

Isospin breaking and chiral symmetry restoration

A. Gómez Nicola^{*} and R. Torres Andrés[†]

Departamento de Física Teórica II, Universidad Complutense, 28040 Madrid, Spain

(Received 14 February 2011; published 12 April 2011)

We analyze quark condensates and chiral (scalar) susceptibilities including isospin-breaking effects at finite temperature T . These include $m_u \neq m_d$ contributions as well as electromagnetic ($e \neq 0$) corrections, both treated in a consistent chiral Lagrangian framework to leading order in $SU(2)$ and $SU(3)$ chiral perturbation theory, so that our predictions are model-independent. The chiral restoration temperature extracted from $\langle \bar{q}q \rangle = \langle \bar{u}u + \bar{d}d \rangle$ is almost unaffected, while the isospin-breaking order parameter $\langle \bar{u}u - \bar{d}d \rangle$ grows with T for the three-flavor case $SU(3)$. We derive a sum rule relating the condensate ratio $\langle \bar{q}q \rangle (e \neq 0) / \langle \bar{q}q \rangle (e = 0)$ with the scalar susceptibility difference $\chi(T) - \chi(0)$, directly measurable on the lattice. This sum rule is useful also for estimating condensate errors in staggered lattice analysis. Keeping $m_u \neq m_d$ allows one to obtain the connected and disconnected contributions to the susceptibility, even in the isospin limit, whose temperature, mass, and isospin-breaking dependence we analyze in detail. The disconnected part grows linearly, diverging in the chiral (infrared) limit as T/M_π , while the connected part shows a quadratic behavior, infrared regular as T^2/M_η^2 , and coming from $\pi^0\eta$ mixing terms. This smooth connected behavior suggests that isospin-breaking correlations are weaker than critical chiral ones near the transition temperature. We explore some consequences in connection with lattice data and their scaling properties, for which our present analysis for physical masses, i.e. beyond the chiral limit, provides a useful model-independent description for low and moderate temperatures.

DOI: 10.1103/PhysRevD.83.076005

PACS numbers: 11.10.Wx, 11.30.Rd, 12.39.Fe

I. INTRODUCTION

The low-energy sector of QCD has been successfully described over recent years within the chiral Lagrangian framework. Chiral perturbation theory (ChPT) is based on the spontaneous breaking of chiral symmetry $SU_L(N_f) \times SU_R(N_f) \rightarrow SU_V(N_f)$ with $N_f = 2, 3$ light flavors and provides a consistent, systematic, and model-independent scheme to calculate low-energy observables [1–3]. The effective ChPT Lagrangian is constructed as an expansion of the form $\mathcal{L} = \mathcal{L}_{p^2} + \mathcal{L}_{p^4} + \dots$, where p denotes a meson energy scale compared to the chiral scale $\Lambda_\chi \sim 1$ GeV. For the $N_f = 3$ case, the vector group symmetry is broken by the strange-light quark mass difference $m_s - m_{u,d}$, although m_s can still be considered as a perturbation compared to Λ_χ , leading to $SU(3)$ ChPT, which reduces formally to $SU(2)$ in the $m_s \rightarrow \infty$ limit [3]. The formalism can also be extended to finite temperature T , in order to describe meson gases and their evolution toward chiral symmetry restoration for T below the critical temperature T_c [4,5], where $T_c \simeq 180\text{--}200$ MeV from lattice simulations [6–9]. The use of ChPT in this context is important in order to provide model-independent results for the evolution of the different observables with T , supporting the original predictions for chiral restoration [10]. The latter are confirmed by lattice simulations, which are consistent with a crossover-like transition for $N_f = 3$

($2 + 1$ flavors in the physical case), which becomes of second order for $N_f = 2$, in the $O(4)$ universality class, and first order in the degenerate case of three equal flavors.

The $SU_V(2)$ vector group is the isospin symmetry, which is a very good approximation to nature. However, there are several examples where isospin-breaking corrections are phenomenologically relevant, such as sum rules for quark condensates [3], meson masses [11], or pion scattering [12,13]. For a recent review see [14]. The two possible sources of isospin breaking are the QCD $m_d - m_u$ light quark mass difference and electromagnetic interactions. Both can be accommodated within the ChPT framework. The expected corrections from the first source are of order $(m_d - m_u)/m_s$ and are encoded in the quark mass matrix, generating also a $\pi^0\eta$ mixing term in the $SU(3)$ Lagrangian [3]. The electromagnetic interactions are included in the ChPT effective Lagrangian via the external source method and give rise to new terms [11–13,15] of order \mathcal{L}_{e^2} , $\mathcal{L}_{e^2 p^2}$ and so on, with e the electric charge. These terms are easily incorporated into the ChPT power counting scheme by considering formally $e^2 = \mathcal{O}(p^2/F^2)$, with F the pion decay constant in the chiral limit.

The purpose of this paper is to study within ChPT isospin-breaking effects related to the thermodynamics of the meson gas. We will be particularly interested in the physical quantities directly related to spontaneous chiral symmetry breaking and its restoration, namely, the quark condensates and their corresponding susceptibilities at finite temperature. The quark condensate is the order parameter of chiral restoration, but since the transition is

^{*}gomez@fis.ucm.es

[†]rtandres@fis.ucm.es

a smooth crossover for the physical case, different observables can yield different transition temperatures. Thus, the susceptibilities, defined as derivatives of the condensates with respect to the quark masses, provide also direct information about the transition and its nature, since they tend to peak around the transition point reflecting the growth of correlations.

Let us mention some of the motivations we have in mind for the present analysis. For the physical values of quark and meson masses, we are interested in the effect of the isospin-breaking terms in the light quark condensate $\langle \bar{u}u + \bar{d}d \rangle$ and therefore on the ChPT estimates of the critical temperature. In addition, in the isospin asymmetric case, one has $\langle \bar{u}u \rangle \neq \langle \bar{d}d \rangle$ and in fact $\langle \bar{u}u - \bar{d}d \rangle$ can be considered an order parameter for isospin breaking. Actually, isospin is not spontaneously broken in QCD [16], which means that this order parameter should vanish for $m_u = m_d$ and $e = 0$. This is an important difference with the scalar condensate $\langle \bar{u}u + \bar{d}d \rangle$, which is nonzero in the chiral limit. It is relevant to estimate the thermal evolution of $\langle \bar{u}u - \bar{d}d \rangle$, since in principle the two condensates melt at different critical temperatures. A further motivation is the analysis of the three independent susceptibilities, directly related to the isosinglet, connected (isotriplet), and disconnected susceptibilities [17] often discussed in lattice analysis [18–21]. Including properly the $m_u - m_d$ dependence of condensates is then essential to analyze the temperature and mass evolution of the connected and disconnected pieces measured in the lattice. In particular, the linear $m_d - m_u$ corrections to condensates survive the $m_u = m_d$ limit in the susceptibilities. The contributions coming from $\pi^0 \eta$ mixing in the $SU(3)$ case belong to this type and are particularly important regarding the temperature dependence. This is not only interesting for physical masses but also to explore the scaling near the chiral limit, which in lattice studies has been used to investigate the nature of the transition [20]. In the lattice works, this scaling may be contaminated by lattice artifacts such as taste breaking in the staggered fermion formalism, which can generate contributions to susceptibilities masking the true scaling behavior [20,21]. Our study provides then a model-independent setup for disentangling these effects and establishes the expected results in the continuum limit.

We will work in ChPT to one loop, considering on the same footing the two sources of isospin breaking. In a previous work [22] we have studied the quark condensates at $T = 0$ and several related phenomenological aspects of the isospin asymmetric case. We will refer to that work for more details about the formalism, the numerical values of the low-energy constants, and other related issues.

The paper is organized as follows. In Sec. II we will review the main aspects of the isospin-breaking ChPT formalism related to the present work. Our results for the quark condensates at finite T both in the $SU(2)$ and $SU(3)$

cases are given and analyzed in Sec. III. In that section we explore the temperature dependence of isospin breaking, as well as that of the sum rule relating condensate ratios. Section IV is devoted to the analysis of the different isospin-breaking scalar susceptibilities and their relation to the connected and disconnected ones. In Sec. IVA we provide an interesting sum rule relating the electromagnetic differences in the condensates with the total susceptibility. We explore the possibility of using that sum rule to estimate the errors in the staggered fermion lattice analysis of the condensates, in connection with the taste-breaking effect. In Sec. IV B we make a thorough study of the connected and disconnected contributions to the susceptibility and their dependence with temperature, the quark mass, and the isospin ratio m_u/m_d . We pay special attention to the connection of our results with different lattice analysis in the literature.

II. FORMALISM

The effective chiral Lagrangian up to fourth order in p (a meson mass, momentum, temperature, or derivative) including electromagnetic interactions proportional to e^2 is given schematically by $\mathcal{L}_{\text{eff}} = \mathcal{L}_{p^2+e^2} + \mathcal{L}_{p^4+e^2p^2+e^4}$. The most general second-order Lagrangian is the familiar nonlinear sigma model, including the gauge coupling of mesons to the electromagnetic field through the covariant derivative, plus an additional term proportional to a low-energy constant C compatible with the $e \neq 0$ symmetries of the QCD Lagrangian [11,15],

$$\mathcal{L}_{p^2+e^2} = \frac{F^2}{4} \text{tr}[D_\mu U^\dagger D^\mu U + 2B_0 \mathcal{M}(U + U^\dagger)] + C \text{tr}[QUQU^\dagger]. \quad (1)$$

Here, $U(x) = \exp[i\Phi/F] \in SU(N_f)$, with Φ the Goldstone boson (GB) matrix field for pions ($N_f = 2$) plus kaons and η ($N_f = 3$), the latter being the octet member with $I_3 = S = 0$. The covariant derivative is $D_\mu = \partial_\mu + iA_\mu[Q, \cdot]$ with A_μ the electromagnetic (EM) field. \mathcal{M} and Q are the quark mass and charge matrices, i.e., in $SU(3)$ $\mathcal{M} = \text{diag}(m_u, m_d, m_s)$ and $Q = (e/3)\text{diag}(2, -1, -1)$. Both the mass term and the charge one proportional to C in (1) break explicitly the chiral symmetry $SU_L(N_f) \times SU_R(N_f)$ under which $U \rightarrow LUR^\dagger$ with $L, R \in SU(N_f)$. The vector symmetry $L = R$ is also broken for unequal quark masses and charges. Thus, in the light sector (u, d) the part of the mass term proportional to $\hat{m} = (m_u + m_d)/2$, the average light quark mass, is also proportional to the identity flavor matrix and therefore invariant under $SU_V(2)$, while the part proportional to the mass difference $m_\delta = (m_u - m_d)/2$ and T_3 , the third isospin generator, is the one carrying out the QCD isospin breaking. The only remaining symmetry of the Lagrangian (1) is the $U(1)$ $L = R = \exp(i\lambda Q)$ corresponding to charge conservation.

