UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE CIENCIAS MATEMÁTICAS
Departamento de Análisis Matemático

FACTORIZATION AND DOMINATION PROPERTIES OF OPERATORS ON BANACH LATTICES.

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Pedro Tradacete Pérez

Bajo la dirección de los doctores

Francisco L. Hernández Rodríguez
Julio Flores Álvarez

Madrid, 2010

ISBN: 978-84-693-7735-2

Factorization and domination properties of operators on Banach lattices

Factorización, mayoración y potencias de operadores positivos en retículos de Banach

Pedro Tradacete Pérez

Directores de Tesis:

Francisco L. Hernández Rodríguez
Julio Flores Álvarez

Memoria para optar al grado de Doctor en Matemáticas.

Febrero de 2009

Departamento de Análisis Matemático.
Universidad Complutense de Madrid.

Contents

Introducción, objetivos y descripción de resultados 5

Introduction 17

Chapter 1. Preliminary results 27
1. Basic facts 27
2. Order continuous Banach lattices 28
3. Convexity, concavity, and related notions 30
4. Equi-integrable sets in Banach lattices 32
5. Operators on Banach lattices 34
6. Some Banach spaces results and notation 38

Chapter 2. Power and domination problems for strictly singular operators 41
1. Domination results 41
2. Powers of dominated operators 46
3. Related questions 49

Chapter 3. A Characterization of strictly singular operators in Banach lattices 53
1. Relation with ℓ_2-singular and DSS operators 53
2. The role of the lower 2-estimate 58

Chapter 4. Invariant subspaces of positive strictly singular operators 67
1. R-condition 68
2. Strictly singular and AM-compact operators 70
3. Invariant subspaces of positive strictly singular operators 71
4. Invariant subspaces of SS-friendly operators 74

Chapter 5. Compact products of strictly singular operators 77
1. Disjointly homogeneous Banach lattices 77
2. Compact squares of strictly singular operators 79
3. Strictly singular operators on Lorentz spaces 83
4. Strictly singular operators on Orlicz spaces 88
5. Duality of strictly singular operators in spaces $L_{p,q}$ 91

Chapter 6. c_0-singular and ℓ_1-singular operators between vector valued Banach lattices 97
1. Tools 97

2. Main results 102
3. Applications 103

Chapter 7. Factorization and domination of Banach-Saks operators 107
1. Factorization of operators through Banach lattices with the Banach-Saks property 107
2. Domination by Banach-Saks operators 112
3. Convex hulls of Banach-Saks sets 114

Chapter 8. Factorization of p-convex and q-concave operators 117
1. The basic construction 117
2. Duality relations 125
3. Complex interpolation of Banach lattices 133
4. Factorization for operators which are both p-convex and q-concave 139
5. Other results 146

Bibliography 149

Introducción, objetivos y descripción de resultados

El tema central de esta Memoria es el estudio de operadores entre retículos de Banach. Recordemos que un retículo de Banach es un espacio de Banach E dotado de una norma $\|\cdot\|$, y un orden parcial $\leq$ tales que:

- $x \leq y$ implica $x + z \leq y + z$, para cualesquiera $x, y, z \in E$,
- $\lambda x \geq 0$, para todo $x \geq 0$ en E y todo número real $\lambda \geq 0$,
- para cualesquiera $x, y \in E$ existe un mínimo y un supremo de x e y en E, denotados respectivamente $x \wedge y$, $x \vee y$,
- si denotamos $|x| = x \vee (-x)$, que verifica $|x| \in E$ por la propiedad anterior, se tiene que $\|x\| \leq \|y\|$ si $|x| \leq |y|$.

Es decir, en un retículo de Banach la estructura lineal, reticular y la topología son compatibles, y el estudio de relaciones entre estas tres estructuras es uno de los objetivos de esta teoría. Por una parte, en un retículo de Banach la estructura adicional de retículo proporciona gran cantidad de herramientas de las que no disponemos en espacios de Banach más generales. Este hecho facilita el estudio de propiedades geométricas de retículos de Banach, así como de las propiedades de operadores que actúan entre retículos.

Por otra parte, la relación de orden en un retículo de Banach induce también un orden en los espacio de operadores que actuan entre dos retículos. Este orden se define de manera natural, considerando un operador positivo como aquél que envía elementos positivos a elementos positivos, y definiendo para dos operadores R y T, $R \leq T$ si la diferencia $T - R$ es un operador positivo. A partir de este hecho, surgen de manera natural una serie de cuestiones sobre la relación entre las propiedades de los operadores y la estructura reticular. Entre estas preguntas, una de las más frecuentes es el llamado *problema de mayoración*, que consiste en dados operadores $S \leq T$ entre retículos de Banach E y F, estudiar qué propiedades hereda S del operador T. Concretamente, dada una familia de operadores $\mathcal{I}$, por ejemplo un ideal en el sentido de Pietsch, se considerará bajo qué condiciones en los retículos E y F, si $0 \leq R \leq T : E \to F$ y $T \in \mathcal{I}$, entonces $R \in \mathcal{I}$.

En este sentido tenemos resultados clásicos de P. G. Dodds y D. H. Fremlin, quienes demostraron en [**37**] para retículos E y F, con E^* y F orden continuos, si $0 \leq R \leq T : E \to F$ y T es compacto, entonces R también es compacto. En la misma línea, A. W. Wickstead ha mostrado en [**136**] que si E^* o F son retículos orden continuos y T es débilmente compacto, entonces R también es débilmente compacto. A su vez, N. Kalton y P. Saab han demostrado en [**77**] que si F es orden continuo y T es Dunford-Pettis, entonces R es también Dunford-Pettis.

Más recientemente, C. C. A. Labuschagne ha obtenido también resultados en esta línea para operadores Asplund y Radon-Nikodym (ver [**83**]).

Los origenes del teorema de Dodds-Fremlin, y de los problemas de mayoración, está en los trabajos de W. A. Luxemburg y A. C. Zaanen sobre operadores integrales [**92**]. El teorema de Dodds-Fremlin fue demostrado previamente para operadores integrales por R. J. Nagel y U. Schlotterbeck en [**97**], independientemente de las versiones existentes en la literatura soviética (ver [**79**, Theorem 5.10]). La validez de dicho teorema para operadores arbitrarios en espacios L_p fue conjeturada a partir de evidencias físicas por los físicos matemáticos J. Avron, I. Herbst y B. Simon en [**16**], y demostrada por L. D. Pitt en [**109**]. Practicamente al mismo tiempo apareció la demostración general de P. G. Dodds y D. H. Fremlin que ha sido la más influyente en la teoría actual de operadores en retículos de Banach.

En el caso particular de endomorfismos en un mismo retículo, es decir si $E = F$, una cuestión interesante relacionada con la anterior es el llamado *problema de la potencia*, que consiste en dados dos operadores $0 \leq R \leq T : E \to E$ saber si alguna potencia de R hereda propiedades de T, en este caso sin pedir condiciones extra al retículo E. Esta pregunta fue inicialmente planteada por C. D. Aliprantis y O. Burkinshaw en [**9**] y [**10**], donde obtuvieron para cualquier retículo de Banach E y operadores $0 \leq R \leq T : E \to E$, si T es compacto, entonces R^3 es compacto, y si T es débilmente compacto, entonces R^2 es débilmente compacto. En la misma línea, para la clase de operadores Dunford-Pettis, N. Kalton y P. Saab, probaron que dado un retículo de Banach E, si $0 \leq R \leq T : E \to E$ y T es Dunford-Pettis, entonces R^2 también es Dunford-Pettis. Además, estos resultados son óptimos, en el sentido de que existen ejemplos de operadores que no cumplen estas propiedades para potencias menores.

En la actualidad, podemos encontrar aplicaciones de estos resultados en análisis de modelos de biología matemática o en aplicaciones de la ecuación del transporte. Concretamente, en biología matemática, la primera aplicación de estos resultados se debe a R. Burger en [**27**], que ha utilizado los resultados de [**9**] para obtener información sobre los autovalores de operadores involucrados en un determinado modelo discreto que describe la evolución de densidades en una población de reproducción asexual bajo la acción de mutación y selección. En una línea similar, A. Rhandi y R. Schnaubelt en [**118**] han utilizado los resultados de mayoración para obtener la existencia y unicidad de soluciones positivas de la ecuación que describe un modelo de poblaciones no autonomas con difusión en L_1. Por otro lado, en el estudio de la ecuación del transporte, estos resultados han sido aplicados recientemente por A. Dehici, A. Jeribi y K. Latrach en [**35**] para estudiar el espectro de operadores de transición que describen la madurez de una población creciente de células, y por M. Sbihi en [**121**] para estudiar la estabilidad del espectro esencial de semigrupos en un espacio de Hilbert. Éstas son sólo algunas de las aplicaciones que pretendemos ilustren la utilidad de estos resultados en otras áreas de las matemáticas, vease [**2**] para más información.

En esta memoria, dos capítulos de la misma giran en torno a estos problemas. En concreto, en el **Capítulo 2**, estudiamos el problema de mayoración y de la potencia para operadores estrictamente singulares. Recordemos que un operador $T : X \to Y$ entre espacios de Banach es *estrictamente singular* (o más brevemente *SS*) si para ningún subespacio (cerrado) $M \subset X$ de dimensión infinita, la restricción $T|_M$ es invertible. Esta clase de operadores fue introducida por T. Kato en [**78**], en conexión con la teoría de perturbación de operadores Fredholm. Los operadores estrictamente singulares forman un ideal cerrado de operadores (en el sentido de Pietsch), que contiene al ideal de los operadores compactos. Además, es conocido que un operador $T : X \to Y$ entre espacios de Banach es estrictamente singular si y sólo si para cada subespacio $M \subset X$ de dimensión infinita, existe otro subespacio de dimensión infinita $N \subset M$, tal que la restricción $T|_N$ es compacta.

El problema de mayoración para operadores estrictamente singulares ha sido considerado por J. Flores y F. L. Hernández en [**50**] y [**51**], y en la tesis doctoral [**48**], donde se obtuvieron resultados positivos para una clase amplia de retículos de Banach. En este trabajo presentamos mejoras de varios resultados dados en [**51**] usando técnicas de factorización. Precisamente, los resultados principales del Capítulo 2 son los siguientes:

TEOREMA 1. *Sean E y F retículos de Banach tales que F tiene la propiedad de Schur positiva. Si $0 \leq R \leq T : E \to F$ y T es estrictamente singular, entonces R es también estrictamente singular.*

TEOREMA 2. *Sea E un retículo de Banach con la propiedad de la división subsecuencial, y F un retículo de Banach orden continuo. Si $0 \leq R \leq T : E \to F$ con T estrictamente singular, entonces R es estrictamente singular.*

TEOREMA 3. *Sea E un retículo de Banach y $0 \leq R \leq T : E \to E$ dos operadores positivos. Si T es estrictamente singular, entonces R^4 también es estrictamente singular.*

Además, si E es orden continuo, entonces R^2 es estrictamente singular.

Estos resultados han sido publicados en [**54**]. Para su demostración, serán de gran utilidad los teoremas de factorización de N. Ghoussoub y W. B. Johnson [**61**]. Estos teoremas proporcionan factorizaciones por un retículo orden continuo para cualquier operador orden débilmente compacto. En particular, podremos aplicar estas técnicas a los operadores estrictamente singulares pues al no ser invertibles en ningún subespacio isomorfo a c_0 son orden débilmente compactos. Además estas factorizaciones tienen buenas propiedades en relación con la mayoración de operadores.

También jugarán un papel importante las propiedades de mayoración de operadores disjuntamente estrictamente singulares, desarrolladas por J. Flores y F. L. Hernández en [**50**], así como las técnicas de equi-integrabilidad, la propiedad de la división subsecuencial y los resultados de aproximación de un operador S con $|S| \leq T$ por elementos del ideal algebraico generado por T, es decir, operadores de la forma $\sum_i A_i T B_i$ (véase el Teorema 1.5.7).

Por otro lado, los resultados anteriores están relacionados con problemas abiertos de teoría espectral, como el problema de mayoración de los operadores de Riesz (operadores con el mismo tipo de espectro que los compactos). Nuestra contribución surge de la observación de que si $0 \leq R \leq T : E \to E$ con T Riesz estrictamente singular, entonces por el teorema anterior R^4 es estrictamente singular, y por tanto R es Riesz. Más resultados en esta línea se pueden encontrar en los trabajos [**103**] y [**112**].

Por otro lado, parte del Capítulo 7 también incluye respuestas a estos problemas, esta vez en cuanto a la clase de operadores Banach-Saks. Recordemos que un operador $T : X \to Y$ entre espacios de Banach se dice *Banach-Saks* si toda sucesión acotada (x_n) en X tiene una subsucesión cuya imagen por T es convergente en sentido Cesàro, es decir la sucesión de medias aritméticas $(\frac{1}{N}\sum_{k=1}^{N} T(x_{n_k}))$ converge en la norma de Y. En concreto, los principales resultados obtenidos son los siguientes:

Teorema 4. *Sean E y F retículos de Banach con F orden continuo. Si tenemos operadores $0 \leq R \leq T : E \to F$ con T Banach-Saks, entonces R también es Banach-Saks.*

Teorema 5. *Sea E un retículo de Banach y $0 \leq R \leq T : E \to E$. Si T es Banach-Saks, entonces R^2 también es Banach-Saks.*

Estos teoremas mejoran resultados previos de J. Flores y C. Ruiz en [**55**], y además mostraremos que son óptimos mediante ejemplos conocidos. Estos resultados han sido publicados en [**56**].

Aparte de los problemas de mayoración y potencia, la clase de los operadores estrictamente singulares es uno de los principales objetos de estudio de esta Memoria. Así, del Capítulo 2 al 6 se presentan propiedades de estos operadores entre retículos de Banach. Concretamente, en el **Capítulo 3** estudiamos caracterizaciones de operadores estrictamente singulares en términos de operadores disjuntamente estrictamente singulares y ℓ_2-singulares. Recordemos que dados un retículo de Banach E y un espacio de Banach Y, un operador $T : E \to Y$ es *disjuntamente estrictamente singular* si no es invertible en el espacio generado por ninguna sucesión de vectores disjuntos. Esta clase fue introducida en [**70**] en relación con el estudio de copias complementadas de ℓ_p en espacios de funciones. Además, ha mostrado ser una util herramienta en el estudio de operadores estrictamente singulares en retículos de Banach, en el problema de mayoración [**51**] y para comparar estructuras de espacios invariantes por reordenamiento por F. L. Hernández, V. M. Sánchez y E. Semenov en [**71**]. Sus propiedades han sido estudiadas por J. Flores y F. L. Hernández en varios trabajos [**49**], [**50**], [**67**]. Los resultados del Capítulo 3 ilustran la importancia de los operadores disjuntamente estrictamente singulares y su relación con los estrictamente singulares.

Los resultados del Capítulo 3 vienen motivados por el hecho siguiente: un endomorfismo en $L_p = L_p[0,1]$, con $1 \leq p < \infty$, es estrictamente singular si y sólo si es ℓ_p-singular y ℓ_2-singular (ver los trabajos de V. Milman [**96**] y L. Weis [**132**]). En otras palabras, un endomorfismo $T : L_p \to L_p$ es estrictamente singular si y sólo si T es disjuntamente estrictamente singular y ℓ_2-singular. Decimos que un operador entre espacios de Banach es *ℓ_p-singular*, para $1 \leq p \leq \infty$, si no es un isomorfismo al restringirlo a ningún subespacio isomorfo a ℓ_p. En [**74**] podemos encontrar resultados recientes sobre operadores ℓ_p-singulares.

Nuestro objetivo en el Capítulo 3 es extender este hecho a retículos de Banach más generales. En concreto demostramos el siguiente

TEOREMA 6. *Sean E y F retículos de Banach tales que E tiene cotipo finito y F satisface una 2-estimación inferior. Un operador $T : E \to F$ es estrictamente singular si y solo si T es disjuntamente estrictamente singular y ℓ_2-singular.*

Además, para la clase de operadores regulares (i.e. diferencia de operadores positivos) esta caracterización sigue cumpliéndose bajo condiciones más débiles en los retículos.

TEOREMA 7. *Sean E y F retículos de Banach tales que E tiene cotipo finito y F es orden continuo. Un operador regular $T : E \to F$ es estrictamente singular si y solo si T es disjuntamente estrictamente singular y ℓ_2-singular.*

Ambos teoremas se obtendrán a partir del siguiente hecho que también demostraremos:

TEOREMA 8. *Si E es un retículo de Banach con cotipo finito, Y es un espacio de Banach y $T : E \to Y$ es disjuntamente estrictamente singular y AM-compacto, entonces T es estrictamente singular.*

Recordemos que un operador $T : E \to Y$ de un retículo de Banach en un espacio de Banach se dice *AM-compacto* si la imagen de un intervalo es siempre un conjunto relativamente compacto.

También mostramos mediante ejemplos que las hipótesis en los Teoremas anteriores no pueden ser debilitadas, en el sentido de que el Teorema 6 no es cierto si el retículo F sólo satisface una q-estimación inferior para algún $q > 2$. Para ello vamos a considerar el retículo de Banach $L_r(\ell_q)$, que consiste en sucesiones $x = (x_1, x_2, \dots)$ de elementos de L_r tales que

$$\|x\|_{L_r(\ell_q)} = \sup_n \Big\| \Big(\sum_{i=1}^n |x_i|^q \Big)^{1/q} \Big\|_{L_r} < \infty.$$

TEOREMA 9. *Sean $1 < r < p < s < 2 < q < \infty$. Existe un operador $T : L_p \to L_r(\ell_q)$ que es ℓ_p-singular y ℓ_2-singular, pero no ℓ_s-singular.*

En particular, el operador T es disjuntamente estrictamente singular y ℓ_2-singular, pero no estrictamente singular.

La demostración de este resultado utilizará varios lemas técnicos que se basan en ciertas estimaciones para sucesiones de variables aleatorias independientes p-estables, obtenidas por W. B. Johnson, B. Maurey, G. Schechtman y L. Tzafriri en [**73**]. Parte del trabajo presentado en este capítulo fue realizada en una visita durante el otoño de 2006 a la University of Missouri-Columbia bajo la supervisión de N. Kalton.

Otra cuestión, que trataremos en el **Capítulo 4**, es el problema del subespacio invariante para operadores estrictamente singulares. Como es bien sabido, el problema del subespacio invariante consiste en determinar si para cualquier operador $T : X \to X$ en un espacio de Banach X, siempre existe un subespacio no trivial $Y \subset X$ (i.e. $Y \neq \{0\}, X$) tal que $T(Y) \subset Y$. Dicho problema fue resuelto negativamente por P. Enflo en [**42**] y simplificado por C. Read en [**114**], quien dio incluso un ejemplo de operador en ℓ_1 sin subespacios invariantes [**115**]. Sin embargo el problema sigue abierto para operadores en espacios reflexivos, e incluso en el caso aparentemente más sencillo: operadores en el espacio de Hilbert. También hay una serie de conjeturas para algunas clases de operadores, vease [**5**], por ejemplo para operadores duales, y el caso que más nos interesa: operadores positivos.

El problema del subespacio invariante para operadores positivos ha sido estudiado principalmente por Y. Abramovich, C. Aliprantis y O. Burkinshaw en [**3**] y [**4**]. Concretamente, estos autores demostraron que todo operador positivo en ℓ_p con $1 \leq p < \infty$, que sea quasinilpotente en algún $x \in (\ell_p)_+$, tiene un subespacio invariante (de hecho un ideal invariante). Más en general, si $B : E \to B$ es un operador positivo en un retículo de Banach, y existe otro operador positivo $S : E \to E$ tal que $SB \leq BS$, S es quasinilpotente en algún $x \in E_+$, y S es mayor que un operador compacto no nulo, entonces B tiene un ideal invariante no trivial.

Es bien conocido, que todo operador compacto en un espacio de Banach tiene subespacios invariantes (el caso de espacios de Hilbert es debido a J. Von Neumann y la extensión para espacios de Banach a N. Aronszajn y K. T. Smith [**15**]). Además, una gran clase de operadores relacionados con los compactos también tienen subespacios invariantes, como por ejemplo todo operador que conmute con un compacto, hecho debido a V. I. Lomonosov [**88**]. Por tanto, una pregunta natural es si esto se puede extender a operadores estrictamente singulares, que son cercanos en cierto sentido a los operadores compactos. Sin embargo, C. Read dio en [**116**] un ejemplo de operador estrictamente singular sin subespacios invariantes. A pesar de este hecho, para operadores estrictamente singulares positivos daremos una respuesta afirmativa en una clase muy amplia de retículos de Banach (aquellos que cumplen la R-condición), que por ejemplo contiene a los retículos con cotipo finito. Así el resultado principal del Capítulo 4 es el siguiente:

TEOREMA 10. *Sea E un retículo de Banach que cumple la R-condición. Si $T : E \to E$ es positivo y estrictamente singular, entonces T tiene un subespacio invariante.*

Además, en la Sección 4 del Capítulo 4, se introducirá la clase de operadores "SS-friendly", clase análoga a la de los operadores "compact-friendly" introducida por Y. Abramovich, C. Aliprantis y O. Burkinshaw en [**4**], y se demostrarán resultados positivos para operadores en esta clase. En particular, demostramos que en un retículo de Banach con la R-condición, todo operador positivo que conmute con un operador estrictamente singular tiene subespacios invariantes. Algunos de los resultados intermedios para demostrar este hecho involucran el análisis del semiconmutador por la derecha de un operador

$$|T\rangle = \{S \geq 0 : ST \geq TS\}.$$

Además, el Teorema 4.3.8 sobre $|T\rangle$ ha sido recientemente utilizado y extendido de manera análoga al semiconmutador por la izquierda $\langle T|$ por H. Gessesse en [**59**].

El trabajo presentado en este capítulo ha sido publicado conjuntamente con J. Flores y V. G. Troitsky en [**57**].

Continuamos en el **Capítulo 5** extendiendo un resultado de V. Milman sobre productos de operadores estrictamente singulares en espacios L_p (véase [**95**], [**96**]). V. Milman demostró que todo operador estrictamente singular en $L_p(0,1)$ tiene cuadrado compacto. El resultado principal de este capítulo extiende este resultado para retículos con una estructura rígida “próxima” a la de los espacios L_p. Concretamente, diremos que un retículo de Banach es *disjuntamente homogéneo* cuando dos sucesiones cualesquiera de elementos disjuntos siempre comparten una subsucesión equivalente. Esta clase incluye los espacios de Lorentz $L_{p,q}(0,1)$ y ciertos espacios de Orlicz $L_\varphi(0,1)$, y otros espacios menos clásicos como el espacio de Tsirelson. El resultado principal de este capítulo es el siguiente:

Teorema 11. *Sea E un retículo de Banach disjuntamente homogéneno con cotipo finito. Si $T : E \to E$ es estrictamente singular, entonces T^2 es compacto.*

Este resultado está en la línea de los obtenidos por G. Androulakis, P. Dodos, G. Sirotkin y V. G. Troitsky en [**13**]. Sin embargo, en ese trabajo las condiciones para obtener un teorema como el anterior son bastante más complicadas pues emplean conceptos como el de equivalencia Schreier-spreading entre sucesiones. Así, hay una diferencia conceptual con nuestro caso, pues al estar interesados principalmente en retículos de Banach, la propia estructura de retículo nos proporciona condiciones más sencillas de formular (y comprobar) como la de ser disjuntamente homogéneo, que tan sólo involucra sucesiones disjuntas.

También obtendremos resultados para casos más concretos de espacios disjuntamente homogéneos. Mencionemos antes que un retículo de Banach se dice p-disjuntamente homogéneo (para $1 \leq p < \infty$) si toda sucesión disjunta normalizada tiene una subsucesión equivalente a la base canónica de ℓ_p.

Teorema 12. *Sea E un retículo de Banach 2-disjuntamente homogéneo. Si $T : E \to E$ es estrictamente singular, entonces T es compacto.*

Teorema 13. *Sea E un retículo de Banach 1-disjuntamente homogéneo. Si $T : E \to E$ es estrictamente singular, entonces T es Dunford-Pettis.*

Tras estudiar como se aplican estos teoremas a operadores en espacios de Lorentz y espacios de Orlicz, proporcionamos varios ejemplos que ilustran las limitaciones de estos resultados. En particular presentamos un espacio de Orlicz L_φ y un operador estrictamente singular $T : L_\varphi(0,1) \to L_\varphi(0,1)$ tal que ninguna potencia T^n es un operador compacto. Finalmente, estudiamos la relación de dualidad dentro de la clase de operadores estrictamente singulares en espacios de Lorentz $L_{p,q}$, obteniendo el siguiente resultado que complementa un resultado similar de L. Weis sobre operadores en espacios L_p [**132**]:

Teorema 14. *Sea $T : L_{p,q} \to L_{p,q}$ con $1 < p < 2$ y $q \notin (p, 2)$, o $2 < p < \infty$ y $q \notin (2, p)$, o $p = 2$ y $1 < q < \infty$. Son equivalentes:*

(1) *T es estrictamente singular.*
(2) *T es $\{\ell_2, \ell_q\}$-singular.*
(3) *No existe ningún subespacio $M \subset L_{p,q}$, isomorfo a ℓ_2 o ℓ_q, tal que $T(M)$ está complementado en $L_{p,q}$ y la restricción $T|_M$ es un isomorfismo.*
(4) *T^* es estrictamente singular.*
(5) *T^* es $\{\ell_2, \ell_{q'}\}$-singular.*

Notese que la relación entre p y q en el teorema es una hipótesis necesaria, como mostraremos en el Ejemplo 5.5.3.

El **Capítulo 6** está dedicado al estudio de ciertas propiedades sobre operadores en espacios de tipo $E(X)$. Recordemos que dado un retículo de Banach E de funciones sobre un espacio de medida (Ω, Σ, μ), y un espacio de Banach X, se define el espacio $E(X)$ como el conjunto de funciones $f : \Omega \to X$ tales que la función $\omega \mapsto \|f(\omega)\|_X$ pertenece al retículo E. El espacio $E(X)$ es un retículo de Banach vectorial dotado de la norma $\|f\|_{E(X)} = \|\|f(\cdot)\|_X\|_E$.

La relación entre las propiedades de X y E con aquéllas de $E(X)$ ha sido objeto de estudio por muchos autores. Así tenemos resultados de S. Kwapien en [**82**], que demostró que X contiene un subespacio isomorfo a c_0 si y sólo si $L_1(X)$ posee la misma propiedad; en la misma línea J. Bourgain [**26**] y G. Pisier [**110**] demostraron independientemente que X contiene un subespacio isomorfo a ℓ_1 si y sólo si $L_p(X)$ tiene la misma propiedad para $1 < p < \infty$. Más tarde, W. Hensgen en [**66**] demostró, entre otras cosas, que $E(X)$ contiene un subespacio isomorfo a c_0, si y sólo si E o X tienen dicha propiedad. Análogamente, G. Emmanuele en [**41**], y L. Drewnowski e I. Labuda en [**39**], han estudiado cuándo estos espacios contienen copias de ℓ_∞, y más recientemente P. Dowling y N. Randranantoanina han obtenido resultados sobre copias asintóticas de c_0 o ℓ_1 en [**38**]. Por otro lado, tenemos importantes resultados sobre conjuntos débilmente compactos en $L_1(X)$ debidos a M. Talagrand en [**128**], y en la misma línea encontramos también los trabajos recientes de M. Nowak [**100**], [**101**] y [**102**].

En el **Capítulo 6** se extienden algunas de estas propiedades al contexto de operadores. Así, dado un operador $T : X \to Y$ entre espacios de Banach, estudiaremos el operador extendido $T_E : E(X) \to E(Y)$ definido por $T_E(f)(\omega) = T(f(\omega))$ para $f \in E(X)$ y $\omega \in \Omega$. Los resultados principales obtenidos son los siguientes:

Teorema 15. *Sea E un retículo de Banach que no contiene nigún subespacio isomorfo a c_0 y sea $T : X \to Y$ un operador entre espacios de Banach. Si el operador extendido $T_E : E(X) \to E(Y)$ preserva copia isomofa de c_0, entonces $T : X \to Y$ también preserva copia isomorfa de c_0.*

TEOREMA 16. *Sea E un retículo de Banach orden continuo tal que E^* es también orden continuo y sea $T : X \to Y$ un operador entre espacios de Banach. Si el operador extendido $T_E : E(X) \to E(Y)$ preserva copia isomofa de ℓ_1, entonces $T : X \to Y$ también preserva copia isomorfa de ℓ_1.*

Mediante ejemplos sencillos se observa que las condiciones en el retículo E son en ambos casos necesarias.

Estos resultados se han podido aplicar para extender al ámbito de operadores resultados de J. Hoffmann-Jørgensen en [**72**], que están relacionados con la convergencia de martingalas y por tanto con la propiedad de Radon-Nykodim.

Por último, como ya hemos mencionado anteriormente los dos capítulos restantes se centran más en las propiedades de factorización de operadores en retículos, que ha resultado ser un tema recurrente a lo largo de todo el trabajo. En el **Capítulo 7**, aparte de considerar el problema de mayoración de operadores Banach-Saks ya mencionado, se estudia el problema de factorización operadores de Banach-Saks positivos, por retículos con la propiedad de Banach-Saks. Este hecho es conocido en el caso de espacios de Banach, para el que B. Beazamy [**19**] demostró que todo operador de Banach-Saks entre espacios de Banach factoriza por un espacio con la propiedad de Banach-Saks. Para operadores positivos en retículos de Banach el resultado principal de este Capítulo es el siguiente:

TEOREMA 17. *Sea $T : E \to F$ un operador positivo entre retículos de Banach con F orden continuo. Si T es un operador de Banach-Saks entonces existe un retículo de Banach G con la propiedad de Banach-Saks, y operadores $T_1 : E \to G$, $T_2 : G \to F$ tales que $T = T_2T_1$.*

Este teorema está relacionado con la siguiente pregunta: ¿Factoriza todo operador compacto positivo por un retículo reflexivo? Esta pregunta permance todavía abierta, conociéndose tan solo resultados parciales debidos a C. Aliprantis y O. Burkinshaw en [**11**]. Sin embargo, para operadores débilmente compactos, M. Talagrand en [**129**] dio un ejemplo de operador débilmente compacto positivo de ℓ_1 en $C(0,1)$ que no factoriza por ningún retículo reflexivo. De hecho, dicho ejemplo es empleado en el Capítulo 7 para mostrar que el Teorema 17 es en cierto sentido óptimo.

En el **Capítulo 8** presentamos dos construcciones para factorizar operadores p-convexos (respectivamente q-cóncavos) por retículos de Banach p-convexos (resp. q-cóncavos). Además, partiendo de la dualidad entre los conceptos de convexidad y concavidad, estudiamos la relación entre los espacios obtenidos al factorizar un operador y su adjunto. Los resultados obtenidos pueden considerarse como una generalización en cierto sentido de la factorización de operadores orden débilmente compactos por retículos orden continuos debida a N. Ghoussoub y W. B. Johnson [**61**]. La motivación parte de los resultados de J. L. Krivine de factorización de operadores que sean a la vez p-convexos y p-concavos, por espacios L_p [**81**]. En esta línea, obtenemos propiedades de factorización para operadores que sean al mismo tiempo p-convexos y q-concavos:

TEOREMA 18. *Sean E y F retículos de Banach y $T : E \to F$ un operador p-convexo y q-concavo. Para cualquier $\theta \in (0,1)$ podemos factorizar T del modo siguiente*

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "\phi_\theta"'] & F \\ E_\theta \arrow[r, "R_\theta"] & F_\theta \arrow[u, "\varphi_\theta"'] \end{tikzcd}$$

donde ϕ_θ y φ_θ son homomorfismos de retículos, E_θ es $(\frac{q}{1-\theta})$-cóncavo y F_θ es $(\frac{p}{\theta+(1-\theta)p})$-convexo.

La demostración de este Teorema utiliza técnicas de interpolación, que han resultado ser una constante en los problemas de factorización. Concretamente generalizamos resultados de V. A. Sestakov [**126**] sobre comparación entre el método de interpolación complejo $[X_0, X_1]_\theta$ y el "método" de Calderón-Lozanovskii $X_0^{1-\theta}X_1^\theta$. Dichos resultados sólo constan en la literatura para retículos de funciones sobre un espacio de medida, y por tanto para nuestro caso necesitamos generalizarlos a retículos de Banach abstractos.

Recordemos que los métodos de interpolación compleja $[X_0, X_1]_\theta$ y $[X_0, X_1]^\theta$ se definen a partir de un par compatible de espacios de Banach complejos (X_0, X_1), es decir, espacios para los que existen inclusiones continuas en un espacio vectorial topológico $i_j : X_j \hookrightarrow X$ para $j = 0, 1$. Esta construcción resulta ser un método de interpolación (ver [**28**]). Sin embargo, la construcción de Calderón-Lozanovskii $X_0^{(1-\theta)}X_1^\theta$ se define para retículos de Banach, que tienen que ser compatibles en un sentido más preciso. Concretamente, diremos que dos retículos de Banach E_0, E_1 forman un par compatible de retículos de Banach, si existe un espacio de Riesz (retículo vectorial topológico) X, e inclusiones continuas $i_j : E_j \hookrightarrow X$ que sean homomorfismos de retículos que preserven intervalos. Notese que a posteriori podemos suponer que X es un retículo de Banach, ya que por las hipótesis en las inclusiones i_0 e i_1, la suma $E_0 + E_1$ será un retículo de Banach.

En este contexto es en el que podemos definir el espacio de Calderón-Lozanovskii $E_0^{(1-\theta)}E_1^\theta$, como el conjunto de elementos de la suma $x \in E_0 + E_1$ tales que $|x| \leq |x_0|^{(1-\theta)}|x_1|^\theta$ con $x_0 \in E_0$, y $x_1 \in E_1$. $E_0^{(1-\theta)}E_1^\theta$ es un retículo de Banach equipado con la norma

$$\|x\| = \inf\{\lambda > 0 : |x| \leq \lambda |x_0|^{(1-\theta)}|x_1|^\theta, \text{ con } \|x_0\|_{E_0} \leq 1, \|x_1\|_{E_1} \leq 1\}.$$

Es importante notar que este espacio no cumple en general un teorema de interpolación ([**90**], [**91**]), aunque sí lo cumple si los retículos son suficientemente "buenos". En general, sin condiciones en los retículos, la construcción de Calderón-Lozanovskii constituye siempre un método de interpolación para operadores positivos.

Finalmente, señalemos que en el Capítulo primero hemos incluido técnicas y resultados de carácter general que serán utilizados en los Capítulos restantes. Todos ellos son resultados conocidos que se pueden encontrar en la mayoría de los libros dedicados a estos temas, pero que incluimos para mayor comodidad del lector.

Quiero concluir esta introducción mostrando mi agradecimiento a todas aquellas personas que me han ayudado en la realización de este proyecto. No puedo sino comenzar por mis directores de Tesis: Francisco Hernández y Julio Flores, sin cuyo apoyo y estímulo esta Tesis habría sido imposible, pues con su paciencia y comprensión, me han guiado en estos años, compartiendo momentos mejores y peores. Por supuesto, va también mi agradecimiento a todos los miembros del Departamento de Análisis Matemático de la Universidad Complutense de Madrid que de uno u otro modo han contribuido a hacer realidad este trabajo. También quiero agradecer el buen recibimiento y trato recibido en el *Department of Mathematics* de la *University of Missouri-Columbia* y en el *Equipe d'Analyse Fonctionnelle* de la *Université Paris VI/Paris VII.*

Por último, quiero agradecer a mis padres y a Idoia su apoyo constante y la confianza que siempre han depositado en mi. Si he llegado hasta aquí es sin duda gracias a ellos.

Introduction

The main topic in this Memoir is the study of operators between Banach lattices. Recall that a Banach lattice is a Banach space E endowed with a norm $\|\cdot\|$, and a partial order $\leq$ such that:

(1) if $x \leq y$, then $x + z \leq y + z$, for every $x, y, z \in E$,
(2) $\lambda x \geq 0$, for every $x \geq 0$ in E and every real number $\lambda \geq 0$,
(3) for every $x, y \in E$ there exist the least upper bound and the greatest lower bound of x, y in E; these are respectively denoted $x \vee y$ and $x \wedge y$,
(4) if we denote $|x| = x \vee (-x)$, it follows that $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$.

This means that in a Banach lattice the linear structure, the lattice operations and the topology are all compatible with each other, and the study of relations among them is one of the aims of this theory. On the one hand, the lattice structure of a Banach lattice provides a big number of tools that are no longer valid in general Banach spaces. This fact makes the study of the geometric properties of Banach lattices, and operators between them, somehow easier.

On the other hand, the order structure in a Banach lattice induces also an order in the space of operators acting between two lattices. This is defined in a natural way by considering an operator positive if it maps positive elements to positive elements, and for operators R and T, setting $R \leq T$ whenever $T - R$ is positive. Several questions regarding the lattice structure and properties of operators are motivated by this simple relation. For instance, a well spread topic is the so-called *domination problem*. This consists in the study of which properties of an operator T are inherited by another operator R with $R \leq T$. Precisely, given a family of operators $\mathcal{I}$, usually an ideal in the sense of Pietsch, the question is to find under which conditions on the lattices E and F, if $0 \leq S \leq T : E \to F$ and T belongs to $\mathcal{I}$, then S also belongs to the class $\mathcal{I}$.

In this direction, there are classical results by P. G. Dodds and D. H. Fremlin, who proved in [**37**] that for Banach lattices E and F, with E^* and F order continuous, if $0 \leq R \leq T : E \to F$ and T is compact, then R is compact as well. In a similar spirit, A. W. Wickstead has shown in [**136**] that if E^* or F is order continuous, and $0 \leq R \leq T : E \to F$ with T is weakly compact, then R is also weakly compact. In addition, N. J. Kalton and P. Saab have proved that if F is order continuous and $0 \leq R \leq T : E \to F$, with T Dunford-Pettis, then R is Dunford-Pettis. More recently, C. C. A. Labuschagne has obtained result in this direction for the classes of Asplund and Radon-Nikodym operators [**83**].

Dodds-Fremlin's Theorem and related domination problems have their origins in the work of W. A. Luxemburg and A. C. Zaanen [**92**]. Dodds-Fremlin's Theorem was previously proved in the case of integral operators by R. J. Nagel and U. Schlotterbeck in [**97**], probably independently from the existent versions in the Soviet literature (see [**79**, Theorem 5.10]). This theorems were first conjectured for operators in L_p spaces by the mathematical physicists J. Avron, I Herbst and B. Simon in [**16**], motivated by some physical evidence, and was finally proved by L. D. Pitt in [**109**]. Almost at the same time the general proof due to P. G. Dodds and D. H. Fremlin appeared. This has become one of the most influential works in the modern theory of operators in Banach lattices.

Another related question, in the context of endomorphism on a Banach lattice, is the so-called *power problem*. To be precise, given operators $0 \leq R \leq T : E \to E$ the problem consists in finding out wether some power of R inherits properties of T, without extra assumptions on the Banach lattice E. This question was posed initially by C. D. Aliprantis and O. Burkinshaw in [**9**] and [**10**], where they obtained that in an arbitrary Banach lattice E, for operators $0 \leq R \leq T : E \to E$, if T is compact, then it always hold that R^3 is compact, while if T is weakly compact, then R^2 is also weakly compact. Similarly, the results of N. Kalton and P. Saab in [**77**] show that if $0 \leq R \leq T : E \to E$ and T is Dunford-Pettis, then R^2 is also Dunford-Pettis. Moreover, all these results are optimal, in the sense that there exists operators which do not satisfy these properties when the powers are lower.

These days, we can find application of the previous results in several fields such as model analysis in mathematical biology or the transport equation. Precisely, in mathematical biology, the first application of these theory are due to R. Burger in [**27**], who has used the results in [**9**] to get information about the eigenvalues of operators arising in a certain discrete model that describes densities evolution in an asexually reproducing population under mutation-selection. In a similar approach, A. Rhandi and R. Schnaubelt, in [**118**], have used the domination results to obtain existence and uniqueness of positive solutions for an equation describing a non-autonomous population model with diffusion in L_1. Besides, in connection with the transport equation, these theorems have been recently applied by A. Dehici, A. Jeribi and K. Latrach in [**35**] in the study of spectra of transition operators that describe the maturity in a growing cell population, and by M. Sbihi, in [**121**], in order to study essential spectra stability of semigroups on a Hilbert space. Anyway, these are just some applications that we mention to shed some light in the utility of these results in other areas of mathematics, see [**2**] for further information.

In this memoir, two chapters deal with these kind of problems. Namely, in **Chapter 2**, the power and domination problems for strictly singular operators are studied. Recall that an operator $T : X \to Y$ between Banach spaces is called strictly singular (or $\mathcal{SS}$ in short) if the restriction $T|_M$ to any infinite dimensional closed subspace $M \subset X$ is never an isomorphism. This class of operators was introduced by T. Kato in [**78**], in connection with the perturbation theory of Fredholm operators. Strictly singular operators form a closed ideal of operators (in the sense of Pietsch) containing the ideal of compact operators. Moreover, it is well known that an operator $T : X \to Y$ is strictly singular if and only if for every infinite dimensional subspace $M \subset X$, there exists another infinite dimensional subspace $N \subset M$, such that the restriction $T|_N$ is a compact operator.

The domination problem for strictly singular operators has already been studied by J. Flores and F. L. Hernández in [**50**] and [**51**], where several positive results were obtained for a large class of Banach lattices. In this work we present improvements of the results given in [**51**] using factorization techniques. Precisely, the main results in Chapter 2 are the following:

THEOREM 1. *Let E and F be Banach lattices such that F has the positive Schur property. If $0 \leq R \leq T : E \to F$ and T is strictly singular, then R is strictly singular.*

THEOREM 2. *Let E be a Banach lattice with the subsequence splitting property, and F an order continuous Banach lattice. If $0 \leq R \leq T : E \to F$ and T is strictly singular, then R is strictly singular too.*

THEOREM 3. *Let E be a Banach lattice and $0 \leq R \leq T : E \to E$ positive operators. If T is strictly singular, then R^4 is strictly singular. Moreover, if E is order continuous, then R^2 is already strictly singular.*

The proofs of these results rely on the factorization theorems due to N. Ghoussoub and W. B. Johnson [**61**]. These theorems allow us to factor every order weakly compact operator through an order continuous Banach lattice. In particular, we can apply these techniques to strictly singular operators, because these operators are order weakly compact since they are never an isomorphism on a subspace isomorphic to c_0. Moreover, as we shall see, these factorization have good properties with respect to domination of operators.

Other facts will play an important role in the proof of the above mentioned results; for instance, the domination properties of disjointly strictly singular operators, developed by J. Flores and F. L. Hernández in [**50**], several properties of equi-integrable sets, the subsequence splitting property and approximation results of an operator S with $|S| \leq T$ by elements in the algebraic ideal generated by T, that is operators of the form $\sum_i A_i T B_i$ (see Theorem 1.5.7).

This work has been published in [**54**]. These results are also related with open questions in spectral theory, such as the domination problem for Riesz operators (those sharing the spectral properties of compact operators). This is because if $0 \leq R \leq T : E \to E$ with T Riesz and strictly singular, then by the previous theorem R^4 is strictly singular, and so R is Riesz. More results along this line can be found in [**103**] and [**112**].

