A solution theorem for implicit differential equations. Applications to parallelism and geodesics for singular metrics

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A Enrique Outerelo, ejemplo, maestro y amigo.

ABSTRACT

Theorems on parallel transport and geodesics for singular metrics given in [1] are shown to be corollaries of a general theorem on implicit differential equations.

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1. Introduction and summary

The usual theorems on existence and uniqueness of solutions of first order ordinary differential equations assume the general form $\dot{x} = F(t,x)$. If instead we consider the "implicit form" $\phi(t,x,\dot{x}) = 0$ and we cannot solve in $\dot{x}$, we lose the powerful theorems we had for the case $\dot{x} = F(t,x)$ as well as the uniqueness of solutions. Still, it is possible to get general results for systems where ϕ is "simple".

This note was inspired in the work of M. Kossowski and M. Kriele [2] on an interesting geometric problem which requires solving implicit differential equations. They considered singular metrics; roughly speaking, those semi-Riemannian metrics g where we allow $g(x)$ to be degenerate for x on a hypersurface Σ. Standard concepts like the Levi-Civita connection, parallel transport and geodesics can be considered in a modified and limited form, going as far as the existence of the g^{ij} is not crucial. If a curve α goes across Σ, parallel transport along α is possible under certain conditions. Also if $p \in \Sigma$ and $\xi \in T_pM$ is transversal to Σ, geodesics with $\dot{\alpha}(0) = \xi$ exist under certain conditions. The work of Kossowski and Kriele is very specific for the geometric problem, but a general and abstract theorem (i.e., independent of the original problem) exists.

Our aim is to prove this theorem, which is in the next section. The other section gives the essential details on singular metrics, as developped in [2] and references therein, without proofs. At the end, the above mentioned theorems on parallel transport and geodesics are seen to be easy consequence of the general theorem.

2. A theorem on implicit differential equations

In this paper "smooth" will mean "class infinity". Let A be an open subset of $\mathbb{R}^n$, $n \geq 2$, and I an open interval containing 0. Given smooth maps $F : I \times A \to \mathbb{R}^n$ and $h : I \times A \to \mathbb{R}$ we consider the following differential equation

$$\begin{cases} \dot{x}^i = F^i(t,x), \quad (1 \leq i \leq n-1), \\ h(t,x)\dot{x}^n = F^n(t,x). \end{cases} \tag{2.1}$$

As usual, a solution of the equation will be a curve $t \to x(t)$, $t \in (-\varepsilon, \varepsilon)$, such that $F(t, x(t))$ is defined for all t and

$$\begin{cases} \dot{x}^i(t) = F^i(t, x(t)), \quad (1 \leq i \leq n-1), \\ h(t, x(t))\dot{x}^n(t) = F^n(t, x(t)). \end{cases}$$

Fix $x_0 \in A$ and suppose $h(0, x_0) = 0$. We want to prove, if possible, the existence and uniqueness of solutions such that $x(0) = x_0$. The usual theorems are not directly applicable because the system is not in the form $\dot{x} = F(t,x)$, but in (a simple) implicit form $\phi(t, x, \dot{x}) = 0$. This is why we call these equations **implicit differential equations.**

In the example with $F(x^1, x^2) = (x^1, x^2)$ and $h(x^1, x^2) = x^2$, we see that through $x_0 = (1, 0)$ we have two solutions $x(t) = (e^t, t)$ and $\xi(t) = (e^t, 0)$. Since we want uniqueness except for the size of the domain of x, we consider **transversal solutions**, defined as those with the property $\frac{d}{dt}\big|_{t=0} h(t, x(t)) \neq 0$.

The essential tool will be the **stable manifold theorem**. We have a field X defined on an open $A \subset \mathbb{R}^n$ and a point $z \in A$ such that $X(z) = 0$. Let $L = DX(z) : \mathbb{R}^n \to \mathbb{R}^n$ be the **linearization** of X at z. We denote by P^+, P^0 and P^- the direct sum of the eigenspaces of L corresponding respectively to the eigenvalues with positive, null and negative real part. *(We warn that the choice of signs is the opposite in the classical book of Abraham-Marsden [1], a standard reference for the theorem.)*

Theorem 2.1 *There are two manifolds S^+, S^- though z which verify*

1. *They are invariant by the flow ϕ_t of X.*
2. *$T_zS^- = P^+$ and $T_zS^+ = P^-$ (In [1] $T_zS^- = P^-$ and $T_zS^+ = P^+$).*
3. *For every $x \in S^+$, the integral curve $\alpha(t) = \phi_t(x)$ is defined for all $t \geq 0$ and $\lim_{t\to\infty} \alpha(t) = z$. For every $x \in S^-$, the integral curve $\alpha(t) = \phi_t(x)$ is defined for all $t \leq 0$ and $\lim_{t\to-\infty} \alpha(t) = z$.*

The manifolds S^+, S^- are locally unique.