Working out the kinetic terms in (1) allows one to relate the low-energy parameters F , $B_0 m_{u,d,s}$, C to the leading-order tree-level values for the decay constants and masses of the pseudo-Goldstone bosons. For $SU(2)$ the masses read

$$M_{\pi^+}^2 = M_{\pi^-}^2 = 2\hat{m}B_0 + 2C\frac{e^2}{F^2}, \quad M_{\pi^0}^2 = 2\hat{m}B_0. \quad (2)$$

In the $SU(3)$ case, the mass term in (1) induces a mixing between the π^0 and the η fields given by $\mathcal{L}_{\text{mix}} = (B_0/\sqrt{3})(m_d - m_u)\pi^0\eta$. This mixing between the two states with $I_3 = S = 0$ will play an important role in what follows. The kinetic term has then to be brought to the canonical form before identifying the GB masses, which can be easily done by the field rotation [3]

$$\pi^0 = \bar{\pi}^0 \cos \varepsilon - \bar{\eta} \sin \varepsilon, \quad \eta = \bar{\pi}^0 \sin \varepsilon + \bar{\eta} \cos \varepsilon, \quad (3)$$

where the mixing angle is given by

$$\tan 2\varepsilon = \frac{\sqrt{3}}{2} \frac{m_d - m_u}{m_s - \hat{m}}. \quad (4)$$

Once the above $\pi^0\eta$ rotation is performed, the $SU(3)$ tree-level meson masses to leading order read

$$\begin{aligned} M_{\pi^+}^2 &= M_{\pi^-}^2 = 2\hat{m}B_0 + 2C\frac{e^2}{F^2}, \\ M_{\pi^0}^2 &= 2B_0 \left[\hat{m} - \frac{2}{3}(m_s - \hat{m}) \frac{\sin^2 \varepsilon}{\cos 2\varepsilon} \right], \\ M_{K^+}^2 &= M_{K^-}^2 = (m_s + m_u)B_0 + 2C\frac{e^2}{F^2}, \\ M_{K^0}^2 &= (m_s + m_d)B_0, \\ M_{\eta}^2 &= 2B_0 \left[\frac{1}{3}(\hat{m} + 2m_s) + \frac{2}{3}(m_s - \hat{m}) \frac{\sin^2 \varepsilon}{\cos 2\varepsilon} \right]. \end{aligned} \quad (5)$$

For pions, the main effect in the $\pi^0 - \pi^+$ mass difference comes from the EM contribution [23], while in the kaon and eta cases the violations of Dashen's theorem $M_{K^\pm}^2 - M_{K^0}^2 = M_{\pi^\pm}^2 - M_{\pi^0}^2$ [24] ($m_u = m_d$ limit) indicate that $m_u - m_d$ corrections are relevant and must be kept on the same footing as the EM ones [11,25]. We emphasize that all the previous expressions hold for tree-level leading-order masses M_a^2 with $a = \pi^\pm, \pi^0, K^\pm, \eta$, in terms of which we will express all our results. They coincide with the physical masses to leading order in ChPT, i.e., $M_{a,\text{phys}}^2 = M_a^2(1 + \mathcal{O}(M^2))$ and so on for the meson decay constants $F_a^2 = F^2(1 + \mathcal{O}(M^2))$.

The fourth-order Lagrangian consists of all possible terms compatible with the QCD symmetries to that order, including the EM ones. The \mathcal{L}_{p^4} Lagrangian is given in [2] for the $SU(2)$ case, $h_{1,2,3}$ (contact terms) and $l_{1\dots 7}$ denoting the dimensionless low-energy constants (LEC) multiplying each independent term, and in [3] for $SU(3)$, the LEC named $H_{1,2}$ and $L_{1\dots 10}$. The electromagnetic $\mathcal{L}_{e^2 p^2}$ and \mathcal{L}_{e^4} for $SU(2)$ are given in [12,13], $k_{1\dots 13}$ denoting the corresponding LEC, and in [11] for $SU(3)$ with the $K_{1\dots 17}$

LEC. The relevant terms needed for this work are given in [22].

The LEC are renormalized in such a way that they absorb all the one-loop ultraviolet divergences coming from \mathcal{L}_{p^2} and \mathcal{L}_{e^2} , according to the ChPT counting, rendering the observables finite and scale-independent. The numerical values of the LEC at a given scale can be fitted to meson experimental data, except the contact h_i and H_i . The latter are needed for renormalization but cannot be directly measured, reflecting an ambiguity in the observables depending on them. The origin of this ambiguity is in the very same definition of the condensates in perturbation theory [2]. It is therefore convenient to define suitable combinations which are independent of those constants and therefore can be determined numerically. We will bear this in mind throughout this work and we will try to provide such combinations when isospin breaking is included. The numerical values we will use for masses and low-energy constants in the $SU(3)$ case are the same as in [22] unless otherwise stated. In $SU(3)$ they come from the fits performed in [26].

III. QUARK CONDENSATES AT FINITE TEMPERATURE

The quark condensates for a given flavor q_i at finite temperature T are given by

$$\langle \bar{q}_i q_i \rangle_T = -\frac{1}{\beta V} \frac{\partial}{\partial m_i} \log Z = \left\langle \frac{\partial \mathcal{L}_{\text{eff}}}{\partial m_i} \right\rangle_T, \quad (6)$$

where $\beta = 1/T$, V is the system volume, Z the partition function, and $\langle \cdot \rangle_T$ denotes a thermal average. We will denote by $\langle \bar{q}q \rangle_T = \langle \bar{u}u + \bar{d}d \rangle_T = -\frac{1}{\beta V} \frac{\partial}{\partial \bar{m}} \log Z$ the order parameter of chiral symmetry, while $\langle \bar{u}u - \bar{d}d \rangle_T = -\frac{1}{\beta V} \frac{\partial}{\partial m_8} \log Z$ behaves as an order parameter of isospin breaking, since it is the expectation value of the part of the mass term in the QCD Lagrangian proportional to $m_u - m_d$ and $e(q_u - q_d)$, respectively. It is still invariant under transformations in the third direction of isospin, which reflects electric charge conservation.

In ChPT to one loop we obtain then the $SU(2)$ finite temperature extension of the $T = 0$ results in [22], which we give also for consistency,

$$\begin{aligned} \langle \bar{q}q \rangle_T &\equiv \langle \bar{u}u + \bar{d}d \rangle_T \\ &= \langle \bar{q}q \rangle_0 + B_0 [g_1(M_{\pi^0}, T) + 2g_1(M_{\pi^\pm}, T)] + \mathcal{O}(p^2), \\ \langle \bar{q}q \rangle_0 &= -2F^2 B_0 \left[1 - \mu_{\pi^0} - 2\mu_{\pi^\pm} + 2\frac{M_{\pi^0}^2}{F^2} (l_3^r(\mu) \right. \\ &\quad \left. + h_1^r(\mu)) + e^2 \mathcal{K}_2^r(\mu) + \mathcal{O}(p^4) \right], \end{aligned} \quad (7)$$

$$\begin{aligned} \langle \bar{u}u - \bar{d}d \rangle_T &= \langle \bar{u}u - \bar{d}d \rangle_0 \\ &= 4B_0^2(m_d - m_u)h_3 - \frac{8}{3}F^2 B_0 e^2 k_7 + \mathcal{O}(p^2), \end{aligned} \quad (8)$$

where

$$\mathcal{K}_2^r(\mu) = \frac{4}{9}[5(k_5^r(\mu) + k_6^r(\mu)) + k_7], \quad (9)$$

and

$$\mu_i = \frac{M_i^2}{32\pi^2 F^2} \log \frac{M_i^2}{\mu^2},$$

$$g_1(M, T) = \frac{1}{2\pi^2} \int_0^\infty dp \frac{p^2}{E_p} \frac{1}{e^{\beta E_p} - 1}, \quad (10)$$

with $E_p^2 = p^2 + M^2$.

The expression (7) contains the leading-order tree-level term from \mathcal{L}_2 given by $\langle \bar{q}q \rangle_0 = -2F^2 B_0$, the one-loop tadpolelike contribution $G_i(x=0)$, with G the free meson thermal propagator, whose finite part yields the combinations $\mu_i + g_1(M_i, T)/(2F^2)$ (we follow the same finite- T notation as in [5]) and the tree level from the fourth-order Lagrangian. The latter shows up only at $T=0$, which contains the LEC renormalized at the scale μ of dimensional regularization in the $\overline{\text{MS}}$ scheme [2,13] so that the full expressions for the condensates are finite and scale-independent. Note that $\langle \bar{q}q \rangle_0$ includes the contact term h_1^r . The isospin breaking in $\langle \bar{q}q \rangle_T$ for $SU(2)$ is purely electromagnetic, showing up explicitly in the e^2 terms and implicitly through the pion mass differences. The temperature dependence is encoded in the functions $g_1(M, T)$ which increase with T and behave near the chiral limit ($T \gg M$) as $g_1(M, T) = \frac{T^2}{12} [1 + \mathcal{O}(M/T)]$.

Note that the effect of the electromagnetic corrections is to decrease the thermal part of $\langle \bar{q}q \rangle_T$, since $M_{\pi^\pm} > M_{\pi^0}$. On the other hand, $\langle \bar{q}q \rangle_0$ increases for the available estimates of the EM LEC, reflecting its ferromagnetic nature [22]. Our first conclusion is then that the critical temperature, estimated as that for which the condensate vanishes, increases with respect to the $e=0$ case, which is also a

ferromagnetic-like behavior induced by the explicit chiral symmetry breaking of the EM quark coupling in the QCD action. A simple estimate of the size of this effect can be obtained by taking the chiral limit $m_u = m_d = 0$ so that $M_{\pi^0} = \mu_{\pi^0} = 0$, $M_{\pi^\pm}^2 = 2Ce^2/F^2$, and $T_c = \sqrt{8F} \sqrt{1 + e^2 \mathcal{K}_2^r - 2\mu_{\pi^\pm}}$, which gives $T_c^{e \neq 0}/T_c^{e=0} \simeq 1.003$ with the parameters used in [22] and setting the involved k_i to their maximum expected “natural” values $k_i = 1/(16\pi^2)$. Thus, in principle we expect rather small corrections to chiral restoration from the electromagnetic breaking. Nevertheless, in Sec. IV A we will go back to this point in connection with a sum rule relating the charge breaking with the susceptibility, suggesting larger corrections either for higher order transitions or for finite lattice spacing.

The two sources of explicit isospin breaking in the Lagrangian show up in the condensate difference (8), which depends linearly on $m_u - m_d$ with the contact h_3 and vanishes for $m_u = m_d$ and $e=0$ in accordance with the absence of spontaneous isospin breaking [16] mentioned in the introduction. Recall that h_3 and k_7 do not need to be renormalized and are therefore finite and scale independent. An important point is that $\langle \bar{u}u - \bar{d}d \rangle$ does not receive pion loop corrections in the two-flavor case and it is therefore temperature-independent to the one-loop order. In other words, isospin breaking in $SU(2)$ does not change with T and the two condensates melt at the same temperature. This picture will change for $N_f = 3$ due to kaon loops and $\pi^0 \eta$ mixing.

In the $SU(3)$ case, we calculate to one loop at finite temperature the light and strange condensates, taking into account both $m_u - m_d$ and $e \neq 0$ corrections. The condensates read now

$$\begin{aligned} \langle \bar{q}q \rangle_T^{SU(3)} &\equiv \langle \bar{u}u + \bar{d}d \rangle_T^{SU(3)} \\ &= \langle \bar{q}q \rangle_0^{SU(3)} + B_0 \left[\frac{1}{3} (3 - \sin^2 \varepsilon) g_1(M_{\pi^0}, T) + 2g_1(M_{\pi^\pm}, T) + g_1(M_{K^0}, T) \right. \\ &\quad \left. + g_1(M_{K^\pm}, T) + \frac{1}{3} (1 + \sin^2 \varepsilon) g_1(M_\eta, T) \right] + \mathcal{O}(p^2), \\ \langle \bar{q}q \rangle_0^{SU(3)} &= -2F^2 B_0 \left\{ 1 + \frac{8B_0}{F^2} [\hat{m}(2L_8^r(\mu) + H_2^r(\mu)) + 4(2\hat{m} + m_s)L_6^r(\mu)] + e^2 \mathcal{K}_{3+}^r(\mu) \right. \\ &\quad \left. - \frac{1}{3} (3 - \sin^2 \varepsilon) \mu_{\pi^0} - 2\mu_{\pi^\pm} - \mu_{K^0} - \mu_{K^\pm} - \frac{1}{3} (1 + \sin^2 \varepsilon) \mu_\eta + \mathcal{O}(p^4) \right\}, \end{aligned} \quad (11)$$

$$\begin{aligned} \langle \bar{u}u - \bar{d}d \rangle_T^{SU(3)} &= \langle \bar{u}u - \bar{d}d \rangle_0^{SU(3)} + B_0 \left\{ \frac{\sin 2\varepsilon}{\sqrt{3}} [g_1(M_{\pi^0}, T) - g_1(M_\eta, T)] + g_1(M_{K^\pm}, T) - g_1(M_{K^0}, T) \right\} + \mathcal{O}(p^2), \\ \langle \bar{u}u - \bar{d}d \rangle_0^{SU(3)} &= 2F^2 B_0 \left\{ \frac{4B_0}{F^2} (m_d - m_u)(2L_8^r(\mu) + H_2^r(\mu)) - e^2 \mathcal{K}_{3-}^r(\mu) + \frac{\sin 2\varepsilon}{\sqrt{3}} [\mu_{\pi^0} - \mu_\eta] + \mu_{K^\pm} - \mu_{K^0} \right\} + \mathcal{O}(p^2), \end{aligned} \quad (12)$$

$$\begin{aligned}
\langle \bar{s}s \rangle_T &= \langle \bar{s}s \rangle_0 + B_0 \left\{ \frac{2}{3} [g_1(M_{\pi^0}, T) \sin^2 \varepsilon + g_1(M_\eta, T) \cos^2 \varepsilon] + g_1(M_{K^\pm}, T) + g_1(M_{K^0}, T) \right\} + \mathcal{O}(p^2), \\
\langle \bar{s}s \rangle_0 &= -F^2 B_0 \left\{ 1 + \frac{8B_0}{F^2} [m_s(2L_8^r(\mu) + H_2^r(\mu)) + 4(2\hat{m} + m_s)L_6^r(\mu)] + e^2 \mathcal{K}_s^r(\mu) \right. \\
&\quad \left. - \frac{4}{3} [\mu_{\pi^0} \sin^2 \varepsilon + \mu_\eta \cos^2 \varepsilon] - 2[\mu_{K^\pm} + \mu_{K^0}] + \mathcal{O}(p^4) \right\}, \tag{13}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{K}_{3+}^r(\mu) &= \frac{4}{9} [6(K_7 + K_8^r(\mu)) + 5(K_9^r(\mu) + K_{10}^r(\mu))], \\
\mathcal{K}_{3-}^r(\mu) &= \frac{4}{3} [K_9^r(\mu) + K_{10}^r(\mu)], \\
\mathcal{K}_s^r(\mu) &= \frac{8}{9} [3(K_7 + K_8^r(\mu)) + K_9^r(\mu) + K_{10}^r(\mu)]. \tag{14}
\end{aligned}$$

