invariant, and in fact for every $e \in \mathrm{Vac}(\mathcal{A}^*)$, $\bar{e} = e$. Let us use Equation 38 with $a = b = d = e \in \mathrm{Vac}(\mathcal{A}^*)$. We obtain that

$$1 = N_{ee}^{e} = N_{\bar{e}e}^{e},$$

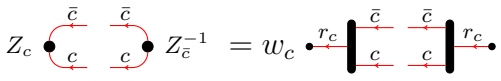
and thus, as $e \in \mathrm{Vac}(\mathcal{A}^*)$, using Equation 24, this latter fusion number can only be non-zero if $e = \bar{e}$. □

Let us show now that in a cosemisimple WHA the matrices Z_a can be expressed with the help of the fusion tensors and the unit of the underlying WBA. Therefore, if in a cosemisimple WBA there exists an antipode making it a WHA, this antipode is uniquely determined by the WBA structure. This statement, in fact, is more general than what we show here, and also holds in the non-cosemisimple case (see e.g. [**14**]).

Let $c \in \mathrm{Irr}(\mathcal{A}^*)$ and let us set $\bar{a} = \bar{c}$, $d = c$ and $b = r_c \in \mathrm{Vac}(\mathcal{A}^*)$ in the left equation in Equation 37, and $d = \bar{c}$, $b = \bar{c}$ and $a = r_c \in \mathrm{Vac}(\mathcal{A}^*)$ in the right equation in Equation 37 and use that $\bar{\bar{c}} = c$. As $N_{\bar{c}c}^{r_c} = 1$, we obtain that there are non-zero complex numbers $C_{cr_c}^{c}$ and $\hat{C}_{r_c\bar{c}}^{\bar{c}}$ such that

$C_{cr_c}^{c}$  and 

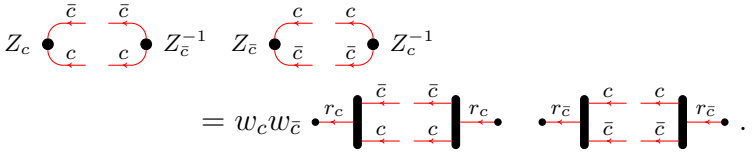
Taking the tensor product of these two equations, applying the vectors describing $b_{r_c}(1)$ and using Equation 25, we conclude that there are non-zero numbers w_c such that

(39) 

We have thus obtained that the matrices Z_c can be expressed with the help of the fusion tensors and the vectors describing the unit of the WBA, and thus the antipode of a cosemisimple WHA is unique and completely determined by the WBA structure.

Let us now define a linear functional that will play a central role in the rest of the dissertation. Before the definition, note that while

the number w_c defined in Equation 39 depends on the concrete choice[3] of the matrices Z_c and $Z_{\bar{c}}^{-1}$, the number $w_c w_{\bar{c}}$ is independent from this choice, and it is also independent from all the choices we have made before (the fusion tensors and the vectors describing the unit of the algebra). This is because all these objects appear in pairs in the following equation:

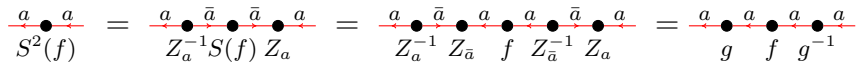


Let d_a be complex numbers such that $d_a^2 = w_a w_{\bar{a}}$ and $d_a = d_{\bar{a}}$. Later we will prove that $w_a w_{\bar{a}} > 0$ and thus one can choose d_a to be positive. Until then, $d_a = d_{\bar{a}}$ is an arbitrary square root of $w_a w_{\bar{a}}$. Let us now define the linear functional $g \in \mathcal{A}^*$ through

(40)

where we have used that for any matrix m of the form $\bigoplus_a m_a$, $m_a \in \mathfrak{B}(V_a)$, one can find a linear functional $f \in \mathcal{A}^*$ such that $\Psi_a(f) = m_a$. The linear functional g defined in this way is independent of all previous choices that we have made, except for the choice of the square root $d_a = d_{\bar{a}}$ of d_a^2. The importance of this linear functional is that the conjugation by it describes the action of S^2. Indeed, applying Equation 31 twice, we obtain that

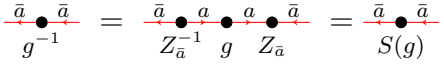
(41)



Let us now show a few additional properties of g. First, note that

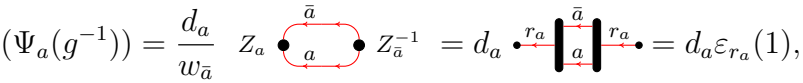
$$\frac{w_{\bar{a}}}{d_a} = \frac{d_a}{w_a} = \frac{d_{\bar{a}}}{w_a},$$

and thus

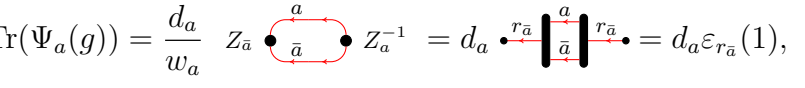


i.e. g satisfies $g^{-1} = S(g)$. Second, the traces of $\Psi_a(g)$ and $\Psi_a(g^{-1})$ evaluate to the following:

(42)



(43)



where $\varepsilon_{r_a} \in \mathcal{A}^*$ is the irreducible representation projector onto the sector r_a and $\varepsilon_{r_{\bar{a}}} \in \mathcal{A}^*$ is the irreducible representation projector onto the sector $r_{\bar{a}}$.

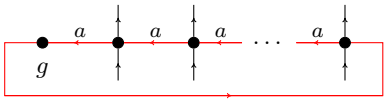
[3] We could have chosen λZ_c to describe the antipode instead of Z_c, for any $\lambda \in \mathbb{C}$. This is in fact the only freedom that we have in the choice of Z_c.

In the following we will encounter algebra elements of the form

$$x = \sum_a (\mathrm{Id} \otimes g) \circ \Delta(\tau_a).$$

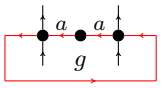
These algebra elements have the following MPO representation:

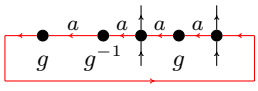
$$\Phi^{\otimes n} \circ \Delta^{n-1}(x) = \sum_a \lambda_a$$



The algebra elements are exactly those that satisfy $\Delta_{\mathrm{op}}(x) = (S^{-1} \otimes \mathrm{Id}) \circ \Delta(x)$. Indeed,

$$\begin{aligned} \Phi^{\otimes 2} \circ \Delta_{\mathrm{op}}(x) &= \sum_a \lambda_a \\ &= \sum_a \lambda_a \\ &= \Phi^{\otimes 2} \circ (S^{-2} \otimes \mathrm{Id}) \circ \Delta(x). \end{aligned} \tag{44}$$





One can easily see that all elements that satisfy the above cocommutation relation are in fact of this form. Such algebra elements will be called *q-traces*:

Definition 5.7. Let $\mathcal{A}$ be a WHA. The algebra element $x \in \mathcal{A}$ is called a *q-trace* if

$$\Delta_{\mathrm{op}}(x) = (S^{-2} \otimes \mathrm{Id}) \circ \Delta(x).$$

In the following we will construct an algebra element that is not only a q-trace, but it is also a *left integral*:

Definition 5.8. A *left integral* of a WHA $\mathcal{A}$ is an element $\Lambda \in \mathcal{A}$ such that for all $x \in \mathcal{A}$

$$(1 \otimes x)\Delta(\Lambda) = (S(x) \otimes 1)\Delta(\Lambda).$$

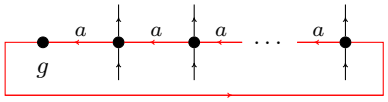
The element $\Lambda \in \mathcal{A}$ is called a *right integral* if $S(\Lambda)$ is a left integral. An integral that is both left and right is a *two-sided integral*. A *non-degenerate* (see Definition 3.7) *integral* Λ is *normalized* if $\Lambda^2 = \Lambda$. Finally, an integral is called a *Haar integral* if it is two-sided, non-degenerate and normalized.

We will now prove the main result of this section: that in a co-semisimple WHA there is a special integral that is non-degenerate, normalized and a q-trace:

THEOREM 5.9. *In a finite dimensional cosemisimple WHA $\mathcal{A}$ over $\mathbb{C}$ the element $\Lambda \in \mathcal{A}$ defined*[4] *by*

(45)

$$\Phi^{\otimes n} \circ \Delta^{n-1}(\Lambda) = \sum_{a \in \mathrm{Irr}(\mathcal{A}^*)} \frac{d_a}{\sum_{x:\, \ell_x = \ell_a} d_x^2}$$

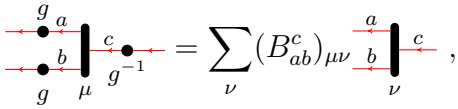


is a non-degenerate normalized left integral that is also a q-trace.

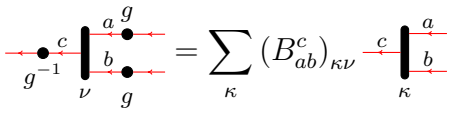
In the proof of Theorem 5.9 we will use the following two lemmas that we prove in Appendix A:

Lemma 5.10. *Let $\mathcal{A}$ be a cosemisimple WHA, and g be the linear functional defined in Equation 40. Then there exists an $N_{ab}^c \times N_{ab}^c$ matrix B_{ab}^c such that the linear functional $g \in \mathcal{A}^*$ satisfies*

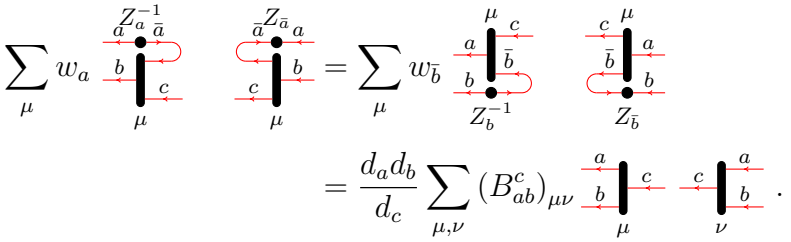
(46a)



(46b)



Moreover, $(B_{ab}^c)^2 = \mathrm{Id}_{N_{ab}^c}$ and the following equation holds as well:



Investigating the trace of B_{ab}^c we can prove that the numbers d_a^2 are positive and thus that the denominator in the r.h.s. of Equation 45 is non-zero. We postpone the proof of this lemma to Appendix A as well.

Lemma 5.11. *For all $a \in \mathrm{Irr}(\mathcal{A}^*)$, $d_a^2 = w_a w_{\bar{a}} > 0$. Let moreover T_{ab}^c be defined by $T_{ab}^c = \sum_\mu (B_{ab}^c)_{\mu\mu}$. Then the following equations hold:*

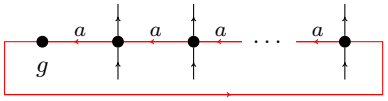
$$\sum_b T_{ab}^c d_b = d_a \delta_{\ell_a \ell_c} d_c \quad \text{and} \quad \sum_{x:\ell_x = \ell_a} d_x^2 = \sum_{x:\ell_x = r_a} d_x^2.$$

With these two lemmas in hand, we can proceed to the proof of Theorem 5.9.

[4] Remember that Φ is w.l.o.g. injective and thus the value of $\Phi^{\otimes n} \circ \Delta^{n-1}(\Lambda)$ defines Λ uniquely.

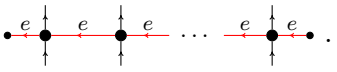
Proof of Theorem 5.9. Let us define L as

$$\Phi^{\otimes n} \circ \Delta^{n-1}(L) = \sum_{a \in \mathrm{Irr}(\mathcal{A}^*)} d_a$$



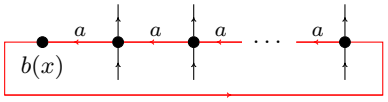
We will prove that L is a non-degenerate left integral that is related to Λ as follows. Let us define N via

$$\Phi^{\otimes n} \circ \Delta^{n-1}(N) = \sum_{e \in \mathrm{Vac}(\mathcal{A}^*)} \sum_{x:\, \ell_x = e} d_x^2 \tag{47}$$



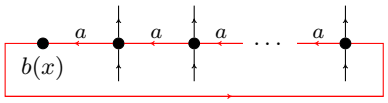
This algebra element N is central, as Nx is described by the MPO

$$\Phi^{\otimes n} \circ \Delta^{n-1}(Nx) = \sum_{a \in \mathrm{Irr}(\mathcal{A}^*)} \sum_{x:\, \ell_x = \ell_a} d_x^2$$



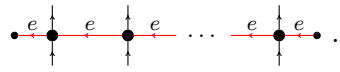
while xN is described by the MPO

$$\Phi^{\otimes n} \circ \Delta^{n-1}(xN) = \sum_{a \in \mathrm{Irr}(\mathcal{A}^*)} \sum_{x:\, \ell_x = r_a} d_x^2$$



and, by Lemma 5.11, $\sum_{x:\ell_x=\ell_a} d_x^2 = \sum_{x:\ell_x=r_a} d_x^2$. Note as well that, as $d_x^2 > 0$ for all x (see again Lemma 5.11), N is invertible with inverse

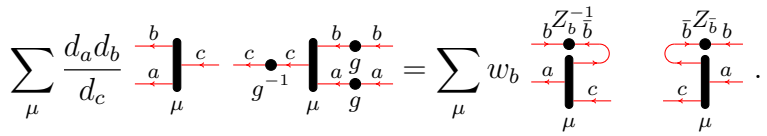
$$\Phi^{\otimes n} \circ \Delta^{n-1}(N^{-1}) = \sum_{e \in \mathrm{Vac}(\mathcal{A}^*)} \frac{1}{\sum_{x:\, \ell_x = e} d_x^2}$$



In fact, $\Lambda = N^{-1}L = LN^{-1}$. Therefore, in order to prove that Λ is a normalized left integral, we only have to prove that L is a left integral, and that $L^2 = NL$. Note here that Λ is of the form Equation 44, and thus it is a q-trace. It is also non-degenerate as g is invertible and the normalization constant is non-zero.

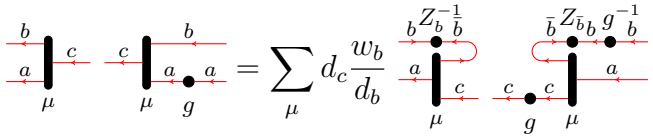
Let us prove now that L is a left integral. Let us start with the following simple corollary of Lemma 5.10:

$$\sum_{\mu} \frac{d_a d_b}{d_c}$$

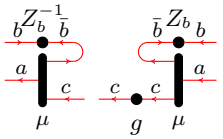


Let us rearrange this equation to obtain

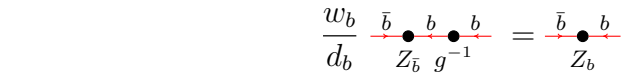
$$\sum_{\mu} d_a$$



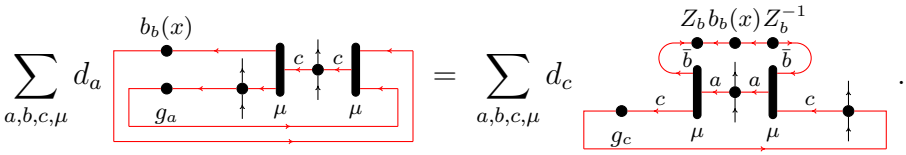
$$= \sum_{\mu} d_c$$



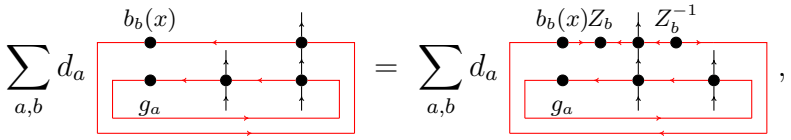
where in the second equality we have used that



This implies that for all $x \in \mathcal{A}$,



We can now use Equation 9 to obtain



or equivalently, using that the representation Φ is w.l.o.g. injective, that $(1 \otimes x)\Delta(L) = (S(x) \otimes 1)\Delta(L)$, i.e. that L is a left integral. This integral is automatically non-degenerate (see Chapter 3) as $b_a(L) = d_a \Psi_a(g)$ is invertible for all $a \in \mathrm{Irr}(\mathcal{A}^*)$.

Let us now show that $L^2 = NL$, and thus that Λ is a normalized integral. The matrices $b_c(L^2)$ describing L^2 are of the form (see Equation 11)

$$
\begin{aligned}
b_c(L^2) &= \sum_{a,b,\mu} d_a d_b \quad \text{(diagram)} \\
&= \sum_{a,b} d_a T^c_{ab} d_b \quad \text{(diagram)} \\
&= \sum_{a:\, \ell_a = \ell_c} d_a^2 \quad \text{(diagram)} = b_c(NL) \ ,
\end{aligned}
$$

where in the second equality we have used the definition of T^c_{ab}, and in the third, the last point of Lemma 5.11. □

A few remarks are in place. First, it is known that every cosemisimple WHA is finite dimensional (see [96] and Theorem 3.13 in [14]), and thus the assumption on $\mathcal{A}$ being finite dimensional is redundant. Second, the normalized integral Λ provides a separability element $I \in \mathcal{A} \otimes \mathcal{A}$ via $I = (S^{-1} \otimes \mathrm{Id}) \circ \Delta(\Lambda)$ for the algebra $\mathcal{A}$. Therefore $\mathcal{A}$ is separable, and in particular, it is semisimple, i.e. we have obtained Theorem 2.26 of [41]. Finally, let us note that Λ is closely related to the canonical left integral $\hat{L}$ defined in [96] (see also [14] and [41]). In [96], the canonical left integral is defined as

$$\hat{L}(f) = \sum_{x \in B} x(f S^2(\delta_x)),$$

where B is a basis of $\mathcal{A}$ and δ_x denotes the dual basis. In a finite dimensional cosemisimple WHA, this expression can be re-expressed using the matrix $b(x)$:

$$\hat{L}(f) = \sum_{x\in B} \sum_{a\in \mathrm{Irr}(\mathcal{A}^*)} \mathrm{Tr}(b_a(x)\Psi_a(f)\Psi_a(S^2(\delta_x)))$$

Using now Equation 41, we can also write

$$\hat{L}(f) = \sum_{a\in \mathrm{Irr}(\mathcal{A}^*)} \sum_{x\in B} \mathrm{Tr}(b_a(x)\Psi_a(f)\Psi_a(g)\Psi_a(\delta_x)\Psi_a(g^{-1}))$$

As we sum over x, the matrix $\Psi_a(\delta_x)$ takes all possible values in $\mathfrak{B}(W_a)$. Conversely, iterating over elements of $\bigoplus_{a\in\mathrm{Irr}(\mathcal{A}^*)} \mathfrak{B}(V_a)$ defines a basis in $\mathcal{A}^*$ through the representation $\Psi = \bigoplus_a \Psi_a$. Let us fix a basis $|ai\rangle$ in each W_a, $i = 1, \ldots, \dim(W_a)$, and choose the basis B such that for all x, there is $a \in \mathrm{Irr}(\mathcal{A}^*)$ and $i, j = 1, \ldots, \dim(W_a)$ such that $\Psi(\delta_x) = |ai\rangle\langle aj|$. It is easy to see that then $b_a(x) = |aj\rangle\langle ai|$. Therefore one can also write

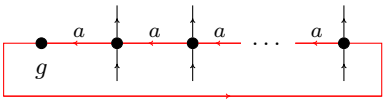
$$\begin{aligned}\hat{L}(f) &= \sum_{a\in \mathrm{Irr}(\mathcal{A}^*)} \sum_{i,j} \mathrm{Tr}(|aj\rangle\langle ai|\Psi_a(fg)|ai\rangle\langle aj|\Psi_a(g^{-1})) \\ &= \sum_{a\in \mathrm{Irr}(\mathcal{A}^*)} \sum_{i} \langle ai|\Psi_a(fg)|ai\rangle \sum_{j} \langle aj|\Psi_a(g^{-1})|aj\rangle.\end{aligned}$$

Using the trace formula Equation 42, $\mathrm{Tr}(\Psi_a(g^{-1})) = d_a\varepsilon_{r_a}(1)$, $\hat{L}(f)$ can be further written as

$$\hat{L}(f) = \sum_{a\in \mathrm{Irr}(\mathcal{A}^*)} d_a\varepsilon_{r_a}(1)\,\mathrm{Tr}(\Psi_a(gf)),$$

i.e. $\hat{L}$ is described by the boundary $\varepsilon_{r_a}(1)d_a\Psi_a(g)$, or equivalently, the MPO representation of $\hat{L}$ is given by

$$\Phi^{\otimes n} \circ \Delta^{n-1}(\hat{L}) = \sum_{a\in \mathrm{Irr}(\mathcal{A}^*)} d_a\varepsilon_{r_a}(1)$$



.

3. Pivotal weak Hopf algebras

In certain WHAs one can define a special cocentral element using the integral Λ defined in the previous section. This element satisfies an equation reminiscent to the definition of an integral, but here, instead of the antipode, another operation appears. The WHAs where the construction works are called *pivotal* WHAs, and their defining property is that the antipode is special: the square of the antipode is an inner automorphism of the algebra realized by a *group-like* element (see the definition below). While such a structure seems to be restrictive, it is conjectured that any semisimple WHA is actually pivotal [**41**].

Definition 5.12. Let $\mathcal{A}$ be a pre-bialgebra. $G \in \mathcal{A}$ is *group-like* if it is invertible and

$$\Delta(G) = (G \otimes G)\Delta(1) = \Delta(1)(G \otimes G).$$

Group-like elements of a WHA $\mathcal{A}$ form a group: if $G, H \in \mathcal{A}$ are group-like, then GH is also group-like; $1 \in \mathcal{A}$ is also group-like, and finally, if $G \in \mathcal{A}$, then G^{-1} is group-like as well. In a WHA

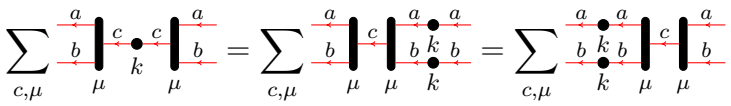
$$S(G) = G^{-1} \tag{48}$$

for any group-like G [14].

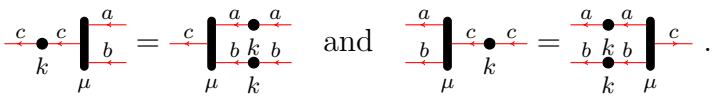
$\mathcal{A}^*$ is a pre-bialgebra if and only if $\mathcal{A}$ is a pre-bialgebra; the unit of $\mathcal{A}^*$ is ε, and thus a linear functional $k \in \mathcal{A}^*$ is group-like if it is invertible and

$$\Delta(k) = (k \otimes k)\Delta(\varepsilon) = \Delta(\varepsilon)(k \otimes k).$$

In a cosemisimple WHA $\mathcal{A}$, group-like elements of $\mathcal{A}^*$ have a nice characterization using the graphical representation of the fusion tensors. An element $k \in \mathcal{A}^*$ is group-like if and only if it is invertible and



Using now the orthogonality relations of the fusion tensors, we conclude that k is group-like if and only if for all $a, b, c \in \mathrm{Irr}(\mathcal{A}^*)$ and $\mu = 1, \ldots, N_{ab}^c$,



In the previous section (Lemma 5.11) we have seen a group-like element: the element $g^2 \in \mathcal{A}^*$ is group-like, where g is defined by Equation 40. Conjugation by this element describes S^4. A *pivotal* WHA is one where not only the fourth power, but also the square of the antipode is realized as a conjugation by a group-like element:

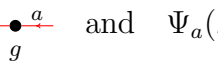
Definition 5.13. A WHA $\mathcal{A}$ is *pivotal* if there is a group-like element G such that for all $x \in \mathcal{A}$,

$$S^2(x) = GxG^{-1}.$$

Such a group-like element G is called a *pivotal element* of $\mathcal{A}$.

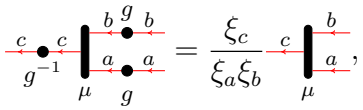
If a WHA $\mathcal{A}$ is such that $\mathcal{A}^*$ is pivotal, then, as S^2 is an inner automorphism of $\mathcal{A}^*$ described by the linear functional g defined in Equation 40, all pivotal elements k of $\mathcal{A}^*$ are of the form $k = \xi g$ for some $\xi \in \mathcal{A}^*$ central element. That is, there are numbers ξ_a such that for all a,

$$\Psi_a(k) = \xi_a \quad \text{and} \quad \Psi_a(k^{-1}) = \frac{1}{\xi_a} \tag{49}$$

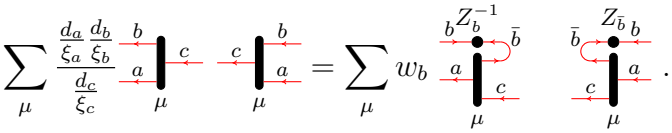


As $S(g) = g^{-1}$ and $S(k) = k^{-1}$, $\xi_{\bar{a}} = 1/\xi_a$. As g^2 is group-like (see Lemma 5.10), the numbers ξ_a satisfy $(\xi_a\xi_b/\xi_c)^2 = 1$ for all a, b, c such that $N_{ab}^c \neq 0$.

Let us consider a WHA $\mathcal{A}$ such that $\mathcal{A}^*$ is pivotal and let us fix the numbers ξ_a such that the element $k \in \mathcal{A}^*$ defined by $k_a = \xi_a g_a$ is pivotal. In this case, as



Lemma 5.10 takes the form

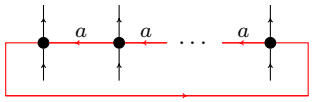
(50)


Let us now notice that the matrix B_{ab}^c defined in Lemma 5.11 is proportional to the identity, more precisely, $B_{ab}^c = \xi_c/(\xi_a\xi_b)\mathrm{Id}_{N_{ab}^c}$, and thus that $T_{ab}^c = N_{ab}^c\xi_c/(\xi_a\xi_b)$. We thus obtain that

$$\sum_b N_{ab}^c \frac{d_b}{\xi_b} = \delta_{\ell_a\ell_c}\frac{d_{\bar{a}}}{\xi_{\bar{a}}}\frac{d_c}{\xi_c}. \tag{51}$$

With these statements we can now prove the following theorem:

THEOREM 5.14. *Let $\mathcal{A}$ be a finite dimensional WHA over $\mathbb{C}$ such that $\mathcal{A}^*$ is semisimple and pivotal with pivotal element $k \in \mathcal{A}^*$ such that $k_a = \xi_a g_a$. Then the element $\Omega \in \mathcal{A}$ defined by*

$$\Phi^{\otimes n} \circ \Delta^{n-1}(\Omega) = \sum_{a\in \mathrm{Irr}(\mathcal{A}^*)} \frac{1}{\sum_{x:\,\ell_x=\ell_a} d_x^2}\frac{d_a}{\xi_a}$$


is non-degenerate and cocommutative, it is a projector, and there exists a linear map $T \in \mathfrak{B}(\mathcal{A})$ such that

$$(1 \otimes x)\Delta(\Omega) = (T(x) \otimes 1)\Delta(\Omega).$$

This[5] *map T moreover satisfies*

$$\Delta \circ T = (T \otimes k^{-1} \otimes T) \circ \Delta_{\mathrm{op}}^2.$$

The number d_a/ξ_a is the *quantum dimension* of the sector a. This number depends on the concrete choice of k (equivalently, on the choice of the numbers ξ_a). Before proceeding to the proof, note that $\Omega = (g^{-1} \otimes \mathrm{Id}) \circ \Delta(\Lambda)$, where Λ is defined in Equation 45. From this, using that g is group-like and the properties of Λ, simple (but tedious) algebraic calculations show all the desired properties. In particular, one can obtain that $T = (S \otimes g) \circ \Delta$. Instead of this algebraic calculation, below we provide a proof based on the graphical notation.

[5] As Ω is non-degenerate, T is uniquely defined, see Chapter 6.

PROOF OF THEOREM 5.14. By definition, $\Omega \in \mathcal{A}$ is cocommutative and as for all $a \in \mathrm{Irr}(\mathcal{A}^*)$, $d_a \neq 0$, it is also non-degenerate. Let us now show that Ω is a projection. As in the proof of Theorem 5.9, note that Ω can be written as $N^{-1}\hat{\Omega} = \hat{\Omega}N^{-1}$, where $\hat{\Omega}$ is given by

$$\hat{\Omega} = \sum_{a \in \mathrm{Irr}(\mathcal{A}^*)} \frac{d_a}{\xi_a} \tau_a, \tag{52}$$

and N is given by Equation 47. Then Ω is a projector if and only if $\hat{\Omega}$ satisfies $\hat{\Omega}^2 = N\hat{\Omega}$. Let us calculate $\hat{\Omega}^2$:

$$\hat{\Omega}^2 = \sum_{a,b} \frac{d_a d_b}{\xi_a \xi_b} \tau_a \tau_b = \sum_{a,b,c} \frac{d_a d_b}{\xi_a \xi_b} N_{ab}^c \tau_c = \sum_{a,c} \frac{d_a}{\xi_a} \left(\sum_b N_{ab}^c \frac{d_b}{\xi_b} \right) \tau_c,$$

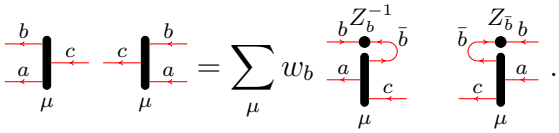
and thus using Equation 51, we obtain that

$$\hat{\Omega}^2 = \sum_{a,c} \delta_{\ell_a \ell_c} \frac{d_a d_{\bar{a}}}{\xi_a \xi_{\bar{a}}} \frac{d_c}{\xi_c} \tau_c = \sum_c \left(\sum_{a : \ell_a = \ell_c} d_a^2 \right) \frac{d_c}{\xi_c} \tau_c = N\hat{\Omega},$$

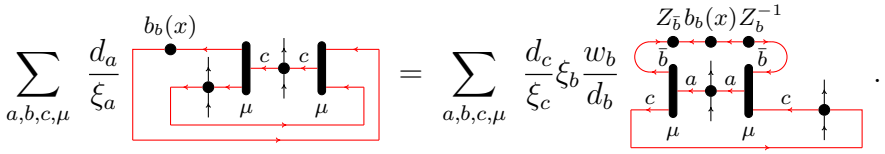
where in the second equation we have used that $\xi_{\bar{a}} = \xi_a^{-1}$ and that $d_a = d_{\bar{a}}$. We have thus obtained that $\Omega = N^{-1}\hat{\Omega}$ is a projector.

Let us now check that $(1 \otimes x)\Delta(\Omega) = (T(x) \otimes 1)\Delta(\Omega)$ holds. Similar to the proof in Theorem 5.9, we start with Equation 50:

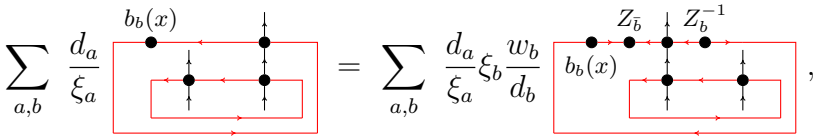
$$\sum_\mu \frac{(d_a/\xi_a)(d_b/\xi_b)}{d_c/\xi_c}$$



This implies that for all $x \in \mathcal{A}$,

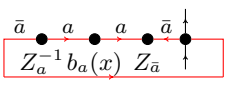


We can now use Equation 9 to obtain



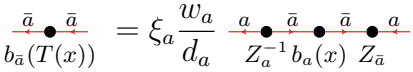
or $(1 \otimes x)\Delta(\Omega) = (T(x) \otimes 1)\Delta(\Omega)$, where T is defined by the formula

$$\Phi(T(x)) = \sum_a \xi_a \frac{w_a}{d_a}$$

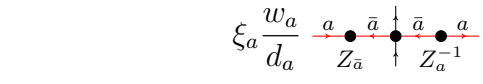


(53)

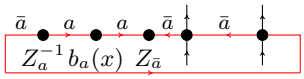
This means, in particular, that

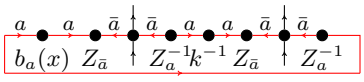


and

$$\xi_a \frac{w_a}{d_a}$$  $$= \sum_x (\Phi \circ T)(x) \otimes \Psi_a(\delta_x) \ .$$

Let us now consider the MPO representation of $\Delta \circ T(x)$ and, using Equation 40, observe that

$$\sum_a \xi_a \frac{w_a}{d_a}$$ 

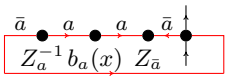
$$= \sum_a \left(\xi_a \frac{w_a}{d_a} \right)^2$$  ,

or, as Φ is w.l.o.g. injective (notice that on the r.h.s. the arrows on the virtual index are reversed),

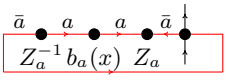
$$T \circ \Delta(x) = (T \otimes k^{-1} \otimes T) \circ \Delta^2_{\mathrm{op}}(x).$$

□

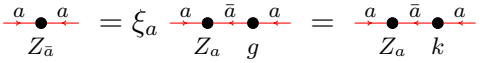
Note that the graphical representation of the action of T, Equation 53, is very similar to that of S:

$$\Phi(T(x)) = \sum_a \xi_a$$ 

and

$$\Phi(S(x)) = \sum_a$$  .

The difference between $T(x)$ and $S(x)$ is that in $T(x)$ the matrix $\xi_a Z_{\bar{a}}$ appears instead of Z_a. As

$$\xi_a$$ 

we directly obtain that $T(x) = (k^{-1} \otimes \mathrm{Id}) \circ \Delta(S(x))$, or, by using that S is an anti-homomorphism and that $k^{-1} \circ S = k$, that

$$T(x) = (S \otimes k) \circ \Delta(x). \tag{54}$$

for all $x \in \mathcal{A}$. The inverse of this relation, as it can be seen from Equation 53, is $S(x) = (k^{-1} \otimes \mathrm{Id}) \circ \Delta \circ T(x)$, or $S(x) = (T \otimes k^{-1}) \circ \Delta(x)$.

4. Spherical weak Hopf algebras

In this section we define spherical weak Hopf algebras as pivotal weak Hopf algebras satisfying an additional property. Then, semisimple spherical weak Hopf algebras are such that their representation category is a spherical multi-fusion category. We specialize Theorem 5.14 to the case where $\mathcal{A}$ is not only pivotal, but also spherical.

