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JEL Classification C71, C78; D71; D78.

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We analyze centralized non-monetary markets for indivisible objects through pairwise exchange when each agent initially owns a single object. We characterize a family of domains of preferences (*minimal reversal domains*) such that there exist pairwise exchange rules that satisfy individual rationality, efficiency, and strategy-proofness. Minimal reversal domains are maximal rich domains for individual rationality, efficiency, and strategy-proofness. Each minimal reversal domain is defined by a common ranking of the set of objects, and agents' preferences over admissible objects coincide with such common ranking but for a specific pair of objects.

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1 Introduction

A house exchange problem is a non-monetary indivisible goods allocation problem where a set of agents own a set indivisible objects that should be assigned to the agents. Each agent initially owns a single object and is entitled to keep that initial object. Monetary transfers among the agents are not allowed. Relevant examples of house exchange problems are assignment of roommates at universitary dorms, allocation of graduate housing with tenants, or kidney exchange.[1] Since agents have initial rights over the objects, final assignments of objects to agents

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[1]See Sönmez and Ünver (2011) for a survey on the literature.

are obtained through exchange cycles, where a group of agents swap their initial objects among themselves. In many real life applications of house exchange markets, as holiday-house swaps, kidney exchange, the number of agents involved in exchange cycles is restricted by logistic or legal constraints. This is the case for kidney exchange programs or holiday-house swaps. Here, we focus on the most stringent feasibility constraint and restrict our attention to pairwise exchanges, and we consider the possibility of designing rules (centralized clearing houses) that select a final assignment of the objects to the agents taking into account the agents' preferences.

We investigate the existence of domains of preferences for which there exist rules for pairwise house exchange problems that satisfy *individual rationality*,[2] *efficiency* (subject to the logistic constraints),[3] and *strategy-proofness*.[4] When preferences are strict and feasible exchanges are not restricted, for each profile of agents' preferences there is a unique core assignment. That is, there is a reallocation of the objects among the agents such that no group of agents can improve upon it by trading their initial endowment objects among them (Shapley and Scarf, 1974; Roth and Postlewaite, 1977; Roth, 1982). The unique core assignment can be obtained through the Gale's Top Trade cycle algorithm and the rule which selects the core allocation for each agents' preference profile is the unique rule that satisfies *individual rationality*, *efficiency*, and *strategy-proofness*. (Ma, 1994). When the number of agents involved in each exchange is limited, there are profiles of preferences for which core assignments either may fail to exist or may be not unique (Chung, 2000). This implies that when preferences are unrestricted, there are no rules that jointly satisfy the proposed axioms (Sönmez, 1999).[5] In a recent paper, Nicolò and Rodríguez-Álvarez (2017) show that those properties are compatible in the domain where given a fixed common ranking over the objects, each agent's preferences coincide with that common ranking on the set of objects that she ranks above her initial object. That result however leaves as an open question whether there are larger domains of preferences where the properties are still compatible.

In this paper, we characterize a family of preference domains –*minimal reversal domains*– that extend the domain restriction proposed by Nicolò and Rodríguez-Álvarez (2017). Given an initial common ranking of objects and a specific pair of objects that occupy adjacent positions in the common ranking, the associated minimal reversal domain is such that agents' preferences

[2] A rule satisfies individual rationality if each agent never prefers her initial object to the object assigned by the rule.

[3] A rule satisfies efficient if it never selects an allocation that is Pareto dominated by another feasible allocation.

[4] A rule satisfies strategy-proofness if agents never have incentives to misrepresent their preferences.

[5] The negative result extends to some restricted domains and incentive compatibility properties weaker than *strategy-proofness*. (Nicolò and Rodríguez-Álvarez, 2012, 2013)

over objects that are ranked above their initial endowment coincide with the common ranking but for the specified pair of objects. Minimal reversal domains are maximal for *individual rationality*, *efficiency*, and *strategy-proofness*, that is, for any rich domain that strictly contains a minimal reversal domain, there is no rule that jointly satisfies the three conditions. For each minimal reversal domain we present a rule – the associated reversal adjusted priority rule– that is characterized by *individual rationality*, *efficiency*, and *strategy-proofness* in that minimal reversal domain. Reversal adjusted priority rules extend the idea of priority selection from the set of assignments that all agents prefer to the initial assignment of objects.[6]

The remainder of the paper is organized as follow. In Section 2, we introduce notation and basic definitions. In Section 3, we present the concept of maximal rich domains of preferences. In Section 4, we collect the main results for strict preferences. In Section 5, we discuss the extension of the analysis to domains with indifferences. In Section 6, we conclude and briefly discuss related works on maximal domains and allocation of indivisible objects under restricted domains of preferences. Finally, in Section 7, we provide all the proofs and ancillary results.

2 Notation and Basic Definitions

Let $N = \{1, \ldots, n\}$ be a finite society consisting of a set of agents ($n \geq 3$). Each agent i has an initial endowment of an object ω_i, and let $\Omega = \{\omega_1, \ldots, \omega_n\}$ be the set of available objects.

A preference relation is a complete, reflexive, and transitive binary relation on Ω. Each agent $i \in N$ is equipped with a preference $\succsim_i$. For each agent $i \in N$ and each $\succsim_i \in \mathcal{R}$, $\succ_i$ is the associated strict preference. We denote by $\mathcal{R}$ the set of all preferences on Ω and by $\mathcal{P} \subset \mathcal{R}$ the set of all strict (antisymmetric) preferences on Ω. A preference profile $\succsim$ is an n-tuples of preferences $\succsim = (\succsim_1, \ldots, \succsim_n)$. For each $i \in N$ and each preference profile $\succsim$, $\succsim_{-i}$ is the complementary profile of preferences of all agents in $N \setminus \{i\}$. For each $i \in N$, let $\mathcal{D}_i$ denote the domain of admissible preferences for agent i, and $\mathcal{D}^N \equiv \times_{i \in N} \mathcal{D}^i$ be the domain of admissible preference profiles.[7] Let $\mathcal{R}^N$ be the domain of all the preference profiles and $\mathcal{P}^N \subset \mathcal{R}^N$ the domain of all strict preference profiles.

An ***assignment*** is a bijection from objects to agents. For each $i \in N$ and each assignment a, a_i is the object assigned to i according to a.

[6]The extension of the analysis to environments allowing for larger cycles of exchange or multiple objects as initial endowment for each agent would lead immediately to impossibility results. See Nicolò and Rodríguez-Álvarez (2017, Theorems 4,6). We address these possible extensions in Section 6.

[7]In this paper we only consider cartesian domains of preference profiles.