In some of the above terms we have preferred to leave the results in terms of quark instead of meson masses. As in the $SU(2)$ case, the results are finite and scale-independent, which concerns only the $T = 0$ part [22].

There are some important differences with respect to the $N_f = 2$ case which deserve to be mentioned. First, the presence of the $\pi^0\eta$ mixing angle ε (4) and the more complicated dependence of meson masses with quark masses (5), imply that now $m_u - m_d$ corrections show up in $\langle \bar{q}q \rangle$, apart from the EM ones. Note also that these corrections in $\langle \bar{q}q \rangle$ and $\langle \bar{s}s \rangle$ are at least $\mathcal{O}(\varepsilon^2)$ in the mixing angle, or equivalently in $m_u - m_d$, except for an $\mathcal{O}(e^2\varepsilon)$ term in the kaon contribution. This is so because, apart from the explicit ε dependence, one has to expand also the meson masses in (5) around $\varepsilon = 0$. All the masses depend quadratically on ε except $M_{K^\pm}^2 \sim -a\varepsilon$, $M_{K^0}^2 \sim a\varepsilon$ with $a = (2B_0/\sqrt{3})(m_s - \hat{m})$. Since, in addition, $M_{K^\pm}^2 = M_{K^0}^2 + 2Ce^2/F^2$ for $\varepsilon = 0$, we end up with the above-mentioned term.

Another important difference between the two cases is that for $SU(3)$ there are loop contributions to $\langle \bar{u}u - \bar{d}d \rangle_T$ in (12). Kaon loops arise from the charged-neutral kaon mass difference, while neutral pion and eta ones from $\pi^0\eta$ mixing. When expanding in ε now, the leading order is $\mathcal{O}(\varepsilon)$ even for $e = 0$. These linear terms will be crucial for our analysis of susceptibilities in Sec. IV. Those loop corrections introduce now a T dependence in $\langle \bar{u}u - \bar{d}d \rangle_T$, unlike the $SU(2)$ case. As it happened in the $SU(2)$ case, we see that $\langle \bar{u}u - \bar{d}d \rangle_T$ in (12) vanishes for e^2 and $m_u = m_d$, in agreement with [16], which still holds including thermal corrections.

At low and moderate temperatures $g_1(M_{\pi^0}, T)$ dominates over the kaon and eta contributions in (12), but it should be remembered that ε in (4) brings up a $1/m_s$ dependence which reduces the size of the pion term. In order to make a crude estimate, let us consider again the chiral limit, but keeping now the leading order in $m_u - m_d$, which we take then very small but nonzero while taking

$\hat{m} \rightarrow 0^+$. In this limit the kaon masses are roughly kept to their physical values, which are well above the critical temperature. Thus, we consider the regime $M_\pi \ll T \ll M_K$, in which the pion term behaves as $B_0(m_d - m_u)T^2/(24m_s) = B_0^2(m_d - m_u)T^2/(18M_\eta^2)$. The kaon and eta contributions go like $B_0T^2[(m_d - m_u)/m_s] \times \sqrt{M_{K,\eta}/T} e^{-M_{K,\eta}/T}$ [5], where we have taken also $e = 0$ for simplicity. The pion term is still dominant due to the exponential suppression of K, η . However, when compared to the $T = 0$ part in that regime, which goes like $(m_d - m_u)B_0^2$, we see that the quadratic growth with temperature is controlled by the scale M_η^2 instead of, say, the chiral restoring behavior of $\langle \bar{q}q \rangle_T$ which is controlled by F^2 in the chiral limit. Therefore, the order parameter for isospin breaking $\langle \bar{u}u - \bar{d}d \rangle_T$ grows with T , although it does so rather softly. Therefore, we do not expect big differences in the melting temperatures of the u and d condensates. This is also consistent with the expectation that in the limit where m_s is arbitrarily large, say compared to \hat{m} , the $SU(2)$ result should be recovered, for which there is no temperature dependence for the condensate difference.

The evolution with temperature of the condensate difference is shown in Fig. 1 for the full case of finite pion mass and both $e \neq 0$ and $m_u \neq m_d$. We have used the same set of low-energy constants and parameters as in [22], in particular $m_u/m_d = 0.46$ and $m_s/\hat{m} = 24$. For the EM LEC K_i involved, we have displayed in the figure the two curves corresponding to their maximum and minimum expected natural values. We also show for comparison the result for $m_u = m_d$, which shows that the charge contribution is actually of the same order as the one proportional to $m_u - m_d$. We see that the T -dependent amplification of the isospin difference is rather large. In fact, this order parameter reaches values comparable to its $T = 0$ value near the critical temperature, which is about $T_c \simeq 265$ MeV in $SU(3)$ ChPT. Nevertheless, due to the additional ε suppressing factor discussed above, this enhancement is not enough to produce a sizable difference in the melting temperature of the u, d condensates, as it is clearly seen in Fig. 1 (right), where we plot the two thermal condensates separately. The two plots shown in Fig. 1 correspond then, respectively, to the two order parameters involved here: isospin breaking and chiral restoration. In turn, note that the curves on the right plot are independent of the choice of LEC since to this order $\langle \bar{q}_i q_i \rangle_T / \langle \bar{q}_i q_i \rangle_0 = 1 - (\langle \bar{q}_i q_i \rangle_T - \langle \bar{q}_i q_i \rangle_0) / (B_0 F^2) + \mathcal{O}(p^4)$ for $i = u, d$.

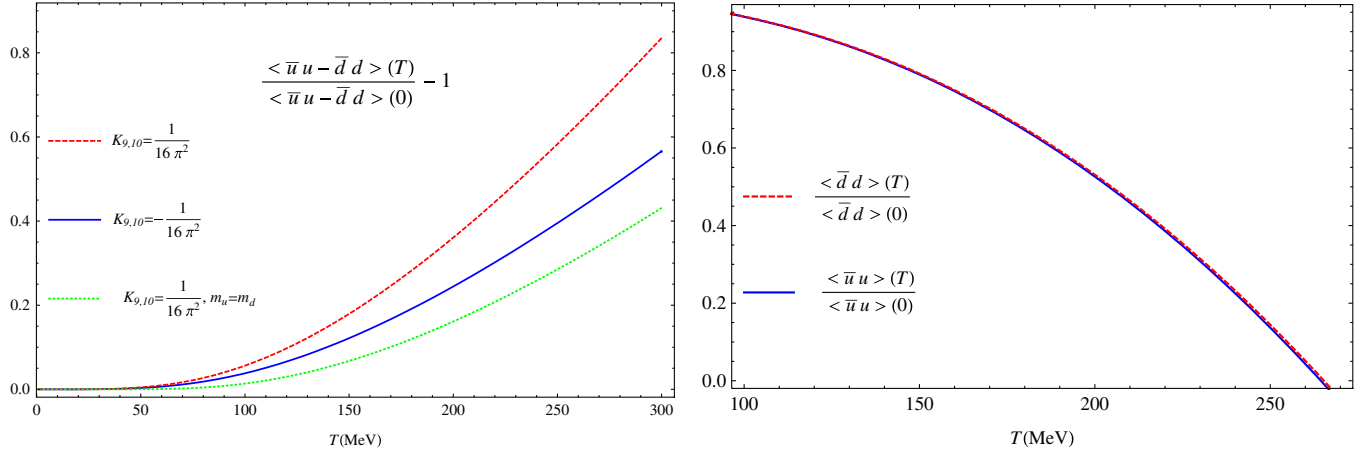


FIG. 1 (color online). Left: the $u - d$ condensate difference (isospin-breaking order parameter) at finite temperature in $SU(3)$, relative to its $T = 0$ value. Right: the two condensates separately.

The individual condensates in (11)–(13) contain the contact terms H_2 . These terms reflect an ambiguity in the quark condensates, inherent to their renormalization in QCD. It is therefore very important to deal with combinations of condensates which are free of this ambiguity. This very same source of ambiguity is also present in lattice simulations at finite T . A simple way to get rid of it is to subtract the $T = 0$ contribution. This is the approach followed by the authors of Ref. [8] both for condensates and for susceptibilities. A different possibility is to consider the combination $\langle \bar{q}q \rangle - (\hat{m})/m_s \langle \bar{s}s \rangle$ [9], or for individual condensates in the isospin-breaking case, $\langle \bar{q}_i q_i \rangle - (m_i/m_s) \times \langle \bar{s}s \rangle$ with $i = u, d$. Another sum rule free of contact ambiguities often used in $T = 0$ phenomenology to relate condensate ratios [3] is the following combination:

$$\begin{aligned} \Delta_{\text{SR}}(T) &\equiv \frac{\langle \bar{d}d \rangle_T}{\langle \bar{u}u \rangle_T} - 1 + \frac{m_d - m_u}{m_s - \hat{m}} \left[1 - \frac{\langle \bar{s}s \rangle_T}{\langle \bar{u}u \rangle_T} \right] \\ &= \Delta_{\text{SR}}(0) + \frac{m_d - m_u}{m_s - \hat{m}} \frac{1}{F^2} [g_1(M_K, T) - g_1(M_\pi, T) \\ &\quad + (M_K^2 - M_\pi^2)g_2(M_K, T)] - \frac{2Ce^2}{F^4} g_2(M_K, T), \end{aligned} \quad (15)$$

where $\mathcal{O}(m_u - m_d)^2$, $\mathcal{O}(e^4)$, $\mathcal{O}(e^2(m_u - m_d)^2)$ have been neglected and $\Delta_{\text{SR}}(0)$ is given in [22] with both sources of isospin breaking contributing at the same order, not only in the chiral counting but also numerically.

We have seen in Sec. III that the $\langle \bar{d}d \rangle_T / \langle \bar{u}u \rangle_T$ ratio receives significant corrections at finite temperature. On the other hand, we expect the strange condensate to vary slowly with T , from chiral symmetry breaking due to the strange quark mass. Therefore, we expect that the thermal corrections to this sum rule are also sizable. These corrections are plotted in Fig. 2 for $K_9^r + K_{10}^r = 1/(8\pi^2)$. They become comparable to the $T = 0$ sum rule near the critical temperature.