Definition 5.15. A pivotal WHA $\mathcal{A}$ is called spherical if the following two conditions hold. First, that for any sector δ from the vacuum, $\varepsilon(1_\delta) \neq 0$, where 1_δ is the projector onto the sector δ. Second, it has a pivotal element G such that for all $\alpha \in \mathrm{Irr}(\mathcal{A})$,

$$\frac{t_\alpha(G)}{\varepsilon(1_{\lambda_\alpha})} = \frac{t_\alpha(G^{-1})}{\varepsilon(1_{\rho_\alpha})},$$

where t_α is the irreducible representation character of the sector α, λ_α is the unique irreducible representation from the vacuum such that $t_{\lambda_\alpha} t_\alpha \neq 0$, and ρ_α is the unique irreducible representation from the vacuum such that $t_\alpha t_{\rho_\alpha} \neq 0$. Such a pivotal element is called a spherical element of $\mathcal{A}$.

Let us now consider a WHA over $\mathbb{C}$ such that $\mathcal{A}^*$ is semisimple and spherical. Let k be a spherical element of $\mathcal{A}^*$ with

$$\Psi_a(k) = \xi_a \quad \text{and} \quad \Psi_a(k^{-1}) = \frac{1}{\xi_a}$$

.

The traces of $\Psi_a(k)$ and $\Psi_a(k^{-1})$ can be then evaluated using Equations 42 and 43:

$$\tau_a(k^{-1}) = \mathrm{Tr}(\Psi_a(k^{-1})) = \frac{1}{\xi_a}\,\mathrm{Tr}(\Psi_a(g^{-1})) = \frac{d_a}{\xi_a}\varepsilon_{r_a}(1),$$
$$\tau_a(k) = \mathrm{Tr}(\Psi_a(k)) = \xi_a\,\mathrm{Tr}(\Psi_a(g)) = \xi_a d_a \varepsilon_{r_{\bar{a}}}(1).$$

Sphericity of k implies that $\xi_a^2 = 1$ for all $a \in \mathrm{Irr}(\mathcal{A}^*)$, i.e. that $\xi_a = \pm 1$. Together with the fact that $\xi_{\bar{a}} = 1/\xi_a$, this implies that $\xi_a = \xi_{\bar{a}}$.

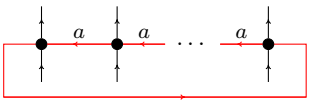
Let us define now Ω as in Theorem 5.14, assuming that the pivotal element $k \in \mathcal{A}^*$ is spherical. Then Ω, using that $\xi_a = \xi_{\bar{a}}$, satisfies additionally that

$$(T \otimes k^{-1}) \circ \Delta(\Omega) = S(\Omega) = \Omega.$$

We have thus seen that

Theorem 5.16. *Let $\mathcal{A}$ be a finite dimensional weak Hopf algebra over $\mathbb{C}$ such that $\mathcal{A}^*$ is semisimple and spherical with spherical element $k \in \mathcal{A}^*$ such that $\Psi_a(k) = \xi_a \Psi_a(g)$. Then the element $\Omega \in \mathcal{A}$ defined by*

$$\Phi^{\otimes n} \circ \Delta^{n-1}(\Omega) = \sum_{a \in \mathrm{Irr}(\mathcal{A}^*)} \frac{1}{\sum_{x:\, \ell_x = \ell_a} d_x^2} \frac{d_a}{\xi_a}$$



is non-degenerate and cocommutative, it is a projector, and there exists a linear map $T \in \mathfrak{B}(\mathcal{A})$ such that

$$(1 \otimes x)\Delta(\Omega) = (T(x) \otimes 1)\Delta(\Omega),$$
$$\Delta \circ T = (T \otimes k^{-1} \otimes T) \circ \Delta^2_{\mathrm{op}}.$$

Moreover, Ω satisfies

$$(T \otimes k^{-1}) \circ \Delta(\Omega) = S(\Omega) = \Omega.$$

5. C*-weak Hopf algebras

In this section we define C*-weak Hopf algebras. We show how the $*$-operation acts on the representing MPOs. We then show that a C*-weak Hopf algebra is spherical, and thus Theorem 5.16 applies.

Definition 5.17. A finite dimensional pre-bialgebra $\mathcal{A}$ over $\mathbb{C}$ is a *$*$-pre-bialgebra* if there is an anti-linear map $* : \mathcal{A} \to \mathcal{A}$ such that it is an involution (i.e. $x^{**} = x$), anti-homomorphism (i.e. $(xy)^* = y^*x^*$) and cohomomorphism (i.e. $\Delta \circ * = (* \otimes *) \circ \Delta$). It is a *C*-pre-bialgebra* if it is a $*$-pre-bialgebra and $\mathcal{A}$ has a faithful $*$-representation. A pre-bialgebra $\mathcal{A}$ is a *$*$-weak Hopf algebra* ($*$-WHA) if it is both a $*$-pre-bialgebra and a WHA, and it is a *C*-weak Hopf algebra* (C*-WHA) if it is both a C*-pre-bialgebra and a WHA.

An equivalent characterization of a finite dimensional C*-WHA is that it is a semisimple $*$-WHA such that it has a complete set of irreducible representations Φ_α ($\alpha \in \mathrm{Irr}(\mathcal{A})$) that are $*$-representations, i.e. $\Phi_\alpha(x^*) = \Phi_\alpha(x)^\dagger$. The dual of a C*-WHA is also a C*-WHA with $*$-operation defined by [14]

$$f^*(x) := \overline{f \circ * \circ S(x)}, \tag{55}$$

where $\overline{\lambda}$ denotes the complex conjugate of λ for all $\lambda \in \mathbb{C}$. Moreover, one can also check that [14]

$$* \circ S \circ * = S^{-1}. \tag{56}$$

As it is a C*-WHA, $\mathcal{A}^*$ also possesses a complete set of $*$-representations. In this section, Ψ_a, $a \in \mathrm{Irr}(\mathcal{A}^*)$, will always denote such a complete set of irreducible representations.

Definitions 5.18. Let $\mathcal{A}$ be a C*-WHA. Then,

$$\mathcal{A}_L := \{x \in \mathcal{A} : x_{(1)} \otimes x_{(2)} = x1_{(1)} \otimes 1_{(2)} = 1_{(1)}x \otimes 1_{(2)}\},$$
$$\mathcal{A}_R := \{y \in \mathcal{A} : y_{(1)} \otimes y_{(2)} = 1_{(1)} \otimes y1_{(2)} = 1_{(1)} \otimes 1_{(2)}y\},$$

are two commuting $*$-subalgebras of $\mathcal{A}$, known as the target and source counital subalgebras of $\mathcal{A}$, respectively. Moreover, $\mathcal{A}_{\min} := \mathcal{A}_L\mathcal{A}_R \subseteq \mathcal{A}$ is the minimal C*-weak Hopf algebra contained in $\mathcal{A}$ containing the unit element. $\mathcal{A}$ is said to be *minimal* if $\mathcal{A} = \mathcal{A}_{\min}$ and *regular* if the squared antipode restricted to $\mathcal{A}_{\min}$ is the identity, i.e. $S^2|_{\mathcal{A}_{\min}} = \mathrm{Id}$.

Let us now observe that the $*$-operation of $\mathcal{A}$ brings the character of Ψ_a into the character of $\Psi_{\bar{a}}$:

Proposition 5.19. *Let $\mathcal{A}$ be a C*-WHA, and $\tau_a \in \mathcal{A}$ ($a \in \mathrm{Irr}(\mathcal{A}^*)$) be the irreducible representation characters of $\mathcal{A}^*$. Then $\tau_a^* = \tau_{\bar{a}}$ for all a.*

Proof. Let us evaluate τ_a^* on a linear functional f:

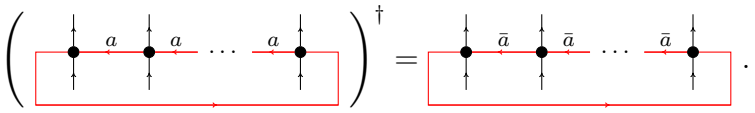
$$\tau_a^*(f) = f(\tau_a^*) = \overline{f^*(S^{-1}(\tau_a))} = \overline{f^*(\tau_{\bar{a}})} = \overline{\tau_{\bar{a}}(f^*)},$$

where in the second equation we have used Equation 55 with $x = S^{-1}(\tau_a)$. Finally note that, as Ψ_a is a $*$-representation,

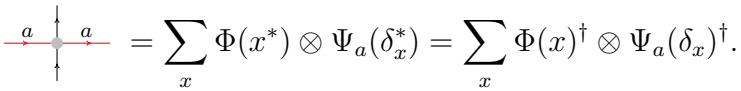
$$\tau_{\bar{a}}(f^*) = \mathrm{Tr}(\Psi_{\bar{a}}(f^*)) = \mathrm{Tr}(\Psi_{\bar{a}}(f)^\dagger) = \overline{\mathrm{Tr}(\Psi_{\bar{a}}(f))} = \overline{\tau_{\bar{a}}(f)},$$

and thus for all $f \in \mathcal{A}^*$, $\tau_a^*(f) = \tau_{\bar{a}}(f)$. □

As the MPO representation of τ_a is the TI MPO defined by the injective block a of the MPO tensor representing the C*-WHA, this proposition states that the injective MPO blocks are permuted under the $*$-operation:

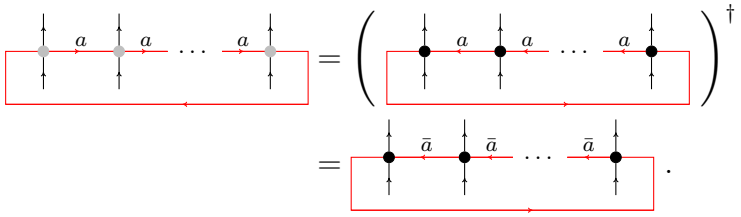


On the l.h.s. of this equation the dagger can be taken component-wise. More precisely, let us define a gray MPO tensor as

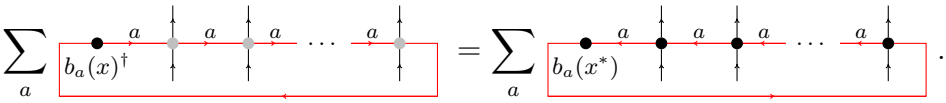


$$= \sum_x \Phi(x^*) \otimes \Psi_a(\delta_x^*) = \sum_x \Phi(x)^\dagger \otimes \Psi_a(\delta_x)^\dagger.$$

Notice that this MPO tensor is oriented in the opposite direction of the original (black) MPO tensor. By construction, the MPO defined by this MPO tensor is the Hermitian conjugate of the MPO describing τ_a:



In fact, this equation holds with arbitrary boundary condition as well, i.e. the MPO representation of x^* can be written as



Let us finally note that the gray and the black tensors can be related to each other using Equation 55. First note that there are two ways of expressing y^*:

$$\sum_x x\delta_x(y^*) = y^* = \sum_x x^*\overline{\delta_x(y)} = \sum_x x^*\delta_x^* \circ S^{-1}(y^*),$$

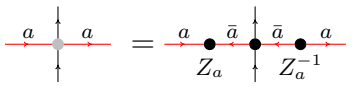
where in the last equation we have used the fact that the conjugation can be expressed with the $*$-operation of $\mathcal{A}$ and $\mathcal{A}^*$ as follows:

$$\overline{f(x)} = \overline{f \circ * \circ S \circ S^{-1} \circ *(x)} = f^* \circ S^{-1}(x^*).$$

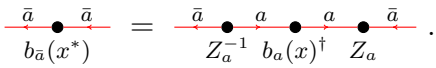
As this equation holds for all y, we have obtained that

$$\sum_x x^* \otimes \delta_x^* = \sum_x x \otimes S(\delta_x),$$

or graphically,



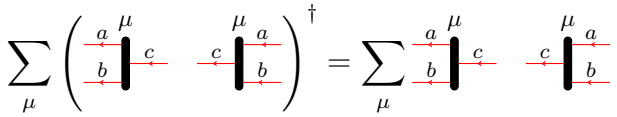
showing that the definition of the gray tensor considered here is the-same as the one defined above in Equation 35. Note finally that then $b(x)$ and $b(x^*)$ is also related:



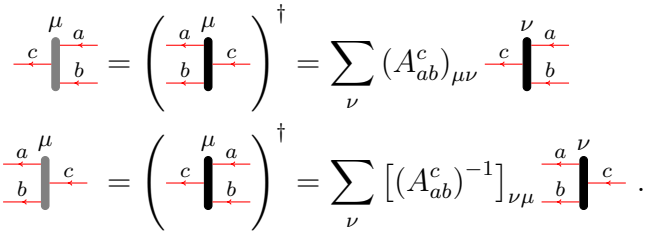
Let us now prove the well-known result that a C*-WHA is pivotal, and in fact, spherical [**14, 41**]:

Proposition 5.20. *Let $\mathcal{A}$ be a C*-WHA. Then $\mathcal{A}^*$ is also a C*-WHA, it is spherical and the linear functional $g \in \mathcal{A}^*$ defined in Equation 40 is a spherical element of it that is also positive.*

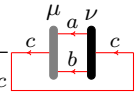
Proof. As mentioned above, we can fix the irreducible representation representatives Ψ_a of $\mathcal{A}^*$ to be $*$-representations. This implies, as the $*$-operation of $\mathcal{A}^*$ is a cohomomorphism, $\Delta = (* \otimes *) \circ \Delta \circ *$, that



This equation implies that for all $a, b, c \in \mathrm{Irr}(\mathcal{A}^*)$ there exists a matrix A_{ab}^c such that the dagger of the fusion tensor, denoted by gray tensors, can be expressed as



These matrices A_{ab}^c are in fact positive, as using the orthogonality relations, they are in the form

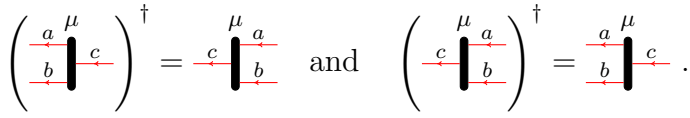


$$(A_{ab}^c)_{\mu\nu} = \frac{1}{D_c} \quad = \sum_\kappa \overline{(o_{ab}^c)_{\kappa\mu}} (o_{ab}^c)_{\kappa\nu}$$
$$= \sum_\kappa ((o_{ab}^c)^\dagger)_{\mu\kappa} (o_{ab}^c)_{\kappa\nu} = ((o_{ab}^c)^\dagger o_{ab}^c)_{\mu\nu}.$$

Therefore changing the fusion tensors to

$$\hat{V}_{ab}^{c\mu} = \sum_\kappa ((o_{ab}^c)^{-1})_{\mu\nu} V_{ab}^{c\nu} \quad \text{and} \quad (\hat{W}_{ab}^c)_{\mu\nu} = \sum_\kappa (o_{ab}^c)_{\mu\kappa} (V_{ab}^c)_{\kappa\nu},$$

we obtain $(\hat{V}_{ab}^{c\mu})^\dagger = \hat{W}_{ab}^{c\mu}$. That is, w.l.o.g. we can assume that the fusion tensors are the hermitian conjugates of each other,

(57) 

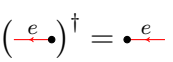
Let us now investigate what restrictions the C*-structure of $\mathcal{A}^*$ imposes on the representation of the unit of the algebra. The unit $1 \in \mathcal{A}$ is invariant under both the $*$-operation and the action of the antipode, therefore

$$1(f) = 1^*(f) = \overline{1(S(f^*))} = \overline{1(f^*)}.$$

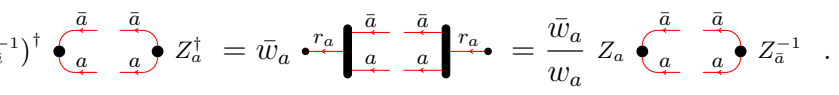
Let us write $|w_e\rangle =$ and $\langle v_e| =$, then this equation reads

$$\begin{aligned}\sum_{e\in \mathrm{Vac}(\mathcal{A}^*)} \langle v_e|\Psi_e(f)|w_e\rangle &= \sum_{e\in \mathrm{Vac}(\mathcal{A}^*)} \overline{\langle v_e|\Psi_e(f^*)|w_e\rangle} \\ &= \sum_{e\in \mathrm{Vac}(\mathcal{A}^*)} \langle w_e|\Psi_e(f^*)^\dagger|v_e\rangle \\ &= \sum_{e\in \mathrm{Vac}(\mathcal{A}^*)} \langle w_e|\Psi_e(f)|v_e\rangle,\end{aligned}$$

where in the last equation we have used that Ψ_e is a $*$-representation. As this equation holds for all f, we have obtained that

(58) 

Using that the fusion tensors and the vectors representing the unit are self-adjoint, we can now prove that the matrices Z_a can be chosen such that $(Z_{\bar{a}}^{-1})^\dagger = Z_a$. To see that, let us first take the dagger of Equation 39 using Equations 57 and 58:

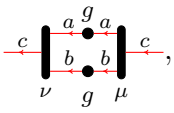


This implies that there is $\lambda_a \in \mathbb{C}$ such that $(Z_{\bar{a}}^{-1})^\dagger = \lambda_a Z_a$ and $Z_a^\dagger = \bar{w}_a/(w_a\lambda_a)Z_{\bar{a}}^{-1}$. Changing a to $\bar{a}$ in the first equation, we obtain that $(Z_a^{-1})^\dagger = \lambda_{\bar{a}} Z_{\bar{a}}$, or, after rearranging, $(Z_{\bar{a}}^{-1})^\dagger = \overline{\lambda_{\bar{a}}} Z_a$, and thus $\lambda_a = \overline{\lambda_{\bar{a}}}$. This implies that if a is such that $\bar{a} \neq a$, then $\lambda_a = \mu_a\overline{\mu_{\bar{a}}}$ can be solved, e.g. by $\mu_a = \lambda_a$ and $\mu_{\bar{a}} = 1$. With this choice, $\lambda_{\bar{a}} = \mu_{\bar{a}}\overline{\mu_a}$ also holds. If $\bar{a} = a$, one has to be more careful. In this case, $\lambda_a = \overline{\lambda_{\bar{a}}}$ implies that λ_a is real. To solve $\lambda_a = \mu_a\overline{\mu_{\bar{a}}} = |\mu_a|^2$, we have to show that λ_a is not only real, but also positive. To show that, note that $(Z_a^{-1})^\dagger = \lambda_a Z_a$, or $\mathrm{Id}_a = \lambda_a Z_a Z_a^\dagger$. As both Id_a, and $Z_a Z_a^\dagger$ are positive, this implies that λ_a is positive, and thus $\lambda_a = \mu_a\overline{\mu_{\bar{a}}}$ can be solved as well. Let $\hat{Z}_a = \mu_a Z_a$, then $(\hat{Z}_{\bar{a}}^{-1})^\dagger = \hat{Z}_a$. This means that we can assume w.l.o.g. that $(Z_{\bar{a}}^{-1})^\dagger = Z_a$.

Let us show now that the linear functional $g \in \mathcal{A}^*$ defined in Equation 40 is a pivotal element that is positive and spherical. First, the

equation $\left(Z_{\bar{a}}^{-1}\right)^\dagger = Z_a$ implies that both w_a and $\Psi_a(g)$ are positive, and thus g is positive as well. Let us show now that the matrix B_{ab}^c defined in Lemma 5.11 is positive as well. This matrix can be obtained by

$$(B_{ab}^c)_{\mu\nu}\, g_c =$$



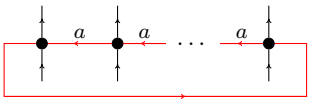
i.e. it is a positive matrix. As B_{ab}^c is positive and it squares to the identity, it is the identity, and thus g is a (positive) pivotal element. Note that it is spherical as well, because

$$\mathrm{Tr}(\Psi_a(g)) = d_a \varepsilon_{r_a}(1) = d_a \varepsilon_{r_{\bar{a}}}(1) = \mathrm{Tr}(\Psi_a(g^{-1})).$$

We have thus derived the well-known result that in a C*-WHA there is a positive spherical element. □

Let us now use this positive pivotal element in the construction in Theorem 5.16. With this choice, the resulting element Ω has MPO representation

$$\Phi^{\otimes n} \circ \Delta^{n-1}(\Omega) = \sum_{a \in \mathrm{Irr}(\mathcal{A}^*)} \frac{1}{\sum_{x:\, \ell_x = \ell_a} d_x^2} \frac{d_a}{\xi_a} d_a$$

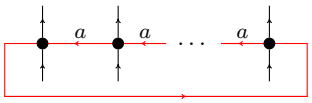


where all d_a are positive and $d_a = d_{\bar{a}}$. As d_a are positive, Ω is a positive linear functional on $\mathcal{A}^*$: for any positive f the representing matrix $\Psi_a(f)$ is positive, and thus $\Omega(f) = \sum_a d_a \,\mathrm{Tr}(\Psi_a(f)) \geq 0$. Similarly, $d_a = d_{\bar{a}}$ implies that $\Omega^* = \Omega$, and therefore, as Ω is also a projector, it is a positive element of $\mathcal{A}$.

We have thus seen that the following theorem holds.

THEOREM 5.21. *Let $\mathcal{A}$ be a finite dimensional C*-weak Hopf algebra over $\mathbb{C}$. Then $\mathcal{A}^*$ is semisimple and spherical with positive spherical element $g \in \mathcal{A}^*$ defined in Equation 40. Then the element $\Omega \in \mathcal{A}$ defined by*

$$\Phi^{\otimes n} \circ \Delta^{n-1}(\Omega) = \sum_{a \in \mathrm{Irr}(\mathcal{A}^*)} \frac{d_a}{\sum_{x:\, \ell_x = \ell_a} d_x^2}$$



is a positive non-degenerate trace-like linear functional on $\mathcal{A}^*$, it is an orthogonal projector, and there exists a linear map $T \in \mathfrak{B}(\mathcal{A})$ such that

$$(1 \otimes x)\Delta(\Omega) = (T(x) \otimes 1)\Delta(\Omega),$$
$$\Delta \circ T = (T \otimes g \otimes T) \circ \Delta_{\mathrm{op}}^2.$$

Moreover, Ω satisfies

$$(T \otimes g^{-1}) \circ \Delta(\Omega) = S(\Omega) = \Omega.$$

If $\mathcal{A}$ is coconnected (see Definition 5.3), the element $\Omega \in \mathcal{A}$ is called the *canonical regular element* of $\mathcal{A}$ [**41**]. See Appendix C for

more details. Let us note that the (positive) numbers d_a satisfy (see Lemma 5.11)

$$\sum_b N_{ab}^c d_b = \delta_{\ell_a \ell_c} d_a d_c.$$

In particular, if $\mathcal{A}$ is coconnected, then $\delta_{\ell_a \ell_c}$ is never zero and thus d_b defines a positive eigenvector for the matrix N_a defined by $(N_a)_b^c := N_{ab}^c$. As N_a is a non-negative matrix, this implies that the corresponding eigenvalue, d_a, is the spectral radius of N_a. This number is also called the Frobenius-Perron dimension of $a \in \mathrm{Irr}(\mathcal{A}^*)$. In this case,

$$\mathcal{D}^2 := \sum_a d_a^2$$

is known as the *total Frobenius-Perron dimension* of $\mathcal{A}$. If $\mathcal{A}$ is biconnected (see Definition 5.3), then the total Frobenius-Perron dimensions of $\mathcal{A}$ and $\mathcal{A}^*$ coincide [**41**].

Note finally that if $\mathcal{A}$ is a C*-weak Hopf algebra, then $\mathcal{A}^*$ is also a C*-weak Hopf algebra. This allows us to define an element $\omega \in \mathcal{A}^*$ that is a positive non-degenerate trace-like linear functional on $\mathcal{A}$. Using this ω then one can define a scalar product on $\mathcal{A}$ by

$$\langle\, x \,|\, y \,\rangle = \omega(x^* y).$$

6. C*-Hopf algebras

In this section we further particularize the results obtained in the previous sections to C*-Hopf algebras. As a C*-Hopf algebra is a special C*-WHA, Theorem 5.21 applies. We will show that the element Ω obtained from this theorem is in fact the Haar integral of the Hopf algebra.

Before stating the definition of a C*-Hopf algebra, note that for a WHA $\mathcal{A}$ the following statements are all equivalent [**14**]:

(1) $\Delta_{\mathcal{A}}(1) = 1 \otimes 1$,
(2) $\Delta_{\mathcal{A}^*}(\varepsilon) = \varepsilon \otimes \varepsilon$,
(3) $S(x_{(1)})x_{(2)} = \varepsilon(x)1$,
(4) $x_{(1)}S(x_{(2)}) = \varepsilon(x)1$.

Keeping this equivalence in mind, one can define Hopf algebras and C*-Hopf algebras as follows:

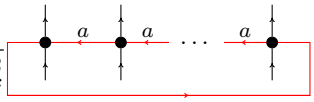
Definition 5.22. A *Hopf algebra* (HA) is a weak Hopf algebra such that $\Delta(1) = 1 \otimes 1$. A *C*-Hopf algebra* (C*-HA) is a C*-weak Hopf algebra such that $\Delta(1) = 1 \otimes 1$.

Any C*-Hopf algebra $\mathcal{A}$ is semisimple, and thus, due to the Larson-Radford theorem, $S^2 = \mathrm{Id}$ [**77, 78**]. This implies that the (unique) positive pivotal element of $\mathcal{A}^*$ is ε, and thus the numbers $\mathrm{Tr}(\Psi_a(g)) = d_a$ appearing in Theorem 5.23 are in fact $d_a = \mathrm{Tr}(\Psi_a(\varepsilon)) = D_a$, the dimension of the sector a. Moreover, Ω in Theorem 5.21 and Λ in

Theorem 5.9 coincide. In particular, as in a C*-Hopf algebra the unique normalized integral is the Haar integral (Larson-Sweedler theorem), Ω is the Haar integral of $\mathcal{A}$. We have thus seen that

THEOREM 5.23. *Let $\mathcal{A}$ be a finite dimensional C*-Hopf algebra over $\mathbb{C}$. Then $\mathcal{A}^*$ is semisimple and spherical and its positive spherical element is $\varepsilon \in \mathcal{A}^*$. The element $\Omega \in \mathcal{A}$ defined by*

$$\Phi^{\otimes n} \circ \Delta^{n-1}(\Omega) = \sum_{a \in \mathrm{Irr}(\mathcal{A}^*)} \frac{D_a}{\sum_{x \in \mathrm{Irr}(\mathcal{A}^*)} D_x^2}$$



where D_a is the dimension of the sector a, is the Haar integral of $\mathcal{A}$.

In particular, as $g = \varepsilon$, the map $T = S$ and the equations

$$(1 \otimes x)\Delta(\Omega) = (T(x) \otimes 1)\Delta(\Omega),$$
$$\Delta \circ T = (T \otimes g \otimes T) \circ \Delta^2_{\mathrm{op}}.$$

simplify to the definition of the integral and to the fact that S is an anti-cohomomorphism:

$$(1 \otimes x)\Delta(\Omega) = (S(x) \otimes 1)\Delta(\Omega),$$
$$\Delta \circ S = (S \otimes S) \circ \Delta_{\mathrm{op}}.$$

Let us finish the chapter with some well known illustrative examples of C*-Hopf algebras and C*-weak Hopf algebras.

Example 5.24. The group C*-algebra $\mathbb{C}G$ of a finite group G is endowed with the structure of a C*-HA by the linear extensions of the maps given by the expressions $\Delta(g) := g \otimes g$, $\varepsilon(g) := 1$ and $S(g) := g^* := g^{-1}$ for all elements $g \in G$.

Example 5.25. The dual vector space $(\mathbb{C}G)^*$ is again a C*-algebra endowed with the multiplication $\phi\psi(g) := \phi(g)\psi(g)$, the unit element $g \mapsto 1$ and the $*$-operation given by $\phi^*(g) := \overline{\phi(g)}$, for all $\phi, \psi \in (\mathbb{C}G)^*$ and all elements $g \in G$. Moreover, it becomes a C*-HA too by virtue of the comultiplication $\Delta(\phi)(g \otimes h) := \phi(gh)$, the counit $\varepsilon(\phi) := \phi(1)$, and the antipode $(S(\phi))(g) := \phi(g^{-1})$ for all $\phi \in (\mathbb{C}G)^*$ and all elements $g, h \in G$.

The following example, due to G. I. Kac and V. G. Paljutkin, describes the smallest C*-HA which is neither cocommutative, i.e. a group algebra, nor commutative, i.e. the dual of a group algebra.

Example 5.26 (see [66])**.** Let H_8 be the C*-algebra generated by three elements x, y and z subject to the relations $x^2 = 1$, $y^2 = 1$, $z^2 = 2^{-1}(1 + x + y - xy)$, $xy = yx$, $zx = yz$, $zy = xz$, $x^* = x$, $y^* = y$ and $z^* = z^{-1}$. It becomes a C*-HA by means of $\Delta(x) := x \otimes x$, $\Delta(y) := y \otimes y$, $\Delta(z) := 2^{-1}(z \otimes z + yz \otimes z + z \otimes xz - yz \otimes xz)$, $\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1$, $S(x) := x$, $S(y) := y$ and $S(z) := z$.

The following example is the smallest proper C*-WHA. It is known as the Lee-Yang C*-WHA as it is reconstructed from the solutions of the pentagon equation arising from the Lee-Yang fusion rules.

Example 5.27 (cf. [13])**.** Let $\mathcal{A}_{\mathrm{LY}}$ be the direct sum $\mathcal{M}_2 \oplus \mathcal{M}_3$ of full-matrix 2×2 and 3×3 C*-algebras with complex coefficients, respectively. Let $\zeta \in \mathbb{R}$ be the unique positive solution to $z^4 + z^2 - 1 = 0$ and fix matrix units e_1^{ij}, $i, j = 1, 2$, in $\mathcal{M}_2$ and $e_2^{k\ell}$, $k, \ell = 1, 2, 3$, in $\mathcal{M}_3$. Then, the comultiplication of $\mathcal{A}_{\mathrm{LY}}$ is defined by the expressions

$$\begin{aligned}
\Delta(e_1^{11}) &:= e_1^{11} \otimes e_1^{11} + e_2^{11} \otimes e_2^{22}, \\
\Delta(e_1^{12}) &:= e_1^{12} \otimes e_1^{12} + \zeta^2 e_2^{12} \otimes e_2^{21} + \zeta e_2^{13} \otimes e_2^{23}, \\
\Delta(e_1^{22}) &:= e_1^{22} \otimes e_1^{22} + \zeta^4 e_2^{22} \otimes e_2^{11} + \\
&\qquad \zeta^3 e_2^{23} \otimes e_2^{13} + \zeta^3 e_2^{32} \otimes e_2^{31} + \zeta^2 e_2^{33} \otimes e_2^{33}, \\
\Delta(e_2^{11}) &:= e_1^{11} \otimes e_2^{11} + e_2^{11} \otimes e_1^{22} + e_2^{11} \otimes e_2^{33}, \\
\Delta(e_2^{12}) &:= e_1^{12} \otimes e_2^{12} + e_2^{12} \otimes e_1^{21} + e_2^{13} \otimes e_2^{32}, \\
\Delta(e_2^{13}) &:= e_1^{12} \otimes e_2^{13} + e_2^{11} \otimes e_1^{22} + \zeta e_2^{12} \otimes e_2^{31} - \zeta^2 e_2^{13} \otimes e_2^{33}, \\
\Delta(e_2^{22}) &:= e_0^{22} \otimes e_2^{22} + e_2^{22} \otimes e_0^{11} + e_2^{33} \otimes e_2^{22}, \\
\Delta(e_2^{23}) &:= e_1^{22} \otimes e_2^{23} + e_2^{23} \otimes e_1^{21} + \zeta e_2^{32} \otimes e_2^{21} - \zeta^2 e_2^{33} \otimes e_2^{23}, \\
\Delta(e_2^{33}) &:= e_1^{22} \otimes e_2^{33} + e_2^{33} \otimes e_1^{22} + \zeta^2 e_2^{22} \otimes e_2^{11} - \\
&\qquad \zeta^3 e_2^{23} \otimes e_2^{13} - \zeta^3 e_2^{32} \otimes e_2^{31} + \zeta^4 e_2^{33} \otimes e_2^{33}
\end{aligned}$$

and the counit $\varepsilon \in (\mathcal{A}_{\mathrm{LY}})^*$ and the antipode $S \in \mathfrak{B}(\mathcal{A}_{\mathrm{LY}})$ are given by

$$\varepsilon(e_1^{ij}) = 1, \quad \varepsilon(e_2^{k\ell}) = 0, \quad S(e_1^{ij}) = e_1^{ji} \quad \text{and} \quad S(e_2^{k\ell}) = \zeta^{\ell - k} e_2^{\sigma(\ell)\sigma(k)}$$

for all $i, j \in \{1, 2\}$ and $k, \ell \in \{1, 2, 3\}$, where $\sigma(1) := 2$, $\sigma(2) := 1$, $\sigma(3) := 3$, endowing $\mathcal{A}_{\mathrm{LY}}$ with the structure of a C*-WHA. This specification has been slightly adapted from [13] as we will propose a tensor network description in Example 8.5 consistent with its string-net model definition.

CHAPTER 6

Pulling-through algebras

In the previous chapter we have seen (Theorem 5.14, and its variants Theorem 5.16 and Theorem 5.21) that in a semisimple pivotal WHA over $\mathbb{C}$ there is a special non-degenerate cocentral element Ω that behaves almost like an integral. In this chapter we abstract this notion and investigate the structure such an element provides, independent from semisimplicity or weak Hopf algebras. The main reason behind this abstraction is that this is exactly the structure we need to use in order to define a PEPS with symmetries, see Chapter 7.