Furthermore, part of Chapter 7 is involved with the power and domination problems of Banach-Saks operators. Recall that an operator between Banach spaces $T : X \to Y$ is Banach-Saks whenever every bounded sequence (x_n) in X has a subsequence whose image under T is Cesàro convergent, that is the sequence of arithmetic means $(\frac{1}{N}\sum_{k=1}^{N} T(x_{n_k}))$ converges in the norm of Y. The results obtained are the following:

THEOREM 4. *Let E and F be Banach lattices with F order continuous. If $0 \leq R \leq T : E \to F$ and T is Banach-Saks, then R is also Banach-Saks.*

THEOREM 5. *Let E be a Banach lattice and $0 \leq R \leq T : E \to E$. If T is Banach-Saks, then R^2 is also Banach-Saks.*

These theorems improve previous results by J. Flores and C. Ruiz in [**55**]. Moreover, we will show that these are optimal adapting some known examples. This work has been published in [**56**].

Apart from domination and power problems, strictly singular operators are one of the main subject of this memoir. Thus, from Chapter 2 to Chapter 6, several properties of this class of operators between Banach lattices are presented. Namely, in **Chapter 3**, some characterizations of strictly singular operators in terms of disjointly strictly singular and ℓ_2-singular operators are studied. Recall that an operator $T : E \to Y$ from a Banach lattice into a Banach space is called *disjointly strictly singular* if it is not an isomorphism on the span of any disjoint sequence in E. This class was introduced by F. L. Hernández and B. Rodríguez-Salinas in [**70**], in connection with the study of ℓ_p-complemented copies in function spaces. Moreover, these operators have turned out to be a useful tool in the study of strictly singular operators between Banach lattices, for instance in the domination and power problems [**51**], and when comparing structures of rearrangement invariant spaces, see the work by F. L. Hernández, V. M. Sánchez and E. Semenov in [**71**]. Properties of disjointly strictly singular operators have been deeply studied by J. Flores and F. L. Hernández in [**49**], [**50**] and [**67**]. The results in Chapter 3 show the importance of disjointly strictly singular operators and its close relation with strictly singular operators.

The results presented in Chapter 3 are motivated by the following fact: an endomorphism on $L_p = L_p[0,1]$, for $1 \leq p < \infty$, is strictly singular if and only if it is both ℓ_p-singular and ℓ_2-singular (see the works of V. Milman [**96**] and L. Weis [**132**]). In other words, an endomorphism $T : L_p \to L_p$ is strictly singular if and only if T is disjointly strictly singular and ℓ_2 singular. Recall that an operator is called *ℓ_p-singular* if it is not an isomorphism on any subspace isomorphic to ℓ_p. In [**74**] more recent results about ℓ_p-singular operators can be found.

Our aim in Chapter 3 is to find an extension of this principle to more general Banach lattices. Precisely, we will prove the following:

THEOREM 6. *Let E and F be Banach lattices such that E has finite cotype and F satisfies a lower 2-estimate. An operator $T : E \to F$ is strictly singular if and only if it is disjointly strictly singular and ℓ_2-singular.*

Furthermore, for the class of regular operators (i.e. those which are difference of positive operators) the previous characterization is still valid under weaker assumptions:

THEOREM 7. *Let E and F be Banach lattices such that E has finite cotype and F is order continuous. A regular operator $T : E \to F$ is strictly singular if and only if T is disjointly strictly singular and ℓ_2-singular.*

Both theorems will be obtained from the following fact that in addition allows us to connect strictly singular and AM-compact operators.

THEOREM 8. *Let E be a Banach lattice with finite cotype, Y a Banach space and $T : E \to Y$ an operator. If T is disjointly strictly singular and AM-compact, then T is strictly singular.*

Recall that an operator $T : E \to Y$ from a Banach lattice to a Banach space is called *AM-compact* if the image of every order interval is a relatively compact set.

We will also show by means of an involved example that the hypothesis in the previous theorems cannot be weakened, in the sense that Theorem 6 is not true if the lattice F satisfies a lower q-estimate for some $q > 2$. To this end we will consider the Banach lattice $L_r(\ell_q)$, whose elements are sequences $x = (x_1, x_2, \ldots)$ of functions in L_r such that

$$\|x\|_{L_r(\ell_q)} = \sup_n \Big\| \Big(\sum_{i=1}^{n} |x_i|^q \Big)^{frac1q} \Big\|_{L_r} < \infty.$$

THEOREM 9. *Let $1 < r < p < s < 2 < q < \infty$. There exists an operator $T : L_p \to L_r(\ell_q)$ which is ℓ_p-singular and ℓ_2-singular, but not ℓ_s-singular.*

In particular, the operator T is strictly singular and ℓ_2-singular but not strictly singular.

The proof of this theorem will involve several technical lemmas which deal with estimates for independent, identically distributed random p-stable variables based on results of W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri in [**73**].

Par of the work presented in this chapter has been done during a visit to the University of Missouri-Columbia in fall 2006 under the supervision of N. Kalton.

A different question, that will be considered in **Chapter 4**, is the invariant subspace problem for strictly singular operators. It is well known that the invariant subspace problem consists in determining if an operator $T : X \to X$ in a Banach space X, has always a non-trivial invariant subspace $Y \subset X$ (i.e. $Y \neq \{0\}, X$) such that $T(Y) \subset Y$. This problem was solved in the negative by P. Enflo in [**42**], and simplified by C. Read in [**114**], who even gave an example of an operator in ℓ_1 without invariant subspaces [**115**]. However, the problem remains open for operators in reflexive spaces, or even the apparently simpler case of operators on a Hilbert space. Moreover, there are also various conjectures for several distinguished classes of operators (see [**5**]), such as adjoint operators, and, the most interesting case for us, positive operators.

The invariant subspace problem for positive operators has been mainly studied by Y. Abramovich, C. Aliprantis and O. Burkinshaw in [**3**] and [**4**]. In these papers, the authors show that a positive operator in ℓ_p with $1 \leq p < \infty$, which is quasinilpotent at some positive element $x \in \ell_p$, always has an invariant subspace (which is in fact an ideal of ℓ_p). More generally, if $B : E \to E$ is a positive operator in a Banach lattice E, and there exists another positive operator $S : E \to S$ such that $SB \leq BS$, S is quasinilpotent at some $x \in E_+$ and S dominates a non-null compact operator, then B has a non-trivial invariant ideal.

It is well known that every compact operator on a Banach space has a non-trivial invariant subspace (the proof of this fact in Hilbert spaces was first given by J. Von Neumann and its extension for Banach spaces is due to N. Aronszajn and K. T. Smith [**15**]). Moreover, a big class of operators related to compact operators also have invariant subspaces, for instance, so does every operator commuting with a compact operator, as was proved by V. I. Lomonosov [**88**]. Hence, a natural question is wether this fact can be extended to strictly singular operators, which are in a certain sense close to the class of compact operators. However, C. Read gave in [**116**] an example of a strictly singular operator without invariant subspaces. Despite this fact, for positive strictly singular operators we will provide a positive answer to the invariant subspace problem in a wide class of Banach lattices (those with the R-condition, see Section 1 of Chapter 4), that for instance contains the lattices with finite cotype. Thus, the main result in Chapter 4 is the following:

THEOREM 10. *Let E be a Banach lattice with the R-condition. If $T : E \to E$ is a positive, strictly singular operator, then T has an invariant subspace.*

In addition, in Section 4 of Chapter 4, we will introduce the class of *SS-friendly* operators which is analogous to the class of compact-friendly operators introduced by Y. Abramovich, C. Aliprantis and O. Burkinshaw in [**4**], and we will prove positive results for operators in this class. In particular, we will show that in a Banach lattice with the R-condition, every positive operator commuting with a strictly singular operator has an invariant subspace. To this end we will study the left semi-commutant of an operator

$$|T\rangle = \{S \geq 0 : \, ST \geq TS\}.$$

Moreover, Theorem 4.3.8 on $|T\rangle$ has been recently used and extended in an analogous way to the left semi-commutant $\langle T|$ by H. Gessesse in [**59**].

The work presented in this chapter has been published as a joint work with J. Flores and V. G. Troistky in [**57**].

We will continue in **Chapter 5** extending a result by V. Milman on products of strictly singular operators on L_p spaces (see [**95**], [**96**]). V. Milman proved that every strictly singular operator on $L_p(0,1)$ has compact square. The main result in this chapter extends this theorem for strictly singular operators in a bigger class of Banach lattices that have a structure similar to that of L_p spaces. Precisely, we will say that a Banach lattice is *disjointly homogeneous* whenever every two disjoint sequences share an equivalent subsequence. Notice that this class contains the Lorentz spaces $L_{p,q}(0,1)$ and some Orlicz spaces $L_\varphi(0,1)$, as well as other not so classical spaces such as Tsirelson's space. The main result of this chapter is the following:

THEOREM 11. *Let E be a disjointly homogeneous Banach lattice with finite cotype. If $T : E \to E$ is strictly singular, then T^2 is compact.*

See also the work by G. Androulakis, P. Dodos, G. Sirotkin and V. G. Troitsky in [**13**], for other results in a similar direction.

We will also obtain results for more particular kinds of disjointly homogeneous Banach lattices. We just mention that for $1 \leq p < \infty$, a Banach lattice is called *p-disjointly homogeneous* if every disjoint normalized sequence has a subsequence equivalent to the unit vector basis of ℓ_p.

THEOREM 12. *Let E be a 2-disjointly homogeneous Banach lattice. If $T : E \to E$ is strictly singular, then T is compact.*

THEOREM 13. *Let E be a 1-disjointly homogeneous Banach lattice. If $T : E \to E$ is strictly singular, then T is Dunford-Pettis.*

After showing applications of these general results to the study of strictly singular operators on Lorentz and Orlicz spaces, we provide some examples showing that the previous theorems cannot be extended much further. In particular, we construct an Orlicz space L_φ and a strictly singular operator $T : L_\varphi(0,1) \to L_\varphi(0,1)$ such that no power T^n is a compact operator.

Finally, we study the duality relation of strictly singular operators in Lorentz spaces $L_{p,q}$. Recall that in general the class of strictly singular operators is not closed under taking adjoints. However, for operators between L_p spaces this is true [**132**]. In particular, we prove the following result that complements a theorem of L. Weis for operators on L_p spaces [**132**].

THEOREM 14. *Let $T : L_{p,q} \to L_{p,q}$ with $1 < p < 2$ and $q \notin (p,2)$, or $2 < p < \infty$ and $q \notin (2,p)$, or $p = 2$ and $1 < q < \infty$. The following are equivalent:*

(1) *T is strictly singular.*
(2) *T is $\{\ell_2, \ell_q\}$-singular.*
(3) *There does not exist a subspace $M \subset L_{p,q}$, isomorphic to ℓ_2 or ℓ_q, such that $T(M)$ is complemented in $L_{p,q}$ and the restriction $T|_M$ is an isomorphism.*
(4) *T^* is strictly singular.*
(5) *T^* is $\{\ell_2, \ell_{q'}\}$-singular.*

Notice that the relation between p and q is a necessary hypothesis as will be shown in Example 5.5.3.

Next, **Chapter 6** is devoted to the study of certain properties of operators between vector valued lattices of the kind $E(X)$. Recall that given a Banach lattice of functions over some measure space (Ω, Σ, μ), and a Banach space X, $E(X)$ is defined as the space of functions $f : \Omega \to X$ such that the function $\omega \mapsto \|f(\omega)\|_X$ belongs to the lattice E. Clearly, $E(X)$ is a vector-valued Banach lattice equipped with the norm $\|f\|_{E(X)} = \|\|f(\cdot)\|_X\|_E$.

The relations between properties of $E(X)$ and those of E and X have been the object of study by several authors. For instance, S. Kwapien, in [**82**], proved that X contains a subspace isomorphic to c_0 if and only if $L_1(X)$ has the same property; in a similar spirit, J. Bourgain [**26**] and G. Pisier [**110**] proved independently that X contains a subspace isomorphic to ℓ_1 if and only if so does $L_p(X)$, for any $1 < p < \infty$. More recently, W. Hensgen in [**66**] has proved, among other facts, that $E(X)$ contains a subspace isomorphic to c_0 if and only if E or X have this property. Analogously, G. Emmanuele in [**41**], and L. Drewnowski and I. Labuda in [**39**], have studied under which conditions these spaces contain subspaces isomorphic to ℓ_∞, and more recently, P. Dowling and N. Randranantoanina in [**38**]have obtained similar results on asymptotic isometric copies of c_0 or ℓ_1. Besides, there are some relevant results about weakly compact sets in $L_1(X)$ due to M. Talagrand in [**128**], and in the same line we can find recent results by M. Nowak in [**100**], [**101**] and [**102**].

The work presented in Chapter 6 extends some of these properties to the context of operators. Thus, given an operator $T : X \to Y$ between Banach spaces, we will study the extended operator $T_E : E(X) \to E(Y)$ defined by $T_E(f)(\omega) = T(f(\omega))$ for $f \in E(X)$ and $\omega \in \Omega$. Our main results are the following:

THEOREM 15. *Let E be a Banach lattice which does not contain an isomorphic copy of c_0 and $T : X \to Y$ be an operator between Banach spaces. If the extended operator $T_E : E(X) \to E(Y)$ preserves an isomorphic copy of c_0, then so does $T : X \to Y$.*

THEOREM 16. *Let E be an order continuous Banach lattice such that E^* is also order continuous, and $T : X \to Y$ be an operator between Banach spaces. If the extended operator $T_E : E(X) \to E(Y)$ preserves an isomorphic copy of ℓ_1, then so does $T : X \to Y$.*

Simple examples show that the hypothesis on the lattice E are in both cases needed.

This results can be applied to obtain an extension to the setting of operators of a theorem by J. Hoffmann-Jørgensen [**72**], which is connected with convergence of martingales, and so with the Radon-Nykodim property.

As we mentioned above, last two Chapters are not related to strictly singular operators, while they focus in *factorization* properties of operators between Banach lattices, which have proved to be a recurring topic throughout this work. In **Chapter 7**, besides considering the already mentioned power and domination problems for Banach-Saks operators, we study the problem of factorization of positive Banach-Saks operators through Banach lattices with the Banach-Saks property. This fact is known for the Banach space case, in which B. Beauzamy [**19**] proved that every Banach-Saks operator factors through a Banach space with the Banach-Saks property. For positive operators between Banach lattices our main result is the following:

THEOREM 17. *Let $T : E \to F$ be a positive Banach-Saks operator between Banach lattice E and F, with F order continuous. Then there exist a Banach lattice G with the Banach-Saks property, and operators $T_1 : E \to G$, $T_2 : G \to F$ such that $T = T_2T_1$.*

This theorem is connected with the following question: Does every positive compact operator factor through a reflexive Banach lattice? This question still remains open, and only partial results due to C. Aliprantis and O. Burkinshaw in [**11**] are known. However, for weakly compact operators, M. Talagrand, in [**129**], gave an example of a positive, weakly compact operator from ℓ_1 to $C(0,1)$ that cannot factor through a reflexive Banach lattice. This example will be used in Chapter 7 to show that Theorem 17 is in a sense optimal.

In **Chapter 8** we present two constructions to factor p-convex (respectively q-concave) operators through p-convex (resp. q-concave) Banach lattices. Moreover, starting from the duality between convexity and concavity, the relation between the factorization of an operator and its adjoint is studied. The results here obtained can be seen in some sense as a generalization of the factorization of order weakly compact operators by order continuous Banach lattices due to N. Ghoussoub and W. B. Johnson [**61**]. The motivation stems from results of J. L. Krivine on factorization of operators which are both p-convex and p-concave through spaces L_p [**81**]. In this direction, we obtain factorization properties or operators which are both p-convex and q-concave:

THEOREM 18. *Let E and F be Banach lattices and $T : E \to F$ be a p-convex and q-concave operator ($1 \leq p, q \leq \infty$). For every $\theta \in (0,1)$, we can factor T in the following way*

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "\phi_\theta"'] & F \\ E_\theta \arrow[r, "R_\theta"] & F_\theta \arrow[u, "\varphi_\theta"'] \end{tikzcd}$$

where ϕ_θ and φ_θ are lattice homomorphisms, E_θ is $(\frac{q}{1-\theta})$-concave and F_θ is $(\frac{p}{\theta+(1-\theta)p})$-convex.

The proof of this result relies on some interpolation techniques, which have turned out to be a constant in factorization problems. Precisely, we generalize a result of V. A. Sestakov [**126**] about comparison between the first complex interpolation method $[X_0, X_1]_\theta$ and Calderón-Lozanovskii's construction $X_0^{(1-\theta)} X_1^\theta$. In the literature, these theorems are considered only in the setting of Banach lattices of measurable functions, hence we will need to generalize them to abstract Banach lattices.

Recall that the complex methods of interpolation $[X_0, X_1]_\theta$ and $[X_0, X_1]^\theta$ are defined for a compatible pair of complex Banach spaces (X_0, X_1), that is, two spaces for which there exist continuous inclusions in a topological vector space $i_j : X_j \hookrightarrow X$ for $j = 0, 1$. This construction turns out to be an interpolation method (see [**28**]). However, the construction of Calderón-Lozanovskii $X_0^{(1-\theta)} X_1^\theta$ is defined for Banach lattices which have to be compatible in a more precise sense. Namely, we will say that two Banach lattices E_0, E_1 form a compatible pair of Banach lattices, if there exists a Riesz space (topological vector lattice) X, and continuous inclusions $i_j : E_j \to X$ for $j = 0, 1$, which are interval preserving lattice homomorphisms. Notice that a posteriori, we can suppose that X is a Banach lattice, since by the hypothesis on the inclusions the sum $E_0 + E_1$ is a Banach lattice.

In this context we can consider the space of Calderón-Lozanovskii $E_0^{(1-\theta)}E_1^{\theta}$, as the set of elements $x \in E_0 + E_1$ such that $|x| \leq |x_0|^{1-\theta}|x_1|^{\theta}$ with $x_0 \in E_0$ and $x_1 \in E_1$. The space $E_0^{(1-\theta)}E_1^{\theta}$ is a Banach lattice equipped with the norm

$$\|x\| = \inf\{\lambda > 0 : \ |x| \leq \lambda|x_0|^{1-\theta}|x_1|^{\theta}, \text{ with } \|x_0\|_{E_0} \leq 1, \, \|x_1\|_{E_1} \leq 1\}.$$

It is important to have in mind that this space in general does not satisfy an interpolation theorem ([**90**],[**91**]), although this can be obtained if the lattice are "good" enough. In general, without extra assumptions on the Banach lattices, the construction of Calderón-Lozanovskii constitutes an interpolation method for positive operators.

I would like to finish this introduction thanking all the people that have helped me in this project. First of all: my Thesis advisors, Francisco Hernández and Julio Flores, whose support and encouragement have made this Thesis possible, since with patience and sympathy they have guided me throughout these years, sharing good and bad times. Let me also thank all the people of the Departamento de Análisis Matemático of the Universidad Complutense de Madrid that have somehow contributed to this work. Moreover, I want to thank for the warm welcome of the *Department of Mathematics of the University of Missouri-Columbia* and the *Equipe d'Analyse Fonctionnelle* of the *Université Paris VI/Paris VII.*

Last, but not least, I want to thank my parents and Idoia for all their constant support and the faith they have always put in me. Without a doubt, it is because of them that I have achieved this goal.

CHAPTER 1

Preliminary results

1. Basic facts

In this Chapter we present the most common definitions regarding Banach lattices and operators between them, as well as several well-known results that will be used throughout the text.

Definition 1.1.1. *A partially ordered Banach space E over $\mathbb{R}$ is a Banach lattice provided*

(1) $x \leq y$ *implies* $x + z \leq y + z$*, for every* $x, y, z \in E$*,*
(2) $\lambda x \geq 0$*, for every* $x \geq 0$ *in* E *and every real number* $\lambda \geq 0$*,*
(3) *for every* $x, y \in E$ *there exist a least upper bound* $x \vee y$*, and a greatest lower bound* $x \wedge y$ *in* E*,*
(4) *if* $|x|$ *denotes* $x \vee (-x)$*, then* $\|x\| \leq \|y\|$ *whenever* $|x| \leq |y|$*.*

The set $E_+ = \{x \in E : x \geq 0\}$ is called the *positive cone* of E. We also define the positive and negative part of $x \in E$ as $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$, and the modulus of x as $|x| = x \vee (-x)$. Two elements $x, y \in E$ are called *disjoint* whenever $|x| \wedge |y| = 0$. We will consider order intervals, $[a, b] = \{x \in E : a \leq x \leq b\}$, and we will say that a subset $A \subset E$ is *order bounded* when it is contained in some interval.

An operator between Banach lattices $T : E \to F$ is *positive* if $T(x) \in F_+$ for every $x \in E_+$. We will say that T is *regular* if it can be written as a difference of two positive operators. Moreover, given two positive operators $R, T : E \to F$, we will say that T dominates R ($R \leq T$) whenever the operator $T - R$ is positive, that is $R(x) \leq T(x)$ for every $x \in E_+$. We will denote the class of positive operators by $\mathcal{L}(E, F)_+$, and that of regular operators by $\mathcal{L}_r(E, F)$. An operator $T : E \to F$ will be called a lattice homomorphism if $|T(x)| = T(|x|)$ for every $x \in E$, and we will say that E and F are order isomorphic if there exists a bijective lattice homomorphism $T : E \to F$ whose inverse T^{-1} is also a lattice homomorphism. Finally, $T \in \mathcal{L}_r(E, F)$ will be called interval preserving if $T([0, x]) = [0, Tx]$ for every $x \in E_+$.

A closed subspace X of a Banach lattice E is called a *sublattice* if $x \wedge y$ and $x \vee y$ belong to X, for every $x, y \in X$. A subset $A \subset E$ will be called *solid* if $x \in A$ whenever $|x| \leq |y|$ and $y \in A$. If a sublattice of E is also solid, then it is called an *ideal* of E. Moreover, if an ideal $I \subset E$ satisfies that every order bounded set $A \subset I$ has its supremum in I, then I is a *band*. Given a band B, if there exists a projection $P : E \to B$ such that $0 \leq P(x) \leq x$ for every $x \in E_+$, then B is called a *projection band*.

Definition 1.1.2. *An element $e \in E_+$ is called a strong unit if $E_e = E$ (where E_e denotes the non-closed ideal generated by e, i.e. $E_e = \bigcup_n [-ne, ne]$), it is a weak unit if $B_e = E$ (where B_e denotes the band generated by e), and it is called quasi-interior point if E_e is norm dense in E.*

We clearly have the following implications:

$$Strong\,unit \;\Rightarrow\; Quasi-interior\,point \;\Rightarrow\; Weak\,unit$$

whose converse are in general not true.

2. Order continuous Banach lattices

Definition 1.2.1.

- *A Banach lattice E is order complete (respectively σ-order complete) if every order bounded set (resp. sequence) in E has a lower upper bound.*
- *A Banach lattice E is order continuous (resp. σ-order continuous) if every downward directed set (resp. sequence) $\{x_i\}_{i\in I}$ in E with $\bigwedge_{i\in I} x_i = 0$ satisfies, $\lim_i \|x_i\| = 0$.*

There exist a big deal of characterizations of order continuity on Banach lattices. We present next some of them, whose proofs can be found for instance in [**12**], [**94**], or [**138**].

Theorem 1.2.2. *Given a Banach lattice E, the following are equivalent:*

1. *E is order continuous,*
2. *Every monotone order bounded sequence of E is convergent.*
3. *E is σ-order complete and σ-order continuous.*
4. *Every order bounded disjoint sequence in E is norm convergent to zero.*
5. *For every $x \in E_+$, the order interval $[-x, x]$ is weakly compact.*
6. *Every closed ideal of E is a band.*
7. *Every closed ideal of E is the range of a positive linear projection.*
8. *Every disjoint sequence (x_n^*) in the unit ball of E^* is weak*-convergent to zero.*
9. *E is an ideal of E^{**}.*

Similar characterizations hold for the order continuity of the dual of a Banach lattice.

Theorem 1.2.3. *Given a Banach lattice E, the following are equivalent:*

1. *E^* is order continuous.*
2. *Every disjoint sequence (x_n) in the unit ball of E is weak-convergent to zero.*
3. *E does not contain any sublattice isomorphic to ℓ_1.*
4. *E^* does not contain any subspace isomorphic to c_0.*
5. *The band generated by E in E^{**} coincides with E^{**}.*

Recall that in an order continuous Banach lattice there are plenty of unconditional basic sequences (see [**87**, Theorem 1.c.9]):

Proposition 1.2.4. *Every infinite dimensional subspace of an order continuous Banach lattice contains a subspace with an unconditional basis.*

A Banach lattice is called KB-space if every bounded monotone sequence is convergent. Clearly, every KB-space is order continuous. Moreover, we have the following (see [**87**, pp. 6-8 and 34-35]).

Proposition 1.2.5. *Let E be a Banach lattice. The following are equivalent:*

(1) *E is a KB-space.*
(2) *E does not contain a subspace isomorphic to c_0.*
(3) *E does not contain a sublattice isomorphic to c_0.*
(4) *E is a band in its bidual E^{**}.*
(5) *E is weakly sequentially complete.*

We will also use the following well known characterization of reflexive Banach lattices (see [**12**, Theorem 14.23] and [**87**, Theorem 1.c.5]).

Theorem 1.2.6. *A Banach lattice E is reflexive if and only if no subspace of E is isomorphic to c_0 or ℓ_1, if and only if no sublattice of E is lattice isomorphic to c_0 or ℓ_1.*

We present now a very useful representation theorem for order continuous Banach lattices (cf. [**87**, Theorem 1.b.14]):

Theorem 1.2.7. *Let E be an order continuous Banach lattice with a weak unit. There exist a probability space (Ω, Σ, μ), an (in general not closed) ideal $\widetilde{E}$ of $L_1(\Omega, \Sigma, \mu)$ and a lattice norm $\|\cdot\|_{\widetilde{E}}$ on $\widetilde{E}$ such that*

(1) *E is order isometric to $(\widetilde{E}, \|\cdot\|_{\widetilde{E}})$.*
(2) *$\widetilde{E}$ is dense in $L_1(\Omega, \Sigma, \mu)$ and $L_\infty(\Omega, \Sigma, \mu)$ is dense in $\widetilde{E}$.*
(3) *$\|f\|_1 \leq \|f\|_{\widetilde{E}} \leq 2\|f\|_\infty$, whenever $f \in L_\infty(\Omega, \Sigma, \mu)$.*
(4) *The dual of the isometry given in (1) maps E^* onto the Banach lattice $\widetilde{E}^*$ of all μ-measurable functions g for which*
$$\|g\|_{\widetilde{E}^*} = \sup\Big\{ \int_\Omega fgd\mu : \|f\|_{\widetilde{E}} \leq 1\Big\} < \infty.$$
The value taken by the functional corresponding to g at $f \in \widetilde{E}$ is $\int_\Omega fgd\mu$.

For practical purposes the existence of a weak unit is usually replaced by separability. This is because if (x_n) denotes a dense sequence in the ball of a separable Banach lattice E, then $e = \sum_{n=1}^\infty \frac{|x_n|}{2^n}$ is clearly a quasi-interior point, so in particular it is a weak unit.

Throughout the text we will also make use of the *Kadeč-Pełczyński disjointification method* in the setting of order continuous Banach lattices (see [**47**]). Recall that by Theorem 1.2.7, an order continuous Banach lattice E with weak unit can be assumed to be included in some $L_1(\Omega, \Sigma, \mu)$, so denote this inclusion by $i : E \hookrightarrow L_1(\Omega, \Sigma, \mu)$

THEOREM 1.2.8. *Let X be any subspace of an order continuous Banach lattice E with weak unit. Then, either*

(1) *X contains an almost disjoint normalized sequence, that is , there exist a normalized sequence $(x_n) \subset X$ and a disjoint sequence $(z_n) \subset E$ such that $\|z_n - x_n\| \to 0$, or,*
(2) *X is strongly embedded. Precisely, the inclusion $i|_X$ restricted to X is an isomorphism.*

Note that more can be said if instead of a subspace we consider a normalized sequence $(x_n) \subset X$; now the alternative is

(1) either $(\|x_n\|_{L_1})$ is bounded away from zero,
(2) or there exist a subsequence (x_{n_k}) and a disjoint sequence $(z_k) \subset X$ such that

$$\|z_k - x_{n_k}\| \underset{k\to\infty}{\longrightarrow} 0.$$

Apart from Theorem 1.2.7, there are other well-known representation theorems for abstract Banach lattices due to S. Kakutani. Recall that for $1 \leq p < \infty$, a Banach lattice E is called an abstract L_p space (or in short, AL-space in the case $p = 1$), if $\|x+y\|^p = \|x\|^p + \|y\|^p$ for every $x, y \in E$ with $x \wedge y = 0$. Analogously, E is called an abstract M space (or AM-space in short), if $\|x + y\| = \max(\|x\|, \|y\|)$, whenever $x, y \in E$ satisfy $x \wedge y = 0$. The proofs of the following theorems can be found in [**87**, 1.b]:

THEOREM 1.2.9. *For $1 \leq p < \infty$, an abstract L_p space E is order isometric to an $L_p(\mu)$ space over some measure space (Ω, Σ, μ). If E has a weak unit, then μ can be chosen to be a finite measure.*

THEOREM 1.2.10. *Any AM-space E is order isometric to a sublattice of a $C(K)$ space, for some compact Hausdorff space K. If, in addition, E has a strong unit, then E is order isometric to a $C(K)$ space.*

The importance of these theorems in Banach lattice theory is clear. As a consequence of the second one, we obtain a local representation of Banach lattices which is useful in many applications. Indeed, for any positive element u in a Banach lattice E, the ideal E_u generated by u is an AM-space under the norm

$$\|x\|_\infty = \inf\{\lambda \geq 0 : \, |x| \leq \lambda u\}.$$

Hence, according to Theorem 1.2.10 it is order isometric to a space $C(K)$ for some compact Hausdorff space K. This allows us to represent the elements in E_u as "real" functions and operate with them as usual.

3. Convexity, concavity, and related notions

Recall that the functional calculus of J. L. Krivine allows us to translate an inequality that holds for the real numbers to the same inequality for elements in a Banach lattice. In particular, this can be used to give a meaning to expressions of the form $\big(\sum_{i=1}^{n} |x_i|^p\big)^{1/p}$ in a general Banach lattice, which will be needed for the following notions (see the discussion in[**87**] around Theorem 1.d.1).

Given a Banach lattice E and a Banach space X, an operator $T : E \to X$ is *q-concave* for $1 \leq q \leq \infty$, if there exists a constant $M < \infty$ so that

$$\Big(\sum_{i=1}^{n} \|Tx_i\|^q\Big)^{\frac{1}{q}} \leq M \Big\|\Big(\sum_{i=1}^{n} |x_i|^q\Big)^{\frac{1}{q}}\Big\|, \qquad \text{if } 1 \leq q < \infty,$$

or

$$\max_{1\leq i\leq n} \|Tx_i\| \leq M \Big\|\bigvee_{i=1}^{n} |x_i|\Big\|, \qquad \text{if } q = \infty,$$

for every choice of vectors $(x_i)_{i=1}^n$ in E (cf. [**87**, 1.d]). The smallest possible value of M is denoted by $M_{(q)}(T)$.

Similarly, an operator $T : X \to E$ is *p-convex* for $1 \leq p \leq \infty$, if there exists a constant $M < \infty$ such that

$$\Big\|\Big(\sum_{i=1}^{n} |Tx_i|^p\Big)^{\frac{1}{p}}\Big\| \leq M \Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{\frac{1}{p}}, \qquad \text{if } 1 \leq p < \infty,$$

or

$$\Big\|\bigvee_{i=1}^{n} |Tx_i|\Big\| \leq M \max_{1\leq i\leq n} \|x_i\|, \qquad \text{if } p = \infty,$$

for every choice of vectors $(x_i)_{i=1}^n$ in X. The smallest possible value of M is denoted by $M^{(p)}(T)$. Recall that a Banach lattice is q-concave (resp. p-convex) whenever the identity operator is q-concave (resp. p-convex).

Recall that a Banach space X has *cotype q* for some $2 \leq q < \infty$ if there exist a constant $C < \infty$ so that, for every finite set of vectors $(x_j)_{j=1}^n$ in X, we have

$$\Big(\sum_{j=1}^{n} \|x_j\|^q\Big)^{\frac{1}{q}} \leq C \int_0^1 \Big\|\sum_{j=1}^{n} r_j(t)x_j\Big\| dt,$$

where r_j denotes the j^{th} Rademacher function. Analogously, we say that X has type p for some $1 < p \leq 2$, if there exists a constant $C < \infty$ o that, for every finite set of vectors $(x_j)_{j=1}^n$ in X, we have

$$\int_0^1 \Big\|\sum_{j=1}^{n} r_j(t)x_j\Big\| dt \leq C \Big(\sum_{j=1}^{n} \|x_j\|^p\Big)^{\frac{1}{p}}.$$

A Banach lattice E satisfies an *upper p-estimate* for some $1 < p < \infty$ if there exists a constant $M < \infty$, such that for every choice of pairwise disjoint elements $(x_j)_{j=1}^n$ in E, we have

$$\Big\|\sum_{j=1}^{n} x_j\Big\| \leq M \Big(\sum_{j=1}^{n} \|x_j\|^p\Big)^{\frac{1}{p}}.$$

In a similar way, we say that E satisfies a *lower q-estimate* for some $1 < q < \infty$ if there exists a constant $M < \infty$, such that for every choice of pairwise disjoint elements $(x_j)_{j=1}^n$ in E, we

have
$$\Big(\sum_{j=1}^n \|x_j\|^q\Big)^{1/q} \leq M\Big\|\sum_{j=1}^n x_j\Big\|.$$

The relations between these notions have been deeply studied. Recall that for $1 \leq p \leq \infty$ an operator T is p-convex (respectively p-concave) if and only if its adjoint T^* is q-concave (respectively q-convex), where $\frac{1}{q}+\frac{1}{p}=1$, (see [**87**, Proposition 1.d.4]). Similarly, for $1<p<\infty$, a Banach lattice E satisfies an upper, respectively, lower p-estimate, if and only if its dual E^* satisfies a lower, respectively, upper q-estimate, where $\frac{1}{p}+\frac{1}{q}=1$ (see [**87**, Proposition 1.f.5]). The duality for type and cotype is not so satisfactory in general, since for instance $X=c_0$ is of no type $p>1$ while $X^*=\ell_1$ is of cotype 2. However, we have that if X is a Banach space of type p for some $p>1$, then its dual X^* is of cotype q, with $\frac{1}{p}+\frac{1}{q}=1$ (see [**87**, Proposition 1.e.17]).

Notice that it is clear that a p-convex (respectively p-concave) Banach lattice always satisfies an upper (respectively lower) p-estimate. Although, in general, the converse is not true, it holds that if a Banach lattice E satisfies an upper, respectively, lower r-estimate for some $1<r<\infty$, then it is p-convex, respectively q-concave, for every $1<p<r<q<\infty$. We will also use frequently the fact that a Banach lattice has finite cotype if and only if it has finite concavity. Notice also that Banach lattices with finite cotype have order continuous norm. We refer to [**87**, 1.f] for the details of these questions and related ones.

The following result of B. Maurey will be repeatedly used (see [**87**, Theorem 1.d.6]):

Theorem 1.3.1. *Let E be a q-concave Banach lattice for some $q<\infty$. Then there exists a constant $C<\infty$ such that, for every sequence $(x_i)_{i=1}^n$ of elements of E, we have*
$$C^{-1}\Big\|\Big(\sum_{i=1}^n |x_i|^2\Big)^{\frac{1}{2}}\Big\| \leq \int_0^1 \Big\|\sum_{i=1}^n r_i(t)x_i\Big\|dt \leq C\Big\|\Big(\sum_{i=1}^n |x_i|^2\Big)^{\frac{1}{2}}\Big\|,$$
where $(r_i)_{i=1}^n$ denote the Rademacher functions on $[0,1]$.

4. Equi-integrable sets in Banach lattices

A bounded subset A of a Banach lattice E is said to be *L-weakly compact* if $\|x_n\|\to 0$ for every disjoint sequence (x_n) contained in the solid hull of A. Equivalently, A is L-weakly compact if for every $\varepsilon>0$ there exists $x\in E_+^a$ such that $A\subset[-x,x]+\varepsilon B_E$ (E^a denotes the maximal order ideal in E on which the induced norm is order continuous, and B_E denotes the closed unit ball of E). A subset M of an order continuous Banach lattice of functions over a measure space (Ω,Σ,μ) (see Theorem 1.2.7) is *equi-integrable* if
$$\sup_{f\in M}\|f\chi_A\|\to 0 \quad \text{when} \quad \mu(A)\to 0.$$

We will make use of the following standard facts (cf. [**51**] and [**94**, §3.6]):

Proposition 1.4.1. *Every L-weakly compact subset A of a Banach lattice E is relatively weakly compact. If E is an AL-space the converse also holds.*

LEMMA 1.4.2. *Let E be a Banach lattice with order continuous norm and a weak unit, and hence representable as an order ideal in $L^1(\Omega, \Sigma, \mu)$ for some probability space (Ω, Σ, μ).*

a) A bounded subset of E is equi-integrable if and only if it is L-weakly compact.

b) A norm bounded sequence (g_n) in E is convergent to zero if and only if (g_n) is equi-integrable and $\| \|_1$-convergent to zero.

PROOF. The proof of (a) is straightforward. For (b), notice that if (g_n) is convergent to zero in E, then it is clearly equi-integrable and convergent in L_1. For the converse, take an equi-integrable sequence (g_n), such that $\|g_n\|_1$ tends to zero, and let $\varepsilon > 0$. By hypothesis, there exists $\delta > 0$ such that $\|\chi_A g_n\|_E < \frac{\varepsilon}{2}$ for all n and for every set $A \in \Sigma$ with $\mu(A) < \delta$. Moreover, since (g_n) is bounded in E, there exists $\alpha > 0$ such that $\mu\{\omega : |g_n(\omega)| > \alpha\} < \delta$ for every n. Hence, $\|\chi_{A_n} g_n\|_E \leq \frac{\varepsilon}{2}$ for every n, where $A_n = \{\omega : |g_n(\omega)| > \alpha\}$.

On the other hand, since $\|g_n\|_1 \to 0$, there exists a subsequence such that $(g_{n_k}) \to 0$ μ-almost everywhere. Let

$$h_j = \sup k \geq j |\chi_{\Omega \setminus A_{n_k}} g_{n_k}|.$$

Clearly, the sequence is bounded by α and decreases to zero. Hence, by the order continuity of E, we have $\|h_j\|_E \to 0$ (see Theorem 1.2.2). In particular, we can consider $i \in \mathbb{N}$ large enough so that $\|\chi_{\Omega \setminus A_{n_i}} g_{n_i}\|_E \leq \frac{\varepsilon}{2}$. Thus, we obtain

$$\|g_{n_i}\|_E \leq \|\chi_{A_{n_i}} g_{n_i}\|_E + \|\chi_{\Omega \setminus A_{n_i}} g_{n_i}\|_E \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This finishes the proof. □

LEMMA 1.4.3. *Let T be a regular operator from a Banach lattice E into a Banach lattice F with order continuous norm. If $A \subset E$ is L-weakly compact, then $T(A)$ is L-weakly compact.*

PROOF. Let $\varepsilon > 0$. By hypothesis, there exists $x \in E_+^a$, such that

$$A \subseteq [-x, x] + \frac{\varepsilon}{\|T\|} B_E.$$

Note that since F is order continuous, it is in particular Dedekind complete, so $\mathcal{L}_r(E, F)$ is a Dedekind complete vector lattice (cf. [**94**, Theorem 1.3.2]). Therefore, the modulus $|T|$ exists, and clearly we have

$$T(A) \subseteq [-|T|x, |T|x] + \varepsilon B_F,$$

where $|T|x \in F_+^a = F_+$. Thus $T(A)$ is L-weakly compact as claimed. □

LEMMA 1.4.4. *Let (Ω, Σ, μ) be a finite measure space and (f_n) be a weakly convergent sequence in $L_1(\mu)$. If (f_n) converges to zero in measure, then (f_n) converges to zero in the $\| \|_1$ norm.*

PROOF. We can assume without loss of generality that $\mu(\Omega) = 1$. The sequence (f_n) is equi-integrable since it is weakly convergent (cf. [**40**, Cor. IV.8.11]). Hence for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\chi_B f_n\|_1 < \varepsilon/2$ for every integer n and every $B \in \Sigma$ with $\mu(B) < \delta$.

Consider $B_n = \{t \in \Omega : |f_n(t)| > \varepsilon/2\}$. By assumption there exists an integer n_0 such that $\mu(B_n) < \delta$ for $n \geq n_0$. Thus, for $n \geq n_0$ we have

$$\|f_n\|_1 = \int_{B_n} |f_n| + \int_{\Omega\setminus B_n} |f_n| \leq \|\chi_{B_n} f_n\|_1 + \frac{\varepsilon}{2}\mu(\Omega\setminus B_n) < \varepsilon.$$

□

The following Theorem gives some characterizations of equi-integrable sets in L_1 (see [**139**, III.C.12]).

THEOREM 1.4.5. *Let μ be a probability measure and let S be a bounded subset of $L_1(\mu)$. The following are equivalent:*

(1) *S is not relatively weakly compact in $L_1(\mu)$.*
(2) *S is not equi-integrable.*
(3) *there exists an $\varepsilon > 0$ such that for every integer N there exist N disjoint sets $A_1, \dots, A_N$ such that*
$$\sup\left\{\int_{A_n} |f| d\mu : f \in S\right\} \geq \varepsilon, \quad n = 1, 2, \dots, N.$$
(4) *there exists a basic sequence $(f_n) \subset S$ equivalent to the unit vector basis in ℓ_1.*

A Banach lattice X with an order continuous norm satisfies the *subsequence splitting property* ([**73**] and [**134**]) if for every bounded sequence (f_n) in X there exist a subsequence (f_{n_k}) and sequences (g_k), (h_k) in X with $|g_k| \wedge |h_k| = 0$ and $f_{n_k} = g_k + h_k$ for all k, such that (g_k) is equi-integrable in X, and (h_k) is disjoint. It is known that every Banach lattice with finite cotype has the subsequence splitting property (see [**73**] or [**134**]).

5. Operators on Banach lattices

Given a Banach lattice E and a Banach space Y, an operator $T : E \to Y$ is *order weakly compact* if $T[-x, x]$ is relatively weakly compact for every $x \in E_+$.

An operator $T : E \to Y$ is order weakly compact if it is not an isomorphism on the span of a positive, disjoint, order bounded sequence of E equivalent to the unit basis of c_0 (see [**94**, Corollary 3.4.5]). Similarly, if X is a Banach space and F a Banach lattice, an operator $T : X \to F$ has an order weakly compact adjoint T^* if and only if T is not an isomorphism on any subspace $M \subset X$ isomorphic to ℓ_1 such that $T(M)$ is a complemented subspace of F (see [**94**, Theorem 3.4.14]).

We recall now two basic constructions of factorization for positive operators (see [**61**, Theorems I.2 and I.6] and [**12**, §5.2]).