In every assignment, objects are allocated by forming exchange cycles. In each exchange cycle, every agent receives an object from some other agent in the cycle and her object is assigned to another agent in the cycle. In this paper, we focus on the most binding feasibility constraints, and consider pairwise assignments such that only exchanges between two agents are admitted. That is, an assignment a is a pairwise assignment if for each $i, j \in N$, $a_i = \omega_j$ implies $a_j = \omega_i$. Let $\mathcal{A}$ be the set of all pairwise assignments.

We are interested in rules that select an assignment for each preference profile. A ***(pairwise exchange) rule*** is a mapping $\varphi : \mathcal{D}^N \to \mathcal{A}$. For each agent i and each preference profile $\succsim$, we denote by $\varphi_i(\succsim)$ the object assigned to i by φ at profile $\succsim$.

The assignment selected by a rule can be interpreted as an optimal recommendation that takes into account the preferences of the agents over objects and that tries to find a compromise between their (perhaps conflicting) interests.

We present a formal definition of the standard conditions for desirable rules.

Individual Rationality. For each $i \in N$ and each $\succsim \in \mathcal{D}^N$, $\varphi_i(\succsim) \succsim_i \omega_i$.

Constrained Efficiency. For each $\succsim \in \mathcal{D}^N$, there is no $a \in \mathcal{A}$ such that for each $i \in N$ $a_i \succsim_i \varphi_i(\succsim)$ and for some $j \in N$, $a_j \succ_j \varphi_j(\succsim)$.

Strategy-Proofness. For each $i \in N$, each $\succsim \in \mathcal{D}^N$, and each $\succsim'_i \in \mathcal{D}_i$, $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim'_i, \succsim_{-i})$.

Individual rationality takes into account an agent's right to refuse any assignment that is worse than her own object. *Constrained efficiency* is the natural version of efficiency that considers the feasibility restrictions on the cardinality of the cycles involved in assignments. *Strategy-Proofness* implies that agents never improve by reporting a false preference.

3 Maximal Rich Domains

The objective of this paper is to provide a systematic analysis of the features of preference domains for which *individual rationality*, *efficiency*, and *strategy-proofness* are compatible. Of course it is always possible to construct examples of rather restricted domains that directly impose constraints on preference profiles, but we are interested in finding the more general restrictions involving individual preference domains. Hence we consider domains that contain a rich variety of preferences.

A domain of preferences $\mathcal{D}^N = \times_{i\in N}\mathcal{D}^i \subset \mathcal{R}^N$ is ***rich*** if it satisfies:

- ***Individuality***: for each $i \in N$, $\succsim_i \in \mathcal{D}_i$, and $\omega \in \Omega$ with $\omega \succ_i \omega_i$, there is $\succsim'_i \in \mathcal{D}_i$ such that

  i) for each ω', $\omega' \succsim_i \omega$ if and only if $\omega' \succsim'_i \omega$,

  ii) and for each ω'', $\omega \succ_i \omega''$ if and only if $\omega \succ'_i \omega''$ and $\omega \succsim'_i \omega_i \succsim'_i \omega''$.

- ***Anonymity***: for each $i, j \in N$ and $\succsim_i \in \mathcal{D}_i$, there is $\succsim_j \in \mathcal{D}_j$ such that for each $\omega, \omega' \notin \{\omega_i, \omega_j\}$, $\omega \succ_i \omega'$ if and only if $\omega \succ_j \omega'$.

Individuality implies that if there is a preference for agent i in the domain such that ω is preferred to ω_i, then there is a preference in the domain that raises ω_i to be immediately below object ω is admissible as well.[8] *Anonymity* is a condition that relates the individual preference domains.

Our objective of finding the largest domains where *individual rationality*, *efficiency*, and *strategy-proofness* are compatible motivates the following definition.

A cartesian domain of preference profiles $\mathcal{D}^N \subseteq \mathcal{R}^N$ is ***maximal for a set of axioms*** $\mathcal{S}$, if there exists a rule that satisfies the axioms $\mathcal{S}$ in the domain $\mathcal{D}^N$, and for each rich cartesian domain $\bar{\mathcal{D}}^N \neq \mathcal{D}^N$ such that $\mathcal{D}^N \subseteq \bar{\mathcal{D}}^N$, there is no rule that satisfies the axioms $\mathcal{S}$ in the domain $\bar{\mathcal{D}}^N$.

4 Main Results: Maximal Domains of Strict Preferences

We start our analysis by presenting two rich domains of strict preferences for which the relevant axioms are compatible.

Example 1. Common Ranking Preferences. *Let $\mathcal{D}^N_C = \times_{i\in N}\mathcal{D}^i_C \subset \mathcal{P}^N$ be such that for each $i \in N$ and $\succsim_i \in \mathcal{D}^i_C$, for each $\omega_j, \omega_{j'} \in \Omega \setminus \{\omega_i\}$ such that $\omega_j \succ_i \omega_i$ and $\omega_{j'} \succ_i \omega_i$, $\omega_j \succsim_i \omega_{j'}$ if and only if $j \leq j'$.*

According to common ranking preferences, agents and objects are naturally ordered.[9] Each agent can rank any set of objects as more preferred than her initial endowment object, but the order in which those objects are ranked is determined by the initial common ranking of the objects.

[8] In our framework, *individuality* is equivalent to Assumption B in Sönmez (1999).

[9] Each relabelling of the agents and objects may define a different common ranking domain. We have opted to consider a fixed order of the agents to keep notation simple.

For each $i \in N$, $\succsim_i \in \mathcal{R}$, and $A \subseteq \mathcal{A}$, the ***top choices*** for agent i from the assignments in A is

$$\text{top}(A, \succsim_i) \equiv \{\omega \in \Omega | \text{there is } a \in A, a_i = \omega, \text{ and for each } b \in A, \omega \succsim_i b_i\}.$$

For each $\succsim \in \mathcal{R}^N$, the ***set of individually rational assignments*** is

$$I(\succsim) \equiv \{a \in \mathcal{A} | \text{for each } i, a_i \succsim_i \omega_i\}.$$

Priority Algorithm: Let $\succsim \in \mathcal{R}^N$,

- $\mathcal{M}_0 \equiv I(\succsim)$.
- For each $t \leq n$, let $\mathcal{M}_t \subseteq \mathcal{M}_{t-1}$ be such that:

$$\mathcal{M}_t \equiv \{a \in \mathcal{M}_{t-1} | \, a_t \in \text{top}(\mathcal{M}_{t-1}, \succsim_t)\}.$$

Starting with $\mathcal{M}_0$ as the set of individually rational assignments, the set $\mathcal{M}_1$ selects the best preferred assignments for agent 1 among all the assignments that each agent considers at least as good as her initial endowment object. The selection proceeds iteratively, and at each step t agent t selects her preferred assignments among those that have survived in the previous steps. The set $\mathcal{M}_n$ is well defined for every $\succsim \in \mathcal{R}^N$, non-empty and single-valued since preferences are strict. In the definition of the priority algorithm, we choose to drop the obvious reference to the preference profile of the sets $\mathcal{M}_t$ whenever it does not induce to confusion.[10]

For each domain $\mathcal{D}^N \subseteq \mathcal{R}^N$, the rule $\varphi : \mathcal{D}^N \to \mathcal{A}$ such that for each $\succsim \in \mathcal{D}^N$, $\varphi(\succsim) = \mathcal{M}_n$ is called the ***priority rule***.