The manifolds S^+, S^- are the **stable manifolds**. We keep the notation of theorem 2.1. Next lemma will characterize a stable manifold if it is one-dimensional.

Lemma 2.1 *Suppose for some $\varepsilon = \pm$ that $\dim(S^\varepsilon) = 1$, with T_zS^ε spanned by w. Let $\sigma : I = (-2a, 2a) \to A$ be a diffeomorphism onto a one-dimensional submanifold L of A such that **(a)** $\sigma(0) = z$ and $\dot\sigma(0) = w$; **(b)** There is $f : I \to \mathbb{R}$, only zero at $t = 0$, such that $f\dot\sigma = X \circ \sigma$; and **(c)** The sign of $f'(0)$ is $\varepsilon = \pm$. Then $L = S^{-\varepsilon}$ near z.*

Proof. Let us say that $\varepsilon = +$. We prove the lemma by checking that L fulfils **1–3** in theorem 2.1 and using the local uniqueness of S^-.

We derive $f\dot\sigma = X \circ \sigma$ at $t = 0$ and we get that $w = \dot\sigma(0)$ is an eigenvector of $DX(z)$ with eigenvalue $f'(0)$. Since we assumed $\varepsilon = +$, $f'(t)$ is always strictly positive. Adding to this that $f(0) = 0$, we see that f is strictly positive on $I^+ = (0, 2a)$ and strictly negative on $I^- = (-2a, 0)$. Let us define for $\delta = \pm$, the diffeomorphisms between intervals

$$q^\delta : I^\delta \to J^\delta \subset \mathbb{R}, \quad q^+(t) = \int_a^t \frac{d\tau}{f(\tau)}, \quad q^-(t) = \int_{-a}^t \frac{d\tau}{f(\tau)}.$$

Let $r^\delta = \left(q^\delta\right)^{-1}$. It is easy to check that $\sigma \circ r^\delta : J^\delta \to M$ is an integral curve of X because

$$\dot r^\delta(s) = \frac{1}{\dot q^\delta(r^\delta(s))} = f(r^\delta(s)),$$

$$\left(\sigma \circ r^\delta\right)' = \left(\dot\sigma \circ r^\delta\right)\dot r^\delta = \frac{\dot r^\delta}{f \circ r^\delta}\left(X \circ \sigma \circ r^\delta\right) = X \circ \sigma \circ r^\delta.$$

Suppose $\delta = +$, so J^+ has the form $(b, q^+(2a))$ for some $b = \lim_{t\to 0} q^+(0) < 0 < q^+(2a)$ (recall that, by definition, $\int_a^b f(x)dx = -\int_b^a f(x)dx$ if $a > b$). Notice that the flow ϕ of X, verifies $\sigma(r^+(s)) = \phi_s(\sigma(a))$ because both curves are integral and $r^+(0) = a$. Now, if $x_0 = \sigma(r^+(s_0)) \in L$ we have

$$\phi_s(x_0) = \phi_s\left(\sigma(r^+(s_0))\right) = \phi_s \circ \phi_{s_0}(\sigma(a)) = \phi_{s+s_0}(\sigma(a)) = \sigma \circ r^+(s + s_0) \in L.$$

We have then seen that $\sigma(I^+)$ is ϕ-invariant, which is part of condition **1** in theorem 2.1. Analogously we get that $\sigma(I^-)$ is ϕ-invariant.

Let us check condition **3** in theorem 2.1. We saw above that if $x_0 = \sigma(r^+(s_0)) \in L$ then $\phi_s(x_0) = \sigma \circ r^+(s+s_0)$ is defined on $J^+ - s_0 = (b - s_0, q^+(2a) - s_0)$. If we had $b = -\infty$, $\phi_s(x_0)$ would be defined for $s \leq 0$ and $\lim_{s\to-\infty} \phi_s(x_0) = \lim_{t\to 0} \sigma(t) = z$ as we wish to get **3**. What if $b > -\infty$? This leads to contradiction. We would still have $\lim_{s\to b} \phi_s(x_0) = \lim_{t\to 0} \sigma(t) = z$. A well-known theorem on extendibility of integral curves proves that $\sigma \circ r^+$ is extendible, as integral curve, to an interval whose left endpoint is strictly less than b. This is impossible because $X(z) = X(\lim_{t\to 0} \sigma(t)) = 0$ would imply that $\sigma \circ r^+$ is constant.