THEOREM 1.5.1. *Let E_1, E_2 be Banach lattices and operators $0 \leq R \leq T : E_1 \to E_2$. There exist a Banach lattice F, a lattice homomorphism $\phi : E_1 \to F$ and operators $0 \leq R^F \leq T^F$*

such that $T = T^F\phi$ *and* $R = R^F\phi$*:*

$$\begin{tikzcd}
E_1 \arrow[rr, "T"] \arrow[rr, dashed, swap, "R"] \arrow[dr, swap, "\phi"] & & E_2 \\
& F \arrow[ur, "T^F"] \arrow[ur, dashed, swap, "R^F"] &
\end{tikzcd}$$

Moreover, F *is order continuous if and only if* $T : E_1 \to E_2$ *is order weakly compact.*

THEOREM 1.5.2. *Let* E_1, E_2 *be Banach lattices and operators* $0 \leq R \leq T : E_1 \to E_2$*. There exist a Banach lattice* G*, a lattice homomorphism* $\psi : G \to E_2$ *and operators* $0 \leq R^G \leq T^G$ *such that* $T = \psi T^G$ *and* $R = \psi R^G$*:*

$$\begin{tikzcd}
E_1 \arrow[rr, "T"] \arrow[rr, dashed, swap, "R"] \arrow[dr, "T^G"] \arrow[dr, dashed, swap, "R^G"] & & E_2 \\
& G \arrow[ur, swap, "\psi"] &
\end{tikzcd}$$

Moreover G^* *is order continuous if and only if* $T^* : E_2^* \to E_1^*$ *is order weakly compact.*

Notice that the Banach lattice F is obtained by completing the normed lattice E_1/I where $I = \{x \in E_1 : T|x| = 0\}$, under the norm $q_T(x + I) = \|T|x|\|$. On the other hand, the Banach lattice G is obtained by interpolating E_2 with its norm and the Minkowski functional of the solid convex hull of $T(B_{E_1})$. Notice that we have included here the version of these theorems that is more suitable for the setting of domination theorems, in general the operator T need not be positive. See [**61**] for details.

Recall that an operator $T : E \to X$ from a Banach lattice to a Banach space is AM-compact, whenever $T[-x, x]$ is a relatively compact set for every $x \in E$.

The following fact is also well known.

LEMMA 1.5.3. *Let* E *be a Banach lattice and* Y *a Banach space. If* $T : E \to Y$ *is AM-compact, then* $T(A)$ *is relatively compact for any* L*-weakly compact set* A *in* E*.*

PROOF. Let A be an L-weakly compact set in E. Then for every $\varepsilon > 0$ there exists $x_\varepsilon \in E_+^a$, such that $A \subset [-x_\varepsilon, x_\varepsilon] + \varepsilon B_E$. Therefore, we have $T(A) \subset T([-x_\varepsilon, x_\varepsilon]) + \varepsilon\|T\|B_E$. Since $T : E \to X$ is AM-compact, $T([-x_\varepsilon, x_\varepsilon])$ is a relatively compact set in X. A standard argument shows then that $T(A)$ is relatively compact, and the proof is finished. $\square$

Recall that an operator between Banach spaces $T : X \to Y$ is *strictly singular* (*SS* in short), if it is not invertible on any (closed) infinite dimensional subspace of X. That is, for every infinite dimensional subspace $M \subset X$, and every $\alpha > 0$ there exists $x \in M$ such that

$$\|Tx\| \leq \alpha\|x\|.$$

An operator $T : X \to Y$ is strictly singular if and only if for every infinite dimensional subspace $M \subset X$, there exists another infinite dimensional subspace $N \subset M$ such that the restriction

$T|_N$ is a compact operator. The class of strictly singular operators forms a closed ideal in the sense of A. Pietsch.

Given a Banach lattice E and a Banach space Y, an operator $T : E \to Y$ is called *disjointly strictly singular* (*DSS* in short) if it is not invertible on any subspace of E generated by a disjoint sequence. Clearly, every strictly singular operator is also disjointly strictly singular. Although this class is not an operator ideal, it only lacks being closed by composition from the right.

In general, given a Banach space Z we will say that an operator $T : X \to Y$ preserves an isomorphic copy of Z if it is an isomorphism on a subspace of X isomorphic to Z. We will say that T is Z-singular if it does not preserve an isomorphic copy of Z.

The following result about operators preserving a copy of c_0 on Banach lattices will be useful (see [**61**]).

Theorem 1.5.4. *Given a Banach lattice E and a Banach space X, an operator $T : E \to X$ preserves an isomorphic copy of c_0 if and only if T preserves a lattice isomorphic copy of c_0.*

For operators preserving copies of ℓ_1, C. Niculescu [**98**] proved that an analogous statement to Theorem 1.5.4 holds for positive operators $T : E \to F$ with F an order continuous Banach lattice. More recently, Z. Chen has given in [**32**] some results on weakly sequentially precompact sets in Banach lattices which yield a more general version of this theorem. First, recall that an operator $T : X \to Y$ between Banach spaces is called weakly sequentially precompact if for every bounded sequence (x_n) in X, (Tx_n) has a weakly Cauchy subsequence. The following Theorem corresponds to [**32**, Theorem 1.1]:

Theorem 1.5.5. *Let E and F be Banach lattices, such that F is order continuous or there exists a weak order unit of F^*_+. If $T : E \to F$ is not weakly sequentially precompact, then T preserves a lattice isomorphic copy of ℓ_1.*

Now we show how this can be used in our context.

Theorem 1.5.6. *Let E and F be Banach lattices with F order continuous, and $T : E \to F$ be a bounded operator. If T preserves an isomorphic copy of ℓ_1, then T preserves a lattice isomorphic copy of ℓ_1.*

Proof. If T preserves a copy of ℓ_1 then there is a normalized sequence (x_n) such that (x_n) and (Tx_n) are both equivalent to the unit vector basis of ℓ_1. Therefore, (Tx_n) has no weakly Cauchy subsequence. Thus, T is not weakly sequentially precompact and the required conclusion follows from Theorem 1.5.5. □

Freudenthal's Theorem states that, under certain conditions, an operator R, such that $|R| \leq T$, can be approximated in the sense of order by components of T (see [**94**, Section 1.2]). This means that there exists a sequence $(S_n)_{n=1}^{\infty}$ of components of T such that

$$0 \leq R - S_n \leq \frac{1}{n} T$$

for each natural number n.

If the operator T has good properties, it is possible to replace the previous approximation with order components by an approximation in norm with operators in the algebraic ideal of T. In this direction, we say that an operator T has order continuous norm whenever every sequence of positive operators with $|T| \geq T_n \downarrow 0$ in $\mathcal{L}(E, F)$ satisfies $\|T_n\| \downarrow 0$. We consider the set

$$I_T = \{S \in \mathcal{L}(E, F) : \text{ there exists } n \in \mathbb{N} \text{ such that } |S| \leq n|T|\},$$

and denote by $Ring(T)$ the closure of the set of operators in $\mathcal{L}(E, F)$ of the form $\sum_{i=1}^n R_i T S_i$ with $S_i \in \mathcal{L}(E)$, $R_i \in \mathcal{L}(F)$. The following theorem shows how this concepts are related (see [**12**, Theorem 5.70] for a proof).

THEOREM 1.5.7. *Let E be a Banach lattice which is either σ-Dedekind complete or has a quasi-interior point, and let F be a Dedekind complete Banach lattice. If T has order continuous norm, then*

$$I_T \subseteq Ring(T).$$

The following theorem due to P. G. Dodds and D. Fremlin [**37**, Theorem 5.1], provides a way to check if an operator has order continuous norm. Recall that an operator between Banach lattices $T : E \to F$ is *M-weakly compact* if $\|Tx_n\| \to 0$ for every norm bounded disjoint sequence (x_n) in E.

THEOREM 1.5.8. *Let E and F be Banach lattices with F Dedekind complete. An operator $T : E \to F$ has order continuous norm if and only if it is both L- and M-weakly compact.*

Another fact we will be using is the following theorem due to L. Weis in [**133**]. This allows us to consider regular endomorphism on order continuous Banach lattice with weak unit as operators defined between L_p spaces.

THEOREM 1.5.9. *Let E be an order complete Banach lattice with a weak unit and a strictly positive linear functional. If $T : E \to E$ is a regular operator, then there is a representation of E as an ideal of functions over a probability space (Ω, Σ, μ), such that $L_\infty(\Omega, \Sigma, \mu) \subset E \subset L_1(\Omega, \Sigma, \mu)$ and $T : L_p(\Omega, \Sigma, \mu) \to L_p(\Omega, \Sigma, \mu)$ is bounded for every $1 \leq p \leq \infty$, with $\|T\|_{\mathcal{L}(L_p)} \leq 4\||T|\|_{\mathcal{L}(E)}$.*

The following fact will also be useful (see [**87**, Proposition 1.d.9 and Theorem 1.f.14]).

THEOREM 1.5.10. *Let E and F be Banach lattices and $T : E \to F$ be an operator. For every choice of $(x_i)_{i=1}^n$ in E we have:*

(1) *If T is positive and $1 \leq p \leq \infty$,*

$$\Big\|\Big(\sum_{i=1}^n |Tx_i|^p\Big)^{\frac{1}{p}}\Big\| \leq \|T\| \Big\|\Big(\sum_{i=1}^n |x_i|^p\Big)^{\frac{1}{p}}\Big\|, \qquad \text{if } p < \infty,$$

$$\text{and} \qquad \Big\| \bigvee_{i=1}^{n} |Tx_i| \Big\| \leq \|T\| \Big\| \bigvee_{i=1}^{n} |x_i| \Big\|, \qquad \text{if } p = \infty.$$

(2) *Without restriction on T,*

$$\Big\| \Big(\sum_{i=1}^{n} |Tx_i|^2 \Big)^{\frac{1}{2}} \Big\| \leq K_G \|T\| \Big\| \Big(\sum_{i=1}^{n} |x_i|^2 \Big)^{\frac{1}{2}} \Big\|,$$

where K_G denotes the universal Grothendieck constant.

Also recall that the concepts of interval preserving operator and lattice homomorphism are related under duality as follows (see [**1**, Thm. 1.35]):

Theorem 1.5.11. *Let $T : E \to F$ be a positive operator between Banach lattices.*

(1) *If T is interval preserving, then its adjoint $T^* : F^* \to E^*$ is a lattice homomorphism.*
(2) *T is a lattice homomorphism if and only if T^* is interval preserving.*

6. Some Banach spaces results and notation

Recall that a Banach space is said to have the *Schur property* whenever every weakly null sequence is norm convergent to zero. The simplest example of a Banach space with the Schur property is ℓ_1. In the context of Banach lattices there is a weaker notion: the positive Schur property. Namely, a Banach lattice has the *positive Schur property* if every positive, weakly null sequence is convergent. Clearly, every Banach lattice with the Schur property has the positive Schur property. Some examples of Banach lattices with the positive Schur property (but not the Schur property) are the $L_1(\mu)$ spaces, the Orlicz function spaces $L^{x\log^p(1+x)}[0,1]$ for $p > 0$, and the Lorentz function spaces $L^{p,1}[0,1]$ for $1 < p < \infty$ (cf. [**137**]).

We will make frequent use of the following results for basic sequences (see [**86**, Proposition 1.a.9 and Proposition 1.a.12]).

Theorem 1.6.1. *Let (x_n) be a normalized basis of a Banach space X with basis constant K. Let (y_n) be a sequence in X with $\sum_{n=1}^{\infty} \|x_n - y_n\| < \frac{1}{2}K$. Then (y_n) is a basis of X which is equivalent to (x_n).*

Proposition 1.6.2. *Let X be a Banach space with a basis (x_n). Let (y_n) be a weakly null sequence such that $\|y_n\|$ does not tend to zero. Then there is a subsequence (y_{n_k}) which is equivalent to a block basis of (x_n), i.e. a sequence (u_k) of the form $u_k = \sum_{n=p_k+1}^{p_{k+1}} a_n x_n$, where (p_k) is an increasing sequence of positive integers.*

The following result, known as Rosenthal's Lemma, provides a useful dichotomy for bounded sequences in Banach spaces (see [**63**, Theorem I.4.10]).

Theorem 1.6.3. *Let (x_n) be a bounded sequence in a Banach space X. Then (x_n) has a subsequence (x_{n_k}) satisfying one of the two mutually exclusive conditions:*

(1) *(x_{n_k}) is equivalent to the unit vector basis of ℓ_1,*

(2) (x_{n_k}) *is weakly Cauchy.*

As usual, the expression $\| \sum_{n=1}^k a_n f_n \| \sim \| \sum_{n=1}^k a_n g_n \|$ will mean that there exist constants c, $C > 0$ such that for any $k \in \mathbb{N}$ and any $(a_n)_{n=1}^k$ we have $c \| \sum_{n=1}^k a_n f_n \| \leq \| \sum_{n=1}^k a_n g_n \| \leq C \| \sum_{n=1}^k a_n f_n \|$.

We refer to [**87**], [**94**] and [**12**], for unexplained terminology from Banach lattices and positive operator theory.

CHAPTER 2

Power and domination problems for strictly singular operators

In this chapter we study the domination and power problems for strictly singular operators. The chapter is organized as follows: the first section is mainly devoted to the proof of two domination theorems for strictly singular operators that will be used afterwards. In the second section we present the proof of the main result concerning the power problem, and its consequences. Finally, in the last section we include some applications to the class of super strictly singular operators and some remarks.

Part of this chapter has been published as a joint work with J. Flores and F. L. Hernández in [**54**].

1. Domination results

In this section we present new domination results for strictly singular operators between Banach lattices, which improve some others obtained in [**51**]. In addition, they will be used in next section for the power problem. We need to recall some previous results first about domination of disjointly strictly singular operators due to J. Flores and F. L. Hernández in [**50**].

Theorem 2.1.1. *Let E and F be Banach lattices such that F is order continuous. If T is disjointly strictly singular and $0 \leq R \leq T : E \to F$, then R is also disjointly strictly singular.*

Here is our first domination result for SS operators.

Proposition 2.1.2. *Let E and F be Banach lattices such that F has the positive Schur property, and operators $0 \leq R \leq T : E \to F$. If T strictly singular, then so is R.*

Proof. Note first that the positive Schur property implies that F does not contain an isomorphic copy of c_0. Otherwise, F would also contain a sequence of positive, pairwise disjoint elements $(e_n)_{n=1}^{\infty}$ equivalent to the unit vector basis of c_0 (see Proposition 1.2.5). This sequence must be weakly null and yet not convergent in norm, which contradicts the fact that F has the positive Schur property. In particular, by Proposition 1.2.5 we get that F is order continuous.

Suppose that R is not strictly singular, therefore, there exists an infinite dimensional subspace X (which can be assumed separable) in E such that $R|_X$ is an isomorphism. Since F does not contain an isomorphic copy of c_0, neither does $R(X)$. Moreover, if $R(X)$ contained an isomorphic copy of ℓ_1, then R would be an isomorphism on the span of a disjoint sequence equivalent to the canonical basis of ℓ_1 (see Theorem 1.5.6); however T is disjointly strictly singular and F order continuous, so by Theorem 2.1.1, this yields a contradiction. Therefore, by Theorem 1.2.6, $R(X)$, and hence X, must be reflexive.

Now, we consider the ideal E_X generated by X in E. We claim that $T|_{E_X}$ is M-weakly compact. Indeed, by [**61**, Thm I.2(c)] we have the following factorization

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "\phi"'] & & F \\ & H \arrow[ur, "T^H"'] & \end{tikzcd}$$

where ϕ is a lattice homomorphism and the Banach lattice H does not contain an isomorphic copy of c_0. Let B_X denote the closed unit ball of X, which is a weakly compact set. Thus, $\phi(B_X)$ is also weakly compact, and since H does not contain c_0, by Proposition 1.2.5 and [**12**, Theorem 4.39], the solid hull $so(\phi(B_X))$ is also weakly compact. Since ϕ is a lattice homomorphism, $\phi(so(B_X)) \subset so(\phi(B_X))$, and therefore $\phi(so(B_X))$ is also weakly compact.

Now, notice that the closed unit ball of E_X coincides with the solid hull of the unit ball B_X. Hence, if (x_n) is a normalized positive disjoint sequence in E_X, by the previous remarks $(\phi(x_n))$ must have a weakly convergent subsequence. We will show that this sequence is in fact weakly null. In order to see this, we make use of the representation Theorem 1.2.7, and consider the closed ideal generated by $e = \sum_{n=1}^{\infty} \phi(x_n)$ in H as a space of functions over (Ω, Σ, μ) included in $L_1(\Omega, \Sigma, \mu)$. Since (x_n) are pairwise disjoint, and ϕ is a lattice homomorphism, $(\phi(x_n))$ are also pairwise disjoint, so in particular the sequence $(\phi(x_n))$ tends to zero in measure. Since $(\phi(x_n))$ was already weakly convergent in H, it is also weakly convergent in $L_1(\Omega, \Sigma, \mu)$. Therefore, by Lemma 1.4.4, $\|\phi(x_n)\|_1 \to 0$ when $n \to \infty$. This implies that the weak limit of $(\phi(x_n))$ in H has to be zero, as desired. In particular, $T(x_n) \to 0$ weakly, and since T is positive, by the positive Schur property of F, we get $T(x_n) \to 0$ in norm. This proves that $T|_{E_X}$ is M-weakly compact, as claimed.

Let us now consider $\hat{X}$, the sublattice of E generated by X, which is separable because X is separable [**12**, pp. 204-205]. It is clear that the restriction

$$T|_{\hat{X}} : \hat{X} \to F$$

is M-weakly compact too, and since F is order continuous, by Theorem 1.5.8, this implies that the operator $T|_{\hat{X}}$ has order continuous norm. Moreover, $\hat{X}$ is separable, so it has a quasi-interior point, and F is Dedekind complete; therefore by Theorem 1.5.7 we get that $R|_{\hat{X}} \in Ring(T|_{\hat{X}})$. Thus, since $T|_{\hat{X}}$ is strictly singular and these form a two sided closed operator ideal, then $Ring(T|_{\hat{X}})$ only consists of strictly singular operators, so in particular $R|_{\hat{X}}$ is strictly singular. But then R cannot be an isomorphism when restricted to X. This is a contradiction that finishes the proof. $\square$

Theorem 2.1.3. *Let E be a Banach lattice with the subsequence splitting property, and F an order continuous Banach lattice. If $0 \leq R \leq T : E \to F$ with T strictly singular, then R is strictly singular.*

Proof. Since T is strictly singular, by [**94**, Theorem 3.4.14], the adjoint T^* is order weakly compact. Thus, by Theorem 1.5.2, we have the following factorization diagram:

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[rr, dashed, "R"'] \arrow[dr, "T^G"] \arrow[dr, dashed, "R^G"'] & & F \\ & G \arrow[ur, "\psi"'] & \end{tikzcd}$$

where G^* is order continuous. Moreover, since F is order continuous, by [**61**, Proposition I.4.d] it follows that G is also order continuous.

We claim that the operator $T^G : E \to G$ is strictly singular. Indeed, since T is strictly singular, for every infinite dimensional subspace M of E there exists another infinite dimensional subspace N of M such that T restricted to N is compact. This means that the set $T(B_N)$ is relatively compact in F, and, by [**12**, Theorem 5.40], this implies that $T^G(B_N)$ is also relatively compact in G (since $T(B_N)$ is trivially contained in the solid convex hull of $T(B_E)$). Hence, T^G is strictly singular.

Since, in particular, T^G is disjointly strictly singular, and G is order continuous, it follows from Theorem 2.1.1 that the operator R^G is also disjointly strictly singular.

We claim that R^G is in fact strictly singular. Suppose the contrary, that is, R^G is an isomorphism when restricted to some separable subspace H of E. Consider $\widehat{H}$, the sublattice of E generated by H, which is also separable. By [**87**, Proposition 1.a.9.], $R^G(\widehat{H})$ is contained in some ideal A of G with weak unit, which by Theorem 1.2.7 can be represented as a function space contained in $L_1(\Omega, \Sigma, \mu)$. Let j denote the inclusion $j : A \hookrightarrow L_1(\Omega, \Sigma, \mu)$. Since $L_1(\Omega, \Sigma, \mu)$ has the positive Schur property, and $0 \leq jR^G \leq jT^G$, Proposition 2.1.2 yields that the operator

$$jR^G|_{\widehat{H}} : \widehat{H} \to A \hookrightarrow L_1$$

is strictly singular. Thus, we can consider an infinite dimensional subspace H' of H such that $jR^G|_{H'}$ is compact. Since E is order continuous, by Proposition 1.2.4, there exists an unconditional basic sequence contained in H'. Let us denote by X the span of this sequence, where R^G is invertible and jR^G is compact.

Consider the subspace $R^G(X)$ of G, and let us apply Theorem 1.2.8. If the norms of G and L_1 were equivalent on $R^G(X)$, then the operator $jR^G : E \to L_1$ would be an isomorphism when restricted to X. However, this is impossible since jR^G is compact when restricted to X. Therefore, by Theorem 1.2.8, $R^G(X)$ contains an almost disjoint sequence.

Let us denote this sequence by $(R^G(f_n))$, which can be assumed to be normalized. Since G^* is order continuous, $R^G(f_n) \to 0$ weakly (see Theorem 1.2.3). Hence, since R^G is an isomorphism on X, (f_n) is also a weakly null, seminormalized unconditional basic sequence in X. Moreover, since $jR^G|_X$ is compact we have

$$\|R^G(f_n)\|_1 \to 0.$$

Since E has the subsequence splitting property, we can extract a subsequence (still denoted (f_n)) and sequences (g_n) and (h_n) such that

$$|g_n|,\ |h_n| \leq |f_n|, \quad \text{and} \quad f_n = g_n + h_n,$$

with (g_n) equi-integrable in X, and (h_n) disjoint. Notice first that the sequence $(\|g_n\|)$ is bounded from below. Otherwise, the operator R^G would be invertible on the span $[h_n]$. Indeed, if $\|g_n\| \to 0$, by the perturbation Theorem 1.6.1, passing to a subsequence we would have that (f_n) and (h_n) are equivalent basic sequences, and since $\|R^G(g_n)\| \to 0$ we would also have that $(R^G(f_n))$ and $(R^G(h_n))$ are equivalent basic sequences. Hence, for scalars $(a_n)_{n=1}^k$ we would have

$$\begin{aligned}
\Big\|\sum_{n=1}^k a_n h_n\Big\| &\leq C_1 \Big\|\sum_{n=1}^k a_n f_n\Big\| \\
&\leq C_1\delta_0 \Big\|R^G\Big(\sum_{n=1}^k a_n f_n\Big)\Big\| \\
&\leq C_1 C_2 \delta_0 \Big\|R^G\Big(\sum_{n=1}^k a_n h_n\Big)\Big\| \\
&\leq C_1 C_2 \delta_0 \|R^G\| \Big\|\sum_{n=1}^k a_n h_n\Big\|
\end{aligned}$$

where $C_1, C_2 < \infty$ are the equivalence constants between (f_n), (h_n) and respectively $(R^G(f_n))$, $(R^G(h_n))$, and δ_0 is a lower bound for the operator R^G restricted to the span $[f_n]$. However, this is not possible because (h_n) are pairwise disjoint and R^G is disjointly strictly singular.

Now, if the sequence of absolute values $(|h_n|)$ has no weak Cauchy subsequence, then, by Rosenthal's Theorem 1.6.3, it has a subsequence $(|h_{n_k}|)$ which is equivalent to the unit vector basis of ℓ_1. Hence, for scalars $(a_k)_{k=1}^{\infty}$ we have:

$$\begin{aligned}
\Big\|\Big(\sum_{k=1}^n a_k f_{n_k}\Big)\Big\| &\geq K^{-1} \int_0^1 \Big\|\Big(\sum_{k=1}^n a_k r_k(t) f_{n_k}\Big)\Big\| dt \\
&\geq A_1 K^{-1} \Big\|\Big(\sum_{k=1}^n |a_k|^2 |f_{n_k}|^2\Big)^{\frac{1}{2}}\Big\| \\
&\geq A_1 K^{-1} \Big\|\Big(\sum_{k=1}^n |a_k|^2 |h_{n_k}|^2\Big)^{\frac{1}{2}}\Big\| \\
&= A_1 K^{-1} \Big\|\sum_{k=1}^n a_k |h_{n_k}|\Big\| \\
&\geq A_1 K^{-1} C \sum_{k=1}^n |a_k|
\end{aligned}$$

where A_1 is the constant given in Theorem 1.3.1, K is the unconditional constant of (f_n), and C is the equivalence constant between $(|h_{n_k}|)$ and the unit vector basis of ℓ_1. Hence, (f_{n_k}) is equivalent to the unit vector basis of ℓ_1, and the operator R^G preserves an isomorphic copy of ℓ_1. However, this implies that R^G preserves a lattice copy of ℓ_1 (see Theorem 1.5.6), but since R^G is disjointly strictly singular, this is a contradiction. Thus, $(|h_n|)$ must have a weakly Cauchy subsequence.

Therefore, since L_1 is weakly sequentially complete, the sequence $(|h_n|)$ has a weakly convergent subsequence in L_1. In fact, its limit must be zero, since being disjoint, it converges to zero in measure, so Lemma 1.4.4 yields that

$$\|h_n\|_1 \to 0.$$

Using Theorem 1.5.9, since R^G is bounded as an operator on L_1, we also have

$$\|R^G(h_n)\|_1 \to 0.$$

Note that (h_n) does not converge to zero in E. Otherwise, the sequence (f_n) would inherit the equi-integrability of the sequence (g_n), and since R^G is positive, the sequence $(R^G(f_n))$ would also be equi-integrable by Lemma 1.4.3. But since this sequence is also almost disjoint, this would imply that $R^G(f_n) \to 0$ in the norm of G, which is a contradiction. Therefore, we can assume that $\|h_n\| > \rho$, for some $\rho > 0$.

We claim that $(R^G(f_n))$ and $(R^G(h_n))$ are equivalent basic sequences in G. Indeed, the sequence (g_n) is norm bounded because $|g_n| \leq |f_n|$ for all n. Moreover, it is equi-integrable, and by Lemma 1.4.3, $(R^G(g_n))$ is equi-integrable too. On the other hand, we have

$$\|R^G(g_n)\|_1 = \|R^G(f_n) - R^G(h_n)\|_1 \leq \|R^G(f_n)\|_1 + \|R^G(h_n)\|_1 \to 0.$$

Hence, $R^G(g_n)$ goes also to zero in the norm of G by Lemma 1.4.2. Thus, by passing to a subsequence, we may assume that $\sum\limits_{n=1}^{\infty} \|R^G(f_n) - R^G(h_n)\|$ is a convergent series. By the perturbation Theorem 1.6.1, there is a constant $\alpha > 0$ such that

$$\alpha^{-1}\Big\|\sum_{n=1}^{\infty} a_n R^G(h_n)\Big\| \leq \Big\|\sum_{n=1}^{\infty} a_n R^G(f_n)\Big\| \leq \alpha\Big\|\sum_{n=1}^{\infty} a_n R^G(h_n)\Big\|.$$

Hence, we have

$$\begin{aligned}
\Big\|R^G\Big(\sum_{n=1}^{\infty} a_n h_n\Big)\Big\| &= \Big\|\sum_{n=1}^{\infty} a_n R^G(h_n)\Big\| \geq \alpha^{-1}\Big\|\sum_{n=1}^{\infty} a_n R^G(f_n)\Big\| \\
&\geq \beta\alpha^{-1}\Big\|\sum_{n=1}^{\infty} a_n f_n\Big\| \geq \beta\alpha^{-1}A_1K^{-1}\Big\|\Big(\sum_{n=1}^{\infty}|a_n|^2|f_n|^2\Big)^{\frac{1}{2}}\Big\| \\
&\geq \beta\alpha^{-1}A_1K^{-1}\Big\|\Big(\sum_{n=1}^{\infty}|a_n|^2|h_n|^2\Big)^{\frac{1}{2}}\Big\| = \beta\alpha^{-1}A_1K^{-1}\Big\|\sum_{n=1}^{\infty} a_n h_n\Big\|
\end{aligned}$$

where A_1 is the above mentioned constant, K is the unconditional constant of (f_n), and β is a lower bound for the operator R^G restricted to X. Therefore, R^G is an isomorphism when restricted to the span of the disjoint sequence (h_n). This is a contradiction with the fact that R^G is disjointly strictly singular.

Hence, R^G cannot be an isomorphism when restricted to any subspace of E, in other words R^G is strictly singular, and so is R. The proof is finished. □

Note that the above result improves [**51**, Theorem 3.1], since the order continuity of the dual E^* is no longer required.

2. Powers of dominated operators

In this section we study the power problem for strictly singular endomorphisms. The key result is the following:

Theorem 2.2.1. *Let*

$$\begin{tikzcd} E_1 \arrow[r, shift left, "T_1"] \arrow[r, shift right, dashed, "R_1"'] & E_2 \arrow[r, shift left, "T_2"] \arrow[r, shift right, dashed, "R_2"'] & E_3 \arrow[r, shift left, "T_3"] \arrow[r, shift right, dashed, "R_3"'] & E_4 \arrow[r, shift left, "T_4"] \arrow[r, shift right, dashed, "R_4"'] & E_5 \end{tikzcd}$$

be operators between Banach lattices, such that $0 \leq R_i \leq T_i$ *for* $i = 1, 2, 3, 4$.

If T_1, T_3 *are strictly singular, and* T_2, T_4 *are order weakly compact, then the composition* $R_4R_3R_2R_1$ *is also strictly singular.*

Proof. Let us suppose that $R_4R_3R_2R_1$ is not strictly singular. Then, there exists an infinite dimensional subspace M of E_1 such that $R_4R_3R_2R_1|_M$ is an isomorphism. Clearly we can suppose that M is separable.

Since T_2 is an order weakly compact operator, by Theorem 1.5.1, we have the following factorizations:

$$\begin{tikzcd} E_2 \arrow[rr, shift left, "T_2"] \arrow[rr, shift right, dashed, "R_2"'] \arrow[dr, "\phi"'] & & E_3 \\ & F \arrow[ur, shift left, "T_2^F"] \arrow[ur, shift right, dashed, "R_2^F"'] & \end{tikzcd}$$

where F is an order continuous Banach lattice, ϕ is a lattice homomorphism and $0 \leq R_2^F \leq T_2^F$.

Let us consider the subspace $X = \phi R_1(M) \subset F$, which is separable, hence it is contained in a closed ideal $A \subset F$ with weak order unit (cf. [**87**, Proposition 1.a.9]) which, by Theorem 1.2.2, is complemented in F by a positive projection say $P : F \to A$. Therefore, by Theorem 1.2.7, A, as an order continuous Banach lattice with weak unit, can be represented as a dense ideal of $L_1(\Omega, \Sigma, \mu)$ for some probability measure μ so that the formal inclusion $j : A \hookrightarrow L_1(\Omega, \Sigma, \mu)$ is continuous.

$$\begin{tikzcd} E_1 \arrow[r, shift left, "T_1"] \arrow[r, shift right, dashed, "R_1"'] & E_2 \arrow[rr, shift left, "T_2"] \arrow[rr, shift right, dashed, "R_2"'] \arrow[dr, "\phi"'] & & E_3 \arrow[r] & E_4 \arrow[r] & E_5 \\ & & F \arrow[ur, shift left, "T_2^F"] \arrow[ur, shift right, dashed, "R_2^F"'] \arrow[d, "P"] & & & \\ & & A \arrow[r, hook, "j"] & L_1(\mu) & & \end{tikzcd}$$

Now we apply the Kadeč-Pełczyński Theorem 1.2.8 to $X \subset F$; then either there exist a normalized sequence $(x_n) \subset X$ and a disjoint sequence $(w_n) \subset F$ such that $\|w_n - x_n\| \to 0$ or the restriction $j|_X$ is an isomorphism.

Suppose first that the restriction $j|_X$ is an isomorphism. We consider the following operators

$$0 \leq jP\phi R_1 \leq jP\phi T_1 : E_1 \to L_1(\Omega, \Sigma, \mu).$$

Since T_1 is strictly singular, then so is $jP\phi T_1$. Now, since $L_1(\Omega, \Sigma, \mu)$ has the positive Schur property, by Proposition 2.1.2, we get that the operator $jP\phi R_1$ is also strictly singular. Since jP is an isomorphism restricted to X, ϕR_1 cannot be an isomorphism when restricted to M. This is a contradiction with the assumption that $R_4R_3R_2R_1|_M$ is an isomorphism. This finishes the proof in this case.

Alternatively, let us assume that there exist a sequence (x_n) in X, and a disjoint sequence (w_n) in F such that $\|w_n - x_n\| \to 0$; then passing to a subsequence, if needed, by Theorem 1.6.1, we can suppose that they are equivalent basic sequences.

Since the operator T_4 is order weakly compact, by Theorem 1.5.1 there exists an order continuous Banach lattice H such that the following factorizations hold:

$$\begin{tikzcd}
E_1 \arrow[r] & E_2 \arrow[rr] \arrow[dr] & & E_3 \arrow[r, shift left, "T_3"] \arrow[r, dashed, shift right, "R_3"'] & E_4 \arrow[rr, shift left, "T_4"] \arrow[rr, dashed, shift right, "R_4"'] \arrow[dr, "\varphi"'] & & E_5 \\
& & F \arrow[ur, shift left, "T_2^F"] \arrow[ur, dashed, shift right, "R_2^F"'] & & & H \arrow[ur, shift left, "T_4^H"] \arrow[ur, dashed, shift right, "R_4^H"'] &
\end{tikzcd}$$

where $0 \leq R_4^H \leq T_4^H$.

Now, let us consider the operator

$$\widetilde{T} = \varphi T_3 T_2^F : F \to H,$$

which is strictly singular because T_3 is. In particular, $\widetilde{T}$ is disjointly strictly singular, and since H is order continuous, and $\widetilde{T}$ dominates the operator

$$\widetilde{R} = \varphi R_3 R_2^F : F \to H,$$

Theorem 2.1.1 implies that $\widetilde{R}$ is disjointly strictly singular.

However, we are assuming that the restriction of the operator $R_4R_3R_2R_1|_M$ is an isomorphism, and so must be $\widetilde{R}|_{\phi R_1(M)}$ (since it is a factor of the former operator). Now since $\|w_n - x_n\| \to 0$, we can find a subsequence of natural numbers (n_j) such that

$$\sum_{j=1}^{\infty} \|w_{n_j} - x_{n_j}\| < \frac{1}{2} C_1,$$

and

$$\sum_{j=1}^{\infty} \|\widetilde{R} w_{n_j} - \widetilde{R} x_{n_j}\| < \frac{1}{2} C_2,$$

where C_1 and C_2 are respectively the basis constants of (x_n) and $(\widetilde{R} x_n)$. This implies that the operator $\widetilde{R}$ is invertible on the span of the disjoint sequence (w_{n_j}), in contradiction with the fact that $\widetilde{R}$ is disjointly strictly singular. The proof is finished. $\square$

As a consequence we get the following:

COROLLARY 2.2.2. *Let E be a Banach lattice, and consider operators $0 \leq R \leq T : E \to E$. If T is strictly singular, then R^4 is also strictly singular.*

PROOF. Since T is strictly singular, it cannot preserve an isomorphic copy of c_0, so, in particular by [**94**, Corollary 3.4.5], it is order weakly compact. It suffices to apply Theorem 2.2.1 with $E_i = E$, $R_i = R$ and $T_i = T$ for $i = 1, 2, 3, 4$. $\square$

COROLLARY 2.2.3. *Let* $0 \leq R \leq T : E \to F$, *and* $0 \leq S \leq V : F \to G$. *If* F *and* G *are order continuous Banach lattices, and* T *and* V *are strictly singular operators, then* SR *is strictly singular.*

In particular, if $0 \leq R \leq T : E \to E$, *with* T *strictly singular and* E *order continuous, then* R^2 *is strictly singular.*

PROOF. Since F is order continuous, by Theorem 1.2.2, the order intervals in F are weakly compact, hence the identity $I_F : F \to F$ is order weakly compact. Let us consider the Banach lattices

$$E_1 = E,\ E_2 = F,\ E_3 = F,\ E_4 = G \text{ and } E_5 = G;$$

and the operators

$$T_1 = T,\ T_2 = I_F,\ T_3 = V \ \text{ and } \ T_4 = I_G.$$

These satisfy the hypothesis of Theorem 2.2.1, hence we obtain that $I_G S I_F R = SR$ is strictly singular, as desired. $\square$

The last assertion of this corollary was proved under stronger assumptions in [**51**, Theorem 3.17].

Notice that, in general, the domination problem for strictly singular endomorphisms is not trivial ([**51**]):

EXAMPLE 2.2.4. *There exist operators* $0 \leq R \leq T : L^2[0,1] \oplus \ell_\infty \to L^2[0,1] \oplus \ell_\infty$ *such that* T *is strictly singular but* R *is not.*

Indeed, consider the rank one operator $Q : L^1[0,1] \to \ell_\infty$ defined by $Q(f) = (\int f, \int f, \ldots)$. Take also an isometry $S : L^1[0,1] \to \ell_\infty$ given by $S(f) = (h_n'(f))$, where (h_n) is a dense sequence in the unit ball of $L^1[0,1]$, and (h_n') is a sequence of norm one functionals such that $h_n'(h_n) = \|h_n\|$ for all $n \in \mathbb{N}$. If $J : L^2[0,1] \hookrightarrow L^1[0,1]$ denotes the canonical inclusion, then the operator $SJ : L^2[0,1] \to \ell_\infty$ is not strictly singular.

Since ℓ_∞ is Dedekind complete we have that $|SJ|$, $(SJ)^+$ and $(SJ)^-$ are also continuous operators between $L^2[0,1]$ and ℓ_∞. It is easy to see that $|SJ| \leq QJ$. Since SJ is not strictly singular, we must have that either $(SJ)^+$ or $(SJ)^-$ is not strictly singular, so let us assume without loss of generality that $(SJ)^+$ is not strictly singular. Now consider the matrices of operators:

$$R = \begin{pmatrix} 0 & 0 \\ (SJ)^+ & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 \\ QJ & 0 \end{pmatrix},$$

which clearly define operators with the required properties.

Note that $L^2[0,1] \oplus \ell_\infty$ is not an order continuous Banach lattice. But the square R^2 is the zero operator, which obviously is strictly singular.

Some natural questions remain open: Do there exist an order continuous Banach lattice E, and operators $0 \leq R \leq T : E \to E$ such that T is strictly singular but R is not? Do there exist a (non order continuous) Banach lattice E and operators $0 \leq R \leq T : E \to E$ such that T is strictly singular but R^3 is not?

3. Related questions

In this section, we give some domination results under weaker conditions on the Banach lattices, by imposing extra conditions on the dominating operator. First, a combination of domination theorems for weakly compact and Dunford-Pettis operators yields the following:

PROPOSITION 2.3.1. *Let E and F be Banach lattices with F order continuous, and operators $0 \leq R \leq T : E \to F$. If T is both weakly compact and Dunford-Pettis, then R is strictly singular.*

PROOF. The domination theorems for weakly compact [**136**] and Dunford-Pettis operators [**77**], give us that $R : E \to F$ is both weakly compact and Dunford-Pettis (because of the order continuity of F). And this implies that R is strictly singular. Indeed, suppose that there exists a subspace X in E such that the restriction $R|_X$ is an isomorphism. Since R is weakly compact, for every bounded sequence (x_n) in X we can find a subsequence (x_{n_k}) such that (Rx_{n_k}) is weakly convergent. Since $R|_X$ is an isomorphism, this implies that (x_{n_k}) is already weakly convergent; but R is Dunford-Pettis, and therefore (Rx_{n_k}) is norm convergent. Thus, the sequence (x_{n_k}) must be norm convergent since the restriction $R|_X$ is an isomorphism. We have shown that every bounded sequence in X has a convergent subsequence, so X must be finite dimensional. □

Notice that the previous proposition is not true without the order continuity hypothesis (see Example 2.2.4).

COROLLARY 2.3.2. *Let E be a Banach lattice, and $0 \leq R \leq T : E \to E$ positive operators. If T is weakly compact and Dunford-Pettis, then R^2 is strictly singular.*

In particular, if T is compact, then R^2 is strictly singular.

PROOF. Since T is weakly compact, in particular it is order weakly compact, so by Theorem 1.5.1 we have the following diagram:

$$\begin{tikzcd}
E \arrow[r, shift left, "T"] \arrow[r, dashed, shift right, "R"'] & E \arrow[rr, shift left, "T"] \arrow[rr, dashed, shift right, "R"'] \arrow[dr, "\phi"'] & & E \\
& & F \arrow[ur, shift left, "T^F"] \arrow[ur, dashed, shift right, "R^F"'] &
\end{tikzcd}$$

with F an order continuous Banach lattice, and $0 \leq \phi R \leq \phi T : E \to F$. Since T is compact, ϕT is weakly compact and Dunford-Pettis. By the previous Proposition, ϕR is strictly singular, and so is R^2. $\square$

Note that along similar lines, Theorems 1.5.1 and 1.5.2, together with Dodds-Fremlin domination theorem for compact operators [**37**], provide an alternative proof for the following theorem due to C. D. Aliprantis and O. Burkinshaw, already mentioned in connection with the power problem for compact operators.

THEOREM 2.3.3. *Let E be a Banach lattice, and $0 \leq R \leq T : E \to E$. If T is compact, then R^3 is compact.*

PROOF. Let $0 \leq R \leq T : E \to E$ be positive operators in an arbitrary Banach lattice, such that T is compact. By Schauder's theorem T^* is also compact, so we have that both T and T^* are order weakly compact operators. By theorems 1.5.1 and 1.5.2 we have factorizations:

$$\begin{tikzcd}
E \arrow[rr, shift left, "T"] \arrow[rr, dashed, shift right, "R"'] \arrow[dr, shift left, "T^G"] \arrow[dr, dashed, shift right, "R^G"'] & & E \arrow[r, shift left, "T"] \arrow[r, dashed, shift right, "R"'] & E \arrow[rr, shift left, "T"] \arrow[rr, dashed, shift right, "R"'] \arrow[dr, "\phi"'] & & E \\
& G \arrow[ur, "\psi"'] & & & F \arrow[ur, shift left, "T^F"] \arrow[ur, dashed, shift right, "R^F"'] &
\end{tikzcd}$$

with F and G^* order continuous Banach lattices. Now we have $0 \leq \phi R\psi \leq \phi T\psi : G \to F$, and $\phi T\psi$ is compact, so by Dodds-Fremlin theorem [**37**], $\phi R\psi$ is also compact. In particular the operator R^3 is compact. $\square$

There exists a local version of strictly singular operators: an operator $T : X \to Y$ between Banach spaces is called *super strictly singular* if there does not exist a number $c > 0$ and a sequence of subspaces E_n of X, with $\dim E_n = n$, such that

$$\|Tx\| \geq c\|x\|$$

for all $x \in \bigcup_n E_n$. This class forms an operators ideal.

Note that compact operators are super strictly singular, and super strictly singular operators are always strictly singular.

Given a Banach space X, and a free ultrafilter $\mathcal{U}$ in $\mathbb{N}$ we denote $X_{\mathcal{U}}$ the ultrapower of X along $\mathcal{U}$; in addition, given an operator between Banach spaces $T : X \to Y$, we denote by $T_{\mathcal{U}} : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ the natural extension given by $T_{\mathcal{U}}([x_n]_{\mathcal{U}}) = [Tx_n]_{\mathcal{U}}$.