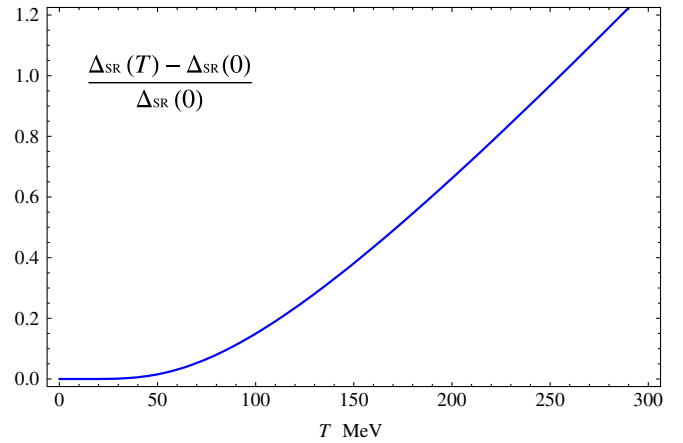


FIG. 2 (color online). Thermal corrections to the sum rule relating condensate ratios.

IV. SCALAR SUSCEPTIBILITIES AND ISOSPIN BREAKING

In the isospin-breaking case, the scalar susceptibilities are defined as

$$\begin{aligned} \chi_{ij} &= -\frac{\partial}{\partial m_i} \langle \bar{q}_j q_j \rangle_T = \frac{1}{\beta V} \frac{\partial^2}{\partial m_i \partial m_j} \log Z \\ &= \int_0^\beta d\tau \int d^3 \vec{x} \langle (\bar{q}_i q_i)(\vec{x}, \tau) (\bar{q}_j q_j)(0, 0) \rangle_T \\ &\quad - \beta V \langle \bar{q}_i q_i \rangle_T \langle \bar{q}_j q_j \rangle_T, \end{aligned} \quad (16)$$

$i, j = u, d, s,$

so that in the light sector, the relevant one concerning chiral restoration, we have three independent scalar susceptibilities χ_{uu} , χ_{dd} , and $\chi_{ud} = \chi_{du}$.

At this point, it is instructive to recall the definition of the connected and disconnected parts of the susceptibility. Consider the isospin limit with two light identical flavors

of mass $\hat{m} = m_u = m_d$ and $e = 0$. There is only one light susceptibility in this case, which can be written as

$$\chi = -\frac{\partial}{\partial \hat{m}} \langle \bar{q}q \rangle_T = \frac{1}{\beta V} \frac{\partial^2}{\partial \hat{m}^2} \log Z = 4\chi_{\text{dis}} + 2\chi_{\text{con}}, \quad (17)$$

with

$$\chi_{\text{dis}} = \langle (\text{Tr} D_l^{-1})^2 \rangle_A - \langle \text{Tr} D_l^{-1} \rangle_A^2, \quad (18)$$

$$\chi_{\text{con}} = -\langle \text{Tr} D_l^{-2} \rangle_A, \quad (19)$$

where $D_l = i\partial - \hat{m}$ is the Dirac operator for every light flavor in the QCD Lagrangian and $\langle \cdot \rangle_A$ denotes integration over the gluon fields, so that formally $Z = \langle \exp \sum_j \text{Tr} \log D_j \rangle_A$, where j runs over flavor and Tr runs over the space-time, Dirac, and color indices. This separation is important for lattice analysis, as we will discuss below, and reflects the contributions with connected and disconnected quark lines, since D_l^{-1} is the quark propagator. However, when considering the low-energy representation for the partition function and the susceptibilities in terms of GB fields, it is not so simple to separate the connected and disconnected parts if we take the isospin limit from the very beginning. A possible approach to perform such separation is to work within the partially quenched ChPT framework, as discussed in [27] for the vacuum polarization. We will however work within the isospin-breaking scenario we are considering here, which is very useful for this purpose, as noted first in [17] for the susceptibilities and used also in [28] for the vacuum polarization. The main point is that for $m_u \neq m_d$

$$\chi_{ud} = \langle (\text{Tr} D_u^{-1})(\text{Tr} D_d^{-1}) \rangle_A - \langle \text{Tr} D_u^{-1} \rangle_A \langle \text{Tr} D_d^{-1} \rangle_A, \quad (20)$$

so that one has $\chi_{\text{dis}} = \lim_{m_u \rightarrow m_d} \chi_{ud}$ and since $\partial_{\hat{m}} = \partial_{m_u} + \partial_{m_d}$, from (17) we have also $\chi_{\text{con}} = \lim_{m_u \rightarrow m_d} [(\chi_{uu} + \chi_{dd})/2 - \chi_{ud}]$.

Therefore, with this observation in mind, we define in the isospin-breaking regime the following basis of total, connected, and disconnected susceptibilities in terms of the ij basis in (16):

$$\chi = \chi_{uu} + \chi_{dd} + 2\chi_{ud}, \quad (21)$$

$$\chi_{\text{con}} = \frac{1}{2}(\chi_{uu} + \chi_{dd}) - \chi_{ud}, \quad (22)$$

$$\chi_{\text{dis}} = \chi_{ud}, \quad (23)$$

which we can therefore obtain directly from our expressions for the isospin-breaking condensates obtained in the previous section. Observe that none of the K_i -dependent terms in the condensates depends on the quark masses and therefore the susceptibilities are independent of the EM LEC.

Note that according to (16), χ in (21) corresponds to the correlator of the isosinglet condensate $\langle \bar{q}q \rangle$, the order parameter of chiral restoration, while the connected

contribution χ_{con} is the correlator of the isotriplet $\bar{u}u - \bar{d}d$, the order parameter for isospin symmetry. A divergence or sudden growth of these susceptibilities would indicate then a phase transition for the corresponding order parameter.

We also remark that the definitions of the connected and disconnected parts in terms of uu , dd , ud ones are not unique. We could as well have defined χ_{dis} as $\alpha(\chi_{uu} - \chi_{dd}) + \chi_{ud}$ for arbitrary α , which also reduces to the combination (18) in the isospin limit. We are following the same convention as [17]. These formulas can be easily extended to N_f identical flavors, for which $\chi = N_f \chi_{\text{con}} + N_f^2 \chi_{\text{dis}}$.

In the following we will analyze several aspects related to the above defined susceptibilities in different limits.

A. Sum rule for EM-like corrections to condensates

Before studying in detail the different susceptibilities, in this subsection we will relate the EM corrections (and actually any chargelike correction to pion masses) to the condensates, found in Sec. III, with the total scalar susceptibility. Consider first the condensate calculated in $SU(2)$ in (7) and let us define the ratio

$$r(T) \equiv \frac{\langle \bar{q}q \rangle_T^{e \neq 0}}{\langle \bar{q}q \rangle_T^{e=0}}. \quad (24)$$

Now note that to one loop, the explicit dependence of the condensate in e^2 is only in the $T = 0$ part, since the charge dependence in \mathcal{L}_2 is contained implicitly in the pion mass differences. Therefore, $r(T) - r(0)$ depends on the charge only through the parameter $\delta_\pi \equiv (M_{\pi^\pm}^2 - M_{\pi^0}^2)/M_{\pi^0}^2$, in which we can further expand (for the EM pion mass difference $\delta_\pi \simeq 0.1$). Taking also into account that the condensate is just the sum of the tadpole contributions for the three pions, we can write

$$r(T) - r(0) = -\frac{M_\pi^2}{2B_0 F^2} \delta_\pi \frac{\partial}{\partial M_{\pi^\pm}^2} [\langle \bar{q}q \rangle_T - \langle \bar{q}q \rangle_0] + \mathcal{O}(\delta_\pi^2) + \mathcal{O}(p^4) \quad (25)$$

$$= -\frac{M_\pi^2}{6B_0^2 F^2} \delta_\pi \frac{\partial}{\partial \hat{m}} [\langle \bar{q}q \rangle_T - \langle \bar{q}q \rangle_0] + \mathcal{O}(\delta_\pi^2) + \mathcal{O}(p^4), \quad (26)$$

which, from the susceptibility definition in (21) can be written, to this order, as

$$r(T) - r(0) = \frac{2}{3} \frac{\hat{m}^2}{M_\pi^2 F^2} \delta_\pi [\chi(T) - \chi(0)]. \quad (27)$$

This sum rule relates then pion mass deviations in the condensate with the total scalar susceptibility. Note that the above result is written only in terms of the quark mass, the pion mass and decay constant, and the charged-neutral mass difference, without specifying if the latter is of electromagnetic origin. It states that, even though the mass

deviation δ_π may be small, the corrections to the condensate may be amplified near the phase transition, where the susceptibility is maximum, if such transition is sufficiently strong. Actually, the quantity proportional to δ_π on the right-hand side of (27) is directly measurable on the lattice [8].

For the case of the electromagnetic mass difference in $SU(2)$ discussed in Sec. II, we have $\delta_\pi M_{\pi^0}^2 = 2Ce^2/F^2$ and

$$r(T) - r(0) = \frac{2Ce^2}{F^4} g_2(M_{\pi^0}, T) + \mathcal{O}(e^4), \quad (28)$$

with

$$r(0) = 1 + e^2 \mathcal{K}_2^r(\mu) - \frac{4Ce^2}{F^4} \nu_{\pi^0}, \quad (29)$$

and

$$\begin{aligned} \nu_i &= F^2 \frac{d}{dM_i^2} \mu_i = \frac{1}{32\pi^2} \left[1 + \log \frac{M_i^2}{\mu^2} \right], \\ g_2(M, T) &= -\frac{dg_1(M, T)}{dM^2} = \frac{1}{4\pi^2} \int_0^\infty dp \frac{1}{E_p} \frac{1}{e^{\beta E_p} - 1}. \end{aligned} \quad (30)$$

Note that $r(T)$ is finite, scale-independent, and also independent of the $e = 0$ LEC l_3 , h_1 , h_3 . In particular, it is free of the contact-term ambiguity, which makes it a quantity suitable for physical predictions. It is also independent of B_0 , unlike the individual quark condensates, which have only physical meaning when multiplied by the appropriate quark masses, since the $m_i B_0$ products give meson masses. In addition, the dependence with the EM LEC disappears in the difference $r(T) - r(0)$, which is the quantity directly related to the susceptibility through (27).

The above relation can also be explored for $SU(3)$. However, the connection with the susceptibility is not direct in that case. The e^2 dependence of $\langle \bar{q}q \rangle_T - \langle \bar{q}q \rangle_0$ enters now through M_{π^\pm} and M_{K^\pm} . However, the condensates (11)–(13) depend on the light quark mass through all the meson masses M_π , M_K , M_η . The result is that $r(T) - r(0)$ can be expressed as the susceptibility term in (27) plus a linear combination of $\partial(\langle \bar{s}s \rangle_T - \langle \bar{s}s \rangle_0)/\partial m$ and $\partial(\langle \bar{s}s \rangle_T - \langle \bar{s}s \rangle_0)/\partial m_s$, to this chiral order and neglecting $\mathcal{O}(e^4)$ and $\mathcal{O}(m_u - m_d)$ isospin-breaking corrections in the right-hand side. Since the strange quark condensate has a much weaker dependence on temperature than the light one (or equivalently, we can approximately neglect the thermal functions evaluated on kaon and eta masses) we expect the T behavior of $r(T) - r(0)$ to be dominated by the light scalar susceptibility also in the $SU(3)$ case and therefore the sum rule (27) should hold approximately. In this case we have to one loop

$$r(T)^{SU(3)} - r(0)^{SU(3)}$$

$$= 1 + \frac{Ce^2}{F^4} [2g_2(M_{\pi^\pm}, T) + g_2(M_{K^\pm}, T)] + \mathcal{O}(e^4),$$

$$r(0)^{SU(3)} = 1 + e^2 \mathcal{K}_{3+}^r(\mu) - \frac{2Ce^2}{F^4} [2\nu_{\pi^\pm} + \nu_{K^\pm}] + \mathcal{O}(e^4),$$

where the expansion in e^2 to leading order allows one to express the result in terms of the π^\pm and K^\pm masses. As in $SU(2)$, $r(T)$ is finite, scale-independent, and independent of the $e = 0$ LEC, so that it is free of contact ambiguities.