Conditions **1** and **3** have been proved and **2** is trivial. We are done. □

Lemma 2.2 *Let $L : \mathbb{E} \to \mathbb{E}$ be the linear map $L(x) = \alpha(x)u + \beta(x)v$, where $u, v \in \mathbb{E}$ are independent and $\alpha, \beta \in \mathbb{E}^*$. We assume that $0, \alpha(u)$ y $\beta(v)$ are different numbers and $\alpha(v) = 0$. Then L has exactly three eigenspaces corresponding (in this order) to the eigenvalues $\beta(v), 0$ and $\alpha(u)$. The first one is spanned by v; the second is $\ker\alpha \cap \ker\beta$; and the third is spanned by $u + cv$, with $c = \beta(u)/ (\alpha(u) - \beta(v))$.*

Proof. Clearly $L(v) = \alpha(v)u+\beta(v)v = \beta(v)v$. If we had $\ker\alpha = \ker\beta$, then $\beta(v) = 0$, which is false. Because of this, $\ker\alpha \cap \ker\beta$ has dimension $\dim(\mathbb{E})-2$ and it is in fact $\ker L$. We consider $x = u + cv$. Condition $L(x) = rx$ is equivalent to

$$\alpha(u)u + \beta(u)v + c\beta(v)v = r\,(u + cv)\,.$$

The independence of u and v gives $r = \alpha(u)$, $\beta(u) + c\beta(v) = \alpha(u)c$ and the right value for c. □

We are now ready to state and prove the main theorem. We see $\mathbb{R}^{n+1}$ as $\mathbb{R} \times \mathbb{R}^n$ and the first coordinate will be the 0-th coordinate.

Theorem 2.2 *Suppose that $h(0, x_0) = F^n(0, x_0) = D_n h(0, x_0) = 0$ and that the numbers*

$$0, \quad D_0 h(0, x_0) + \sum_{i=1}^{n-1} F^i(0, x_0)\, D_i h(0, x_0) \quad \text{and} \quad D_n F^n(0, x_0)$$

are different. Then the system (2.1) has a transversal solution x with $x(0) = x_0$ which is the only transversal solution except for the size of its domain.

Proof. **Existence.** Consider the field on $I\times\ A \subset \mathbb{R}^{n+1}$,

$$W(t, x) = \big(h(t, x)\,, h(t, x)\,F^1(t, x)\,, \ldots, h(t, x)\,F^{n-1}(t, x)\,, F^n(t, x)\big)\,,$$

which is 0 at $(0, x_0)$. Write $(0, x_0) = y_0$. We have that $L = DW(y_0)$ has the matrix

$$L = \begin{pmatrix} D_0 h(y_0) & D_1 h(y_0) & \cdots & D_n h(y_0) \\ D_0 h(y_0) F^1(y_0) & D_1 h(y_0) F^1(y_0) & \cdots & D_n h(y_0) F^1(y_0) \\ \vdots & \vdots & \ddots & \vdots \\ D_0 h(y_0) F^{n-1}(y_0) & D_1 h(y_0) F^{n-1}(x_{y0}) & \cdots & D_n h(y_0) F^{n-1}(y_0) \\ D_0 F^n(y_0) & D_1 F^n(y_0) & \cdots & D_n F^n(y_0) \end{pmatrix}$$

Alternatively, if $(e_0, e_1, \ldots, e_n)$ is the natural basis of $\mathbb{R}^{n+1}$,

$$L(z) = \alpha(z)u + \beta(z)v,$$

where

$$\alpha = Dh(y_0), \ \beta = DF^n(y_0), \ u = e_0 + \sum_{i=1}^{n-1} F^i(y_0) e_i, \ v = e_n. \tag{2.2}$$

Our hypothesis say that lemma 2.2 is applicable with the definitions (2.2) for α, β, u, v. We may apply now theorem 2.1 and we will consider the stable manifold S through y_0 tangent to

$$u + cv = e_0 + \sum_{i=1}^{n-1} F^i(x_0) e_i + ce_n, \quad c = \beta(u)/(\alpha(u) - \beta(v)).$$

Let $\sigma : (-\varepsilon, \varepsilon) \to S$ be a diffeomorphism with $\sigma(0) = x_0$ and $\dot{\sigma}(0) = u + cv$. The number ε and $S = \mathrm{im}(\sigma)$ will be chosen small enough for some local conditions below to hold. Since S is invariant by the flow ϕ_t of W, we may define for small enough ε a function $f : (-\varepsilon, \varepsilon)^2 \to \mathbb{R}$ such that $\phi_t(\sigma(s)) = \sigma(f(t, s))$ and $f(0, s) = s$. If we derive at $t = 0$,

$$W(\sigma(s)) = k(s)\dot{\sigma}(s) \quad \text{for } k(s) = \frac{\partial f}{\partial t}(0, s).$$

Recall that $\dot{\sigma}(0) = u + cv$; therefore $\dot{\sigma}^0(0) = (u + cv)^0 = 1$. For small enough ε, σ^0 gives a diffeomorphism on its image; let us say that $q = (\sigma^0)^{-1}$. Substituting, if needed, σ by $\sigma \circ q$ and $k(s)$by $k(q(t))/\dot{q}(t)$ we may suppose, without losing generality, that σ has the special form $\sigma(t) = (t, x(t))$, $x(t) \in \mathbb{R}^n$. Now, condition $W^0(\sigma(s)) = k(s)\dot{\sigma}^0(s)$ is just $h(t, x(t)) = k(t)$ and the other conditions become

$$\begin{cases} h(t, x(t)) F^i(t, x(t)) = h(t, x(t)) \dot{x}^i(t), \ (1 \le i \le n-1) \\ F^n(t, x(t)) = h(t, x(t)) \dot{x}^n(t). \end{cases} \tag{2.3}$$