An operator between Banach spaces $T : X \to Y$ is super strictly singular if and only if, for any free ultrafilter $\mathcal{U}$, the operator $T_{\mathcal{U}} : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ is strictly singular ([**93**]). More properties of this class of operators have been studied in [**53**] and [**111**]. Regarding the domination of super strictly singular operatros we have the following:

COROLLARY 2.3.4. *Let E be a Banach lattice, and $0 \leq R \leq T : E \to E$. If T is super strictly singular, then R^4 is also super strictly singular.*

PROOF. Let $\mathcal{U}$ be a free ultrafilter in $\mathbb{N}$. It is known that the ultrapower $E_{\mathcal{U}}$ is also a Banach lattice, and $0 \leq R_{\mathcal{U}} \leq T_{\mathcal{U}} : E_{\mathcal{U}} \to E_{\mathcal{U}}$. Since T is super strictly singular, by the previous remark the operator $T_{\mathcal{U}}$ is strictly singular; therefore, $(R_{\mathcal{U}})^4$ is also strictly singular by Corollary 2.2.2. Since $(R_{\mathcal{U}})^4 = (R^4)_{\mathcal{U}}$, we obtain that R^4 is super strictly singular. $\square$

Using Theorem 2.1.3, and the fact that every ultrapower $E_{\mathcal{U}}$ of a Banach lattice E is p-concave for some $p < \infty$ if and only if E is p-concave (cf. [**64**, p. 39]), we get the following:

COROLLARY 2.3.5. *Let E be a p-concave Banach lattice for some $p < \infty$, and $0 \leq R \leq T : E \to E$. If T is super strictly singular, then R is super strictly singular.*

Note that the hypothesis of Corollary 2.3.5 cannot be weakened as the following example shows (cf. [**53**])

EXAMPLE 2.3.6. *Let $E = (\oplus_{n=1}^{\infty} \ell_1^n)_p \oplus (\oplus_{n=1}^{\infty} \ell_\infty^{2^n})_q$, be the usual direct sum spaces with the p-norm and q-norm respectively and $1 < p < q < \infty$. There exist operators*

$$0 \leq P \leq Q : E \to E$$

such that Q is super strictly singular, but P is not.

PROOF. For every n consider the rank one operator

$$\begin{array}{cccc} T_n : & \ell_1^n & \longrightarrow & \ell_\infty^{2^n} \\ & (a_1, \ldots, a_n) & \longmapsto & (\sum a_n, \ldots, \sum a_n) \end{array}$$

Consider also for every n the isometry $R_n : \ell_1^n \to \ell_\infty^{2^n}$ represented by the $(2^n \times n)$ matrix with $\{1, -1\}$-entries defined as follows

$$R_n \equiv (x_{k,\ell}) = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & \ldots & 1 & -1 \\ 1 & 1 & \ldots & -1 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -1 & -1 & \ldots & -1 & -1 \end{pmatrix}$$

(we set $x_{k,\ell} = \epsilon_k(\ell)$, where ϵ_k, for $k = 1, ..., 2^n$ is an enumeration of $\{-1, 1\}^n$). Consider now the operators

$$T = \oplus T_n \qquad \text{and} \qquad R = \oplus R_n$$

from $(\oplus_{n=1}^{\infty} \ell_1^n)_p$ into $(\oplus_{n=1}^{\infty} \ell_\infty^{2^n})_q$. The operator T is positive and factors through the natural injection $i : \ell_p \hookrightarrow \ell_q$. Indeed, the operator

$$\varphi : (\oplus_{n=1}^{\infty} \ell_1^n)_p \to \ell_p$$

defined as $\varphi(x_n) = (\sigma(x_n))$, where

$$\sigma(x_n) = \sum_{i=1}^{n} x_{n,i} \ ,$$

is well defined and bounded. Consider next the bounded operator

$$\psi : \ell_q \to (\oplus_{n=1}^{\infty} \ell_{\infty}^{2^n})_q$$

defined as $\psi(a_n) = \oplus a_n 1_{2^n}$, where 1_{2^n} is the unit of $\ell_{\infty}^{2^n}$. Notice that $T = \psi\ i\ \varphi$. Since i is super strictly singular as observed by Plichko (cf. [**111**, Cor.1]), the operator T itself is super strictly singular.

On the other hand, it is clear that the operator R cannot be super strictly singular since R_n is an isometry for every n. Standard facts show that R is regular and that the inequalities $0 \leq R^-, R^+ \leq |R| \leq T$ hold true. From this we obtain that R^+ and R^- cannot be both super strictly singular since R is not. Suppose without loss of generality that R^+ is not super strictly singular, and consider the operators P and Q defined by the matrices:

$$P = \begin{pmatrix} 0 & 0 \\ R^+ & 0 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}$$

the conclusion follows. $\square$

CHAPTER 3

A Characterization of strictly singular operators in Banach lattices

In this chapter we give some characterizations of strictly singular operators between Banach lattices, in terms of AM-compactness and disjoint strict singularity. Since strictly singular operators are disjointly strictly singular, we are mainly interested in converse statements.

The first section is devoted to the proofs of these characterizations of strictly singular operators both for regular operators and the general case. The second one mainly presents an example to show that the hypotheses in the characterization theorems are necessary.

Notice that for an order continuous Banach lattice E, it holds that if an operator $T : E \to Y$ is disjointly strictly singular and ℓ_p-singular, for *every* $1 \leq p \leq 2$, then T is strictly singular. This is shown by using Kadeč-Pełczyński Theorem 1.2.8, and a theorem of D. J. Aldous which states that every subspace of L_1 contains a subspace isomorphic to some ℓ_p for $1 \leq p \leq 2$ ([**7**]). Furthermore, in the special case of X (or Y) being a Banach lattice with type 2, a similar argument shows that if $T : X \to Y$ is disjointly strictly singular and ℓ_2-singular, then T is strictly singular. An analogous statement also holds for inclusion operators between rearrangement invariant spaces (see [**68**]).

The connection between AM-compact operators and ℓ_2-singular operators is studied in Section 1 (see Propositions 3.1.4 and 3.1.5). Our motivation stems from [**120**], where it was proved that for endomorphisms on L_1 being AM-compact and ℓ_2-singular coincide.

Part of the work of this chapter was done while the author was visiting the University of Missouri-Columbia Mathematics Department in fall 2006, under the supervision of N. Kalton.

1. Relation with ℓ_2-singular and DSS operators

We will make use of the following:

Lemma 3.1.1. *Let (f_n) be a weakly null normalized sequence in $L_p(\mu)$, $1 < p < \infty$, which is uniformly bounded (i.e. there exists $M < \infty$ such that $|f_n| \leq M$ for every $n \in \mathbb{N}$). Then, there is a subsequence (f_{n_k}) equivalent to the unit vector basis of ℓ_2.*

Proof. Since it is uniformly bounded and normalized, by Lemma 1.4.2 we have $\inf \|f_n\|_{L_1} > 0$. Hence, (f_n) is seminormalized in every L_q, $1 \leq q \leq \infty$. Since it is weakly null in L_p, it has to be weakly null in every L_q with $1 \leq q < \infty$. So, for every $1 < q < \infty$, passing to a subsequence, it is equivalent to a block basis of the Haar system in L_q (see Proposition 1.6.2), which is an unconditional basis by [**87**, Theorem 2.c.5]. Hence, there is a subsequence (f_{n_k}) which is an unconditional basic sequence in L_q.

Let $1 < q \leq p$ with $1 < q \leq 2$, and consider a subsequence (f_{n_k}) which is seminormalized and unconditional in both L_q and L_p. Thus, for scalars $(a_i)_{i=1}^n$ we have:

$$\begin{aligned}\Big\|\sum_{i=1}^n a_i f_{n_i}\Big\|_{L_p} &\geq \Big\|\sum_{i=1}^n a_i f_{n_i}\Big\|_{L_q} \geq \frac{1}{K_1}\int_0^1 \Big\|\sum_{i=1}^n r_i(t) a_i f_{n_i}\Big\|_{L_q} dt \\ &\geq \frac{1}{K_1} C_1 \Big(\sum_{i=1}^n \|a_i f_{n_i}\|_{L_q}^2\Big)^{\frac{1}{2}} \geq \frac{1}{K_1} C_1 \inf \|f_i\|_{L_q} \Big(\sum_{i=1}^n |a_i|^2\Big)^{\frac{1}{2}},\end{aligned}$$

where C_1 is the cotype 2 constant of L_q and K_1 is the unconditional constant of (f_{n_i}) in L_q. On the other hand, we have

$$\begin{aligned}\Big\|\sum_{i=1}^n a_i f_{n_i}\Big\|_{L_p} &\leq K_2 \int_0^1 \Big\|\sum_{i=1}^n r_i(t) a_i f_{n_i}\Big\|_{L_p} dt \leq K_2 C_2 \Big\|\Big(\sum_{i=1}^n |a_i f_{n_i}|^2\Big)^{\frac{1}{2}}\Big\|_{L_p} \\ &\leq 2K_2 C_2 M \Big(\sum_{i=1}^n |a_i|^2\Big)^{\frac{1}{2}},\end{aligned}$$

where C_2 is the constant appearing in Theorem 1.3.1, and K_2 is the unconditional constant of (f_{n_i}) in L_p. This finishes the proof. □

Proposition 3.1.2. *If E is a Banach lattice with a lower 2 estimate, then every strictly singular operator T from ℓ_2 to E is compact.*

Proof. Clearly there is no loss of generality in assuming E separable. Now, since E satisfies a lower 2 estimate, in particular it is order continuous and by Theorem 1.2.7 we can represent E as an order dense ideal in $L_1(\mu)$. Let us consider the operator T as an operator into $L_1(\mu)$.

Let us see that $T : \ell_2 \to E$ is compact. It clearly suffices to prove that $\|Te_n\|_E \to 0$, where (e_n) is any weakly null normalized sequence in ℓ_2. Suppose not, by Theorem 1.2.8, either $(\|Te_n\|_{L_1})$ is bounded away from zero or (Te_n) has a subsequence equivalent to a disjoint sequence.

Assume first that $(\|Te_n\|_{L_1})$ is bounded away from zero, then by [**8**, §6.Theorem], there exists $\delta > 0$ and a subsequence (Te_{n_k}) such that for scalars $(a_k)_{k=1}^m$ we have

$$\|T\|\Big(\sum_{k=1}^m |a_k|^2\Big)^{\frac{1}{2}} \geq \Big\|\sum_{k=1}^m a_k T e_{n_k}\Big\|_E \geq \Big\|\sum_{k=1}^m a_k T e_{n_k}\Big\|_{L_1} \geq \delta\Big(\sum_{k=1}^m |a_k|^2\Big)^{\frac{1}{2}}.$$

On the other hand, if (Te_{n_j}) is equivalent to a disjoint sequence we would have

$$\|T\|\Big(\sum_{j=1}^k |a_j|^2\Big)^{\frac{1}{2}} \geq \Big\|\sum_{j=1}^k a_j T(e_{n_j})\Big\|_E \geq M^{-1} \inf \|Te_n\| \Big(\sum_{j=1}^k |a_j|^2\Big)^{\frac{1}{2}},$$

where M is the constant appearing in the lower 2 estimate. Thus, in both cases this would imply that T is not strictly singular, and we reach a contradiction. □

Theorem 3.1.3. *Let E be a Banach lattice with finite cotype and Y a Banach space. If an operator $T : E \to Y$ is disjointly strictly singular and AM-compact, then it is strictly singular.*

PROOF. Suppose $T : E \to Y$ is *DSS*, AM-compact and not *SS*. Then T is an isomorphism when restricted to the span of some normalized unconditional basic sequence (f_n) in E (see Proposition 1.2.4). Moreover, if this sequence had a subsequence equivalent to the unit vector basis of ℓ_1, then, by Theorem 1.5.6, T would also be an isomorphism on the span of a disjoint sequence equivalent to the unit vector basis of ℓ_1, which contradicts the fact that T is *DSS*. Hence, by Rosenthal's Theorem 1.6.3, (f_n) can be assumed weakly Cauchy, and since E does not contain a subspace isomorphic to c_0, by Theorem 1.2.5, (f_n) is weakly convergent. Since it is a basic sequence, a standard argument yields that (f_n) is weakly null.

Let $\alpha > 0$ be such that for every sequence of scalars (a_n) we have

$$\Big\|T\Big(\sum_{n=1}^{\infty} a_n f_n\Big)\Big\| \geq \alpha \Big\|\sum_{n=1}^{\infty} a_n f_n\Big\|.$$

Since E has finite cotype, in particular it has the subsequence splitting property. Thus, we can extract a subsequence (still denoted (f_n)) and sequences (g_n) and (h_n) such that $|g_n|$, $|h_n| \leq |f_n|$, $f_n = g_n + h_n$, (g_n) is equi-integrable in E and (h_n) is disjoint.

Suppose first that $\|h_n\| \to 0$; then, passing to a subsequence if needed, the sequence (f_n) would inherit the equi-integrability from the sequence (g_n). Therefore, since T is AM-compact, (Tf_n) would have a convergent subsequence by Lemma 1.5.3; but since T is invertible on $[f_n]$, this would imply that (f_n) must also have a convergent subsequence. This is a contradiction since (f_n) is weakly null and normalized.

Alternatively, let us suppose that $\|h_n\| \geq \rho$. We consider the operator $V : [f_n] \to E$ defined by

$$V\Big(\sum_{n=1}^{\infty} a_n f_n\Big) = \sum_{n=1}^{\infty} a_n h_n,$$

which is bounded. Indeed, since $|h_n| \leq |f_n|$, by Theorem 1.3.1, for some constant C, we have

$$\begin{aligned}\Big\|V\Big(\sum_{i=1}^{n} a_i f_i\Big)\Big\| &= \Big\|\sum_{i=1}^{n} a_i h_i\Big\| = \Big\|\Big(\sum_{i=1}^{n} |a_i|^2 |h_i|^2\Big)^{\frac{1}{2}}\Big\| \\ &\leq \Big\|\Big(\sum_{i=1}^{n} |a_i f_i|^2\Big)^{\frac{1}{2}}\Big\| \leq C\int_0^1 \Big\|\sum_{i=1}^{n} a_i f_i r_i(t)\Big\| dt.\end{aligned}$$

Now, let K denote the unconditional constant of (f_n). We have

$$\int_0^1 \Big\|\sum_{i=1}^{n} a_i f_i r_i(t)\Big\| dt \leq K \Big\|\sum_{i=1}^{n} a_i f_i\Big\|,$$

so the operator V is bounded with $\|V\| \leq KC$.

Therefore, the restriction operator $T|_{[f_n]} : [f_n] \to Y$ can be decomposed as

$$T|_{[f_n]} = TV + T(I_{[f_n]} - V),$$

where $I_{[f_n]} : [f_n] \hookrightarrow E$ is the identity inclusion. Since T is *DSS*, it is clear that the operator TV is strictly singular. Hence, since $T|_{[f_n]}$ is an isomorphism, by [**86**, Proposition 2.c.10], the operator $T(I_{[f_n]} - V)$ has finite dimensional kernel and closed range.

Now, by Lemma 1.5.3, $T(g_n) = T(I_{[f_n]} - V)(f_n)$ has a convergent subsequence. Let us consider

$$[f_n] = \ker(T(I_{[f_n]} - V)) \oplus X.$$

Since $T(I_{[f_n]} - V)$ has closed range, by the open mapping Theorem $T(I_{[f_n]} - V)$ is invertible on X. Therefore, since $\ker(T(I_{[f_n]} - V))$ is finite dimensional, (f_n) would also have a convergent subsequence. Again, this is a contradiction with the fact that (f_n) is weakly null and normalized.

□

Proposition 3.1.4. *Let E and F be Banach lattices such that E has finite cotype. If $T : E \to F$ is an ℓ_2-singular operator, then T is also AM-compact under any of the following conditions:*

(1) *F satisfies a lower 2 estimate.*
(2) *F is order continuous and T is regular.*

Proof. Let (f_n) be a uniformly bounded sequence, that is $|f_n| \leq M$ for some $M < \infty$. Since the order intervals in E are weakly compact, without loss of generality we can assume that $f_n \to 0$ weakly. Now, since E is q-concave for some $q < \infty$, we have $L_q(\mu) \hookrightarrow E \hookrightarrow L_1(\mu)$ for a certain probability measure μ (see [**73**, p. 14]).

The sequence (f_n) must be weakly null also in $L_q(\mu)$, therefore it has an unconditional subsequence in $L_q(\mu)$. Thus, passing to a subsequence we have the following

$$\begin{aligned}\Big\|\sum_{i=1}^n a_i f_i\Big\|_X &\leq \Big\|\sum_{i=1}^n a_i f_i\Big\|_{L_q} \leq K\int_0^1 \Big\|\sum_{i=1}^n a_i r_i(t) f_i\Big\|_{L_q} dt \\ &\leq KC\Big\|\Big(\sum_{i=1}^n |a_i f_i|^2\Big)^{\frac{1}{2}}\Big\|_{L_q} \leq 2KCM\Big(\sum_{i=1}^n |a_i|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

where K is the unconditional constant of (f_n) and C the constant appearing in Theorem 1.3.1.

Moreover, passing to a further subsequence (still denoted (f_n)), by Lemma 3.1.1, the sequence (f_n) spans in $L_p(\mu)$ a subspace isomorphic to ℓ_2, for every $1 < p < \infty$. Therefore, by Theorem 1.2.8, $[f_n]$ is strongly embedded in $L_p(\mu)$, which implies that on $[f_n]$ the topologies of L_p and L_1 coincide. Thus for certain constant $\alpha > 0$

$$\alpha\Big(\sum_{i=1}^n |a_i|^2\Big)^{\frac{1}{2}} \leq \Big\|\sum_{i=1}^n a_i f_i\Big\|_{L_1} \leq \Big\|\sum_{i=1}^n a_i f_i\Big\|_E.$$

Thus, (f_n) has a subsequence whose span in X is isomorphic to ℓ_2.

On the one hand, suppose that F satisfies a lower 2 estimate. Consider the restricted operator $T|_{[f_n]} : [f_n] \to F$, which is strictly singular since $[f_n]$ is isomorphic to ℓ_2. Hence, by Proposition 3.1.2, $(T(f_n))$ must have a subsequence that goes to zero.

On the other hand, if F is order continuous and T is regular, then the sequence $(T(f_n))$ is equi-integrable by Lemma 1.4.3. Hence, Theorem 1.2.8 yields that $(\|T(f_n)\|_{L_1})$ is bounded away from zero. This implies that the operator R defined by the following diagram

$$\begin{tikzcd} \ell_2 \arrow[r, "R"] \arrow[d, "i"'] & L_1(\nu) \\ {[f_{n_k}]} \arrow[r, "T"] & F \arrow[u, hook, "j"'] \end{tikzcd}$$

is not compact. Here i is an isomorphism, and j the formal inclusion of Y in L_1, or just the inclusion of a separable sublattice containing $(T(f_n))$ (cf. Theorem 1.2.7). By Proposition 3.1.2 we obtain that T is not ℓ_2-singular. This is a contradiction.

So far, we have shown that, in both cases, $T[-M,M]$ is a relatively compact set for every positive M. For an arbitrary $x \in E_+$, since by the representability of E as a function space between $L_q(\mu)$ and $L_1(\mu)$ we have that $L_\infty(\mu)$ is dense in E, given $\varepsilon > 0$ we can consider $M_\varepsilon < \infty$ such that $x \in [0, M_\varepsilon] + \varepsilon\|T\|^{-1}B_E$. Therefore, for every $\varepsilon > 0$, $T[-x,x] \subset T[-M_\varepsilon, M_\varepsilon] + \varepsilon B_F$ and since $T[-M_\varepsilon, M_\varepsilon]$ is relatively compact, so is $T[-x,x]$. $\square$

Proposition 3.1.4 has a partial converse:

PROPOSITION 3.1.5. *Let E be a Banach lattice and Y a Banach space. Suppose that E has finite cotype and does not contain any sequence of disjoint elements which span a subspace isomorphic to ℓ_2. If $T : E \to Y$ is AM-compact, then T is also ℓ_2-singular.*

PROOF. Suppose $T : E \to Y$ is AM-compact, but not ℓ_2-singular. Therefore, there exist a sequence (f_n) in E which spans a subspace isomorphic to ℓ_2 and T is an isomorphism when restricted to $[f_n]$.

Since E has the subsequence splitting property [**134**], passing to a subsequence we have $f_n = h_n + g_n$ with (g_n) equi-integrable, (h_n) disjoint, and $|h_n|, |g_n| \leq |f_n|$ for all n. Again, arguing as in the proof of Theorem 3.1.3 the operator $V : [f_n] \to E$ defined by $V(f_n) = h_n$ is bounded. Hence, since (f_n) is equivalent to the unit vector basis of ℓ_2, in particular it is weakly null. Hence, so are (h_n) and (g_n), since $h_n = V(f_n)$, and $g_n = f_n - h_n$.

Since T is AM-compact, by Lemma 1.5.3, the sequence $(T(g_n))$ must have a subsequence that goes to zero. Therefore, $(T(h_n))$ has a subsequence, say $(T(h_{n_k}))$, equivalent to $(T(f_{n_k}))$, which is equivalent to the unit vector basis of ℓ_2. Then, there exist constants α and β, such that for any n and scalars $(a_k)_{k=1}^n$ we have

$$\begin{aligned} \alpha\Big(\sum_{k=1}^n |a_k|^2\Big)^{\frac{1}{2}} &\leq \Big\|T\Big(\sum_{k=1}^n a_k h_{n_k}\Big)\Big\| \leq \|T\|\Big\|\sum_{k=1}^n a_k h_{n_k}\Big\| \\ &\leq \|T\|\|V\|\Big\|\sum_{k=1}^n a_k f_{n_k}\Big\| \leq \beta\|T\|\|V\|\Big(\sum_{k=1}^n |a_k|^2\Big)^{\frac{1}{2}}. \end{aligned}$$

This means that $[h_{n_k}]$ is isomorphic to ℓ_2, and this is impossible according to the hypothesis on E. $\square$

Now we can state and prove the main result of this Chapter:

THEOREM 3.1.6. *Let E and F be Banach lattices such that E has finite cotype.*

(1) *If F satisfies a lower 2-estimate, then every operator $T : E \to F$ is strictly singular if and only if it is both disjointly strictly singular and ℓ_2-singular.*
(2) *If F is order continuous, then every regular operator $T : E \to F$ is strictly singular if and only if it is both disjointly strictly singular and ℓ_2-singular.*

PROOF. Clearly strictly singular operators are disjointly strictly singular and ℓ_2-singular. Conversely, if $T : E \to F$ is ℓ_2-singular, then by Proposition 3.1.4 T is also AM-compact. The conclusion follows from Theorem 3.1.3. □

2. The role of the lower 2-estimate

In this section we prove that Theorem 3.1.6 cannot be extended to the case when the Banach lattice F has a lower q-estimate, for some $q > 2$. To this end we consider the Banach lattice $L_r(\ell_q)$ (see [**87**, pp. 46-47]), which consists of sequences $x = (x_1, x_2, \ldots)$ of elements in L_r such that

$$\|x\|_{L_r(\ell_q)} = \sup_n \left\| \left(\sum_{i=1}^n |x_i|^q \right)^{1/q} \right\|_{L_r} < \infty.$$

Notice that since L_r $(1 \leq r < \infty)$ is weakly sequentially complete, $L_r(\ell_q)$ is spanned by the eventually zero sequences of elements in L_r.

THEOREM 3.2.1. *Let $1 < r < p < s < 2 < q < \infty$. There exists an operator $T : L_p \to L_r(\ell_q)$ such that it is ℓ_p-singular and ℓ_2-singular, but not ℓ_s-singular.*

In particular, the operator T is disjointly strictly singular and ℓ_2-singular, but not strictly singular.

Before the proof, we need some preliminary Lemmas. The first of them will be deduced from the following:

PROPOSITION 3.2.2 ([**73**] Lemma 3.10). *Let X be a finite dimensional Banach space with a 1-unconditional basis $(x_i)_{i=1}^n$ and let Y be a q-concave Banach lattice, for some $1 < q < \infty$. Let T be an isomorphism from X into Y and let $(g_i)_{i=1}^n$ be a sequence of independent and identically distributed (i.i.d.) p-stable random variables over a probability space (Ω, Σ, μ), for some $1 < p < 2$. Then, for every scalars $(a_i)_{i=1}^n$ for which*

$$\int_\Omega \left\| \sum_{i=1}^n a_i g_i(\omega) x_i \right\|_X d\mu(\omega) \leq \left\| \sum_{i=1}^n a_i x_i \right\|_X,$$

the following inequalities hold

$$K \left\| \sum_{i=1}^n a_i x_i \right\|_X \leq \left\| \max_{1 \leq i \leq n} |a_i T x_i| \right\|_Y \leq \left\| \left(\sum_{i=1}^n |a_i T x_i|^p \right)^{1/p} \right\|_Y \leq \|T\| \|g_1\|_{L_1}^{-1} \left\| \sum_{i=1}^n a_i x_i \right\|_X,$$

for certain constant K (depending only on the q-concavity constant of Y).

Lemma 3.2.3. *Let $1 < q < s < 2$. If $T : \ell_s \to L_q(\mu)$ is an isomorphic embedding, and $f_n = T(e_n)$, where (e_n) is the canonical basis of ℓ_s, then for $2 \leq r < \infty$*

$$\Big\|\sum_{n=1}^{k} a_n f_n\Big\|_{L_q} \sim \Big\|\Big(\sum_{n=1}^{k} |a_n f_n|^r\Big)^{1/r}\Big\|_{L_q} \sim \Big\|\max_{1\leq n\leq k} |a_n f_n|\Big\|_{L_q},$$

for any scalars $(a_n)_{n=1}^k$.

Proof. Take $n \in \mathbb{N}$ and let $X = \ell_s^n$, $Y = L_q(\mu)$, and $T = T|_X$ in Proposition 3.2.2. If $(g_i)_{i=1}^n$ is a sequence of i.i.d. p-stable random variables with $s < p < 2$, then

$$\begin{aligned}
\int_\Omega \Big\|\sum_{i=1}^{n} a_i g_i(\omega) e_i\Big\|_{\ell_s} d\mu(\omega) &= \int_\Omega \Big(\sum_{i=1}^{n} |a_i g_i(\omega)|^s\Big)^{1/s} d\mu(\omega)\\
&\leq \Big(\int_\Omega \sum_{i=1}^{n} |a_i g_i(\omega)|^s d\mu(\omega)\Big)^{1/s}\\
&= \Big(\sum_{i=1}^{n} |a_i|^s \int_\Omega |g_i(\omega)|^s d\mu(\omega)\Big)^{1/s}\\
&= \|g_1\|_{L_s}\Big\|\sum_{i=1}^{n} a_i e_i\Big\|_{\ell_s}.
\end{aligned}$$

Since $\|g_1\|_{L_s} < \infty$, using Proposition 3.2.2 we get

$$K\|g_1\|_{L_s}^{1/2}\Big\|\sum_{i=1}^{n} a_i e_i\Big\|_{\ell_s} \leq \Big\|\max_{1\leq i\leq n} |a_i f_i|\Big\|_{L_q} \leq \Big\|\Big(\sum_{i=1}^{n} |a_i f_i|^p\Big)^{1/p}\Big\|_{L_q} \leq \|T\|\|g_1\|_{L_1}^{-1}\|g_1\|_{L_s}^{1/2}\Big\|\sum_{i=1}^{n} a_i e_i\Big\|_{\ell_s}$$

for certain constant K independent of n.

Since $\|\max_{1\leq i\leq n} |a_i f_i|\|_{L_q} \leq \|(\sum_{i=1}^n |a_i f_i|^2)^{1/2}\|_{L_q} \leq \|(\sum_{i=1}^n |a_i f_i|^p)^{1/p}\|_{L_q}$, by the previous inequality, we immediately get

$$\Big\|\max_{1\leq i\leq n} |a_i f_i|\Big\|_{L_q} \sim \Big\|\Big(\sum_{i=1}^{n} |a_i f_i|^2\Big)^{1/2}\Big\|_{L_q} \sim \Big\|\Big(\sum_{i=1}^{n} |a_i f_i|^p\Big)^{1/p}\Big\|_{L_q}.$$

On the other hand, Theorem 1.3.1 yields $\|\sum_{n=1}^k a_n f_n\|_{L_q} \sim \|(\sum_{n=1}^k |a_n f_n|^2)^{1/2}\|_{L_q}$. Therefore, for every $2 \leq r \leq \infty$, we have

$$\Big\|\sum_{n=1}^{k} a_n f_n\Big\|_{L_q} \sim \Big\|\Big(\sum_{n=1}^{k} |a_n f_n|^r\Big)^{1/r}\Big\|_{L_q}.$$

$\square$

Next result shows why Lemma 3.2.3 cannot be extended to the case $s = 2$. Given functions $f, g : \mathbb{N} \to \mathbb{R}$, by $f = o(g)$ we mean, as usual, that $\frac{f(m)}{g(m)} \underset{m\to\infty}{\longrightarrow} 0$. For a subset $A \subset \mathbb{N}$, $|A|$ denotes the cardinality of A.

Lemma 3.2.4. *Let $1 < p < 2$. If $T : \ell_2 \to L_p(\mu)$ is a bounded operator, and $f_n = T(e_n)$ where (e_n) denotes the canonical basis of ℓ_2, then for each natural number m,*

$$\inf_{|A|=m} \Big\|\max_{j\in A} |f_j|\Big\|_{L_p} = o(\sqrt{m}).$$

Moreover, for any $2<q<\infty$,

$$\inf_{|A|=m}\Big\|\Big(\sum_{j\in A}|f_j|^q\Big)^{1/q}\Big\|_{L_p}=o(\sqrt{m}).$$

Proof. By Krivine's Theorem (see Theorem 1.5.10), given any finite family $(x_i)_{i=1}^n$ of vectors in ℓ_2 we have

$$\Big\|\Big(\sum_{i=1}^n|Tx_i|^2\Big)^{1/2}\Big\|_{L_p}\leq K_G\|T\|\Big\|\Big(\sum_{i=1}^n|x_i|^2\Big)^{1/2}\Big\|_{\ell_2}=K_G\|T\|\Big(\sum_{i=1}^n\|x_i\|_{\ell_2}^2\Big)^{1/2}$$

where K_G is Grothendieck's constant. Now, by Maurey's factorization theorem (cf. [**6**, Theorem 7.1.2]), there exists a density function h on Ω (i.e. $h>0$ and $\int hd\mu=1$) such that we can factor T in the following way:

$$\begin{tikzcd} \ell_2 \arrow[r, "T"] \arrow[d, "\widetilde{T}"'] & L_p(\mu) \\ L_2(hd\mu) \arrow[r, hook, "i"] & L_p(hd\mu) \arrow[u, "J"'] \end{tikzcd}$$

where $\widetilde{T}(x)=h^{-1/p}T(x)$ for every $x\in\ell_2$, i denotes the canonical inclusion, and J is the isometry mapping each $f\in L_p(hd\mu)$ to $J(f)=fh^{1/p}$.

Let us denote $\widetilde{f_n}=\widetilde{T}(e_n)\in L_2(hd\mu)$. Since $L_2(hd\mu)$ satisfies the subsequence splitting property, there exist a subsequence $(\widetilde{f_{n_k}})$ and sequences (g_k), (h_k), with (h_k) disjoint and (g_k) equi-integrable in $L_2(hd\mu)$ such that $\widetilde{f_{n_k}}=g_k+h_k$ and $|g_k|\wedge|h_k|=0$. Therefore, $(J(h_k))$ is disjoint in $L_p(\mu)$, and for scalars $(a_k)_{k=1}^n$ we have

$$\begin{aligned}\Big\|\sum_{k=1}^n a_kJ(h_k)\Big\|_{L_p}&\leq\Big\|\sum_{k=1}^n a_kh_k\Big\|_{L_2(h)}=\Big\|\Big(\sum_{k=1}^n|a_kh_k|^2\Big)^{1/2}\Big\|_{L_2(h)}\\&\leq\Big\|\Big(\sum_{k=1}^n|a_k\widetilde{f_{n_k}}|^2\Big)^{1/2}\Big\|_{L_2(h)}\leq K_G\|\widetilde{T}\|\Big\|\Big(\sum_{k=1}^n|a_ke_{n_k}|^2\Big)^{1/2}\Big\|_{\ell_2}\\&=C\Big(\sum_{k=1}^n|a_k|^2\Big)^{1/2},\end{aligned}$$

with C independent of n (see Theorem 1.5.10). Now, if $\inf_k\|J(h_k)\|_{L_p}>0$, then for some constant c and for all $(a_k)_{k=1}^n$ we have

$$c\Big(\sum_{k=1}^n|a_k|^p\Big)^{1/p}\leq\Big\|\sum_{k=1}^n a_kJ(h_k)\Big\|_{L_p}\leq C\Big(\sum_{k=1}^n|a_k|^2\Big)^{1/2},$$

which is impossible (since $1<p<2$).

Thus, passing to a subsequence we can assume that $\|J(h_k)\|_{L_p}\underset{k\to\infty}{\longrightarrow}0$. Now, for each $m\in\mathbb{N}$, let us denote

$$\phi(m)=\Big\|\max_{1\leq k\leq m}|g_k|\Big\|_{L_2(h)},$$

and let us take disjoint measurable sets $A_1^m, A_2^m, \dots, A_m^m$ in Ω such that

$$\max_{1\le k\le m} |g_k| = \sum_{k=1}^m |g_k|\chi_{A_k^m}.$$

Claim: *It holds that* $\frac{\phi(m)}{\sqrt{m}} \to 0$ *when* $m \to \infty$.

Assume the contrary; then, there exist $\varepsilon > 0$ and an increasing sequence $m_n \to \infty$ such that $\frac{\phi(m_n)}{\sqrt{m_n}} \ge \varepsilon$. That is, for all $n \in \mathbb{N}$ we can choose an integer m_n and disjoint sets $A_1^{m_n}, A_2^{m_n}, \dots, A_{m_n}^{m_n}$ such that

$$(1) \qquad \frac{1}{m_n}\sum_{k=1}^{m_n}\int_{A_k^{m_n}} |g_k|^2 h d\mu = \frac{1}{m_n}\Big\|\sum_{k=1}^{m_n}|g_k|\chi_{A_k^{m_n}}\Big\|^2_{L_2(h)} = \left(\frac{\phi(m_n)}{\sqrt{m_n}}\right)^2 \ge \varepsilon^2 \qquad \forall n \in \mathbb{N}.$$

From this fact we conclude that for every $N \in \mathbb{N}$ we can find $B_1, \dots, B_N$ disjoint sets in Ω such that

$$\sup_k \Big\{\int_{B_n} |g_k|^2 h d\mu\Big\} \ge \frac{\varepsilon^2}{2},$$

for all $n = 1, \dots, N$.

Otherwise, suppose that there exists N such that for n large enough the set

$$S_n = \{k \le m_n : \int_{A_k^{m_n}} |g_k|^2 h d\mu \ge \varepsilon^2/2\}$$

always has cardinality less than N. Then for n large enough so that

$$\frac{m_n - N}{m_n} < 1, \quad \text{and} \quad \frac{N \sup_k \|g_k\|_{L_2}}{m_n} < \frac{\varepsilon^2}{2},$$

we have

$$\frac{1}{m_n}\sum_{k=1}^{m_n}\int_{A_k^{m_n}} |g_k|^2 h d\mu < \frac{1}{m_n}\Big((m_n - N)\frac{\varepsilon^2}{2} + N\sup_k \|g_k\|_{L_2}\Big) < \varepsilon^2$$

which is a contradiction with (1).

However, by Theorem 1.4.5, we reach a contradiction with the fact that (g_k) is equi-integrable in $L_2(hd\mu)$. Therefore, $\frac{\phi(m)}{\sqrt{m}} \to 0$ when $m \to \infty$, and the claim is proved.

Now, since $\|J(h_k)\|_{L_p} \underset{k\to\infty}{\longrightarrow} 0$, for every $m \in \mathbb{N}$ we have that, for every $\varepsilon > 0$ there exists a set $A_\varepsilon = \{k_1, k_2 \dots, k_m\}$ of natural numbers, such that $\|J(h_k)\|_{L_p} < \varepsilon/m$ for all $k \in A_\varepsilon$. Therefore,

$$\begin{aligned}\inf_{|A|=m} \|\max_{j\in A}|f_j|\|_{L_p} &\le \inf_{|A|=m} \|\max_{j\in A} J(|g_j| + |h_j|)\|_{L_p} \\ &\le \phi(m) + \|\max_{j\in A_\varepsilon} J(|h_j|)\|_{L_p} \le \phi(m) + \varepsilon\end{aligned}$$

and since this inequality holds for all $\varepsilon > 0$ we get that $\inf_{|A|=m} \|\max_{j\in A}|f_j|\|_{L_p} \le \phi(m)$, which implies

$$\inf_{|A|=m} \|\max_{j\in A}|f_j|\|_{L_p} = o(\sqrt{m}).$$

The second assertion of the Lemma is obtained by a Hölder type inequality ([**87**, 1.d.2]). Indeed, given $m \in \mathbb{N}$, for any $A \subset \mathbb{N}$ with $|A| = m$,

$$\Big\|\Big(\sum_{j\in A}|f_j|^q\Big)^{1/q}\Big\|_{L_p} \leq \Big\|\Big(\Big(\sum_{j\in A}|f_j|^2\Big)^{1/2}\Big)^{\theta}\Big(\max_{j\in A}|f_j|\Big)^{1-\theta}\Big\|_{L_p}$$
$$\leq \Big\|\Big(\sum_{j\in A}|f_j|^2\Big)^{1/2}\Big\|_{L_p}^{\theta}\Big\|\max_{j\in A}|f_j|\Big\|_{L_p}^{1-\theta},$$

for $\theta = \frac{2}{q} \in (0,1)$. Now, by the first part of the Lemma, the function $\varphi(m) = \inf_{|A|=m}\|\max_{j\in A}|f_j|\|_{L_p}$ satisfies $\frac{\varphi(m)}{\sqrt{m}} \underset{m\to\infty}{\to} 0$. Moreover, by passing to a subsequence, (f_j) can be assumed to be unconditional, so we have

$$\Big\|\Big(\sum_{j\in A}|f_j|^2\Big)^{1/2}\Big\|_{L_p} \sim \Big\|\sum_{j\in A}f_j\Big\|_{L_p} \leq \|T\|\sqrt{m}.$$

Thus, for $q > 2$

$$\inf_{|A|=m}\Big\|\Big(\sum_{j\in A}|f_j|^q\Big)^{1/q}\Big\|_{L_p} \leq (\|T\|\sqrt{m})^{\theta}(\varphi(m))^{1-\theta},$$

and clearly

$$\frac{(\|T\|\sqrt{m})^{\theta}\varphi(m)^{1-\theta}}{\sqrt{m}} \underset{m\to\infty}{\to} 0.$$

□

LEMMA 3.2.5. *Let $1 < p < 2$. If a sequence $(f_n) \subset L_p$ satisfies*

$$\Big\|\Big(\sum_{n=1}^{\infty}|a_n|^2|f_n|^2\Big)^{1/2}\Big\|_{L_p} \leq C\Big(\sum_{n=1}^{\infty}|a_n|^2\Big)^{1/2},$$

for some constant $C > 0$ and every finitely non zero scalar sequence (a_n), then for $2 < q < \infty$,

$$\inf_{|A|=m}\Big\|\Big(\sum_{j\in A}|f_j|^q\Big)^{1/q}\Big\|_{L_p} = o(\sqrt{m}).$$

PROOF. Let us consider the operator $T : \ell_2 \to L_p([0,1]\times[0,1])$ defined by $T(e_n) = f_n \otimes r_n$, where (e_n) denotes the canonical basis of ℓ_2, (r_n) are the Rademacher functions on $[0,1]$, and $f_n \otimes r_n(s,t) = f_n(s)r_n(t)$. By Theorem 1.3.1 and the hypothesis, this operator is bounded. Thus, by Lemma 3.2.4, we have

$$\inf_{|A|=m}\Big\|\Big(\sum_{j\in A}|f_j\otimes r_j|^q\Big)^{1/q}\Big\|_{L_p} = o(\sqrt{m}).$$

But, since $|f_j| = |f_j \otimes r_j|$ we are done. □

Now we can give the proof of Theorem 3.2.1:

Proof of Theorem 3.2.1. Given $1 < r < p < s < 2 < q < \infty$, consider the formal inclusion $J : L_p(0,1) \hookrightarrow L_r(0,1)$. Let us denote by H_r the atomic lattice isomorphic to $L_r(0,1)$ and whose lattice structure comes from the unconditional Haar basis (h_n) in $L_r(0,1)$. Let $L : L_r(0,1) \to H_r$ denote this isomorphism.

Now, let (f_n) be a sequence of i.i.d. s-stable random variables in $L_p(0,1)$. In particular, the span $[f_n]$ is isometrically isomorphic to ℓ_s both in $L_p(0,1)$ and $L_r(0,1)$. Now, since $f_n \xrightarrow{w} 0$ in $L_p(0,1)$ there exists a block basis (w_n) of (h_n), of the form

$$w_n = \sum_{i=q_{n-1}+1}^{q_n} a_i h_i,$$

where (q_n) is an increasing sequence in $\mathbb{N}$, and such that (w_n) is equivalent in $L_p(0,1)$ and $L_r(0,1)$ to a subsequence of (f_n) (still denoted (f_n)).

We claim that the following operator is bounded:

$$\begin{array}{rccc} R: & H_r & \longrightarrow & L_r(\ell_q) \\ & (c_i)_{i=1}^{\infty} & \mapsto & \left(\sum_{i=q_{n-1}+1}^{q_n} c_i h_i\right)_{n=1}^{\infty} \end{array}$$

Indeed, since $q > 2$ and (h_n) is unconditional we have

$$\begin{aligned} \Big\| R((c_i)) \Big\|_{L_r(\ell_q)} &= \left\| \left(\sum_{n=1}^{\infty} \Big| \sum_{i=q_{n-1}+1}^{q_n} c_i h_i \Big|^q \right)^{1/q} \right\|_{L_r} \\ &\leq \left\| \left(\sum_{n=1}^{\infty} \Big| \sum_{i=q_{n-1}+1}^{q_n} c_i h_i \Big|^2 \right)^{1/2} \right\|_{L_r} \\ &\leq C \left\| \sum_{i=1}^{\infty} c_i h_i \right\|_{L_r} = C \|(c_i)\|_{H_r}, \end{aligned}$$

for certain constant C.

Let us consider now the operator T defined by

$$\begin{tikzcd} L_p \arrow[r, "T"] \arrow[d, "J"'] & L_r(\ell_q) \\ L_r \arrow[r, "L"] & H_r \arrow[u, "R"'] \end{tikzcd} .$$

T is an isomorphism when restricted to the span $[w_n]$ in L_p, which is isomorphic to ℓ_s. Indeed, by Lemma 3.2.3, we have

$$\begin{aligned} \left\| T\Big(\sum_{n=1}^{\infty} b_n w_n \Big) \right\|_{L_r(\ell_q)} &= \left\| \left(\sum_{n=1}^{\infty} \Big| b_n \sum_{i=q_{n-1}+1}^{q_n} a_i h_i \Big|^q \right)^{1/q} \right\|_{L_r} \\ &\sim \left\| \sum_{n=1}^{\infty} b_n w_n \right\|_{L_r} \sim \left\| \sum_{n=1}^{\infty} b_n w_n \right\|_{L_p} \\ &\sim \left(\sum_{n=1}^{\infty} |b_n|^s \right)^{1/s}. \end{aligned}$$

Therefore, T is not strictly singular. On the other hand, the operator T is ℓ_p-singular because so is the inclusion $J : L_p \hookrightarrow L_r$.