We compare the above expression with the susceptibility in the $SU(3)$ case. As will become clear in Sec. IV B, the corrections to the total susceptibility χ from both sources of isospin breaking are small. Actually, we will see that the next-to-leading order correction in the QCD breaking is $\mathcal{O}(m_u - m_d)^2$. The one-loop result is

$$\begin{aligned} \frac{\chi(T) - \chi(0)}{B_0^2} &= \frac{4\hat{m}^2}{M_\pi^4} [\chi(T) - \chi(0)] \\ &= 2 \left[3g_2(M_\pi, T) + g_2(M_K, T) + \frac{1}{9}g_2(M_\eta, T) \right] \\ &\quad + \mathcal{O}(p^2) + \mathcal{O}(e^2) + \mathcal{O}\left(\frac{m_d - m_u}{m_s}\right)^2, \end{aligned} \quad (31)$$

$$\begin{aligned} \frac{\chi(0)}{B_0^2} &= \frac{4\hat{m}^2}{M_\pi^4} \chi(0) \\ &= 16[8L_6^r(\mu) + 2L_8^r(\mu) + H_2^r(\mu)] \\ &\quad - 4 \left[3\nu_\pi + \nu_K + \frac{1}{9}\nu_\eta \right] + \mathcal{O}(p^2) + \mathcal{O}(e^2) \\ &\quad + \mathcal{O}\left(\frac{m_d - m_u}{m_s}\right)^2. \end{aligned} \quad (32)$$

The two quantities are compared in Fig. 3. The deviations between them are negligible for the range of relevant temperatures. Therefore, although for physical masses the electromagnetic corrections are relatively small, they grow with the susceptibility, which is a model-independent prediction. For comparison, taking the value of $(\hat{m}^2)/(M_\pi^4)[\chi(T_c) - \chi(0)]$ from the lattice simulations in [8] for 2 + 1 flavors with the lattice T_c value gives $r(T_c) - r(0) \simeq 0.013$, not far from the higher temperature values in Fig. 3, although the ChPT curve cannot reproduce the susceptibility peak, only the low and moderate T behavior. These small EM corrections for the condensate are in accordance with our simple estimates made in Sec. III and translate into a few MeV difference in the determination of the critical temperature from the order parameter.

The sum rule (27) has another interesting consequence, regarding lattice simulations. In the staggered fermion lattice formalism, the need to introduce four different copies (tastes) for every quark flavor leads to the so-called taste violation [8,19,29]. This is a lattice artifact which in some aspects is similar to the isospin or flavor violations

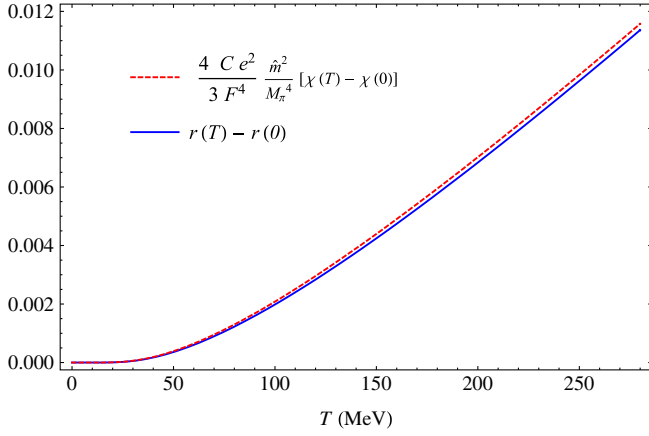


FIG. 3 (color online). The $r(T)$ function in $SU(3)$ encoding the EM corrections of the quark condensate. We compare it with the normalized light susceptibility for physical quark and meson masses. For these values, $r(0) \simeq 1.01$.

we are analyzing here. The new tastes enlarge the chiral symmetry group to $SU(4N_f) \times SU(4N_f)$, producing then 15 pseudo-Goldstone bosons plus one massive state (η' -like) for every quark flavor. All these new meson states become degenerate in the continuum limit, where taking the fourth root of the Dirac fermion determinant is enough to remove all the spurious copies. However, for finite lattice spacing a , the tree-level masses of those states receive $\mathcal{O}(a^2)$ contributions, which break explicitly the chiral group in the Lagrangian, only one Goldstone boson remaining massless in the chiral limit, leaving then a residual $O(2)$ or $U(1)$ symmetry. The mechanism is similar to the electric charge one we are analyzing here, by which the charged states receive $\mathcal{O}(e^2)$ corrections and the $U(1)$ EM symmetry remains. In fact, for the staggered case one can construct a generalized chiral Lagrangian including all possible terms compatible with the new symmetry. This is called staggered chiral perturbation theory [29,30]. Among the new terms one recognizes contributions of the form $\text{Tr}[\xi U \xi U^\dagger]$ with ξ a given combination of $SU(4)$ generators, i.e., like the charge term in (1) in $SU(3)$. Obviously, the staggered case includes additional operators and the spectrum of states is more complicated. However, we can use the sum rule (27) to estimate roughly the expected differences between the lattice staggered condensate and the continuum one, considering the lightest states. For fine-enough lattices, one has $\delta_\pi^a = (M_{\pi,a}^2 - M_\pi^2)/M_\pi^2 \simeq ca^2$ for the lightest tastes of squared mass $M_{\pi,a}^2$ [8], where from the two smallest lattices in [8] we get $c \simeq 140 \text{ fm}^{-2}$. With this δ_π^a we can then use (27) to estimate $r^a(T) - r^0(0)$. Consequently, we expect the larger errors coming from this taste violation effect to appear near T_c . That is indeed the case when we compare lattices of decreasing temporal extent $N_t = a/T$ for the condensate data given in [8]. More quantitatively, taking also the susceptibility values of [8], we get

$r^a(T_c) - r^a(0) \simeq 0.07ca^2$. Estimating the $T = 0$ part using (29) with $e^2 \rightarrow \delta_\pi^a M_\pi^2 F^2 / (2C)$, we get a relative correction for the condensates near T_c with respect to the continuum of about 20% for the $N_t = 12$ data in [8] and about 12% for the $N_t = 16$ ones in [31]. Following the same idea, we get a relative difference between the $N_t = 10$ and $N_t = 12$ lattices of around 8% near T_c , which is actually in good agreement with lattice data [8]. A direct translation into an error for the critical temperature is not easy to obtain. If we simply extrapolate the one-loop chiral limit expression $\langle \bar{q}q \rangle_T = \langle \bar{q}q \rangle_0 (1 - T^2/T_c^2)$, writing the left-hand side of (27) in the one-loop equivalent form (25), we get a very rough estimate $\Delta T_c \simeq 10 \text{ MeV}$ for $N_t = 12$ with respect to the continuum, although the chiral limit is not always numerically accurate, as we will actually see in the next section.

Estimating taste-violation effects is important, since they are one of the main sources of the discrepancies between different lattice groups for the determination of the critical temperature. An important effort has been made over recent years to minimize these effects, not only by considering finer lattices, but also by introducing lattice actions where taste symmetry is reduced [31].

B. Temperature and mass dependence of connected and disconnected susceptibilities. Relation with chiral restoration and lattice analysis

The behavior of condensates and susceptibilities with temperature and quark masses is crucial in order to understand the nature of the chiral phase transition when approaching the chiral region $(m_q, T) \rightarrow (0^+, T_c)$. Lattice simulations have addressed the question of how those quantities scale with m_q and T until very recently [20]. An essential part of this program concerns the scaling of the connected and disconnected parts of the scalar susceptibility. The disconnected piece is given in terms of closed quark lines and is therefore directly related to $\langle \bar{q}q \rangle$ and expected to be sensitive to chiral restoration. Actually, near the chiral limit, i.e., the infrared (IR) behavior, it is known to scale as $\chi_{\text{dis}}^{\text{IR}} \sim \log M_\pi^2$ for $T = 0$ and $\chi_{\text{dis}}^{\text{IR}} \sim T/M_\pi$ at finite temperature [17]. The infrared contribution is then controlled by the GB loop contributions. The situation is not so clear for the connected part, since its infrared divergent piece is proportional to $n_f^2 - 4$, with n_f the number of identical light flavors [17], and therefore it vanishes for $n_f = 2$. However, its IR finite part contributes in the physical case of massive pions and is actually an important difference between QCD and $O(N)$ models, on which the lattice scaling fits are based [20]. Besides, the connected contribution receives important “false” GB-like corrections coming from taste violation [20,21]. It is therefore important for lattice studies to provide the continuum result for the disconnected and connected susceptibilities in the physical case of 2 + 1 flavors and massive pions.

Our present ChPT one-loop analysis allows us to obtain a model-independent prediction for the low-temperature and small-mass behavior of the susceptibilities. The inclusion of isospin-breaking effects is crucial. In fact, it will be useful for the following discussion to note that

$$\begin{aligned}\chi_{\text{con}} &= \frac{1}{2}(\chi_{uu} - \chi_{ud}) + \frac{1}{2}(\chi_{dd} - \chi_{ud}) \\ &= -\frac{1}{2}\partial_{m_\delta}\langle\bar{u}u - \bar{d}d\rangle, \\ \chi_{\text{dis}} &= -\frac{1}{4}[\partial_{\hat{m}}\langle\bar{u}u + \bar{d}d\rangle - 2\partial_{m_\delta}\langle\bar{u}u - \bar{d}d\rangle],\end{aligned}\quad (33)$$

with $m_\delta = (m_u - m_d)/2$, so that χ_{con} comes only from the condensate difference in (12) and its leading order is obtained from the linear terms in $m_u - m_d$.

Now, from our considerations in Sec. III, if we neglect for the moment the charge corrections and we take the quark mass derivatives in (11) we have, to leading order in m_δ , $\chi_{uu} \simeq \chi_{dd}$, $\chi_{uu} + \chi_{ud} \simeq \chi/2$, and $\chi_{uu} - \chi_{ud} \simeq \chi_{\text{con}}$, where the leading terms χ , χ_{con} , and $\chi_{\text{dis}} = \chi/4 - \chi_{\text{con}}/2$ are $\mathcal{O}(1)$ in the m_δ counting and are given in $SU(3)$ by (31) and (32) and

$$\begin{aligned}\frac{\chi_{\text{dis}}(T) - \chi_{\text{dis}}(0)}{B_0^2} &= \frac{1}{18}[27g_2(M_\pi, T) + g_2(M_\eta, T)] \\ &\quad - \frac{g_1(M_\pi, T) - g_1(M_\eta, T)}{3(M_\eta^2 - M_\pi^2)} + \mathcal{O}(p^2) \\ &\quad + \mathcal{O}\left(\frac{m_d - m_u}{m_s}\right)^2 + \mathcal{O}(e^2),\end{aligned}\quad (34)$$

$$\begin{aligned}\frac{\chi_{\text{dis}}(0)}{B_0^2} &= \frac{1}{9}(288L_6^r(\mu) - \nu_\eta - 27\nu_\pi) - \frac{2F^2(\mu_\pi - \mu_\eta)}{3(M_\eta^2 - M_\pi^2)} \\ &\quad + \mathcal{O}(p^2) + \mathcal{O}\left(\frac{m_d - m_u}{m_s}\right)^2 + \mathcal{O}(e^2),\end{aligned}\quad (35)$$

$$\begin{aligned}\frac{\chi_{\text{con}}(T) - \chi_{\text{con}}(0)}{B_0^2} &= g_2(M_K, T) + \frac{2[g_1(M_\pi, T) - g_1(M_\eta, T)]}{3(M_\eta^2 - M_\pi^2)} \\ &\quad + \mathcal{O}(p^2) + \mathcal{O}\left(\frac{m_d - m_u}{m_s}\right)^2 + \mathcal{O}(e^2),\end{aligned}\quad (36)$$

$$\begin{aligned}\frac{\chi_{\text{con}}(0)}{B_0^2} &= 2(4H_2^r(\mu) + 8L_8^r(\mu) - \nu_K) + \frac{4F^2(\mu_\pi - \mu_\eta)}{3(M_\eta^2 - M_\pi^2)} \\ &\quad + \mathcal{O}(p^2) + \mathcal{O}\left(\frac{m_d - m_u}{m_s}\right)^2 + \mathcal{O}(e^2).\end{aligned}\quad (37)$$

Several remarks are in order about the previous expressions. First, we emphasize that all of them are finite and scale-independent, which can be explicitly checked from the scale dependence of the LEC [22]. We also note that,

unlike the disconnected part, the connected susceptibility does not receive contributions from the mass derivative of pion tadpoles, i.e., ν_π or $g_2(M_\pi, T)$. These turn out to be the dominant ones in the chiral limit (see below) and this is what we expected from our previous discussion on the infrared behavior. Thus, we identify the difference $\mu_\pi - \mu_\eta$ in (37) and the corresponding g_1 one in (36) as the contribution of $\pi^0\eta$ mixing, while the ν_K , $g_2(M_K)$ terms come from the expansion of the kaon contribution in the right-hand side of (12) around the isospin limit. From (33), the disconnected part receives in addition a contribution from the sum (11) and hence it incorporates the ν_π , $g_2(M_\pi)$ terms. Note also that the $T = 0$ part of χ_{con} depends on the contact term H_2 , while that dependence cancels in χ_{dis} . This means that only quantities such as $\chi_{\text{con}}(T) - \chi_{\text{con}}(0)$ can be unambiguously determined, similar to the quark condensate case (see our discussion at the end of Sec. III). This is a relevant comment for lattice evaluations of this quantity. In fact, as a consequence of the vanishing pion terms, we see that $\chi_{\text{con}}(T) - \chi_{\text{con}}(0)$ vanishes formally in the $m_s \rightarrow \infty$ limit, recovering the pure $SU(2)$ result that we would get from (8), which holds also for the other susceptibilities taking into account the conversion between the $SU(2)$ and $SU(3)$ LEC [22] (see below).