We have, by construction, that $\frac{d}{dt}\big|_{t=0} h(t, x(t)) = u + cv$. Therefore,

$$\frac{d}{dt}\bigg|_{t=0} h(t, x(t)) = Dh(y_0)(u + cv) = \alpha(u + cv) \neq 0$$

by hypothesis. Therefore, $h(t, x(t))$ is zero only at $t = 0$ and can be cancelled at (2.3), showing that x is a transversal solution. This ends the proof of existence.

Uniqueness. Suppose first that $x : (-\varepsilon, \varepsilon) \to A$ is a transversal solution of (2.2) with $x(0) = x_0$. We assume $\varepsilon > 0$ small enough for some local conditions around $0 \in (-\varepsilon, \varepsilon)$ to hold. First, $\sigma(t) = (t, x(t))$ will be a diffeomorphism on a submanifold L through $y_0 = (0, x_0)$. We will show with lemma 2.1 that L is a stable submanifold of $L = DW(y_0)$. Clearly σ verifies $(h \circ \sigma)\dot{\sigma} = W \circ \sigma$, hence, by derivation at $t = 0$ we get $(h \circ \sigma)'(0)\dot{\sigma}(0) = L(\dot{\sigma}(0))$. Therefore, $\dot{\sigma}(0)$ is an eigenvector and, by transversality, with non-zero eigenvalue $(h \circ \sigma)'(0)$. We may apply lemma 2.1 to $X = W$, $f = h \circ \sigma$, etc., to get that for small enough ε, L is a stable manifold. We remark that L is the stable manifold for the eigenvalue $\alpha(u)$. Indeed, if not, the eigenvalue should be $\beta(v)$ whose eigenspace is spanned by $v = e_n$. This is impossible because $\dot{\sigma}^0(0) = 1$.

If we have two transversal solutions $x_q : (-\varepsilon, \varepsilon) \to A$, $q = 1, 2$, for small enough ε we may suppose that both σ_q parametrize the same one dimensional manifold (in fact a stable manifold). Therefore, there is a diffeomorphism $\psi : (-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$ with $\psi(0) = 0$ such that $\psi(0) = 0$ and $\sigma_1 = \sigma_2 \circ \psi$. Equating the 0-th coordinates, $\psi = \mathrm{id}$, showing uniqueness. □

We can solve (2.1) by elementary means under *much stronger hypothesis.* The idea is to guarantee that $F^n(t, x)$ can be factored as $F^n(t, x) = h(t, x)G(t, x)$ and that $h(t, x)$ can be cancelled at both sides of the last equation in (2.1). In this way, the solutions of a non-implicit differential equation are solutions of our system. We say that we need stronger hypothesis because the relations between h and F^n do not hold at a single x_0 but for general x (see details below).

Lemma 2.3 *Let A be open in $\mathbb{R}^n$ and I an open interval containing* 0.

1. *Given $\phi : I \times A \to \mathbb{R}$ smooth there is a smooth $\psi : I \times A \to \mathbb{R}$ such that $\phi(t, x) = \phi(0, x) + t\psi(t, x)$.*
2. *Let $f : (-\varepsilon, \varepsilon) \times A \to \mathbb{R}$ be smooth with $f(0, x) = 0$ for all x. If $g : (-\varepsilon.\varepsilon) \to \mathbb{R}$ is a diffeomorphism onto its image J and $g(0) = 0$, there is $q : I \times A \to \mathbb{R}$ smooth such that $f(t, x) = g(t)q(t, x)$.*

Proof. Define $h : [0, 1] \to \mathbb{R}$ by $h(\lambda) = \phi(\lambda t, x)$. Obviously

$$\phi(t, x) - \phi(0, x) = h(1) - h(0) = \int_0^1 h'(\lambda)\, d\lambda = \int_0^1 \phi'(\lambda t, x)\, t d\lambda = t\psi(t, x),$$

where $\psi(t, x) = \int_0^1 \phi'(\lambda t, x)\, d\lambda$. This proves **1**.