Let us prove now that T does not preserve an isomorphic copy of ℓ_2. To see this it suffices to show that RL preserves no isomorphic copy of ℓ_2. Indeed, if this were not the case, let (g_n) be a sequence equivalent to the unit vector basis of ℓ_2 in L_r, so that $(RL(g_n))$ is also equivalent to the unit vector basis of ℓ_2. Since $g_n \overset{w}{\to} 0$, without loss of generality, we can suppose that (g_n) is a block basis of the Haar system. In fact, we can extract a subsequence (still denoted (g_n)) such that

$$g_n = \sum_{k=p_{n-1}+1}^{p_n} \psi_k^n,$$

where each $\psi_k^n \in [h_{q_{j_k-1}+1}, \dots, h_{q_{j_k}}]$ for certain increasing sequence (j_k) in $\mathbb{N}$ (notice that the sequence (q_n) was already fixed in the definition of the operator R).

Now, the sequence (ψ_k^n) for $n = 1, 2, \dots$, and $k = p_{n-1} + 1, \dots, p_n$ is an unconditional basic sequence since they are blocks of the Haar basis, which is unconditional in $L_r(0, 1)$. Therefore, for every finitely non zero sequence of scalars (a_n) we have

$$\begin{aligned}\Big\| \Big(\sum_{n=1}^{\infty} \sum_{k=p_{n-1}+1}^{p_n} |a_n \psi_k^n|^2 \Big)^{1/2} \Big\|_{L_r} &\sim \Big\| \sum_{n=1}^{\infty} \sum_{k=p_{n-1}+1}^{p_n} a_n \psi_k^n \Big\|_{L_r} \\ &= \Big\| \sum_{n=1}^{\infty} a_n g_n \Big\|_{L_r} \sim \Big(\sum_{n=1}^{\infty} a_n^2 \Big)^{1/2}\end{aligned}$$

(see Theorem 1.3.1). Let us consider

$$f_n = \Big(\sum_{k=p_{n-1}+1}^{p_n} |\psi_k^n|^q \Big)^{1/q}.$$

Since $q > 2$ we have

$$\begin{aligned}\Big\| \Big(\sum_{n=1}^{\infty} a_n^2 f_n^2 \Big)^{1/2} \Big\|_{L_r} &= \Big\| \Big(\sum_{n=1}^{\infty} a_n^2 \Big(\sum_{k=p_{n-1}+1}^{p_n} |\psi_k^n|^q \Big)^{2/q} \Big)^{1/2} \Big\|_{L_r} \\ &\leq \Big\| \Big(\sum_{n=1}^{\infty} a_n^2 \sum_{k=p_{n-1}+1}^{p_n} |\psi_k^n|^2 \Big)^{1/2} \Big\|_{L_r} \sim \Big(\sum_{n=1}^{\infty} a_n^2 \Big)^{1/2}.\end{aligned}$$

Now, by Lemma 3.2.5 we get

$$\inf_{|A|=m} \Big\| \Big(\sum_{n \in A} |f_n|^q \Big)^{1/q} \Big\|_{L_r} = o(\sqrt{m}).$$

Therefore, by hypothesis there exist some constant $C > 0$ such that

$$
\begin{aligned}
C\sqrt{m} &\leq \inf_{|A|=m} \Big\| \sum_{n\in A} RLg_n \Big\|_{L_r(\ell_q)} \\
&= \inf_{|A|=m} \Big\| \Big(\sum_{n\in A} \sum_{k=p_{n-1}+1}^{p_n} |\psi_k^n|^q \Big)^{1/q} \Big\|_{L_r} \\
&= \inf_{|A|=m} \Big\| \Big(\sum_{n\in A} |f_n|^q \Big)^{1/q} \Big\|_{L_r} = o(\sqrt{m}).
\end{aligned}
$$

However, this is a contradiction. Thus, the operator RL is ℓ_2-singular, and so is $T = RLJ$. $\square$

Remark 3.2.6. The hypothesis of order continuity of the Banach lattice Y in the second part of Theorem 3.1.6 cannot be removed. Indeed, consider the operator $T : L_p \to L_r(\ell_q)$ constructed in Theorem 3.2.1 and the canonical isomorphic embedding $j : L_r(\ell_q) \to \ell_\infty$. Note that the composition operator $jT : L_p \to \ell_\infty$ is regular (cf. [**94**, Theorem 1.5.11]), *DSS* and ℓ_2-singular, but it is not *SS* (since T is not *SS* and j is an isomorphism).

CHAPTER 4

Invariant subspaces of positive strictly singular operators

Read [**116**] presented an example of a strictly singular operator with no (closed non-zero and proper) invariant subspaces. It remains as an open question whether every positive strictly singular operator on a Banach lattice has an invariant subspace. The present chapter contains several results in this direction.

Throughout this chapter, E will be a fixed order continuous Banach lattice with a weak order unit. We can assume, by Theorem 1.2.7 that there is a probability space (Ω, Σ, μ) such that E is an order ideal of $L_1(\mu)$. We fix the probability space (Ω, Σ, μ) throughout the chapter. We will also make use of the fact due to L. Weis that every regular operator $T\colon E \to E$ can be extended to a bounded operator $\widetilde{T}\colon L_1(\mu) \to L_1(\mu)$ (see Theorem 1.5.9).

We say that a Banach lattice E satisfies the *R-condition* if every sequence, which, viewed as a sequence in $L_1(\mu)$, is equivalent to the unit vector basis of ℓ_2, has a subsequence which remains equivalent to the unit vector basis of ℓ_2 when viewed as a sequence in E. We show in Section 1 that every p-concave Banach lattice with $1 \leq p < \infty$ satisfies the R-condition. In particular, if E contains $L_p(\mu)$ then E satisfies the R-condition.

In Section 2 we establish certain connections between some special classes of operators on Banach lattices. We show that a regular operator T on E is AM-compact if and only if its extension to $L_1(\mu)$ is a Dunford-Pettis operator. This extends the known fact that on L_1 the classes of AM-compact and Dunford-Pettis operators coincide. In connection with the results of the previous chapter, we also show that if E satisfies the R-condition then every regular ℓ_2-singular operator is AM-compact. Then, in Section 3 we use the obtained results to show that, in spaces with the R-condition, strictly singular operators as well as other operators related to them have invariant subspaces.

Recall that the concept of a *compact-friendly* operator was introduced in [**4**], where the existence of invariant subspaces for these operators under certain additional assumptions were proved. In Section 4 we define *strictly singular-friendly* operators in a similar fashion, and show that under the same assumptions plus the R-condition, strictly singular-friendly operators have invariant subspaces.

Part of this chapter has been published as a joint work with J. Flores and V. G. Troitsky in [**57**].

1. R-condition

We will say that E satisfies the *R-condition* if every bounded sequence (f_n) in $L_\infty(\mu)$ which is equivalent in $L_1(\mu)$ to the unit basis of ℓ_2, has a subsequence which is equivalent in E to the unit basis of ℓ_2. In this section we show that many Banach lattices enjoy the R-condition.

LEMMA 4.1.1. *Suppose that the inclusion $i\colon L_\infty(\mu) \hookrightarrow E$ factors, with positive factors, through $L_p(\nu)$ for some probability measure ν and $1 \le p < \infty$. Then E satisfies the R-condition.*

PROOF. Let (f_n) be a bounded sequence in $L_\infty(\mu)$ which is equivalent in $L_1(\mu)$ to the unit vector basis of ℓ_2. By hypothesis, we have the following factorization

$$\begin{tikzcd} L_\infty(\mu) \arrow[rr, hook, "i"] \arrow[dr, "T_1"'] & & E \\ & L_p(\nu) \arrow[ur, "T_2"'] & \end{tikzcd}$$

Since (f_n) viewed as a sequence in $L_1(\mu)$ is equivalent to the unit vector basis of ℓ_2, it is weakly null in $L_1(\mu)$. Since (f_n) is order bounded in E, (f_n) is also weakly null in E by Theorem 1.2.2.

The sequence $(T_1 f_n)$ has a subsequence which converges weakly to some $g \in L_p(\nu)$; therefore $T_2 g = 0$. Consider the sequence $y_n = T_1 f_n - g$; it has a weakly null subsequence. It cannot be null in $L_p(\nu)$ because (f_n) is not null in E.

Therefore, by passing to a subsequence, we may assume that (y_n) is weakly null, seminormalized, and $(y_n) \subset [-y, y]$ for some y in $L_p(\nu)$. Since $L_p(\nu)$ has an unconditional basis, we can extract a subsequence (y_{n_k}) which is unconditional with constant K. By Theorem 1.3.1, there exists $C > 0$ such that

$$\Big\|\sum_{k=1}^m a_k y_{n_k}\Big\|_p \le K \int_0^1 \Big\|\sum_{k=1}^m r_k(s) a_k y_{n_k}\Big\|_p ds \le KC \Big\|\Big(\sum_{k=1}^m |a_k y_{n_k}|^2\Big)^{\frac{1}{2}}\Big\|_p \le KC\|y\|_p \big\|(a_i)\big\|_{\ell_2^m}$$

for all $(a_k)_{k=1}^m$, where (r_k) denotes the Rademacher functions. Then, for some $c > 0$ we have

$$\begin{aligned} c\big\|(a_i)\big\|_{\ell_2^m} &\le \Big\|\sum_{k=1}^m a_k f_{n_k}\Big\|_1 \le \Big\|\sum_{k=1}^m a_k f_{n_k}\Big\|_E \\ &= \Big\|\sum_{k=1}^m a_k T_2 y_{n_k}\Big\|_E \le \|T_2\| \Big\|\sum_{k=1}^m a_k y_{n_k}\Big\|_p \\ &\le KC\|T_2\| \big\|(a_i)\big\|_{\ell_2^m}. \end{aligned}$$

Therefore (f_{n_k}) is equivalent in E to the unit vector basis of ℓ_2. $\square$

We claim that if $L_p(\mu) \subseteq E$ for some $1 \le p < \infty$ then E satisfies the R-condition. This is a special case of a more general fact: *if E is p-concave for some $1 \le p < \infty$ then E has R-condition.* This follows from Lemma 4.1.1 together with Krivine's factorization theorem [**87**, Theorem 1.d.11] since the inclusion map $i\colon L_\infty(\mu) \to E$ is p-convex (an inspection of the proof of [**87**, Theorem 1.d.11] reveals that in our setting the factors are also positive).

Actually, this last statement is a consequence of the fact that every p-concave Banach lattice ($1 \leq p < \infty$) has property (U_2) [**113**]. Recall that property (U_2) was introduced by F. Räbiger in [**113**]: a Banach lattice F has *property* (U_2) if for every seminormalized weakly null order bounded sequence (x_n) in F there is a subsequence (x_{n_i}) and a constant $C > 0$ such that

$$\Big\|\sum_{i=1}^{m} a_i x_{n_i}\Big\| \leq C\big\|(a_i)\big\|_{\ell_2^m}$$

for any coefficients $a_1, \dots, a_m$.

Proposition 4.1.2. *If E has property (U_2) then it satisfies the R-condition.*

Proof. Let (x_n) be a bounded sequence in $L_\infty(\mu)$ equivalent in $L_1(\mu)$ to the unit vector basis of ℓ_2. Again, (x_n) is weakly null in E. By property (U_2), there exists a constant $C > 0$ such that, after passing to a subsequence, we have

$$\Big\|\sum_{i=1}^{m} a_i x_{n_i}\Big\|_E \leq C\big\|(a_i)\big\|_{\ell_2^m}$$

for any $(a_i)_{i=1}^m$. On the other hand, by our choice of (x_n) there exists another constant c such that

$$c\|(a_i)\|_{\ell_2^m} \leq \Big\|\sum_{i=1}^{m} a_i x_{n_i}\Big\|_1 \leq \Big\|\sum_{i=1}^{m} a_i x_{n_i}\Big\|_E$$

for any coefficients $a_1, \dots, a_m$. Hence (x_{n_i}) is also equivalent to the unit vector basis of ℓ_2 in E. $\square$

As mentioned above, it was proved in [**113**] that every Banach lattice which is p-concave for some $1 \leq p < \infty$ has property (U_2). However, the following example shows that the converse is false.

Example 4.1.3. *The Banach lattice $\ell_2(\ell_\infty^{2^n})$ has property (U_2) but it is not p-concave for any $1 \leq p < \infty$.*

Proof. Let us write every element $x \in \ell_2(\ell_\infty^{2^n})$ as a sequence $(x_i)_{i=1}^\infty$, with $x_1 \in \ell_\infty^1$, $(x_2, x_3) \in \ell_\infty^2$, $(x_4, x_5, x_6, x_7) \in \ell_\infty^4$, and so on. Therefore,

$$\|x\| = \left(\sum_{n=0}^{\infty} \max_{2^n \leq i < 2^{n+1}} |x_i|^2\right)^{\frac{1}{2}}.$$

For $x \in \ell_2(\ell_\infty^{2^n})$ we consider $\overline{x} = (\overline{x}_i)_{i=1}^\infty$, defined by $\overline{x}_i = \max\big\{|x_j| \ : \ 2^n \leq j < 2^{n+1}\big\}$ whenever $2^n \leq i < 2^{n+1}$.

Let $(x^{(n)})$ be a seminormalized order bounded weakly null sequence in $\ell_2(\ell_\infty^{2^n})$. For each n, put $y^{(n)} = \overline{x^{(n)}}$. It is clear that $(y^{(n)})$ is also seminormalized and order bounded by some positive $z \in \ell_2(\ell_\infty^{2^n})$. We claim that $(y^{(n)})$ is also weakly null. Indeed, let $f \in \ell_2(\ell_1^{2^n}) = (\ell_2(\ell_\infty^{2^n}))^*$. Since $f = f^+ - f^-$, we may assume that $f \geq 0$. For every $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that $z_i \leq \frac{\varepsilon}{2\|f\|}$ for all $i > N$. Since $(x^{(n)})$ is weakly null, $(x^{(n)})$ and, therefore, $(y^{(n)})$, converge to zero coordinatewise. Hence, we can find $M \in \mathbb{N}$ such that $f_i y_i^{(n)} < \frac{\varepsilon}{2N}$ for $i \leq N$ and for all $n \geq M$. Therefore,

$$\langle f, y^{(n)} \rangle = \sum_{i=1}^{\infty} f_i y_i^{(n)} = \sum_{i=1}^{N} f_i y_i^{(n)} + \sum_{i=N+1}^{\infty} f_i y_i^{(n)} \leq \sum_{i=1}^{N} \frac{\varepsilon}{2N} + \sum_{i=N+1}^{\infty} f_i z_i \leq \frac{\varepsilon}{2} + \|f\| \frac{\varepsilon}{2\|f\|} = \varepsilon,$$

for $n \geq M$. Thus, $(y^{(n)})$ is weakly null.

Notice that $(y^{(n)})$ belongs to the closed linear span of (e^n), where

$$e_k^n = \begin{cases} 1 & \text{if } 2^n \leq k < 2^{n+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the closed linear span of (e^n) is isomorphic to ℓ_2, and since $(y^{(n)})$ is weakly null, it must have a subsequence $(y^{(n_k)})$ equivalent to the unit vector basis of ℓ_2. Therefore, we have:

$$\Big\| \sum_{k=1}^{\infty} a_k x^{n_k} \Big\| \leq \Big\| \sum_{k=1}^{\infty} |a_k| y^{(n_k)} \Big\| \leq C \Big(\sum_{k=1}^{\infty} |a_k|^2 \Big)^{\frac{1}{2}}$$

for some constant $C > 0$ independent of the sequence $(a_k)_{k=1}^{\infty}$. Therefore, $\ell_2(\ell_\infty^{2^n})$ has property (U_2). On the other hand, this space contains ℓ_∞^n's uniformly, hence it is not p-concave for any $1 \leq p < \infty$. $\square$

2. Strictly singular and AM-compact operators

We start by showing that a regular operator T on E is AM-compact if and only if its extension $\widetilde{T}$ to $L_1(\mu)$ is Dunford-Pettis. This result is related to [**12**, Theorem 5.97] which asserts that a regular operator from E to $L_1(\mu)$ is Dunford-Pettis if and only if it maps order intervals onto norm compact sets. We will use the following observation due to J. J. Uhl (see [**120**], [**24**]).

Theorem 4.2.1. *An operator $T\colon L_1(\mu) \to L_1(\mu)$ is Dunford-Pettis if and only if its restriction to $L_\infty(\mu)$ is compact as an operator from $L_\infty(\mu)$ to $L_1(\mu)$.*

Theorem 4.2.2. *Let T be a regular operator on E. Then T is AM-compact if and only if $\widetilde{T}\colon L_1(\mu) \to L_1(\mu)$ is Dunford-Pettis.*

Proof. Suppose that $\widetilde{T}$ is Dunford-Pettis. It suffices to show that if (x_n) is a sequence in $[0, x]$ for some $x \in E_+$ then (Tx_n) has a convergent subsequence. Without loss of generality we can take (x_n) normalized. Since order intervals in $L_1(\mu)$ are weakly compact, there exists a subsequence (x_{n_k}) which converges weakly to some g in $L_1(\mu)$. Then $Tx_{n_k} = \widetilde{T}x_{n_k} \xrightarrow{\|\cdot\|_1} \widetilde{T}g$ because $\widetilde{T}$ is Dunford-Pettis. Since T is regular then (Tx_n) is contained in $\big[-|T|x, |T|x\big]$, and in addition $\widetilde{T}g \in \big[-|T|x, |T|x\big]$. By Amemiya's Theorem [**94**, 2.4.8] it follows that $\widetilde{T}g \in E$ and $Tx_{n_k} \to \widetilde{T}g$ in E.

Conversely, suppose that $T : E \to E$ is AM-compact. Then T is also AM-compact as an operator from $L_\infty(\mu)$ to $L_1(\mu)$, because $L_\infty(\mu)$ is a (non-closed) ideal in E and the inclusion

$E \hookrightarrow L_1(\mu)$ is continuous. Since the unit ball in $L_\infty(\mu)$ is an order interval, it follows that this operator is in fact compact. Now, by Theorem 4.2.1 we obtain that $\widetilde{T}$ is Dunford-Pettis, as desired. $\square$

REMARK 4.2.3. It was shown in [**120**] that an operator $S\colon L_1(\mu) \to L_1(\mu)$ is Dunford-Pettis if and only if it is ℓ_2-singular. Moreover, if S is not Dunford-Pettis then one can find a sequence (f_n) bounded in $L_\infty(\mu)$ such that (f_n) viewed as a sequence in $L_1(\mu)$ is weakly null and equivalent to the unit vector basis of ℓ_2; in addition, the restriction of S to the span of (f_n) in $L_1(\mu)$ is an isomorphism.

Next proposition is another version for regular operators of Proposition 3.1.4. Here we are using the more general notion of R-condition, instead of finite cotype.

PROPOSITION 4.2.4. *Suppose that E satisfies the R-condition and $T : E \to E$ is regular and ℓ_2-singular. Then T is AM-compact.*

PROOF. In view of Theorem 4.2.2 it suffices to show that $\widetilde{T}\colon L_1(\mu) \to L_1(\mu)$ is Dunford-Pettis. Suppose it is not. Let (f_n) be as in Remark 4.2.3 for $S = \widetilde{T}$. Since E satisfies the R-condition, after passing to a subsequence we have the following chain of inequalities with appropriate constants:

$$\Big\|T\Big(\sum_{n=1}^{\infty}\alpha_n f_n\Big)\Big\|_E \geq \Big\|T\Big(\sum_{n=1}^{\infty}\alpha_n f_n\Big)\Big\|_1 \geq C_1\Big\|\sum_{n=1}^{\infty}\alpha_n f_n\Big\|_1 \geq C_2\big\|(\alpha_n)\big\|_2 \geq C_3\Big\|\sum_{n=1}^{\infty}\alpha_n f_n\Big\|_E$$

for any $\sum_{n=1}^{\infty}\alpha_n f_n$ in E. This contradicts T being ℓ_2-singular. $\square$

COROLLARY 4.2.5. *Suppose that X is an arbitrary Banach lattice, $T\colon X \to X$ is strictly singular and factors with positive factors through E, and E satisfies the R-condition. Then T^3 is AM-compact.*

PROOF. Suppose that we can factor $T = RS$ where

$$X \xrightarrow{S} E \xrightarrow{R} X$$

and $S, R \geq 0$. Then $STR\colon E \to E$ is positive and strictly singular, hence AM-compact by Proposition 4.2.4. Since AM-compact operators form an algebraic ideal among regular operators, it follows from $T^3 = R(STR)S$ that T^3 is AM-compact $\square$

3. Invariant subspaces of positive strictly singular operators

In this section we apply the results of the preceding sections to the Invariant Subspace Problem. Invariant subspaces are always assumed to be non-zero and proper. A subspace is said to be *hyperinvariant* under an operator T if it is invariant under every operator commuting with T. Recall that T is said to be *quasinilpotent* if its spectrum is $\{0\}$ or, equivalently, if $\lim\limits_{n\to\infty}\sqrt[n]{\|T^n\|} = 0$. We will use the following standard lemma.

LEMMA 4.3.1. *Suppose that T is an operator on a Banach space. If T is not quasinilpotent and some power of T is strictly singular then T has a finite-dimensional hyperinvariant subspace.*

PROOF. Suppose that T is an operator on a Banach space X such that T is not quasinilpotent and T^n is strictly singular for some n. Clearly, T^n is not quasinilpotent.

Suppose first that X is a Banach space over $\mathbb{C}$. Then T^n has non-trivial eigenspaces by [**1**, Theorem 7.11]. Let Z be a non-trivial eigenspace of T^n. Since T^n is strictly singular, we have $\dim Z < \infty$. It is easy to see that Z is hyperinvariant under T.

Now suppose that X is a Banach space over $\mathbb{R}$. The complexification $T^n_{\mathbb{C}}$ of T^n is still strictly singular by [**1**, p. 177]. Again, let Z be a non-trivial eigenspace of $T^n_{\mathbb{C}}$ in $X_{\mathbb{C}}$. Then $\dim Z < \infty$ and Z is hyperinvariant under $T_{\mathbb{C}}$. Let $(x_1 + iy_1), \dots, (x_m + iy_m)$ be a basis of Z, put $M = \mathtt{Span}\{x_1, \dots, x_m, y_1, \dots, y_m\}$ in X. Clearly, $0 < \dim M \leq 2m$. We claim that M is hyperinvariant under T. Indeed, suppose that S is an operator such that $ST = TS$. Then $S_{\mathbb{C}}T_{\mathbb{C}} = T_{\mathbb{C}}S_{\mathbb{C}}$. Since Z in hyperinvariant under $T_{\mathbb{C}}$, for every $k \leq m$ we have $S_{\mathbb{C}}(x_k + iy_k) \in Z$, so that Sx_k and Sy_k are both in M. Hence, $S(M) \subseteq M$. $\square$

We make use of the following statement, which is a special case of Theorem 10.26 of [**1**].

THEOREM 4.3.2. *Every quasinilpotent AM-compact positive operator on a Banach lattice has an invariant subspace.*

Combining Theorem 4.3.2 with Lemma 4.3.1 we obtain the following result.

PROPOSITION 4.3.3. *Suppose that T is a positive AM-compact operator on a Banach lattice. If T^n is strictly singular for some n then T has an invariant subspace.*

Together with Theorem 4.2.4, this yields the following.

COROLLARY 4.3.4. *If E satisfies the R-condition and $T \in \mathcal{L}(E)_+$ is strictly singular then T is AM-compact and has an invariant subspace.*

COROLLARY 4.3.5. *Suppose that E satisfies the R-condition and $S, T \in \mathcal{L}(E)$ are such that $0 \leq S \leq T$. If T is strictly singular then S has an invariant subspace.*

PROOF. We use the results of Chapter 2. Since E is order continuous, by Corollary 2.2.3, S^2 is strictly singular. Theorem 4.2.4 yields that T is AM-compact, hence $0 \leq S \leq T$ implies that S is AM-compact. The conclusion now follows from Proposition 4.3.3. $\square$

Recall that if T is a positive operator on a Banach lattice then its left and right semi-commutants are defined as follows:

$$\langle T] = \bigl\{S \geq 0 \,:\, ST \leq TS\bigr\} \text{ and } [T\rangle = \bigl\{S \geq 0 \,:\, ST \geq TS\bigr\}.$$

The following is a theorem on the technique of minimal vectors that is being used fruitfully within the context of invariant subspaces. We refer to [**14**] and [**60**] for a proof.

THEOREM 4.3.6. *Suppose that T is a positive quasinilpotent operator on a Banach lattice X. Suppose that there exists a closed ball $B(x_0, r)$ in X centered at some $x_0 \geq 0$, of positive radius $r < \|x_0\|$ such that for every sequence (x_n) in $B(x_0, r) \cap [0, x_0]$ there exists a subsequence (x_{n_i}) and a sequence of operators (K_i) such that $0 \leq K_i \leq T$ for each i and $(K_i x_{n_i})$ converges to a non-zero vector. Then $\langle T]$ has a (common) invariant closed order ideal. In particular, T has an invariant subspace.*

THEOREM 4.3.7. *Suppose that E satisfies R-condition and T is a positive quasinilpotent strictly singular operator on E. Then $\langle T]$ has an invariant closed ideal.*

PROOF. Choose $x_0 \in X_+$ and $r > 0$ so that $B(x_0, r) \cap \ker T = \varnothing$. Suppose that (x_n) is a sequence in $B(x_0, r) \cap [0, x_0]$. We claim that there is a subsequence (x_{n_i}) such that (Tx_{n_i}) converges in norm to a non-zero vector; then the result will follow from Theorem 4.3.6. We may assume without loss of generality that (x_n) has no norm convergent subsequences. Since order intervals are weakly compact in E, we may assume by passing to a subsequence that (x_n) converges weakly to some x. Since $B(x_0, r)$ is weakly closed and $B(x_0, r) \cap \ker T = \varnothing$, we have $x \neq 0$ and $Tx \neq 0$. Notice that $\|Tx_n - Tx\|_1 \to 0$ since the extension $\widetilde{T}\colon L_1(\mu) \to L_1(\mu)$ is Dunford-Pettis by Theorem 4.2.2 and Corollary 4.3.4. Again, since the sequence (Tx_n) is order bounded (T is positive) we obtain $\|Tx_n - Tx\|_E \to 0$ by Amemiya's theorem [**94**, Theorem 2.4.8]. $\square$

To produce a similar result for $[T\rangle$ we use the following version of a theorem of Drnovšek [**1**, Theorem 10.50]. We start by recalling a few definitions and notation. Given a Banach space X, for a collection $\mathcal{C}$ of operators in $\mathcal{L}(X)$, let $\|\mathcal{C}\| = \sup\{\|T\| : T \in \mathcal{C}\}$. For each $x \in X$, let $\mathcal{C}x = \{Tx : T \in \mathcal{C}\}$, and so $\|\mathcal{C}x\| = \sup\{\|Tx\| : T \in \mathcal{C}\}$. For $n \in \mathbb{N}$, we denote $\mathcal{C}^n = \{T_1 \dots T_n : T_1, \dots, T_n \in \mathcal{C}\}$. Recall, that an operator T on a Banach lattice F is *locally quasinilpotent* at a point x if $\lim\limits_{n\to\infty} \sqrt[n]{\|T^n x\|} = 0$. Similarly, a family $\mathcal{C}$ of operators in $\mathcal{L}(X)$ is called *(locally) quasinilpotent at a point $x \in X$* if $\lim\limits_{n\to\infty} \sqrt[n]{\|\mathcal{C}^n x\|} = 0$, and is called *finitely quasinilpotent at a point $x \in X$* if every finite subcollection of $\mathcal{C}$ is (locally) quasinilpotent at x.

THEOREM 4.3.8. *Suppose that T is a positive operator on a Banach lattice F with a quasi-interior point w such that*

(1) *T is locally quasinilpotent at some $x_0 > 0$, and*
(2) *there is $S \in [T\rangle$ and a non-zero AM-compact operator K such that $|Kx| \leq S|x|$..*

Then $[T\rangle$ has an invariant closed ideal.

PROOF. Since the null ideal $N_T = \{x \in F : T|x| = 0\}$ of T is $[T\rangle$-invariant [**1**, Lemma 10.23], we may assume that $N_T = \{0\}$. Let $z \in F$ such that $Kz \neq 0$. We may assume that $|Kz| \leq w$, otherwise we can replace w with $w \vee |Kz|$. By [**1**, Lemma 4.16(1)] there exists an operator M dominated by the identity operator such that $MKz > 0$. Put $K_1 = MK$. Since

$N_T = 0$, it follows that $TK_1 z \neq 0$, hence $TK_1 \neq 0$. It is clear that TK_1 is AM-compact and is dominated by TS.

Let $\mathcal{J}$ be the algebraic ideal in $[T\rangle$ generated by TS, i.e., $\mathcal{J} = \{ATSB \,:\, A, B \in [T\rangle\}$. A straightforward verification shows that $\mathcal{J}$ is finitely quasinilpotent at x_0. Since $TS \in \mathcal{J}$ and TS dominates a non-zero AM-compact operator, $\mathcal{J}$ has an invariant closed ideal by [**1**, Theorem 10.44]. Now [**1**, Lemma 10.49] yields that $[T\rangle$ has an invariant closed ideal. □

Corollary 4.3.9. *Suppose that E has the R-condition and T is a positive quasinilpotent strictly singular operator on E. Then $[T\rangle$ has an invariant closed ideal.*

Proof. By Corollary 4.3.4, T is AM-compact. Now apply Theorem 4.3.8. □

Corollary 4.3.10. *Suppose that E has R-condition and T is a positive strictly singular operator on E. Then every positive operator commuting with T has an invariant subspace.*

Proof. If T is quasinilpotent then the conclusion follows from Theorem 4.3.7 or Corollary 4.3.9. If T is not quasinilpotent, then the result follows from Lemma 4.3.1. □

Proposition 4.3.11. *Suppose that X is an arbitrary Banach lattice, $T\colon X \to X$ is strictly singular and factors with positive factors through E, and E satisfies the R-condition. If T is locally quasinilpotent at a positive vector then $[T\rangle$ has an invariant closed ideal.*

Proof. Corollary 4.2.5 yields that T^3 is AM-compact. The result now follows from Theorem 4.3.8. □

4. Invariant subspaces of *SS*-friendly operators

It is well known that compact operators enjoy good properties concerning the Invariant Subspace Problem. The compactness properties were relaxed in [**4**], where the class of compact-friendly operators was introduced, showing that these operators also have invariant subspaces. We present here the analogous concept for strictly singular operators.

We call an operator $B \in \mathcal{L}(E)_+$ *strictly singular-friendly* (or *SS*-friendly in short) if there is a positive operator that commutes with B and dominates a non-zero operator which is dominated by a strictly singular positive operator. In particular, every operator dominating a positive strictly singular operator is strictly singular-friendly.

Theorem 4.4.1. *Suppose that E satisfies the R-condition. If $B \in \mathcal{L}(E)_+$ is a non-zero SS-friendly operator which is locally quasinilpotent at some $x_0 > 0$, then B has a non-trivial closed invariant ideal. Moreover, if (T_n) is a sequence in $[B\rangle$, then there exist a non-trivial closed ideal that is invariant under B and each T_n.*

Proof. Our argument is similar to the one in [**4**]. We can suppose without loss of generality that $\|B\| < 1$. Pick small enough scalars $\alpha_n > 0$ such that the positive operator $T = \sum_{n=1}^{\infty} \alpha_n T_n$

exists and $\|B+T\|<1$. It is clear that $T\in[B\rangle$, and $(B+T)^n\in[B\rangle$ for every n, so that the positive operator $A=\sum_{n=0}^{\infty}(B+T)^n$ also belongs to $[B\rangle$.

For each $x>0$, let J_x be the principal ideal generated by Ax, that is,

$$J_x=\big\{y\in E\ :\ |y|\leq \lambda Ax \text{ for some } \lambda>0\big\}.$$

Since $x\leq Ax$, we have that $x\in J_x$, so this is a non-zero ideal.

Note that J_x is $(B+T)$-invariant. Indeed, if $y\in J_x$, then $|y|\leq\lambda Ax$ for some $\lambda>0$ so we have

$$\big|(B+T)y\big|\leq(B+T)|y|\leq\lambda(B+T)\sum_{n=0}^{\infty}(B+T)^nx=\lambda\sum_{n=1}^{\infty}(B+T)^nx\leq\lambda Ax.$$

Clearly J_x is also invariant under B and T, since $0\leq B,\,T\leq B+T$, so it is also T_n-invariant for each n.

Therefore, for our purposes, it suffices to prove that there exists a positive $x\in E$ such that the ideal J_x is not norm dense in E. Suppose the contrary, that is, Ax is a quasi-interior point in E for each $x>0$. By assumption, there exist operators R, S, and C in $\mathcal{L}(E)$ such that R and S are positive, S is strictly singular, $C\neq 0$, $RB=BR$, and C is dominated by both R and S.

Since $C\neq 0$, there exists some $x_1>0$ such that $Cx_1\neq 0$. Then $A|Cx_1|$ is a quasi-interior point satisfying $A|Cx_1|\geq|Cx_1|$. By [**1**, Lemma 4.16], there exists an operator $M_1\in\mathcal{L}(E)$ dominated by the identity operator such that $x_2=M_1Cx_1>0$. Let $U_1=M_1C$. Note that U_1 is dominated by S and by R.

Now we have $\overline{J_{x_2}}=E$. Therefore, since $C\neq 0$, there exists $0<y<Ax_2$ such that $Cy\neq 0$. Because $A|Cy|$ is a quasi-interior point and $|Cy|\leq A|Cy|$, then, as before, there exists $M_2\in\mathcal{L}(E)$ dominated by the identity operator such that $x_3=M_2Cy>0$. Since $|y|\leq Ax_2$ and Ax_2 is a quasi-interior point, it follows that there is an operator $M\in\mathcal{L}(E)$ dominated by the identity such that $MAx_2=y$. So $x_3=M_2Cy=M_2CMAx_2$. And the operator $U_2=M_2CMA$ is dominated by SA and by RA.

Consider the operator U_2U_1. From $U_2U_1x_1=x_3>0$, we see that U_2U_1 is a non-zero operator. Since both U_1 and U_2 are dominated by strictly singular positive operators, Corollary 2.2.3 yields that U_2U_1 is strictly singular. Moreover,

$$|U_2U_1x|=|M_2CMAM_1Cx|\leq RAR|x|$$

for each $x\in E$.

Let $V=RAR$. Since A and R belong to $[B\rangle$ then V also belongs to $[B\rangle$. Observe that V dominates U_2U_1 which is strictly singular, therefore AM-compact. The result now follows from Theorem 4.3.8. □

It remains open wether *SS*-friendly and compact-friendly operators are in fact different classes of operators.

CHAPTER 5

Compact products of strictly singular operators

In this chapter we generalize a result of V. Milman [**96**] on products of *SS* operators on spaces L_p. Precisely, if $T \in SS(L_p)$, then T^2 is compact.

To this end we introduce first the concept of disjointly homogeneous Banach lattices, which behave in a certain sense like the spaces L_p. Namely, this spaces have a rigid lattice structure, since every pair of disjoint sequences share an equivalent subsequence.

In Section 1, we study several properties of disjointly homogeneous Banach lattice, and the second section contains the main results. Applications are given for Lorentz spaces and certain classes of Orlicz spaces on $[0, 1]$. We also provide examples of strictly singular operators on rearrangement invariant spaces none of whose powers are compact, showing that Milman's result cannot be generalized much further.

Finally, in the same spirit as in [**132**], some duality results for strictly singular operators on $L_{p,q}$ spaces are given.

1. Disjointly homogeneous Banach lattices

A Banach lattice E is said to be *disjointly homogeneous* if given two seminormalized disjoint sequences (x_i), (y_j) in E, there exist equivalent subsequences, i.e.

$$\Big\| \sum_{k=1}^{n} a_k x_{i_k} \Big\| \sim \Big\| \sum_{k=1}^{n} a_k y_{j_k} \Big\|.$$

Examples of disjointly homogeneous spaces include the spaces $L_p(\mu)$ for $1 \leq p \leq \infty$ and every measure μ, since every normalized disjoint sequence in $L_p(\mu)$ is equivalent to the unit vector basis of ℓ_p. Moreover, in [**47**] it was shown that every disjoint sequence in the Lorentz function spaces $\Lambda(W, p)$, contains a subsequence equivalent to the unit vector basis of ℓ_p. Hence, these spaces are also disjointly homogeneous.

Motivated by these examples, we say that a Banach lattice is *p-disjointly homogeneous* if every normalized disjoint sequence has a subsequence equivalent to the unit vector basis of ℓ_p (c_0 in the case $p = \infty$). Clearly, the spaces $\ell_p(X_n)$ where X_n is a sequence of finite dimensional Banach lattices, are p-disjointly homogeneous. So are the spaces B_p introduced by C. J. Seifert in connection with Baernstein's space(see [**30**, p. 7]).

Remark 5.1.1.

- A Banach lattice E is disjointly homogeneous if for any pair of disjoint positive normalized sequences (x_n) and (y_n), there exist subsequences which are equivalent.

- Note that the definition depends on the lattice structure, that is, it is not preserved under isomorphisms in general. For instance, for any $1 < p < \infty$, $p \neq 2$, the function space $L_p[0,1]$ is isomorphic to the atomic Banach lattice H_p given by the unconditional Haar basis, and this lattice has disjoint sequences equivalent to ℓ_2 and ℓ_p; thus, with its atomic structure, H_p is not disjointly homogeneous.

PROPOSITION 5.1.2. *Suppose that E is a disjointly homogeneous Banach lattice. Then, E or E^* is a KB-space.*

(1) *E is not a KB-space if and only if E is ∞-disjointly homogeneous.*
(2) *E^* is not a KB-space if and only if E is 1-disjointly homogeneous.*

PROOF. The equivalence in (1) follows immediately from the definition of a KB-space. By Theorem 1.2.3 E^* is not a KB-space if and only if E contains a lattice copy of ℓ_1, this yields the equivalence in (2). Finally, since no subsequence of the unit vector basis of c_0 is equivalent to the unit vector basis of ℓ_1 and vice versa, the two pairs of conditions are incompatible, hence one of the two spaces has to be a KB-space. □

PROPOSITION 5.1.3. *Let E be disjointly homogeneous. Every sublattice of E is disjointly homogeneous. If I is a closed ideal of E, then the Banach lattice E/I is also disjointly homogeneous.*

PROOF. The first assertion is immediate. For the second one, suppose first that E is a disjointly homogeneous KB-space. If I is a closed ideal of E, then it is also a projection band (see Theorem 1.2.2). Hence, E/I is lattice isomorphic to the orthogonal complement $I^{\perp}$, which is also a band in E. Therefore, E/I is also disjointly homogeneous.

Now, if E is not a KB-space, then by Proposition 5.1.2, every disjoint sequence of E has a subsequence equivalent to the unit vector basis of c_0. Let $(x_n + I)$ be a disjoint normalized sequence in E/I. Clearly, we can take $x_n \in E$ pairwise disjoint and seminormalized. Hence, some subsequence (x_{n_k}) must be equivalent to the unit vector basis of c_0. Therefore, there exists a constant $C > 0$ such that for scalars $a_1, \ldots, a_n$ we have

$$\Big\| \sum_{k=1}^{n} a_k x_{n_k} \Big\|_E \leq C \sup_{1 \leq k \leq n} |a_k|.$$

Since, the sequence $(x_n + I)$ is basic unconditional it follows that for some constant $C' > 0$ and scalars $a_1, \ldots, a_n$ we have

$$C' \sup_{1 \leq k \leq n} |a_k| \leq \Big\| \sum_{k=1}^{n} a_k (x_{n_k} + I) \Big\|_{E/I} \leq \Big\| \sum_{k=1}^{n} a_k x_{n_k} \Big\|_E \leq C \sup_{1 \leq k \leq n} |a_k|.$$

Thus, every disjoint normalized sequence in E/I has a subsequence equivalent to the unit vector basis of c_0, and so E/I is also disjointly homogeneous. □

One could ask whether every disjointly homogeneous Banach lattice is p-disjointly homogeneous for some $p \in [1, \infty]$. The following example shows that this is false. Let T denote

Tsirelson's space (cf. [**30**]), which is in fact the dual of the original space constructed by B. S. Tsirelson in [**130**]. Tsirelson's space T is the completion of the space of eventually null sequences of real numbers under the norm $\|x\|_T = \lim_n \|x\|_n$, where $\|x\|_n$ is defined inductively. Let (t_n) denote the unit vector basis. For $x = \sum_{k=1}^m a_k t_k$, we set

$$\|x\|_0 = \max_k |a_k|$$

and for $n \geq 0$

$$\|x\|_{n+1} = \max\Big\{\|x\|_n, \frac{1}{2}\max\sum_{j=1}^{k}\Big\|\sum_{n=p_j+1}^{p_{j+1}} a_n t_n\Big\|_n\Big\}$$

where the inner max is taken over all choices of $k \leq p_1 \leq p_2 \leq \ldots \leq p_{k+1}$, $k = 1, 2, \ldots$.

The space T does not contain any isomorphic copy of c_0 or ℓ_p for any $1 \leq p < \infty$ [**130**]. We show next that T with the order given by its unconditional basis is a disjointly homogeneous Banach lattice, which is not p-disjointly homogeneous for any $p \in [1, \infty]$.

EXAMPLE 5.1.4. *Tsirelson's space T, with the lattice structure given by its unconditional basis (t_n) is disjointly homogeneous, and clearly does not contain any disjoint sequence equivalent to the unit vector basis of ℓ_p or c_0.*

PROOF. If $x \in T$ with $x = \sum_{i=1}^\infty \alpha_i t_i$, then we denote $\mathtt{supp}x = \{i \in \mathbb{N} : \alpha_i \neq 0\}$. For $x, y \in T$ we write $\mathtt{supp}x < \mathtt{supp}y$ if $i < j$ whenever $i \in \mathtt{supp}x$ and $j \in \mathtt{supp}y$. Given two normalized disjoint sequences in T, (x_n) and (y_n), we will show that they have equivalent subsequences.

By truncating each x_n sufficiently far, by Theorem 1.6.1, we may assume that each x_n has finite support. By passing to a subsequence, we may further assume that $\mathtt{supp}x_n < \mathtt{supp}x_{n+1}$ for all n. Similarly, we may assume that $\mathtt{supp}y_n < \mathtt{supp}y_{n+1}$ for all n. Now it is easy to construct subsequences (x_{n_k}) and (y_{n_k}) so that

$$\mathtt{supp}x_{n_1} < \mathtt{supp}y_{n_1} < \mathtt{supp}x_{n_2} < \mathtt{supp}y_{n_2} \ldots$$

It follows from [**30**, Proposition II.4] that (x_{n_k}) and (y_{n_k}) are equivalent. □

2. Compact squares of strictly singular operators

We will make use of the following fact for strictly singular operators.