Regarding the expansion in $(m_u - m_d)$, as can be seen from (33), from the condensate expressions (11) and (12) and from the meson masses and mixing angle dependence on \hat{m} and m_δ , that the linear order in m_δ cancels both in the connected and the disconnected parts and so we have written in the previous expressions and in the total susceptibility (31) and (32). It is important to remark that the $\mathcal{O}(m_\delta/m_s)^2 \sim \mathcal{O}(\varepsilon)^2$ corrections contain "tadpole mass derivative" terms ν_π , $g_2(M_\pi)$ in both χ_{con} and χ_{dis} . However, an important difference between them is that those IR-dominant terms do not appear to leading order in the connected contribution. In fact, since $\nu_{\pi^0} = \nu_\pi + (\partial\nu_\pi/\partial M_\pi^2)\mathcal{O}(m_\delta/m_s)^2$, where the subscript π indicates just the $m_u = m_d$ and $e^2 = 0$ pion, the disconnected part receives an $\mathcal{O}(m_\delta/m_s)^2$ proportional to the second derivative of the pion tadpole and hence is more IR divergent, and so on for the thermal part $g_2(M_{\pi^0}, T)$. This contribution is not present in the connected part to that order. Thus, we expect the isospin-breaking corrections to be larger in the disconnected than in the connected susceptibility.

Another pertinent comment is that we have been able to obtain the leading order for χ_{con} and χ_{dis} only after considering properly all the $m_u \neq m_d$ contributions and then taking the $m_u = m_d$ limit. However, one can be led to misleading results by setting $m_u = m_d$ from the very beginning. For instance, for two equal masses one could think naively that $\chi_{uu} = \chi_{ud} = \chi_{dd} = \chi/4$, from the definition (21). However, from our previous analysis we see that in the isospin limit what we get actually is $\chi_{ud} = \chi/4 - \chi_{\text{con}}/2$ and $\chi_{uu} = \chi_{dd} = \chi/4 + \chi_{\text{con}}/2$ with χ_{con}

given in (36) and (37), i.e., not vanishing for $m_u = m_d$ and physical m_s , although formally suppressed in the $m_s \rightarrow \infty$ limit. Thus, in the isospin limit taking just one flavor susceptibility and multiplying by four does not give the total scalar susceptibility, which should be obtained instead by considering the derivative of the full sum of u and d condensates as given by (21). Note that this correction does not affect condensates, for which the $\langle \bar{u}u \rangle$ and $\langle \bar{d}d \rangle$ difference vanishes for $m_u = m_d$ and $e^2 = 0$ and therefore $\langle \bar{q}q \rangle = 2\langle \bar{u}u \rangle = 2\langle \bar{d}d \rangle$ in the isospin limit. This comment may be relevant for certain lattice analysis, where working within the one-flavor equivalent framework is often done, since in that way it is easy to discuss for instance the taste-breaking effect. This observation could then help to explain the worse $O(N)$ scaling properties of the susceptibility with respect to those of the condensate [20]. In that work, the lattice data for the susceptibility scaling function in the $2 + 1$ case suffer from a sizable increase as the strange quark mass is decreased relative to the light one. That increase is not seen in the quark condensate data and could be due partly to the definition used as we have just explained, since the positive term proportional to χ_{con} increases with $1/m_s$, from (36) and (37). This is not the only effect that may cause this “wrong” scaling because, as pointed out in [20,21], taste breaking induces an artificial infrared pion contribution in χ_{con} which is not present in the continuum, as our above expressions show. What we are pointing out here is that the one-flavor χ_{uu} and χ_{dd} are sensitive to isospin-breaking terms even for $m_u = m_d$ and in the continuum, unlike considering for instance the total susceptibility $\chi \sim \chi_{uu} + \chi_{ud}$ in (21) for which the χ_{con} term cancels. In addition, as we will see below, χ_{con} is dominated numerically by its $T = 0$ part, which would explain why the anomalous scaling is reduced for the subtracted susceptibility.

The charge corrections to susceptibilities in our previous expressions (32) and (37) arise only through the mass of the charged mesons π^\pm and K^\pm . Thus, including the charge amounts to replace $3\nu_\pi \rightarrow \nu_{\pi^0} + 2\nu_{\pi^\pm}$ in (32) and (35) and $2\nu_K \rightarrow \nu_K + \nu_{K^\pm}$ in (32) and (37), and so on for the thermal parts $3g_2(M_\pi, T) \rightarrow g_2(M_{\pi^0}, T) + 2g_2(M_{\pi^\pm}, T)$ and $2g_2(M_K, T) \rightarrow g_2(M_K, T) + g_2(M_{K^\pm}, T)$. Although for physical values of the electric charge and masses these represent small perturbative corrections, the fact that near the chiral limit the coefficient of the IR-dominant ν_π and $g_2(M_\pi, T)$ reduces in $1/3$ for $e^2 \neq 0$ is the reflection in the scalar susceptibility of the behavior of the condensate in terms of δ_π corrections to the masses analyzed in Sec. IV A. Thus, when the mass corrections δ_π become sizable, as in the staggered lattice formalism, the susceptibility is reduced by that factor, which eventually would imply that the transition peak or maximum is displaced to a higher temperature, consistent with our analysis in the previous section about the increasing of T_c in the condensates. Recall that one cannot just expand the pion

terms in e^2 and then take the chiral limit, since that expansion assumes that the charge part of M_{π^\pm} is small compared to the quark mass one.

Before continuing, we also remark that our above results are compatible with the recent observation [21] that the connected and disconnected susceptibilities can be inferred from the zero momentum limit of the a_0 and f_0 correlators calculated previously in staggered ChPT [32]. The motivation of those works is precisely to estimate the contribution of heavy pion-like tastes to the IR part of χ_{con} , which could mask the scaling behavior. The continuum limit of the results in [21] reveals the same π , K , η loop contributions as in our expressions (34)–(37). We provide the full ChPT result, including the LEC contribution necessary to guarantee the finiteness and scale independence of the results, as well as the analysis of the higher order corrections in isospin breaking.

As discussed above, the behavior of the susceptibilities near the chiral limit (IR regime) is very illuminating regarding their approach to chiral restoration within the $O(4)$ or $SU(2)$ pattern in the continuum, i.e., without taste-breaking effects. Let us consider this regime first for $T = 0$ and only for the leading-order terms in the isospin expansion in (35) and (36), i.e., we set $m_u = m_d = \hat{m}$ and $e^2 = 0$ in the $T = 0$ susceptibilities and consider $\hat{m} \ll m_s$. We denote by a superscript IR the nonvanishing terms in that limit,

$$\frac{\chi_{\text{dis}}^{\text{IR}}(T = 0)}{B_0^2} = -\frac{3}{32\pi^2} \log \frac{M_\pi^2}{\mu^2} + 32L_6^r(\mu) + \frac{1}{288\pi^2} \left(-28 + 5 \log \frac{M_\eta^2}{\mu^2} \right), \quad (38)$$

$$\frac{\chi_{\text{con}}^{\text{IR}}(T = 0)}{B_0^2} = 8[H_2^r(\mu) + 2L_8^r(\mu)] - \frac{1}{16\pi^2} \left(1 + \log \frac{M_K^2}{\mu^2} + \frac{2}{3} \log \frac{M_\eta^2}{\mu^2} \right). \quad (39)$$

The IR divergent $\log M_\pi^2$ term in (38) coincides with the one obtained in [17], where a cutoff regularization was used. Multiplied by 4, this is also the IR-dominant part in the total susceptibility χ . In addition to that term, we obtain here the regular part, not IR divergent but not vanishing in the chiral limit, which provides the dependence with the LEC and together with the logarithm gives the consistent scale-independent ChPT prediction. As for the connected part, again only the IR-divergent contribution is given in [17], which as commented above turns out to vanish exactly for two light fermions. Here we also give the regular contribution, also scale independent, which unlike the disconnected part depends on the contact term H_2^r . As an interesting consistency check, we can recover the $SU(2)$ limit from the above expressions, using the conversion between the LEC of $SU(2)$ and $SU(3)$ [3] $l_3^r + h_1^r - h_3 = 16L_6^r + 5\nu_\eta/18 - 1/(96\pi^2)$ and

$h_3^r = 4L_8^r + 2H_2^r - \nu_K/2 - \nu_\eta/3 + 1/(96\pi^2)$, so that $\chi_{\text{dis}}^{SU(2)} = -3\nu_\pi + 2(l_3^r + h_1^r - h_3)$ and $\chi_{\text{con}}^{SU(2)} = 4h_3^r$, which is the same result that we would have starting directly from the $SU(2)$ expressions in (7) and (8).

Let us consider now the dominant IR thermal contribution in the $2 + 1$ flavor case, i.e., apart from $\hat{m} \ll m_s$ we also consider temperatures $M_\pi \ll T \ll M_K$, so that we neglect all the Boltzmann exponentials $\exp(-M_{K,\eta}/T)$ and expand $g_1(M_\pi, T) = \frac{T^2}{12}[1 - 3M_\pi/(\pi T) + \mathcal{O}(M_\pi^2 \log M_\pi^2)]$ and $g_2(M_\pi, T) = T/(8\pi M_\pi) + \mathcal{O}(\log M_\pi^2)$ [5]. Thus, we get

$$\frac{[\chi_{\text{dis}}(T) - \chi_{\text{dis}}(0)]^{\text{IR}}}{B_0^2} = \frac{3T}{16\pi M_\pi}, \quad (40)$$

$$\frac{[\chi_{\text{con}}(T) - \chi_{\text{con}}(0)]^{\text{IR}}}{B_0^2} = \frac{T^2}{18M_\pi^2}. \quad (41)$$

The disconnected part (40) is again the one obtained in [17] in the IR limit. It diverges more strongly than the $T = 0$ contribution in (38) in this limit, revealing its critical behavior. The growth with T is linear over the GB mass scale. Recall that, apart from the thermally suppressed exponentials, we are neglecting also $\log M_\pi$ terms in the disconnected part (40). The situation is completely different for the connected contribution, which is regular, albeit not vanishing, in the chiral limit. The quadratically growing term in (41) survives for $M_\pi \rightarrow 0$ against neglected $\mathcal{O}(M_\pi)$ and is dominant over $\exp(-M_{K,\eta}/T)$. It vanishes formally as $m_s \rightarrow \infty$, recovering the $SU(2)$ limit. For physical masses though, a specific and model-independent difference between the $N_f = 2$ and $N_f = 2 + 1$ cases is the (soft) temperature dependence of the connected susceptibility, the scale that controls its growth being M_η^2 instead of the M_π^2 of the connected part. This is a consequence of χ_{dis} measuring the fluctuations of the chiral restoration order parameter, while χ_{con} is related to those of the isospin-breaking one, i.e., $\langle \bar{u}u - \bar{d}d \rangle$, which as we have seen in Sec. III increases moderately. Note that near the chiral limit we could as well have written the T^2 term divided by M_K^2 just by changing the multiplying factor. Keeping M_η^2 recalls its $\pi^0\eta$ origin, as is clearly seen in the original expressions in (36) and (37).