The priority rule is the only rule that satisfies our set of axioms in the domain of common ranking preferences $\mathcal{D}_C^N$ (Nicolò and Rodríguez-Álvarez (2012)[Theorem 2]).

It is easy to see that the domain of common ranking preferences is not a maximal rich strict domain for our axioms.

Example 2. Free Pair Preferences. *Let $N = \{1, 2, 3\}$. Let $\mathcal{D}_F^N = \times_{i \in N} \mathcal{D}_F^i$ such that for each $i \in \{1, 2\}$, $\mathcal{D}_F^i \equiv \mathcal{D}_C^i$, and $\mathcal{D}_F^3 \equiv \mathcal{P}$. Note that $\mathcal{D}_F^N$ satisfies individuality and anonymity.*

In the domain $\mathcal{D}_F^N$, agents 1 and 2 have common ranking preferences according to the natural order, but agent 3 faces no restriction regarding how to rank ω_1 and ω_2. Hence, agents i and j may rank ω_3 as

[10] We make the same notational choice in the definitions of the subsequent algorithms. We only need to make explicit reference to the specific profiles in the proof of Theorem 1 in Section 6.

the best preferred object when it's the only object they prefer to keep their respective initial endowment objects. Note that $\mathcal{D}_C^N \subset \mathcal{D}_F^N$.

Let φ^+ be such that for each $\succsim \in \mathcal{D}_F^N$, $\varphi^+(\succsim) \in I(\succsim)$ and:

- *If $\omega_2 \succ_1 \omega_2$ and $\omega_1 \succ_2 \omega_1$, then $\varphi_1^+(\succsim) = \omega_2$, $\varphi_2^+(\succsim) = \omega_1$, and $\varphi_3^+(\succsim) = \omega_3$*
- *otherwise, $\varphi_3^+(\succsim) \in \text{top}(I(\succsim), \succsim_3)$, and for each $i \in \{1, 2\}$, if $\omega_i \notin \text{top}(I(\succsim), \succsim_3)$, $\varphi_i(\succsim) = \omega_i$*

The rule φ^+ satisfies individual rationality, efficiency, *and* strategy-proofness *in the domain $\mathcal{D}_F^N$. We can think of the rule as a priority rule where 1 and 2 propose simultaneously for an exchange among the individually rational assignments. When they make proposal that are not compatible (each proposes an exchange with agent 3), then agent 3 selects between ω_1 and ω_2 her most preferred exchange.*

We present now a new class of domains that combine the restrictions introduced by common ranking and free pair preferences.

Let $k, k+1 \in N$. The domain $\mathcal{D}_{\{k,k+1\}}^N = \times_{i \in N} \mathcal{D}^i \subset \mathcal{P}^N$ is the ***minimal reversal domain for*** $\{k, k+1\}$ if for each $i \in N$, $\succsim_i \in \mathcal{D}_{\{k,k+1\}}^i$, $\omega_j, \omega_{j'} \in \Omega \setminus \{\omega_i\}$ with $\{j, j'\} \neq \{k, k+1\}$, $\omega_j \succ_i \omega_i$, $\omega_{j'} \succ_i \omega_i$ and $j < j'$ imply $\omega_j \succ_i \omega_{j'}$. The domain $\mathcal{D}^N$ is a ***minimal reversal domain*** if there are $k, k+1 \in N$ such that $\mathcal{D}^N = \mathcal{D}_{\{k,k+1\}}^N$.

In a minimal reversal domain, each agent's preferences over objects that are preferred to her initial endowment objects are determined by the natural order (common ranking) but for a pair of objects. The way in which the agents may order that pair of objects is not restricted.

Theorem 1. *Every minimal reversal domain is a maximal rich domain for* individual rationality, efficiency, *and* strategy-proofness.

The proof follows a series of steps. First we prove that in rich preference domains that strictly contain minimal reversal domains, there are preference profiles that either present a three-object cycle, or there are at most two pairs of objects that agents can rank in any arbitrary order. Then we check, that in such domains no rule may satisfy our axioms (Lemmata 1–2). Finally, for every minimal reversal domain we can construct a rule that satisfies our axioms in that domain.

We conclude this section describing the rules that satisfy our axioms in minimal reversal domains.

Reversal Adjusted Priority Algorithm. Let $k, k+1 \in N$ and $\succsim \in \mathcal{R}^N$.

- $\mathcal{M}_0^* \equiv I(\succsim)$.
- For each $t \leq n$, to define $\mathcal{M}_t^* \subseteq \mathcal{M}_{t-1}^*$ we consider two cases:
  - If $t \notin \{k, k+1\}$,
  $$\mathcal{M}_t^* \equiv \left\{a \in \mathcal{M}_{t-1}^* \mid a_t \in \text{top}(\mathcal{M}_{t-1}^*, \succsim_t)\right\}.$$
  - if $t \in \{k, k+1\}$ and $\text{top}(\mathcal{M}_{k-1}^*, \succsim_k) \neq \text{top}(\mathcal{M}_{k-1}^*, \succsim_{k+1})$, then
  $$\mathcal{M}_t^* \equiv \left\{a \in \mathcal{M}_{t-1}^* \mid a_t \in \text{top}(\mathcal{M}_{t-1}^*, \succsim_t)\right\}.$$
  - if $t \in \{k, k+1\}$ and $\text{top}(\mathcal{M}_{k-1}^*, \succsim_k) = \text{top}(\mathcal{M}_{k-1}^*, \succsim_{k+1}) = \{\omega_j\}$, let's define $\bar{k} \equiv k$ if $\omega_k \succ_j \omega_{k+1}$ and $\bar{k} \equiv (k+1)$ if $\omega_{k+1} \succ_j \omega_k$, and $\underline{k} \equiv \{k, k+1\} \setminus \{\bar{k}\}$, and let
  $$\mathcal{M}_k^* \equiv \left\{a \in \mathcal{M}_{k-1}^* \mid a_{\bar{k}} \in \text{top}(\mathcal{M}_{k-1}^*, \succsim_{\bar{k}})\right\}, \ and$$
  $$\mathcal{M}_{k+1}^* \equiv \left\{a \in \mathcal{M}_k^* \mid a_{\underline{k}} \in \text{top}(\mathcal{M}_k^*, \succsim_{\underline{k}})\right\}.$$