Definition 6.1. A finite dimensional pre-bialgebra over $\mathbb{C}$ is called a *pulling-through algebra* if there is a cocentral, non-degenerate element $\Omega \in \mathcal{A}$, a linear map $T \in \mathfrak{B}(\mathcal{A})$ and a group-like linear functional $g \in \mathcal{A}^*$ such that for all $x \in \mathcal{A}$,

$$(1 \otimes x)\Delta(\Omega) = (T(x) \otimes 1)\Delta(\Omega), \tag{59}$$

$$\Delta \circ T = (T \otimes g \otimes T) \circ \Delta^2_{\mathrm{op}}. \tag{60}$$

Equation 59 will be called *pulling-through* equation. The rationale behind the name will become clear when we give a tensor network description in Equation 62 below. In a pulling-through algebra, Ω uniquely determines T: If there was another map $\hat{T} \in \mathfrak{B}(\mathcal{A})$ also satisfying Equation 59, then

$$(T(x) \otimes 1)\Delta(\Omega) = (1 \otimes x)\Delta(\Omega) = (\hat{T}(x) \otimes 1)\Delta(\Omega),$$

and thus by non-degeneracy of Ω, $\hat{T}(x) = T(x)$. Using that $\Delta(\Omega) = \Delta_{\mathrm{op}}(\Omega)$, one can show that T is an involution:

$$\begin{aligned}(1 \otimes x)\Delta(\Omega) &= (T(x) \otimes 1)\Delta(\Omega)\\ &= (T(x) \otimes 1)\Delta_{\mathrm{op}}(\Omega)\\ &= (1 \otimes T^2(x))\Delta_{\mathrm{op}}(\Omega)\\ &= (1 \otimes T^2(x))\Delta(\Omega),\end{aligned}$$

where the third equation is the pulling-through equation for $y = T(x)$. As above, due to non-degeneracy of Ω, $x = T^2(x)$, i.e. T is an involution. In particular, T is invertible, and thus there can be at most one linear functional $g \in \mathcal{A}^*$ satisfying Equation 60. Let us show now that T is an anti-homomorphism by using the pulling-through equation

Equation 59 twice:

$$\begin{aligned}(T(xy)\otimes 1)\Delta(\Omega) &= (1\otimes xy)\Delta(\Omega)\\ &= (T(y)\otimes x)\Delta(\Omega)\\ &= (T(y)T(x)\otimes 1)\Delta(\Omega),\end{aligned}$$

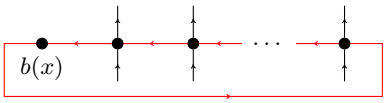
and thus non-degeneracy of Ω implies that $T(xy) = T(y)T(x)$.

Let us note that in a pre-bialgebra $\mathcal{A}$ there might be several different elements Ω that make it a pulling-through algebra. For example, it is easy to check that given an element $\Omega \in \mathcal{A}$ defining a pulling-through structure on $\mathcal{A}$ and any central group-like element $c \in \mathcal{A}^*$ (if there is any), the element $\hat{\Omega} := (\mathrm{Id}\otimes c)\circ\Delta(\Omega)$ defines another pulling-though structure. This is the case if in the construction of Theorem 5.14 we use two different pivotal elements g and $\hat{g}$ to arrive at the cocentral elements Ω and $\hat{\Omega}$, respectively.

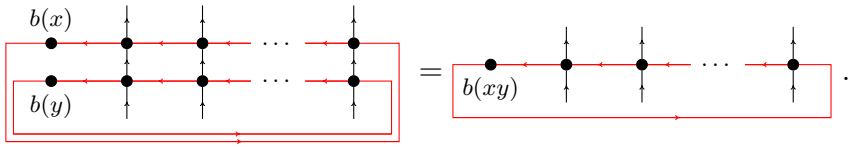
Let $\mathcal{A}$ be a pulling-through algebra. As $\mathcal{A}$ is a pre-bialgebra, it has MPO representations: given any representation Φ of $\mathcal{A}$ and an injective representation Ψ of $\mathcal{A}^*$ on a vector space W, let us define an MPO tensor by


$$:= \sum_{x\in B} \Phi(x)\otimes\Psi(\delta_x),$$

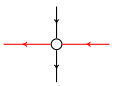
where B is a basis of $\mathcal{A}$, and δ_x denotes the dual basis, i.e. $\delta_x(y) = \delta_{xy}$. Given this MPO tensor, for all $x \in \mathcal{A}$ there is a matrix $b(x) \in \mathfrak{B}(W)$ such that

$$\Phi^{\otimes n}\circ\Delta^{n-1}(x) =$$


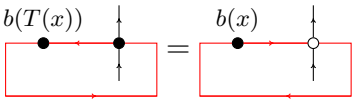
Notice that in this chapter we do not assume semisimplicity of $\mathcal{A}^*$. The MPOs, nevertheless, are still multiplicative,



Let us introduce a new MPO tensor, denoted by white dot, as


$$:= \sum_{x\in B} (\Phi\circ T)(x)\otimes\Psi(\delta_x).$$

Note that as $T \in \mathfrak{B}(\mathcal{A})$ is an algebra anti-homomorphism, this tensor is flipped upside down w.r.t. the tensor denoted by full dot, i.e. the input index of Φ is on the top of the tensor, not on the bottom. $T(x) \in \mathcal{A}$ can be expressed both with the original tensor and this new tensor:

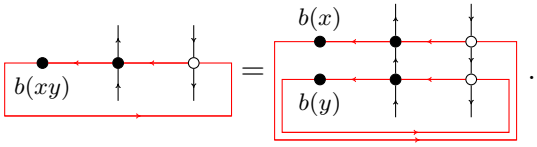
$$\Phi(T(x)) =$$


Note that the orientation of the red lines in the second MPO is the opposite of the orientation of the red lines in the first MPO. This is because to be able to compare the tensor networks on the two sides of the equation, the white tensor had to be rotated.

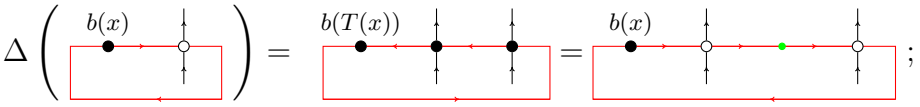
With the help of this new tensor, the identity

$$(xy)_{(1)} \otimes T((xy)_{(2)}) = x_{(1)}y_{(1)} \otimes T(y_{(2)})T(x_{(2)})$$

has a particularly nice MPO representation:

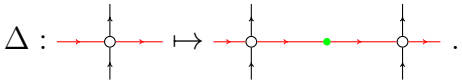


Let small green dot denote the (representation of the) group-like element g; then the tensor network representation of the identity $\Delta \circ T(x) = (T \otimes g \otimes T) \circ \Delta^2_{op}(x)$ is

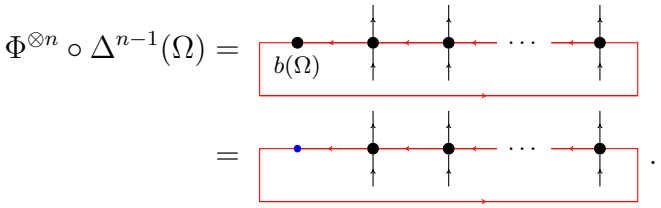


notice here that the third MPO is oriented in the opposite direction as the second one. This is due to the fact that it is the MPO corresponding to Δ_{op}, and not to Δ. This equation describes the coproduct of the white tensor:

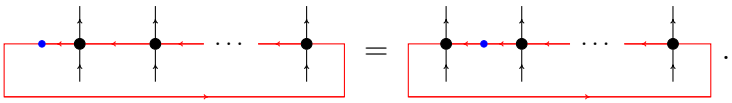
(61)



We will denote the boundary corresponding to the pulling-through element Ω by a small blue dot without label:

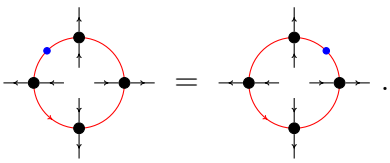


As Ω is co-central, the corresponding MPO is translation invariant, that is,

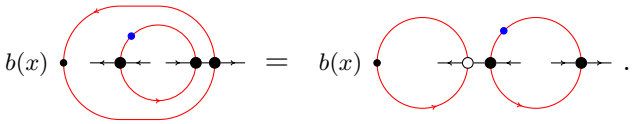


Let us now explain how these MPO representations will be used when depicting two-dimensional tensor networks in the next chapter. Keeping in mind that the orientation of each MPO tensor has to be kept, we can draw MPOs in arbitrary form such as on a circle. For

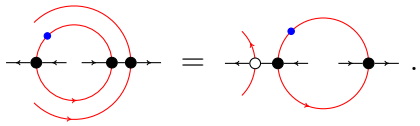
example, the fact that Ω is cocentral can be represented by the equation

$\Phi^{\otimes 4} \circ \Delta^3(\Omega) =$  .

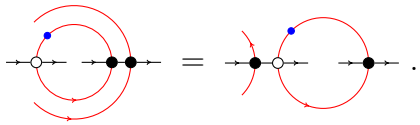
The pulling through equation $(1 \otimes x)\Delta(\Omega) = (T(x) \otimes 1)\Delta(\Omega)$ is represented by the following tensor network notation:



As this equation holds for all $x \in \mathcal{A}^*$, it also holds with open indices:

(62) 

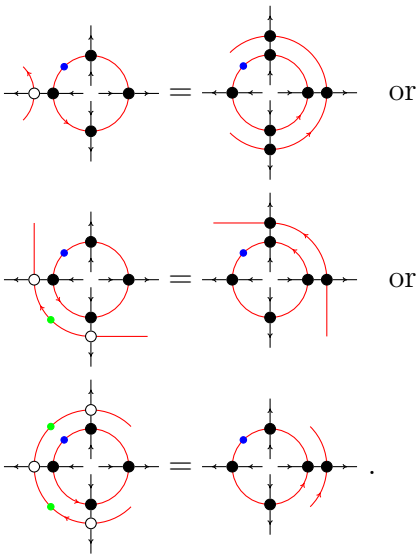
Applying $T \in \mathfrak{B}(\mathcal{A})$ on the left tensor component of this equation,



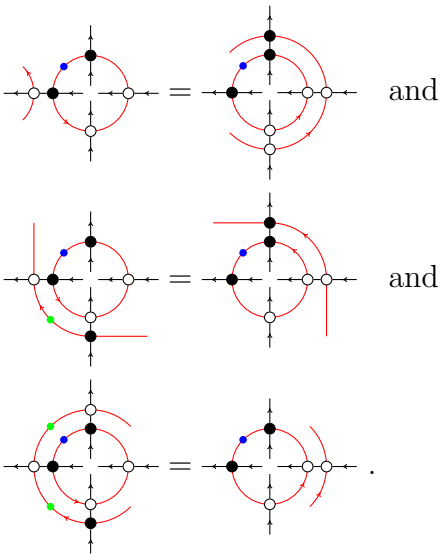
This equation is in fact why $(1 \otimes x)\Delta(\Omega) = (T(x) \otimes 1)\Delta(\Omega)$ is called pulling-through equation: it states that the MPO described by the black MPO tensors can be pulled through the circular MPO describing $(T \otimes \mathrm{Id}) \circ \Delta(\Omega)$. Let us remark that this type of equation is the key ingredient behind the notion of topological order in 2D is the breakthrough insight of [**108**], built on the previous work [**105**].

To familiarize ourselves with the notation, we derive some equations that we will use in the following chapter. Taking different coproducts

of Equation 62, and using Equation 61, we arrive at, for example,



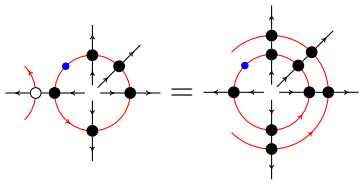
Applying $T \in \mathfrak{B}(\mathcal{A})$ on some of the tensor components reverses the orientation of the lines corresponding to the physical indices of those tensor components, and changes the black tensors into white ones and vice versa, that is, for example, the following holds:



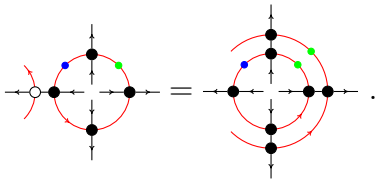
These identities are obtained from the previous ones by applying $T \in \mathfrak{B}(\mathcal{A})$ on the two tensor components on the bottom and on the right.

As $g \in \mathcal{A}^*$ is group-like, applying g on the product of MPO tensors is the same as applying it on the individual tensors. For example, applying the linear functional g on the upper left tensor component of

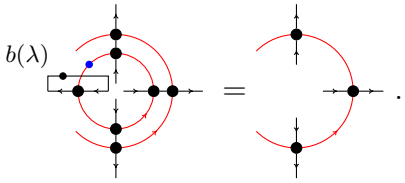
the equation



results in the equation



Non-degeneracy of Ω means that the action of the linear functional $\lambda \in \mathcal{A}^*$ defined by $(\lambda \otimes \mathrm{Id}) \circ \Delta(\Omega) = 1$ removes closed loops:



Finally note that the above equations also hold for non-translation invariant MPO representations of the pulling-through algebra.

Let us finish this chapter by rewritting Theorem 5.14 at the light of Definition 6.1:

THEOREM 6.2. *Every co-pivotal and cosemisimple weak Hopf algebra is a pulling-through algebra.*

CHAPTER 7

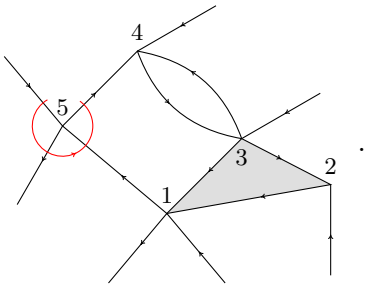
MPO-injective PEPS

In this chapter we define projected entangled pair states that possess certain symmetries described by a pulling-through algebra $\mathcal{A}$. Such a PEPS will also be called $\mathcal{A}$-injective PEPS or MPO-injective PEPS. As the primary examples for pulling-through algebras are pivotal cosemisimple weak Hopf algebras, and the representations of such algebras form a pivotal fusion category, our definition can be translated into a category theoretical language. In fact, when the pulling-through algebra corresponds to a C*-WHA, we believe that our definition is equivalent to the formalism presented in [85]. We expect that the pulling-through structure is actually more general than cosemisimple weak Hopf algebras; for example, certain non-semisimple pivotal Hopf algebras will also admit a pulling-through structure. States corresponding to those models have unusual properties. We show that MPO-injectivity is a topological property in the sense that it is invariant under the blocking of tensors (and thus under renormalization); in fact, our definition has been designed to satisfy this property. We also show, in the special case where $\mathcal{A}$ is a C*-Hopf algebra, the relation between this PEPS and the generalization of Kitaev's toric code to C*-Hopf algebras [19]. Following [20], we then construct a set of states that we call local excitations and that are states that differ only locally from the PEPS. We also construct a set of local operators that form a representation of the Drinfeld double $\mathcal{D}(\mathcal{A}^*)$ of the Hopf algebra $\mathcal{A}^*$ and show that these local operators transform the local excitations among each other. We identify the anyons of the model as the sectors of the local excitations under the action of these local operators.

Let us first recall the definition of two dimensional PEPS. A 2D PEPS is defined on a directed pseudo-graph[1] $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ that can be drawn on an orientable 2-manifold such that no edges intersect (like in the case of a planar graph). The figure below shows a local part of such a graph (locally it *is* a planar graph). For convenience we have numbered the vertices and we have displayed a circular arrow around

[1] by pseudo-graph we mean a graph where multiple edges are allowed between vertices

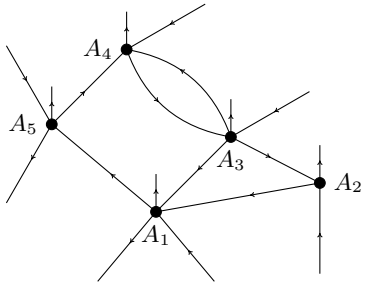
vertex 5 depicting the orientation of the 2-manifold:



.

The areas enclosed by minimal (unoriented) cycles in the graph are called plaquettes. Such a plaquette is denoted by light gray shading between the vertices 1, 2 and 3. Let $\mathcal{E}^{op}$ denote the set of edges in $\mathcal{E}$ with reversed orientation, and for any $e \in \mathcal{E}$ or $e \in \mathcal{E}^{op}$ let $\bar{e}$ denote the edge e with the opposite orientation.

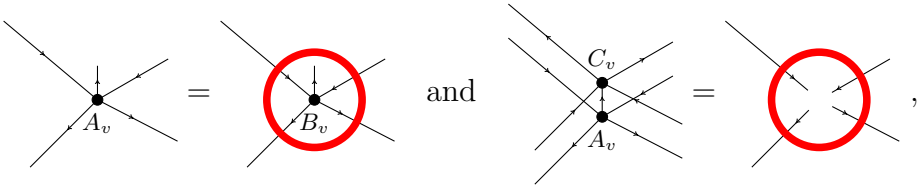
To every edge $e \in \mathcal{E}$ we assign a finite dimensional complex vector space, V_e. Let us also assign a vector space $V_{\bar{e}}$ to every oppositely oriented edge $\bar{e} \in \mathcal{E}^{op}$ such that $V_{\bar{e}} = V_e^*$. Note that this relation is symmetric, i.e. $V_e = V_{\bar{e}}^* = V_{\bar{\bar{e}}}$. To every vertex we assign a (finite dimensional) Hilbert space, $\mathcal{H}_v$. Finally, to each vertex v we assign a tensor $A_v \in \mathcal{H}_v \otimes \bigotimes_{e \in \mathcal{N}_v} V_e$, where $\mathcal{N}_v$ denotes the set of edges $e \in \mathcal{E} \cup \mathcal{E}^{op}$ that connect v with another vertex such that the orientation of e points away from v. Given all these data, the state defined by the PEPS is obtained by contracting $\bigotimes_v A_v$ along the edges of the graph; note that this contraction is possible, because if e is an edge between v and w, then the tensor component corresponding to e in the tensor A_v is V_e, while in the tensor A_w it is $V_{\bar{e}} = V_e^*$. The state defined by the PEPS can be represented, using the graphical notation of tensor calculus, as

$|\psi\rangle =$ 

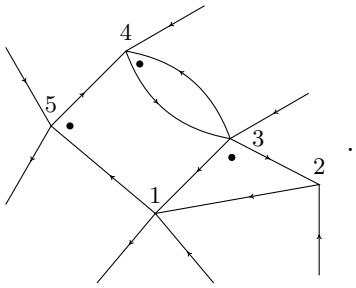
.

Let us now define MPO-injective PEPS. These are PEPS such that the PEPS tensors are invariant under certain symmetry operations acting on their virtual degrees of freedom, i.e. on the tensor components $\bigotimes_{e \in \mathcal{N}_v} V_e$ of the tensor A_v. More precisely, a PEPS tensor $A_v \in \mathcal{H}_v \otimes \bigotimes_{e \in \mathcal{N}_v} V_e$ is called O_v-injective for an operator $O_v \in \mathfrak{B}(\bigotimes_{e \in \mathcal{N}_v} V_e)$,

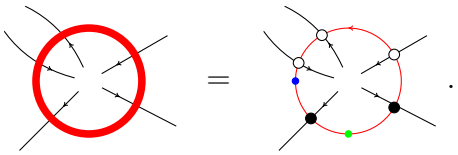
if there are tensors B_v and C_v such that



where the red circle denotes the operator O_v. Note that if O_v is a projector, then one can choose $B_v = A_v$. If, on the other hand, O_v is nilpotent, B_v and A_v are different. This is the case when the symmetries of the tensor form a non-semisimple algebra. In the following we will define the symmetry operators O_v for each vertex $v \in \mathcal{V}$. To do so, we need the following additional data. First, a pulling-through algebra $(\mathcal{A}, \Omega, T, g)$, and second, representations $\Phi_e : \mathcal{A} \to \mathfrak{B}(V_e)$ on the vector spaces V_e such that $\Phi_{\bar{e}} = \bar{\Phi}_e$, where $\bar{\Phi}(x) = (\Phi \circ T(x))^T$ for all $x \in \mathcal{A}$. Note that as T is idempotent, this relation is symmetric: if $\Phi_{\bar{e}} = \bar{\Phi}_e$, then $\Phi_e = \bar{\Phi}_{\bar{e}}$ as well. Finally, for every plaquette in the graph, we will choose a vertex from the ones surrounding the plaquette. Below we denote such a choice by putting a black dot close to the vertex chosen inside each plaquette:



The symmetry operator O_v around each vertex v is an MPO representation of the element Ω of the pulling-through algebra $\mathcal{A}$. The concrete form of the MPO is designed such that it is a generalization of G-injective PEPS [112] and such that the PEPS remains MPO-injective even after blocking. Let us explain through an example what this MPO representation exactly is. The symmetry operator O_v, around the vertex 3, takes the following form:



As stated above, this MPO is an MPO representation of Ω. The blue dot represents the boundary $b(\Omega)$; remember that as Ω is cocommutative, the placement of the boundary is not relevant. The orientation of the virtual index of the MPO follows the orientation of the surface, as the arrow on it shows. The MPO consists of two different MPO

tensors. For an outgoing edge e, we use the MPO tensor denoted by black dots given by

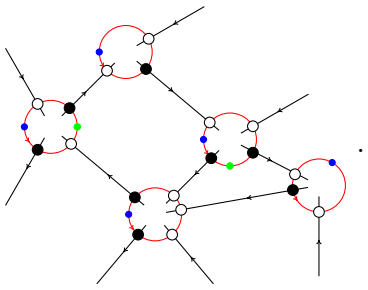
$$= \sum_x \Phi_e(x) \otimes \Psi(\delta_x),$$

where x runs over a basis of $\mathcal{A}$ and Ψ is a representation of $\mathcal{A}^*$ on the virtual vector space denoted by red indices. For an incoming edge e, we use the white MPO tensor given by

$$= \sum_x \Phi_e \circ T(x) \otimes \Psi(\delta_x) = \sum_x \Phi_{\bar{e}}(x)^T \otimes \Psi(\delta_x).$$

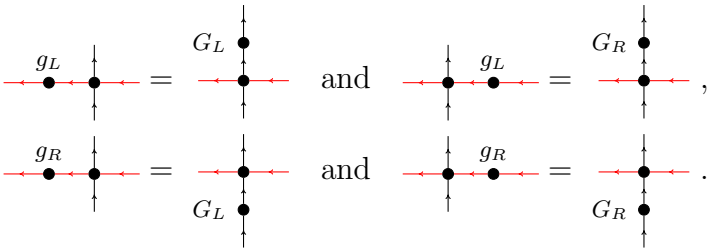
Note that this white tensor is constructed the same way as the black tensor, but for the construction we use the wrong representation: $\Phi_{\bar{e}}$ instead of Φ_e (in the formula transpose appears because on the l.h.s. we read the tensor as a linear map from the bottom to the top). Changing the orientation of the edge e (i.e. replacing it with $\bar{e}$) changes the black tensors to white and white tensors to black. Finally, as the vertex 3 was selected for the plaquette surrounded by vertices 1, 2 and 3, we insert the linear functional g between the edges connecting the vertex 3 to the vertices 1 and 2.

Using the above definition of an MPO-injective PEPS tensor, an MPO-injective PEPS – after applying the inverse tensors C_v at each vertex v – can be written as the following tensor network:

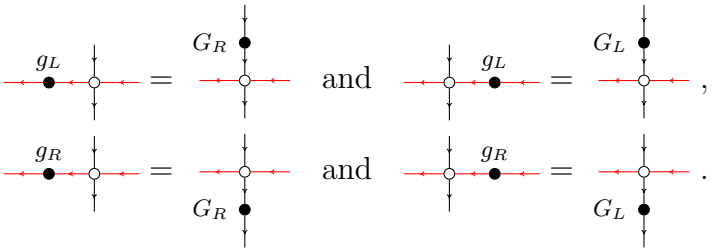


In general pulling-through algebras (such as a pulling-through algebra that originates from a pivotal WHA that is not a C*-WHA) different placements of the group-like elements g lead to different states. In a pulling-through algebra that originates from a C*-WHA, however, all these states are related to each other by local operations, and in fact, one can define MPO-injective PEPS in a more translation invariant way. To understand why, recall [14] that in a C*-WHA the positive spherical element $g \in \mathcal{A}^*$ can be written as $g = g_L g_R^{-1} = g_R^{-1} g_L$ such that $g_R = S(g_L)$ and such that there are algebra elements $G_L, G_R \in \mathcal{A}$

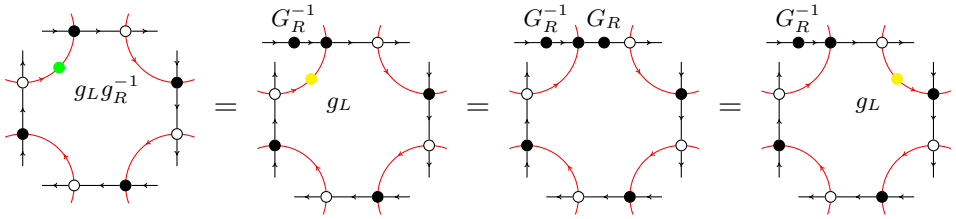
satisfying the following relations:



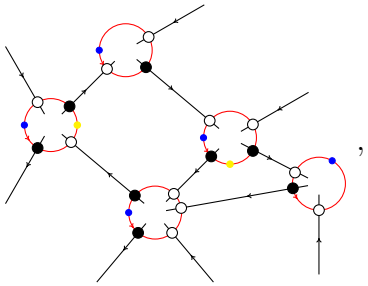
The equation $G_L = S(G_R)$ also holds and it is easy to check that $G_L = T(G_R)$ holds as well. Applying thus T on the previous relations, we obtain that the white tensors satisfy



Let us write now $g = g_L g_R^{-1}$ in the definition of the MPO-injective PEPS. Using the relations above, we obtain then that g_R^{-1} translates to a local operator acting on the vertex which g belongs to, while g_L is delocalized around the plaquette. Let us illustrate this fact by depicting one of the plaquettes of the previous PEPS:

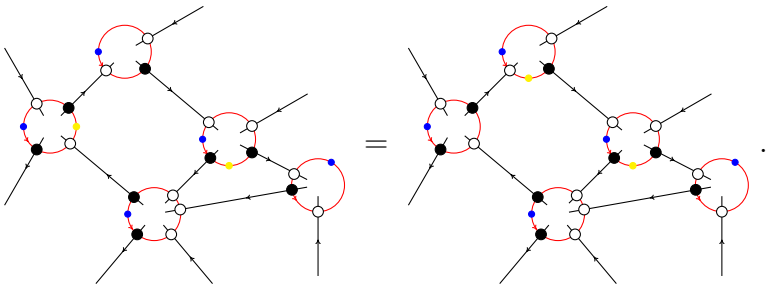


Applying thus G_R on the appropriate tensor components, we obtain that the state defined by



where the yellow dot now denotes g_L instead of $g = g_L g_R^{-1}$, is also MPO-injective and in this state the yellow dots move freely around the

plaquette:

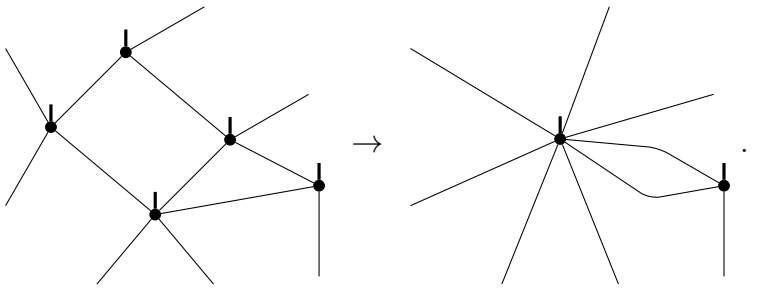


As we have seen above, the yellow dot does not even have to appear on the virtual index of the MPO, it can be instead inserted between two PEPS tensors anywhere around the plaquette.

1. Scale independence

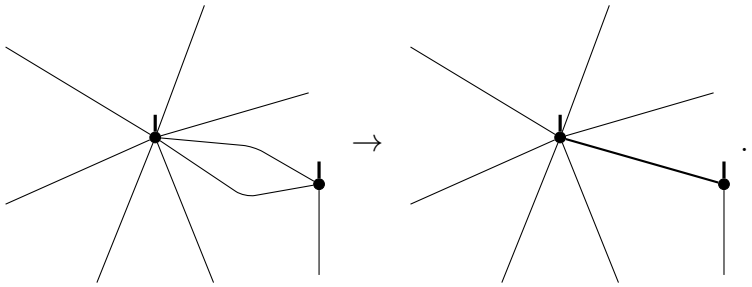
In this section we introduce an operation on states that we call *blocking* and show that an MPO-injective PEPS stays MPO-injective even after blocking. Blocking is the basis of renormalization (there it is followed by an isometry that gets rid of certain degrees of freedom) and has a natural representation in tensor networks. Blocking simply means that we treat certain neighboring particles together: for example, given a three-partite state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, the blocking of particles 2 and 3 means that we reinterpret $|\psi\rangle$ as a two-partite state in $\mathcal{H}_1 \otimes \mathcal{H}_{23}$, where $\mathcal{H}_{23} = \mathcal{H}_2 \otimes \mathcal{H}_3$.

In PEPS, blocking is a partial contraction of the tensor network, i.e. that in a given region we replace all tensors by one tensor that is the result of the contraction of the tensors in that region. The blocked tensor network is then another, coarser tensor network. For example, in the tensor network below blocking of four tensors results in the following new tensor network:



As the example above shows, sometimes two edges e and f that were distinct edges in the original PEPS become a single double edge in the blocked tensor network. We can block these two edges together, i.e. consider $V_e \otimes V_f$ as a single vector space. Pictorially we express this as

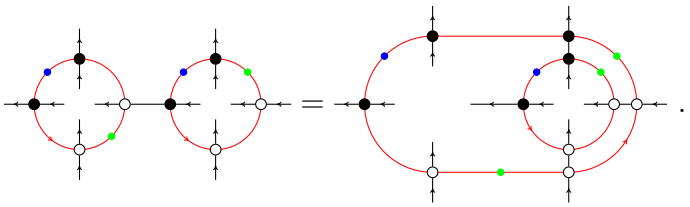
writing a single edge instead of the two distinct ones:



In fact, every blocking can be broken down into a series of simple steps: in each step we either block two neighboring tensors that are connected by a singe edge, possibly leading to double edges in the PEPS, or block two edges together to remove a double edge from the PEPS. Note that in the first case the number of plaquettes does not change (if there are double edges after blocking, the area enclosed between the two edges is considered as a plaquette), while in the second case the number of plaquettes decreases by one.

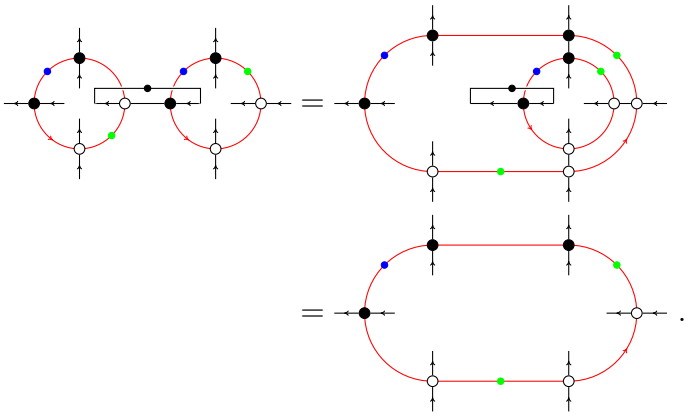
In the following we will show that an MPO-injective PEPS stays MPO-injective even after blocking. Let us consider a pulling-through algebra $\mathcal{A}$ and an $\mathcal{A}$-injective PEPS. We first show that blocking two neighboring tensors in the $\mathcal{A}$-injective PEPS results in another $\mathcal{A}$-injective PEPS, and then, that removing double edges does not change the $\mathcal{A}$-injectivity property either.

Let us now consider the blocking of two neighboring $\mathcal{A}$-injective PEPS tensors that are connected to each other with a single edge. After applying the inverse tensors on the two PEPS tensors and using the pulling-through property, we obtain



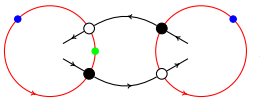
Due to non-degeneracy, applying a suitable linear functional to the inner two indices, the inner MPO disappears leading to a single MPO

on the boundary:

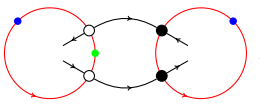


We have thus constructed an inverse tensor (the inverse tensors of the individual PEPS tensors contracted with the matrix describing the linear functional used above) for the blocked tensor, and showed that it is MPO-injective. Let us note that in the process neither the "outgoing arrow is black tensor, incoming arrow is white tensor" nor the "one green dot per plaquette" property have changed, and thus not only the blocked tensor is MPO-injective but the whole PEPS remains MPO-injective as well.

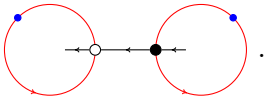
Let us now consider the blocking of two neighboring edges e and f. Assume that the edges are oriented in the opposite direction and that the "one green dot per plaquette" is on the left vertex:



Here we did not indicate the other MPO tensors on the two vertices. In order to block these two edges together, let us first reorient the lower edge, f. The reorientation of the arrow implies that we change the corresponding vector space V_f to $V_{\bar{f}}$ and the representation Φ_f to $\Phi_{\bar{f}}$. This also changes the black and white tensors and thus the tensor network notation becomes:



Blocking the edges e and f now simply means replacing the two individual $\mathcal{A}$-modules V_e and V_f by their tensor product $V_f \boxtimes V_e$. After blocking, we obtain a single edge with two MPO tensors at the two ends:



Here the black MPO tensor is built using the representation $\Phi_{fe} = \Phi_f \boxtimes \Phi_e$, while the white MPO tensor is built using the transpose of

the representation $\bar{\Phi}_{fe}$,

$$\Phi_{fe} \circ T = (\Phi_f \otimes \Phi_e) \circ \Delta \circ T = (\Phi_f \circ T \otimes g \otimes \Phi_e \circ T) \circ \Delta^2_{\mathrm{op}}.$$

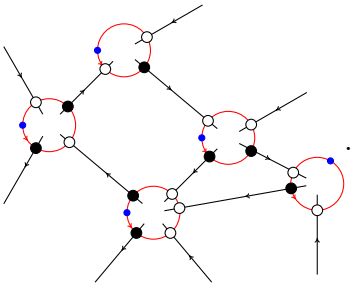
This equation shows that the single (blocked) white tensor is the concatenation of the two white tensors with the green dot in the middle. The fact that the equation contains Δ_{op} instead of Δ simply reflects the fact that as the orientation of the two circles are the same, the arrow on the virtual index (red line) on the left is oriented in the opposite direction as on the right.