In Fig. 4 we plot our numerical ChPT results for the susceptibilities, including all the isospin-breaking corrections. The plots in that figure show the difference with respect to the $T = 0$ results, which are collected in Table I. At $T = 0$ we use the same LEC values as in [22], which are quoted in the table. Remember that the susceptibilities are independent of the EM LEC and that the disconnected one is independent of contact terms. The contact LEC H_2^r appearing in the connected contribution is estimated from resonance saturation arguments [15,26]. The normalization used $B_0^2 = M_\pi^4/(4\hat{m}^2)$ is the same one used in some lattice works [8].

We see in the plots that the general features explored in our previous analytical discussion are well reproduced. First, the $\mathcal{O}(e^2)$ and $\mathcal{O}(m_d - m_u)^2$ terms neglected in (31), (32), and (34)–(37) are numerically small for the relevant temperature range, for physical values of quark and meson masses. In fact, as anticipated in our previous discussion, we see that those isospin corrections are larger for the disconnected than for the connected part and are also larger with temperature, all due to the appearance of IR terms proportional to the second derivative of the tadpole in χ_{dis} . Remember that the leading order in the isospin limit comes actually from the $\mathcal{O}(m_d - m_u)$ terms in the condensates. Second, we appreciate qualitatively the linear and quadratic growth with temperature of the disconnected and connected parts, respectively, as expected from the infrared analysis. In fact, we see that although the connected term grows faster, its absolute value is much smaller due to the M_η^2 scale compared to the M_π^2 of the disconnected part. However, it is important to note that the IR limit expressions (40) and (41) are numerically rather far from the exact ones for the physical pion mass. The difference is larger for the disconnected contribution since, as stated above, in (40) we are neglecting $\mathcal{O}(\log M_\pi)$ terms, while in (41) the neglected terms are $\mathcal{O}(M_\pi)$. In fact, this justifies further our present analysis, since we provide the full expressions beyond the chiral limit. Also as discussed above, the infrared limit expressions for $T = 0$ given in (38) and (39) survive not only the chiral limit but also the $m_s \rightarrow \infty$ one, χ_{dis} still diverging but only logarithmically, which for physical masses makes the two susceptibilities numerically comparable. This is clearly seen in the values given in the first two rows of Table I. Actually, for this very same reason, and following our previous discussion, the deviations of χ_{uu} from the naive isospin-limit expectation $\chi/4$ is much more pronounced at $T = 0$ than for finite T . This can be seen by comparing the last two columns in Table I which give about a 30% relative difference, while the last plot in Fig. 4, where we compare their thermal differences, gives only corrections below 10%. In fact, this is consistent with our previous discussion about the influence of the connected part in the scaling properties observed in the lattice. If we consider the subtracted susceptibilities as defined in [20] from the subtracted condensate $\langle \bar{u}u \rangle - (m_u/m_s)\langle \bar{s}s \rangle$, we see that the dependence on the LEC disappears in the $\hat{m}/m_s \rightarrow 0$ limit. Remember that in this limit all the $T = 0$ contribution of the connected part is absorbed in h_3^r (see our previous comments) and therefore considering the subtracted susceptibility is equivalent to switching off the dominant $T = 0$ part of the connected susceptibility. This is indeed observed in the lattice [20] since the subtracted susceptibility fits better the expected $\mathcal{O}(N)$ scaling behavior than the unsubtracted one.

The variation with the quark mass is displayed for $T \neq 0$ in Fig. 5 and for $T = 0$ in Table I. It is important to remark that we have chosen to keep fixed the ratio $m_u/m_d \simeq 0.46$

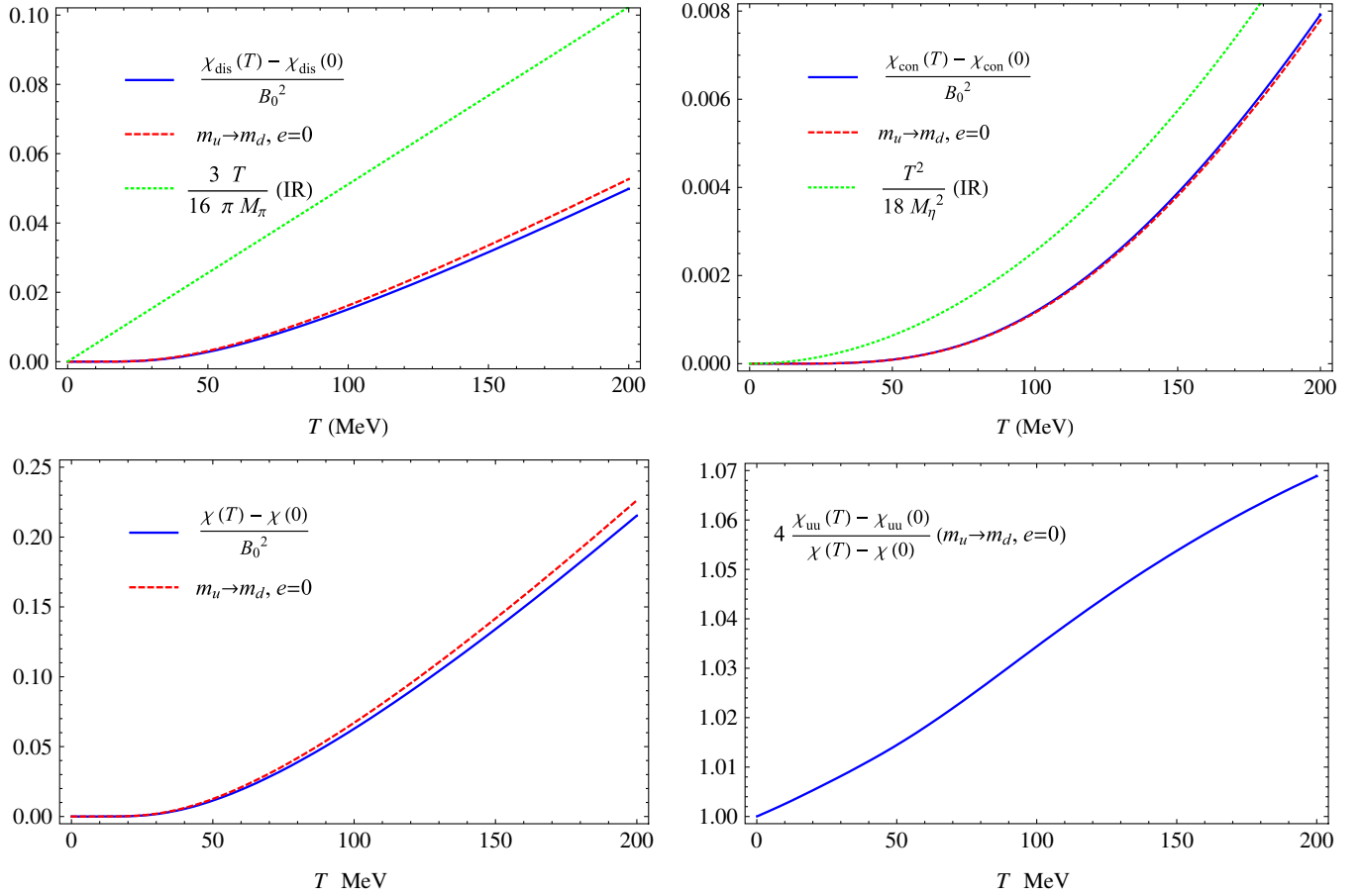


FIG. 4 (color online). Temperature dependence of disconnected, connected, and total susceptibilities in three-flavor ChPT, for physical quark and meson masses. The solid blue curves show the results with all the isospin-breaking corrections of higher order included, while in the dashed red ones we display the leading order in the isospin limit. We also show the IR expressions for the connected and disconnected parts, as well as the deviations of $4\chi_{uu}$ from χ_{tot} in the isospin limit (last plot). The $T = 0$ results are given in Table I.

(same value used in our previous analysis [22]) and m_s , while we vary \hat{m}/m_s above and below the physical quark mass ratio. In other words, when $\hat{m} \rightarrow 0^+$, $M_\pi \rightarrow 0^+$ while $M_{K,\eta}$ remain fixed. This is meant to be the relevant limit when approaching chiral restoration. In addition, since we

TABLE I. $T = 0$ values for the different susceptibilities in ChPT. The isospin limit (IL) values correspond to $m_u \rightarrow m_d$ and $e = 0$. For the third to sixth rows we fix m_s/m_d and m_s and vary the light to heavy quark mass ratio. The first and second rows correspond to the physical values. The LEC values used are $H_2^r = 2L_8^r = 1.24 \times 10^{-3}$, $L_6^r = 0$ at the scale $\mu = 770$ MeV.

	χ_{dis}/B_0^2	χ_{con}/B_0^2	χ/B_0^2	$4\chi_{uu}/B_0^2$
$m_s/\hat{m} = 24$	0.024	0.025	0.146	0.196
$m_s/\hat{m} = 24$, IL	0.025	0.025	0.148	0.197
$m_s/\hat{m} = 10$	0.016	0.024	0.113	0.163
$m_s/\hat{m} = 10$, IL	0.017	0.023	0.114	0.161
$m_s/\hat{m} = 100$	0.036	0.026	0.194	0.245
$m_s/\hat{m} = 100$, IL	0.038	0.025	0.203	0.254

can write $(m_d - m_u)/m_s = 2(\hat{m}/m_s)(1 - m_u/m_d) \times (1 + m_u/m_d)^{-1}$, ε scales in this limit as $\mathcal{O}(\hat{m})$. Recall that, although the values of the L_i^r are fitted to low-energy data with physical masses [26], those LEC are formally independent of the quark masses [3]. The same applies to the tree-level value of F we are using.

As we expected from our previous IR analysis in Eqs. (40) and (41), the light quark mass dependence of the thermal disconnected susceptibility is much stronger than the connected one, as seen clearly in Fig. 5 and as long as we take the limit in the order specified above. In terms of chiral restoration, this anticipates a much stronger growth or peak near T_c for the disconnected part. From the same arguments, the behavior of the connected part is expected to be softer near the transition, although growing with T^2 for low and moderate temperatures. We also show in the figure the comparison with the infrared limit for the smaller \hat{m}/m_s case, where it can be seen that the curves are now closer than for the physical pion mass case in Fig. 4.

In addition, the isospin corrections are also more important for the disconnected part, where they actually