To prove **2** we apply **1** to $\phi : J \times A \to \mathbb{R}$, $\phi(s, x) = f\left(g^{-1}(s), x\right)$. For some ψ we obtain $\phi(s, x) = s\psi(s, x)$. Now, for $s = g(t)$ and $q(t, x) = \psi(g(t), x)$, $f(t, x) = g(t)q(t, x)$. □

Theorem 2.3 *Consider the system (2.1).*

1. *Suppose that h depends only on t and **(a)** $h(0) = 0$ and $h'(0) \neq 0$; **(b)** $F^n(0, x) = 0$ for all $x \in A$. Then, for small enough ε there is a smooth $G : (-\varepsilon, \varepsilon) \times A \to \mathbb{R}$ such that $F^n(t, x) = h(t)\, G(t, x)$ and the solutions of the system $\dot{x}^i = F^i(t, x)$ $(i < n)$ and $\dot{x}^n = G(t, x)$ are solutions of (2.1).*

2. *Suppose that h and F^n depend only on x and that $h(x) = 0$ implies $F^n(x) = 0$ and $dh(x) \neq 0$. Then there is a smooth $G : A \to \mathbb{R}$ such that $F^n(x) = h(x)\, G(x)$ and the solutions of the system $\dot{x}^i = F^i(t, x)$ $(i < n)$ and $\dot{x}^n = G(x)$ are solutions of (2.1).*

Proof. **1** is straightforward from the preceding lemma. There is a folk theorem which says that if we have a hypersurface H whose points verify $h(x) = 0$ and $\mathrm{d}h(x) \neq 0$, then any f such that $f|_H = 0$ can be factored as $f(x) = g(x)\, h(x)$ for some function g. Now **2** is obvious if we apply this result to $F^n = f$. $\square$

3. Singular metrics and dual connections

Let M be a m-dimensional smooth manifold and g a 2-covariant symmetric tensor, which will be called a **singular metric (**if g were non-degenerate we would have an ordinary semi-riemannian metric). The map $\mathbf{g} : \mathfrak{X}(M) \to \mathfrak{X}^*(M)$ denotes the natural homomorphism given by g. We have a dual analog to the Levi-Civita connection as follows.

Theorem 3.1 *There is a unique operator $\Delta : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}^*(M)$, $\lambda = \Delta_X Y$, to be called the **dual connection**, determined by*

$$\Delta_X(Y_1 + Y_2) = \Delta_X Y_1 + \Delta_X Y_2, \quad \Delta_{(X_1+X_2)} Y = \Delta_{X_1} Y + \Delta_{X_2} Y,$$

$$\Delta_{fX} Y = f \Delta_X Y, \quad \Delta_X(fY) = (X.f)\, \mathbf{g}(Y) + f \Delta_X Y,$$

$$\mathbf{g}([X, Y]) = \Delta_X Y - \Delta_Y X, \quad X.g(Y, Z) = \Delta_X Y(Z) + \Delta_Y X(Z),$$

where $X, Y, Z \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. The formula for Δ is

$$\begin{aligned} 2\triangle_X Y(Z) &= X.g(Y, Z) + Y.g(X, Z) - Z.g(X, Y) \\ &\quad - \{g(X, [Y, Z]) + g(Y, [X, Z]) - g(Z, [X, Y])\}. \end{aligned}$$

Let (E_a) be a frame on $U \subset M$ with (E^a) as dual frame. We have $\varepsilon^a : TM \to \mathbb{R}$, $\varepsilon^a(\xi) = E^a(x)(\xi)$ for $\xi \in T_x M$. Define $\Gamma_{cab} : U \to \mathbb{R}$ by $\Delta_{E_a} E_b = \sum_c \Gamma_{cab} E^c$. Then,

$$(\Delta_X Y)\,|_U = \sum_{c=1}^{m} (X.Y^c)\, \mathbf{g}(E_c) + \sum_{a,b,c=1}^{m} \Gamma_{cab} X^a Y^b E^c, \quad X^a = \varepsilon^a \circ X,\ Y^b = \varepsilon^b \circ Y.$$

It follows that if $X_1(p) = X_2(p)$, then $\Delta_{X_1}Y(p) = \Delta_{X_2}Y(p)$. Therefore $\Delta_\xi Y \in T_p^*M$ for $\xi \in T_pM$ and $Y \in \mathfrak{X}(M)$ can be correctly defined by $\Delta_\xi Y = \Delta_X Y(p)$, $X(p) = \xi$. Let $\mathfrak{X}(\alpha)$ and $\mathfrak{X}^*(\alpha)$ be the set of fields and 1-forms along the curve $\alpha : I \to M$. We may derive fields A along a curve, in particular $A = \dot\alpha$, analogously as we do with standard connections.

