Bulk Viscosity and the Conformal Anomaly in the Pion Gas

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We calculate the bulk viscosity of the massive pion gas within unitarized chiral perturbation theory. We obtain a low-temperature peak arising from explicit conformal breaking due to the pion mass and another peak near the critical temperature, dominated by the conformal anomaly through gluon condensate terms. The correlation between bulk viscosity and conformal breaking supports a recent QCD proposal. We discuss the role of resonances, heavier states, and large-N_c counting.

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The matter produced after thermalization in relativistic heavy ion collisions behaves nearly as a perfect fluid [1]. Deviations are seen mainly in elliptic flow and can be reasonably explained with a small shear viscosity over entropy density ratio $\eta/s < 0.5$ [2], whereas bulk viscosity ζ is generically assumed to be negligible. However, it has been recently proposed [3] that ζ might be large near the QCD phase transition. If ζ/s is comparable to η/s near the critical point (where indeed the latter is expected to have a minimum) interesting physical possibilities arise such as radial flow suppression, modifications of the hadronization mechanism [3] or clustering at freeze-out [4]. The argument of [3] is that, following the QCD sum rules in [5], one can relate ζ with the trace anomaly:

$$\zeta(T) = \frac{1}{9\omega_0(T)}\left[T^5 \frac{\partial}{\partial T}\frac{\langle\theta\rangle_T - \langle\theta\rangle_0}{T^4} + 16|\epsilon_0|\right], \qquad (1)$$

with $\langle\theta\rangle_T \equiv \langle T^\mu_\mu\rangle_T = \epsilon - 3P$, T^μ_ν the energy-momentum tensor, ϵ the energy density, P the pressure, and $\epsilon_0 = \langle\theta\rangle_0/4$ in vacuum. To derive (1), a particular ansatz has been used for $\rho(\omega)$, the $\langle\theta\theta\rangle$ spectral function at zero spatial momentum, with $(\rho/\omega)(0) = 9\zeta/\pi$ and $9\zeta\omega_0 = 2\int_0^\infty(\rho/\omega)d\omega$. Equation (1) implies then a large bulk viscosity near the QCD transition, from the $\langle\theta\rangle_T$ peak observed in the lattice [6], more or less pronounced depending on the transition order [3]. However, this argument has been recently criticized on the basis of the $\int_0^\infty(\rho/\omega)$ convergence and parametric dependence with the QCD coupling constant [7]. On the other hand, estimates of ζ from lattice data show that $\omega\delta(\omega)$ terms and large-ω nonthermal contributions have to be properly accounted for in spectral functions [8,9].

It is therefore of great importance to study QCD regimes where one can rely on analytic calculations, in order to clarify the validity of the above proposal without appealing directly to lattice data. In the weak coupling regime, valid for very high temperatures, ζ/η has been found to be parametrically small [10]. Another regime where one can perform analytic calculations is low-energy QCD, where the system consists primarily of a meson gas and, for low temperatures, one can rely on chiral perturbation theory (ChPT) [11]. In this regime, we have recently shown [12,13], within linear response theory (Kubo's formula), that the usual ChPT power counting must be extended to account for $1/\Gamma_p$ contributions arising in transport coefficients. Here, Γ_p is the thermal width of a pion with three-momentum $\boldsymbol{p}$, in which the $\pi\pi$ total elastic cross section enters linearly in the dilute gas regime [14]. Performing the power counting, which includes a detailed analysis of ladder-type diagrams considered in [15], the leading-order ChPT contribution comes from a one-loop meson diagram with $\Gamma_p \neq 0$ internal lines. An essential point is to include unitarity corrections in Γ_p to describe correctly the temperature behavior as the system approaches chiral restoration. We neglect inelastic $2\pi \leftrightarrow 4\pi$ reactions restoring particle number equilibrium, which are suppressed in our counting and yield chemical relaxation times about 10 times larger than the plasma lifetime [16]. Thus, our bulk viscosity is meaningful for the pion gas formed in heavy ion collisions, which conserves approximately pion number between chemical and thermal freeze-out, as confirmed by particle spectra data analyses with a pion chemical potential [17]. If ζ is defined in complete chemical equilibrium, then particle-changing processes dominate [15]. The dominance of elastic processes for ζ in the pion gas holds also in kinetic theory [18–20]. With our approach we have also obtained η/s developing a minimum compatible with AdS/CFT bounds with values in good agreement with kinetic theory [19,21] and phenomenological estimates on elliptic flow. This is the theoretical basis of the present work, where we will analyze within ChPT the correlation between bulk viscosity and the conformal anomaly in the pion gas regime, studying the origin of the different contributions to conformal breaking for physical massive pions. Thus, we start with Kubo's formula