As the blocking leaves the rest of the PEPS invariant, blocking two edges keeps the PEPS MPO-injective. We have thus shown that MPO-injectivity is invariant both under the blocking of neighboring vertices and under the blocking of neighboring edges. As blocking any number of tensors in a simply connected region can be decomposed into a series of such simple blocking steps, we have proven that MPO-injectivity is invariant under blocking.

2. Relation to the Kitaev model

The tensor networks defined above are strongly linked to the generalized Kitaev models defined in [19]. In this section we show the concrete connection. First, let us consider an MPO-injective PEPS constructed from a C*-Hopf algebra $\mathcal{A}$. Based on the construction of this PEPS, we can define another multi-partite state $|\psi\rangle$ where the individual degrees of freedom are described on the Hilbert space $\mathcal{A}$. Second, we construct a parent Hamiltonian for $|\psi\rangle$ – the Kitaev Hamiltonian – that is the sum of commuting projectors. Finally, we define the Drinfeld double $\mathcal{D}(\mathcal{B})$ of any Hopf algebra $\mathcal{B}$ and construct a set of local operators that form a representation of $\mathcal{D}(\mathcal{A}^*)$; we also construct a set of local deformations of $|\psi\rangle$ and show that this set of states S is invariant under the action of $\mathcal{D}(\mathcal{A}^*)$. We identify the anyons of the model as the sectors of S under the action of $\mathcal{D}(\mathcal{A}^*)$.

Let us first show how the state $|\psi\rangle$ is defined. For that, consider the PEPS defined by a C*-Hopf algebra. The construction of these states is easier to understand than in the general case, because there are no group-like elements g present in it (as $g = \varepsilon$). The PEPS, after applying the inverse tensors, read as

$|\phi\rangle =$ 

This vector $|\phi\rangle$ lives in the vector space $\bigotimes_{v\in\mathcal{V}}\left(\bigotimes_{e\in\mathcal{N}_v} V_e\right)$, because at every vertex v the degrees of freedom are $\bigotimes_{e\in\mathcal{N}_v} V_e$. Note that in this tensor product every edge appears exactly twice, once with the orientation defined by the graph and once with opposite orientation. Let us now rearrange this tensor product and group together the degrees of freedom corresponding to the two orientations of each edge. After this regrouping, we can interpret the previous vector space as $\bigotimes_{e\in\mathcal{E}}(V_e\otimes V_{\bar{e}})$. Finally note that as $V_{\bar{e}}=V_e^*$ and for every finite dimensional vector space V, the tensor product $V\otimes V^*$ is isomorphic to $\mathfrak{B}(V)$, this vector space can also be interpreted as $\bigotimes_{e\in\mathcal{E}}\mathfrak{B}(V_e)$, i.e. one can think of $|\phi\rangle$ as

$$|\phi\rangle\in\bigotimes_{e\in\mathcal{E}}\mathfrak{B}(V_e).$$

Note that, by construction, $|\phi\rangle$ is not supported on the whole Hilbert space $\bigotimes_{e\in\mathcal{E}}\mathfrak{B}(V_e)$, but instead only on the subspace $\bigotimes_{e\in\mathcal{E}}\Phi_e(\mathcal{A})$; here Φ_e is the representation of $\mathcal{A}$ used on the edge e. This implies that we can define another vector, $|\psi\rangle$, as

$$|\psi\rangle=\left(\bigotimes_{e\in\mathcal{E}}\Phi_e^{-1}\right)|\phi\rangle\in\bigotimes_{e\in\mathcal{E}}\mathcal{A}.$$

If $\mathcal{A}$ is a C*-Hopf algebra, there is a scalar product on $\mathcal{A}$, and thus the vector space $\bigotimes_{e\in\mathcal{E}}\mathcal{A}$ is a finite dimensional Hilbert space. Therefore $|\psi\rangle$ can be interpreted as a (possibly unnormalized) state. The maps Φ_e^{-1} act locally, therefore this state is again a PEPS – the virtual legs of the PEPS tensor are the red lines in the figure above. This PEPS is then the same (up to a choice of orientation of the arrows on both the red and black lines) as the one described in [19].

2.1. The Kitaev Hamiltonian. Building on the results of [19], in this section we explicitly construct the Kitaev parent Hamiltonian for the state $|\psi\rangle$ defined in the previous section. To make the reading easier, we will use a graphical language to depict the action of the defined Hamiltonian. This graphical language is nothing but the tensor network representation of the state using the representation $\bigotimes_{e\in\mathcal{E}}\Phi_e$. For simplicity, let us restrict ourselves to a square lattice, and fix the orientation of the lattice such that all vertical edges point from bottom to top and all horizontal ones from right to left (i.e. the product of two elements on the vertical edge reads from top to bottom and on the horizontal edges from left to right).

The Kitaev Hamiltonian consists of two type of terms, the plaquette terms B_p and vertex terms A_v. The total Hamiltonian is the sum of these terms,

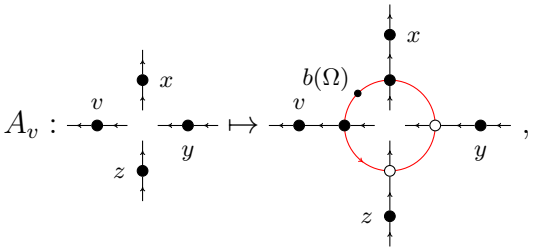
$$H=-\sum_{p\in\text{plaquettes}}B_p-\sum_{v\in\text{vertices}}A_v.$$

Each plaquette term B_p acts on the edges surrounding the plaquette p, while each vertex term A_v act on the edges connected to the vertex v. We will define B_p and A_v such that they are orthogonal projectors and any two such terms commute. The MPO-injective PEPS $|\psi\rangle$ is a frustration-free ground state of this Hamiltonian.

Let us now define the operator A_v for a given vertex v. As stated above, this operator acts on the edges surrounding the vertex v. On these four particles, its action is given by:

$$A_v : x \otimes y \otimes z \otimes v \mapsto x\Omega_{(1)} \otimes S(\Omega_{(2)})y \otimes S(\Omega_{(3)})z \otimes v\Omega_{(4)},$$

where Ω is the Haar integral of $\mathcal{A}$, x is the particle above the vertex, y is the one on its right, z is the one below and v is the one on the left. The concrete form of A_v depends on the orientation of the lattice. Using the graphical representation, it is easier to visualize the action of A_v:



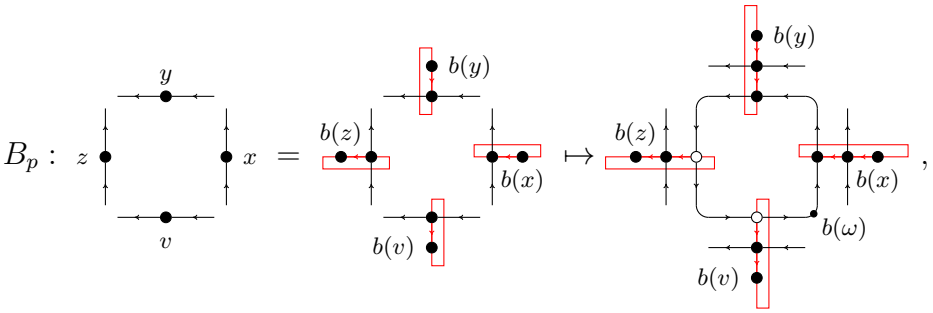
i.e. it multiplies the four particles by the (translation invariant) MPO $\Delta^3(\Omega)$, each from the side that is closer to the vertex. As Ω is a projector, it is clear that A_v is a projector as well. As both representations $x \mapsto (y \mapsto yx)$ and $x \mapsto (y \mapsto S(x)y)$ of $\mathcal{A}^{op}$ are $*$-representations, A_v is also self-adjoint. Note that for any two different vertices v_1 and v_2 the Hamiltonian terms A_{v_1} and A_{v_2} clearly commute: if v_1 and v_2 are not neighboring vertices, they act on different particles; if v_1 and v_2 are neighboring, there is a single particle on which both of them acts, but if A_{v_1} acts from the left, then A_{v_2} acts from the right of the particle.

Let us now define the operator B_p for a given plaquette p. As stated above, this operator acts on the edges surrounding the plaquette. On these four particles, its action is given by

$$B_p : x \otimes y \otimes z \otimes v \mapsto \omega(S(x_{(1)})S(y_{(1)}z_{(2)}v_{(2)}))x_{(2)} \otimes y_{(2)} \otimes z_{(1)} \otimes v_{(1)},$$

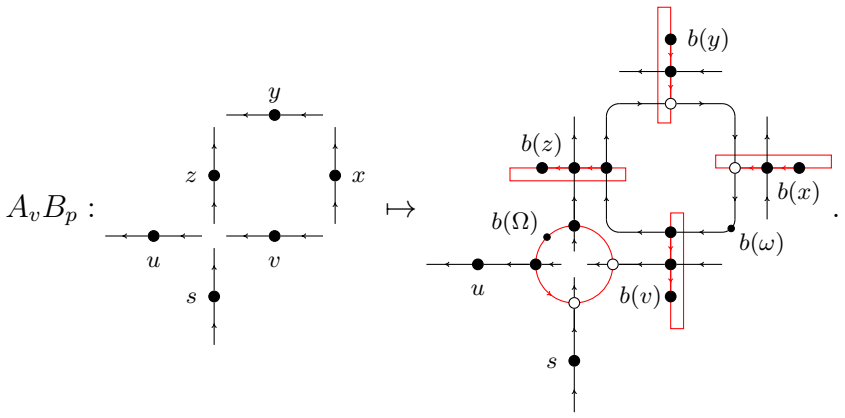
where ω is the Haar integral of $\mathcal{A}^*$ and x is the particle on the right of the plaquette, y is the one on top, z is the one on the left and v is the particle below the plaquette. Again, the action of B_p is easier to

understand using the graphical representation:

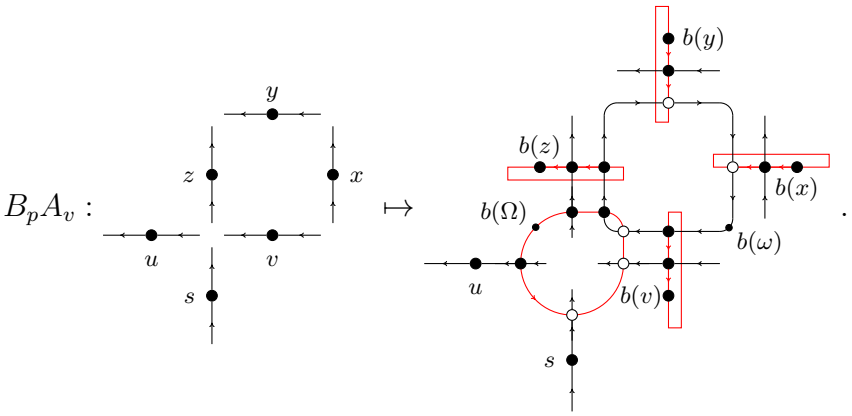


where the matrix $b(\omega)$ is the boundary describing the MPO representation of ω, see Equation 15. Similar to the vertex terms, the operators B_p are projectors, and as the representations $f \mapsto (x \mapsto x_{(1)} f(x_{(2)}))$ and $f \mapsto (x \mapsto f \circ S(x_{(1)}) x_{(2)})$ are both $*$-representations of $\mathcal{A}^*$, they are also self-adjoint. If p_1 and p_2 are plaquettes that are not neighboring, then B_{p_1} and B_{p_2} act on different particles, and thus they commute. If p_1 and p_2 are neighboring, then there is one particle both act on. One of the operators, however, acts from the right, the other from the left, and thus even in this case, B_{p_1} and B_{p_2} commute.

Let us now show that the operators A_v and B_p commute, i.e. that the Kitaev Hamiltonian is indeed a sum of commuting orthogonal projectors. If the vertex v is not a vertex on the plaquette p, then A_v and B_p are acting on different particles and thus they trivially commute. If the vertex v is one of the vertices around the plaquette, we first calculate the graphical representation of the action of $A_v B_p$ on the six particles surrounding both the plaquette and the vertex:

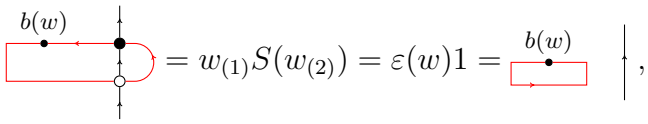


Let us now calculate the action of $B_p A_v$:

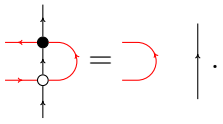


The two tensors in the middle, closed by arbitrary boundary condition $b(w)$ on the virtual index and after straightening the physical index, read as:

(63)

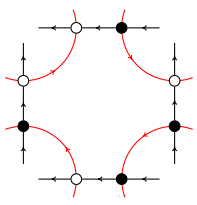


or equivalently,

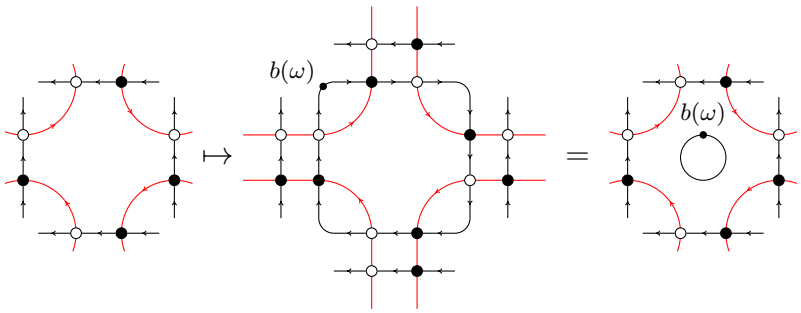


Using this identity in the expression for $B_p A_v$, we obtain that the two loops, the virtual and physical one, can be untangled, and thus $B_p A_v = A_v B_p$, i.e. the Hamiltonian terms A_v and B_p commute.

Let us finally show that the state $|\psi\rangle$ is a ground state of the Hamiltonian H. First, as Ω is a projector, $|\psi\rangle$ is clearly invariant under each term A_v. Let us now show that it is also invariant under all B_p, then this will mean that $|\psi\rangle$ is a frustration-free ground state of H. To see that $|\psi\rangle$ is invariant under the action of B_p, note that the state, locally around a plaquette, looks like



The action of B_p is thus



Where in the last equation we have used Equation 63 in each corner. The result is therefore just a multiplication with the complex number $\omega(1) = 1$, i.e. the state $|\psi\rangle$ is invariant under B_p.

2.2. The Drinfeld double and anyons. In this section we define the Drinfeld double $\mathcal{D}(\mathcal{A})$ of a finite dimensional Hopf algebra $\mathcal{A}$ and, for each pair of plaquette p and neighboring vertex v, a set of local operators including the Hamiltonian terms A_v and B_p that forms a representation of $\mathcal{D}(\mathcal{A}^*)$. We also define a set of sates that differ only locally from the state $|\psi\rangle$. The defined local operators transform these states amongst each other, and thus these states form a $\mathcal{D}(\mathcal{A}^*)$-module. We identify the anyons of the model as the subsets of states that form *irreducible* $\mathcal{D}(\mathcal{A}^*)$-modules (see also [**20**]).

Let us first define the Drinfeld double of a Hopf algebra.

Definition 7.1. Let $\mathcal{A}$ be a finite dimensional Hopf algebra. The *Drinfeld double* $\mathcal{D}(\mathcal{A})$ is a Hopf algebra constructed as follows. As a vector space, it is $\mathcal{A}^* \otimes \mathcal{A}$. Given $f \in \mathcal{A}^*$ and $x \in \mathcal{A}$ we will write $f \bowtie x$ for their tensor product. The comultiplication in $\mathcal{D}(\mathcal{A})$ is given by

$$\Delta(f \bowtie x) = (f_{(2)} \bowtie x_{(1)}) \otimes (f_{(1)} \bowtie x_{(2)}).$$

The multiplication in $\mathcal{D}(\mathcal{A})$ is given by

$$(f \bowtie x)(g \bowtie y) = g_{(1)} \circ S^{-1}(x_{(3)}) g_{(3)}(x_{(1)}) f g_{(2)} \bowtie x_{(2)} y. \tag{64}$$

One can verify that the above multiplication and comultiplication indeed define a Hopf algebra. The unit of $\mathcal{D}(\mathcal{A})$ is $\varepsilon \bowtie 1$. Linear functionals on $\mathcal{D}(\mathcal{A})$ are of the form $x \bowtie f$ ($x \in \mathcal{A}$ and $f \in \mathcal{A}^*$) and in particular, the counit is given by $1 \bowtie \varepsilon$. Finally, the antipode in $\mathcal{D}(\mathcal{A})$ is given by $S(f \bowtie x) = S^{-1}(f) \bowtie S(x)$. If $\mathcal{A}$ is a C*-Hopf algebra, then $\mathcal{D}(\mathcal{A})$ is also a C*-Hopf algebra with $*$ operation $f \bowtie x \mapsto f^* \bowtie x^*$. Note that, as $\varepsilon_{(1)} \otimes \varepsilon_{(2)} \otimes \varepsilon_{(3)} = \varepsilon \otimes \varepsilon \otimes \varepsilon$, the map

$$\mathcal{A} \to \mathcal{D}(\mathcal{A}),\ x \mapsto \varepsilon \bowtie x$$

is both a homomorphism and a cohomomorphism. Similarly, as $1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} = 1 \otimes 1 \otimes 1$, the map

$$\mathcal{A}^* \to \mathcal{D}(\mathcal{A}),\ f \mapsto f \bowtie 1$$

is a homomorphism and an anti-cohomomorphism. Using the images of these maps, all elements in the Drinfeld double can be written as

$$f \bowtie x = (f \bowtie 1)(\varepsilon \bowtie x).$$

The elements $(f \bowtie 1)$ and $(\varepsilon \bowtie x)$ satisfy the following commutation relation:

$$(\varepsilon \bowtie x)(f \bowtie 1) = f_{(1)} \circ S^{-1}(x_{(3)}) f_{(3)}(x_{(1)})(f_{(2)} \bowtie 1)(\varepsilon \bowtie x_{(2)}).$$

Let $\mathcal{A}$ now be a finite dimensional Hopf algebra and $\mathcal{A}^*$ be its dual Hopf algebra. Let us now construct the Drinfeld double of $\mathcal{A}^*$. As a vector space, it is $\mathcal{A} \otimes \mathcal{A}^*$, and thus (as a vector space) it is canonically isomorphic to $\mathcal{D}(\mathcal{A})$. Let us make use of this isomorphism and write the elements of $\mathcal{D}(\mathcal{A}^*)$ as $f \bowtie x$ instead of $x \bowtie f$. In this notation, the comultiplication of the Drinfeld double $\mathcal{D}(\mathcal{A}^*)$ is given by

$$\Delta(f \bowtie x) = (f_{(1)} \bowtie x_{(2)}) \otimes (f_{(2)} \bowtie x_{(1)}),$$

and the multiplication is given by

$$(f \bowtie x)(g \bowtie y) = f_{(3)} \circ S^{-1}(y_{(1)}) f_{(1)}(y_{(3)}) f_{(2)} g \bowtie x y_{(2)}. \tag{65}$$

Again, the maps

$$\mathcal{A} \to \mathcal{D}(\mathcal{A}^*) : \ x \mapsto \varepsilon \bowtie x \quad \text{and} \quad \mathcal{A}^* \to \mathcal{D}(\mathcal{A}^*) : \ f \mapsto f \bowtie 1$$

are homomorphisms. Similar as above, every element of $\mathcal{D}(\mathcal{A}^*)$ can be written as

$$f \bowtie x = (\varepsilon \bowtie x)(f \bowtie 1).$$

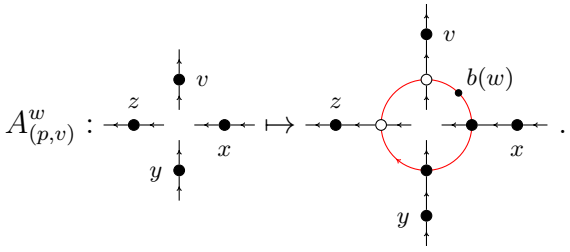
The elements $(\varepsilon \bowtie x)$ and $(f \bowtie 1)$ of $\mathcal{D}(\mathcal{A}^*)$ satisfy the commutation relation

$$(f \bowtie 1)(\varepsilon \bowtie x) = f_{(3)} \circ S^{-1}(x_{(1)}) f_{(1)}(x_{(3)})(\varepsilon \bowtie x_{(2)})(f_{(2)} \bowtie 1). \tag{66}$$

Let us now define a set of local operators acting on an $\mathcal{A}$-injective PEPS and show that they form a representation of the Drinfeld double $\mathcal{D}(\mathcal{A}^*)$. All of these operators will act on the particles surrounding a neighboring plaquette and vertex pair (p, v). We will define two types of operators. The first type is denoted by $A^w_{(p,v)}$ for any $w \in \mathcal{A}$, it acts only on the particles surrounding the vertex and represents the element $\varepsilon \bowtie w \in \mathcal{D}(\mathcal{A}^*)$. The second type is denoted by $B^f_{(p,v)}$ for any $f \in \mathcal{A}^*$, it acts only on the particles surrounding the plaquette and it represents the element $f \bowtie 1 \in \mathcal{D}(\mathcal{A}^*)$.

Let us first define the operators $A^w_{(p,v)}$ for a given plaquette p, vertex v and algebra element $w \in \mathcal{A}$. We define such an operator by its action on the four particles around the vertex; this action is defined by the graphical representation, using the injective representations $\Phi_{(v,w)}$ on

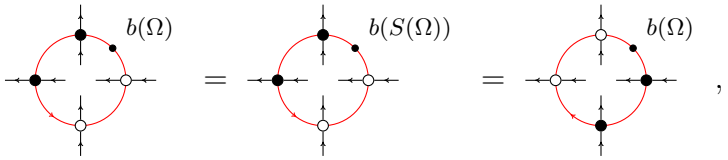
each edge. In the figure below, the plaquette p is in the upper right corner of the vertex. The action of $A^w_{(p,v)}$ is given by



Note that the MPO representing this operator is oriented in the opposite way as in the definition of the PEPS and in the definition of the vertex term of the Hamiltonian. This expression reads as

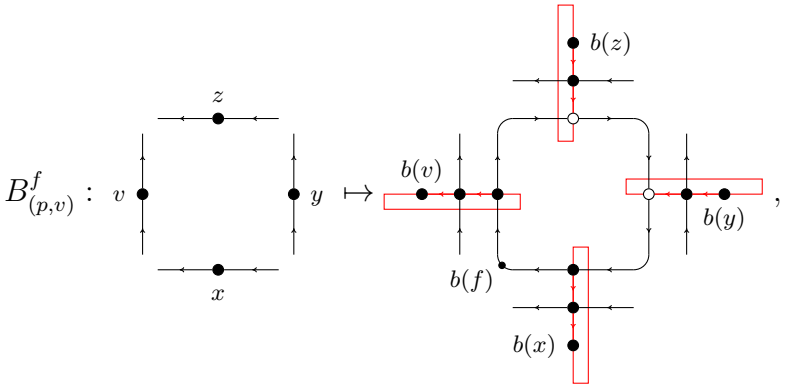
$$A^w_{(p,v)} : x \otimes y \otimes z \otimes v \mapsto w_{(4)}x \otimes w_{(3)}y \otimes zS(w_{(2)}) \otimes vS(w_{(1)}) \ .$$

Here and in what follows we omit the representations Φ when translating between the figures and the algebraic formulas. It is easy to check that the operators $A^w_{(p,v)}$ form a representation of $\mathcal{A}$ and that this representation is a $*$-representation. Notice that the vertex term A_v is exactly $A^\Omega_{(p,v)}$, because the Haar integral Ω is invariant under the antipode S:



where the last equation is Equation 36. The plaquette term A_v is the multiplication with the MPO on the left, while $A^\Omega_{(p,v)}$ is multiplication with the MPO on the right. As Ω is cocommutative, the particular choice of the plaquette p does not matter, i.e. $A_v = A^\Omega_{(p,v)}$ holds for any choice of the plaquette p next to the vertex v.

Let us now define the operator $B^f_{(p,v)}$ for the plaquette p, vertex v and linear functional $f \in \mathcal{A}^*$. In the figure below, the vertex v is in the lower left corner of the plaquette p. The action of $B^f_{(p,v)}$ is given by



or equivalently, by

$$B^f_{(p,v)} : x \otimes y \otimes z \otimes v \mapsto f(x_{(2)}S(y_{(1)})S(z_{(1)})v_{(2)})x_{(1)} \otimes y_{(2)} \otimes z_{(2)} \otimes v_{(1)}.$$

It is easy to check that these operators form a $*$-representation of $\mathcal{A}^*$. The plaquette term B_p of the Kitaev Hamiltonian is exactly the operator $B^\omega_{(p,v)}$, where ω is the Haar integral of $\mathcal{A}^*$. As ω is cocommutative, the particular choice of v does not matter, i.e. $B_p = B^\omega_{(p,v)}$ for any vertex v around the plaquette p.

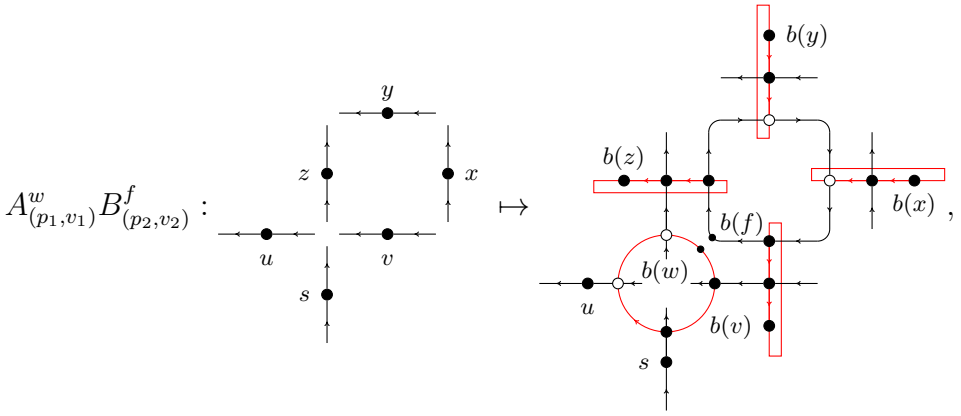
Let us now define a linear map $\mathcal{D}(\mathcal{A}^*) \to \mathfrak{B}(\mathcal{A}^6)$ for a given pair of plaquette p and vertex v by

$$f \bowtie w \mapsto A^w_{(p,v)} B^f_{(p,v)}, \tag{67}$$

for all $f \in \mathcal{A}^*$ and $w \in \mathcal{A}$, and by linear extension, on the whole $\mathcal{D}(\mathcal{A})$. Below we show that this map defines a representation of the Drinfled double $\mathcal{D}(\mathcal{A})$ (see also [19]). As the maps

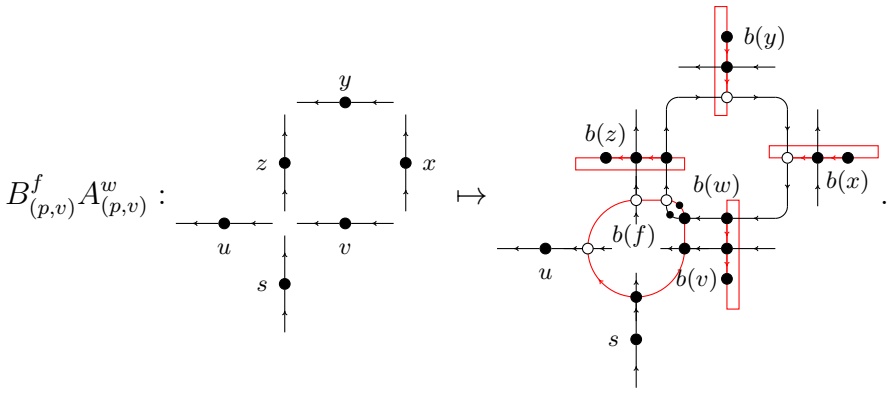
$$f \bowtie \varepsilon \mapsto B^f_{(p,v)} \quad \text{and} \quad \varepsilon \bowtie w \mapsto A^w_{(p,v)}$$

form representations of $\mathcal{A}^*$ and $\mathcal{A}$, respectively, and $(f \bowtie w) = (\varepsilon \bowtie w)(f \bowtie 1)$, we only have to check that the commutation relation Equation 66 holds. Let us therefore compare the action of $A^w_{(p,v)} B^f_{(p,v)}$ and the action of $B^f_{(p,v)} A^w_{(p,v)}$. The graphical representation of the action of the operator $A^w_{(p,v)} B^f_{(p,v)}$ is the following:



where v is the vertex with all four edges drawn, and p is the plaquette with all four bordering edges drawn. Let us now depict the action of the operator $B^f_{(p,v)} A^w_{(p,v)}$. To do that, note that as we act first with the operator $A^w_{(p,v)}$, the coproduct in the operator $B^f_{(p,v)}$ also applies to the MPO representation of w. The graphical representation of this action

is thus

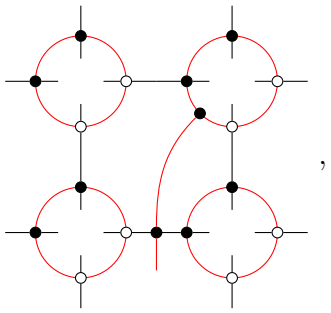


This operator describes the action of the operator

$$f_{(3)} \circ S(w_{(1)}) f_{(1)}(w_{(3)}) A^{w_{(2)}}_{(p,v)} B^{f_{(2)}}_{(p,v)},$$

and thus, as this commutation relation is the same[2] as Equation 66, the map in Equation 67 describes a representation of the Drinfeld double $\mathcal{D}(\mathcal{A}^*)$.

Let us now construct a set of states that differ from $|\psi\rangle$ only on a given pair of plaquette p and vertex v. The states, in the bulk, are given by

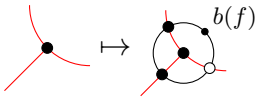


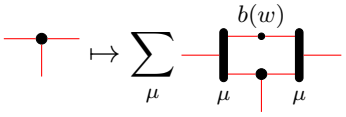
,

i.e. at the position defined by the vertex and plaquette we insert a rank-three tensor in $W \otimes \Psi(\mathcal{A}) \subset W \otimes \mathfrak{B}(W)$, where W is the virtual vector space of the MPO and Ψ is the corresponding representation of $\mathcal{A}^*$, and we continue with the MPO starting from the given point. If the state is defined by open boundary, then the boundary has one more index than the boundary describing the ground state $|\psi\rangle$ of the PEPS. If the state is defined on closed boundary, then the string has to terminate at some point; at this termination we will insert another rank-three tensor the same way as above. That is, on closed boundary, we do not define a state with a single defect in it, instead, only states that differ from $|\psi\rangle$ at least in two different positions (note, however, that one can construct periodic boundary states with an odd number of defects).

[2] remember that $\mathcal{A}$ is a C*-Hopf algebra, and thus $S^{-1} = S$

Both operators $B^f_{(p,v)}$ and $A^w_{(p,v)}$ map a state with a defect at the pair of plaquette p and vertex v to another state with a defect at the same position, but this state might be described by a different rank-three tensor. The action of the operators $B^f_{(p,v)}$ and $A^w_{(p,v)}$ on the rank-three tensor can be depicted as

$B^f_{(p,v)}$: 

and $A^w_{(p,v)}$: 

Or, by formulas, if the tensor is given by $|v\rangle \otimes b(x)$, then

$$B^f_{(p,v)} : |v\rangle \otimes b(x) \mapsto \Psi(f_{(2)})|v\rangle \otimes f_{(1)}(x_{(1)})f_{(3)} \circ S(x_{(3)})b(x_{(2)}),$$
$$A^w_{(p,v)} : |v\rangle \otimes b(x) \mapsto |v\rangle \otimes b(xw).$$

These states thus form a $\mathcal{D}(\mathcal{A}^*)$-module. We identify the *anyons* of the model as the sectors of this module, i.e. an anyon is a set of states, each of which locally differ from the ground state of the Hamiltonian and such that the above defined local operators do not mix the different anyons. Note that as the Hamiltonian, in general, is not in the center of $\mathcal{D}(\mathcal{A}^*)$, an anyon might not have a definite energy (i.e. the different states in the anyonic sector might have different energies, see also [74]).

CHAPTER 8

Renormalization Fixed Point MPDOs

In this chapter we define a distinguished family of MPOs starting from a biconnected C*-WHA (see Definition 5.17 and Definition 5.3) and show that they are Renormalization Fixed Point (RFP) MPDOs, as defined in [**31**]. More concretely, we provide explicit expressions for both local coarse-graining and local fine-graining quantum channels $\mathfrak{T}$ and $\mathfrak{S}$ for which the generating rank-four tensor is a fixed point under the corresponding induced flows, very much in the spirit of standard renormalization. The generating tensor of the RFP MPDOs is obtained here by appropiately weighting the tensor from the original MPO algebra, described in the previous Chapters 3 to 5, obtaining:

This weighting is done by means of the element constructed in Theorem 5.21 of the dual C*-WHA, known as the *canonical regular element*. See Appendix C for a review on its description. To this end, let us examine first the properties of this linear functional, which formally plays the role in C*-WHAs of the character of the usual left-regular representation.