For each pair of adjacent agents $\{k, k+1\}$, the rule $\varphi : \mathcal{D}^N_{\{k,k+1\}} \to \mathcal{A}$ such that for each $\succsim \in \mathcal{D}^N_{\{k,k+1\}}$, $\varphi(\succsim) = \mathcal{M}_n^*$ is called the k*- **reversal adjusted priority rule***.

For any agent $k \neq n$, the associated k-reversal adjusted priority rule follows the logic of the priority rule but for the agents $k, k+1$. At the stage k of the priority algorithm, agents k and $k+1$ each propose an exchange to other agents. If they both propose an exchange to the same agent, this agent chooses with whom to perform the exchange.

We close this section with a characterization of k-reversal adjusted priority rules in minimal reversal domains. In any minimal reversal domain, the associated reversal adjusted priority rule is the only rule that satisfies *individual rationality*, *efficiency*, and *strategy-proofness*.

Theorem 2. *Let $\mathcal{D}^N_{\{k,k+1\}}$ be the minimal reversal domain for $\{k, k+1\}$. A rule $\varphi : \mathcal{D}^N_{\{k,k+1\}} \to \mathcal{A}$ satisfies* individual rationality, efficiency, *and* strategy-proofness *if and only if φ is the k*-reversal adjusted priority rule*.*

5 Discussion: Weak Preferences

In environments with indifferences, the extension of minimal reversal priority rules is not immediate because agents may have multivalued top choices sets. In environments where objects are not initially attached to any agent, Nicolò and Rodríguez-Álvarez (2017) analyze domains with indifferences that are naturally related to common ranking preference domains –age based domains– where rules that satisfy *individual rationality*, *efficiency*, and *strategy-proofness* exist. In age based domains, there is an ordered partition of the set of objects, and each agent ranks the objects she prefers to her initial endowment object according to the order over the elements of the partition, being indifferent among objects belonging to the same element of the partition. Of course, common ranking preferences correspond to age based preferences according the the finest partition of the set of objects. Age based preferences do not extend minimal reversal domains because they do not admit any preference reversal. Moreover, age based preferences do not admit strict preferences between objects in the same element of the partition.

We present an extension of minimal reversal domains that admit indifferences among objects that are adjacent in the natural order.

Let $k, k+1 \in N$, the domain $\widetilde{\mathcal{D}}^N_{\{k,k+1\}} = \times_{i\in N}\widetilde{\mathcal{D}}^i_{\{k,k+1\}} \subset \mathcal{R}^N$ is the ***minimal reversal weak preference domain*** for $\{k, k+1\}$ if for each $i \in N$, $\succsim_i \in \widetilde{\mathcal{D}}^i_{\{k,k+1\}}$, there is no $\omega \in \Omega \setminus \{\omega_i\}$ such that $\omega \sim_i \omega_i$ and for each $\omega_j, \omega_{j'} \in \Omega \setminus \{\omega_i\}$ with $\{j, j'\} \neq \{k, k+1\}$, $\omega_j \succ_i \omega_i$, and $\omega_{j'} \succ_i \omega_i$

- $j < j'$ implies $\omega_j \succsim_i \omega_{j'}$, and
- whenever there is l with $j < l < j'$, then $\omega_j \succ_i \omega_{j'}$.

We say that a domain $\mathcal{D}^N = \times_{i\in N}\mathcal{D}_i \subset \mathcal{R}^N$ is a ***minimal reversal domain with indifferences*** if there are $k, k+1 \in N$ such that $\mathcal{D}^N = \widetilde{\mathcal{D}}^N_{\{k,k+1\}}$.

Priority rules fail to satisfy *strategy-proofness* in age-based domains that admit indifferences. It is illustrative to show that even in minimal reversal domain with indifferences reversal adjusted priority rules do not satisfy *strategy-proofness*.[11]

Example 3. *Let* $N = \{1, 2, 3, 4\}$ *and* $\succsim \in \widetilde{\mathcal{D}}^N_{\{3,4\}}$ *be such that* $\omega_3 \sim_1 \omega_4 \succ_1 \omega_1$, $\omega_3 \succ_2 \omega_2$, $\omega_1 \succ_3 \omega_2 \succ_3 \omega_3$, *and* $\omega_1 \succ_4 \omega_4$. *Let* φ *be the* 3*-reversal priority rule. Notice that* $\varphi(\succsim) = (\omega_4, \omega_3, \omega_2, \omega_1)$. *Let* $\succsim' \in \mathcal{D}^N_{\{3,4\}}$ *be such that* $\succsim'_{-3} = \succsim_{-3}$ *and* $\omega_1 \succ_3 \omega_3 \succ_3 \omega_2$. *Then* $\varphi(\succsim') = (\omega_3, \omega_2, \omega_1, \omega_4)$. *Since* $\varphi^*_3(\succsim') \succ_3 \varphi^*_3(\succsim)$, φ *violates* strategy-proofness.

Example 3 highlights the difficulties of extending (minimal reversal) priority rules to domains with indifferences. Under the priority rule and minimal reversal rules, agent 1 in indifferent among two exchanges involving different agents and the tie is broken using the preferences of another agent that is not directly involved in the exchange. An arbitrary tie-breaker would help to restore the incentives to reveal the true preferences for the agents, but further

[11] Example 3 is a convenient rephrasing of Example 2 in Nicolò and Rodríguez-Álvarez (2017).

flexibility is required to satisfy *efficiency*. When the indifferences among objects are restricted to involve only two objects that are adjacent in the natural order, we can construct rules that satisfy the requirement of our axioms.

Reversal Adjusted Priority Algorithm with Indifferences. Let $k, k+1 \in N$ and $\succsim \in \mathcal{R}^N$.