LEMMA 5.2.1. *Let $T : E \to E$ be a strictly singular operator on a Banach lattice with finite cotype. Then, every equi-integrable sequence (g_n) has a subsequence such that $(T(g_{n_k}))$ converges in the norm of L_1.*

PROOF. Since $T : E \to E$ is strictly singular and L_1 satisfies a lower 2 estimate, by Proposition 3.1.4, it follows that $T : E \to E \hookrightarrow L_1$ is AM-compact. Lemmas 1.4.2 and 1.5.3 yield the claimed result. □

THEOREM 5.2.2. *Let E be a reflexive disjointly homogeneous Banach lattice with finite cotype and unconditional basis. Every strictly singular operator $T : E \to E$ satisfies that the square T^2 is compact.*

PROOF. Let (x_n) be a weakly null sequence in E. Recall that a Banach lattice with finite cotype satisfies the subsequence splitting property [**134**]. Hence, we have $x_{n_k} = g_k + h_k$ for some equi-integrable sequence (g_k), and a disjoint sequence (h_k). Moreover, since E is reflexive we have that both (g_k) and (h_k) are also weakly null. Let us prove that $(T^2(g_k))$ and $(T^2(h_k))$ tend to zero in norm.

First, if $(T^2(g_k))$ does not tend to zero in the norm of E, then passing to a subsequence we can assume that $(T(g_k))$ and $(T^2(g_k))$ are seminormalized. Since T is strictly singular, by Lemma 5.2.1 we have that both sequences $(T(g_k))$ and $(T^2(g_k))$ tend to zero in the norm of L_1. Now, by Kadeč-Pełczyński Theorem 1.2.8, either $(T^2(g_k))$ tends to zero in the norm of E or both $(T(g_k))$ and $(T^2(g_k))$ have subsequences equivalent to disjoint sequences in E. Since E is disjointly homogeneous, the latter case would imply that T is an isomorphism on the span of some subsequence $[T(g_{k_j})]$, which is a contradiction with the fact that T is strictly singular. Hence, we must have that $T^2(g_k) \to 0$ in the norm of E.

Now, for the disjoint part, using again the subsequence splitting property, passing to a subsequence we can write $T(h_k) = u_k + v_k$ with (u_k) equi-integrable and (v_k) disjoint. If (v_k) is seminormalized, then since E is disjointly homogeneous, there exist a subsequence (k_j) and a constant $C > 0$ such that for any $n \in \mathbb{N}$ and scalars $(a_j)_{j=1}^n$

$$C^{-1}\Big\|\sum_{j=1}^n a_j v_{k_j}\Big\| \leq \Big\|\sum_{j=1}^n a_j h_{k_j}\Big\| \leq C\Big\|\sum_{j=1}^n a_j v_{k_j}\Big\|.$$

Moreover, since E has an unconditional basis and $(T(h_k))$ is weakly null, we can assume passing to a further subsequence that $(T(h_k))$ is basic unconditional. Hence using the fact that $|v_k| \leq |T(h_k)|$ and Theorem 1.3.1, we would have

$$\begin{aligned}
\Big\|\sum_{j=1}^n a_j h_{k_j}\Big\| &\leq C\Big\|\sum_{j=1}^n a_j v_{k_j}\Big\| \\
&= C\Big\|\Big(\sum_{j=1}^n |a_j v_{k_j}|^2\Big)^{\frac{1}{2}}\Big\| \\
&\leq C\Big\|\Big(\sum_{j=1}^n |a_j T(h_{k_j})|^2\Big)^{\frac{1}{2}}\Big\| \\
&\leq CB\Big\|\sum_{j=1}^n a_j T(h_{k_j})\Big\|
\end{aligned}$$

where B is the constant given in Theorem 1.3.1. This means that T is invertible on the span of $[h_{k_j}]$ in contradiction with the fact that T is strictly singular. Therefore, $(T(h_k))$ is equi-integrable, and by Lemma 5.2.1, $(T^2(h_k))$ tends to zero in the norm of L_1. Again by

Kadeč-Pełczyński Theorem 1.2.8, either $(T^2(h_k))$ is equivalent to a disjoint sequence or tends to zero in the norm of E. Note that since E is disjointly homogeneous, if $(T^2(h_k))$ is equivalent to a disjoint sequence, passing to a further subsequence we would have for any scalars $(a_k)_{k=1}^n$,

$$\Big\|\sum_{k=1}^n a_k T^2(h_k)\Big\| \sim \Big\|\sum_{k=1}^n a_k h_k\Big\|.$$

This is a contradiction with the fact that T is strictly singular. Therefore, $(T^2(h_k))$ tends to zero in E.

Finally, we have

$$\|T^2(x_{n_k})\| \leq \|T^2(g_k)\| + \|T^2(h_k)\|,$$

which means that $(T^2(x_{n_k}))$ tends to zero in E. Thus, T^2 is compact as claimed. $\square$

THEOREM 5.2.3. *Let E be a 2-disjointly homogeneous Banach lattice. Every strictly singular operator $T : E \to E$ is compact.*

PROOF. Since every disjoint sequence in E has a subsequence equivalent to ℓ_2, then E does not contain any sublattice isomorphic to c_0 nor ℓ_1. In particular, by Theorem 1.2.6, E is reflexive. We claim that the operator T is both AM-compact and M-weakly compact.

Indeed, to show that T is AM-compact, let (g_n) be a sequence in $[-x, x]$ for some $x \in E_+$. Since E is reflexive, passing to a subsequence and taking differences we can assume that (g_n) is weakly null. We will prove that Tg_n converges to zero in E. Thus, suppose that $\|T(g_n)\| \geq \alpha > 0$, for every $n \in \mathbb{N}$. Since (g_n) is equi-integrable, for every $\varepsilon > 0$ there exists $M < \infty$, such that $\|g_n \chi_{\{|g_n| \geq M\}}\| < \varepsilon$ for all $n \in \mathbb{N}$. Since the sequence $g_n^M = g_n \chi_{\{|g_n| < M\}}$ is contained in the order interval $[-M, M]$, passing to some subsequence we can assume that (g_n^M) converges weakly to a certain $g \in [-M, M]$. Moreover, since $\|g_n \chi_{\{|g_n| \geq M\}}\| \leq \varepsilon$, and (g_n) is weakly null, it follows that $\|g\| \leq \varepsilon$.

Let $z_n = g_n^M - g$. Clearly, (z_n) is weakly null. Moreover, it holds that

$$\|Tz_n\| \geq \|Tg_n^M\| - \|Tg\| \geq \|Tg_n\| - (\|T(g_n \chi_{\{|g_n| \geq M\}})\| + \|T\|) \geq \alpha - 2\|T\|\varepsilon,$$

which is bounded below for ε small enough. Since (z_n) is weakly null, passing to a subsequence we can assume that it is an unconditional basic sequence in E, and since $|z_n| \leq 2M$, by Theorem 1.3.1, it follows that

$$\Big\|\sum_{n=1}^k a_n z_n\Big\| \leq K \int_0^1 \Big\|\sum_{n=1}^k a_n r_n(t) z_n\Big\| dt \leq KC \Big\|\Big(\sum_{n=1}^k |a_n z_n|^2\Big)^{\frac{1}{2}}\Big\| \leq KC2M \Big(\sum_{n=1}^k |a_n|^2\Big)^{\frac{1}{2}}.$$

Therefore, we can define the operator $R : \ell_2 \to E$ by linear extension of $R(e_n) = z_n$. Hence, we can consider the composition

$$\ell_2 \xrightarrow{R} E \xrightarrow{T} E \hookrightarrow L_1$$

which is strictly singular because so is T. Thus, by Proposition 3.1.2, $\|TR(e_n)\|_{L_1} = \|T(z_n)\|_{L_1} \to 0$. Hence, if (Tz_n) does not tend to zero in the norm of E, by Theorem 1.2.8, we can extract a subsequence (still denoted (Tz_n)) which is equivalent to the unit vector basis of ℓ_2. However, this would yield the following estimation

$$\Big(\sum_{n=1}^{k}|a_n|^2\Big)^{\frac{1}{2}}\leq A\Big\|\sum_{n=1}^{k}a_nT(z_n)\Big\|\leq A\|T\|\Big\|\sum_{n=1}^{k}a_nz_n\Big\|\leq B\Big(\sum_{n=1}^{k}|a_n|^2\Big)^{\frac{1}{2}},$$

for certain constants A and B. This yields that T is an isomorphism on a subspace isomorphic to ℓ_2, in contradiction with the strict singularity of T.

Therefore, we can assume that $\|T(z_n)\|_E\to 0$, so for n large enough, $\|T(z_n)\|\leq\varepsilon$. In particular, we get

$$\|T(g_n)\|\leq\|T(z_n)\|+\|T(g)\|+\|T(g_n-g_n^M)\|\leq\varepsilon+2\|T\|\varepsilon$$

for n large enough. Since ε was arbitrary this shows that $(T(g_n))$ tends to zero in the norm of E. Therefore, T is AM-compact.

Now, we show that T is M-weakly compact. Hence, consider a normalized disjoint sequence (h_n) in E. By hypothesis, we can assume, passing to a subsequence, that it is equivalent to the unit vector basis of ℓ_2. Hence, using again Proposition 3.1.2, we get that $(T(h_n))$ tends to zero in the norm of L_1. If $(T(h_n))$ were not convergent to zero in the norm of E, then by Kadeč-Pełczyński Theorem 1.2.8 we could extract a subsequence equivalent to a disjoint sequence, hence equivalent to the unit vector basis of ℓ_2. This implies that T is an isomorphism on a subspace isomorphic to ℓ_2, in contradiction with the fact that T is strictly singular. This proves that $(T(h_n))$ converges to zero in E, and so T is M-weakly compact.

Finally, since the operator T is AM-compact and M-weakly compact, by [**94**, Prop. 3.7.4], we conclude that T is compact. This finishes the proof. □

THEOREM 5.2.4. *Let E be a 1-disjointly homogeneous Banach lattice with finite cotype. Every strictly singular operator $T:E\to E$ is Dunford-Pettis.*

PROOF. Let (x_n) be a weakly null sequence in E. We claim that (x_{n_k}) is equi-integrable, for some subsequence (n_k). Indeed, since finite cotype implies the subsequence splitting property ([**134**]), passing to a subsequence we would have $x_n=g_n+h_n$ where (g_n) is equi-integrable, (h_n) disjoint and $|g_n|\wedge|h_n|=0$. If (h_n) were seminormalized, then since E is 1-disjointly homogeneous, (h_n) would have a subsequence equivalent to the unit vector basis of ℓ_1. However, since (g_n) must have some subsequence converging weakly to $g\in E$, this would imply that $h_n=x_n-g_n$ has a subsequence converging weakly to g. This is impossible since the unit vector basis of ℓ_1 is not even weakly Cauchy.

Therefore, (x_{n_k}) is equi-integrable and Lemma 5.2.1 implies that $T(x_{n_k})$ tends to zero in L_1. If $T(x_{n_k})$ does not tend to zero in the norm of E, then by Kadeč-Pełczyński $(T(x_{n_k}))$ has a subsequence, still denoted (n_k), which is equivalent to a disjoint sequence, hence equivalent to the unit vector basis of ℓ_1. This yields that for scalars (a_k) and some constant $C>0$ we have

$$C\sum_{k=1}^{\infty}|a_k|\leq\Big\|\sum_{k=1}^{\infty}a_kT(x_{n_k})\Big\|\leq\|T\|\Big\|\sum_{k=1}^{\infty}a_kx_{n_k}\Big\|\leq\|T\|\sup\|x_{n_k}\|\sum_{k=1}^{\infty}|a_k|.$$

Therefore, T is an isomorphism on a subspace isomorphic to ℓ_1, in contradiction with the fact that T is strictly singular. Thus, $(T(x_{n_k}))$ tends to zero in E, and T is Dunford-Pettis. $\square$

As a consequence we get the following:

COROLLARY 5.2.5. *Let E be a 1-disjointly homogeneous Banach lattice with finite cotype. Every strictly singular operator $T : E \to E$ has compact square.*

PROOF. Since E is 1-disjointly homogeneous, in particular, it cannot contain a subspace isomorphic to c_0. Thus, by Proposition 1.2.5 E is weakly sequentially complete. Let (x_n) be a sequence in the unit ball of E. By Rosenthal's Theorem 1.6.3, the sequence $(T(x_n))$ has either a subsequence equivalent to the unit vector basis of ℓ_1 or a weakly Cauchy subsequence.

If $(T(x_{n_k}))$ is equivalent to the unit vector basis of ℓ_1, we would have for scalars $(a_k)_{k=1}^n$

$$\Big\|\sum_{k=1}^n a_k x_{n_k}\Big\| \le \sum_{k=1}^n |a_k| \le C\Big\|\sum_{k=1}^n a_k T(x_{n_k})\Big\|,$$

which implies that T is invertible on the span $[x_{n_k}]$, in contradiction with the fact that T is strictly singular. Hence, $(T_(x_{n_k}))$ must be weakly Cauchy, and since E is weakly sequentially complete, it must be weakly convergent in fact.

Theorem 5.2.4 yields that $(T^2(x_{n_k}))$ must be convergent in the norm of E. Therefore, T^2 is compact. $\square$

3. Strictly singular operators on Lorentz spaces

Let us recall first the definition of the Lorentz spaces (see [**89**]):

DEFINITION 5.3.1. *Let $1 \le q < \infty$ and W be a continuous, positive, non-increasing function in $[0,1]$, such that $\lim_{t\to 0} W(t) = \infty$, $W(1) > 0$, $\int_0^1 W(t)dt = 1$. The Lorentz function space over a measure space (Ω, Σ, μ), is defined by*

$$\Lambda(W,q) = \Big\{ f : \Omega \to \mathbb{R} : \ f \textit{ is } \Sigma - \textit{measurable, and } \|f\| = \Big(\int_0^1 f^*(t)^q W(t)dt\Big)^{1/q} < \infty \Big\},$$

where f^* denotes the decreasing rearrangement of f.

Recall that the decreasing rearrangement of a function f is defined by

$$f^*(s) = \inf\{t > 0 : \mu_f(t) \le s\},$$

where μ_f denotes the distribution function of f:

$$\mu_f(t) = \mu(\{\omega \in \Omega : f(\omega) > t\}).$$

The space $\Lambda(W,q)$ is a Banach lattice ([**89**]) equipped with the norm

$$\|f\| = \Big(\int_0^1 f^*(t)^q W(t)dt\Big)^{1/q}.$$

Now, let us recall the definition of the spaces $L_{p,q}$:

DEFINITION 5.3.2. *Let (Ω, Σ, μ) be a measure space. For $1 \leq p < \infty$ and $1 \leq q < \infty$, $L_{p,q}(\Omega, \Sigma, \mu)$ is the space of locally integrable functions in Ω, such that*

$$\|f\|_{p,q} = \left(\frac{q}{p}\int_0^\infty (t^{1/p}f^*(t))^q \frac{dt}{t}\right)^{1/q} < \infty.$$

For $1 \leq p \leq \infty$, $L_{p,\infty}(\Omega, \Sigma, \mu)$ is the space of locally integrable functions on Ω, for which

$$\|f\|_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t) < \infty.$$

Note that $L_{p,p} = L_p$ isometrically. Moreover, for $1 \leq q < p$ and $\Omega = [0, 1]$, $L_{p,q}$ is a Lorentz function space of the type $\Lambda(W, q)$, where $W(t) = \frac{q}{p} t^{q/p-1}$. Notice also, that for $q > p$, the above expression $\| \, \|_{p,q}$ does not define a norm, since it fails the triangle inequality (this is because the corresponding $W(t)$ is increasing in this case). However, for $p > 1$ we can consider a norm $\||\, \||_{p,q}$ which makes $L_{p,q}$ a Banach space, and that satisfies $\|f\|_{p,q} \leq \||f\||_{p,q} \leq C(p, q)\|f\|_{p,q}$, for a certain constant $C(p, q)$ (cf. [**87**, p. 142]).

The main point in order to make use of the results in the previous section is the following result for disjoint sequences ([**29**] and [**47**, 5.1]).

PROPOSITION 5.3.3. *Given $1 \leq q < \infty$, $1 < p < \infty$, let (f_n) be a sequence of disjoint normalized functions in $\Lambda(W, q)[0, 1]$ (respectively $L_{p,q}[0, 1]$). For each $\varepsilon > 0$, there exists a subsequence (f_{n_k}) which is $(1 + \varepsilon)$-equivalent to the unit vector basis of ℓ_q, whose span is a complemented subspace of $\Lambda(W, q)[0, 1]$ (resp. $L_{p,q}[0, 1]$) with projection constant smaller than $(1 + \varepsilon)$.*

An immediate consequence of this proposition and the results in Section 2 are the following.

COROLLARY 5.3.4. *For any W as in Definition 5.3.1 and $1 \leq q < \infty$ every strictly singular operator $T : \Lambda(W, q)[0, 1] \to \Lambda(W, q)[0, 1]$ has a compact square. Moreover, if $q = 2$, then T is already compact. And if $q = 1$, then T is Dunford-Pettis.*

COROLLARY 5.3.5. *For $1 < p < \infty$, $1 \leq q < \infty$ every strictly singular operator T : $L_{pq}[0, 1] \to L_{pq}[0, 1]$ has a compact square. Moreover, if $q = 2$, then T is already compact. And if $q = 1$, then T is Dunford-Pettis.*

The situation in the non-separable case is much different. For $q = \infty$ there are strictly singular operators on $L_{p,\infty}$ whose square is not compact (even the cube is not compact). To show this we first need the following fact:

PROPOSITION 5.3.6. *Given $1 \leq p < \infty$, there exists a disjoint normalized sequence (f_n) in $L_{p,\infty}$ whose span is isomorphic to ℓ_p and complemented in $L_{p,\infty}$.*

PROOF. Let (t_n) be a decreasing sequence of numbers in the interval $[0, 1]$ such that $t_n \downarrow 0$, and let us consider the functions

$$f_n(t) = \frac{p-1}{p}(t - t_n)^{-\frac{1}{p}} \chi_{(t_n, t_{n+1})}(t),$$

for $t \in [0,1]$. We claim that the closed linear span $[f_n]$ in $L_{p,\infty}$ is isomorphic to ℓ_p.

Indeed, since

$$\|f\|_{L_{p,\infty}} = \sup_{s>0} s(\mu_f(s))^{\frac{1}{p}},$$

where $\mu_f(s) = \mu\{t \in (0,1) : |f(t)| > s\}$ is the distribution function of f, then for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \mu_{f_n}(s) &= \mu\{t \in (0,1) : |f_n(t)| > s\} = \mu\{t \in (t_n, t_{n+1}) : \frac{p-1}{p}(t-t_n)^{-\frac{1}{p}} > s\} \\ &= \mu\Big\{t \in (t_n, t_{n+1}) : t < t_n + \Big(\frac{p-1}{p}\Big)^p \frac{1}{s^p}\Big\} \\ &= \begin{cases} t_{n+1} - t_n & \text{if } s \le \frac{p-1}{p(t_{n+1}-t_n)^{\frac{1}{p}}}, \\ \Big(\frac{p-1}{p}\Big)^p \frac{1}{s^p} & \text{if } s > \frac{p-1}{p(t_{n+1}-t_n)^{\frac{1}{p}}}. \end{cases} \end{aligned}$$

This clearly implies that (f_n) is a seminormalized sequence in $L_{p,\infty}$. Now, given scalars a, b let us see that for $i \neq j$, $\|af_i + bf_j\|_{L_{p,\infty}} \sim (|a|^p + |b|^p)^{\frac{1}{p}}$. Hence, since f_i and f_j are disjoint, we have

$$\begin{aligned} \|af_i + bf_j\|_{L_{p,\infty}} &= \sup_{s>0} s\Big(\mu_{af_i+bf_j}(s)\Big)^{\frac{1}{p}} \\ &= \sup_{s>0} s\Big(\mu_{af_i}(s) + \mu_{bf_j}(s)\Big)^{\frac{1}{p}} \\ &= \sup_{s>0} s\Big(\mu_{f_i}(\frac{s}{|a|}) + \mu_{f_j}(\frac{s}{|b|})\Big)^{\frac{1}{p}} \\ &\ge s_0\Big(\mu_{f_i}(\frac{s_0}{|a|}) + \mu_{f_j}(\frac{s_0}{|b|})\Big)^{\frac{1}{p}} \\ &= s_0\Big[\Big(\frac{p-1}{p}\Big)^p \frac{|a|^p}{s_0^p} + \Big(\frac{p-1}{p}\Big)^p \frac{|b|^p}{s_0^p}\Big]^{\frac{1}{p}} \\ &= \frac{p-1}{p}(|a|^p + |b|^p)^{\frac{1}{p}} \end{aligned}$$

where s_0 is any number greater than $\max\Big\{\frac{|a|p-1}{p(t_{i+1}-t_i)^{\frac{1}{p}}}, \frac{|b|p-1}{p(t_{j+1}-t_j)^{\frac{1}{p}}}\Big\}$. Since $L_{p,\infty}$ satisfies an upper p-estimate, then we also get automatically that $\|af_i + bf_j\|_{L_{p,\infty}} \le C(|a|^p + |b|^p)^{\frac{1}{p}}$ for certain constant $C > 0$. The statement that $[f_n]$ is isomorphic to ℓ_p follows by induction.

Now, to construct a projection onto $[f_n]$, let $B \in \ell_\infty^*$ be a Banach limit, that is $B(1,1,\dots) = 1$ and $B(x_1, x_2, \dots) = B(x_2, x_3, \dots)$ for (x_n) in ℓ_∞. In [**125**] it was proved that the operator given by

$$Rx(t) = B\Big(n^{(1-\frac{1}{p})} \int_t^{t+\frac{1}{n}} x(s)ds\Big)$$

for $x \in L_{p,\infty}$ and $t \in [0,1]$, is bounded from $L_{p,\infty}$ to $\ell_p(0,1)$, with norm one. We define $P : L_{p,\infty} \to L_{p,\infty}$ by

$$Px(t) = \sum_{k=1}^{\infty} Rx(t_k) f_k(t).$$

Notice that P can be seen as the composition

$$\begin{tikzcd} L_{p,\infty} \arrow[r, "P"] \arrow[d, "R"'] & L_{p,\infty} \\ \ell_p(0,1) \arrow[r, "T"] & \ell_p \arrow[u, hook, "J"'] \end{tikzcd}$$

where $Tx(t) = x(t)\chi_{\{t_k\}}(t)$ and $J((a_k)_{k=1}^{\infty}) = \sum_{k=1}^{\infty} a_k f_k$. Hence the operator P is bounded. Moreover, we have

$$Rf_k(t) = \begin{cases} 0 & \text{if } t \neq t_k \\ 1 & \text{if } t = t_k \end{cases}$$

Indeed, for $t \neq t_k$ we have that

$$n^{(1-\frac{1}{p})} \int_t^{t+\frac{1}{n}} f_k(s)ds \leq n^{(1-\frac{1}{p})} \frac{p-1}{p} \int_t^{t+\frac{1}{n}} (t-t_k)^{-\frac{1}{p}} ds \leq n^{-\frac{1}{p}} (t-t_k)^{-\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0.$$

While for $t = t_k$, and $n > \frac{1}{t_k}$, it holds that

$$n^{(1-\frac{1}{p})} \int_{t_k}^{t_k+\frac{1}{n}} f_k(s)ds = n^{(1-\frac{1}{p})} \frac{p-1}{p} \int_0^{\frac{1}{n}} s^{-\frac{1}{p}} ds = 1.$$

Therefore, $P(f_k) = f_k$, which yields that P is a projection onto $[f_k]$ as claimed. □

It is known that the Lorentz sequence space $\ell_{p,\infty}$ embeds as a complemented sublattice into $L_{p,\infty}$ (see [**84**] for details). Moreover, for $1 < p < \infty$ the Rademacher functions span a complemented subspace of $L_{p,\infty}$ isomorphic to ℓ_2, which is complemented. Also, we can consider a subspace isomorphic to ℓ_∞ [**87**, Prop. 1.a.7], which is clearly complemented in $L_{p,\infty}$ (cf. [**86**, p. 105]). Therefore, for $p \neq 2$ we can consider a complemented subspace of $L_{p,\infty}$ isomorphic to $\ell_p \oplus \ell_{p,\infty} \oplus \ell_2 \oplus \ell_\infty$. This allows us to define the operator T given by

$$\begin{tikzcd} L_{p,\infty} \arrow[r, "T"] \arrow[d, "P"'] & L_{p,\infty} \\ \ell_p \oplus \ell_{p,\infty} \oplus \ell_2 \oplus \ell_\infty \arrow[r, "S"] & \ell_p \oplus \ell_{p,\infty} \oplus \ell_2 \oplus \ell_\infty \arrow[u, "J"'] \end{tikzcd}$$

where P is a projection, J an isomorphic embedding, and for $(x, y, z, w) \in \ell_p \oplus \ell_{p,\infty} \oplus \ell_2 \oplus \ell_\infty$,

$$S(x, y, z, w) = \begin{cases} (0, x, y, z) & \text{if } p < 2, \\ (z, x, 0, y) & \text{if } p > 2. \end{cases}$$

Clearly, T is strictly singular, and T^3 is not compact. A natural question is: Does always hold that if $T : L_{p,\infty} \to L_{p,\infty}$ is strictly singular, then T^4 is compact?

However, for the order continuous part $(L_{p,\infty})_o$, every strictly singular operator on $(L_{p,\infty})_o$ has compact square. Recall that the order continuous part $(L_{p,\infty})_o$ consists of the closure in the norm of $L_{p,\infty}$ of the span of the simple functions. The proof of this result has some differences with those of the above theorems, mainly because $(L_{p,\infty})_o$ does not have finite cotype.

PROPOSITION 5.3.7. *Let $1 < p < \infty$. If $T \in SS((L_{p,\infty})_o)$, then $T^* \in SS(L_{p',1})$ and the square T^2 is compact.*

PROOF. Indeed, if T^* were not strictly singular, then there exists an infinite dimensional subspace $X \subset L_{p',1}$ such that the restriction $T^*|_X$ is an isomorphism. Using [**29**, Thm 2.5] either $T^*(X)$ is a strongly embedded subspace of $L_{p',1}$ or there exist a normalized sequence in $T^*(X)$ and a disjoint sequence (z_n) in $L_{p',1}$ such that $\|z_n - y_n\| \to 0$, and so passing to a further subsequence, $[y_n]$ is a complemented subspace of $L_{p',1}$ isomorphic to ℓ_1. If $1 < p \leq 2$, then since $L_{p',1} \subset L_2$, we get that every strongly embedded subspace of $L_{p',1}$ is complemented and isomorphic to ℓ_2. On the other case, if $2 < p < \infty$, then supposing that $T^*(X)$ is strongly embedded, by [**29**, Thm. 2.5] we also get that X must be strongly embedded. From this it can be proved that T^* is an isomorphism on a subspace M isomorphic to ℓ_2 such that both M and $T^*(M)$ are strongly embedded (for the details see the proof of implication (3) $\Rightarrow$ (1) of Theorem 5.5.1 in Section 5).

Hence, in any case we have that T^* is invertible in a subspace M such that $T^*(M)$ is complemented in $L_{p',1}$ and either

(1) $M \simeq T^*(M) \simeq \ell_2$ with M and $T^*(M)$ strongly embedded, or
(2) $M \simeq T^*(M) \simeq \ell_1$ with $M = [f_n]$ and $T^*(f_n)$ disjoint.

Any of these cases yields a contradiction with the fact that T is strictly singular. Indeed, if (1) holds, then this implies that T^{**} is an isomorphism on a complemented subspace X isomorphic to ℓ_2 which is identified with the dual of the subspace $T(M) \subset L_{p',1}$, and the projection is the adjoint of the projection onto $T(M)$, $P : L_{p',1} \to L_{p',1}$. However, since $T(M)$ is strongly embedded we can factor P through the formal inclusion $L_{p',1} \hookrightarrow L_r$, for some $1 < r < p'$. This implies that X is in fact complemented in $(L_{p,\infty})_o$, hence T is not strictly singular.

Now, if case (2) holds, then as in the proof of [**47**, Thm. 5.1] we can find functionals F_n on $L_{p',1}$ with $F_n(Tf_n) = 1$ and $F_n(Tf_m) = 0$ for $n \neq m$. This functionals are defined by

$$F_n(f) = \int f(\tau(t))\chi_{[\varepsilon_n,|A_n|]}\mathtt{sgn}T^*f_n(\tau(t))t^{\frac{1}{p'}-1}dt \Big/ \int_{\varepsilon_n}^{|A_n|} |T^*f_n(\tau(t))|t^{\frac{1}{p'}-1}dt,$$

where $A_n = \mathtt{supp}T^*f_n$, $\tau_n : [0, |A_n|] \to A_n$ are measure preserving such that

$$\int_0^1 |T^*f_n(\tau_n(t))|t^{\frac{1}{p'}-1}dt = \|T^*f_n\|,$$

and $\varepsilon_n > 0$ are sufficiently small (see [**47**, Thm. 5.1]). It follows that F_n belong to the separable part $(L_{p,\infty})_o$ and is equivalent to the unit vector basis of c_0. Moreover, $\|TF_n\| \geq \langle F_n, T^*f_n\rangle \frac{1}{\|f_n\|} > \alpha$ for every $n \in \mathbb{N}$, and some $\alpha > 0$. Hence, passing to a further subsequence we have

$$c \max |a_i| \leq \Big\| \sum a_i TF_i \Big\| \leq \|T\| \Big\| \sum a_i F_i \Big\| \leq C \max |a_i|,$$

for certain constants $c, C > 0$ and scalars $a_1, \dots, a_n$. This implies that T is an isomorphism on a subspace isomorphic to c_0, in contradiction with the fact that T is strictly singular.

Hence, we have shown that the operator T^* is strictly singular, and by Corollary 5.3.5, the square $(T^*)^2$ is compact. This clearly means that T^2 is also compact as claimed. $\square$

4. Strictly singular operators on Orlicz spaces

Given an Orlicz function φ, we consider the Orlicz space $L_\varphi(0,1)$ consisting of measurable functions f on $[0,1]$ such that

$$\int_0^1 \varphi\Big(\frac{|f(t)|}{\rho}\Big)dt < \infty,$$

for some $\rho > 0$. The norm is defined by

$$\|f\| = \inf\Big\{\rho > 0 : \int_0^1 \varphi\Big(\frac{|f(t)|}{\rho}\Big)dt \leq 1\Big\}.$$

Let us consider the sets associated to the Orlicz function φ in the space $C(0,\infty)$ (cf. [**85**],[**69**]):

$$E^\infty_{\varphi,s} = \overline{\Big\{\frac{\varphi(rt)}{\varphi(r)} : r \geq s\Big\}}, \qquad E^\infty_\varphi = \bigcap_{s>0} E^\infty_{\varphi,s}\,, \quad \text{and} \quad C^\infty_\varphi = \overline{co}(E^\infty_\varphi),$$

where as usual $\overline{co}(A)$ denotes the closed convex hull of the set A.

Proposition 5.4.1. *Let φ be an Orlicz function such that $E^\infty_\varphi = \{t^p\}$, for some $1 \leq p < \infty$, then $L_\varphi[0,1]$ is p-disjointly homogeneous. Hence, if $T \in SS(L_\varphi)$, then the square T^2 is compact. And for $p = 2$, T is already compact.*

Proof. By [**85**, Prop. 3], for every disjoint sequence (f_n) in L_φ, there exists a subsequence (f_{n_k}) equivalent to the unit vector basis of ℓ_p, because $C^\infty_\varphi = E^\infty_\varphi = \{t^p\}$.

Now, for $1 \leq p < \infty$ and $p \neq 2$, by Theorem 5.2.2 and Corollary 5.2.5, if $T \in SS(L_\varphi)$, then the square T^2 is compact. And for $p = 2$, Theorem 5.2.3 yields that every $T \in SS(L_\varphi)$ is compact. $\square$

Recall the definition of the superior and inferior indices for a Banach lattice E:

$$s(E) = \sup\{p > 1 : E \text{ satisfies an upper } p\text{-estimate}\}$$

$$\sigma(E) = \inf\{p > 1 : E \text{ satisfies a lower } p\text{-estimate}\}.$$

The condition $E^\infty_\varphi = \{t^p\}$ implies that the associated Banach lattice indices satisfy

$$s(L_\varphi) = \sigma(L_\varphi) = p.$$

Many regular Orlicz functions satisfy the condition $E^\infty_\varphi = \{t^p\}$. For instance, the class of all Orlicz functions such that $\lim_{t\to\infty} \frac{t\varphi'(t)}{\varphi(t)} = p$. However, we cannot weaken this condition on E^∞_φ, as the following examples show.

Example 5.4.2. *There exist Orlicz spaces L_φ with index $2 < p < \infty$ such that there are strictly singular operators $T : L_\varphi \to L_\varphi$ with T^2 non compact.*

Proof. Let $2 < p < \infty$, and consider L_φ for $\varphi(t) = \varphi_{p,q}(t) = t^p \exp\{qf(\log t)\}$ for $t > 0$, where $f(x) = \sum_{k=1}^{\infty} \left(1 - \cos\left(\frac{\pi x}{2^k}\right)\right)$. This functions are minimal [**70**], with index p. Hence, L_φ has a complemented subspace isomorphic to ℓ_φ [**69**]. Moreover, $\varphi(t) \geq t^p$, so the inclusion $\ell_\varphi \hookrightarrow \ell_p$ is bounded, and since for q large enough ℓ_φ has no complemented subspace isomorphic to ℓ_p [**70**, Corollary 1.7], then the inclusion $\ell_\varphi \hookrightarrow \ell_p$ is a strictly singular operator.

Consider the decomposition $L_\varphi[0,1] = L_\varphi[0,\frac{1}{3}] \oplus L_\varphi[\frac{1}{3},\frac{2}{3}] \oplus L_\varphi[\frac{2}{3},1]$, and denote by $P_R : L_\varphi[0,\frac{1}{3}] \to [r_n]$ the projection onto de Rademacher functions on $[0,\frac{1}{3}]$, and by $P_\varphi : L_\varphi[\frac{1}{3},\frac{2}{3}] \to \ell_\varphi$ the projection onto ℓ_φ.

Define the operator $T : L_\varphi \to L_\varphi$ by the following factorization diagram

$$\begin{tikzcd}
L_\varphi[0,\frac{1}{3}] \arrow[d, "P_R"'] & \oplus & L_\varphi[\frac{1}{3},\frac{2}{3}] \arrow[d, "P_\varphi"'] & \oplus \; L_\varphi[\frac{2}{3},\frac{1}{3}] \arrow[r, "T"] & L_\varphi[0,\frac{1}{3}] \; \oplus & L_\varphi[\frac{1}{3},\frac{2}{3}] & \oplus & L_\varphi[\frac{2}{3},\frac{1}{3}] \\
\ell_2 & \oplus & \ell_\varphi & \arrow[r, "S"] & & \ell_\varphi \arrow[u, "J_\varphi"'] & \oplus & \ell_p \arrow[u, "J_p"']
\end{tikzcd}$$

where S denotes the formal inclusion, and J_φ and J_p are isomorphic embeddings.

T is well defined, and strictly singular since so are the inclusions $\ell_2 \hookrightarrow \ell_\varphi$ and $\ell_\varphi \hookrightarrow \ell_p$. However, T^2 is not compact, since it maps the Rademacher functions on $[0,\frac{1}{3}]$ to the canonical basis of ℓ_p. $\square$

Example 5.4.3. *An Orlicz space L_φ with indices $s(L_\varphi) = \sigma(L_\varphi) = 2$ and an operator $T \in SS(L_\varphi)$ such that T is not compact.*

Proof. Consider the function $\varphi = \varphi_{2,q}$ for big $q > 0$ as defined in the above example. L_φ contains complemented copies of ℓ_φ and ℓ_2. Moreover, since $\varphi_{2,q}(t) \geq t^2$ and ℓ_φ has no complemented copy of ℓ_2, we can deduce that the inclusion $\ell_\varphi \hookrightarrow \ell_2$ is strictly singular. Therefore, we can consider the operator

$$\begin{tikzcd}
L_\varphi \arrow[r, "T"] \arrow[d, "P"'] & L_\varphi \\
\ell_\varphi \oplus \ell_2 \arrow[r, "S"] & \ell_\varphi \oplus \ell_2 \arrow[u, hook]
\end{tikzcd}$$

where P is a projection and $S(x,y) = (0,x)$. Clearly, T is strictly singular but not compact. $\square$

In general, without assuming the condition $s(L^F) = \sigma(L^F)$, something stronger can be proved.

Example 5.4.4. *An Orlicz space $L^F[0,1]$, and an operator $T \in SS(L^F[0,1])$, but T^k is not compact for any $k \in \mathbb{N}$.*

Proof. Given any increasing sequence (p_n) contained in $[1,\infty)$, we can consider an Orlicz function space $L^F(0,1)$ containing complemented copies of ℓ_{p_n} for every n (cf. [**70**]). Let us denote by $P_n : L^F(0,1) \to L^F(0,1)$, the projection onto each ℓ_{p_n}, which will be spanned by a sequence of functions supported in $[2^{-n}, 2^{-n+1}]$. Now, for every $k \in \mathbb{N}$, denote $m_k = \sum_{n=1}^{k} n = \frac{k(k+1)}{2}$. Let us consider the operator given by

$$\begin{tikzcd} L^F(\mu) \arrow[r, "T_k"] \arrow[d, "R_k"'] & L^F(\mu) \\ \bigoplus_{j=m_k+1}^{m_{k+1}} \ell_{p_j} \arrow[r, "S_k"] & \bigoplus_{j=m_k+1}^{m_{k+1}} \ell_{p_j} \arrow[u, hook, "i_k"'] \end{tikzcd}$$

where $R_k(f) = (P_{m_k+1}(f), \dots, P_{m_{k+1}}(f))$, $S_k(f_1, \dots, f_k) = (0, f_1, \dots, f_{k-1})$ is a “shift” operator and i_k is just the isomorphic embedding. Clearly, T_k is a bounded operator in L^F, acting only on functions supported in $[2^{-m_{k+1}}, 2^{-m_k}]$. In particular, $T_iT_j = 0$ unless $i = j$. Moreover, T_k is strictly singular but $(T_k)^{k-1}$ is not compact (although $(T_k)^k = 0$).

Let us consider the operator

$$T = \sum_{k=1}^{\infty} \frac{T_k}{2^k \|T_k\|}.$$

Clearly, T is bounded, and since

$$\Big\| T - \sum_{k=1}^{n} \frac{T_k}{2^k \|T_k\|} \Big\| \to 0$$

when $n \to \infty$, we have that T is strictly singular because so is $\sum_{k=1}^{n} \frac{T_k}{2^k\|T_k\|}$ for every $n \in \mathbb{N}$.

Let us see that T^k is not compact for any $k \in \mathbb{N}$. To this end, let (e_n) denote the sequence of norm one functions in L^F, supported in $[2^{-m_{k+2}}, 2^{-m_{k+1}}]$, which correspond to the unit vector basis of $\ell_{p_{m_{k+1}}}$. Hence, by construction $((T_{k+1})^k(e_n))$ correspond to the unit vector basis of $\ell_{p_{m_{k+2}}}$, which is weakly null and of norm one.

Now, if $N > k$ is sufficiently large so that $\Big\| T^k - \Big(\sum_{n=1}^{N} \frac{T_n}{2^n\|T_n\|} \Big)^k \Big\| < \frac{1}{2(2^k\|T_{k+1}\|)^k}$, then using the fact that T_n acts only on functions supported in $[2^{-m_{n+1}}, 2^{-m_n}]$, it follows that

$$\begin{aligned} \|T^k(e_n)\| &\geq \Big\| \Big(\sum_{n=1}^{N} \frac{T_n}{2^n\|T_n\|} \Big)^k (e_n) \Big\| - \Big\| T^k(e_n) - \Big(\sum_{n=1}^{N} \frac{T_n}{2^n\|T_n\|} \Big)^k (e_n) \Big\| \\ &\geq \frac{\|(T_{k+1})^k(e_n)\|}{(2^k\|T_{k+1}\|)^k} - \frac{1}{2(2^k\|T_{k+1}\|)^k} \\ &= \frac{1}{2(2^k\|T_{k+1}\|)^k}. \end{aligned}$$

This means that $(T^k(e_n))$ is bounded away from zero for every $n \in \mathbb{N}$, so T^k is not compact.

□

We don’t know if there is a similar example for an Orlicz space L_φ with $s(L_\varphi) = \sigma(L_\varphi) = p$?

The following result allows us to extend the proved results for iterations of strictly singular operators on disjointly homogeneous r.i. spaces to composition of operators.

Proposition 5.4.5. *Given a rearrangement invariant space X on $[0,1]$, and $n \in \mathbb{N}$ the following statements are equivalent:*

(1) *For every strictly singular operator $T : X \to X$, the power T^n is compact.*
(2) *If $T_1, \ldots, T_n$ belong to $SS(X)$, then the composition $T_n \circ \ldots \circ T_1$ is compact.*

PROOF. The implication (2) ⇒ (1) is trivial. We will prove (1) ⇒ (2). To this end, let

$$X_i = \left\{ x \in X : x(t) = 0 \, \forall t \notin \left[\frac{i}{n+1}, \frac{i+1}{n+1} \right] \right\}$$

for $i = 0, 1, \ldots, n$. Clearly, each X_i is isomorphic to X, so we can denote this isomorphisms by $J_i : X \to X_i$, and we can decompose $X = \bigoplus_{i=0}^n X_i$. Now, if $T_1, \ldots, T_n$ belong to $SS(X)$, then we can consider the operator $T : \bigoplus_{i=0}^n X_i \to \bigoplus_{i=0}^n X_i$ given by the following matrix

$$T = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ J_1 T_1 J_0^{-1} & 0 & 0 & \cdots & 0 \\ 0 & J_2 T_2 J_1^{-1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & J_n T_n J_{n-1}^{-1} & 0 \end{pmatrix}$$

Since the operators T_i are strictly singular, T is strictly singular. By hypothesis, T^n is compact, and in matrix form this operator is the following

$$T^n = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ J_n T_n \cdots T_1 J_0^{-1} & 0 & \cdots & 0 \end{pmatrix}$$

Since J_i are isomorphisms, this implies that the composition $T_n \cdots T_1$ is compact, and the proof is finished. □

5. Duality of strictly singular operators in spaces $L_{p,q}$

In the same spirit as in [**132**] we have the following result.

THEOREM 5.5.1. *Let $T : L_{p,q} \to L_{p,q}$, with $1 < p < 2$ and $q \notin (p, 2)$, or $2 < p < \infty$ and $q \notin (2, p)$, or $p = 2$ and $1 < q < \infty$. The following are equivalent:*

(1) *T is strictly singular.*
(2) *T is $\{\ell_2, \ell_q\}$-singular.*
(3) *There is no subspace $M \subset L_{p,q}$, isomorphic to ℓ_q or ℓ_2, with $T(M)$ complemented in $L_{p,q}$, such that $T|_M$ is an isomorphism.*
(4) *T^* is strictly singular.*
(5) *T^* is $\{\ell_2, \ell_{q'}\}$-singular.*

Before the proof, we need a well-known Lemma, whose proof can be found in [**132**], but we include it here for further reference.

LEMMA 5.5.2. *Let $1 \leq r \leq 2$ and (f_n) be a seminormalized basic sequence in $L_r(\mu)$, whose closed linear span is a strongly embedded subspace of $L_r(\mu)$. Then for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\mu(A) < \delta(\varepsilon) \Rightarrow \sup_n \Big(\int_A |f_n|^r d\mu \Big)^{\frac{1}{r}} < \varepsilon.$$

PROOF. Assume the contrary. Then there exist a subsequence (f_{n_k}), a sequence of measurable sets (A_k) with $\mu(A_k) < \frac{1}{2^k}$, and some $\alpha > 0$ such that

$$\int_{A_k} |f_{n_k}|^r d\mu > \alpha,$$

for all $k \in \mathbb{N}$.