Theorem 3.2 *There is an operator $\mathfrak{X}(\alpha) \to \mathfrak{X}^*(\alpha)$ associated to Δ, still denoted by Δ, uniquely determined by the properties*

$$\Delta(A+B) = \Delta(A) + \Delta(B), \quad \Delta(fA) = f'A + f\Delta(A), \quad A, B \in \mathfrak{X}(\alpha),\ f : I \to \mathbb{R}$$

and $\Delta(C)(t) = \Delta_{\dot\alpha(t)}Y$ if $C = Y \circ \alpha$ for some $Y \in \mathfrak{X}(M)$. If $\alpha(I) \subset U$, and (E_a) is a frame on U then

$$\Delta V = \sum_{c=1}^{m} \dot V^c \mathbf{g}\left(E_c \circ \alpha\right) + \sum_{a,b,c} \left(\Gamma_{cab} \circ \alpha\right) A^a \dot V^b \left(E^c \circ \alpha\right), \quad V^a = \varepsilon^a \circ V,\ A^a = \varepsilon^a \circ \dot\alpha.$$

The field $V \in \mathfrak{X}(\alpha)$ is **parallel** if $\Delta V = 0$ and α is a **geodesic** si $\Delta\dot\alpha = 0$. The local equation for geodesics is

$$\sum_{c=1}^{m} \dot A^c \mathbf{g}\left(E_c \circ \alpha\right) + \sum_{a,b,c} \left(\Gamma_{cab} \circ \alpha\right) A^a A^b \left(E^c \circ \alpha\right) = 0, \quad A^c = \varepsilon^c \circ \dot\alpha$$

Let Σ be the set of $x \in M$ such that $g(x)$ is degenerate. *We assume for the rest of the paper the following hypothesis:* For every $x \in \Sigma$ there is at least a frame (E_a) defined on a neighbourhood U of x, such that if $f = \det(g_{ab}) : U \to \mathbb{R}$, then $df(x) \neq 0$. It is clear that Σ is a hypersurface and that $df(x) \neq 0$ holds at $x \in \Sigma$ for *any* frame. We define the **radical** $\mathrm{Rad}\,(x) = \{\xi \in T_xM \mid g_x(\xi, \eta) = 0,\ \forall \eta \in T_xM\}$. If $x \notin \Sigma$, $\mathrm{Rad}\,(x) = 0$ and if $x \in \Sigma$ it can be proved that $\mathrm{Rad}\,(x)$ is a line.

For technical reasons we are interested in the so called **adapted frames**. A frame $(E_1, \ldots, E_m)$ is adapted if **(a)** $g(E_a, E_b) = 0$ for $a \neq b$; **(b)** $g(E_i, E_i) = \varepsilon_i = \pm 1$ for $1 \leq i \leq m-1$; and **(c)** $E_m|_\Sigma$ spans the radical. It can be proved that adapted frames exist on a neighbourhood U of each $x_0 \in \Sigma$. *We simplify in what follows* $\tau = g(E_m, E_m)$. It is clear that $U \cap M = \{q \in U \mid \tau(q) = 0\}$ and that $E_m(x)$ spans $\mathrm{Rad}\,(x)$ if $\in \Sigma$.

We define at $x \in \Sigma$ the *symmetric* tensor

$$\mathbb{I}_x : T_xM \times T_xM \times \mathrm{Rad}\,(x) \longrightarrow \mathbb{R}, \quad \mathbb{I}_x(\xi, \eta, \zeta) = \Delta_X Y(Z)(x),$$

where X, Y, Z are fields such that $X(x) = \xi$, $Y(x) = \eta$, $Z(x) = \zeta$. The definition is correct; i.e., independent of the chosen extensions. In any frame

$$\begin{aligned}\mathbb{I}_x(\xi,\eta,\zeta) &= \sum_{c=1}^{m}(\xi.Y^c)\, g(E_c, Z)(x) + \sum_{a,b,c=1}^{m}\Gamma_{cab}(x)X^a(x)Y^b(x)Z^c(x)\\ &= \sum_{a,b,c=1}^{m}\Gamma_{cab}(x)\xi^a\eta^b\zeta^c.\end{aligned}$$

The tensor field $\mathbb{I}$ along Σ is closely related to extendibility conditions from $M-\Sigma$ to all M. It can be proved that given X, Y fields on M, the field $\nabla_X Y$ on $M-\Sigma$ is extendible to Σ if and only if $\mathbb{I}(X(x), Y(x), Z(x)) \equiv 0$ where Z is any field along Σ spanning the radical.

On $M-\Sigma$ the singular metric g is an ordinary (i.e., non-degenerate) metric. Therefore its Levi-Civita connection ∇ is defined on $M-\Sigma$. The dual connection Δ is related to ∇ by $\Delta_X Y(Z) = g(\nabla_X Y, Z)$ for fields X, Y, Z on $M-\Sigma$. The Christoffel symbols Γ^c_{ab} of ∇ are related to the symbols Γ_{cab} by $\Gamma^c_{ab} = \sum_d g^{cd}\Gamma_{dab}$ where $\left(g^{ab}\right)$ is the inverse matrix of (g_{ab}). Of course, the previous equations only make sense on $U-\Sigma$, where U is the domain of the frame.