- $\widetilde{\mathcal{M}}_0 \equiv I(\succsim)$.
- For each $t \leq n$, to define $\widetilde{\mathcal{M}}_t \subseteq \widetilde{\mathcal{M}}_{t-1}$ we consider two cases:
  - If $t \notin \{k, k+1\}$, let $\mu(t) = \min\{i \,|\, \omega_i \in \text{top}(\widetilde{\mathcal{M}}_{t-1}, \succsim_t)\}$, and
  $$\widetilde{\mathcal{M}}_t = \left\{ a \in \widetilde{\mathcal{M}}_{t-1}(\succsim) \middle| \begin{array}{c} a_k \succsim_k \omega_{\mu(t)}, \\ a_{\mu(t)} \succsim_{\mu(t)} \omega_k \end{array} \right\}.$$
  - If $t \in \{k, k+1\}$ and $\min\{i \,|\, \omega_i \in \text{top}(\widetilde{\mathcal{M}}_{k-1}, \succsim_k)\} \neq \min\{i \,|\, \omega_i \in \text{top}(\widetilde{\mathcal{M}}_{k-1}, \succsim_{k+1})\}$ let $\mu(t) = \min\{i \,|\, \omega_i \in \text{top}(\widetilde{\mathcal{M}}_{t-1}, \succsim_t)\}$, and
  $$\widetilde{\mathcal{M}}_t = \left\{ a \in \widetilde{\mathcal{M}}_{t-1}(\succsim) \middle| \begin{array}{c} a_k \succsim_k \omega_{\mu(t)}, \\ a_{\mu(t)} \succsim_{\mu(t)} \omega_k \end{array} \right\}.$$
  - If $t \in \{k, k+1\}$ and there is $j \in N$ such that
  $$j = \mu(k) \equiv \min\{i \,|\, \omega_i \in \text{top}(\widetilde{\mathcal{M}}_{k-1}, \succsim_k)\} = \min\{i \,|\, \omega_i \in \text{top}(\widetilde{\mathcal{M}}_{k-1}, \succsim_{k+1})\},$$
  define $\bar{k}, \underline{k} \in \{k, k+1\}$ in such a way that $\bar{k} = k$ if $\omega_k \succsim_j \omega_{k+1}$, $\bar{k} = k+1$ if $\omega_{k+1} \succ_j \omega_k$, and $\underline{k} = \{k, k+1\} \setminus \{\bar{k}\}$, $\mu(k) = \omega_{\bar{k}}$ and let
  $$\widetilde{\mathcal{M}}_k = \left\{ a \in \widetilde{\mathcal{M}}_{t-1}(\succsim) \middle| \begin{array}{c} a_{\bar{k}} \succsim_{\bar{k}} \omega_{\mu(k)}, \\ a_{\mu(k)} \succsim_{\mu(k)} \omega_{\bar{k}} \end{array} \right\},$$
  $\mu(k+1) \equiv \min\{i \,|\, \omega_i \in \text{top}(\widetilde{\mathcal{M}}_k, \succsim_{\underline{k}})\}$, and
  $$\widetilde{\mathcal{M}}_{k+1} = \left\{ a \in \widetilde{\mathcal{M}}_k \middle| \begin{array}{c} a_{\underline{k}} \succsim_{\underline{k}} \omega_{\mu(k+1)}, \\ a_{\mu(k+1)} \succsim_{\mu(k+1)} \omega_{\underline{k}} \end{array} \right\}$$

Note that for each $\succsim \in \widetilde{\mathcal{D}}^N_{\{k,k+1\}}$, $\widetilde{\mathcal{M}}_n$ is non-empty and essentially single-valued.[12] For each pair of agents $\{k, k+1\}$, the rule $\varphi : \widetilde{\mathcal{D}}^N_{\{k,k+1\}} \to \mathcal{A}$ such that for each $\succsim \in \widetilde{\mathcal{D}}^N_{\{k,k+1\}}$, $\varphi(\succsim) \in \widetilde{\mathcal{M}}_n$ is called the ***k-reversal adjusted priority rule with indifferences***.

[12] The set $\widetilde{\mathcal{M}}_n$ may be multivalued only if there are two pairs of agents $j, j+1$ and $l, l+1$ such that $\omega_l \sim_j \omega_{l+1}$, $\omega_l \sim_{j+1} \omega_{l+1}$, $\omega_j \sim_l \omega_{j+1}$, and $\omega_j \sim_{l+1} \omega_{j+1}$.

To satisfy *strategy-proofness*, when the agent selecting at stage t top choice set is multivalued, ***k*-reversal adjusted priority rules with indifferences** break ties following the natural order independently of the preferences of the following agents and propose a tentative selection for agent t. When indifferences are only admitted among adjacent objects, *efficiency* is also satisfied because Pareto improvements over the tentative selection obtained at a stage t can be performed in the next stage $t+1$.

Proposition 1. *Let* $k, k+1 \in N$. *The* k-reversal adjusted priority rule with indifferences *satisfies* individual rationality*,* efficiency*, and* strategy-proofness *in the domain* $\widetilde{\mathcal{D}}^N_{\{k,k+1\}}$.

6 Conclusion and Related Literature

In this paper we have analyzed the pairwise house exchange problems to find maximal extensions of common ranking domains where centralized rules that satisfy *individual rationality*, *efficiency*, and *strategy-proofness* exist. We show that those properties do not admit great departures from common ranking preferences. The maximal domains, minimal reversal domains, just admit preference reversal from a common ranking for two objects that are adjacent in the common ranking. In those minimal reversal domains, reversal adjusted priority rules satisfy *individual rationality*, *efficiency*, and *strategy-proofness*, and basically propose a sequential priority selection of individually rational pairwise exchanges for all but for the agents that initially own the objects that admit preference reversals. Since for general problems of allocation of indivisible objects where core assignment exist, there are rules that satisfy *individual rationality*, *efficiency*, and *strategy-proofness*, only if the set of core assignments is (essentially) single-valued (Sönmez, 1999), our results may be interpreted as the maximal domain for the existence of unique core assignments in pairwise house exchange problems.

Some remarks on possible extensions of the model are in order. In this paper we obtained positive results under the most stringent logistic constraint on the number of agents involved in exchange cycles. The positive results do not extend to the framework with less stringent logistic constraints. When cycles involving at least three agents are allowed, there is no rule defined in the common ranking domain that satisfies *individual rationality*, *efficiency*, and *strategy-proofness*(Nicolò and Rodríguez-Álvarez, 2017, Theorem 4).[13] Finally, throughout this paper we assume that all agents are initially endowed with a unique object. Note that in this case we can treat all the objects owned by an agent as a single object. Hence, the main results of the paper

[13] Nicolò and Rodríguez-Álvarez (2017) show that rules that satisfy *individual rationality*, *strategy-proofness*, and an appropriate version of *efficiency* can be constructed in domains with common ranking with indifferences when large exchange cycles are restricted to involve objects in the same indifference class of the common ranking.

would go through in minimal reversal domains where all the objects initially owned by each agent occupy adjacent positions in the common ranking. Unfortunately, the results do not extend to more general common rankings. When the different objects initially owned by an agent do not occupy adjacent positions in the remaining agents preferences, no rule satisfies *individual rationality*, *efficiency*, and *strategy-proofness* (Nicolò and Rodríguez-Álvarez, 2017, Theorem 6).