Let us consider the sets $B_k = \bigcup_{j=k}^{\infty} A_j$. Since

$$\int_{B_k - B_l} |f_{n_k}|^r d\mu \underset{l\to\infty}{\longrightarrow} \int_{B_k} |f_{n_k}|^r d\mu > \alpha,$$

then there is a further subsequence (k_i) satisfying

$$\int_{B_{k_i} - B_{k_{i+1}}} |f_{n_{k_i}}|^r d\mu > \frac{\alpha}{2}.$$

Since $(B_{k_i} - B_{k_{i+1}})$ are pairwise disjoint, it follows from [**75**, Lemma 2], that $(f_{n_{k_i}})$ is equivalent to the unit vector basis of ℓ_r. By [**43**, Theorem 2.2], ℓ_r is not strongly embedded in $L_r(\mu)$. Hence, we have reached a contradiction and the proof is finished. □

PROOF OF THEOREM 5.5.1. It is clear that (1) ⇒ (2) ⇒ (3). Let us see first that (3) ⇒ (1). To this end, suppose that T is not SS, then there exists an infinite dimensional subspace $X \subset L_{p,q}$ such that $T|_X$ invertible. We will find a subspace of $T(X)$ isomorphic to ℓ_2 or ℓ_q which is complemented in $L_{p,q}$, thus reaching a contradiction with (3).

First, in the case $2 < p < \infty$, by [**29**, Corollary 2.4] and the fact that $L_{p,q} \subset L_2$, it follows that $T(X)$ contains a subspace isomorphic to ℓ_q or ℓ_2 which is complemented in $L_{p,q}$, and we are done. Note, that the same argument also works in the case $p = 2$ and $1 < q \leq 2$.

Now, for the case $1 < p < 2$ by [**29**] it follows that $T(X)$ either contains a subspace isomorphic to ℓ_q and complemented in $L_{p,q}$ or $T(X)$ is strongly embedded in $L_{p,q}[0,1]$. If $T(X)$ contains ℓ_q complemented we are done, so suppose that $T(X)$ is strongly embedded in $L_{p,q}[0,1]$.

We claim that this forces X not to contain a subspace isomorphic to ℓ_q. Indeed, depending on q, we distinguish four cases:(i) $q = 2$; (ii) $q = p$; (iii) $1 < q < p$; and (iv) $q > 2$.

In case (i), since $L_{p,2}$ is 2-concave and has an unconditional basis, every subspace of $L_{p,2}$ isomorphic to ℓ_2 has a subspace complemented in $L_{p,2}$ (see [**106**, Theorem 3.1 and Remark 4]). Hence, in this case, if X contained a subspace isomorphic to ℓ_2, then $T(X)$ would contain a subspace isomorphic to ℓ_2 and complemented, which contradicts (3).

Case (ii) follows from the fact that ℓ_p is not strongly embedded in $L_p[0,1]$ (see [**43**, Theorem 2.2]), which is isometric to $L_{p,p}[0,1]$.

In case (iii), consider r with $q < r < p$. Therefore, ℓ_q is not strongly embedded in $L_r[0,1]$. Hence, if X contained a subspace isomorphic to ℓ_q, then the same would hold for $T(X)$, which is strongly embedded in $L_{p,q}[0,1]$, and in particular also strongly embedded in $L_r[0,1]$. This is clearly impossible.

Finally, in case (iv) consider r with $1 < r < p$. Now ℓ_q does not embed in $L_r[0,1]$, hence if X contained a subspace isomorphic to ℓ_q, then so would $T(X)$ which is strongly embedded in $L_{p,q}[0,1] \subset L_r[0,1]$. Again a contradiction. Notice here that the same argument works for the case $p = 2$ and $2 < q < \infty$.

Therefore, in any of these cases, X does not contain a subspace isomorphic to ℓ_q, and by [**29**] we can assume that X is strongly embedded in $L_{p,q}[0,1]$, as it holds for $T(X)$.

Now, let (f_n) be a normalized unconditional basic sequence in X with $\|T(f_n)\|_{L_{p,q}} > C$, for some $C > 0$. Given $1 < r < p$, we have $L_{p,q}[0,1] \subset L_r[0,1]$ [**87**, p. 143]. By Lemma 5.5.2, given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\mu(A) < \delta(\varepsilon) \Rightarrow \sup_n \Big(\int_A |f_n|^r d\mu \Big)^{\frac{1}{r}} < \varepsilon.$$

Since (f_n) is bounded in L_r, for every $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that $\mu(\{|f_n| > M_\varepsilon\}) < \delta(\varepsilon)$.

For each $n \in \mathbb{N}$, let us consider $g_n = f_n \chi_{\{|f_n| > M_\varepsilon\}}$. Clearly, $\|g_n\|_{L_r} \leq \varepsilon$. Extracting a subsequence we can assume that g_n converges weakly to some $g \in L_r[0,1]$, with $\|g\|_{L_r} \leq \varepsilon$. Choose a measurable set B and $N < \infty$, such that $\mu(B^c) < \delta(\varepsilon)$ and $|g(t)| \leq N$ for $t \in B$, and define

$$h_n = (f_n - g_n - g)\chi_B.$$

If ε is small enough, the sequence (h_n) satisfies the following properties:

(1) h_n is seminormalized and weakly null in L_r.
(2) $|h_n(t)| \leq M$ almost everywhere for some $M < \infty$.
(3) $\|T(h_n)\|_{L_r} > C'$ for some constant $C' > 0$.

These imply that (h_n) has a subsequence (h_{n_k}) which is an unconditional basic sequence in $L_2[0,1]$. Therefore, for every $m \in \mathbb{N}$ and scalars $a_1, \ldots, a_m$, we have:

$$\Big\|\sum_{i=1}^m a_i T(h_{n_i})\Big\|_{L_{p,q}} \leq \|T\| \Big\|\sum_{i=1}^m a_i h_{n_i}\Big\|_{L_{p,q}} \leq \|T\| \Big\|\sum_{i=1}^m a_i h_{n_i}\Big\|_{L_2} \leq \|T\| C_1 \Big(\sum_{i=1}^m |a_i|^2\Big)^{\frac{1}{2}},$$

for a certain constant C_1.

On the other hand, extracting a further subsequence we can assume that $(T(h_{n_k}))$ is also an unconditional basic sequence in $L_r[0,1]$. Hence, it follows that

$$\begin{aligned}
\Big\|\sum_{i=1}^{r} a_i T(h_{n_i})\Big\|_{L_{p,q}} &\geq \Big\|\sum_{i=1}^{r} a_i T(h_{n_i})\Big\|_{L_r} \\
&\geq K\int_0^1 \Big\|\sum_{i=1}^{r} a_i r_i(u) T(h_{n_i})\Big\|_{L_r} du \\
&\geq KD\Big\|\Big(\sum_{i=1}^{r} |a_i T(h_{n_i})|^2\Big)^{\frac{1}{2}}\Big\|_{L_r} \\
&\geq KDL\Big(\sum_{i=1}^{r} \|a_i T(h_{n_i})\|_{L_r}^2\Big)^{\frac{1}{2}} \\
&\geq KDLC'\Big(\sum_{i=1}^{r} |a_i|^2\Big)^{\frac{1}{2}},
\end{aligned}$$

where K is the unconditional constant of $(T(h_{n_k}))$, D is the constant appearing in Theorem 1.3.1, L is the 2-concavity constant of L_r and C' the constant satisfying $\|T(g_n)\|_{L_r} > C'$.

Hence, M, the closed linear span of (g_{n_k}) in $L_{p,q}$ is isomorphic to ℓ_2, where T is invertible. Now in the case $q = 2$, [**106**, Thm. 3.1 and Remark 4] implies that $T(M)$ contains a subspace complemented in $L_{p,2}$ and isomorphic to ℓ_2, which contradicts (3). While in the case $q \neq 2$, then both M and $T(M)$ are strongly embedded in $L_{p,q}$. By [**106**, Thm. 3.1] $T(M)$ contains a subspace, still isomorphic to ℓ_2 which is complemented in L_r. Since $T(M)$ is strongly embedded in $L_{p,q}$ and $L_{p,q} \subset L_r$, it follows that there is a subspace of $T(M)$ complemented in $L_{p,q}$, in contradiction with (3). Thus, we have shown that (3) $\Rightarrow$ (1).

Hence, we have proved (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Now, notice that the statement (3) for $T : L_{p,q} \to L_{p,q}$ is equivalent to the same statement for $T^* : L_{p',q'} \to L_{p',q'}$. Therefore, by the proved equivalence we also get (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5), and the proof is finished. $\square$

The following example shows that the above equivalence is no longer true if the hypothesis are not satisfied.

EXAMPLE 5.5.3. *Let $1 < p < q < 2$. There exists an operator $T : L_{p,q} \to L_{p,q}$ such that T^* is strictly singular, but T is not.*

Indeed, since $1 < q < p < 2$ we can consider (g_n), a sequence of independent q-stable random variables in $L_{p,q}$. Moreover, let (f_n) be a normalized sequence of disjoint elements in $L_{p,q}$ whose span is isomorphic to ℓ_q and complemented in $L_{p,q}$. Let $P : L_{p,q} \to [f_n]$ denote this projection.

Notice, that the subspace $[g_n]$ of $L_{p,q}$ is strongly embedded in L_p (see [**29**, Corollary 2.9]). In particular, $[g_n]$ is a closed subspace of $L_{p,r}$ isomorphic to ℓ_q, for any fixed r with $p < r < q$.

Let us consider the following operator

$$\begin{tikzcd} L_{p,q} \arrow[rr, "T"] \arrow[d, "P"'] & & L_{p,q} \\ {[f_n]} \arrow[r, "R"] & {[g_n]} \arrow[r, "S"] & L_{p,r} \arrow[u, hook, "I_r"'] \end{tikzcd}$$

where R is an isomorphism mapping each f_n to g_n, S is the isomorphic embedding of $[g_n]$ in $L_{p,r}$, and I_r denotes the canonical inclusion from $L_{p,r}$ to $L_{p,q}$.

Clearly, T is an isomorphism on a subspace isomorphic to ℓ_q, thus it is not strictly singular. However, the adjoint operator $T^* : L_{p',q'} \to L_{p',q'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$ is strictly singular. Indeed, notice first that T^* factors through $L_{p',r'}$, hence T^* cannot be an isomorphism on any subspace isomorphic to $\ell_{q'}$, because $L_{p',r'}$ does not contain any subspace isomorphic to $\ell_{q'}$. On the other hand, T^* factors through $[g_n]^* \simeq \ell_{q'}$, hence T^* cannot be an isomorphism on any subspace isomorphic to ℓ_2. Since $2 < p' < q' < \infty$, every subspace of $L_{p',q'}$ contains either a subspace isomorphic to $\ell_{q'}$ or ℓ_2. Therefore, T^* is strictly singular as claimed.

CHAPTER 6

c_0-singular and ℓ_1-singular operators between vector valued Banach lattices

In this Chapter, we present a version of a result by W. Hensgen about containment of c_0 in vector valued Banach lattices, extended to the operator setting. Hensgen's result extends the previous results by J. Bourgain, G. Pisier and S. Kwapien for spaces $L_p(X)$ to more general vector-valued Banach lattices. The first Section is devoted to the proofs of the tools that will be needed for the main results. Namely, an extension of Kadeč-Pełczyński disjointness method (Theorem 1.2.8) to vector valued Banach lattices, and an adaptation of a result by J. Bourgain for comparison of seminorms.

In the second Section we present the proofs of the main results and examples that show these are best possible, while the third one is devoted to an application to obtain a version for operators of a result by J. Hoffmann-Jørgensen.

Part of this chapter has been presented as a communication in the international conference "Vector measures, integration and applications" held in Eichstätt in September 2008.

1. Tools

Recall that an order continuous Banach lattice E with a weak unit can be considered as an (in general not closed) order ideal of $L_1(\Omega, \Sigma, \mu)$ for certain probability space (Ω, Σ, μ), such that $E \hookrightarrow L_1(\Omega, \Sigma, \mu)$ is continuous with norm smaller than or equal to one (see Theorem 1.2.7).

Given a Banach lattice E of functions over a measure space (Ω, Σ, μ), and a Banach space X, we define the space $E(X)$ consisting of Σ-measurable functions $f : \Omega \to X$, such that the function $\omega \mapsto \|f(\omega)\|_X$ belongs to E. $E(X)$ is a vector valued Banach lattice equipped with the norm

$$\|f\|_{E(X)} = \|\|f(\cdot)\|_X\|_E$$

.

We will need the following vector valued version of Kadeč-Pełczyński disjointness method (see Theorem 1.2.8).

Proposition 6.1.1. *Given a Banach space X. Let E be an order continuous Banach lattice over a finite measure space (Ω, Σ, μ), and let M be a separable subspace of $E(X)$. If we consider the formal inclusion $i : E(X) \hookrightarrow L_1(X)$, then one of the following holds:*

(1) *the restriction $i|_M$ is an isomorphic embedding,*

(2) *or there exist a normalized sequence* $(f_n)_{n=1}^{\infty}$ *in* M *and a disjoint sequence* $(g_n)_{n=1}^{\infty}$ *in* $E(X)$*, such that* $\|f_n - g_n\|_{E(X)} \to 0$ *when* $n \to \infty$.

Proof. Let us start by considering the following sets: for $\varepsilon > 0$, and $f \in E(X)$, let

$$\sigma(f,\varepsilon) = \{\omega \in \Omega : \|f(\omega)\|_X \geq \varepsilon \|f\|_{E(X)}\}.$$

Now for $\varepsilon > 0$, let us consider the Kadeč-Pełczyński sets

$$\mathcal{KP}(\varepsilon) = \{f \in E(X) : \mu(\sigma(f,\varepsilon)) \geq \varepsilon\}.$$

Suppose first that $M \subset \mathcal{KP}(\varepsilon)$, for some $\varepsilon > 0$. Hence for every $f \in M$ we have

$$\|f\|_{E(X)} \geq \|f\|_{L_1(X)} = \int_{\Omega} \|f(\omega)\|_X d\mu \geq \int_{\sigma(f,\varepsilon)} \|f(\omega)\|_X d\mu \geq \varepsilon^2 \|f\|_{E(X)}.$$

Therefore, in this case, the inclusion $i : E(X) \hookrightarrow L_1(X)$ when restricted to the subspace M is an isomorphic embedding.

Suppose now that M is not contained in $\mathcal{KP}(\varepsilon)$, for any $\varepsilon > 0$. Therefore, there exists f_1 in M with $f_1 \notin \mathcal{KP}(4^{-2})$ and $\|f_1\|_{E(X)} = 1$. Thus,

$$\mu(\sigma(f_1, 4^{-2})) < 4^{-2},$$

and

$$\|\chi_{\Omega \setminus \sigma(f_1, 4^{-2})} f_1\|_{E(X)} \leq 4^{-2}.$$

Since E is order continuous, there exists $\delta_1 > 0$ such that $\|\chi_A f_1\|_{E(X)} < 4^{-3}$, whenever $\mu(A) < \delta_1$. Let $j_2 < 2 = j_1$ be such that $4^{-j_2} < \delta_1$. Hence, there exists $f_2 \in M$ with $\|f_2\|_{E(X)} = 1$ and $f_2 \notin \mathcal{KP}(4^{-j_2})$, which in turn means that

$$\mu(\sigma(f_2, 4^{-j_2})) < 4^{-j_2} < \delta_1,$$

and so

$$\|\chi_{\sigma(f_2, 4^{-j_2})} f_1\| \leq 4^{-(j_1+1)}.$$

Moreover,

$$\|\chi_{\Omega \setminus \sigma(f_2, 4^{-j_2})} f_2\|_{E(X)} \leq \|4^{-j_2} \chi_{\Omega \setminus \sigma(f_2, 4^{-j_2})}\|_{E(X)} \leq 4^{-j_2}.$$

We can continue this construction inductively and we get a normalized sequence (f_n) in M, and a sequence (j_n) of natural numbers such that

(1) $\mu(\sigma(f_n, 4^{-j_n})) < 4^{-j_n}$,
(2) $\|\chi_{\Omega \setminus \sigma(f_n, 4^{-j_n})} f_n\|_{E(X)} \leq 4^{-j_n}$,
(3) $\|\chi_{\sigma(f_n, 4^{-j_n})} f_i\|_{E(X)} \leq 4^{-(j_{n-1}+1)}$, for $i = 1, \ldots, n-1$.

Now, if we consider

$$\sigma_n = \sigma(f_n, 4^{-j_n}) - \bigcup_{i=n+1}^{\infty} \sigma(f_i, 4^{-j_i}),$$

then $\sigma_n \cap \sigma_m = \emptyset$ whenever $n \neq m$. Let us define $g_n = \chi_{\sigma_n} f_n$, which is a disjoint sequence in $E(X)$. Moreover, it holds

$$\begin{aligned}\|f_n - g_n\|_{E(X)} &= \|\chi_{\Omega\setminus\sigma_n} f_n\|_{E(X)} \leq \|\chi_{\Omega\setminus\sigma(f_n,4^{-j_n})} f_n\|_{E(X)} + \|\chi_{\bigcup_{i=n+1}^{\infty}\sigma(f_i,4^{-j_i})} f_n\|_{E(X)} \\ &\leq 4^{-j_n} + \sum_{i=n+1}^{\infty} \|\chi_{\sigma(f_i,4^{-j_i})} f_n\|_{E(X)} \leq 4^{-j_n} + \sum_{i=n+1}^{\infty} 4^{-(j_{i-1}+1)} \\ &\leq \tfrac{1}{3} 4^{-(j_n-1)}.\end{aligned}$$

Therefore, $\|f_n - g_n\| \to 0$ when $n \to \infty$, as claimed. $\square$

Notice that if E is an order continuous Banach lattice defined over an infinite measure space (Ω, Σ, μ), and M is a separable subspace of $E(X)$, then there exists a closed order ideal I of E, which can be considered as a function space over a finite measure space $(\Omega_1, \Sigma_1, \mu_1)$, such that M is a subspace of $I(X) \subset E(X)$ (see [**87**, Proposition 1.a.9]).

The following property of disjoint sequences in $E(X)$ will be useful.

LEMMA 6.1.2. *Let E be a Köthe Function space over (Ω, Σ, μ), and X a Banach space. Suppose that $(f_n)_{n=1}^{\infty}$ is a normalized disjoint sequence in $E(X)$, and denote $\varphi_n(\omega) = \|f_n(\omega)\|$ which is also disjoint and normalized. Then $(f_n)_{n=1}^{\infty}$ and $(\varphi_n)_{n=1}^{\infty}$ are 1-equivalent unconditional basic sequences.*

PROOF. For each natural number n, since $f_1, \ldots, f_n$ are disjoint elements of $E(X)$, we can consider $B_1, \ldots, B_n \in \Sigma$ such that $\bigcup_{i=1}^{n} B_i = \Omega$ and f_i is supported on B_i, for each $i = 1, \ldots, n$. Hence for scalars $(a_i)_{i=1}^{n}$, we have:

$$\begin{aligned}\Big\|\sum_{i=1}^{n} a_i f_i\Big\|_{E(X)} &= \Big\|\Big\|\sum_{i=1}^{n} a_i f_i(\omega)\Big\|_X\Big\|_E = \Big\|\sum_{j=1}^{n}\chi_{B_j}(\omega)\Big\|\sum_{i=1}^{n} a_i f_i(\omega)\Big\|_X\Big\|_E \\ &= \Big\|\sum_{j=1}^{n}\Big\|\sum_{i=1}^{n} a_i \chi_{B_j}(\omega) f_i(\omega)\Big\|_X\Big\|_E = \Big\|\sum_{j=1}^{n} |a_j| \|f_j(\omega)\|_X\Big\|_E \\ &= \Big\|\sum_{j=1}^{n} a_j \varphi_j(\omega)\Big\|_E.\end{aligned}$$

Since this holds for every scalars $(a_i)_{i=1}^{n}$, the proof is finished. $\square$

Notice that for an operator between Banach spaces $T : X \to Y$, there is a natural extension $T_E : E(X) \to E(Y)$ given by

$$\begin{aligned}T_E(f) : \quad & \Omega \longrightarrow Y \\ & \omega \mapsto T(f(\omega))\end{aligned}$$

Our aim is to relate the invertibility properties of the operator T with its extensions T_E.

The next ingredient for the proof of our main result in this Chapter is the following extension of S. Kwapien's Theorem [**82**]. Recall that Kwapien's Theorem states that $L_1(X)$ contains a subspace isomorphic to c_0 if and only if X does.

PROPOSITION 6.1.3. *Given $T : X \to Y$, if the operator $T_{L_1} : L_1(\mu; X) \to L_1(\mu; Y)$ is an isomorphism on a subspace isomorphic to c_0, then so is $T : X \to Y$.*

The proof is based on the following version of J. Bourgain's theorem on c_0-sequences obtained by averaging of seminorms [**25**]. Recall that a c_0-sequence in a normed space is a sequence (x_n) such that $\|\sum a_k x_k\| \sim \max|a_k|$. Here, c_{00} denotes the space of sequences of real numbers which are eventually zero, equipped with the supremum norm. We recall a previous Lemma [**25**].

LEMMA 6.1.4. *Let $\|\cdot\|$ be a seminorm in c_{00}, satisfying*

$$\sup_n \int_0^1 \Big\|\sum_{i=1}^n r_i(\omega)e_i\Big\| d\omega = B < \infty \text{ and } \limsup \|e_i\| > 0.$$

Then (e_i) has a c_0-subsequence for $\|\cdot\|$.

LEMMA 6.1.5. *Let (Ω, Σ, μ) be a probability measure space. For every $\omega \in \Omega$ we consider two seminorms in c_{00}, ρ_ω and ϱ_ω, such that the functions $\omega \mapsto \rho_\omega(x)$ and $\omega \mapsto \varrho_\omega(x)$ are integrable in (Ω, Σ, μ) for every $x \in c_{00}$, and there exists a finite constant $C > 0$ such that*

$$\rho_\omega(x) \leq C\varrho_\omega(x),$$

for all $x \in c_{00}$.

Let us define two seminorms on c_{00} by

$$\|x\|_1 = \int_\Omega \rho_\omega(x)d\mu(\omega), \qquad \text{and} \qquad \|x\|_2 = \int_\Omega \varrho_\omega(x)d\mu(\omega)$$

for every $x \in c_{00}$. If (x_i) is a sequence in c_{00} which is equivalent to the c_0-basis for both $\|\cdot\|_1$ and $\|\cdot\|_2$, then there exists a set of positive measure $A \in \Sigma$ such that for every $\omega \in A$ there is a subsequence of (x_i) which is equivalent to the c_0-basis for both ρ_ω and ϱ_ω.

PROOF. First note that the set

$$A = \{\omega \in \Omega : \limsup \rho_\omega(x_i) > 0\}$$

is clearly measurable and has positive measure [**25**, Lemma 2]. It also follows from [**25**] (see also [**31**, p. 53] for a more detailed explanation) that

$$\sup_n \int_0^1 \varrho_\omega\Big(\sum_{i=1}^n r_i(t)x_i\Big)dt = B < \infty$$

for almost all $\omega \in \Omega$.

Thus, for every $\omega \in A$ we have

$$\limsup \rho_\omega(x_i) > 0, \qquad \text{and} \qquad \sup_n \int_0^1 \rho_\omega\Big(\sum_{i=1}^n r_i(t)x_i\Big)dt \leq CB < \infty.$$

Therefore, applying Lemma 6.1.4 to the seminorm ρ_ω, we obtain a subsequence (x_{i_k}) which is a c_0-sequence for ρ_ω, and satisfies $\rho_\omega(x_{i_k}) > \alpha$ for some $\alpha > 0$ and every natural number k. It follows from the fact that $\rho_\omega(x) \le C\varrho_\omega(x)$ and [**31**, Proposition 2.1.1. and Proposition 2.1.2.] that

$$\varrho_\omega(x_{i_k}) > C^{-1}\alpha > 0, \qquad \text{and} \qquad \sup_n \int_0^1 \varrho_\omega\Big(\sum_{k=1}^n r_k(t)x_{i_k}\Big)dt \le B < \infty,$$

for every k. Hence, another application of Lemma 6.1.4 gives a further subsequence (still denoted (x_{i_k})) which is a c_0-sequence for ϱ_ω. Thus, this sequence is a c_0-sequence for both ρ_ω and ϱ_ω, and the proof is finished. $\square$

Now we can prove Proposition 6.1.3.

Proof of Proposition 6.1.3. Let (f_i) be a sequence in $L_1(\mu; X)$ such that for some $\delta > 0$ and $M > 0$, and for every $(a_1, \dots, a_n, 0\dots)$ in c_{00} we have

$$\begin{aligned}\delta \max_{1\le i\le n}|a_i| &\le \Big\|\sum_{i=1}^n a_i T_{L_1}(f_i)\Big\|_{L_1(\mu;Y)} = \int_\Omega \Big\|\sum_{i=1}^n a_i T(f_i(\omega))\Big\| d\mu(\omega)\\ &\le \|T\|\int_\Omega \Big\|\sum_{i=1}^n a_i f_i(\omega)\Big\| d\mu(\omega) = \|T\|\Big\|\sum_{i=1}^n a_i f_i\Big\|_{L_1(\mu;X)}\\ &\le \|T\| M \max_{1\le i\le n}|a_i|.\end{aligned}$$

Let us define for each $\omega \in \Omega$, and each $x = \sum_{i=1}^n a_i e_i \in c_{00}$

$$\rho_\omega(x) = \Big\|\sum_{i=1}^n a_i T(f_i(\omega))\Big\|_Y,$$

and

$$\varrho_\omega(x) = \Big\|\sum_{i=1}^n a_i f_i(\omega)\Big\|_X.$$

Clearly, for every $x \in c_{00}$, the functions $\omega \mapsto \rho_\omega(x)$ and $\omega \mapsto \varrho_\omega(x)$ are integrable on (Ω, Σ, μ), and satisfy

$$\rho_\omega(x) \le \|T\|\varrho_\omega(x).$$

Hence, we can consider the seminorms $\|\cdot\|_1$ and $\|\cdot\|_2$, as defined in Lemma 6.1.5. It follows that the unit vector sequence (e_i) in c_{00} is a c_0-sequence for both $\|\cdot\|_1$ and $\|\cdot\|_2$, because

$$\Big\|\sum_{i=1}^n a_i e_i\Big\|_1 = \int_\Omega \rho_\omega\Big(\sum_{i=1}^n a_i e_i\Big)d\mu(\omega) = \int_\Omega \Big\|\sum_{i=1}^n a_i T(f_i(\omega))\Big\| d\mu(\omega) = \Big\|\sum_{i=1}^n a_i T_{L_1}(f_i)\Big\|_{L_1(\mu;Y)}$$

$$\Big\|\sum_{i=1}^n a_i e_i\Big\|_2 = \int_\Omega \varrho_\omega\Big(\sum_{i=1}^n a_i e_i\Big)d\mu(\omega) = \int_\Omega \Big\|\sum_{i=1}^n a_i f_i(\omega)\Big\| d\mu(\omega) = \Big\|\sum_{i=1}^n a_i f_i\Big\|_{L_1(\mu;X)}$$

Now, Lemma 6.1.5 implies that the set of points $\omega \in \Omega$ such that (e_i) has a subsequence which is a c_0-basis for both ρ_ω and ϱ_ω is a non null set. Thus, for every ω in this set, there exists an increasing sequence (i_k) such that $(f_{i_k}(\omega))$ and $(T(f_{i_k}(\omega)))$ are (non null) c_0-sequences. This implies that T is an isomorphism when restricted to the span of $(f_{i_k}(\omega))$ in X. $\square$

2. Main results

Now, we can give the proofs of our main results.

THEOREM 6.2.1. *Let E be a Banach lattice which does not contain a subspace isomorphic to c_0. Let $T : X \to Y$ be an operator between Banach spaces. If the operator*

$$T_E : E(X) \to E(Y)$$

is invertible on a subspace isomorphic to c_0, then the same holds for the operator T.

PROOF. Let $T : X \to Y$ be a bounded operator. And let $T_E : E(X) \to E(Y)$ be such that there exists a subspace M of $E(X)$, which is isomorphic to c_0, and the restriction $T_E|_M : M \to E(Y)$ is an isomorphic embedding.

Since c_0 is not contained in E, in particular E is order continuous. Hence, by Proposition 6.1.1 applied to $M \subset E(X)$, it follows that either $i : E(X) \hookrightarrow L_1(X)$ is an isomorphism when restricted to M or M contains a normalized sequence (f_n), such that there exists a disjoint sequence (g_n) in $E(X)$ with $\|f_n - g_n\|_{E(X)} \to 0$ when $n \to \infty$.

Suppose that $i : E(X) \hookrightarrow L_1(X)$ is not an isomorphism when restricted to M. Therefore, passing to a further subsequence we can assume that the basic sequences (f_n) and (g_n) are equivalent. Since M is isomorphic to c_0, this means that $E(X)$ contains a disjoint sequence equivalent to the unit vector basis of c_0. Hence, by Lemma 6.1.2, E would also contain a disjoint sequence equivalent to the unit vector basis of c_0, which is a contradiction with the hypothesis on E.

Thus, we can assume that $i : E(X) \hookrightarrow L_1(X)$ is an isomorphism when restricted to M. The same argument shows that $i : E(Y) \hookrightarrow L_1(Y)$ is an isomorphism when restricted to $T_E(M)$ (which is also isomorphic to c_0).

Therefore, the operator $T_{L_1} : L_1(X) \to L_1(Y)$ is an isomorphism when restricted to M, which is isomorphic to c_0. Hence, by Proposition 6.1.3, we can conclude that $T : X \to Y$ is an isomorphism on a subspace isomorphic to c_0. $\square$

REMARK 6.2.2. Notice that if the Banach lattice E contains a subspace isomorphic to c_0, then the statement of Theorem 6.2.1 may fail to be true. Indeed, the identity $I : c_0 \to c_0$ can be seen as the extension T_{c_0} of the identity map on the scalar field $T : \mathbb{R} \to \mathbb{R}$, which clearly is not an isomorphism on a subspace isomorphic to c_0 (c_0 is just too big!).

This theorem has a natural analogue for operators preserving ℓ_1.

THEOREM 6.2.3. *Let E be an order continuous Banach lattice, such that E^* is also order continuous. Let $T : X \to Y$ be an operator between Banach spaces. If $T_E : E(X) \to E(Y)$ is invertible on a subspace isomorphic to ℓ_1, then so is T.*

PROOF. Let M be a subspace of $E(X)$ isomorphic to ℓ_1, such that $T_E : E(X) \to E(Y)$ is an isomorphism when restricted to M. Both M and $T_E(M)$ satisfy one of the alternatives of

Proposition 6.1.1. Since E^* is order continuous, it follows that E cannot contain a sequence of disjoint elements whose closed linear span is isomorphic to ℓ_1 (see 1.2.3). Therefore, by Lemma 6.1.2, M cannot contain a normalized sequence equivalent to a disjoint sequence. Hence, the inclusion $i_X : E(X) \hookrightarrow L_1(X)$ is an isomorphism when restricted to M, and similarly $i_Y : E(Y) \hookrightarrow L_1(Y)$ is an isomorphism when restricted to $T_E(M)$.

Let (f_n) in M be equivalent to the unit vector basis of ℓ_1. Since E and E^* are order continuous $(\|f_n(\cdot)\|_X)$ and $(\|T(f_n(\cdot))\|_Y)$ are uniformly integrable sequences in L_1 (cf. [**12**, Theorem 4.25] and [**6**, Theorem 5.2.9]). Hence, by [**31**, Theorem 2.2.1(a)], the set A of all $\omega \in \Omega$ such that $(T(f_n(\omega)))_{n=1}^{\infty}$ has a subsequence equivalent to the unit vector basis of ℓ_1 is a measurable set with positive measure.

Let $\alpha = \mu(A)$. Since E is order continuous, there exists $M < \infty$ such that $\|f_n(\omega)\|_X \leq M$ except on a set of measure $\frac{\alpha}{2}$. Let $B = \{\omega \in \Omega : \|f_n(\omega)\|_X \leq M\}$. Since $\mu(B) = 1 - \frac{\alpha}{2}$ and $\mu(A) = \alpha$, we must have that $\mu(A \cap B) \geq \frac{\alpha}{2} > 0$. Therefore, for any $\omega \in A \cap B$ there is a subsequence $(T(f_{n_k}(\omega)))$ equivalent to the unit vector basis of ℓ_1, and for every scalars $(a_k)_{k=1}^m$ we have

$$\Big\| \sum_{k=1}^m a_k f_{n_k}(\omega) \Big\|_X \leq M \sum_{k=1}^m |a_k| \leq MC \Big\| \sum_{k=1}^m a_k T(f_{n_k}(\omega)) \Big\|_Y .$$

Hence, $T : X \to Y$ preserves a copy of ℓ_1, as claimed.

□

REMARK 6.2.4. As for Theorem 6.2.1, the identity on ℓ_1, seen as the extension T_{ℓ_1} of the identity map $T : \mathbb{R} \to \mathbb{R}$ on the scalar field, shows that the hypothesis of order continuity on E^* cannot be removed from Theorem 6.2.3.

Note that in fact, we have proved more than it was claimed. It was shown that if $T_E : E(X) \to E(Y)$ is an isomorphism on the span of a sequence $[f_n] \subset E(X)$, which is isomorphic to c_0, respectively to ℓ_1, then, under the proper assumptions, the set of all $\omega \in \Omega$ such that $T : X \to Y$ is an isomorphism on the span of a subsequence of $(f_n(\omega))$ (which is isomorphic to c_0, respectively to ℓ_1) is a set of positive measure.

3. Applications

In connection with Theorem 6.2.1, we have a version for operators of Hoffmann-Jorgensen's result (see [**72**]).

THEOREM 6.3.1. *Let $T : X \to Y$ be an operator between Banach spaces, and let (Ω, Σ, μ) be a probability space. The following are equivalent:*

(1) *For every sequence (X_n) of independent, symmetric, X-valued random variables on (Ω, Σ, μ), if the partial sums*

$$S_m = \sum_{n=1}^m X_n$$

are bounded almost everywhere, then the sequence $(T(S_m))$ *converges almost everywhere.*

(2) T *is not an isomorphism on any subspace isomorphic to* c_0.

PROOF. (1) $\Rightarrow$ (2) is easy to see. For the implication (2) $\Rightarrow$ (1), let $(\epsilon_j)_{j=1}^{\infty}$ be a Bernoulli sequence on (Ω, Σ, μ), that is, a sequence of independent random variables so that $\mu(\epsilon_j = 1) = \mu(\epsilon_j = -1) = \frac{1}{2}$ for all $j \geq 1$. By [**72**, Proposition 2.8] it suffices to prove that the sets

$$A = \left\{(x_j) \subset X : \left(\sum_{j=1}^{n} \epsilon_j x_j\right) \text{ is bounded in } L_p(X)\right\},$$

and

$$B = \left\{(x_j) \subset X : \sum_{j=1}^{\infty} \epsilon_j T x_j \text{ is convergent in } L_p(Y)\right\},$$

coincide (notice that by [**72**, Thm 3.1], there is no difference in the choice of $0 \leq p < \infty$).

So, suppose that there exists (x_j) in A and not in B. Since in particular, $\sum_{j=1}^{\infty} \epsilon_j T x_j$ is not convergent in $L_1(Y)$, there exist $\delta > 0$ and a subsequence such that

$$\int_{\Omega} \Big\| \sum_{n_k \leq j < n_{k+1}} \epsilon_j T x_j \Big\|_Y d\mu \geq \delta,$$

for $k \in \mathbb{N}$. Now, let

$$X_k = \sum_{n_k \leq j < n_{k+1}} \epsilon_j x_j, \quad \text{and} \quad Y_k = \sum_{n_k \leq j < n_{k+1}} \epsilon_j T x_j,$$

for $k \in \mathbb{N}$. Clearly, the sequence $(X_k(\omega))$ belongs to A μ-a.e. However, [**72**, Theorem 3.1] yields that $\mu(Y_k \nrightarrow 0) > 0$.

Therefore, by scaling, we can consider (z_j) in A such that $\|Tz_j\| = 1$, for every $j \in \mathbb{N}$. Now, by [**72**, Theorem 2.6], we have

$$|a_j| = \left(\int_{\Omega} \|a_j \epsilon_j(\omega) T z_j\|_Y^p d\mu\right)^{\frac{1}{p}} \leq \left(\int_{\Omega} \Big\| \sum_{j=1}^{n} a_j \epsilon_j(\omega) T z_j \Big\|_Y^p d\mu\right)^{\frac{1}{p}},$$

for $1 \leq j \leq n$ and scalars $(a_j)_{j=1}^{n}$. While [**72**, Lemma 4.1] yields

$$\left(\int_{\Omega} \Big\| \sum_{j=1}^{n} a_j \epsilon_j(\omega) z_j \Big\|_Y^p d\mu\right)^{\frac{1}{p}} \leq \max_{1 \leq j \leq n} |a_j| \left(\int_{\Omega} \Big\| \sum_{j=1}^{n} \epsilon_j(\omega) z_j \Big\|_Y^p d\mu\right)^{\frac{1}{p}} \leq \max_{1 \leq j \leq n} |a_j| K,$$

where

$$K = \sup_n \left(\left(\int_{\Omega} \Big\| \sum_{j=1}^{n} \epsilon_j(\omega) z_j \Big\|_Y^p d\mu\right)^{\frac{1}{p}}\right) < \infty,$$

since $(z_j) \in A$.

Hence, if we consider $T_{L_p} : L_p(X) \to L_p(Y)$ defined as usual, then we have

$$\max_{1 \leq j \leq n} |a_j| \leq \Big\| T_{L_p}\Big(\sum_{j=1}^{n} a_j \epsilon_j z_j\Big) \Big\|_{L_p(Y)} \leq \|T\| \Big\| \sum_{j=1}^{n} a_j \epsilon_j z_j \Big\|_{L_p(X)} \leq \|T\| K \max_{1 \leq j \leq n} |a_j|.$$

This shows that the operator T_{L_p} is an isomorphism on the subspace generated by the sequence $(\epsilon_j z_j)$ which is isomorphic to c_0. Therefore, by Theorem 6.2.1, $T : X \to Y$ is also an isomorphism on a subspace isomorphic to c_0. This finishes the proof. $\square$

CHAPTER 7

Factorization and domination of Banach-Saks operators

This chapter is devoted to two problems related to Banach-Saks operators on Banach lattices: factorization and domination.

Recall that an operator between Banach spaces $T : X \to Y$ is *Banach-Saks* if every bounded sequence (x_n) in X has a subsequence such that (Tx_{n_k}) is Cesàro convergent, that is, the sequence of arithmetic means $(\frac{1}{N}\sum_{k=1}^{N} T(x_{n_k}))$ is convergent in the norm of Y. A Banach space is said to have the *Banach-Saks property* if the identity operator is Banach-Saks, that is every bounded sequence has a Cesàro convergent subsequence. We will say that a Banach space has the *weak Banach-Saks property* if every weakly null sequence has a subsequence Cesàro convergent to zero. In reflexive spaces, both properties are equivalent.

Definition 7.0.2. *Given a Banach space X, we say that $S \subset X$ is a Banach-Saks set if for every sequence (x_n) in S, there exists a subsequence (x_{n_k}) whose arithmetic means are convergent in the norm of X.*

Clearly an operator $T : X \to Y$ is Banach-Saks if and only if $T(B_X)$ is a Banach-Saks set, where B_X is the closed unit ball of X.

In the first section a factorization theorem for positive Banach-Saks operators is obtained. Moreover, by means of an example of M. Talagrand [**129**], we show that the hypothesis in this theorem are necessary.

The second section focusses on domination properties of Banach-Saks operators, improving results in [**55**]. Finally, we close the Chapter with some results on convex hulls of Banach-Saks sets, answering a question of M. González and J. Gutiérrez [**62**].

Part of this Chapter will be published as a joint work with J. Flores in [**56**].

1. Factorization of operators through Banach lattices with the Banach-Saks property

The main result of this Section is the following:

Theorem 7.1.1. *Let E and F be Banach lattices and $T : E \to F$ a positive Banach-Saks operator. If F is order continuous, then there exist a Banach lattice H with the Banach-Saks property, and operators $T_1 : E \to H$, $T_2 : H \to F$, such that the following factorization diagram* *holds:*

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "T_1"'] & & F \\ & H \arrow[ur, "T_2"'] & \end{tikzcd}$$

PROOF OF THEOREM 7.1.1. First of all, we claim that the convex solid hull of $T(B_E)$ is a Banach-Saks set.

Indeed, since T is Banach-Saks, $T(B_E)$ is a Banach-Saks set. Let W denote the solid hull of the set $T(B_E)$ (note that $T(B_E)$ is already convex). Take an arbitrary sequence (z_k) in W. This sequence satisfies $|z_k| \leq |Tx_k|$ for certain $x_k \in B_E$. For every k, take $r_k = |x_k|$ in B_E, and then

$$|z_k| \leq |Tx_k| \leq Tr_k.$$

Since T is Banach-Saks, there exist $g \in F$ and some subsequence (k_s) such that

$$\frac{1}{m}\sum_{s=1}^{m} Tr_{k_s} \to g$$

in the norm of F. In fact, by [**44**] we can assume that the same property holds for every subsequence of (k_s).

Take M, the closed ideal of F generated by $x = g + \sum\limits_{s=1}^{\infty} \frac{z_{k_s}}{2^s}$. Denote $i : M \hookrightarrow L_1(\Omega, \Sigma, \mu)$ the continuous inclusion of M as an order continuous Banach lattice with weak unit into some $L_1(\Omega, \Sigma, \mu)$, and let $P : F \to M$ be a positive projection onto M.

Clearly, the composition

$$iPT : E \to L_1(\Omega, \Sigma, \mu)$$

is a Banach-Saks operator. In particular, $iPT(B_E)$ is relatively weakly compact and so is its convex solid hull, because $L_1(\Omega, \Sigma, \mu)$ is a band in its bidual [**12**, Theorem 4.39]. Hence, there exists $f \in L_1(\Omega, \Sigma, \mu)$ such that $i(z_{k_s}) \to f$ in the weak topology of $L_1(\Omega, \Sigma, \mu)$. Since $L_1(\Omega, \Sigma, \mu)$ has the weak Banach-Saks property [**127**], passing to a further subsequence we can assume that

$$\frac{1}{m}\sum_{s=1}^{m} i(z_{k_s}) \to f$$

in the norm of $L_1(\Omega, \Sigma, \mu)$.

Notice that

$$\left|\frac{1}{m}\sum_{s=1}^{m} z_{k_s}\right| \leq \frac{1}{m}\sum_{s=1}^{m} Tr_{k_s},$$

and $\frac{1}{m}\sum_{s=1}^{m} Tr_{k_s} \to g$ in the norm of F. It follows that $\left(\frac{1}{m}\sum_{s=1}^{m} z_{k_s}\right)_m$ is equi-integrable in F, and therefore convergent in F by the previous lines and Lemma 1.4.2. Thus, W is a Banach-Saks set as claimed.

Consider now F_0, the completion of the space $\{z \in F : \exists \lambda < \infty, z \in \lambda W\}$ under the norm induced by the Minkowski functional of W. Since W is solid, the space F_0 is in fact a Banach lattice. Hence, the space $(F_0, F)_{\theta,p}$ $(0 < \theta < 1,\ 1 < p < \infty)$, obtained by Lions-Peetre interpolation of F_0 and F, is a Banach lattice too [**87**, 2.g]. Moreover, by [**19**, Theorem 2], $(F_0, F)_{\theta,p}$ has the Banach-Saks property.