Lemma 8.1. *Let $\mathcal{A}$ be a connected C*-WHA. Then, the canonical regular element $\omega \in \mathcal{A}^*$ of the dual C*-WHA $\mathcal{A}^*$ is the unique trace-like, faithful, positive linear functional on $\mathcal{A}$ that is idempotent, i.e. $(\omega \otimes \omega) \circ \Delta = \omega$.*

Proof. Recall Theorem 5.21 and Proposition 5.20. We have already seen that $\omega \in \mathcal{A}^*$ is a trace-like linear functional since it is a cocentral element of $\mathcal{A}^*$. Also, it is a faithful and positive linear functional by construction. In addition, it satisfies the eigenvalue equation $\hat{\tau}_\alpha \cdot \omega = \omega \cdot \hat{\tau}_\alpha = d_\alpha \omega$ for all sectors $\alpha \in \mathrm{Irr}(\mathcal{A})$, where $\hat{\tau}_\alpha$ stands for the character of the sector α; see Section 2 from Chapter 5, Appendix C or [**43**, Section 3]. We note that this implies, in particular, that $\omega \in \mathcal{A}^*$ is idempotent. Assume now that $f \in \mathcal{A}^*$ is any linear functional satisfying the properties above. Since it is trace-like, it can be expanded in the form $f = \sum_\alpha f_\alpha \hat{\tau}_\alpha$ for some numbers $f_\alpha \in \mathbb{C}$, $\alpha \in \mathrm{Irr}(\mathcal{A})$. By evaluating f on the primitive central idempotents of $\mathcal{A}$ it is easy to check that $f_\alpha > 0$ for all $\alpha \in \mathrm{Irr}(\mathcal{A})$, since f is assumed to be also a faithful positive linear functional. Define the $|\mathrm{Irr}(\mathcal{A})| \times |\mathrm{Irr}(\mathcal{A})|$ matrix

N_f with complex coefficients $(N_f)_{\beta\gamma} := \sum_\alpha f_\alpha N_{\alpha\beta}^\gamma$, which implements the left-multiplication by $f \in \mathcal{A}^*$ in the basis $\{\hat{\tau}_\alpha : \alpha \in \mathrm{Irr}(\mathcal{A})\}$, i.e. it satisfies $N_f\psi = f\psi$ for all $\psi \in \mathcal{A}^*$. Then, $N_f f = f^2 = f$ and $N_f\omega = f\omega = \sum_\alpha f_\alpha d_\alpha \omega \propto \omega$, where the first equation holds since $f \in \mathcal{A}^*$ is idempotent by hypothesis and the second equation follows from the eigenvalue equation. Since $\mathcal{A}$ is connected, the Grothendieck ring (see Proposition 4.3) is, in particular, a transitive ring, and hence N_f has strictly positive entries; see e.g. Section 5 from Chapter 5, [**96**] and [**43**, Section 3]. By virtue of the Frobenius-Perron theorem, it has only one eigenvector with strictly positive entries, up to a constant. Therefore, $f = \omega$, since both are positive idempotents. □

Now, given a faithful $*$-representation of the C*-WHA, we define the appropiate weight extending the previous linear functional to the representation space.

Remark 8.2. Let $\mathcal{A}$ be a connected C*-WHA and let (V, Φ) be a faithful $*$-representation of $\mathcal{A}$. Let $b(f)$ denote the boundary weight for the matrix product operators arising from the dual C*-WHA $\mathcal{A}^*$, for all $f \in \mathcal{A}^*$. It turns out that $b(\omega) = \Phi(c_\omega)$ for some strictly positive central element $c_\omega \in \mathcal{A}$. It provides an extension of $\omega \in \mathcal{A}^*$ to the representation space $\mathfrak{B}(V)$ in the sense that

$$\mathrm{Tr}(b(\omega)\Phi(x)) = \mathrm{Tr}(\Phi(c_\omega x)) = \omega(x)$$

for all elements $x \in \mathcal{A}$.

Proof. For all sectors $\alpha \in \mathrm{Irr}(\mathcal{A})$, let $e_\alpha \in \mathcal{A}$ be the corresponding primitive central idempotent of $\mathcal{A}$ and let $\nu_\alpha \in \mathbb{C}$ denote the multiplicity of (V_α, Φ_α) within (V, Φ). Then, define the element

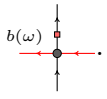
$$c_\omega := \mathcal{D}^{-2} \sum_\alpha d_\alpha \nu_\alpha^{-1} e_\alpha \in \mathcal{A}.$$

Trivially, it is a central invertible positive element and satisfies

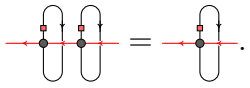
$$\begin{aligned}\mathrm{Tr}(\Phi(c_\omega x)) &= \mathcal{D}^{-2} \sum_\alpha \nu_\alpha^{-1} d_\alpha \mathrm{Tr}(\Phi(x e_\alpha)) \\ &= \mathcal{D}^{-2} \sum_\alpha \nu_\alpha^{-1} d_\alpha \nu_\alpha \hat{\tau}_\alpha(x) = \omega(x)\end{aligned}$$

for all elements $x \in \mathcal{A}$, as we wanted to prove. □

Let us now consider the tensor obtained by multiplying the MPO tensor in Chapter 4 by $b(\omega) = \Phi(c_\omega)$ in the physical space:



Idempotence of $\omega \in \mathcal{A}^*$ implies that this tensor generates an MPO with zero correlation length; see [31]:



It is clear that computations of correlation functions using the MPDOs generated by the previous tensor will be length-independent. In particular, it induces the following family of mixed states:

THEOREM 8.3. *Let $\mathcal{A}$ be a biconnected C*-WHA and let (V, Φ) be a faithful $*$-representation of $\mathcal{A}$. Then, the operators*

$$\rho(x, n) := \omega(x)^{-1} b(\omega)^{\otimes n} \Phi^{\otimes n}(\Delta^{(n-1)}(x)) \in \mathfrak{B}(V^{\otimes n})$$

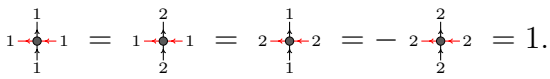
are RFP MPDOs for all positive non-zero elements $x \in \mathcal{A}$ and all $n \in \mathbb{N}$. Specifically, there are quantum channels $\mathfrak{T} : \mathfrak{B}(V) \to \mathfrak{B}(V \otimes V)$ and $\mathfrak{S} : \mathfrak{B}(V \otimes V) \to \mathfrak{B}(V)$, known as local fine-graining and coarse-graining maps, respectively, such that

$$\mathfrak{T}(\rho(x, 1)) = \rho(x, 2) \quad \textit{and} \quad \mathfrak{S}(\rho(x, 2)) = \rho(x, 1)$$

for all positive non-zero elements $x \in \mathcal{A}$ and all $n \in \mathbb{N}$.

Let us illustrate the construction with an extremely modest example.

Example 8.4. Let $\mathcal{A} := \mathbb{C}\mathbb{Z}_2$ be the C*-HA arising from the group $G := \mathbb{Z}_2$ generated by $g \in G$; see Example 5.24. It posseses only two sectors, namely the equivalence classes of the trivial representation and the sign representation, each one-dimensional. Consider that both physical and virtual spaces are $V := W := \mathbb{C}^2$, with basis elements $|1\rangle, |2\rangle$, and consider the faithful $*$-representation of $\mathcal{A}$ $\Phi : \mathcal{A} \to \mathfrak{B}(\mathbb{C}^2)$ defined by $\Phi(g) := \sigma_z$, the usual Pauli-Z matrix. It is easy to see that both Frobenius-Perron dimensions are 1 and hence the canonical regular elements of $\mathcal{A}$ and $\mathcal{A}^*$ are given by $\Omega = 2^{-1}(e + g)$ and $\omega(x) = (x, e)_V$, for all $x \in \mathcal{A}$, respectively. A tensor generating the corresponding MPOs is specified by the non-zero coefficients

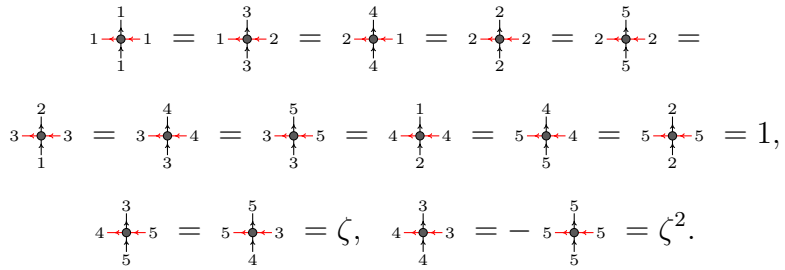


Moreover, in this case the weight is trivially given by $c_\omega = 2^{-1}e$ and thus

$$\rho(x, n) = \frac{1}{2^n}(\mathbf{1}^{\otimes n} + \frac{(x, g)_V}{(x, e)_V}\sigma_z^{\otimes n}),$$

are the induced RFP MPDOs, for all positive non-zero $x \in \mathcal{A}$. In particular, $\rho(\Omega, n) = 2^{-n}(\mathbf{1}^{\otimes n} + \sigma_z^{\otimes n})$ is the boundary state of the toric code; see [31].

Example 8.5. Let $\mathcal{A}_{\mathrm{LY}}$ be the Lee-Yang C*-WHA from Example 5.27. It possesses only two sectors, denoted 1 and τ, for which it is easy to check that $d_1 = 1$ and $d_\tau = \zeta^{-2} = 2^{-1}(1+\sqrt{5})$, respectively. Consider that both physical and virtual spaces are $V := W := \mathbb{C}^5$ and let $\Phi : \mathcal{A}_{\mathrm{LY}} \to \mathfrak{B}(\mathbb{C}^5)$ be the faithful $*$-representation arising from the string-net specification; see Appendix B and [**20**] for a derivation. A tensor generating the corresponding MPOs is then specified by the non-zero coefficients



Finally, it is easy to check that $\Phi(c_\omega) = 2(5+5^{1/2})^{-1}\mathbf{1}_2 \oplus 5^{-1/2}\mathbf{1}_3$.

With the aim of giving explicit definitions of both quantum channels and prove Theorem 8.3, we introduce the following auxiliary result.

Lemma 8.6. *Let $\mathcal{A}$ be a biconnected C*-WHA. There exists a unique element $\xi \in \mathcal{A}$ such that $\omega(\xi T(\Omega_{(1)}))\Omega_{(2)} = 1$. Furthermore, it satisfies the following properties:*

(1) *it is strictly positive and $\xi^{-1} = \omega(\Omega_{(1)})\Omega_{(2)} = \omega(T(\Omega_{(1)}))\Omega_{(2)}$;*
(2) *it is invariant under $T \in \mathfrak{B}(\mathcal{A})$, i.e. $T(\xi) = \xi$;*
(3) *it satisfies $T(x)^* = \xi T(x^*)\xi^{-1}$ for all elements $x \in \mathcal{A}$;*
(4) *$\hat{\tau}_\alpha(\xi^{-1}) = d_\alpha \omega(\Omega)$ for all sectors $\alpha \in \mathrm{Irr}(\mathcal{A})$, where $\hat{\tau}_\alpha$ stands for the character of the sector α;*
(5) *it can be decomposed as $\xi = \xi_L \xi_R$ for two positive elements $\xi_L \in \mathcal{A}_L$ and $\xi_R = S(\xi_L) = S^{-1}(\xi_L) \in \mathcal{A}_R$;*

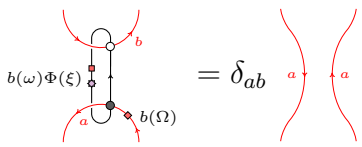
Dually, if we denote $\hat{\xi} = \hat{\xi}_L \hat{\xi}_R \in \mathcal{A}^*$, then:

(6) *$x_{(1)}\hat{\xi}_L(x_{(2)}) = \xi_L^{-1}x$ and $x_{(1)}\hat{\xi}_R^{-1}(x_{(2)}) = x\xi_L^{-1}$ for all $x \in \mathcal{A}$.*

Finally, if $\mathcal{A}$ is a C-HA, then $\xi_L^2 = \xi_R^2 = \xi = \mathcal{D}^2\varepsilon(1)1 = \omega(\Omega)^{-1}1$.*

See Appendix D for a proof. The fundamental property of the definition of $\xi \in \mathcal{A}$ here, interpreted in terms of tensor networks, is provided by the following result.

Lemma 8.7. *Let $\mathcal{A}$ be a biconnected C*-WHA. Then,*

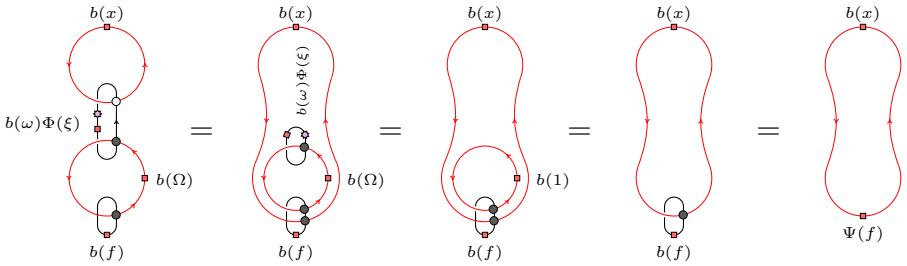


for all sectors $a, b \in \mathrm{Irr}(\mathcal{A})$, where δ_{ab} stands for the Kronecker delta.

PROOF. Note that

$$\begin{aligned} f(\Omega_{(1)})\omega(\xi T(x)\Omega_{(2)}) &= f(T(\xi T(x))\Omega_{(1)})\omega(\Omega_{(2)}) && \text{by Equation 59} \\ &= f(x\xi\Omega_{(1)})\omega(\Omega_{(2)}) && \text{by Theorem 5.21} \\ &= f(x\xi\xi^{-1}) = f(x) && \text{by Lemma 8.6} \end{aligned}$$

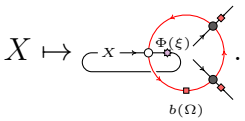
for any two elements $x \in \mathcal{A}$ and $f \in \mathcal{A}^*$. Pictorially:



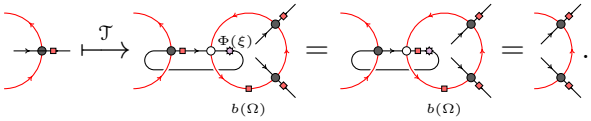
The results follows from the surjectivity of b and Ψ in each block. □

We are now in the position to partially prove that the MPOs generated by the MPO tensor presented above are RFP.

PROOF OF THEOREM 8.3. Define $\mathfrak{T} : \mathfrak{B}(V) \to \mathfrak{B}(V \otimes V)$ by



Trivially, it has the property of duplicating the tensor defining the MPDO:

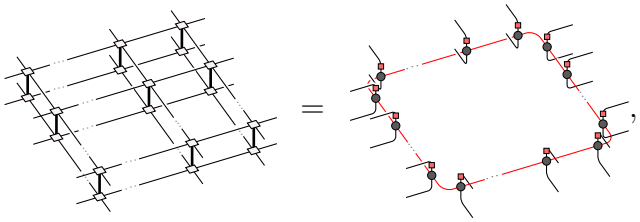


In the first equality we have used that the weight $\Phi(c_\omega) \in \mathfrak{B}(V)$ can be freely moved along the physical indices since $c_\omega \in \mathcal{A}$ is a central element. The second equality follows from Lemma 8.7. We postpone the proof of the fact that it is a quantum channel and the definition of the quantum channel $\mathfrak{S}$ to Appendix E. □

CHAPTER 9

RFP MPDOs are boundary states of topological 2D RFP PEPS

In this chapter we show that RFP MPDOs $\rho(\Omega, n)$ defined in Theorem 8.3 arise as boundary states of MPO-injective 2D PEPS with certain properties, see Chapter 7. For simplicity, we will restrict here to PEPS defined in a rectangular lattice. The boundary state associated to a 2D PEPS is obtained by contracting the physical indices of the 2D PEPS with open boundaries and its conjugate transpose. Pictorially:

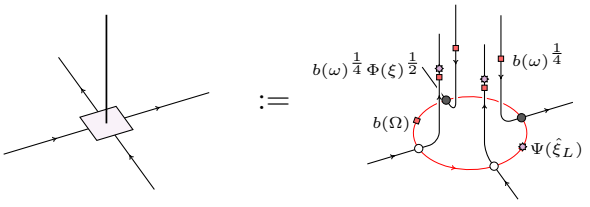


The boundary state is known to encode relevant properties of the 2D PEPS, such as for instance the gap of its parent Hamiltonian [**30, 67**], making the Li-Haldane [**81**] bulk-boundary correspondence very explicit in the context of PEPS. Let us prove the following theorem.

THEOREM 9.1. *For any regular biconnected C*-WHA $\mathcal{A}$, RFP MPDOs defined in Theorem 8.3 are boundary states of $\mathcal{A}$-injective 2D PEPS fulfilling a renormalization fixed point property.*

PROOF. Fix a regular biconnected C*-WHA $\mathcal{A}$ and a faithful $*$-representation (V, Φ). As commented in Chapters 3 and 4, the associate MPO tensors are described in terms of another $*$-representation (W, Ψ) of $\mathcal{A}^*$ in the virtual level. Let us first construct the ansatz tensor for the 2D PEPS whose boundary state is the given matrix product density operator $\rho(\Omega, n)$. For the sake of simplicity, we will restrict to underlying geometries described by square lattices, although the proof works for any 2D PEPS defined on any directed pseudo-graph. In this case, we will consider the 2D PEPS tensor depicted as follows:

(68)



Here, the physical space is given by the tensor product $V \otimes V \otimes V^* \otimes V^*$ and there are four virtual indices, each of them corresponding to V or V^*. If read in clockwise direction starting from the virtual weight $b(\Omega) \in \mathfrak{B}(W)$, the tensor is algebraically described by the expresion

$$(b(\omega)^{\frac{1}{4}})^{\otimes 4}\big(\Phi(\Omega_{(1)}) \otimes \Phi(\Omega_{(2)}) \otimes \hat{\xi}_L(\Omega_{(3)})\Phi(\xi^{\frac{1}{2}}T(\Omega_{(4)})) \otimes \Phi(\xi^{\frac{1}{2}}T(\Omega_{(5)}))\big)$$

as an operator from physical to virtual spaces. Recall that $b(\omega) \in \mathfrak{B}(V)$ is an invertible positive central operator and hence it can be freely moved along the physical indices. Also, it defines an $\mathcal{A}$-injective PEPS as constructed in Chapter 7; see Appendix D. Let us now show that the boundary operator is the desired operator.

Step 1. Let us first simplify the transfer operator associated to the previous 2D PEPS tensor, $\mathbb{E} = \mathbb{E}(\mathcal{A}, V, \Phi) \in \mathfrak{B}(V \otimes V \otimes V^* \otimes V^*)$ obtained by contracting the physical indices of the 2D PEPS tensor and its corresponding conjugate transpose if regarded as an operator. Algebraically, it is given by the expression

$$\begin{aligned}\mathbb{E} = (b(\omega)^{\frac{1}{2}})^{\otimes 4}\big(&\Phi(\Omega_{(1)})\Phi(\Omega_{(1')})^\dagger \otimes \Phi(\Omega_{(2)})\Phi(\Omega_{(2')})^\dagger \\ &\otimes \overline{\hat{\xi}_L^{-1}(\Omega_{(3')})}\hat{\xi}_L^{-1}(\Omega_{(3')})\Phi(T(\Omega_{(4')}))^\dagger\Phi(\xi)\Phi(T(\Omega_{(4)})) \\ &\otimes \Phi(T(\Omega_{(5')}))^\dagger\Phi(\xi)\Phi(T(\Omega_{(5)}))\big),\end{aligned}$$

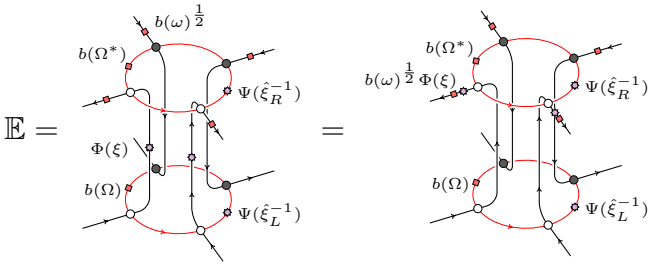
where we have employed that $b(\omega) \in \mathfrak{B}(V)$ is positive and central and $\Phi(\xi) \in \mathfrak{B}(V)$ is positive, since $\Phi : \mathcal{A} \to \mathfrak{B}(V)$ is a $*$-representation and $\xi \in \mathcal{A}$ is positive. Note that the order of composition is reversed for the terms associated to white tensors. In order to fully describe $\mathbb{E}$ in terms of tensor networks, note that

$$\overline{\hat{\xi}_L^{-1}(x)} = ((\hat{\xi}_L^{-1})^*)(S(x)^*) = \hat{\xi}_L^{-1}(S(x)^*) = \hat{\xi}_L^{-1}(S^{-1}(x^*)) = \hat{\xi}_R^{-1}(x^*)$$

for all $x \in \mathcal{A}$, where the first equality is due to Equation 55, the second equality follows from the positivity of $\hat{\xi}_L \in \mathcal{A}^*$, the third equality is due to Equation 56 and the fourth equality follows from the definition of $\hat{\xi}_R \in \mathcal{A}^*$, see Lemma 8.6. In addition, recall that $\Phi : \mathcal{A} \to \mathfrak{B}(V)$ is a $*$-representation and $T(x)^*\xi = \xi T(x^*)$ for all $x \in \mathcal{A}$, see Lemma 8.6. Therefore:

$$\begin{aligned}\mathbb{E} = (b(\omega)^{\frac{1}{2}})^{\otimes 4}\big(&\Phi(\Omega_{(1)}\Omega_{(1')}^*) \otimes \Phi(\Omega_{(2)}\Omega_{(2')}^*) \\ &\otimes \hat{\xi}_R^{-1}(\Omega_{(3')}^*)\hat{\xi}_L^{-1}(\Omega_{(3)})\Phi(\xi T(\Omega_{(4')}^*)T(\Omega_{(4)})) \otimes \Phi(\xi T(\Omega_{(5')}^*)T(\Omega_{(5)}))\big).\end{aligned}$$

Hence, the transfer operator can be represented graphically as follows:

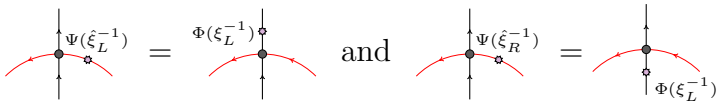


On the other hand, $\Psi(\hat{\xi}_L^{-1})$ and $\Psi(\hat{\xi}_R^{-1})$ can be "moved" from the virtual to the physical spaces using the following identities:

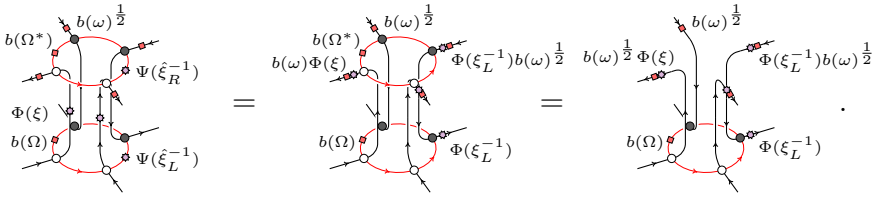
$$x_{(1)}\hat{\xi}_L^{-1}(x_{(2)}) = \xi_L^{-1}x \quad \text{and} \quad x_{(1)}\hat{\xi}_R^{-1}(x_{(2)}) = x\xi_L^{-1}$$

for all elements $x \in \mathcal{A}$; see Lemma 8.6. In graphical notation, the previous formulas are rephrased in the following form:

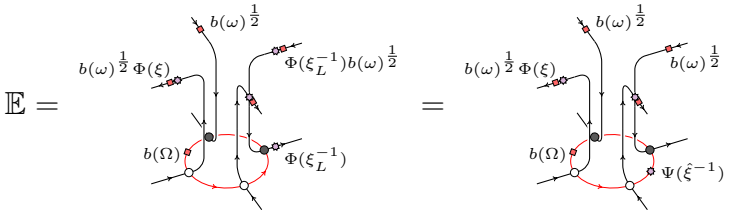
(69)



By virtue of these identities, the fact that $T \in \mathfrak{B}(\mathcal{A})$ is an algebra anti-homomorphism and $\Omega^* = \Omega = \Omega^2$ by Theorem 5.21, it follows that



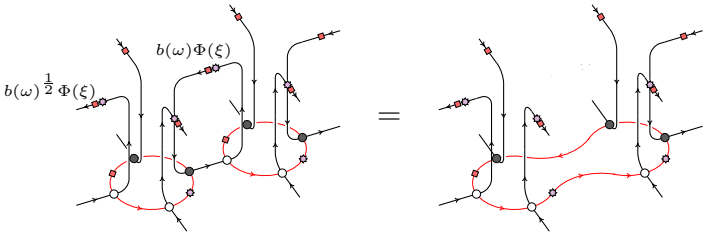
Applying again Equation 69, we conclude that the transfer operator takes the form:



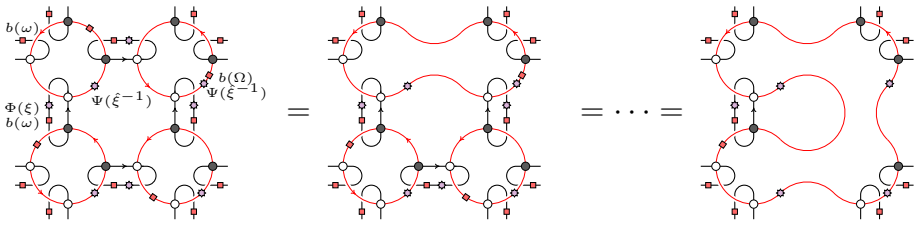
since $\Psi \in \mathfrak{B}(\mathcal{A}^*, \mathfrak{B}(W))$ is a $*$-representation and $\xi^{-1} = \xi_L^{-1}\xi_R^{-1}$.

Step 2. Let us consider the concatenation of two transfer operators. By virtue of the pulling-through identity Equation 59,

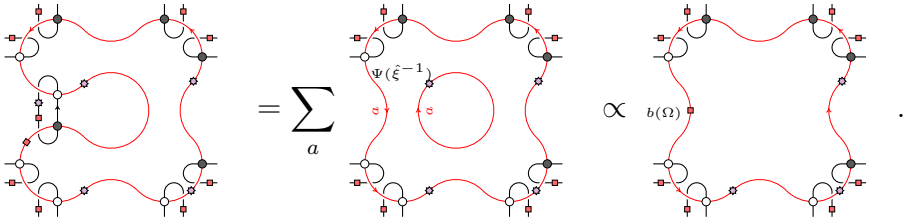
(70)



Step 3. Let us consider the concatenation of four transfer operators around the vertices of a plaquette. In particular, we prove that the ansatz 2D PEPS tensors gives rise to a *normalized* PEPS and its boundary state is the RFP MPDO defined in Theorem 8.3. Consider, from a top view, the procedure of simplifying the concatenation of transfer operators that form a whole plaquette:



To close a plaquette, recall Lemma 8.7 and Lemma 8.6:



Note that in the previous equations the inner circle representing $\tau_a(\hat{\xi}^{-1})$ is not independent of the outer shape and hence it gives rise to possibly different constant in each sector, as it is a sum over all sectors $a \in \mathrm{Irr}(\mathcal{A}^*)$. As showed in Lemma 8.6, these are precisely the Frobenius-Perron dimensions which define, in each sector, the canonical regular element $\Omega \in \mathcal{A}$. Therefore we can rewrite it in terms of the weight $b(\Omega) \in \mathfrak{B}(W)$, as done in the last equality. Iterating this procedure for each plaquette of the lattice proves that matrix product density operators defined in the previous chapter arise naturally as boundary states of topological 2D PEPS.

Note also that Equation 70 is nothing but a natural 2D generalization of the renormalization fixed point condition for MPS defined in [**31**]. In that sense, we can conclude that the RFP MPDOs considered in Theorem 8.3 are boundary states of PEPS fulfilling this renormalization fixed point property. □

CHAPTER 10

Classification of RFP MPDOs via circuits of quantum channels

In this chapter we prove that RFP MPDOs arising from C*-HAs belong to the trivial phase. Namely, we provide explicit definitions of depth-two circuits of finite-range quantum channels that map the maximally mixed state to these RFP MPDOs. Finally, we show that our construction cannot be extended to arbitrary biconnected C*-WHAs, which lead us to the conjecture that there are non-trivial phases in this context.

1. An illustrative example

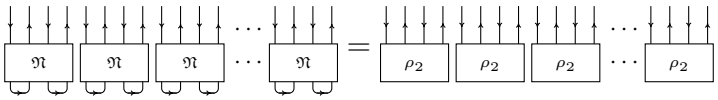
In order to deepen the intuition towards the general case of C*-HAs, let us first examine the simplest non-trivial example.

Example 10.1. RFP MPDOs arising from the group C*-HA $\mathcal{A} := \mathbb{C}\mathbb{Z}_2$, introduced in Examples 5.24 and 8.4 are in the trivial phase. Specifically, we build

$$\rho(\Omega, n) = \tfrac{1}{2^n}(\mathbf{1}^{\otimes n} + \sigma_z^{\otimes n})$$

via a depth-two circuit of range-two quantum channels from the maximally mixed state $\mathrm{Tr}(\mathbf{1})^{-n}\mathbf{1}^{\otimes n}$. We assume without loss of generality that $n \in \mathbb{N}$ is even and propose the following procedure:

Step 1 ("initialization"). We first construct $n/2$ copies $(\rho_2)^{\otimes n/2}$ of the mixed state ρ_2 between pairs of nearest neighbors by replacing the product states separately. This is easily done by means of the quantum channel $\mathfrak{N} : X \otimes Y \mapsto \mathrm{Tr}(X \otimes Y)\rho_2$. In the Choi-Jamiołkowski picture, this process can be depicted as follows:



When the system size is an odd natural number simply replace three of them with the mixed state ρ_3, for example.

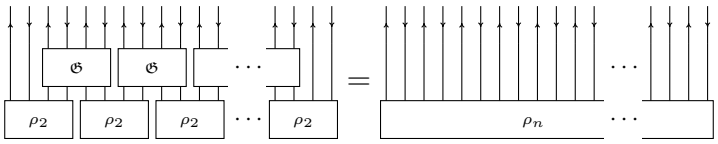
Step 2 ("gluing"). Now, we "glue" together all these copies of ρ_2 in order to obtain the target mixed state ρ_n. This is done inductively by means of the following quantum channel, called from now on *gluing* map:

$$\mathfrak{G} : X \otimes Y \mapsto \tfrac{1}{2^2}(\mathrm{Tr}(X \otimes Y)\mathbf{1} \otimes \mathbf{1} + \mathrm{Tr}(X\sigma_z \otimes Y\sigma_z)\sigma_z \otimes \sigma_z) \tag{71}$$

for all $X, Y \in \mathfrak{B}(\mathbb{C}^2)$. It is easy to check that it is a quantum channel and that

$$\mathrm{Id} \otimes \mathfrak{G} \otimes \mathrm{Id}(\tfrac{1}{2^2}(\mathbf{1}^{\otimes 2} + \sigma_z^{\otimes 2}) \otimes \tfrac{1}{2^2}(\mathbf{1}^{\otimes 2} + \sigma_z^{\otimes 2})) = \tfrac{1}{2^4}(\mathbf{1}^{\otimes 4} + \sigma_z^{\otimes 4}).$$

By induction, it is clear that simultaneous applications of these quantum channels lead to the mixed state ρ_n. Again, in the Choi-Jamiołkowski picture this procedure can be depicted as follows:



2. Phase classification of C*-Hopf algebras

The previous construction can be generalized to arbitrary C*-HAs as follows. In the first place, the role of the previous element is replaced by the RFP MPDO associated to the canonical regular element. In addition, we introduce a family of quantum channels that "glue" together two RFP MPDOs associated to the canonical regular element $\Omega \in \mathcal{A}$ into a larger one, associated to any arbitrary positive non-zero element of $\mathcal{A}$.

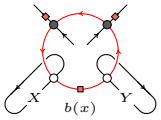
Lemma 10.2. *Let $\mathcal{A}$ be a C*-HA and let (V, Φ) be a faithful $*$-representation of $\mathcal{A}$. Then, for all positive non-zero elements $x \in \mathcal{A}$ there exists a quantum channel $\mathfrak{G}_x : \mathfrak{B}(V \otimes V) \to \mathfrak{B}(V \otimes V)$, called "gluing" map, such that*

$$(\mathrm{Id}^{\otimes m-1} \otimes \mathfrak{G}_x \otimes \mathrm{Id}^{\otimes n-1})(\rho(\Omega, m) \otimes \rho(\Omega, n)) = \rho(x, m+n) \tag{72}$$

for all $m, n \in \mathbb{N}$.

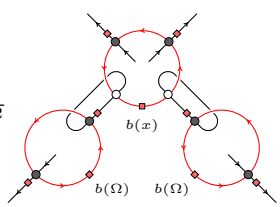
See Appendix F for a proof, but let us propose now an explicit expression for the gluing map and check using graphical notation that Equation 72 holds. To this end, fix any positive non-zero element $x \in \mathcal{A}$ and assume without loss of generality that $m = n = 2$. Define the map $\mathfrak{G}_x : \mathfrak{B}(V \otimes V) \to \mathfrak{B}(V \otimes V)$ by the expression

$$X \otimes Y \mapsto \tfrac{1}{\omega(x)}$$

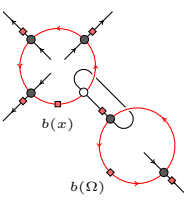


for all $X, Y \in \mathfrak{B}(V)$. To prove that Equation 72 holds, recall first that $\Phi(c_\omega) \in \mathfrak{B}(V)$ can be moved freely along the physical vector spaces. By virtue of Lemma 8.7:

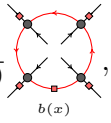
$$\tfrac{1}{\omega(x)}\tfrac{1}{\omega(\Omega)^2}$$



$$= \tfrac{1}{\omega(x)}\tfrac{1}{\omega(\Omega)}$$



$$= \tfrac{1}{\omega(x)}$$



,

since $\Phi(\xi) = \omega(\Omega)^{-1}1$ by Lemma 8.6.

Similar to the construction described for the boundary state of the toric code, the existence of such a quantum channel immediately induces a finite-depth circuit of quantum channels manifesting the triviality of these states.