We close this concluding section by examining the related literature. There are a growing number of works that analyze maximal domains for rules that satisfy *strategy-proofness* in different g environments, like selection of multiple objects (Barberà, Sonnenschein and Zhou , 1991; Le Breton and Sen , 1999; Hatsumi, Berga and Serizawa , 2014), public good provision (Berga and Serizawa , 2000; Ching and Serizawa , 1998), or the division and allotment problems (Barberà, Massó and Neme , 1999; Massó and Neme , 2001; Mizobuchi and Serizawa , 2001; Wakayama , 2017). Beyond those papers that focus on maximal domains for general voting and public good provision settings, the most closely related papers analyze conditions on agents preferences that allow for rules that satisfy *strategy-proofness* in different problems of allocation of indivisible objects. Alcalde and Barberà (1994) consider two-sided marriage and college admission markets, and propose a necessary and sufficient condition (not related to minimal reversal domains) on domains of preferences for one side of the market that allow for rules that satisfy *individual rationality*, *efficiency*, and *strategy-proofness*. Ehlers (2002) considers the allocation of indivisible objects without property rights when indifference are admitted. In that context *individual rationality* is not relevant, and for the unrestricted domain of preferences hierarchically dictatorial rules satisfy *efficiency* and *strategy-proofness*. This paper finds the unique maximal domain extending the universal domain of strict preferences for *efficiency* and *group-strategy-proofness* when indifferences are admitted. Ergin (2003) considers school choice problems, that is two-sided matching markets where the preferences (priorities) of one side of the market (schools) are not taken into account to determine *efficiency*. Ergin (2003) shows that *individual rationality*, *efficiency*, and *strategy-proofness* are compatible if the preferences (priorities) of the members of the welfare irrelevant side of the market (schools) satisfy an acyclicity condition.

We finally refer to Alcalde (1995) and Abizada (2019). These papers study roommate problems, which correspond to pairwise house exchange problems with additional feasibility restrictions. Specifically, agents are not allowed to keep their initial objects. Alcalde (1995) proposes a stability condition for roommate problems –exchange proofness– and a sufficient condition on preference profiles (α -reducible preferences) for the existence of exchange-proof allocations. Abizada (2019) shows that exchange-proof allocation always exist under weaker sufficient conditions on agents preferences (r-level mixed condition). Those proposed conditions are consistent with the existence of a common ranking of the objects, but since those papers focus on a different but related framework and do not focus on maximality of domains, and our results and theirs are not directly comparable.

7 Proofs

We start with a pair of auxiliary lemmata.

Lemma 1. ***(No Cycle.)*** *Let $\mathcal{D}^N \subseteq \mathcal{R}^N$ be a rich domain. If there are $i, j, k \in N$ and $\succsim \in \mathcal{D}^N$ such that $\omega_j \succ_i \omega_k \succ_i \omega_i$, $\omega_k \succ_j \omega_i \succ_j \omega_j$, and $\omega_i \succ_k \omega_j \succ_i \omega_i$ and for each $l \notin \{i, j, k\}$ and $\omega \in \{\omega_i, \omega_j, \omega_k\}$, $\omega_l \succ_l \omega$, then there is no rule $\varphi : \mathcal{D}^N \to$ that satisfies* individual rationality, efficiency, *and* strategy-proofness.

Proof. The proof replicates the argument in the proof of Theorem 1 Nicolò and Rodríguez-Álvarez (2012). Assume, by way of contradiction, that there is a rule φ that satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in the domain $\mathcal{D}^N$. By *individual rationality*, we can focus on the assignment to agents $\{i, j, k\}$. By *individual rationality* and *efficiency*, φ selects an assignment in which two agents exchange their objects while the remaining agent keeps her initial endowment object. Without loss of generality let $\varphi_i(\succsim) = \omega_j$, $\varphi_j(\succsim) = \omega_i$, and $\varphi_k(\succsim) = \omega_k$.

By richness of $\mathcal{D}^N$, there is $\succsim' \in \mathcal{D}^N$ be such that $\omega_j \succ'_i \omega_i \succ'_i \omega_k$ and $\succsim'_{-i} = \succsim_{-i}$. By *strategy-proofness*, $\varphi_i(\succsim') \succsim'_i \varphi_i(\succsim) = \omega_j$. Hence, $\varphi_i(\succsim') = \omega_j$. Finally, consider $\succsim'' \in \mathcal{D}$ such that $\omega_k \succ''_j \omega_j \succ''_j \omega_i$ and $\succsim''_{-j} = \succsim'_{-j}$. By *individual rationality*, $\varphi_j(\succsim'') \in \{\omega_j, \omega_k\}$. By *strategy-proofness*, $\varphi_j(\succsim') = \omega_i \succsim'_j \varphi_j(\succsim'')$. Hence, $\varphi_j(\succsim'') = \omega_j$. Because, by *individual rationality*, $\varphi_i(\succsim'') \in \{\omega_i, \omega_j\}$, $\varphi_i(\succsim'') = \omega_i$, and therefore, $\varphi_3(\succsim'') = \omega_3$. Note that there exists an assignment $a \in \mathcal{A}$ such that $a_i = \omega_i$, $a_j = \omega_k$, $a_k = \omega_j$ and for each $l \notin \{i, j, k\}$, $a_l = \varphi_l(\succsim'')$. For each agent $l \in N$, $a_l \succsim''_i \varphi_l(\succsim'')$ and $a_j \succ''_j \varphi_j h(\succsim'')$, which contradicts φ's *efficiency*. □

Lemma 2. *Let $\mathcal{D}^N \subseteq \mathcal{R}^N$ be a rich domain, if there are $i, j, k, l \in N$ such that there are $\succsim_i, \succsim'_i \in \mathcal{D}_i$, $\succsim_k, \succsim'_k \in \mathcal{D}_k$ with $\omega_k \succ_i \omega_l$, $\omega_l \succ'_i \omega_k$, and $\omega_i \succ_k \omega_j$, $\omega_j \succ'_k \omega_i$, then then there is no rule $\varphi : \mathcal{D}^N \to \mathcal{A}$ that satisfies* individual rationality, efficiency, *and* strategy-proofness.