$$\zeta(T) = \frac{1}{2}\lim_{\omega\to 0^+}\frac{\partial}{\partial\omega}\int d^4x e^{i\omega x^0}\langle[\hat{\mathcal{P}}(x), \hat{\mathcal{P}}(0)]\rangle, \qquad (2)$$

where the modified pressure operator $\hat{\mathcal{P}} \equiv -T^i_i/3 - c_s^2 T_{00}$, the squared speed of sound $c_s^2 = \partial P/\partial\epsilon = s/c_v$, $s = \partial P/\partial T$ and the specific heat $c_v = \partial\epsilon/\partial T = T\partial s/\partial T$. We follow the conventions of [22], where ζ is defined as the change in the pressure produced by a gradient in the flow velocity, relative to equilibrium. This leads to the correlator in (2), which is the adequate one to be used within perturbation theory [7,10]. In lattice analyses, one works with the Lorentz invariant θ instead. In our approach these two correlators are not equivalent, since the leading order in $1/\Gamma_p$ for perturbative T_{00} commutators does not vanish for zero spatial momentum. As we shall see, sticking to the original definition (2) leads naturally to the expected conformal properties and asymptotic behavior of the bulk viscosity. Following [12], we calculate then the spectral function ($\hat{\mathcal{P}}$ commutator) in (2) in the imaginary-time formalism, picking up the dominant contribution in $1/\Gamma_p$ (pinching pole) of the analytically continued retarded correlator. That term is purely imaginary and gives the dominant effect in the spectral function at zero momentum and small energy. Thus, to leading order:

$$\zeta(T) = \int_0^\infty dp \frac{3p^2(p^2/3 - c_s^2 E_p^2)^2}{4\pi^2 T E_p^2 \Gamma_p} n_B(E_p)[1 + n_B(E_p)], \tag{3}$$

with $n_B(x) \equiv 1/[\exp(x/T) - 1]$ the Bose-Einstein distribution function, $E_p \equiv \sqrt{p^2 + M_\pi^2}$ and where the leading $\mathcal{O}(p^2)$ order in $T^{\mu\nu}$ has been retained in the vertex. Now, we get c_s^2 in (3) from P calculated up to $\mathcal{O}(T^8)$ in [11]. In Fig. 1 we see that to $\mathcal{O}(T^6)$, both the specific heat and the speed of sound increase monotonically, c_s^2 approaching the ultrarelativistic limit of $1/3$ corresponding to a gas of free massless pions. To that order, since the distribution function is peaked around $p \sim T$ for $T \gg m_\pi$, we see that (3) vanishes asymptotically for large temperatures, as expected for conformally invariant systems [10,13,18,19,22]. In fact, from (3) we get for massless pions (chiral limit) $\zeta = 15(1/3 - c_s^2)^2 \eta$, consistently with [23] and parametrically with high-T QCD [10]. The crucial point here is that taking one more order in the pressure c_v grows, reaching a maximum at about $T_c \simeq 220$ MeV. The speed of sound attains then a minimum at T_c which will alter the behavior of $\zeta(T)$. This is the critical behavior of a $O(4)$-like crossover, as expected for two massive flavors at zero chemical potential. A physical interpretation is that, although temperature tends to erase mass scales, chiral interactions are enhanced and produce in the critical region a significant, nonperturbative, conformal breaking reflected in $c_s^2 \neq 1/3$. Note that, although in the massive case T_c is near the chiral restoration temperature T_c^χ where the order parameter $\langle \bar{q}q \rangle_T$ vanishes [11], in the chiral limit $T_c^\chi \simeq 170$ MeV, while T_c is almost unchanged.