THEOREM 10.3. *Let $\mathcal{A}$ be a C*-HA and let (V, Φ) be a faithful $*$-representation of $\mathcal{A}$. Then, for all positive non-zero elements $x \in \mathcal{A}$ and all $n \in \mathbb{N}$ there exists a depth-two circuit of bounded-range quantum channels that maps $\mathrm{Tr}(\mathbf{1})^{-n}\mathbf{1}^{\otimes n}$ into $\rho(x, n)$. That is, the sequence $(\rho(x, n))_{n=1}^{\infty}$ is in the trivial phase.*

PROOF. Assume without loss of generality that $n \in \mathbb{N}$ is even. The circuit consists of two layers, as presented above in Example 10.1. In the first layer, we replace the maximally mixed state $\mathrm{Tr}(\mathbf{1})^{-n}\mathbf{1}^{\otimes n}$ with the sequence of $n/2$ tensor products $\rho(\Omega, 2)\otimes\cdots\otimes\rho(\Omega, 2)$ as previously done. Now, by virtue of Lemma 10.2, let $\mathrm{Id}\otimes\mathfrak{G}_\Omega\otimes\cdots\otimes\mathfrak{G}_\Omega\otimes\mathfrak{G}_x\otimes\mathrm{Id}$ be the second layer of quantum channels, where all subindices are $\Omega \in \mathcal{A}$ except for one, which is $x \in \mathcal{A}$. This second layer of channels then glues together all local MPDOs into the single MPDO $\rho(x, n)$. □

3. Phase classification of C*-weak Hopf algebras

For general RFP MPDOs constructed from biconnected C*-WHAs a straightforward generalization of the previous procedure is not possible anymore, since the comultiplication is no longer unit-preserving.

Remark 10.4. There are no trace-preserving gluing maps for general biconnected C*-WHAs such that Equation 72 holds for all elements $x \in \mathcal{A}$.

See Appendix G for a proof. Unfortunately, the description of the phases in this general case is still an open problem. Nevertheless, some evidence indicates the existence of non-trivial phases, as we conjecture here.

Conjecture 10.5. *RFP MPDOs arising from the Lee-Yang C*-WHA of Example 5.27 do not belong to the trivial phase.*

However, these obstructions can be circumvented if one restricts to the trivial sector. The following result establishes the existence of a special gluing map, motivated by the characterization of simple RFP MPDO tensors in [**31**].

Lemma 10.6. *Let $\mathcal{A}$ be a biconnected C*-WHA and let (V, Φ) be a faithful $*$-representation of $\mathcal{A}$. There is a quantum channel $\mathfrak{G}_1 : \mathfrak{B}(V\otimes V) \to \mathfrak{B}(V \otimes V)$, called "gluing" map, such that*

$$(\mathrm{Id}^{\otimes m-1} \otimes \mathfrak{G}_1 \otimes \mathrm{Id}^{\otimes n-1})(\rho(1, m) \otimes \rho(1, n)) = \rho(1, m + n) \tag{73}$$

for all $m, n \in \mathbb{N}$.

A proof is given in Appendix G. As an immediate corollary, similar to the case of C*-HAs, we obtain the following result.

THEOREM 10.7. *Let $\mathcal{A}$ be a biconnected C*-WHA and let (V, Φ) be a faithful $*$-representation of $\mathcal{A}$. Then, for all $n \in \mathbb{N}$ there exist two depth-two circuits of bounded-range quantum channels that map $\mathrm{Tr}(\mathbf{1})^{-n}\mathbf{1}^{\otimes n}$ into $\rho(1, n)$ and $\rho(\tau_1, n)$. That is, the sequences $(\rho(1, n))_{n=1}^{\infty}$ and $(\rho(\tau_1, n))_{n=1}^{\infty}$ are in the trivial phase.*

CHAPTER 11

Conclusion and discussion

In this thesis we have shown how Matrix Product Operators are inextricably connected to the representation of different types of prebialgebras, such as weak-Hopf algebras. As an application of the results and techniques used to uncover such connection, we have obtained new results in the context of topological phases of matter in 2D and, very particularly, in 1D, where we analyze the phase diagram of open quantum systems. However, many questions are still unanswered. In this chapter we briefly discuss some of the most interesting projects that we believe these questions open up in the immediate future.

First, we stated in Conjecture 10.5 that RFP MPDOs arising from the Lee-Yang C*-weak Hopf algebra (or equivalently, the Fibonacci fusion category) cannot be constructed via a finite-depth quantum circuit. There is some theoretical and numerical evidence that this is indeed the case. The main challenge we face here is to find invariants that are preserved under the considered notion of phase.

Second, it is interesting on its own to determine whether every rank-four tensor generating RFP MPDOs, as regarded in Chapter 8, arises by weighting the MPO tensor from the representation of a C*-weak Hopf algebra. The RFP condition has already been characterized by the "algebras" generated by these MPDOs in the vertical direction in [**31**], and hence it is interesting to adapt our work to this structure theorem to find out what algebraic structure underlies.

Additionally, in Chapter 9 we investigated RFP MPDOs constructed from C*-weak Hopf algebras and proved that they arise as boundary states of certain 2D MPO-injective PEPS satisfying a natural RFP condition, up to minor technical details. This problem of proposing compelling generalizations of the RFP condition to PEPS has already been mentioned in e.g. [**31, 33**], although none of them match the assumption needed in our work so far. Thus, it is also interesting to relate this description to the ones existing.

Besides, we expect the tensor network description of MPO-injective PEPS presented in Chapter 7 to be extremely useful to generalize the results of [**86**] beyond the quantum double of a group (e.g. for weaker structures such as C*-Hopf algebras), showing that these models thermalize quickly and thus they are useless as self-correcting quantum memories.

On the other hand, the bridge between tensor networks and pre-bialgebras allows to tackle e.g. the classification of symmetries of Matrix Product Unitaries (MPU) in [32]. For instance, in the case of MPUs generated by time-reversal symmetric tensors, these are connected to very simple pre-bialgebras after some twist. It is expected that one can find invariants associated to the twists and the second cohomology group, proving that these MPUs cannot be connected through a smooth path of time-reversal symmetric MPUs.

Also, it would be interesting to explore whether the results and techniques of this thesis allow to extend the construction of 1D and 2D topological models beyond the semisimple case. A first interesting example to examine is the MPO-injective PEPS constructed using the 4-dimensional Sweedler's Hopf algebra.

Finally, it seems there is a close connection between MPO algebras and Jones theory of subfactors in operator algebras. This has been uncovered by Y. Kawahigashi in the recent papers [68, 69, 70, 71]. This opens up the possibility of using tensor network techniques in the core theory of operator algebras, something that deserves further exploration in the future.

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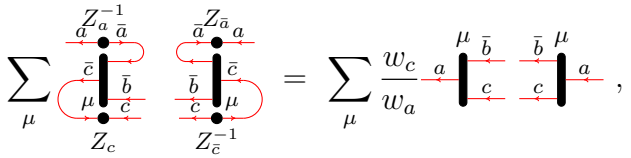
APPENDIX A

Proof of Lemma 5.10 and of Lemma 5.11

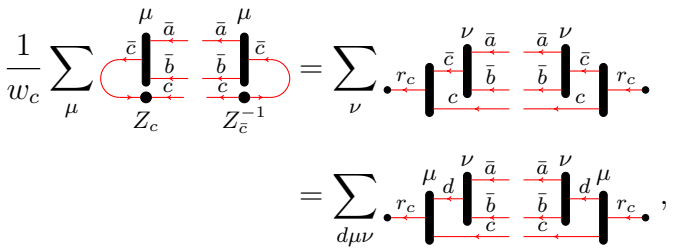
Before proving Lemma 5.10, we need the following lemma:

Lemma A.1. *In a cosemisimple WHA, the following holds for all* $a, b, c \in \mathrm{Irr}(\mathcal{A}^*)$*:*

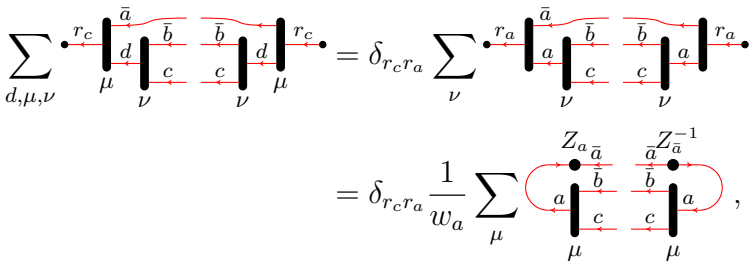


where the constants w_c *and* w_a *are defined in Equation 39.*

Proof. Using Equation 39, one can write



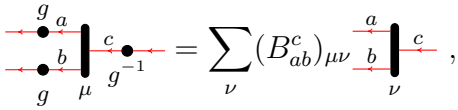
where in the second equality we have used that $N_{dc}^{r_c} = N_{r_c\bar{c}}^{d}$ (see Equation 38) and that $r_c = \ell_{\bar{c}}$ (see Proposition 5.6), and thus that the sum over μ on the r.h.s. runs from 1 to $N_{dc}^{r_c} = N_{r_c\bar{c}}^{d} = N_{\ell_{\bar{c}}\bar{c}}^{d} = \delta_{d\bar{c}}$, i.e. it is an empty sum if $d \neq \bar{c}$ and it consists of a single term if $d = \bar{c}$. Due to associativity (Equation 7), the r.h.s. can be further rewritten as

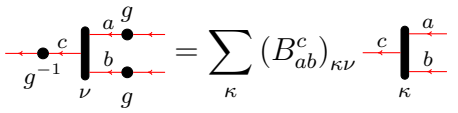


where in the first equation we have used that $N_{\bar{a}d}^{e} = N_{ae}^{d}$ (see Equation 38), and that it is non-zero for $e \in \mathrm{Vac}(\mathcal{A}^*)$ if and only if $e = r_a$, and in this case it is δ_{da} (see Equation 24); in the second equation we have used again Equation 39. Finally note that $N_{\bar{a}\bar{b}}^{\bar{c}} = N_{\bar{b}c}^{a}$ is non-zero only if $r_c = r_a$, and thus one can drop the prefactor $\delta_{r_c r_a}$. Rearranging the resulting equation leads to the desired result. □

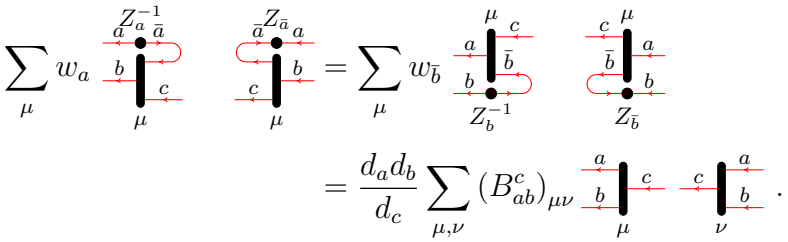
Let us now restate and prove Lemma 5.10:

Lemma A.2. *Let $\mathcal{A}$ be a cosemisimple WHA, and g be the linear functional defined in Equation 40. Then there exists an $N_{ab}^c \times N_{ab}^c$ matrix B_{ab}^c such that the linear functional $g \in \mathcal{A}^*$ satisfies*

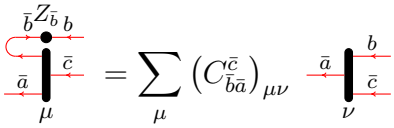
(46a)


(46b)


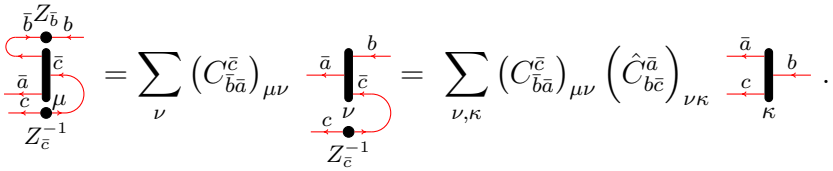
Moreover, $(B_{ab}^c)^2 = \mathrm{Id}_{N_{ab}^c}$ and the following equation holds as well:



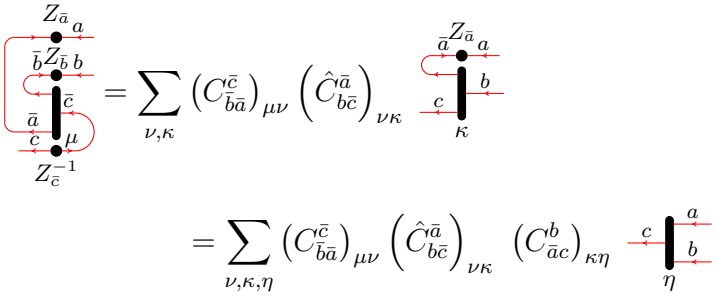
Proof. Let us apply Equation 37 three times. The first application yields



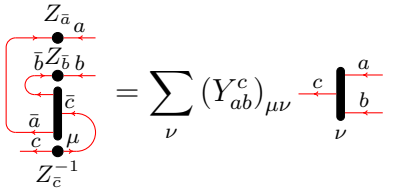
The second application yields



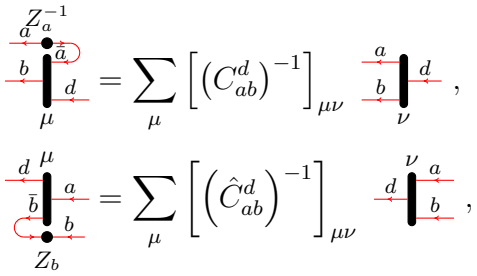
Finally, the third application yields



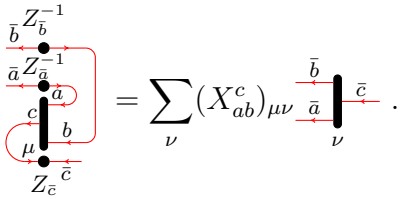
We have obtained that there exists an invertible matrix Y_{ab}^c such that



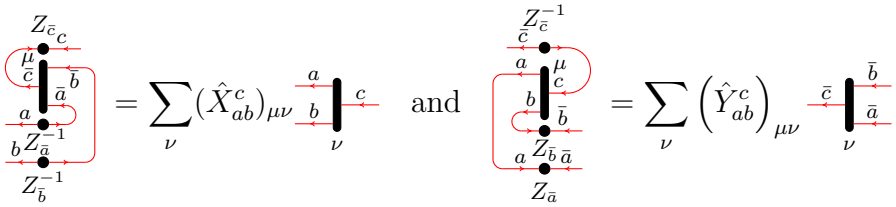
and in fact, $Y_{ab}^c = C_{\bar{b}\bar{a}}^{\bar{c}} \hat{C}_{b\bar{c}}^{\bar{a}} C_{\bar{a}c}^{b}$. Similarly, by three consecutive applications of the inverse relations of Proposition 5.5,



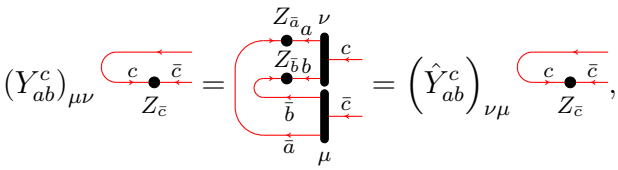
we conclude that there is an invertible matrix X_{ab}^c such that



In fact, $X_{ab}^c = (C_{\bar{a}c}^{b})^{-1} (\hat{C}_{b\bar{c}}^{\bar{a}})^{-1} (C_{\bar{b}\bar{a}}^{\bar{c}})^{-1}$, i.e. $X_{ab}^c Y_{ab}^c = \mathrm{Id}$. Using similar arguments, we conclude that there are invertible matrices $\hat{X}$ and $\hat{Y}$ such that



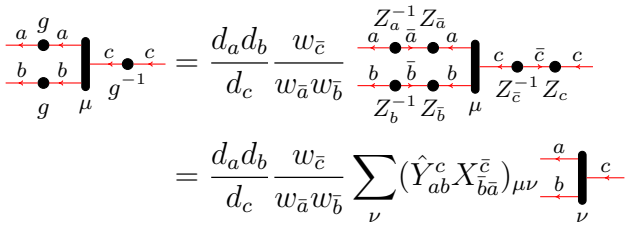
Just as above, $\hat{X}_{ab}^c \hat{Y}_{ab}^c = \mathrm{Id}$. Moreover, by orthogonality of the fusion tensors, $\hat{Y}_{ab}^c = (Y_{ab}^c)^T$:

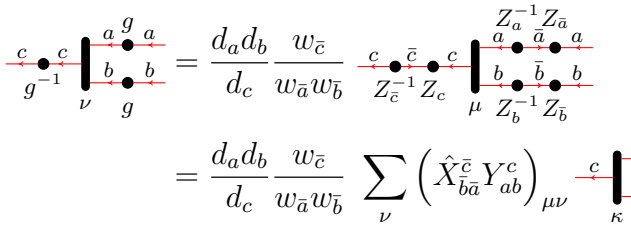


where in the first equality we have used the definition of Y_{ab}^c and the orthogonality of the fusion tensors, while in the second the definition

of $\hat{Y}^c_{ab}$ and again the orthogonality of the fusion tensors. By a similar argument, we obtain $\hat{X}^c_{ab} = (X^c_{ab})^T$ as well.

By definition of the linear functional g, we obtain that

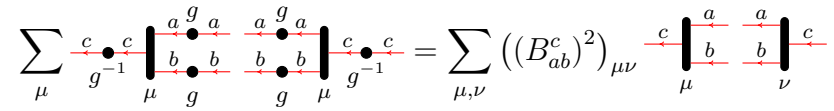
(74)


(75)


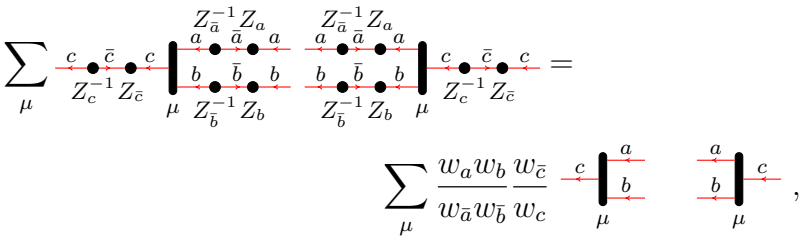
We have thus obtained that Equation 46 holds with

(76)
$$B^c_{ab} = \frac{d_a d_b}{d_c} \frac{w_{\bar{c}}}{w_{\bar{a}} w_{\bar{b}}} \hat{Y}^c_{ab} X^{\bar{c}}_{\bar{b}\bar{a}}.$$

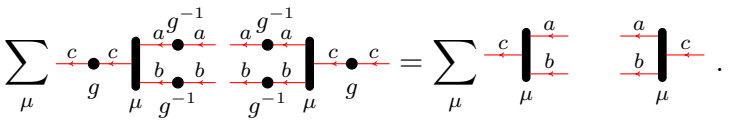
Combining the two equations in Equation 46, we obtain that

(77)


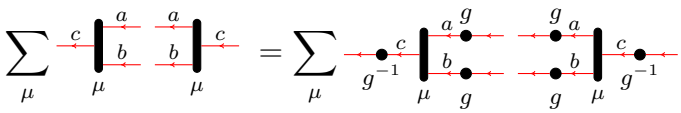
On the other hand, using Lemma A.1 three times, we obtain that



or equivalently, as $(d_a/w_{\bar{a}})^2 = w_a/w_{\bar{a}}$, that

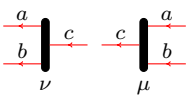
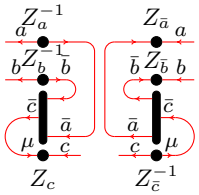


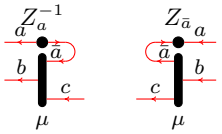
This equation can be rearranged as



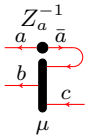
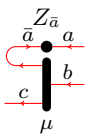
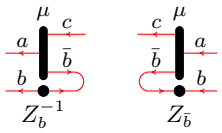
and therefore, comparing this equation to Equation 77, we obtain that $(B^c_{ab})^2 = \mathrm{Id}$. Finally, using Equation 76 and then Lemma A.1, we can

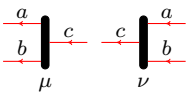
write

$$\sum_{\mu,\nu} (B^c_{ab})_{\mu\nu}$$  $$= \sum_{\mu} \frac{d_a d_b}{d_c} \frac{w_{\bar{c}}}{w_{\bar{a}} w_{\bar{b}}}$$ 

$$= \sum_{\mu} \frac{d_c}{d_b} \frac{d_a}{w_{\bar{a}}}$$ 

Part of the last statement follows using that $d_a/w_{\bar{a}} = w_a/d_a$. Finally, using again Lemma A.1, we conclude that

$$\sum_{\mu} w_a$$   $$= \sum_{\mu} w_{\bar{b}}$$ 

$$= \frac{d_a d_b}{d_c} \sum_{\mu,\nu} (B^c_{ab})_{\mu\nu}$$ 

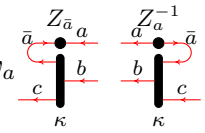
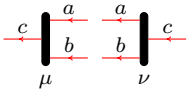
□

Let us restate and prove now Lemma 5.11:

Lemma A.3. *For all $a \in \mathrm{Irr}(\mathcal{A}^*)$, $d_a^2 = w_a w_{\bar{a}} > 0$. Let moreover T^c_{ab} be defined by $T^c_{ab} = \sum_\mu (B^c_{ab})_{\mu\mu}$. Then the following equations hold:*

$$\sum_b T^c_{ab} d_b = d_a \delta_{\ell_a \ell_c} d_c \quad \text{and} \quad \sum_{x:\ell_x=\ell_a} d_x^2 = \sum_{x:\ell_x=r_a} d_x^2.$$

Proof. Using Proposition 5.5,

$$\sum_{\kappa} w_a$$  $$= w_a \sum_{\kappa,\mu,\nu} \left(C^b_{\bar{a}c}\right)_{\kappa\mu} \left((C^c_{ab})^{-1}\right)_{\kappa\nu}$$ 

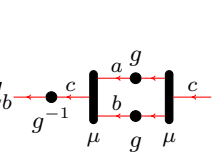
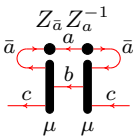
Therefore, using Lemma 5.10, we obtain that B^c_{ab} can be expressed as

$$B^c_{ab} = w_a \frac{d_c}{d_a d_b} \left(C^b_{\bar{a}c}\right)^T (C^c_{ab})^{-1}. \tag{78}$$

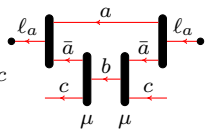
As B^c_{ab} squares to the identity, the eigenvalues of B^c_{ab} are ± 1, and thus T^c_{ab}, the trace of B^c_{ab}, is an integer with $|T^c_{ab}| \leq N^c_{ab}$. Note now that Equation 78 can be used to obtain the following expression for $B^b_{\bar{a}c}$:

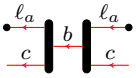
$$\begin{aligned} B^b_{\bar{a}c} &= w_{\bar{a}} \frac{d_b}{d_a d_c} (C^c_{ab})^T \left(C^b_{\bar{a}c}\right)^{-1} \\ &= C^b_{\bar{a}c} \left[(B^c_{ab})^{-1}\right]^T \left(C^b_{\bar{a}c}\right)^{-1} \\ &= C^b_{\bar{a}c} (B^c_{ab})^T \left(C^b_{\bar{a}c}\right)^{-1}, \end{aligned}$$

where in the first equation we have used that $d_a = d_{\bar{a}}$, in the second equation we have used that $w_{\bar{a}} d_b/(d_a d_c) = (w_a d_c/(d_a d_b))^{-1}$ and in the third that B^c_{ab} squares to the identity. Taking the trace of the two sides in this equation, we obtain that $T^b_{\bar{a}c} = T^c_{ab}$. By a similar argument, we obtain that $T^c_{ab} = T^a_{c\bar{b}}$ as well. The definition of T^c_{ab} and Lemma 5.10 implies that

$$\sum_b T^c_{ab} d_b \mathrm{Id}_c = \sum_{b,\mu} d_b$$  $$= \sum_{b,\mu} \frac{d_c}{d_a} w_a$$  .

Using Equation 39, the fact that $d_a^2 = w_a w_{\bar{a}}$, and the counit axiom Equation 27, the r.h.s. can be further written as

$$\sum_b T^c_{ab} d_b \mathrm{Id}_c = \sum_{b,\mu} d_a d_c$$ 

$$= d_a d_c \sum_b$$ 

$$= \begin{cases} d_a d_c \mathrm{Id}_c, & \text{if } \ell_a = \ell_c, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the equation $\sum_b T^c_{ab} d_b = d_a \delta_{\ell_a \ell_c} d_c$ holds.

Following the proof of Theorem 2.3 in [41] (see also [11]), we can now prove that $d_a^2 = w_a w_{\bar{a}}$ is a positive number. Notice first that the matrix $T_a^T T_a = T_{\bar{a}} T_a$ is positive semidefinite (as T_a is a real matrix), and thus all of its eigenvalues are non-negative. Let us show that $d_a^2 = w_a w_{\bar{a}}$ is one of its eigenvalues, then, as neither w_a nor $w_{\bar{a}}$ is 0, this implies the the positivity of d_a^2. To see that it is an eigenvalue, we can check that $\delta_{r_a \ell_b} d_b$ is the corresponding eigenvector:

$$\sum_{b,c} T^d_{\bar{a}c} T^c_{ab} \delta_{r_a \ell_b} d_b = \sum_{b,c} T^d_{\bar{a}c} T^c_{ab} d_b = d_a \sum_c T^d_{\bar{a}c} \delta_{\ell_a \ell_c} d_c$$

$$= d_a \sum_c T^d_{\bar{a}c} d_c = d_a d_{\bar{a}} \delta_{r_a \ell_d} d_d,$$

where in the first equation we have used that $T^c_{ab} = 0$ if $\delta_{r_a \ell_b} = 0$ (as then $N^c_{ab} = 0$ and $|T^c_{ab}| < N^c_{ab}$); the same relation is used in the third equation, together with $r_{\bar{a}} = \ell_a$. In the last equation we have used $r_a = \ell_{\bar{a}}$. Finally note that $d_{\bar{a}} = d_a$ and that the vector defined by $\delta_{r_a \ell_b} d_b$ is non-zero.

Finally, let us prove that $\sum_{x:\ell_x=\ell_a} d_x^2 = \sum_{x:\ell_x=r_a} d_x^2$. For that, note that $T_{ab}^c = T_{\bar{a}c}^b = T_{b\bar{c}}^{\bar{a}} = T_{\bar{b}\bar{a}}^{\bar{c}}$, and thus

$$\begin{aligned} d_a \sum_{x:\ell_x=\ell_a} d_x^2 &= \sum_{x,b} T_{xb}^a d_x d_b = \sum_{x,b} T_{\bar{b}\bar{x}}^{\bar{a}} d_x d_b \\ &= \sum_{\bar{x}\bar{b}} T_{\bar{b}\bar{x}}^{\bar{a}} d_{\bar{x}} d_{\bar{b}} = d_{\bar{a}} \sum_{\bar{b}:\ell_{\bar{b}}=\ell_{\bar{a}}} d_{\bar{b}}^2 = d_a \sum_{x:\ell_x=r_a} d_x^2. \end{aligned}$$

As $d_a \neq 0$, the statement follows. □

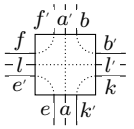
APPENDIX B

String-net models

In the following we create WHAs from fusion categories that appear in the construction of string-net models. For simplicity, we will restrict to fusion categories where the fusion multiplicities are all 0 or 1, $N_{ab}^{c} \leq 1$ for all a, b, c. With this assumption, the pentagon equations for the F-symbols simplify to the equation:

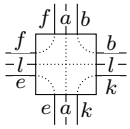
$$\left[F_{fcd}^{e}\right]_{l}^{g}\left[F_{abl}^{e}\right]_{k}^{f} = \sum_{h}\left[F_{abc}^{g}\right]_{h}^{f}\left[F_{ahd}^{e}\right]_{k}^{g}\left[F_{bcd}^{k}\right]_{l}^{h}. \tag{79}$$

The MPO tensor used in the construction of the string-net models is then defined as



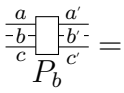
$$= \left[F_{abl}^{e}\right]_{k}^{f}\,\delta_{aa'}\delta_{bb'}\delta_{ll'}\delta_{ee'}\delta_{ff'}\delta_{kk'},$$

where the dotted lines serve as a visual reminder of the δ prefactors. We will often make use of this visual reminder of the δ prefactors and only write the non-zero components of the MPO tensor,



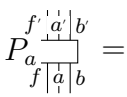
$$= \left[F_{abl}^{e}\right]_{k}^{f}.$$

Each line is labeled by the simple objects of the category, i.e. it is an N-dimensional vector space (N is the number of simple objects in the fusion category); thus the bond dimension of this tensor is N^3. This bond dimension can be reduced as the tensor is block diagonal and it contains a zero block. The projectors reducing the tensor (those that correspond to a non-zero block) are



$$= N_{ab}^{c}\delta_{bk}\delta_{aa'}\delta_{bb'}\delta_{cc'}. \tag{80}$$

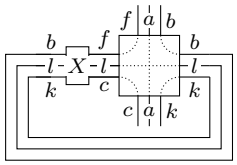
These are indeed projectors as $N_{fl}^{e} \in \{0, 1\}$ and it is easy to see that they commute with the MPO tensor (the F-symbols satisfy $[F_{abl}^{e}]_{k}^{f} = 0$ unless $N_{fl}^{e} \neq 0$ and $N_{bl}^{k} \neq 0$). The bond dimension of each block is then $D_l = \sum_{f,e} N_{fl}^{e}$. A similar decomposition holds for the vertical direction as well. There the projectors that decompose the MPO tensor are



$$= N_{ab}^{f}\delta_{bk}\delta_{aa'}\delta_{bb'}\delta_{cc'}.$$

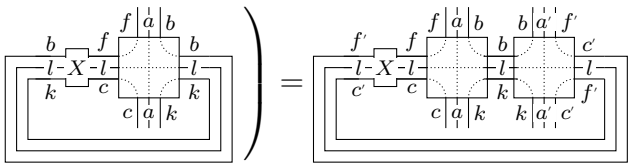
Starting from this MPO tensor, let us define a linear space $\mathcal{A}$ as

$$\mathcal{A} = \left\{ \left. \begin{array}{c} \\ \\ \\ \\ \end{array} \right| X \in \bigoplus_l \mathcal{M}_{D_l} \right\}.$$



This linear space $\mathcal{A}$ has a natural coalgebra structure, with the coproduct given by

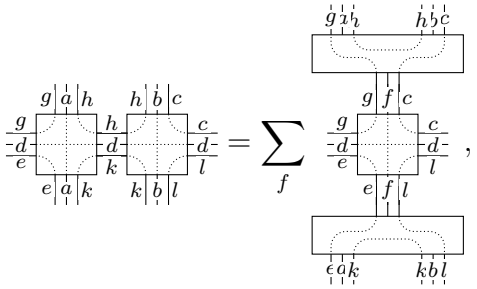
$$\Delta \left(\begin{array}{c} \\ \\ \\ \\ \end{array} \right) = \begin{array}{c} \\ \\ \\ \\ \end{array} .$$



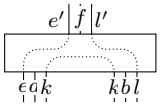
As above, taking repeated coproducts of a coalgebra element is the same as growing the size of the MPO. In particular, the operation Δ defined by this equation is associative. Note that as the pentagon equation Equation 79 can be rearranged as

$$[F^e_{ahd}]^g_k \left[F^k_{bcd}\right]^h_l = \sum_f \left[F^e_{fcd}\right]^g_l [F^e_{abl}]^f_k \left[(F^g_{abc})^{-1}\right]^h_f,$$

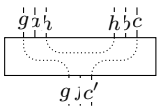
the coproduct can also be expressed as



where we have defined the fusion tensors of the physical level as



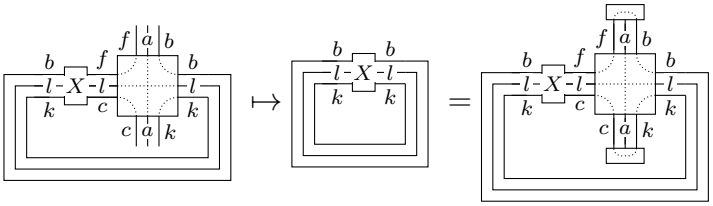
$$= [F^e_{abl}]^f_k \, \delta_{ff'} \delta_{gg'} \delta_{ee'},$$



$$= \left[(F^g_{abc})^{-1}\right]^h_f \delta_{ff'} \delta_{gg'} \delta_{ee'}.$$

As the construction is built on a fusion category, there is a unique vacuum label that we denote by 1. The counit of this coalgebra is given by

$$\varepsilon : \begin{array}{c} \\ \\ \\ \\ \end{array} \mapsto \begin{array}{c} \\ \\ \\ \\ \end{array} = \begin{array}{c} \\ \\ \\ \\ \end{array} ,$$

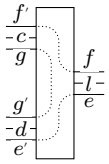


where we have used that $\left[F^k_{1bl}\right]^b_k = \alpha_b/\alpha_k$ is independent of l and we have defined

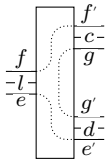
$= \alpha_b^{-1}\delta_{fb}\delta_{a1}$ and 

$= \alpha_k\delta_{ck}\delta_{a1}$.

Let us now show that $\mathcal{A}$ is not only a coalgebra, but it also has a pre-bialgebra structure. For that, we define fusion tensors corresponding to this MPO tensor as

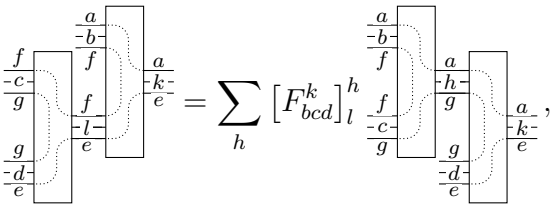


$= \left[F^e_{fcd}\right]^g_l \delta_{ff'}\delta_{gg'}\delta_{ee'}$,

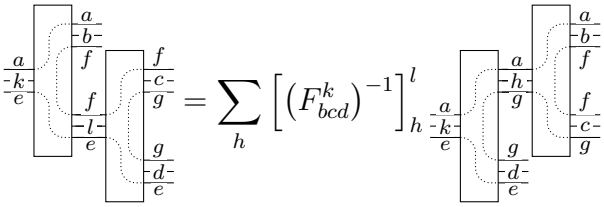


$= \left[\left(F^e_{fcd}\right)^{-1}\right]^l_g \delta_{ff'}\delta_{gg'}\delta_{ee'}$.