Proof. Assume, by way of contradiction, that there is a rule φ that satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in the domain $\mathcal{D}^N$. If $\{i, j\} \cap \{k, l\} \neq \varnothing$, the result follows from Lemma 1. Hence, we consider the case $\{i, j\} \cap \{k, l\} = \varnothing$. The result follows from the impossibility of obtaining a rule that satisfies *individual rationality*, *efficiency*, and *strategy-proofness* for two-sided marriage problems (Alcalde and Barberà, 1994). We present the proof for the sake of completeness. By richness, there is $\succsim \in \mathcal{D}$ be such that for each $i' \notin \{i, j, k, l\}$, $\omega \in \{\omega_i, \omega_j, \omega_k, \omega_l\}$, $\omega i' \succ_{i'} \omega$, and:

$\succsim_i$	$\succsim_j$	$\succsim_k$	$\succsim_l$
ω_k	ω_l	ω_j	ω_i
ω_l	ω_k	ω_i	ω_j
ω_i	ω_j	ω_k	ω_l

By *individual rationality* and *efficiency*, there are two cases.

Case i). $\varphi_i(\succsim) = \omega_k$, $\varphi_j(\succsim) = \omega_l$, $\varphi_k(\succsim) = \omega_i$, and $\varphi_l(\succsim) = \omega_j$.

Case ii). $\varphi_i(\succsim) = \omega_l$, $\varphi_j(\succsim) = \omega_k$, $\varphi_k(\succsim) = \omega_j$, and $\varphi_l(\succsim) = \omega_i$.

We focus on Case i). A parallel argument applies to Case ii). By richness of $\mathcal{D}^N$, there is $\succsim' \in \mathcal{D}^N$, such that $\omega_j \succ'_k \omega_k \succ'_k \omega_i$ and $\succ'_{-k} = \succ_{-k}$. By *individual rationality* and *strategy-proofness*, $\varphi_k(\succsim') = \omega_k$, and either $\varphi_i(\succsim') = \omega_i$ and $\varphi_j(\succsim') = \omega_l$, or $\varphi_i(\succsim') = \omega_l$ and $\varphi_j(\succsim') = \omega_j$. Assume first that $\varphi_i(\succsim') = \omega_i$ and $\varphi_j(\succsim') = \omega_l$. Let $\succsim'' \in \mathcal{D}^N$ be such that $\omega_i \succ''_l \omega''_l \succ''_l \omega_j$ and $\succsim''_{-l} = \succsim'_{-l}$. By *strategy-proofness*, $\varphi_l(\succsim'') \neq \omega_i$. By *individual rationality*, $\varphi_l(\succsim'') \neq \omega_j$. Hence, $\varphi_l(\succsim'') = \omega_l$, and by *efficiency*, $\varphi_k(\succsim'') = \omega_j$. Let $\succsim''' \in \mathcal{D}^N$ be such that $\succsim'''_k = \succsim_j$ and $\succsim'''_{-k} = \succsim''_k$. By *strategy-proofness*, $\varphi_k(\succsim''') = \omega_j$ and by *efficiency*, $\varphi_l(\succsim''') = \omega_l$. Since $\succsim'''_{-l} = \succsim_{-l}$ and $\varphi_l(\succsim''') \succ_l \varphi_l(\succsim)$, this contradicts *strategy-proofness*. If $\varphi_i(\succsim') = \omega_l$ and $\varphi_j(\succsim') = \omega_j$ a parallel argument applies to complete the proof. □

Proof of Theorem 1. Let $k, k+1 \in N$ and φ be the *k-reversal adjusted priority rule*. We start by showing that φ satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in the domain $\mathcal{D}^N_{\{k,k+1\}}$. In this proof, with a slight abuse of notation, for each $t \in N$ and $\succsim \in \mathcal{R}^N$, the notation $\mathcal{M}^*_t(\succsim)$ refers to the set $\mathcal{M}^*_t$ when $\mathcal{M}^*_0 = \mathcal{I}(\succsim)$.

By construction, φ satisfies *individual rationality* and *efficiency*. Consequently, we only focus on checking that φ satisfies *strategy-proofness*. Let $i \in N$, and $\succsim, \succsim' \in \mathcal{D}^N_{\{k,k+1\}}$ be such that $\succsim'_{-i} = \succsim_{-i}$. Assume first that, $\varphi_i(\succsim) = \varphi_i(\succsim')$, then $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim')$. Assume now that $\varphi_i(\succsim) \neq \varphi_i(\succsim')$. Let $j, j' \in N$ be such that $\varphi_i(\succsim) = \omega_j$ and $\varphi_i(\succsim') = \omega_{j'}$. There are two cases:

Case i). $i = j$. Assume first $i \notin \{k, k+1\}$, then $\varphi_i(\succsim) = \omega_i$ implies that for each $a \in \mathcal{M}^*_{i-1}(\succsim)$, $a_i = \omega_i$. Since only agent i has changed her preferences, $\varphi_i(\succsim') \in \mathcal{M}^*_{i-1}(\succsim')$ and $\varphi_i(\succsim') = \omega_{j'}$ imply $\omega_i \succ_i \omega_{j'}$. Hence, $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim')$. Finally, assume that $i \in \{k, k+1\}$, $\varphi_i(\succsim) = \omega_i$ implies that either for each $a \in \mathcal{M}^*_{k-1}(\succsim)$, $a_i = \omega_i$, or that $i = \underline{k}$ at profile $\succsim$ and for each $b \in \mathcal{M}^*_k(\succsim)$, $b_i = \omega_i$. In both cases, the argument for $i \notin \{k, k+1\}$ immediately applies and $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim')$.