We plot our result for the bulk viscosity in Fig. 2. The effect of including the $\mathcal{O}(T^8)$ in c_s^2 effectively produces a peak around T_c, not present to $\mathcal{O}(T^6)$ [13]. The speed of sound is not the only relevant effect yielding a sizable peak: unitarization of the cross section entering Γ_p [12] is also crucial to $\mathcal{O}(T^8)$. Considering unitarized partial waves for $\pi\pi$ scattering (ChPT is only perturbatively unitary) improves the high energy behavior (and therefore the high temperature one) and generates dynamically the $f_0(600)$ and $\rho(770)$ resonance poles. Consistently, we have chosen the values of the low-energy constants $\bar{l}_i$ entering pion scattering (they can be found in [24]) so that the mass and width of the ρ are at their physical values for $T = 0$. As we discuss below, the $\bar{l}_i$ dependence is crucial in the present analysis. In the chiral limit, the transition peak is almost unchanged and so is T_c, unlike T_c^χ, which indicates that chiral restoration is not the main source of this effect. Our massless results are in reasonable agreement with a recent kinetic theory analysis [20]. We also obtain a low-T peak, which disappears in the chiral limit. In our regime and for $T \ll m_\pi$, $n_B(E_p) \simeq e^{-m_\pi/T} e^{-p^2/2m_\pi T}$ so that three-momenta $p = \mathcal{O}(\sqrt{m_\pi T})$ and taking the leading order for Γ_p [12] and $c_s^2 \simeq T/m_\pi + \ldots$ [11], Eq. (3) becomes

$$\zeta(T) \simeq 13.3 \frac{f_\pi^4 \sqrt{T}}{m_\pi^{3/2}} \quad \text{for } T \ll m_\pi, \tag{4}$$

where f_π is the pion decay constant. The above behavior is consistent with nonrelativistic kinetic theory [18] where ζ and η are expected to be comparable at low T. Thus, $\zeta(T)$ increases for very low T and has to decrease at some point to match the asymptotic vanishing behavior, thus explaining the low-T maximum.

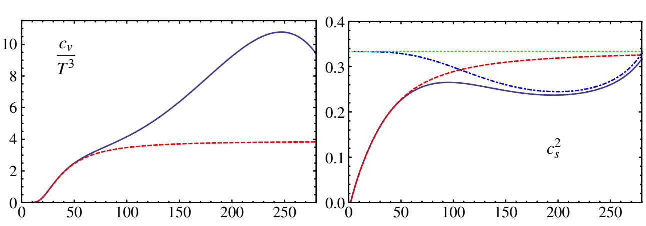

FIG. 1 (color online). Specific heat (left) and speed of sound squared (right) for the pion gas. The red dashed line is the $\mathcal{O}(T^6)$ calculation, and the continuous blue line the $\mathcal{O}(T^8)$ one. The green dotted line is the ultrarelativistic limit $c_s^2 = 1/3$. The dash-dotted blue line is the chiral limit result to $\mathcal{O}(T^8)$.

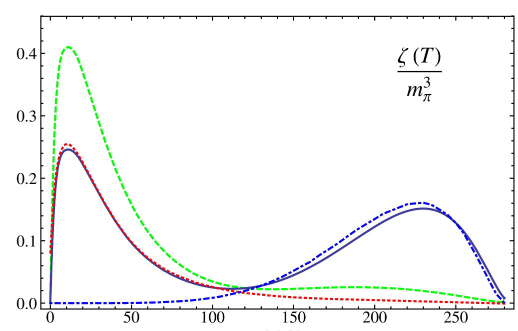

FIG. 2 (color online). Bulk viscosity of the pion gas. The full blue line is the unitarized result with c_s^2 to $\mathcal{O}(T^8)$ and the dash-dotted blue one is the same calculation in the chiral limit. The dashed green line is the nonunitarized result at the same order. The dotted red line is unitarized with c_s^2 to $\mathcal{O}(T^6)$ and lies very close to the nonunitarized curve, which is not displayed.

Let us now evaluate conformal-breaking contributions for the pion gas. First, it is instructive to recall the QCD result for the trace anomaly [25]:

$$(T^\mu_\mu)_{\rm QCD} = \frac{\beta(g)}{2g} G^a_{\mu\nu} G_a^{\mu\nu} + (1 + \gamma_m(g))\bar{q} M q, \qquad (5)$$

where the renormalization group functions are, perturbatively, $\beta(g) = \mathcal{O}(g^3)$, $\gamma_m(g) = \mathcal{O}(g^2)$. The first term is the conformal anomaly proportional to the gluon condensate. The second one comes from the explicit breaking in the QCD Lagrangian, M being the quark mass matrix. For the pion gas, using the thermodynamic identity

$$\langle\theta\rangle_T = T^5 \frac{d}{dT}\left(\frac{P}{T^4}\right), \qquad (6)$$

we represent in Fig. 3 the trace anomaly to different orders in the pressure, as well as the T function appearing in the right-hand side of (1). We observe clearly the same two-peak structure as the bulk viscosity, with similar features.