These tensors satisfy the associativity equations Equation 7, because the equations



$= \sum_h \left[F^k_{bcd}\right]^h_l$

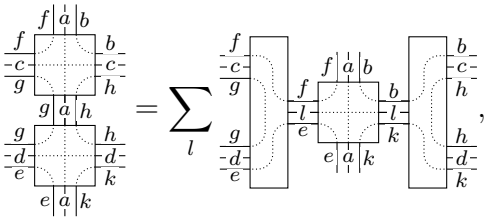


$= \sum_h \left[\left(F^k_{bcd}\right)^{-1}\right]^l_h$

both follow from the pentagon equation Equation 79. More precisely, the left equation is exactly Equation 79, while the right equation is its inverse,

$$\left[\left(F^e_{fcd}\right)^{-1}\right]^l_g \left[\left(F^e_{abl}\right)^{-1}\right]^k_f = \sum_h \left[\left(F^g_{abc}\right)^{-1}\right]^h_f \left[\left(F^e_{ahd}\right)^{-1}\right]^k_g \left[\left(F^k_{bcd}\right)^{-1}\right]^l_h .$$

We can now check for the key equation Equation 9. The r.h.s. of Equation 9 is

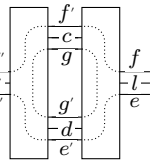


$= \sum_l$

where we have only written out the components that are not automatically zero due to the delta functions. This equation is the pentagon equation rearranged:

$$[F^{g}_{abc}]^{f}_{h}\,[F^{e}_{ahd}]^{g}_{k} = \sum_{l} \left[F^{e}_{fcd}\right]^{g}_{l}\,[F^{e}_{abl}]^{f}_{k}\left[\left(F^{k}_{bcd}\right)^{-1}\right]^{l}_{h}.$$

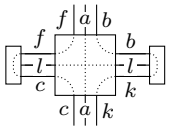
Let us now check for the l.h.s. of Equation 9, i.e. the orthogonality of the fusion tensors:

$$\sum_{f',g,g',e'}$$ 

$$= \sum_{f',g,g',e',c,d} \left[\left(F^{e}_{fcd}\right)^{-1}\right]^{l'}_{g} \delta_{ff'}\delta_{gg'}\delta_{ee'}\left[F^{e}_{fcd}\right]^{g}_{l}\delta_{f'f''}\delta_{ee''}$$

$$= \delta_{ee''}\delta_{ff''}\delta_{ll''}N^{e}_{fl},$$

where again N^{e}_{fl} is one or zero, which gives the r.h.s. of Equation 9. We have thus checked that $\mathcal{A}$ admits a pre-bialgebra structure with the usual multiplication, except that we have not shown that $\mathcal{A}$ has a unit. For that, note that the unit is of the form

$$1 =$$ 

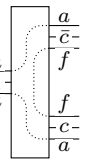
,

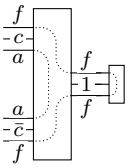
where we have defined

$$\boxed{\begin{smallmatrix}a\\ b\\ c\end{smallmatrix}} = \alpha_a^{-1}\delta_{ac}\delta_{b1} \qquad \text{and} \qquad \boxed{\begin{smallmatrix}a\\ b\\ c\end{smallmatrix}} = \alpha_a\delta_{ac}\delta_{b1}.$$

As the unit is described by a rank-one boundary that is supported only in the vacuum sector, the pre-bialgebra $\mathcal{A}$ automatically satisfies the unit axiom. Dually, as the counit is also described by a rank-one boundary that is supported only in the vacuum sector, $\mathcal{A}$ also satisfies the counit axiom. Therefore $\mathcal{A}$ is a WBA.

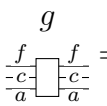
Let us now show that $\mathcal{A}$ is a WHA as well. A cosemisimple WBA is a WHA if and only if for every sector a there is another sector $\bar{a}$ such that the symmetries $N^{c}_{ab} = N^{b}_{\bar{a}c}$ and $N^{c}_{ab} = N^{a}_{c\bar{b}}$ hold. In our case, as the fusion multiplicities originate from a fusion category, these symmetries trivially hold. Therefore $\mathcal{A}$ is a WHA. The matrices Z_c and Z_c^{-1} describing the antipode can be expressed with the help of the fusion tensors as Equation 39:

$$Z_c =$$ 

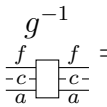
and $Z_c^{-1} =$ $\dfrac{1}{[F^{\bar{c}}_{\bar{c}c\bar{c}}]^{1}_{1}}$ 

.

It is easy to see that these matrices are the inverses of each other. The square of the antipode is realized by the matrices

(81a) 

$$= \frac{1}{\left[F^{\bar{c}}_{\bar{c}c\bar{c}}\right]^1_1} \left[F^f_{fc\bar{c}}\right]^a_1 \left[\left(F^f_{fc\bar{c}}\right)^{-1}\right]^1_a,$$

(81b) 

$$= \frac{1}{\left[F^{c}_{c\bar{c}c}\right]^1_1} \left[F^a_{a\bar{c}c}\right]^f_1 \left[(F^a_{a\bar{c}c})^{-1}\right]^1_f .$$

1. Fibonacci anyons

The fusion rules are given by:

$$N^1_{11} = N^\tau_{\tau 1} = N^\tau_{1\tau} = N^1_{\tau\tau} = N^1_{\tau\tau} = 1,$$
$$N^\tau_{11} = N^1_{\tau 1} = N^1_{1\tau} = 0.$$

The F-symbol $\left[F^e_{fcd}\right]^g_l$ is proportional to $N^g_{fc} N^l_{cd} N^e_{fg} N^e_{ld}$. Therefore the following entries are the only non-zero ones and are given by

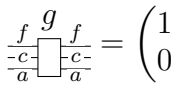
$$\left[F^1_{111}\right]^1_1 = [F^\tau_{11\tau}]^1_\tau = [F^\tau_{\tau 11}]^\tau_1 = [F^\tau_{1\tau 1}]^\tau_\tau = \left[F^1_{1\tau\tau}\right]^\tau_1 = [F^\tau_{1\tau\tau}]^\tau_\tau =$$
$$[F^\tau_{\tau 1\tau}]^\tau_\tau = \left[F^1_{\tau 1\tau}\right]^\tau_\tau = [F^\tau_{\tau\tau 1}]^\tau_\tau = \left[F^1_{\tau\tau 1}\right]^1_\tau = \left[F^1_{\tau\tau\tau}\right]^\tau_\tau = 1$$
$$[F^\tau_{\tau\tau\tau}]^1_1 = \varphi, \ .[F^\tau_{\tau\tau\tau}]^\tau_\tau = -\varphi, \ [F^\tau_{\tau\tau\tau}]^\tau_1 = [F^\tau_{\tau\tau\tau}]^1_\tau = \sqrt{\varphi},$$

where $\varphi = (\sqrt{5}-1)/2$. We also have $\left(\left(F^d_{abc}\right)^{-1}\right)^f_e = \left(F^d_{abc}\right)^e_f$.

1.1. The MPO tensor. The MPO tensor given in Equation 80 has bond- and physical dimension 8. It is block-diagonal both in the horizontal and vertical direction, with the two blocks labeled by 1 and τ. The block 1 corresponds to the subspace spanned by $|111\rangle, |\tau 1\tau\rangle$ and the block τ to the subspace spanned by $|1\tau\tau\rangle, |\tau\tau 1\rangle, |\tau\tau\tau\rangle$. The two simple cocommutative elements are given by

$$a_1 \equiv \begin{pmatrix} [F^1_{111}]^1_1 & [F^\tau_{111}]^1_\tau \\ [F^1_{1\tau 1}]^\tau_1 & [F^\tau_{1\tau 1}]^\tau_\tau \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & [F^\tau_{\tau\tau 1}]^\tau_\tau \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$a_\tau \equiv \begin{pmatrix} [F^1_{11\tau}]^1_1 & [F^\tau_{11\tau}]^1_\tau \\ [F^1_{1\tau\tau}]^\tau_1 & [F^\tau_{1\tau\tau}]^\tau_\tau \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & [F^\tau_{\tau\tau\tau}]^\tau_\tau \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\varphi \end{pmatrix},$$

where the basis is ordered as given above. The square of the antipode, as given by Equation 81, is implemented by



$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} \varphi^{-1} & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

APPENDIX C

The Canonical Regular Element

Here we review the description of the element Ω, known as the *canonical regular element* in the particular case of biconnected C*-WHAs (see Definition 5.17 and Definition 5.3), introduced in Theorems 5.14 and 5.21 for more general WHAs. First, it is well-known that in any C*-WHA $\mathcal{A}$ there exists a unique non-degenerate two-sided normalized integral $h \in \mathcal{A}$, known as the *Haar integral of* $\mathcal{A}$; see Definition 3.24 and Theorem 4.5 in [14]. In particular,

$$h^2 = h^* = h = S(h). \tag{82}$$

By self-duality, let $\hat{h} \in \mathcal{A}^*$ denote the Haar integral of the dual C*-WHA. We also recall the existence of $\Lambda \in \mathcal{A}$, known as the *dual left-integral of* $\hat{h}$, such that

$$\hat{h}(\Lambda_{(1)})\Lambda_{(2)} = 1 \quad \text{and} \quad S(\Lambda_{(1)}) \otimes \Lambda_{(2)} = \Lambda_{(2)} \otimes \Lambda_{(1)}; \tag{83}$$

see e.g. Theorem 3.18 and Lemma 3.20 in [14]. Second, there is a unique positive element $g \in \mathcal{A}$ implementing the antipode squared as an inner automorphism, i.e.

$$S^2(x) = gxg^{-1} \tag{84}$$

for all elements $x \in \mathcal{A}$, among other properties, known as the *canonical group-like element of* $\mathcal{A}$; see Proposition 4.9 in [14]. As its name implies, it is a *group-like element*, i.e. it satisfies the following property:

$$g_{(1)} \otimes g_{(2)} = g1_{(1)} \otimes g1_{(2)} = 1_{(1)}g \otimes 1_{(2)}g. \tag{85}$$

Moreover, it can be decomposed in the form $g = g_L g_R^{-1}$ for two elements $g_L, g_R > 0$ given by

$$g_L := (\hat{h}(h_{(1)})h_{(2)})^{\frac{1}{2}} \in \mathcal{A}_L \quad \text{and} \quad g_R := S(g_L) = S^{-1}(g_L) \in \mathcal{A}_R. \tag{86}$$

By self-duality, we denote by $\hat{g} \in \mathcal{A}^*$ the canonical group-like element of the dual C*-WHA. Finally, let us recall the following formula.

Proposition C.1. *For any C*-WHA,*

$$x_{(1)}\hat{g}(x_{(2)}) = g_R x g_R^{-1} \quad \textit{and} \quad \hat{g}(x_{(1)})x_{(2)} = g_L x g_L^{-1}$$

for all elements $x \in \mathcal{A}$. In particular,

$$1_{(1)}\hat{g}(1_{(2)}) = 1 = \hat{g}(1_{(1)})1_{(2)}.$$

Proof. See Scholium 2.7 and Lemma 4.13 in [14] for a proof. □

Proposition C.2 (see [96]). *For any connected C*-WHA $\mathcal{A}$,*

$$\hat{\tau}_\alpha(g) = \varepsilon(1)d_\alpha$$

for all sectors $\alpha \in \mathrm{Irr}(A)$, where $\hat{\tau}_\alpha$ stands for the character of the sector α.

Proposition C.3. *In any connected C*-WHA*

$$\omega(x) = \mathcal{D}^{-2}\varepsilon(1)^{-1}\hat{h}(g_L^{-1}g_R^{-1}x) = \mathcal{D}^{-2}\varepsilon(1)^{-1}\hat{h}(xg_L^{-1}g_R^{-1})$$

for all elements $x \in \mathcal{A}$. Equivalently, for any coconnected C-WHA,*

$$\Omega = \mathcal{D}^{-2}\varepsilon(1)^{-1}\Lambda_{(1)}\hat{g}^{-1}(\Lambda_{(2)}) = \mathcal{D}^{-2}\varepsilon(1)^{-1}\hat{g}^{-1}(\Lambda_{(1)})\Lambda_{(2)}.$$

Proof. Assume first that $\mathcal{A}$ is a connected C*-WHA. There exists a well-known element, called the S-invariant trace of $\mathcal{A}$, see [14], given by the expression $\sum_\alpha \hat{\tau}_\alpha(g)\hat{\tau}_\alpha$, where $\hat{\tau}_\alpha$ stands for the character of the sector $\alpha \in \mathrm{Irr}(\mathcal{A})$. By virtue of Theorem 5.21 and Proposition C.2, one easily checks that both elements are proportional. □

Remark C.4 (see Equation 54). The linear map $T \in \mathfrak{B}(\mathcal{A})$ in Theorem 5.21 is given by

$$T(x) = S(x_{(1)})\hat{g}(x_{(2)}) = \hat{g}(x_{(1)})S^{-1}(x_{(2)})$$

Remark C.5. In any coconnected C*-WHA, $\omega \circ S = \omega = \omega \circ T$.

Proof. It trivially follows from the fact that $\hat{h}$ is S-invariant, the definition of both g_L and g_R in Equation 86 and Propositions C.1 and C.3 and Remark C.4. □

Finally, let us particularize the previous notions and results in the context of C*-Hopf algebras. We refer the reader to [94] for more details.

Proposition C.6. *Let $\mathcal{A}$ be a C*-HA. Then:*

(1) $S^2 = \mathrm{Id}$ *and the canonical grouplike element is $g = 1$;*
(2) $d_\alpha = \dim_{\mathbb{C}}(V_\alpha)$ *for all sectors $\alpha \in \mathrm{Irr}(\mathcal{A})$;*
(3) *the dual left integral of the Haar measure $\hat{h} \in \mathcal{A}^*$ is $t = \mathcal{D}^2\Omega$;*
(4) *the canonical regular element and the Haar integral coincide, i.e. $\Omega = h$;*
(5) *the map $T \in \mathfrak{B}(\mathcal{A})$ coincides with the antipode $S \in \mathfrak{B}(\mathcal{A})$;*
(6) $g_L = g_R = \mathcal{D}^{-1}1$.

Proof. (1) It is well-known that for any C*-HA it holds that $S^2 = \mathrm{Id}$ [77, 78]. Since the unit element $1 \in \mathcal{A}$ satisfies the defining properties of the canonical group-like element too, which is unique, we can conclude that $g = 1$. (2) Consider that $\varepsilon(1) = 1$ by Definition 5.22 and hence Proposition C.2 proves that $\dim_{\mathbb{C}}(V_\alpha) = \hat{\tau}_\alpha(1) = \hat{\tau}_\alpha(g) = \varepsilon(1)d_\alpha = d_\alpha$ for all sectors $\alpha \in \mathrm{Irr}(\mathcal{A})$, where $\hat{\tau}_\alpha$ stands for the character of the sector α. (3) Since the axioms of C*-HAs are self-dual $\hat{g} = \varepsilon$ and hence $\Omega = \mathcal{D}^{-2}\varepsilon(1)^{-1}t = \mathcal{D}^{-2}t$, where the first expression

follows from Proposition C.3. (4) Every C*-HA is *unimodular*, see [**94**], i.e. every left integral is a two-sided integral, and the subspace of two-sided integrals is one-dimensional. Hence $t = \eta h$ for some $\eta \in \mathbb{C}$. Since $\Omega^2 = \Omega$ and $h^2 = h$, the only possibility left is $\eta = \mathcal{D}^2$. (5) This follows trivially as a consequence of Remark C.4 since $\hat{g} = \varepsilon$. (6) Recall the definition of g_L and g_R in Equation 86 and consider both steps (3) and (4). □

APPENDIX D

Proof of Lemma 8.6

Here we restate and prove the following result.

Lemma D.1. *Let $\mathcal{A}$ be a biconnected C*-WHA. There exists a unique element $\xi \in \mathcal{A}$ such that $\omega(\xi T(\Omega_{(1)}))\Omega_{(2)} = 1$. Furthermore, it satisfies the following properties:*

(1) *it is strictly positive and $\xi^{-1} = \omega(\Omega_{(1)})\Omega_{(2)} = \omega(T(\Omega_{(1)}))\Omega_{(2)}$;*
(2) *it is invariant under $T \in \mathfrak{B}(\mathcal{A})$, i.e. $T(\xi) = \xi$;*
(3) *it satisfies $T(x)^* = \xi T(x^*)\xi^{-1}$ for all elements $x \in \mathcal{A}$;*
(4) *$\hat{\tau}_\alpha(\xi^{-1}) = d_\alpha \omega(\Omega)$ for all sectors $\alpha \in \mathrm{Irr}(\mathcal{A})$, where $\hat{\tau}_\alpha$ stands for the character of the sector α;*
(5) *it can be decomposed as $\xi = \xi_L \xi_R$ for two positive elements $\xi_L \in \mathcal{A}_L$ and $\xi_R = S(\xi_L) = S^{-1}(\xi_L) \in \mathcal{A}_R$;*

Dually, if we denote $\hat{\xi} = \hat{\xi}_L \hat{\xi}_R \in \mathcal{A}^*$, then:

(6) *$x_{(1)}\hat{\xi}_L(x_{(2)}) = \xi_L^{-1} x$ and $x_{(1)}\hat{\xi}_R^{-1}(x_{(2)}) = x\xi_L^{-1}$ for all $x \in \mathcal{A}$.*

Finally, if $\mathcal{A}$ is a C-HA, then $\xi_L^2 = \xi_R^2 = \xi = \mathcal{D}^2\varepsilon(1)1 = \omega(\Omega)^{-1}1$.*

Proof. Since $\Omega \in \mathcal{A}$ is non-degenerate there exists a linear functional $f \in \mathcal{A}^*$ such that $f(\Omega_{(1)})\Omega_{(2)} = 1$; see Definition 3.7 and Theorem 5.21. On the other hand, since $\omega \in \mathcal{A}^*$ is non-degenerate, there exists an element $\xi \in \mathcal{A}$ such that $\omega(\xi x) = (f \circ T)(x)$ for all elements $x \in \mathcal{A}$. Therefore,

$$\omega(\xi T(\Omega_{(1)}))\Omega_{(2)} = (f \circ T)(T(\Omega_{(1)}))\Omega_{(2)} = f(\Omega_{(1)})\Omega_{(2)} = 1,$$

where in the second equality we have used that $T \in \mathfrak{B}(\mathcal{A})$ is involutive, i.e. $T \circ T = \mathrm{Id}$. Recall that $\omega \in \mathcal{A}^*$ is cocentral, it is a trace-like linear functional of $\mathcal{A}^*$. It follows by the pulling-through identity in Equation 59 that

$$1 = \omega(\xi T(\Omega_{(1)}))\Omega_{(2)} = \omega(T(\Omega_{(1)})\xi)\Omega_{(2)} = \omega(T(\Omega_{(1)}))\xi\Omega_{(2)},$$

and hence $\xi \in \mathcal{A}$ is invertible. Its inverse is then trivially given by the expression

$$\xi^{-1} = \omega(T(\Omega_{(1)}))\Omega_{(2)} = \omega(\Omega_{(1)})\Omega_{(2)}.$$

where the last equality follows from Remark C.5. Let us prove now (4). By virtue of Proposition C.3, it is easy to conclude by its defining property $\omega(\xi T(\Omega_{(1)}))\Omega_{(2)} = 1$ that $\xi \in \mathcal{A}$ is necessarily given by the expression

$$\xi = \mathcal{D}^4\varepsilon(1)^2 g_L g_R. \tag{87}$$

Consequently, a natural choice of positive elements $\xi_L \in \mathcal{A}_L$ and $\xi_R \in \mathcal{A}_R$ is

$$\xi_L := \mathcal{D}^2\varepsilon(1)g_L \quad \text{and} \quad \xi_R := \mathcal{D}^2\varepsilon(1)g_R = S(\xi_L). \tag{88}$$

Since $g_L, g_R > 0$, ξ is strictly positive, as we wanted to prove. We now prove (2), i.e. that $T(\xi) = \xi$, note by the previous expressions that it turns out to be enough to check that $T(g_L) = g_R$ and $T(g_R) = g_L$. We refer to Equations 99a and 99b below for elementary proofs of these facts. In addition, note that (4) is straightforward by the eigenvalue equation $\Omega \cdot \tau_a = \tau_a \cdot \Omega = d_a\Omega$. See Scholium 2.7 and Lemma 4.13 from [**14**] for a proof of (6). Let us now move to the proof of (3). For simplicity, we prove the equivalent formula $\xi T(x)\xi^{-1} = T(x^*)^*$ for all $x \in \mathcal{A}$. To this end, we recall first that

$$\xi y \xi^{-1} = g_L g_R y g_L^{-1} g_R^{-1} = \hat{g}(y_{(1)})y_{(2)}\hat{g}(y_{(3)}) \tag{89}$$

for all elements $y \in \mathcal{A}$, see Proposition C.1. On the other hand, by virtue of the the fact that $S^{-1}(\hat{g}) = \hat{g}^{-1}$ and the positivity of $\hat{g} \in \mathcal{A}^*$,

$$\hat{g}^{-1}(y) = \hat{g}(S^{-1}(y)) = \hat{g}^*(S^{-1}(y)) = \overline{\hat{g}(S(S^{-1}(y))^*)} = \overline{\hat{g}(y^*)} \tag{90}$$

for all elements $y \in \mathcal{A}$. Thus,

$$\begin{aligned}
&\xi T(x)\xi^{-1} \\
&= \xi S(x_{(1)})\xi^{-1}\hat{g}(x_{(2)}) && \text{by Remark C.4} \\
&= \hat{g}(S(x_{(1)})_{(1)})S(x_{(1)})_{(2)}\hat{g}(S(x_{(1)})_{(3)})\hat{g}(x_{(2)}) && \text{by Equation 89} \\
&= \hat{g}(S(x_{(3)}))S(x_{(2)})\hat{g}(S(x_{(1)}))\hat{g}(x_{(4)}) && \text{by Equation 48} \\
&= \hat{g}^{-1}(x_{(1)})S(x_{(2)})\hat{g}^{-1}(x_{(3)})\hat{g}(x_{(4)}) && \text{by Equation 86} \\
&= \hat{g}^{-1}(x_{(1)})S(x_{(2)}) \\
&= \hat{g}^{-1}(x_{(1)})S^{-1}(S^2(x_{(2)})) \\
&= \hat{g}^{-1}(x_{(1)})\hat{g}(x_{(2)})S^{-1}(x_{(3)})\hat{g}^{-1}(x_{(4)}) && \text{by Definition 5.13} \\
&= S^{-1}(x_{(1)})\hat{g}^{-1}(x_{(2)}) \\
&= S^{-1}(x_{(1)})\overline{\hat{g}(x^*_{(2)})} && \text{by Equation 90} \\
&= S(x^*_{(1)})^*\overline{\hat{g}(x^*_{(2)})} \\
&= S((x^*)_{(1)})^*\overline{\hat{g}((x^*)_{(2)})} && \text{by Definition 5.17} \\
&= T(x^*)^*, && \text{by Remark C.4}
\end{aligned}$$

for all elements $x \in \mathcal{A}$, as we wanted to prove. Finally, if $\mathcal{A}$ is a C*-HA, it is already known by Proposition C.6 that $g_L = g_R = \mathcal{D}^{-1}1$. This, together with the definition of $\xi \in \mathcal{A}$ in Lemma 8.6 and the fact that $\varepsilon(1) = 1$, leads to the expressions $\xi_L = \xi_R = \mathcal{D}1$ and $\xi = \mathcal{D}^2 1$, as we wanted to prove. □

APPENDIX E

Proof of Theorem 8.3

We now provide algebraic explicit expressions for both local coarse-graining and fine-graining quantum channels. We restate and prove the following theorem now.

THEOREM 8.3. *Let $\mathcal{A}$ be a biconnected C*-WHA and let (V, Φ) be a faithful $*$-representation of $\mathcal{A}$. Then, the operators*

$$\rho(x, n) := \omega(x)^{-1} b(\omega)^{\otimes n} \Phi^{\otimes n}(\Delta^{(n-1)}(x)) \in \mathfrak{B}(V^{\otimes n})$$

are RFP MPDOs for all positive non-zero elements $x \in \mathcal{A}$ and all $n \in \mathbb{N}$. Specifically, there are quantum channels $\mathfrak{T} : \mathfrak{B}(V) \to \mathfrak{B}(V \otimes V)$ and $\mathfrak{S} : \mathfrak{B}(V \otimes V) \to \mathfrak{B}(V)$, known as local fine-graining and coarse-graining maps, respectively, such that

$$\mathfrak{T}(\rho(x, 1)) = \rho(x, 2) \quad \textit{and} \quad \mathfrak{S}(\rho(x, 2)) = \rho(x, 1)$$

for all positive non-zero elements $x \in \mathcal{A}$ and all $n \in \mathbb{N}$.

PROOF. As previously done, let us define the local coarse-graining quantum channel

$$\mathfrak{T}(X) := \mathrm{Tr}(\Phi(\xi T(\Omega_{(1)}))X)\Phi(c_\omega \Omega_{(2)}) \otimes \Phi(c_\omega \Omega_{(3)}) \tag{91}$$

for all $X \in \mathfrak{B}(V)$. First, let us check that $\mathfrak{T}(\rho(x, 1)) = \rho(x, 2)$ for all positive non-zero $x \in \mathcal{A}$. Indeed,

$$\begin{aligned}
&\mathfrak{T}(\rho(x, 1)) \\
&= \tfrac{1}{\omega(x)} \mathrm{Tr}(\Phi(\xi T(\Omega_{(1)}) c_\omega x)) \Phi(c_\omega \Omega_{(2)}) \otimes \Phi(c_\omega \Omega_{(3)}) \\
&= \tfrac{1}{\omega(x)} \omega(\xi T(\Omega_{(1)}) x) \Phi(c_\omega \Omega_{(2)}) \otimes \Phi(c_\omega \Omega_{(3)}) && \text{by Remark 8.2} \\
&= \tfrac{1}{\omega(x)} \omega(\xi T(\Omega_{(1)})) \Phi(c_\omega x_{(1)} \Omega_{(2)}) \otimes \Phi(c_\omega x_{(2)} \Omega_{(3)}) && \text{by Equation 59} \\
&= \tfrac{1}{\omega(x)} \Phi(c_\omega x_{(1)} 1_{(1)}) \otimes \Phi(c_\omega x_{(2)} 1_{(2)}) && \text{by Lemma 8.6} \\
&= \tfrac{1}{\omega(x)} \Phi(c_\omega x_{(1)}) \otimes \Phi(c_\omega x_{(2)}) = \rho(x, 2) && \text{by Definition 5.17}
\end{aligned}$$

Second, this map is trace-preserving:

$$\begin{aligned}
&\mathrm{Tr}(\mathfrak{T}(X)) \\
&= \mathrm{Tr}(\Phi(\xi T(\Omega_{(1)}))X)\mathrm{Tr}(\Phi(c_\omega\Omega_{(2)}))\mathrm{Tr}(\Phi(c_\omega\Omega_{(3)})) \\
&= \mathrm{Tr}(\Phi(\xi T(\Omega_{(1)}))X)\omega(\Omega_{(2)})\omega(\Omega_{(3)}) && \text{by Remark 8.2} \\
&= \mathrm{Tr}(\Phi(\xi T(\Omega_{(1)}))X)\omega(\Omega_{(2)}) && \text{by Lemma 8.1} \\
&= \mathrm{Tr}(\Phi(\xi T(\xi^{-1}))X) && \text{by Lemma 8.6} \\
&= \mathrm{Tr}(\Phi(\xi\xi^{-1})X)\mathrm{Tr}(X) && \text{by Lemma 8.6}
\end{aligned}$$

Finally, since $\Omega = \Omega^2 = \Omega\Omega^*$ (in fact, only positivity of Ω is needed), we can rewrite the map in the following form:

$$\begin{aligned}
\mathfrak{T}(X) &= \mathrm{Tr}(\Phi(\xi T(\Omega_{(1)}(\Omega^*)_{(1')})X)\Phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\Omega_{(2)}(\Omega^*)_{(2')})) \\
&= \mathrm{Tr}(\Phi(\xi T(\Omega_{(1)}(\Omega^*)_{(1')})X)\Phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\Omega_{(2)})\Delta((\Omega^*)_{(2')})) \\
&= \mathrm{Tr}(\Phi(\xi T(\Omega_{(1)}\Omega^*_{(1')})X)\Phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\Omega_{(2)})\Delta(\Omega_{(2')})^*) \\
&= \mathrm{Tr}(\Phi(\xi T(\Omega^*_{(1')})T(\Omega_{(1)}))X)\Phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\Omega_{(2)})\Delta(\Omega_{(2')})^*) \\
&= \mathrm{Tr}(\Phi(T(\Omega_{(1')})^*\xi T(\Omega_{(1)}))X)\Phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\Omega_{(2)})\Delta(\Omega_{(2')})^*) \\
&= \mathrm{Tr}(\Phi(T(\Omega_{(1')})^*\xi^{\frac{1}{2}}\xi^{\frac{1}{2}}T(\Omega_{(1)}))X) \\
&\qquad \Phi^{\otimes 2}((c_\omega^{\frac{1}{2}})^{\otimes 2}\Delta(\Omega_{(2)})\Delta(\Omega_{(2')})^*(c_\omega^{\frac{1}{2}})^{\otimes 2}) \\
&= \mathrm{Tr}(\Phi(T(\Omega_{(1')})^*\xi^{\frac{1}{2}})\Phi(\xi^{\frac{1}{2}}T(\Omega_{(1)}))X) \\
&\qquad \Phi^{\otimes 2}((c_\omega^{\frac{1}{2}})^{\otimes 2}\Delta(\Omega_{(2)}))\Phi^{\otimes 2}(\Delta(\Omega_{(2')})^*(c_\omega^{\frac{1}{2}})^{\otimes 2}) \\
&= \mathrm{Tr}(\Phi(\xi^{\frac{1}{2}}T(\Omega_{(1)}))X\Phi(\xi^{\frac{1}{2}}T(\Omega_{(1')}))^\dagger) \\
&\qquad \Phi^{\otimes 2}((c_\omega^{\frac{1}{2}})^{\otimes 2}\Delta(\Omega_{(2)}))\Phi^{\otimes 2}((c_\omega^{\frac{1}{2}})^{\otimes 2}\Delta(\Omega_{(2')}))^\dagger \\
&= (\mathrm{Tr}\otimes\mathrm{Id}\otimes\mathrm{Id})(Q(X\otimes\mathbf{1}\otimes\mathbf{1})Q^\dagger)
\end{aligned}$$

where

$$Q := \Phi^{\otimes 3}(\xi^{\frac{1}{2}}T(\Omega_{(1)})\otimes c_\omega^{\frac{1}{2}}\Omega_{(2)}\otimes c_\omega^{\frac{1}{2}}\Omega_{(3)})$$

Thus, $\mathfrak{T}$ is completely positive. Now, let us define a local fine-graining quantum channel $\mathfrak{S}$. Consider first the following hermitian projectors

$$P := \Phi^{\otimes 2}(\Delta(1)), \quad P^\perp := \Phi^{\otimes 2}(1\otimes 1 - \Delta(1)), \quad P + P^\perp = \mathbf{1}\otimes\mathbf{1} \tag{92}$$

and let $\rho_0 \in \mathfrak{B}(V)$ be any mixed state. Define

$$\mathfrak{S}(X) := \mathrm{Tr}(\Phi(\Delta(\xi T(\Omega_{(1)})))X)\Phi(c_\omega\Omega_{(2)}) + \mathrm{Tr}(P^\perp X)\rho_0 \tag{93}$$

for all elements $X \in \mathfrak{B}(V\otimes V)$. We first check that it satisfies $\mathfrak{S}(\rho(x,2)) = \rho(x,1)$ for all positive non-zero $x \in \mathcal{A}$. Notice that the second summand in the right-hand side of Equation 93 simply vanishes, i.e. $P^\perp\rho(x,2) = 0$, since $\rho(x,2)$ is supported on the orthogonal

subspace $P \cdot \mathfrak{B}(V^{\otimes 2})$. Thus,

$$
\begin{aligned}
&\mathfrak{S}(\rho(x,2)) \\
&= \tfrac{1}{\omega(x)}\mathrm{Tr}(\Phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\xi T(\Omega_{(1)})x)))\Phi(c_\omega\Omega_{(2)}) && \text{by Definition 5.17} \\
&= \tfrac{1}{\omega(x)}(\omega\otimes\omega)(\Delta(\xi T(\Omega_{(1)})x))\Phi(c_\omega\Omega_{(2)}) && \text{by Remark 8.2} \\
&= \tfrac{1}{\omega(x)}\omega(\xi T(\Omega_{(1)})x)\Phi(c_\omega\Omega_{(2)}) && \text{by Lemma 8.1} \\
&= \tfrac{1}{\omega(x)}\omega(\xi T(\Omega_{(1)}))\Phi(c_\omega x\Omega_{(2)}) && \text{by Equation 59} \\
&= \tfrac{1}{\omega(x)}\Phi(c_\omega x) = \rho(x,1) && \text{by Lemma 8.6}
\end{aligned}
$$

for all positive non-zero elements $x \in \mathcal{A}$, as we wanted to prove. Secondly, let us check that it is trace-preserving:

$$
\begin{aligned}
&\mathrm{Tr}(\mathfrak{S}(X)) \\
&= \mathrm{Tr}(\Phi^{\otimes 2}(\Delta(\xi T(\Omega_{(1)})))X)\mathrm{Tr}(\Phi(c_\omega\Omega_{(2)})) + \mathrm{Tr}(P^\perp X) \\
&= \mathrm{Tr}(\Phi^{\otimes 2}(\Delta(\xi T(\Omega_{(1)})))X)\omega(\Omega_{(2)}) + \mathrm{Tr}(P^\perp X) && \text{by Remark 8.2} \\
&= \mathrm{Tr}(\Phi^{\otimes 2}(\Delta(\xi T(\xi^{-1})))X) + \mathrm{Tr}(P^\perp X) && \text{by Lemma 8.6} \\
&= \mathrm{Tr}(\Phi^{\otimes 2}(\Delta(\xi\xi^{-1}))X) + \mathrm{Tr}(P^\perp X) && \text{by Lemma 8.6} \\
&= \mathrm{Tr}(PX) + \mathrm{Tr}(P^\perp X) = \mathrm{Tr}((P+P^\perp)X) = \mathrm{Tr}(X) && \text{by Equation 92}
\end{aligned}
$$

for all $X \in \mathfrak{B}(V \otimes V)$. That $\mathfrak{S}$ is completely positive can be proved analogously and we do not include it here: simply notice that the second summand in Equation 93 is clearly a completely positive map, and a similar argument to that for $\mathfrak{T}$ applies to the first summand. □

APPENDIX F

Proof of Lemma 10.2

In this appendix we derive a proof of Lemma 10.2. We first provide the following auxiliary result, related to the trace-preserving condition of the gluing map.