We write down some formulas we will need on an adapted frame $(E_1, \ldots, E_m)$ defined on U. The main simplifications are:

1. $(g_{ij}(x))_{1\le i,j\le m-1}$ is everywhere invertible and $g_{im}(x) = 0$ if $i < m$ (even if $x \in \Sigma$). Therefore, the coefficients of the matrix $\left(g^{ab}(x)\right)$ are defined for all $x \in U$, except $g^{mm}(x) = 1/\tau(x) = 1/g_{mm}(x)$, only defined if $x \notin \Sigma$. In any case $g^{im}(x) = 0$ for $i \le m-1$.
2. All Christoffel symbols Γ^i_{ab}, $i \le m-1$ are smoothly extendible to Σ.
3. The local equations of a parallel field V along $\alpha : I \to M$ and a Δ-geodesic α are

$$\Delta V = 0 \iff \begin{cases} \dot V^i + \sum_{a,b=1}^m \left(\Gamma^i_{ab}\circ\alpha\right) A^a V^b = 0, \quad (1 \le i \le m-1),\\ (\tau\circ\alpha)\,\dot V^m + \sum_{a,b=1}^m (\Gamma_{mab}\circ\alpha)\, A^a V^b = 0.\end{cases} \tag{3.1}$$

$$\Delta \dot\alpha = 0 \iff \begin{cases} \dot A^i + \sum_{a,b=1}^m \left(\Gamma^i_{ab}\circ\alpha\right) A^a A^b = 0, \quad (1 \le i \le m-1),\\ (\tau\circ\alpha)\,\dot A^m + \sum_{a,b=1}^m (\Gamma_{mab}\circ\alpha)\, A^a A^b = 0.\end{cases} \tag{3.2}$$

   where, as in the general formulas, $V^c = \varepsilon^c \circ V$ and $A^c = \varepsilon^c \circ \dot\alpha$, both scalar functions of $t \in I$.
4. The local equation of Σ is $\tau(x) = g_{mm}(x) = 0$ and

$$d\tau(x)(\xi) = 2\mathbb{I}_x(\xi, E_m(x), E_m(x)) \quad \text{if } x \in \Sigma.$$

   Hence, $\xi \in T_xM$ is transversal to $T_x\Sigma$ if and only if $\mathbb{I}_x(\xi, E_m(x), E_m(x)) \neq 0$.

Theorem 3.3 *Suppose that $\alpha : (-a, a) \to M$ is transversal to Σ at $p = \alpha(0)$ and that $\xi \in T_pM$ verifies $\mathbb{I}_p(\xi, \dot\alpha(0), E_m(p)) = 0$. Then, for some $0 < \varepsilon \le a$ there is a Δ-parallel field V along α such that $V(0) = \xi$ which is unique except for the size of its domain.*

Proof. Let $\xi = \sum_{a=1}^m \xi^a E_a(p)$. The existence of V is equivalent to the existence of a solution of (3.1) with initial condition $V^a(0) = V_0^a = \xi^a$, $1 \le a \le m$. But (3.1) is a particular case of (2.1), where, with V instead of x,

$$\begin{cases} F^i(t, V) = -\sum_{a,b=1}^m \Gamma^i_{ab}(\alpha(t)) A^a(t) V^b, \ (1 \le i \le m-1), \\ F^m(t, V) = -\sum_{a,b=1}^m \Gamma_{mab}(\alpha(t)) A^a(t) V^b, \end{cases}$$

and $h(t) = \tau(\alpha(t))$. This equation has the required solution because the hypothesis of theorem 2.2 hold. Indeed, $h(0, V_0) = \tau(p) = 0$ and $D_m h(0, V_0) = 0$ because h only depends on t. Also

$$F^m(0, V_0) = -\sum_{a,b=1}^m \Gamma_{mab}(p) A^a(0) \xi^b = -\mathbb{I}_p(\xi, \dot\alpha(0), E_m(p)) = 0.$$

As for the condition on different values we have

$$D_0 h(0, x_0) + \sum_{i=1}^{m-1} F^i(0, x_0) D_i h(0, x_0) = d\tau(p)(\dot\alpha(0)),$$

$$\begin{aligned} D_m F^m(0, x_0) &= -\sum_{a,b=1}^m \Gamma_{mam}(p) A^a(0) = -\mathbb{I}_p(\dot\alpha(0), E_m(0), E_m(0)) \\ &= (1/2)\, d\tau(p)(\dot\alpha(0)). \end{aligned}$$

By the transversality of α, $d\tau(p)(\dot\alpha(0)) \ne 0$; therefore these numbers are different and non-zero. $\square$

Theorem 3.4 *Let $p \in \Sigma$ and $\xi \in T_pM$ be transversal to Σ (which is equivalent to $\mathbb{I}_p(\xi, E_m(p), E_m(p)) \ne 0$). If $\mathbb{I}_p(\xi, \xi, E_m(p)) = 0$ there is a Δ-geodesic $\alpha : (-\varepsilon, \varepsilon) \to M$ such that $\alpha(0) = p$ and $\dot\alpha(0) = \xi$, which is unique except for the size of its domain.*