The low-T peak disappears in the chiral limit. Its contribution comes then from explicit conformal breaking. Calculating only the first nonvanishing order in ChPT, either using (6) or evaluating directly the energy-momentum correlators, we get

$$\begin{aligned}\langle\theta\rangle_T - \langle\theta\rangle_0 &= 3m_\pi^2 g_1(m_\pi, T) + \mathcal{O}(f_\pi^{-2}) \\ &= 2m_q(\langle\bar{q}q\rangle_T - \langle\bar{q}q\rangle_0) + \mathcal{O}(f_\pi^{-2}), \qquad (7)\end{aligned}$$

where $m_q = m_u = m_d$ and we formally account for different chiral orders by their f_π power. The function g_1 is the thermal correction to the free pion propagator $G(x=0)$ [11]. Comparing with the QCD expression (5) the factor of 2 in (7) for the quark condensate is perfectly consistent with the result [26] showing that the quark and gluon contributions to the trace anomaly are identical at low temperatures. Now, $g_1(T)/T^4$ has a maximum at $T \simeq 2m_\pi/5 \simeq 60$ MeV, which is the low-T peak in Fig. 3 and the source for the first peak of the bulk viscosity.

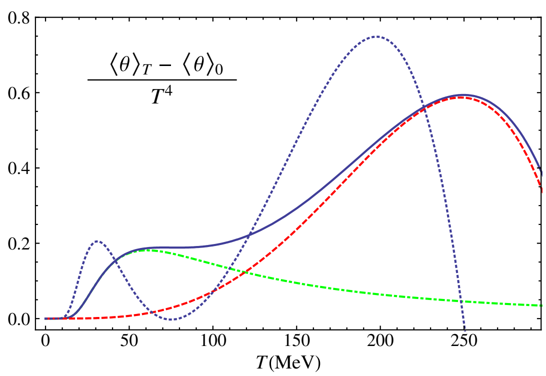

FIG. 3 (color online). Thermal expectation value of the trace anomaly for the pion gas. The dash-dotted green and continuous blue lines are, respectively, the $\mathcal{O}(T^6)$ and $\mathcal{O}(T^8)$ results. The dashed red line corresponds to the $\mathcal{O}(T^8)$ result for massless pions [the $\mathcal{O}(T^6)$ order vanishes for $m_\pi = 0$]. The dotted blue line is $T\frac{\partial}{\partial T}\frac{\langle\theta\rangle_T - \langle\theta\rangle_0}{T^4}$.

The transition peak only shows up at $\mathcal{O}(T^8)$ and survives in the chiral limit, where its origin is purely anomalous. It comes from ChPT interactions involving dimensionful couplings, such as f_π, and is therefore suppressed at low temperatures [27]. For massive pions, the value of the peak and its position are almost unchanged with respect to the chiral limit, the difference being even smaller than the simple extrapolation of the quark condensate contribution in (7) with $\langle\bar{q}q\rangle_T$ to $\mathcal{O}(T^8)$, which represents around a 10% correction in the critical region. The fermion contribution is also subdominant in lattice analyses [6]. These results show again that the nature of this effect is not likely to be related to chiral symmetry restoration but rather to other QCD critical effects such as deconfinement. The correlation with the bulk viscosity is again clear. In fact, in the chiral limit the function between brackets in (1) and $15(c_s^2 - 1/3)^2 = \zeta/\eta$ have their maximum at the same $T_c = e^{-5/8}\Lambda_p$ with Λ_p given in [11] in terms of $\bar{l}_1 + 4\bar{l}_2$. We recall that in order to establish the possible correlations between the conformal anomaly and the bulk viscosity, we have used the same set of $\bar{l}_i$ in both figures. For those unitarized values, $T_c \simeq 220$ MeV. Using perturbative values, for instance those given in [11] fixed to reproduce pion scattering lengths, the critical peak is about 3 times smaller and $T_c \simeq 148$ MeV, while T_c^χ varies only about 10 MeV from one set to another. We get exactly the same drastic reduction of the critical peak and shift of T_c in the bulk viscosity. The presence of resonances is then crucial to yield a sizable effect in the transition peak, whose dominant contribution comes from the gluon condensate.