Lemma F.1. *Let $\mathcal{A}$ be a C*-HA. Then,*

$$x_{(1)} \otimes \omega(x_{(2)})x_{(3)} = \omega(x)1 \otimes 1$$

for all elements $x \in \mathcal{A}$.

Proof. Fix any $x \in \mathcal{A}$. Since $\Omega \in \mathcal{A}$ is non-degenerate, there exists $f \in \mathcal{A}^*$ such that

$$x = \Omega_{(1)} f(\Omega_{(2)}). \tag{94}$$

As an immediate consequence,

$$\omega(x) = \omega(\Omega_{(1)})f(\Omega_{(2)}) = \mathcal{D}^{-2} f(1), \tag{95}$$

where the last equality follows from Lemma 8.6. Then, it is easy to conclude that

$$\begin{aligned}
x_{(1)}\omega(x_{(2)}) \otimes x_{(3)} &= \Omega_{(1)} \otimes \omega(\Omega_{(2)})\Omega_{(3)} f(\Omega_{(4)}) && \text{by Equation 94} \\
&= \Omega_{(4)} \otimes \omega(\Omega_{(1)})\Omega_{(2)} f(\Omega_{(3)}) && \text{by Theorem 5.21} \\
&= \mathcal{D}^{-2} 1_{(3)} \otimes 1_{(1)} f(1_{(2)}) && \text{by Lemma 8.6} \\
&= \mathcal{D}^{-2} f(1) 1 \otimes 1 && \text{by Definition 5.22} \\
&= \omega(x)1 \otimes 1, && \text{by Equation 95}
\end{aligned}$$

as we wanted to prove. □

Lemma F.2. *Let $\mathcal{A}$ be a C*-HA and let (V, Φ) be a faithful $*$-representation of $\mathcal{A}$. Then, for all positive non-zero elements $x \in \mathcal{A}$ there exists a quantum channel $\mathfrak{G}_x : \mathfrak{B}(V \otimes V) \to \mathfrak{B}(V \otimes V)$, called "gluing" map, such that*

$$(\mathrm{Id}^{\otimes m-1} \otimes \mathfrak{G}_x \otimes \mathrm{Id}^{\otimes n-1})(\rho(\Omega, m) \otimes \rho(\Omega, n)) = \rho(x, m+n) \tag{72}$$

for all $m, n \in \mathbb{N}$.

Proof. Fix any positive non-zero element $x \in \mathcal{A}$. We recall first the definition of the gluing map previously given in Chapter 10. For simplicity, let $\mathfrak{G}_x := \mathfrak{T} \circ \mathfrak{G}$ for the linear map $\mathfrak{G} \in \mathfrak{B}(\mathfrak{B}(V \otimes V), \mathfrak{B}(V))$ defined by the expression

$$\mathfrak{G}(X \otimes Y) := \tfrac{1}{\omega(x)} \mathrm{Tr}(\Phi(S(x_{(1)}))X)\Phi(c_\omega x_{(2)})\mathrm{Tr}(\Phi(S(x_{(3)}))Y) \tag{96}$$

for all $X, Y \in \mathfrak{B}(V)$. Then, it is enough to check that $\mathfrak{G}(\rho(\Omega,2) \otimes \rho(\Omega,2)) = \rho(\Omega,3)$. To this end, let us recall that, in the case of C*-HAs,

$$\omega(\Omega_{(1)})\Omega_{(2)} = \tfrac{1}{\mathcal{D}^2}1 = \omega(\Omega)1, \tag{97}$$

where the first equality is stated in Lemma 8.6 and the second equality follows by applying the counit in the first one, since $\varepsilon(1) = 1$. Then,

$$\begin{aligned}
&(\mathrm{Id} \otimes \mathfrak{G} \otimes \mathrm{Id})(\rho(\Omega,2) \otimes \rho(\Omega,2)) = \\
&= \tfrac{1}{\omega(x)}\tfrac{1}{\omega(\Omega)^2}\Phi(c_\omega \Omega_{(1)}) \otimes \omega(S(x_{(1)})\Omega_{(2)})\Phi(c_\omega x_{(2)}) \\
&\qquad \otimes \omega(S(x_{(3)})\Omega_{(1')})\Phi(c_\omega \Omega_{(2')}) \\
&= \tfrac{1}{\omega(x)}\tfrac{1}{\omega(\Omega)^2}\Phi(c_\omega x_{(1)}\Omega_{(1)}) \otimes \omega(\Omega_{(2)})\Phi(c_\omega x_{(2)}) \\
&\qquad \otimes \omega(S(x_{(3)})\Omega_{(1')})\Phi(c_\omega \Omega_{(2')}) \\
&= \tfrac{1}{\omega(x)}\tfrac{1}{\omega(\Omega)^2}\Phi(c_\omega x_{(1)}\Omega_{(1)}) \otimes \omega(\Omega_{(2)})\Phi(c_\omega x_{(2)})\omega(\Omega_{(1')}) \otimes \Phi(c_\omega x_{(3)}\Omega_{(2')}) \\
&= \tfrac{1}{\omega(x)}\Phi(c_\omega x_{(1)}1) \otimes \Phi(c_\omega x_{(2)}) \otimes \Phi(c_\omega x_{(3)}1) = \rho(x,3).
\end{aligned}$$

This calculation can be explained as follows. In the first place, we have replaced the trace with the canonical regular element $\omega \in \mathcal{A}^*$ since by Remark 8.2 the weight $c_\omega \in \mathcal{A}$, which is central, defines a linear extension of ω to the representation space. In the second and third steps we have applied the pulling-through identity; see Equation 59. Finally, we apply twice Equation 97 to get rid of Ω and the coefficients $\omega(\Omega)^{-1}$. As an aside, note that $\omega(\Omega_{(1)})\Omega_{(2)} = \Omega_{(1)}\omega(\Omega_{(2)})$ since Ω is cocentral; see Theorem 5.21. Since $\mathfrak{T}$ is a quantum channel it only remains to prove that $\mathfrak{G}$ is also a quantum channel. On the one hand, that $\mathfrak{G}$ is trace-preserving is a straightforward consequence of Lemma F.1:

$$\begin{aligned}
&\mathrm{Tr}(\mathfrak{G}(X \otimes Y)) \\
&= \tfrac{1}{\omega(x)}\mathrm{Tr}(\Phi(S(x_{(1)}))X)\omega(x_{(2)})\mathrm{Tr}(\Phi(S(x_{(3)}))Y) && \text{by Remark 8.2} \\
&= \tfrac{1}{\omega(x)}\omega(x)\mathrm{Tr}(\Phi(S(1))X)\mathrm{Tr}(\Phi(S(1))Y) && \text{by Lemma F.1} \\
&= \mathrm{Tr}(X)\mathrm{Tr}(Y) = \mathrm{Tr}(X \otimes Y)
\end{aligned}$$

for all $X, Y \in \mathfrak{B}(V)$. On the other hand, in order to prove that $\mathfrak{G}$ is completely positive, let $x = yy^*$ for some element $y \in \mathcal{A}$. Then, we can

rewrite it as follows

$$
\begin{aligned}
&\mathfrak{G}(X\otimes Y)=\\
&\quad=\tfrac{1}{\omega(x)}\mathrm{Tr}(\Phi(S((yy^*)_{(1)}))X)\Phi(c_\omega(yy^*)_{(2)})\mathrm{Tr}(\Phi(S(yy^*)_{(3)})Y)\\
&\quad=\tfrac{1}{\omega(x)}\mathrm{Tr}(\Phi(S(y_{(1)}y^*_{(1')}))X)\Phi(c_\omega y_{(2)}y^*_{(2')})\mathrm{Tr}(\Phi(S(y_{(3)}y^*_{(3')}))Y)\\
&\quad=\tfrac{1}{\omega(x)}\mathrm{Tr}(\Phi(S(y^*_{(1')})S(y_{(1)}))X)\Phi(c_\omega y_{(2)}y^*_{(2')})\\
&\qquad\mathrm{Tr}(\Phi(S(y^*_{(3')})S(y_{(3)}))Y)\\
&\quad=\tfrac{1}{\omega(x)}\mathrm{Tr}(\Phi(S(y_{(1')})^*S(y_{(1)}))X)\Phi(c_\omega y_{(2)}y^*_{(2')})\\
&\qquad\mathrm{Tr}(\Phi(S(y_{(3')})^*S(y_{(3)}))Y)\\
&\quad=\tfrac{1}{\omega(x)}\mathrm{Tr}(\Phi(S(y_{(1)}))X\Phi(S(y_{(1')})^*))\Phi(c_\omega y_{(2)}y^*_{(2')})\\
&\qquad\mathrm{Tr}(\Phi(S(y_{(3)}))Y\Phi(S(y_{(3')})^*))\\
&\quad=(\mathrm{Tr}\otimes\mathrm{Id}\otimes\mathrm{Tr})(Q(X\otimes\mathbf{1}\otimes Y)Q^\dagger)
\end{aligned}
$$

where we have defined

$$
Q:=\tfrac{1}{\omega(x)^{1/2}}\Phi^{\otimes 4}(S(y_{(1)})\otimes c_\omega^{\frac{1}{2}}y_{(2)}\otimes S(y_{(3)})). \tag{98}
$$

Therefore, $\mathfrak{G}$ is completely positive. Indeed, in the first step we have applied that the comultiplication is multiplicative and the $*$-operation is a coalgebra homomorphism. In the second and third steps we have used that $S\in\mathfrak{B}(\mathcal{A})$ is an algebra anti-homomorphism and the relation between the antipode and the $*$-operation; see Section 5 from Chapter 5. Note that, for C*-HAs, $S=S^{-1}$; see Proposition C.6. The fourth step is a simple consequence of the fact that Φ is a $*$-representation and the cyclic property of the trace. Finally, the middle term can be rewritten in the form $\Phi(c_\omega y_{(2)}y^*_{(2')})=\Phi(c_\omega^{1/2}y_{(2)})\Phi(c_\omega^{1/2}y_{(2')})^\dagger$ since $c_\omega\in\mathcal{A}$ is positive central element and Φ is a $*$-representation. □

APPENDIX G

Proof of Lemma 10.6

In this appendix we prove Lemma 10.6. In order to perform an analogous construction of this gluing map to the one given in the C*-HA case, we first derive an appropiate version of the usual pulling-through identity in Equation 59 to the trivial sector.

Lemma G.1. *Let $\mathcal{A}$ be a biconnected C*-WHA. Then,*

$$x_L S(1_{(1)}) \otimes 1_{(2)} \otimes S(1_{(3)}) y_R = S(1_{(1)}) \otimes y_R 1_{(2)} x_L \otimes S(1_{(3)})$$

for all elements $x_L \in \mathcal{A}_L$ and $y_R \in \mathcal{A}_R$.

Proof. First, recall Equations 2.31a and 2.31b from [**14**]:

$$x_L S(1_{(1)}) \otimes 1_{(2)} = S(1_{(1)}) \otimes 1_{(2)} x_L,$$
$$y_R 1_{(1)} \otimes S(1_{(2)}) = 1_{(1)} \otimes S(1_{(2)}) y_R.$$

for all $x_L \in \mathcal{A}_L$ and $y_R \in \mathcal{A}_R$. This, together with Definitions 5.18, leads by taking coproducts accordingly to the following identities:

$$x_L S(1_{(1)}) \otimes 1_{(2)} \otimes 1_{(3)} = S(1_{(1)}) \otimes 1_{(2)} x_L \otimes 1_{(3)},$$
$$1_{(1)} \otimes y_R 1_{(2)} \otimes S(1_{(3)}) = 1_{(1)} \otimes 1_{(2)} \otimes S(1_{(3)}) y_R,$$

respectively, for all elements $x_L \in \mathcal{A}_L$ and $y_R \in \mathcal{A}_R$. Finally, since $\mathcal{A}_L$ and $\mathcal{A}_R$ commute, we conclude the result by combining both identities. □

In addition, we adapt slightly Lemma 8.6 to the trivial sector, which is a key property concerning complete positivity of the gluing map in Lemma 10.6. The following result solves this problem.

Lemma G.2. *Let $\mathcal{A}$ be a biconnected C*-WHA. Then,*

$$\xi_R S(x_L^*) = S(x_L)^* \xi_R \quad \text{and} \quad S(y_R) \xi_L = \xi_L S(y_R^*)^*$$

for all elements $x_L \in \mathcal{A}_L$ and $y_R \in \mathcal{A}_R$.

Proof. In the first place, note that $T \in \mathfrak{B}(\mathcal{A})$ coincides with S and S^{-1} restricted to $\mathcal{A}_L$ and $\mathcal{A}_R$, respectively. Indeed, by virtue of Remark C.4, Proposition C.1 and Section 5,

$$T(x_L) = S(x_L 1_{(1)}) \hat{g}(1_{(2)}) = S(x_L), \tag{99a}$$
$$T(y_R) = \hat{g}(1_{(1)}) S^{-1}(1_{(2)} y_R) = S^{-1}(y_R) = S(y_R^*)^*, \tag{99b}$$

for all $x_L \in \mathcal{A}_L$ and $y_R \in \mathcal{A}_R$. Then, recall Lemma 8.6 to conclude that

$$\begin{aligned} S(x_L^*) = T(x_L^*) &= \xi_L^{-1}\xi_R^{-1}T(x_L)^*\xi_L\xi_R \\ &= \xi_L^{-1}\xi_R^{-1}S(x_L)^*\xi_L\xi_R = \xi_R^{-1}S(x_L)^*\xi_R, \end{aligned}$$

where in the last step we have used that $S(x_L) \in \mathcal{A}_R$ and $\mathcal{A}_L$ and $\mathcal{A}_R$ commute. The remaining identity is proved similarly. □

The following auxiliary results arise naturally in the course of the derivation of the properties of the gluing map.

Lemma G.3. *Let $\mathcal{A}$ be a bicoconnected C*-WHA. Then,*

$$\hat{h}(\Omega_{(1)})\Omega_{(2)} = \frac{1}{\mathcal{D}^2\varepsilon(1)}1.$$

Proof. It is easy to check that

$$\mathcal{D}^2\varepsilon(1)\hat{h}(\Omega_{(1)})\Omega_{(2)} = \hat{h}(t_{(1)})t_{(2)}\hat{g}(t_{(3)}) = 1_{(1)}\hat{g}(1_{(2)}) = 1,$$

where the first step is a consequence of the characterization of $\Omega \in \mathcal{A}$ in Proposition C.3, the second follows from the definition of dual left integral in Equation 83 and the third equality is due to Proposition C.1. □

Lemma G.4. *Let $\mathcal{A}$ be a biconnected C*-WHA. Then,*

$$1_{(1)}\hat{h}(1_{(2)}) \otimes 1_{(3)} = \frac{1}{\varepsilon(1)}1 \otimes 1. \tag{100}$$

Proof. Equivalently, we will check that

$$(\phi\hat{h}\psi)(1) = \frac{1}{\varepsilon(1)}\phi(1)\psi(1)$$

for all $\phi, \psi \in \mathcal{A}^*$. Recall that $\hat{h} \in \mathcal{A}^*$ is a one-dimensional projector supported on the trivial sector [**14**, Lemma 4.8]. Hence,

$$(\phi\hat{h}\psi\hat{h})(\tau_1) = (\phi\hat{h})(\tau_1)(\psi\hat{h})(\tau_1) \quad \text{and} \quad (\phi\hat{h})(\tau_a) = \delta_{a1} \tag{101}$$

for all $\phi, \psi \in \mathcal{A}^*$ and all sectors $a \in \mathrm{Irr}(\mathcal{A}^*)$. In particular

$$(f\hat{h})(\tau_1) = (f\hat{h})(\sum_a d_a\tau_a) = \mathcal{D}^2(f\hat{h})(\Omega) = \varepsilon(1)^{-1}f(1) \tag{102}$$

for all $f \in \mathcal{A}^*$. Thus, we conclude that:

$$\frac{1}{\varepsilon(1)}(\phi\hat{h}\psi)(1) = (\phi\hat{h}\psi\hat{h})(\tau_1) = (\phi\hat{h})(\tau_1)(\psi\hat{h})(\tau_1) = \frac{1}{\varepsilon(1)^2}\phi(1)\psi(1),$$

where the first equality follows from Equation 102 using $f := \phi\hat{h}\psi$, the second equality is simply Equation 101 and the third equality follows from Equation 102 considering $f := \phi, \psi$. □

Lemma G.5. *Let $\mathcal{A}$ be a biconnected C*-WHA. Then,*

$$1_{(1)} \otimes \omega(1_{(2)})1_{(3)} = \mathcal{D}^2\xi_R^{-1} \otimes \xi_L^{-1}.$$

PROOF. Note by the definition of $\mathcal{A}_L$ and $\mathcal{A}_R$ in Definitions 5.18 and the decomposition $\xi^{-1} = \xi_L^{-1}\xi_R^{-1}$ in Lemma 8.6, that

$$(\xi^{-1})_{(1)} \otimes (\xi^{-1})_{(2)} \otimes (\xi^{-1})_{(3)} = \xi_L^{-1} 1_{(1)} \otimes 1_{(2)} \otimes \xi_R^{-1} 1_{(3)} \tag{103}$$

Then, the statement follows from the following calculation:

$$\begin{aligned}
&1_{(1)} \otimes \omega(1_{(2)}) 1_{(3)} \\
&= \mathcal{D}^2 \varepsilon(1) \hat{h}(\Omega_{(1)}) \Omega_{(2)} \otimes \omega(\Omega_{(3)}) \Omega_{(4)} && \text{by Lemma G.3} \\
&= \mathcal{D}^2 \varepsilon(1) \hat{h}(\Omega_{(3)}) \Omega_{(4)} \otimes \omega(\Omega_{(1)}) \Omega_{(2)} && \text{by Theorem 5.21} \\
&= \mathcal{D}^2 \varepsilon(1) \hat{h}((\xi^{-1})_{(2)}) (\xi^{-1})_{(3)} \otimes (\xi^{-1})_{(1)} && \text{by Lemma 8.6} \\
&= \mathcal{D}^2 \varepsilon(1) \hat{h}(1_{(2)}) \xi_R^{-1} 1_{(3)} \otimes \xi_L^{-1} 1_{(1)} && \text{by Equation 103} \\
&= \mathcal{D}^2 \xi_R^{-1} \otimes \xi_L^{-1} && \text{by Lemma G.4}
\end{aligned}$$

as we wanted to prove. □

Lemma G.6. *Let $\mathcal{A}$ be a biconnected C*-WHA. Then,*

$$\omega(1_{(1)}) 1_{(2)} \omega(1_{(3)}) = \mathcal{D}^2 \omega(1) \xi^{-1}. \tag{104}$$

PROOF. First, it will be useful to compute the constant $\omega(1)$ in a more operative way. The following calculation is a direct consequence of Proposition C.3 and Equation 88:

$$\omega(1) = \tfrac{1}{\mathcal{D}^2 \varepsilon(1)} \hat{h}(g_L^{-1} g_R^{-1}) = \tfrac{\mathcal{D}^4 \varepsilon(1)^2}{\mathcal{D}^2 \varepsilon(1)} \hat{h}(\xi_L^{-1} \xi_R^{-1}) = \mathcal{D}^2 \varepsilon(1) \hat{h}(\xi_L^{-1} \xi_R^{-1}). \tag{105}$$

Now, by an analogous reasoning as in the previous proof:

$$\begin{aligned}
&\omega(1_{(1)}) 1_{(2)} \omega(1_{(3)}) \\
&= \mathcal{D}^2 \varepsilon(1) \hat{h}(\Omega_{(1)}) \omega(\Omega_{(2)}) \Omega_{(3)} \omega(\Omega_{(4)}) && \text{by Lemma G.3} \\
&= \mathcal{D}^2 \varepsilon(1) \hat{h}(\Omega_{(4)}) \omega(\Omega_{(1)}) \Omega_{(2)} \omega(\Omega_{(3)}) && \text{by Theorem 5.21} \\
&= \mathcal{D}^2 \varepsilon(1) \hat{h}(\xi_R^{-1} 1_{(3)}) \xi_L^{-1} 1_{(1)} \omega(1_{(2)}) && \text{by Lemma 8.6} \\
&= \mathcal{D}^4 \varepsilon(1) \hat{h}(\xi_R^{-1} \xi_L^{-1}) \xi_L^{-1} \xi_R^{-1} && \text{by Lemma G.5} \\
&= \mathcal{D}^2 \omega(1) \xi_L^{-1} \xi_R^{-1} = \mathcal{D}^2 \omega(1) \xi^{-1} && \text{by Equation 105}
\end{aligned}$$

as we wanted to prove. □

Remark G.7. There are no trace-preserving gluing maps for general biconnected C*-WHAs such that Equation 72 holds for all elements $x \in \mathcal{A}$.

PROOF. Suppose by contradiction that there exists a trace-preserving linear map $\mathfrak{G} \in \mathfrak{B}(\mathfrak{B}(V \otimes V))$ that is a "gluing map". In particular,

$$(\mathrm{Id} \otimes \mathfrak{G} \otimes \mathrm{Id})(\rho(\Omega, 2) \otimes \rho(\Omega, 2)) = \rho(\Omega, 4).$$

On the one hand, after performing a partial trace on the second and third subsystems, the left-hand side would be trivially given by the

product state

$$\begin{aligned}&(\mathrm{Id}\otimes\mathrm{Tr}\otimes\mathrm{Tr}\otimes\mathrm{Id})(\rho(\Omega,2)\otimes\rho(\Omega,2))\\&=\omega(\Omega)^{-2}\Phi(c_\omega\Omega_{(1)})\omega(\Omega_{(2)})\otimes\omega(\Omega_{(1')})\Phi(c_\omega\Omega_{(2')})\\&=\omega(\Omega)^{-2}\Phi(c_\omega\xi^{-1})\otimes\Phi(c_\omega\xi^{-1})\end{aligned}$$

by virtue of Remark 8.2 and Lemma 8.6. However, the right-hand side would take the following form:

$$\begin{aligned}&(\mathrm{Id}\otimes\mathrm{Tr}\otimes\mathrm{Tr}\otimes\mathrm{Id})(\rho(\Omega,4))\\&=\omega(\Omega)^{-2}\Phi(c_\omega\Omega_{(1)})\otimes\omega(\Omega_{(2)})\omega(\Omega_{(3)})\Phi(c_\omega\Omega_{(4)})&&\text{by Remark 8.2}\\&=\omega(\Omega)^{-2}\Phi(c_\omega\Omega_{(1)})\otimes\omega(\Omega_{(2)})\Phi(c_\omega\Omega_{(3)})&&\text{by Lemma 8.1}\\&=\omega(\Omega)^{-2}\Phi(c_\omega\Omega_{(2)})\otimes\omega(\Omega_{(3)})\Phi(c_\omega\Omega_{(1)})&&\text{by Theorem 5.21}\\&=\omega(\Omega)^{-2}\Phi(c_\omega\xi_R^{-1}1_{(2)})\otimes\Phi(c_\omega\xi_L^{-1}1_{(1)})&&\text{by Lemma 8.6}\end{aligned}$$

which is not a product state. This contradicts the previous equation. □

Lemma G.8. *Let $\mathcal{A}$ be a biconnected C*-WHA and let (V,Φ) be a faithful $*$-representation of $\mathcal{A}$. There is a quantum channel $\mathfrak{G}_1:\mathfrak{B}(V\otimes V)\to\mathfrak{B}(V\otimes V)$, called "gluing" map, such that*

$$(\mathrm{Id}^{\otimes m-1}\otimes\mathfrak{G}_1\otimes\mathrm{Id}^{\otimes n-1})(\rho(1,m)\otimes\rho(1,n))=\rho(1,m+n)\tag{73}$$

for all $m,n\in\mathbb{N}$.

Proof. For simplicity, let $\mathfrak{G}_1:=\mathfrak{T}\circ\mathfrak{G}$, where $\mathfrak{T}:\mathfrak{B}(V)\to\mathfrak{B}(V\otimes V)$ stands for the local coarse-graining quantum channel from Chapter 8 and $\mathfrak{G}:\mathfrak{B}(V\otimes V)\to\mathfrak{B}(V)$ is given by

$$\mathfrak{G}(X\otimes Y):=\tfrac{1}{\mathcal{D}^2}\mathrm{Tr}(\Phi(S(1_{(1)})\xi_L)X)\Phi(c_\omega 1_{(2)})\mathrm{Tr}(\Phi(\xi_R S(1_{(3)}))Y)$$

for all $X,Y\in\mathfrak{B}(V)$. First, assume that $m=n=2$ without loss of generality and let us check that it fulfills $\mathfrak{G}(\rho(1,2)\otimes\rho(1,2))=\rho(1,3)$. To this end, it turns out to be enough to prove:

$$\mathfrak{G}(\Phi(c_\omega x_L)\otimes\Phi(c_\omega x_R))=\omega(1)\Phi(c_\omega x_L x_R)\tag{106}$$

for all $x_L\in\mathcal{A}_L$ and $x_R\in\mathcal{A}_R$. Indeed, in that case,

$$\begin{aligned}&(\mathrm{Id}\otimes\mathfrak{G}\otimes\mathrm{Id})(\rho(1,2)^{\otimes 2})\\&=\tfrac{1}{\omega(1)^2}\Phi(c_\omega 1_{(1)})\otimes\mathfrak{G}(\Phi(c_\omega 1_{(2)})\otimes\Phi(c_\omega 1_{(1')}))\otimes\Phi(c_\omega 1_{(2')})\\&=\tfrac{1}{\omega(1)}\Phi(c_\omega 1_{(1)})\otimes\Phi(c_\omega 1_{(2)}1_{(1')})\otimes\Phi(c_\omega 1_{(2')})\\&=\tfrac{1}{\omega(1)}\Phi(c_\omega 1_{(1)})\otimes\Phi(c_\omega 1_{(2)})\otimes\Phi(c_\omega 1_{(3)})=\rho(1,3).\end{aligned}$$

by the weak comultiplicativity of the counit and the fact that $1_{(1)}\otimes 1_{(2)}\in\mathcal{A}_R\otimes\mathcal{A}_L$; see Definition 5.17 and [**14**]. Thus, let us move to the

proof of Equation 106:

$$
\begin{aligned}
&\mathfrak{G}(\Phi(c_\omega x_L) \otimes \Phi(c_\omega x_R)) \\
&= \tfrac{1}{\mathcal{D}^2}\omega(S(1_{(1)})\xi_L x_L)\Phi(c_\omega 1_{(2)})\omega(\xi_R S(1_{(3)})x_R) && \text{by Remark 8.2} \\
&= \tfrac{1}{\mathcal{D}^2}\omega(\xi_L x_L S(1_{(1)}))\Phi(c_\omega 1_{(2)})\omega(S(1_{(3)})x_R\xi_R) \\
&= \tfrac{1}{\mathcal{D}^2}\omega(S(1_{(1)}))\Phi(c_\omega x_R\xi_R 1_{(2)}\xi_L x_L)\omega(S(1_{(3)})) && \text{by Lemma G.1} \\
&= \tfrac{1}{\mathcal{D}^2}\omega(1_{(1)})\Phi(c_\omega x_R\xi_R 1_{(2)}\xi_L x_L)\omega(1_{(3)}) && \text{by Remark C.5} \\
&= \omega(1)\Phi(c_\omega x_R\xi_R\xi_R^{-1}\xi_L^{-1}\xi_L x_L) && \text{by Equation 104} \\
&= \omega(1)\Phi(c_\omega x_R x_L)
\end{aligned}
$$

as we wanted to prove. Additionally, $\mathfrak{G}$ is trace-preserving as an immediate consequence of Lemma G.5:

$$
\begin{aligned}
&\mathrm{Tr}(\mathfrak{G}(X \otimes Y)) \\
&= \tfrac{1}{\mathcal{D}^2}\mathrm{Tr}(\Phi(S(1_{(1)})\xi_L)X)\omega(1_{(2)})\mathrm{Tr}(\Phi(\xi_R S(1_{(3)}))Y) && \text{by Remark 8.2} \\
&= \mathrm{Tr}(\Phi(S(\xi_R^{-1})\xi_L)X)\mathrm{Tr}(\Phi(\xi_R S(\xi_L^{-1}))Y) && \text{by Lemma G.5} \\
&= \mathrm{Tr}(\Phi(\xi_L^{-1}\xi_L)X)\mathrm{Tr}(\Phi(\xi_R\xi_R^{-1})Y) && \text{by Equation 88} \\
&= \mathrm{Tr}(X)\mathrm{Tr}(Y) = \mathrm{Tr}(X \otimes Y).
\end{aligned}
$$

Finally, in order to check that $\mathfrak{G}$ is a completely positive linear map, let us first consider the following two calculations:

$$
\begin{aligned}
&\mathrm{Tr}(\Phi(S(x_R y_R^*)\xi_L)X) \\
&= \mathrm{Tr}(\Phi(S(y_R^*)S(x_R)\xi_L)X) \\
&= \mathrm{Tr}(\Phi(S(y_R^*)\xi_L S(x_R^*)^*)X) && \text{by Lemma G.2} \\
&= \mathrm{Tr}(\Phi(S(y_R^*)\xi_L^{\frac{1}{2}}\xi_L^{\frac{1}{2}}S(x_R^*)^*)X) && \text{by Equation 88} \\
&= \mathrm{Tr}(\Phi(S(y_R^*)\xi_L^{\frac{1}{2}})\Phi(\xi_L^{\frac{1}{2}}S(x_R^*)^*)X) \\
&= \mathrm{Tr}(\Phi(\xi_L^{\frac{1}{2}}S(x_R^*)^*)X\Phi(S(y_R^*)\xi_L^{\frac{1}{2}})) \\
&= \mathrm{Tr}(\Phi(\xi_L^{\frac{1}{2}}S(x_R^*)^*)X\Phi((\xi_L^{\frac{1}{2}}S(y_R^*))^*)) && \text{by Equation 88} \\
&= \mathrm{Tr}(\Phi(\xi_L^{\frac{1}{2}}S(x_R^*)^*)X\Phi(\xi_L^{\frac{1}{2}}S(y_R^*)^*)^\dagger)
\end{aligned}
$$

for all $x_R, y_R \in \mathcal{A}_R$ and, analogously,

$$\begin{aligned}
&\mathrm{Tr}(\Phi(\xi_R S(x_L y_L^*))Y) \\
&= \mathrm{Tr}(\Phi(\xi_R S(y_L^*)S(x_L))Y) \\
&= \mathrm{Tr}(\Phi(S(y_L)^*\xi_R S(x_L))Y) && \text{by Lemma G.2} \\
&= \mathrm{Tr}(\Phi(S(y_L)^*\xi_R^{\frac{1}{2}}\xi_R^{\frac{1}{2}}S(x_L))Y) && \text{by Equation 88} \\
&= \mathrm{Tr}(\Phi(S(y_L)^*\xi_R^{\frac{1}{2}})\Phi(\xi_R^{\frac{1}{2}}S(x_L))Y) \\
&= \mathrm{Tr}(\Phi(\xi_R^{\frac{1}{2}}S(x_L))Y\Phi(S(y_L)^*\xi_R^{\frac{1}{2}})) \\
&= \mathrm{Tr}(\Phi(\xi_R^{\frac{1}{2}}S(x_L))Y\Phi((\xi_R^{\frac{1}{2}}S(y_L))^*)) && \text{by Equation 88} \\
&= \mathrm{Tr}(\Phi(\xi_R^{\frac{1}{2}}S(x_L))Y\Phi(\xi_R^{\frac{1}{2}}S(y_L))^\dagger)
\end{aligned}$$

for all $x_L, y_L \in \mathcal{A}_L$. Now, recall that $1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} \in \mathcal{A}_R \otimes \mathcal{A} \otimes \mathcal{A}_L$; see [14]. This allows us to rewrite $\mathfrak{G}$ in the following form:

$$\begin{aligned}
&\mathfrak{G}(X \otimes Y) \\
&= \tfrac{1}{\mathcal{D}^2}\mathrm{Tr}(\Phi(S((1\cdot 1^*)_{(1)})\xi_L)X)\Phi(c_\omega(1\cdot 1^*)_{(2)}) \\
&\qquad \mathrm{Tr}(\Phi(\xi_R S((1\cdot 1^*)_{(3)}))Y) \\
&= \tfrac{1}{\mathcal{D}^2}\mathrm{Tr}(\Phi(S(1_{(1)}(1^*)_{(1')})\xi_L)X)\Phi(c_\omega 1_{(2)}(1^*)_{(2')}) \\
&\qquad \mathrm{Tr}(\Phi(\xi_R S(1_{(3)}1_{(3')}))Y) \\
&= \tfrac{1}{\mathcal{D}^2}\mathrm{Tr}(\Phi(S(1_{(1)}1^*_{(1')})\xi_L)X)\Phi(c_\omega 1_{(2)}1^*_{(2')}) \\
&\qquad \mathrm{Tr}(\Phi(\xi_R S(1_{(3)}1^*_{(3')}))Y) \\
&= (\mathrm{Tr}\otimes \mathrm{Id}\otimes \mathrm{Tr})(Q(X\otimes \mathbf{1}\otimes Y)Q^\dagger)
\end{aligned}$$

where the last step follows from the previous calculations, and we have defined

$$Q := \tfrac{1}{\mathcal{D}}\Phi^{\otimes 3}(\xi_L^{\frac{1}{2}}S(1^*_{(1)}) \otimes c_\omega^{\frac{1}{2}}1_{(2)} \otimes \xi_R^{\frac{1}{2}}S(1_{(3)})). \tag{107}$$

This concludes the proof. □