Proof. We choose around p an adapted frame $(E_1, \ldots, E_m)$ and a chart $\left(u^1, \ldots, u^m\right)$ such that $u(p) = 0$ and $\left(\partial/\partial u^i\right)(p) = E_i(p)$. Therefore tangent vectors at p like ξ and $\dot\alpha(0)$ have the same coefficients on the chart and the frame. For some invertible matrix $\left(C_i^j(x)\right)$, which is $\left(\delta_i^j\right)$ at $x = p$, we have

$$\left(u^i \circ \alpha\right)'(t) = \dot\alpha^i(t) = \sum_{j=1}^m C_j^i(\alpha(t)) A^j(t), \quad \text{for } \dot\alpha(t) = \sum_{j=1}^m A^j(t) E_j(\alpha(t))$$

(equivalent to $A^j(t) = \varepsilon^j(\dot{\alpha}(t))$). The system (3.2) is equivalent to a system (2.1) where

$$n = 2m, \quad x = \left(\alpha^1, \dots, \alpha^m, A^1, \dots, A^m\right),$$

$$x_0 = \left(u^1(p), \dots, u^m(p), A^1(0), \dots, A^m(0)\right) = \left(0, \dots, 0, \xi^1, \dots, \xi^m\right),$$

$$F^i\left(t, \alpha^1, \dots, \alpha^m, A^1, \dots, A^m\right) = \sum_{j=1}^m \bar{C}^i_j\left(\alpha^1, \dots, \alpha^m\right) A^j, \quad (1 \le i \le m),$$

$$F^{m+i}\left(t, \alpha^1, \dots, \alpha^m, A^1, \dots, A^m\right) = -\sum_{a,b=1}^m \bar{\Gamma}^i_{ab}\left(\alpha^1, \dots, \alpha^m\right) A^a A^b = 0, \quad (i < m),$$

$$F^{2m}\left(t, \alpha^1, \dots, \alpha^m, A^1, \dots, A^m\right) = -\sum_{a,b=1}^m \bar{\Gamma}_{mab}\left(\alpha^1, \dots, \alpha^m\right) A^a A^b,$$

$$h\left(t, \alpha^1, \dots, \alpha^m, A^1, \dots, A^m\right) = \bar{\tau}\left(\alpha^1, \dots, \alpha^m\right).$$

In these formulas, the overbar means "composed with u^{-1}"; e.g., $\bar{\tau} = \tau \circ u^{-1}$, $\bar{\Gamma}^i_{ab} = \Gamma^i_{ab} \circ u^{-1}$, etc.. We see that the hypothesis of theorem 2.2 hold. First,

$$h(0, x_0) = \tau(p) = 0, \quad D_n h(0, x_0) = \frac{\partial \bar{\tau}\left(\alpha^1, \dots, \alpha^m\right)}{\partial A^m} = 0,$$

$$F^n(0, x_0) = -\sum_{a,b=1}^m \bar{\Gamma}_{mab}(0, \dots, 0)\, \xi^a \xi^b = -\mathbb{I}_p(\xi, \xi, E_m(p)) = 0,$$

As for the condition on different values we have

$$\begin{aligned} D_0 h(0, x_0) + \sum_{i=1}^{n-1} F^i(0, x_0)\, D_i h(0, x_0) &= \sum_{i=1}^n \sum_{j=1}^m \bar{C}^i_j(0, \dots, 0)\, \xi^j D_i \bar{\tau}(0, \dots, 0) \\ &= \sum_{i=1}^n \sum_{j=1}^m \delta^i_j \xi^j \frac{\partial \tau}{\partial u^i}(p) = d\tau(p)(\xi), \end{aligned}$$

$$\begin{aligned} D_n F^n(0, x_0) &= -\sum_{a,b=1}^m \bar{\Gamma}_{mab}(0, \dots, 0)\left(\delta^a_m \xi^b + \xi^a \delta^b_m\right) = -2\sum_{a,b=1}^m \Gamma_{mmb}(p)\, \xi^b \\ &= -2\mathbb{I}_p(\xi, \xi, E_m(p)) = -d\tau(p)(\xi), \end{aligned}$$

and it is clear that these numbers are different and non-zero by the transversality of ξ. □

It is easy to prove that the conditions $\mathbb{I}_p(\xi, \dot{\alpha}(0), E_m(p)) = 0$ and $\mathbb{I}_p(\xi, \xi, E_m(p)) = 0$ in the preceding theorems are in fact necessary for the existence of V and α.

References

[1] R. Abraham, J. Marsden: *Foundations of mechanics (Second Edition).* Addison–Wesley, 1978.

[2] M. Kossowski, M. Kriele: *Transverse, type changing, pseudo–riemannian metrics and the extendability of geodesics.* Proc. R. Soc. Lond. A **444** (1994), 297–306.