Regarding the $\omega_0(T)$ function defined through (1), in the chiral limit it grows linearly with T, reaching $\omega_0 \sim 400$ MeV at the transition. In the massive case, taking $|\epsilon_0| = f_\pi^2 m_\pi^2$, the ChPT lowest order, we get $\omega_0(T_c) \sim 1$ GeV, almost constant from $T \sim 150$ MeV onwards. These values are in reasonable agreement with the estimates in [3]. On the other hand, from (4) we get $\omega_0(T) \simeq 0.13 m_\pi^{7/2}/(f_\pi^2\sqrt{T})$ for $T \to 0^+$.

The numerical values of the trace anomaly in Fig. 3 are not far from the lattice values [6] for low T, but they are about a factor of 10 smaller near T_c. The increasing of degrees of freedom due to heavier states, not included in our approach, is clearly important in that region. For instance, the $\mathcal{O}(T^8)$ pressure in the chiral limit is proportional to $N_f^2(N_f^2 - 1)$ [27] so that changing from two to three flavors, which are not Boltzmann suppressed near the transition, significantly increases the anomaly. In addition, using a simple free hadron resonance gas approach [3], the π, ρ, σ contributions to the anomaly amount only to a 5% of all baryon and meson states up to 2.5 GeV. In fact, although we get $\zeta/s \simeq 0.02$ at the transition peak, still smaller than $\eta/s \simeq 0.25$, we would get a larger value if we assume that the introduction of heavier states increase the anomaly, and that implies an increase of the transition strength and a strong reduction of c_s^2 [3]. As an indication, setting $c_s^2 = 0$ in (3) we get $\zeta/s \sim 1$ at T_c.

We have seen that it is crucial to include correctly the effect of the ρ resonance. On the other hand, the $f_0(600)/\sigma$ is expected to be related to chiral restoration. Regarding bulk viscosity, it has been suggested in [28], within mean field theory, that any dynamic scalar field σ should contribute to $\zeta \propto \Gamma_\sigma/m_\sigma^2$, which may be large near the critical region by mass reduction, for instance in the linear sigma model (LSM) context. Within unitarized ChPT, the dynamically generated $f_0(600)$ pole undergoes a significant mass reduction towards $2m_\pi$ governed by chiral restoration, remaining a broad state with sizable width near the transition [24]. Interestingly, from [24], we find that Γ_σ/m_σ^2 has a peak at $T \sim 180$ MeV, where the pole mass reaches threshold. For higher T the width still decreases (by phase space reduction) while the mass remains close to threshold. This critical value is very close to the one obtained in [28] for the LSM assuming a T-independent width. However, as discussed above, these chiral restoration effects are likely to be subdominant.

The large-N_c limit is also revealing. The counting of the $\bar{l}_i$ can be extracted from the L_i ($N_f = 3$) [29] while $f_\pi^2 = \mathcal{O}(N_c)$. We get $\Gamma_p \sim \mathcal{O}(N_c^{-2})$ and, in the chiral limit, $\langle\theta\rangle_T \sim \mathcal{O}(N_c^{-1}) \sim (c_s^2 - 1/3)$ so that $\zeta \sim \mathcal{O}(1)$ and $\zeta/\eta \sim \mathcal{O}(N_c^{-2})$, parametrically suppressed as expected. Now, taking into account the critical behavior, $T_c \sim \mathcal{O}(e^{N_c})$ and $\langle\theta\rangle_{T_c} \sim \mathcal{O}(e^{N_c}/N_c^2)$. This large dependence is another indication of the dominance of confinement over chiral restoration, comparing with the chiral $T_c^\chi = \mathcal{O}(N_c)$. Also, $\langle\theta\rangle \propto L_3$, which in large N_c includes a term proportional to the gluon condensate [30]. Comparing with the QCD expressions in [10], we agree except for the overall $\mathcal{O}(N_c^2)$ constants in the pressure which count the degrees of freedom. For massive pions, the above chiral limit scaling is only reached asymptotically for large T, while for any T we get $\zeta/\eta \sim \mathcal{O}(1) \sim \langle\theta\rangle_T - \langle\theta\rangle_0$ with $\zeta \sim \mathcal{O}(N_c^2)$, compatible with (4).

Summarizing, we have shown, within unitarized ChPT, that the massive pion gas develops a strong correlation between bulk viscosity and the conformal anomaly. Both quantities show a low-temperature peak coming from mass conformal breaking and another one at the critical temperature remaining in the chiral limit and mainly dominated by gluon condensate contributions not related to chiral restoration. The dynamically generated light resonances are essential to obtain sizable effects at the transition. Different estimates indicate that heavier states could yield a larger bulk viscosity near the transition, leading to observable effects in heavy ion collisions.

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