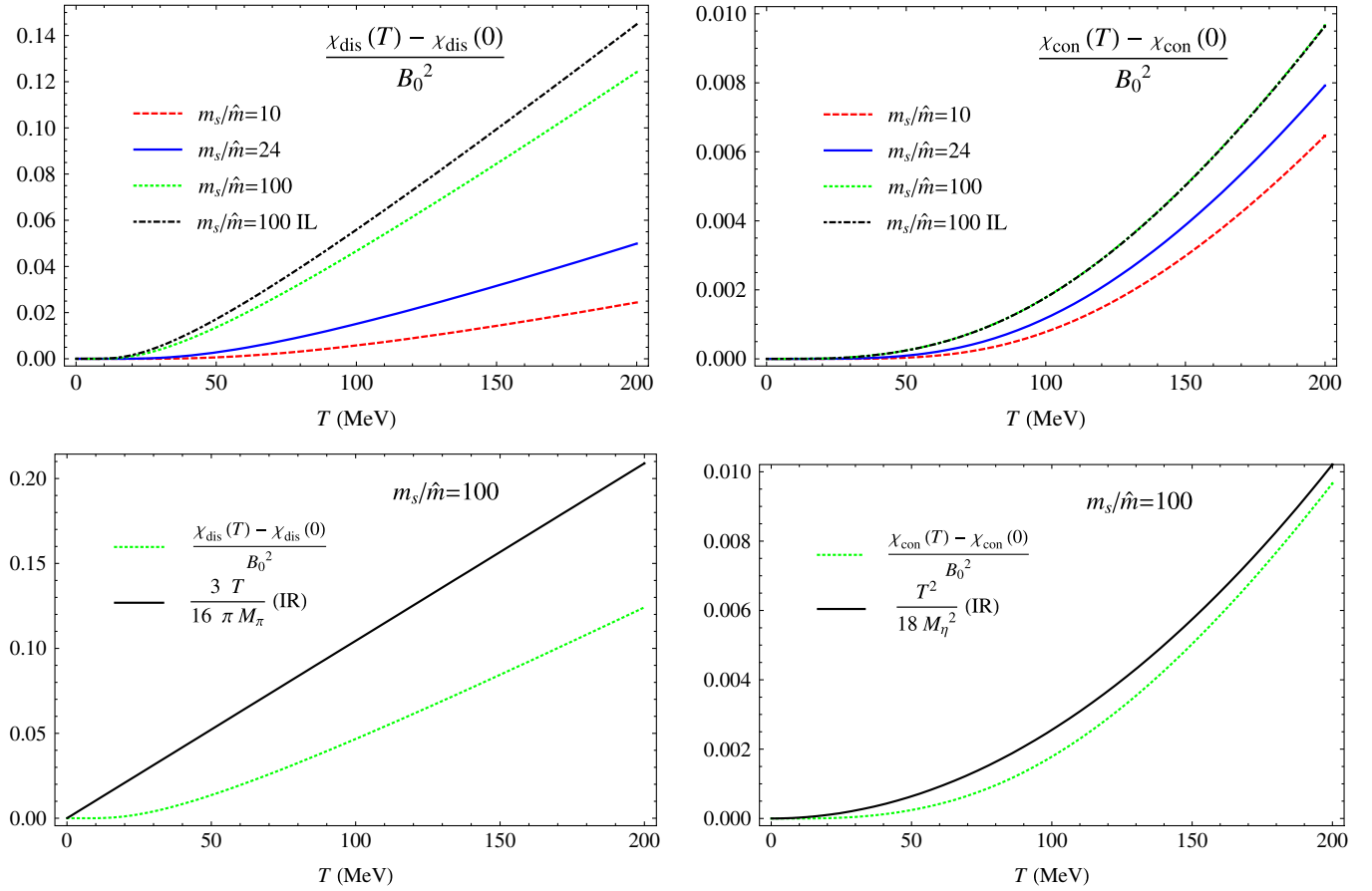


FIG. 5 (color online). Quark mass dependence of the thermal disconnected, connected, and total susceptibilities in three-flavor ChPT, for fixed m_s and fixed m_u/m_d . We also show the isospin limit (IL) $m_u \rightarrow m_d$ and $e^2 = 0$ in the smaller light mass case. The $T = 0$ results are given in Table I. For comparison, we also display the infrared limit in the $m_s/\hat{m} = 100$ case.

increase as \hat{m} is decreased, than for the connected one, where the isospin limit and complete curves are almost indistinguishable in the figure. The same holds for the $T = 0$ contributions in Table I. According to our previous discussion, this behavior of the isospin corrections arises from the dominant IR terms $\chi_{\text{dis}} \sim B_0^2(TM_\pi^2/M_\pi^3)\epsilon^2 = \mathcal{O}(\sqrt{\hat{m}})$ as compared to $\chi_{\text{con}} \sim B_0^2(T/M_\pi)\epsilon^2 = \mathcal{O}(\hat{m}^{3/2})$. This effect is weaker for the $T = 0$ contributions since the IR leading corrections $\nu_\pi\epsilon^2$ (in χ_{con}) and $(\partial\nu_\pi/\partial M_\pi^2)\epsilon^2$ (in χ_{dis}) diverge softly, namely, as $\hat{m}^2 \log \hat{m}$ and \hat{m} respectively. Thus, although the isospin corrections are amplified in the disconnected susceptibility for large temperatures and small masses, they are still perturbatively under control in the chiral limit.

The limit where \hat{m}/m_s vanishes not by taking $\hat{m} \rightarrow 0^+$ but keeping \hat{m} fixed and taking $m_s \rightarrow \infty$, is where we recover the pure $SU(2)$ results, as discussed before. In that case, it is the connected part which is more sensitive to the quark mass variation, vanishing for large M_η^2 , while the disconnected one remains invariant. Although this is formally interesting for connecting the $SU(2)$ and $SU(3)$ cases, it is not so relevant for studying the critical behavior.

V. CONCLUSIONS

In this work we have analyzed the relevant observables regarding chiral symmetry restoration, namely, quark condensates and scalar susceptibilities, in the presence of isospin breaking. We have considered on the same footing the QCD (m_u/m_d mass difference) and electromagnetic corrections, to one loop in chiral perturbation theory, both in the $SU(2)$ and $SU(3)$ sectors. Our analysis provides useful and model-independent results regarding several relevant aspects of isospin breaking and chiral restoration, which may be particularly interesting for lattice studies.

The sum $\langle \bar{u}u + \bar{d}d \rangle_T$, the order parameter for chiral restoration, receives small isospin-breaking corrections for the physical values of masses and electric charge. These corrections affect only slightly (less than 1%) the value of the critical temperature, which they increase as a ferromagnetic response. The difference $\langle \bar{u}u - \bar{d}d \rangle_T$ is the order parameter of isospin breaking. It is temperature-independent in the $SU(2)$ limit, but when kaons and the eta are included, it shows an increasing behavior, which in the chiral limit is given by $(m_u - m_d)T^2/M_\eta^2$. The deviations with respect to its $T = 0$ value become sizable as the

temperature is increased, but they are controlled by a larger energy scale M_η^2 than the typical F_π^2 of $\langle\bar{u}u + \bar{d}d\rangle_T$. This large growth of isospin breaking does not reflect in the chiral restoration temperatures of $\langle\bar{u}u\rangle_T$ and $\langle\bar{d}d\rangle_T$, which remain close to each other, consistently with the idea that chiral restoration is little affected. We have also evaluated the temperature corrections to the sum rule relating the $\langle\bar{s}s\rangle_T/\langle\bar{u}u\rangle_T$ and $\langle\bar{u}u\rangle_T/\langle\bar{d}d\rangle_T$ ratios, which is useful because it does not involve undetermined contact low-energy constants. The corrections in this case come directly from the $\langle\bar{d}d\rangle_T/\langle\bar{u}u\rangle_T$ ratio and are therefore rather large for the temperatures of interest.

A very important part of the present work has been the analysis of scalar susceptibilities in the isospin asymmetric scenario. We have related the different flavor susceptibilities with the total, quark connected and quark disconnected susceptibilities often used in lattice analysis. Electromagnetic corrections to the quark condensate turn out to be directly related by a sum rule to the total susceptibility and then to the growth of fluctuations, which is meant to be maximum near the critical point. This sum rule is valid for any small deviation of the pion masses, as for instance the one arising in the staggered lattice formalism due to taste-breaking effects. Actually, we have made rough estimates of the corrections to condensates expected from this source, comparing lattices of different sizes among them and with the continuum limit. These estimates are in good agreement with the errors quoted in the lattice works.

The isospin asymmetric calculation allows for a direct extraction of the connected and disconnected susceptibilities, even in the isospin symmetric limit. The terms in $\langle\bar{u}u - \bar{d}d\rangle_T$ linearly proportional to $m_u - m_d$ give contributions to the connected part not vanishing in the isospin limit and which affect for instance the naive extrapolation of a given flavor susceptibility to the total one. Our analysis provides model-independent predictions for the mass,

temperature, and isospin dependence of those quantities, which should be recovered in lattice analysis as they approach the continuum limit. In accordance with the behavior of the corresponding order parameters, the disconnected susceptibility shows a linear growth at low and moderate temperatures, infrared divergent near the chiral limit as T/M_π , whereas the connected one is infrared regular but survives the chiral limit as a growing T^2/M_η^2 behavior. The chiral or infrared limit gives qualitatively the behavior as the temperature approaches chiral restoration but numerically is not a good approximation for physical pion masses. The higher order isospin-breaking corrections are quadratic in $m_u - m_d$ and are enhanced in the chiral limit for the disconnected susceptibility, as long as m_s and m_u/m_d remain fixed. The ChPT susceptibilities reproduce the growing T -dependence at low and moderate temperatures in a model-independent way. Although they do not show the peaks expected near the transition, our small mass analysis allows one to infer that the disconnected part should have a more pronounced peak than the connected one, the latter expected to present a rather soft behavior. This difference can be interpreted from the different order parameters that fluctuate in each case: the chiral quark condensate for the disconnected piece and the isospin-breaking one in the connected case. In the formal $SU(2)$ limit $m_s \rightarrow \infty$ the connected contribution becomes temperature-independent, like $\langle\bar{u}u - \bar{d}d\rangle_T$. Our analysis for the susceptibilities is consistent with previous related work in the literature.

ACKNOWLEDGMENTS

We are grateful to W. Unger for useful comments. Work partially supported by Spanish research Contracts No. FPA2008-00592, FIS2008-01323, UCM-BSCH GR58/08 910309, and the FPI program (BES-2009-013672).

-
- [1] S. Weinberg, *Physica (Amsterdam)* **96A**, 327 (1979).
 - [2] J. Gasser and H. Leutwyler, *Ann. Phys. (N.Y.)* **158**, 142 (1984).
 - [3] J. Gasser and H. Leutwyler, *Nucl. Phys.* **B250**, 465 (1985).
 - [4] J. Gasser and H. Leutwyler, *Phys. Lett. B* **184**, 83 (1987).
 - [5] P. Gerber and H. Leutwyler, *Nucl. Phys.* **B321**, 387 (1989).
 - [6] C. Bernard *et al.* (MILC Collaboration), *Phys. Rev. D* **71**, 034504 (2005).
 - [7] Y. Aoki, G. Endrodi, Z. Fodor, S. D. Katz, and K. K. Szabo, *Nature (London)* **443**, 675 (2006).
 - [8] Y. Aoki, S. Borsanyi, S. Durr, Z. Fodor, S. D. Katz, S. Krieg, and K. K. Szabo, *J. High Energy Phys.* **06** (2009) 088.
 - [9] M. Cheng *et al.*, *Phys. Rev. D* **81**, 054504 (2010).
 - [10] R. D. Pisarski and F. Wilczek, *Phys. Rev. D* **29**, 338 (1984).
 - [11] R. Urech, *Nucl. Phys.* **B433**, 234 (1995).
 - [12] U. G. Meissner, G. Muller, and S. Steininger, *Phys. Lett. B* **406**, 154 (1997); **407**, 454(E) (1997).
 - [13] M. Knecht and R. Urech, *Nucl. Phys.* **B519**, 329 (1998).
 - [14] A. Rusetsky, *Proc. Sci.*, CD09 (2009) 071.
 - [15] G. Ecker, J. Gasser, A. Pich, and E. de Rafael, *Nucl. Phys.* **B321**, 311 (1989).
 - [16] C. Vafa and E. Witten, *Nucl. Phys.* **B234**, 173 (1984).
 - [17] A. V. Smilga and J. J. M. Verbaarschot, *Phys. Rev. D* **54**, 1087 (1996).

- [18] C. E. Detar and R. Gupta (HotQCD Collaboration), Proc. Sci., LAT2007 (2007) 179.
- [19] C. DeTar and U. M. Heller, *Eur. Phys. J. A* **41**, 405 (2009).
- [20] S. Ejiri *et al.*, *Phys. Rev. D* **80**, 094505 (2009).
- [21] W. Unger (RBC-Bielefeld Collaboration), Proc. Sci., LAT2009 (2009) 180.
- [22] A. G. Nicola and R. T. Andres, [arXiv:1009.2170](https://arxiv.org/abs/1009.2170).
- [23] T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low, and J. E. Young, *Phys. Rev. Lett.* **18**, 759 (1967).
- [24] R. F. Dashen, *Phys. Rev.* **183**, 1245 (1969).
- [25] J. Bijnens and J. Prades, *Nucl. Phys.* **B490**, 239 (1997).
- [26] G. Amoros, J. Bijnens, and P. Talavera, *Nucl. Phys.* **B602**, 87 (2001).
- [27] M. Della Morte and A. Juttner, *J. High Energy Phys.* **11** (2010) 154.
- [28] A. Juttner and M. Della Morte, Proc. Sci., LAT2009 (2009) 143.
- [29] W. J. Lee and S. R. Sharpe, *Phys. Rev. D* **60**, 114503 (1999).
- [30] C. Aubin and C. Bernard, *Phys. Rev. D* **68**, 034014 (2003).
- [31] S. Borsanyi, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, C. Ratti, and K. K. Szabo (Wuppertal-Budapest Collaboration), *J. High Energy Phys.* **09** (2010) 073.
- [32] C. Bernard, C. E. DeTar, Z. Fu, and S. Prelovsek, *Phys. Rev. D* **76**, 094504 (2007).