Case ii). $i \neq j$. Since $\varphi_i(\succsim) \neq \omega_i$, by *individual rationality* $\omega_j \succ_i \omega_i$. If $j' = i$, then $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim')$. Hence, we focus on the case $j' \neq i$. Assume first that $i \notin \{k, k+1\}$. If $\varphi(\succsim') \in \mathcal{M}^*_{i-1}(\succsim)$, then by definition of $\mathcal{M}^*_{i-1}(\succsim)$, $\varphi_i(\succsim) \succsim \varphi_i(\succsim')$. If $\varphi(\succsim') \notin \mathcal{M}^*_{i-1}(\succsim)$, since only agent i changes her preferences $\omega_{j'} = \text{top}(\mathcal{M}^*_{i-1}(\succsim'), \succsim'_i)$ implies that $\omega_i \succ_i \omega_{j'}$ and $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim')$. Finally, assume that $i \in \{k, k+1\}$. If $\text{top}(\mathcal{M}^*_{k-1}(\succsim), \succsim_k) = \text{top}(\mathcal{M}^*_{k-1}(\succsim), \succsim_{k+1})$ and $i = \underline{k}$ at profile $\succsim$, then the argument for the case $i \notin \{k, k+1\}$ applies directly. If either $\text{top}(\mathcal{M}^*_{k-1}(\succsim), \succsim_k) \neq \text{top}(\mathcal{M}^*_{k-1}(\succsim), \succsim_{k+1})$ or $i = \bar{k}$ at profile $\succsim$, then the argument for the case $i \notin \{k, k+1\}$ applies just noting that $\varphi_i(\succsim) = \text{top}(\mathcal{M}^*_{k-1}(\succsim), \succsim_i)$.

Once we have shown that *k-reversal adjusted priority rule* satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in $\mathcal{D}^N_{\{k,k+1\}}$, we show that $\mathcal{D}^N_{\{k,k+1\}}$ is maximal. Assume there is a rich domain $\mathcal{D}^N = \times_{i \in N} \mathcal{D}i$ such that $\mathcal{D}^N_{\{k,k+1\}} \subset \mathcal{D}^N \subseteq \mathcal{P}^N$ and there is a rule $\varphi : \mathcal{D}^N \to \mathcal{A}$ that satisfies *individual rationality*, *efficiency*, and *strategy-proofness*. Hence, there is $i \in N \succsim_i \in \mathcal{D}^i$ such that for some $j, j' \in N$, $j, j' \neq \{k, k+1\}$, $j < j'$ and $\omega_{j'} \succ_i \omega_j$. We have to consider two exhaustive cases:

Case i). $\{j, j'\} \cap \{k, k+1\} \neq \{\varnothing\}$. Assume that $j' = k$ and therefore $j < j' = k < (k+1)$ (a parallel argument applies if $k < (k+1) = j < j'$). Since $\mathcal{D}^N_{\{k,k+1\}} \subset \mathcal{D}^N$ and $\mathcal{D}^N$ is rich, there is a profile of preferences $\succsim' \in \mathcal{D}^N$ such that $\omega_k \succ'_{k+1} \omega_j \succ_{k+1} \omega_{k+1}$, $\omega_j \succ'_k \omega_{k+1} \succ'_k \omega_k$, $\omega_{k+1} \succ_j \omega_k \succ'_j \omega_j$, and for each $l \notin \{j, k, k'\}$ and $l' \in \{j, k, k'\}$, $\omega_l \succ'_l \omega_{l'}$. By Lemma 1, this contradicts that φ satisfies *individual rationality*, *efficiency*, and *strategy-proofness*.

Case ii). $\{j, j'\} \cap \{k, k+1\} = \{\varnothing\}$. By Lemma 2, this contradicts that φ satisfies *individual rationality*, *efficiency*, and *strategy-proofness*.

□

Before proceeding with the proof of Theorem 2, we need to introduce some useful definitions. Let $a \in \mathcal{A}$ and $\succsim \in \mathcal{R}^N$. Let $C \subseteq N$, we say that the coalition C blocks assignment a with assignment b at profile $\succsim$ if for each $i \in C$, there is $i' \in C$ with $b_i = \omega_{i'}$, and for each $j \in C$, $b_j \succ_j a_j$. The assignment $a \in \mathcal{A}$ is a ***core assignment*** at profile $\succsim$ if there is no coalition C that blocks a at $\succsim$.

Proof of Theorem 2. By Theorem 1, the k-reversal adjusted priority rule satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in $\mathcal{D}^N_{\{k,k+1\}}$, which proves sufficiency. Thus, we focus on the converse statement.

Let φ be the k-reversal adjusted priority rule. First note that, for each $\succsim \in \mathcal{D}^N_{\{k,k+1\}}$, $\varphi(\succsim)$ corresponds to the unique core assignment. That is, there are no blocking coalition formed by two agents $i, j \in N$ and an assignment $b \in \mathcal{A}$ such that $b_i = \omega_j$, $b_j = \omega_i$, $b_i \succ_i \varphi_i(\succsim)$, and

$b_j \succ_j \varphi_j(\succsim)$. This follows from the fact that agent 1 and the agent that obtains ω_1 do not block $\varphi(\succsim)$ because they both obtain their best preferred assignment in the set of individually rational assignments. The argument can be iterated for pairs of agents matched along subsequent steps of the reversal adjusted priority algorithm, to show that no coalition of agents blocks $\varphi(\succsim)$. Finally, for each $i \in N$ and $\succsim_i \in \mathcal{D}^N_{\{k,k+1\}}$, since $\mathcal{D}^i_{\{k,k+1\}} \subset \mathcal{P}$, for each $a \in \mathcal{A}$, $a_i \sim_i \omega_i$ if and only if $a_i = \omega_i$. Moreover, since $\mathcal{D}^N_{\{k,k+1\}}$ is rich, $\mathcal{D}^N_{\{k,k+1\}}$ satisfies anonymity. Hence, $\mathcal{D}^N_{\{k,k+1\}}$ satisfies Assumptions A and B on domains of preferences proposed by Sönmez (1999). By Sönmez (1999, Theorem 1), if there is a rule that satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in $\mathcal{D}^N_{\{k,k+1\}}$, then for each $\succsim$ there is a (essentially) unique core assignment and such rule always selects core assignments. Since by Theorem 1, the k-reversal adjusted priority rule satisfies *individual rationality*, *efficiency*, and *strategy-proofness* in $\mathcal{D}^N_{\{k,k+1\}}$, this suffices to prove necessity. □

Proof of Proposition 1. By construction, the *k-reversal adjusted priority rule with indifferences* satisfies *individual rationality*. Note that at each stage $t < n$ of the adjusted priority algorithm, a temporary match is proposed for agents t and $\mu(t)$ (the same argument applies for $t \in \{k, k+1\}$). Both t and $\mu(t)$ are assigned the best object available at stage t. This temporary match may change if there's another match under which they are both indifferent but allows an improvement for the next agent in the priority order. Since agents cannot be indifferent among three objects, the algorithm is welldefined and the *k-reversal adjusted priority rule with indifferences* satisfies *efficiency*. Since ties are broken using the natural order, the arguments in the proof of Theorem 1 apply to check that *k-reversal adjusted priority rule with indifferences* satisfies *strategy-proofness*. □

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