Finally, since $T(B_E) \subset W$, the operator $T : E \to F_0$ is bounded. Thus, by the interpolation theorem [**87**, Proposition 2.g.15], T is bounded from E to $(F_0, F)_{\theta,p}$. Let $T_1 : E \to (F_0, F)_{\theta,p}$ denote this operator. Since $W \subset \|T\| B_F$, we also have that the inclusion $i : (F_0, F)_{\theta,p} \hookrightarrow F$ is bounded. Therefore, we have the factorization

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "T_1"'] & & F \\ & (F_0, F)_{\theta,p} \arrow[ur, hook, "i"'] & \end{tikzcd}$$

Take $H = (F_0, F)_{\theta,p}$, and $T_2 = i$ to conclude the proof. □

COROLLARY 7.1.2. *Let E be a Banach lattice. If $0 \leq T : E \to E$ is Banach-Saks, then T^2 factors through a Banach lattice with the Banach-Saks property.*

PROOF. Since c_0 does not have the Banach-Saks property, T cannot be an isomorphism on any subspace of E isomorphic to c_0. Hence, by Theorem 1.5.1, there exist an order continuous Banach lattice F, and positive operators R, S such that

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "R"'] & & E \\ & F \arrow[ur, "S"'] & \end{tikzcd}$$

Therefore, since F is order continuous, Theorem 7.1.1 yields that $RT : E \to F$ factors through a Banach lattice H with the Banach-Saks property. Hence, $T^2 = SRT$ also factors through H as claimed. □

Note that, in general, every Banach-Saks operator between Banach spaces factors through a Banach space with the Banach-Saks property [**19**, Theorem 1]. However, if the operator acts between Banach lattices it is not true in general that the space obtained in such a factorization has to be a lattice. To see this we will benefit from the well-known example provided by M. Talagrand [**129**] of a positive weakly compact operator between Banach lattices which fails to factor through any reflexive Banach lattice. Since Banach-Saks property implies reflexivity it suffices to prove that Talagrand's operator is in fact Banach-Saks. Thus, Theorem 7.1.1 and Corollary 7.1.2 turn out to be optimal in a sense.

Let us briefly recall the construction of Talagrand's operator. First, let

$$L = \{h : \mathbb{N} \cup \{\infty\} \to \{0, 1\};\ \exists\ p \leq i_1 < \ldots < i_p;\ h(i) = 0 \text{ for } i \neq i_1, \ldots, i_p\}.$$

Then $L \subset C(\mathbb{N} \cup \{\infty\})$ is weakly compact. For every $l \geq 1$ consider the map

$$\theta_l : L^l \to C((\mathbb{N} \cup \{\infty\})^l),$$

defined as $\theta_l(h_1, \ldots, h_l)(n_1, \ldots, n_l) = 1$ if the number of indexes i for which $h_i(n_i) = 1$ is even, and $\theta_l(h_1, \ldots, h_l)(n_1, \ldots, n_l) = 0$ otherwise. Notice that since θ_l is continuous for the topology of point-wise convergence in $C((\mathbb{N} \cup \{\infty\})^l)$, the set $K_l = \theta_l(L^l)$ is weakly compact.

Call M the Alexandroff compactification of the discrete sum of the sets $(\mathbb{N} \cup \{\infty\})^l$. Each K_l can be considered as a subset of $C(M)$ by extending the functions of K_l to zero outside $(\mathbb{N} \cup \{\infty\})^l$. Let $K = \bigcup_l K_l$. By construction, K consists of $\{0,1\}$-valued functions, so K is contained in the positive cone of $C(M)$.

LEMMA 7.1.3. *The closed convex hull of K, $\overline{co}K$, is a Banach-Saks set.*

PROOF. Indeed, take (y_n) arbitrarily in $\overline{co}K$. We want to show that there is a subsequence of (y_n) whose arithmetic means are convergent. For each $n \in \mathbb{N}$ write

$$y_n = \sum_{j=1}^{\infty} \lambda_{n,j} w_{n,j},$$

where $\sum_{j=1}^{\infty} \lambda_{n,j} = 1$, $\lambda_{n,j} \geq 0$, and $w_{n,j}$ belongs to $W_j = \overline{co}K_j$, the closed convex hull of K_j.

Passing to a subsequence of (y_n), we can assume that for all $j \in \mathbb{N}$, there exists λ_j such that $\lambda_{n,j} \xrightarrow{n} \lambda_j$, with $\sum_{j=1}^{\infty} \lambda_j = 1$. Let

$$y'_n = \sum_{j=1}^{\infty} \lambda_j w_{n,j},$$

and

$$e_n = y_n - y'_n.$$

Since K_j is weakly compact, so is W_j; hence, passing to a further subsequence we can assume that for each $j \in \mathbb{N}$ there is some $z_j \in W_j$ such that $w_{n,j} \xrightarrow{n} z_j$ weakly. Note that for each $j \in \mathbb{N}$, W_j is weakly compact in $C((\mathbb{N} \cup \{\infty\})^j)$, which is isomorphic to c_0. Since c_0 has the weak Banach-Saks property [**45**], we obtain that W_j is a Banach-Saks set. Hence, using [**44**] and a diagonal process, we can extract a subsequence (n_i) such that for each $j \in \mathbb{N}$ there exists $f_j : \mathbb{N} \to \mathbb{R}$ satisfying

$$\left\| \sum_{i=1}^{k} w_{n_i,j} - k z_j \right\| \leq f_j(k)$$

and $\frac{f_j(k)}{k} \to 0$ when $k \to \infty$. Since the W_j are disjointly supported on M we get

$$\left\| \sum_{i=1}^{k} y'_{n_i} - k \sum_{j=1}^{\infty} \lambda_j z_j \right\| = \left\| \sum_{j=1}^{\infty} \lambda_j \left(\sum_{i=1}^{k} w_{n_i,j} - k z_j \right) \right\| \leq \max_j \lambda_j f_j(k),$$

which implies

$$\frac{1}{k} \sum_{i=1}^{k} y'_{n_i} \xrightarrow{k} \sum_{j=1}^{\infty} \lambda_j z_j$$

in the norm of $C(M)$. Hence, (y'_{n_i}) has convergent arithmetic means (and also every subsequence of it).

A gliding hump argument yields that (e_n) has a subsequence equivalent to the unit vector basis of c_0. Indeed, taking an appropriate subsequence we can assume that $\lambda_{n,j} \to \lambda_j$ fast enough, so that the following construction can be carried out. First, set $n_1 = 1$ and let j_1 be such that

$$\Big\| \sum_{j=j_1}^{\infty} (\lambda_{n_1,j} - \lambda_j) w_{n_1,j} \Big\| < \frac{1}{2}.$$

Next, take n_2 such that

$$\Big\| \sum_{j=1}^{j_1} (\lambda_{n_2,j} - \lambda_j) w_{n_2,j} \Big\| < \frac{1}{2^3},$$

and then choose j_2 such that

$$\Big\| \sum_{j=j_2}^{\infty} (\lambda_{n_2,j} - \lambda_j) w_{n_2,j} \Big\| < \frac{1}{2^3}.$$

In this way, we construct inductively a pair of sequences (n_k) and (j_k) such that

$$\sum_{k=1}^{\infty} \Big\| e_{n_k} - \sum_{j=j_{k-1}}^{j_k - 1} (\lambda_{n_k,j} - \lambda_j) w_{n_k,j} \Big\| \leq 1.$$

Thus, (e_{n_k}) is equivalent to $\big(\sum_{j=j_{k-1}}^{j_k} (\lambda_{n_k,j} - \lambda_j) w_{n_k,j}\big)_k$ which is a disjoint sequence in $C(M)$ equivalent to the unit vector basis of c_0.

Finally, note that every subsequence of the unit vector basis of c_0 has convergent arithmetic means. Therefore, both (e_{n_k}) and (y'_{n_k}) have subsequences with the same property. This implies that the same is true for some subsequence of (y_n) and the proof is finished. □

EXAMPLE 7.1.4. *There exists a positive operator $U : \ell_1 \to C[0,1]$ which is Banach-Saks but it fails to factor through a Banach lattice with the Banach-Saks property.*

PROOF. Note that K can be seen as a subset of $C[0,1]$ by taking a positive embedding of $C(M)$ into $C([0,1])$, such that its image is complemented. Take (x_n) a dense sequence in K and consider the operator

$$\begin{array}{rccc} U: & \ell_1 & \longrightarrow & C([0,1]) \\ & (a_n)_{n=1}^{\infty} & \longmapsto & \sum_{n=1}^{\infty} a_n x_n. \end{array}$$

Clearly U is positive. Moreover, $U(B_{\ell_1}) = \overline{co}(K)$ is a Banach-Saks set by Lemma 7.1.3, and therefore the operator U is Banach-Saks.

By [**129**, Theorem A] the operator $U : \ell_1 \to C([0,1])$ does not factor through any reflexive Banach lattice. Since every space with the Banach-Saks property is reflexive, the proof is finished. □

This shows that the hypothesis on Theorem 7.1.1 are necessary. Moreover, if we consider the operator $\widetilde{U} : \ell_1 \oplus C([0,1]) \to \ell_1 \oplus C([0,1])$, given by $\widetilde{U}(x,y) = (0, U(x))$, then one notices that Corollary 7.1.2 cannot be improved.

2. Domination by Banach-Saks operators

In this section we look at the problem of domination for Banach-Saks operators on Banach lattices. Observe that given $0 \leq R_1 \leq T_1 : E_1 \to E_2$ and $0 \leq R_2 \leq T_2 : E_2 \to E_3$, with T_1 Banach-Saks and T_2 order weakly compact, the proof of Theorem 7.1.1 can be adapted to obtain a factorization

$$\begin{tikzcd}
E_1 \arrow[rr, shift left, "T_1"] \arrow[rr, dashed, shift right, "R_1"'] \arrow[drr, shift left, "P_1"] \arrow[drr, dashed, shift right, "Q_1"'] & & E_2 \arrow[rr, shift left, "T_2"] \arrow[rr, dashed, shift right, "R_2"'] & & E_3 \\
& & H \arrow[urr, shift left, "P_2"] \arrow[urr, dashed, shift right, "Q_2"'] & &
\end{tikzcd}$$

where H is a Banach lattice with the Banach-Saks property, $0 \leq Q_1 \leq P_1$ and $0 \leq Q_2 \leq P_2$. From here a domination result for Banach-Saks operators is easily obtained. However, we provide an alternative proof of this fact which does not depend on interpolation. This is the content of the following theorem which improves previous results of J. Flores and C. Ruiz in [**55**].

THEOREM 7.2.1. *Let E_1, E_2 and E_3 be Banach lattices and $0 \leq R_i \leq T_i : E_i \to E_{i+1}$ be positive operators for $i = 1, 2$. If T_1 is a Banach-Saks operator, and T_2 is order weakly compact then the composition R_2R_1 is a Banach-Saks operator.*

PROOF. Since T_2 is order weakly compact, by Theorem 1.5.1, we have the factorization:

$$\begin{tikzcd}
E_1 \arrow[r, shift left, "T_1"] \arrow[r, dashed, shift right, "R_1"'] & E_2 \arrow[rr, shift left, "T_2"] \arrow[rr, dashed, shift right, "R_2"'] \arrow[dr, "\phi"'] & & E_3 \\
& & F \arrow[ur, shift left, "P"] \arrow[ur, dashed, shift right, "Q"'] &
\end{tikzcd}$$

where F is an order continuous Banach lattice, and $0 \leq Q \leq P$.

Take an arbitrary sequence (x_n) in B_{E_1} and consider M the closed ideal in F generated by $x = \sum_{n=1}^{\infty} \frac{|\phi T_1(x_n)|}{2^n}$.

Since T_1 is Banach-Saks and

$$|\phi R_1 x_{n_k}| \leq \phi T_1 |x_{n_k}|,$$

there exists a subsequence (n_k) such that the arithmetic means $\frac{1}{m}\sum_{k=1}^{m} \phi T_1 |x_{n_k}|$ converge to some $x \in F$. Thus, if we denote $c_m = \frac{1}{m}\sum_{k=1}^{m} \phi R_1 x_{n_k}$, then the sequence (c_m) is equi-integrable in F (and the same holds for any subsequence of (n_k) by [**99**]).

Since ϕT_1 is weakly compact, Gantmacher's theorem implies in particular that its adjoint $(\phi T_1)^*$ is order weakly compact, so we get a factorization for ϕT_1 and ϕR_1 through a Banach lattice G, such that both G and G^* are order continuous [**61**, Proposition I.4 and Theorem I.6]:

$$\begin{tikzcd}
E_1 \arrow[rr, "T_1"] \arrow[rr, dashed, "R_1"'] \arrow[dr, "U"] \arrow[dr, dashed, "V"'] & & E_2 \arrow[rr, "T_2"] \arrow[rr, dashed, "R_2"'] \arrow[dr, "\phi"'] & & E_3 \\
& G \arrow[rr, "\psi"'] & & F \arrow[ur, "P"] \arrow[ur, dashed, "Q"'] &
\end{tikzcd}$$

By passing to some subsequence, [**12**, Theorem 4.25] yields that $(\phi R_1 x_{n_k})$ is weakly Cauchy. Since F is order continuous, by Theorem 1.2.7, we can represent the ideal M of F as a Banach lattice of functions on a probability space (Ω, Σ, μ), with $M \hookrightarrow L_1(\Omega, \Sigma, \mu)$. Since $L_1(\Omega, \Sigma, \mu)$ is weakly sequentially complete, the sequence $(\phi R_1 x_{n_k})$, whose elements belongs to M, is weakly weakly convergent in $L_1(\Omega, \Sigma, \mu)$. Now, by [**127**], $(\phi R_1 x_{n_k})$ has a subsequence whose arithmetic means converge in the norm of $L_1(\Omega, \Sigma, \mu)$ to some function $f \in L_1(\Omega, \Sigma, \mu)$. However, since

$$\left|\frac{1}{m}\sum_{k=1}^{m}\phi R_1 x_{n_k}\right| \leq \frac{1}{m}\sum_{k=1}^{m}\phi T_1 |x_{n_k}|,$$

and

$$\frac{1}{m}\sum_{k=1}^{m}\phi T_1 |x_{n_k}| \to x$$

for some $x \in F$, we must have $|f| \leq x$, which implies that $f \in F$. Therefore the sequence of arithmetic means, (c_m), must be convergent in the norm of F (see Lemma 1.4.2). This implies that ϕR_1 and consequently $R_2 R_1$ are Banach-Saks operators. This finishes the proof. $\square$

Corollary 7.2.2. *Let E be a Banach lattice and $0 \leq R \leq T : E \to E$ be positive operators. If T is Banach-Saks, then R^2 is also Banach-Saks.*

Proof. Since T is Banach-Saks, it is also weakly compact [**99**], and in particular order weakly compact. Theorem 7.2.1 yields the result. $\square$

Note that in [**55**, Ex. 2.9] it was shown that there exist operators

$$0 \leq R \leq T : \ell_1 \to \ell_\infty$$

such that T is Banach-Saks, but R is not. This shows that Corollary 7.2.2 is sharp; indeed, consider the operators $0 \leq \widetilde{R} \leq \widetilde{T} : \ell_1 \oplus \ell_\infty \to \ell_1 \oplus \ell_\infty$ defined by

$$\widetilde{R} = \begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix} \qquad \widetilde{T} = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}.$$

Clearly $\widetilde{T}$ is Banach-Saks, but $\widetilde{R}$ is not. Notice that $\widetilde{R}^2 = 0$.

We also have the following improvement to [**55**, Theorem 1.1].

Corollary 7.2.3. *Let E and F be Banach lattices, such that F is order continuous. If $0 \leq R \leq T : E \to F$, with T Banach-Saks, then R is also a Banach-Saks operator.*

PROOF. Use Theorem 7.2.1 and the fact that order intervals in an order continuous Banach lattice are weakly compact (Theorem 1.2.2). □

The following question remains open: Can order continuity on F be replaced with order continuity on E^* in Corollary 7.2.3?

3. Convex hulls of Banach-Saks sets

With respect to convex hulls of Banach-Saks sets in Banach spaces, the following question was posed in [**62**]: Are Banach-Saks sets stable under convex hulls? apparently, no examples of Banach-Saks sets without a Banach-Saks convex hull can be found in the literature. Here we give one such example, communicated by N. Kalton, answering the question in the negative.

EXAMPLE 7.3.1. *A Banach-Saks set, whose closed convex hull is not Banach-Saks.*

PROOF. Let $Y = \ell_2(\ell_1^{2^n})$, and let $(e_{n,j})_{n=1,j=1,\dots,2^n}^{\infty}$ be the basis vectors of Y. The elements

$$f_n = \frac{1}{2^n}\sum_{j=1}^{2^n} e_{n,j}$$

form a basic sequence which spans a complemented subspace of Y. Let $P : Y \to [f_n]$ denote this projection. We have

$$P(\xi) = \sum_{n=1}^{\infty} a_n(\xi) f_n,$$

for certain coefficients $(a_n(\xi))$ depending on $\xi \in Y$. We can define a new norm in Y by

$$\|\xi\|_X = \|\xi\|_Y + \Big\|\sum_{n=1}^{\infty} a_n(\xi) b_n\Big\|_B,$$

where (b_n) denotes the coordinate basis of Baernstein's space, and $\| \, \|_B$ denotes its norm (see [**17**])

$$\Big\|\sum_{n=1}^{\infty}\lambda_n b_n\Big\|_B = \sup\Big\{\Big(\sum_{k=1}^{\infty}\Big(\sum_{j\in\gamma_k}|\lambda_j|\Big)^2\Big)^{\frac{1}{2}}\Big\},$$

where the supremum is taken over all $\gamma_k \subset \mathbb{N}$ such that the cardinality of γ_k is smaller than or equal to the smallest element of γ_k.

Notice that he set $S = \{e_{n,j} : n = 1, 2, \dots,\ j = 1, \dots, 2^n\}$ is Banach-Saks. Indeed, let (x_n) be a sequence in S, we can extract a subsequence such that $x_{n_k} = e_{p_k,j_k}$ with $p_k < p_{k+1}$. Since $P(e_{n,j}) = \frac{1}{2^n} f_n$, we have

$$a_n\Big(\sum_{k=1}^{m} e_{p_k,j_k}\Big) = \begin{cases} \frac{1}{2^n} & \text{for } n \in \{p_1, \dots, p_m\}, \\ 0 & \text{in any other case.} \end{cases}$$

Therefore, we have

$$\begin{aligned}\Big\|\tfrac{1}{m}\sum_{k=1}^{m} e_{p_k,j_k}\Big\|_X &= \tfrac{1}{m}\Big(\Big\|\sum_{k=1}^{m} e_{p_k,j_k}\Big\|_Y + \Big\|\Big(a_n\Big(\sum_{k=1}^{m} e_{p_k,j_k}\Big)\Big)\Big\|_B\Big)\\ &= \tfrac{1}{m}\Big(\sqrt{m}+\sum_{k=1}^{m}\tfrac{1}{2^{p_k}}\Big)\\ &\leq \tfrac{\sqrt{m}+1}{m},\end{aligned}$$

which clearly tends to zero when $m \to \infty$.

However, the sequence (f_n), which belongs to the convex hull of S, has no Cesaro convergent subsequence. Indeed, notice first that since

$$a_n(f_k) = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{if } n \neq k, \end{cases}$$

for scalars (a_n) we have

$$\Big\|\sum_{k=1}^{\infty} a_n f_n\Big\|_X = \Big(\sum_{k=1}^{\infty} |a_k|^2\Big)^{\frac{1}{2}} + \Big\|\sum_{k=1}^{\infty} a_n b_n\Big\|_B.$$

Hence, the operator

$$\begin{aligned} T:\ \ \ell_2 \oplus_1 B &\longrightarrow [f_n]_X \\ e_n \oplus b_n &\longmapsto f_n \end{aligned}$$

is bounded, so in particular (f_n) is weakly null.

Now, for any subsequence (f_{n_k}) and every m, we obtain

$$\begin{aligned}\Big\|\tfrac{1}{m}\sum_{k=1}^{m} f_{n_k}\Big\|_X &= \Big\|\tfrac{1}{m}\sum_{k=1}^{m} f_{n_k}\Big\|_X + \Big\|\sum_{n=1}^{\infty} a_n\Big(\tfrac{1}{m}\sum_{k=1}^{m} f_{n_k}\Big) b_n\Big\|_B\\ &= \tfrac{\sqrt{m}}{m} + \Big\|\tfrac{1}{m}\sum_{k=1}^{m} b_{n_k}\Big\|_B,\end{aligned}$$

where (b_i) denotes the coordinate basis of Baernstein's space B. By [**17**], we have that

$$\Big\|\frac{1}{m}\sum_{k=1}^{m} b_{n_k}\Big\|_B \geq \alpha$$

for a certain $\alpha > 0$, and every $m \in \mathbb{N}$. In particular, (f_{n_k}) is not Cesaro convergent to zero in X. Since (f_n) is weakly null the proof is finished. $\square$

However, in certain cases, the convex hull of a Banach-Saks set is also Banach-Saks.

PROPOSITION 7.3.2. *Let X be a Banach space with the weak Banach-Saks property. For every Banach-Saks set $S \subset X$, the closed convex hull of S is also a Banach-Saks set.*

PROOF. Let S be a Banach-Saks set. Since S is relatively weakly compact, the same holds for $\overline{co}(S)$. Hence every sequence in $\overline{co}(S)$ has a weakly convergent subsequence, which has a Cesaro convergent subsequence, since X has the weak Banach-Saks property. $\square$

Notice that the converse of Proposition 7.3.2 is not true. Indeed, let us consider Schreier's space S, obtained by completion of the space of finite sequences with respect to the norm

$$\|x\|_S = \sup\left\{ \sum_{j\in A} |x_j| : A = \{n_1, \dots, n_k\} \text{ with } k \leq n_1 \right\}.$$

By [**123**], S does not have the weak Banach-Saks property, but the convex hull of a Banach-Saks set in S is always a Banach-Saks set [**62**, Corollary 17].

CHAPTER 8

Factorization of p-convex and q-concave operators

The present Chapter is devoted to factorization of q-concave and p-convex operators. In a similar spirit to [**61**], we show that a q-concave operator from a Banach lattice to a Banach space always factors in a natural way through a q-concave Banach lattice. An analogous result is also proved for p-convex operators.

We study this factorizations, showing that in a certain sense they are "maximal", i.e. best possible. And in the second Section we study their relation by duality.

In Section 3, we extend results of G. Lozanovskii and V. Sestakov for interpolation of arbitrary Banach lattices. And in the fourth Section, we apply this results for operators which are both p-convex and q-concave, to obtain simultaneous factorizations.

The content of this Chapter was done during two visits to the *Equipe d'Analyse Fonctionnelle* in the *Université de Paris VI/ParisVII* in fall 2007 and fall 2008, under the supervision of Y. Raynaud.

1. The basic construction

The following result was essentially proved by S. Reisner in [**117**, Lemma 6]. However, we include a simpler proof for completeness.

THEOREM 8.1.1. *Let E be a Banach lattice, X a Banach space and $1 \le q \le \infty$. An operator $T : E \to X$ is q-concave if and only if there exist a q-concave Banach lattice V, a positive operator $\phi : E \to V$ (in fact, a lattice homomorphism with dense image), and another operator $S : V \to X$ such that $T = S\phi$.*

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "\phi"'] & & X \\ & V \arrow[ur, "S"'] & \end{tikzcd}$$

PROOF. Let us suppose $q < \infty$. The proof for the case $q = \infty$ is trivial because every Banach lattice is ∞-concave. However, the precise construction carried out here for $q < \infty$ has its analogue for $q = \infty$.

For the direct implication, let $(x_i)_{i=1}^n$ in E. Since V is q-concave and ϕ is positive, by Theorem 1.5.10, we have

$$\begin{aligned}\Big(\sum_{i=1}^n \|Tx_i\|^q\Big)^{\frac{1}{q}} &\leq \|S\|\Big(\sum_{i=1}^n \|\phi x_i\|^q\Big)^{\frac{1}{q}}\\ &\leq \|S\| M_{(q)}(I_V)\Big\|\Big(\sum_{i=1}^n |\phi x_i|^q\Big)^{\frac{1}{q}}\Big\|\\ &\leq \|S\| M_{(q)}(I_V)\|\phi\|\Big\|\Big(\sum_{i=1}^n |x_i|^q\Big)^{\frac{1}{q}}\Big\|.\end{aligned}$$

This yields that T is q-concave.

Now, for the converse implication, given $x \in E$, let us consider

$$\rho(x) = \sup\Big\{\Big(\sum_{i=1}^n \|Tx_i\|^q\Big)^{\frac{1}{q}} : \Big(\sum_{i=1}^n |x_i|^q\Big)^{\frac{1}{q}} \leq |x|\Big\}.$$

If $M_{(q)}(T)$ denotes the q-concavity constant of T, then for $(x_i)_{i=1}^n$ in E, we have

$$\Big(\sum_{i=1}^n \|Tx_i\|^q\Big)^{\frac{1}{q}} \leq M_{(q)}(T)\Big\|\Big(\sum_{i=1}^n |x_i|^q\Big)^{\frac{1}{q}}\Big\|.$$

In particular, for all $x \in E$

$$\|Tx\| \leq \rho(x) \leq M_{(q)}(T)\|x\|.$$

Moreover, ρ is a lattice semi-norm on E. Indeed, for any $x \in E$ and $\lambda \geq 0$ it is clear that $\rho(\lambda x) = \lambda\rho(x)$. In order to prove the triangle inequality, let $x, y \in E$ and $z = |x| + |y|$, and denote $I_z \subset E$ the ideal generated by z in E, which is identified with a space $C(K)$ in which z corresponds to the function identically one. Now, for every $\varepsilon > 0$ let $z_1, \ldots, z_n \in E$ such that $\big(\sum_{i=1}^n |z_i|^q\big)^{\frac{1}{q}} \leq |z|$ and

$$\rho(z) \leq \big(\sum_{i=1}^n \|Tz_i\|^q\big)^{\frac{1}{q}} + \varepsilon.$$

Since $x, y \in I_z$, they correspond to functions $f, g \in C(K)$ such that $|f(t)| + |g(t)| = 1$ for every $t \in K$. Similarly, z_i corresponds to $h_i \in C(K)$ with $\big(\sum_{i=1}^n |h_i(t)|^q\big)^{\frac{1}{q}} \leq 1$ for every $t \in K$. Hence we can consider

$$\begin{cases} f_i(t) = h_i(t)f(t),\\ g_i(t) = h_i(t)g(t),\end{cases}$$

which belong to $C(K)$ and satisfy $\big(\sum_{i=1}^n |f_i(t)|^q\big)^{\frac{1}{q}} \leq |f(t)|$ and $\big(\sum_{i=1}^n |g_i(t)|^q\big)^{\frac{1}{q}} \leq |g(t)|$. This means that we can consider $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ such that $\big(\sum_{i=1}^n |x_i|^q\big)^{\frac{1}{q}} \leq |x|$ and $\big(\sum_{i=1}^n |y_i|^q\big)^{\frac{1}{q}} \leq |y|$ in E. Thus, we finally have

$$\begin{aligned}\rho(x+y) &\leq \big(\sum_{i=1}^n \|Tz_i\|^q\big)^{\frac{1}{q}} + \varepsilon\\ &\leq \big(\sum_{i=1}^n \|Tx_i\|^q\big)^{\frac{1}{q}} + \big(\sum_{i=1}^n \|Ty_i\|^q\big)^{\frac{1}{q}} + \varepsilon\\ &\leq \rho(x) + \rho(y) + \varepsilon.\end{aligned}$$

Since this holds for every $\varepsilon > 0$, the triangle inequality is proved.

Now, if $|y| \leq |x|$, then for any $(x_i)_{i=1}^n$ such that $\left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}} \leq |y|$, it holds that $\left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}} \leq |x|$, hence for any such $(x_i)_{i=1}^n$, $\left(\sum_{i=1}^n \|Tx_i\|^q\right)^{\frac{1}{q}} \leq \rho(x)$. This implies that $\rho(y) \leq \rho(x)$, and so $\rho^{-1}(0)$ is an ideal.

Let V denote the Banach lattice obtained by completing $E/\rho^{-1}(0)$ with the norm induced by ρ. Let ϕ denote the quotient map from E to $E/\rho^{-1}(0)$, seen as a map to V. Now, for $x \in E$ let us define $S(\phi(x)) = T(x)$. Since $\|Tx\| \leq \rho(x)$, S is well defined and extends to a bounded operator $S : V \to X$, such that $T = S\phi$.

Now, let $(x_i)_{i=1}^n$ in E. For every $\varepsilon > 0$ and for every $i = 1, \ldots, n$ there exist $(y_j^i)_{j=1}^{k_i}$ in E such that

$$\left(\sum_{j=1}^{k_i} |y_j^i|^q\right)^{\frac{1}{q}} \leq |x_i|$$

and

$$\rho(x_i)^q = \sup\left\{\sum_{j=1}^{k} \|Ty_j\|^q : \left(\sum_{j=1}^{k} |y_j|^q\right)^{\frac{1}{q}} \leq |x_i|\right\} \leq \sum_{j=1}^{k_i} \|Ty_j^i\|^q + \frac{\varepsilon^q}{n},$$

for every $i = 1, \ldots, n$. Therefore, we have

$$\begin{aligned}\left(\sum_{i=1}^n \rho(x_i)^q\right)^{\frac{1}{q}} &\leq \left(\sum_{i=1}^n \left(\sum_{j=1}^{k_i} \|Ty_j^i\|^q + \tfrac{\varepsilon^q}{n}\right)\right)^{\frac{1}{q}}\\ &\leq \left(\sum_{i=1}^n \sum_{j=1}^{k_i} \|Ty_j^i\|^q\right)^{\frac{1}{q}} + \varepsilon\\ &\leq \rho\left(\left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}}\right) + \varepsilon.\end{aligned}$$

Since this holds for every $\varepsilon > 0$, the normed lattice $E/\rho^{-1}(0)$ is q-concave; hence, the same holds for its completion V. $\square$

Since the lattice V constructed in the proof depends on the operator $T : E \to X$ and q, we will denote it by $V_{T,q}$ whenever needed. Similarly we will denote ρ_T (or $\rho_{T,q}$) for the expression defining the norm of $V_{T,q}$.

Remark 8.1.2. Recall that in Theorem 1.5.1, an order continuous Banach lattice F was constructed to factor order weakly compact operators (those mapping intervals to weakly compact sets). Given $T : E \to Y$, the Banach lattice F is constructed by means of the expression $\|x\|_F = \sup\{\|Ty\| : |y| \leq |x|\}$, for $x \in E$, producing a Banach lattice in the usual way. Notice that if $T : E \to Y$ is q-concave, which implies that T is order weakly compact, then $\|x\|_F \leq \rho_T(x)$, hence we can consider a natural map $V_{T,q} \xrightarrow{i} F$ such that we can factor T as follows:

$$\begin{array}{ccc} E & \xrightarrow{T} & Y \\ {\scriptstyle\phi}\downarrow & & \uparrow{\scriptstyle\widetilde{T}} \\ V_{T,q} & \overset{i}{\hookrightarrow} & F \end{array}$$

Moreover, F coincides with $V_{T,\infty}$, so in a sense the previous Theorem is an extension of Theorem 1.5.1.

The factorization given in Theorem 8.1.1 is in a certain sense maximal, as the following Proposition shows.

PROPOSITION 8.1.3. *Let $T : E \to X$ be a q-concave operator. Suppose that T factors through a q-concave Banach lattice $\widetilde{V}$ with factors $A : E \to \widetilde{V}$ and $B : \widetilde{V} \to X$, such that A is a lattice homomorphism whose image is dense in $\widetilde{V}$, and $T = B \circ A$. Then there is a lattice homomorphism $u : \widetilde{V} \to V_{T,q}$ such that $\phi = u \circ A$ and $S \circ u = B$.*

$$\begin{tikzcd}
E \arrow[rr, "T"] \arrow[dr, "A"] \arrow[ddr, "\phi"'] & & X \\
& \widetilde{V} \arrow[ur, "B"'] \arrow[d, dashed, "u"'] & \\
& V_{T,q} \arrow[uur, "S"'] &
\end{tikzcd}$$

PROOF. Let us consider for $x \in E$, $u(A(x)) = \phi(x)$. Notice that for $\{x_i\}_{i=1}^n$ in E, such that $\left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}} \leq |x|$, since A is a lattice homomorphism, we have

$$\begin{aligned}
\left(\sum_{i=1}^n \|Tx_i\|^q\right)^{\frac{1}{q}} &= \left(\sum_{i=1}^n \|BAx_i\|^q\right)^{\frac{1}{q}} \leq \|B\| \left(\sum_{i=1}^n \|Ax_i\|^q\right)^{\frac{1}{q}} \\
&\leq \|B\| M_{(q)}(I_{\widetilde{V}}) \left\| \left(\sum_{i=1}^n |A(x_i)|^q\right)^{\frac{1}{q}} \right\| = \|B\| M_{(q)}(I_{\widetilde{V}}) \left\| A\left(\left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}}\right) \right\| \\
&\leq \|B\| M_{(q)}(I_{\widetilde{V}}) \|A(x)\|.
\end{aligned}$$

Therefore,

$$\|u(A(x))\| = \|\phi(x)\| = \rho_T(x) \leq \|B\| M_{(q)}(I_{\widetilde{V}}) \|A(x)\|.$$

Since A has dense image, the preceding inequality implies that u can be extended to a bounded operator $u : \widetilde{V} \to V_{(T,q)}$, which is clearly a lattice homomorphism and satisfies the required properties. $\square$

There is an analogous version of Theorem 8.1.1 for p-convex operators.

THEOREM 8.1.4. *Let E be a Banach lattice, X a Banach space and $1 \leq p \leq \infty$. An operator $T : X \to E$ is p-convex if and only if there exist a p-convex Banach lattice W, a positive operator (an injective interval preserving lattice homomorphism) $\varphi : W \to E$ and another operator $R : X \to W$ such that $T = \varphi R$.*

$$\begin{tikzcd}
X \arrow[rr, "T"] \arrow[dr, "R"'] & & E \\
& W \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Proof. Let us suppose $p < \infty$. The proof for the case $p = \infty$ is analogous, with the usual changes.

As in the proof of Theorem 8.1.1, Theorem 1.5.10 yields one implication.

For the non-trivial one, let $T : X \to E$ be p-convex. Let us consider the following set

$$S = \{y \in E : \, |y| \leq \left(\sum_{i=1}^{k} |Tx_i|^p\right)^{\frac{1}{p}}, \text{ where } \sum_{i=1}^{k} \|x_i\|^p \leq 1 \text{ and } k \in \mathbb{N}\}.$$

We can consider the Minkowski functional defined by its closure $\overline{S}$ in E

$$\|z\|_W = \inf\{\lambda > 0 : z \in \lambda \overline{S}\}.$$

Clearly $\overline{S}$ is solid, and since T is p-convex, it is also a bounded set of E. Let us consider the space $W = \{z \in E : \|z\|_W < \infty\}$. We claim that for any $z_1, \dots, z_n$ in W, it holds that

$$\left\| \left(\sum_{i=1}^{k} |z_i|^p\right)^{\frac{1}{p}} \right\|_W \leq \left(\sum_{i=1}^{n} \|z_i\|_W^p\right)^{\frac{1}{p}}.$$

Indeed, given $z_1, \dots, z_n$ in W, for every $\varepsilon > 0$ and for every $i = 1, \dots, n$ there exist λ_i with $z_i \in \lambda_i \overline{S}$, such that

$$\lambda_i^p \leq \inf\left\{\mu^p : \, z_i \in \mu \overline{S}\right\} + \frac{\varepsilon^p}{n},$$

for each $i = 1, \dots, n$.

This means that for every $i = 1, \dots, n$, and for every $\delta > 0$ there exists y_i^δ in E with $\|z_i - y_i^\delta\|_E \leq \delta$, and

$$|y_i^\delta| \leq \left(\sum_{j=1}^{m_{i,\delta}} |Tx_j^{i,\delta}|^p\right)^{\frac{1}{p}},$$

where $(x_j^{i,\delta})_{j=1}^{m_{i,\delta}}$ satisfy

$$\left(\sum_{j=1}^{m_{i,\delta}} \|x_j^{i,\delta}\|^p\right)^{\frac{1}{p}} \leq \lambda_i,$$

for each $i = 1, \dots, n$, and each $\delta > 0$.

Now, for each $\delta > 0$ let

$$w_\delta = \left(\sum_{i=1}^{n} |y_i^\delta|^p\right)^{\frac{1}{p}}.$$

Notice that

$$\left\| \left(\sum_{i=1}^{n} |z_i|^p\right)^{\frac{1}{p}} - w_\delta \right\|_E \leq \left\| \left(\sum_{i=1}^{n} |z_i - y_i^\delta|^p\right)^{\frac{1}{p}} \right\|_E \leq \sum_{i=1}^{n} \|z_i - y_i^\delta\|_E \leq n\delta.$$

Moreover, note that for every $\delta > 0$ w_δ belongs to $\left(\sum_{i=1}^{n} \lambda_i^p\right)^{\frac{1}{p}} S$. Indeed,

$$|w_\delta| = \left(\sum_{i=1}^{n} |y_i^\delta|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} \sum_{j=1}^{m_{i,\delta}} |Tx_j^{i,\delta}|^p\right)^{\frac{1}{p}},$$

and

$$\Big(\sum_{i=1}^{n}\sum_{j=1}^{m_{i,\delta}}\|x_j^{i,\delta}\|^p\Big)^{\frac{1}{p}} \leq \Big(\sum_{i=1}^{n}\lambda_i^p\Big)^{\frac{1}{p}}.$$

Hence, $\Big(\sum\limits_{i=1}^{n}|z_i|^p\Big)^{\frac{1}{p}} \in \Big(\sum\limits_{i=1}^{n}\lambda_i^p\Big)^{\frac{1}{p}}\overline{S}$.

Therefore, it follows that

$$\begin{aligned}\Big\|\Big(\sum_{i=1}^{n}|z_i|^p\Big)^{\frac{1}{p}}\Big\|_W &= \inf\{\mu>0 : \Big(\sum_{i=1}^{n}|z_i|^p\Big)^{\frac{1}{p}} \in \mu\overline{S}\}\\ &\leq \Big(\sum_{i=1}^{n}\lambda_i^p\Big)^{\frac{1}{p}}\\ &\leq \Big(\sum_{i=1}^{n}\big(\inf\{\mu^p : z_i \in \mu\overline{S}\} + \tfrac{\varepsilon^p}{n}\big)\Big)^{\frac{1}{p}}\\ &\leq \Big(\sum_{i=1}^{n}\|z_i\|_W^p\Big)^{\frac{1}{p}} + \varepsilon.\end{aligned}$$

Since this holds for every $\varepsilon > 0$, we finally have

$$\Big\|\Big(\sum_{i=1}^{n}|z_i|^p\Big)^{\frac{1}{p}}\Big\|_W \leq \Big(\sum_{i=1}^{n}\|z_i\|_W^p\Big)^{\frac{1}{p}}.$$

Now, in the vector lattice W, we can consider the operations

$$z_1 \oplus z_2 = (z_1^p + z_2^p)^{\frac{1}{p}}, \quad \alpha \odot z = \alpha^{\frac{1}{p}} z,$$

for z, z_1, z_2 in E_W^+ and $\alpha \in \mathbb{R}^+$, and define $|||z||| = \|z\|_W^p$ (see [**87**, pp.53-54]). By the preceding inequalities, we have

$$|||z_1 \oplus z_2||| = \|(z_1^p + z_2^p)^{\frac{1}{p}}\|_W^p \leq \|z_1\|_W^p + \|z_2\|_W^p = |||z_1||| + |||z_2|||,$$

and

$$|||\alpha \odot z||| = \|\alpha^{\frac{1}{p}} z\|_W^p = \alpha\|z\|_W^p = \alpha|||z|||,$$

for all z, z_1, z_2 in W^+ and $\alpha \geq 0$.

We can consider now $W_{(p)}$: the space W, endowed with the above operations, and the order provided by E. Clearly, $W_{(p)}$ is a vector lattice for which $|||\cdot|||$ is a norm. Since $\|\cdot\|_W$ coincides with the norm of the p-convexification of $(W_{(p)}, |||\cdot|||)$ (see [**87**, pp.53-54]), it follows that $\|\cdot\|_W$ is actually a norm. Therefore, $(W, \|\cdot\|_W)$ is a p-convex normed lattice. We claim that W is complete, and hence a p-convex Banach lattice.

Indeed, let (w_i) be a Cauchy sequence in W. Since for every $z \in E$ it holds that

$$\|z\|_E \leq M^{(p)}(T)\|z\|_W,$$

it follows that (w_i) is also a Cauchy sequence in E. Let $w \in E$ be its limit. Notice that since w_i are bounded in W, there exists some $\lambda < \infty$ such that $w_i \in \lambda\overline{S}$ for every $i = 1, 2, \ldots$ and since $\overline{S}$ is closed in E, we must have $w \in \lambda\overline{S}$. Thus, w belongs to W, and we will show that

(w_i) converges to w also in W. To this end, let $\varepsilon > 0$. Since (w_i) is a Cauchy sequence, there exists N such that $w_i - w_j \in \varepsilon\overline{S}$ when $i, j \geq N$. Thus, if $i \geq N$ we can write

$$w - w_i = (w - w_j) + (w_j - w_i)$$

for every $j \in \mathbb{N}$, and letting $j \to \infty$ we obtain that $w - w_i \in \varepsilon\overline{S}$. This shows that $w_i \to w$ in W, and hence W is complete, as claimed.

Clearly, by the definition of W we have

$$\|Tx\|_W \leq \|x\|_X$$

for every $x \in X$. Moreover, as noticed above it also holds that

$$\|z\|_E \leq M^{(p)}(T)\|z\|_W$$

for each $z \in E$, therefore the formal inclusion $\varphi : W \hookrightarrow E$ is clearly an injective interval preserving lattice homomorphism, and we have the following diagram

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "R"'] & & E \\ & W \arrow[ur, hook, "\varphi"'] & \end{tikzcd}$$

where R is defined by $Rx = Tx$ for $x \in X$. This finishes the proof. $\square$

As with the Banach lattice constructed in Theorem 8.1.1, we will denote by $W_{T,p}$ the Banach lattice obtained in the proof of Theorem 8.1.4.

Remark 8.1.5. Note that if $T : X \to E$ is p-convex, then it is also p'-convex for every $1 \leq p' \leq p$. Hence, if we consider the factorization spaces $W_{T,p}$ and $W_{T,p'}$ it always holds that

$$W_{T,p'} \hookrightarrow W_{T,p},$$

with norm smaller than or equal to one.

Indeed, this follows from the following two facts. First, the set

$$S = \{y \in E : |y| \leq \left(\sum_{i=1}^k |Tx_i|^p\right)^{\frac{1}{p}}, \text{ with } \sum_{i=1}^k \|x_i\|^p \leq 1\}$$

can be equivalently described by

$$S = \{y \in E : |y| \leq \left(\sum_{i=1}^k a_i|Tw_i|^p\right)^{\frac{1}{p}}, \text{ with } \|w_i\| \leq 1, a_i \geq 0, \sum_{i=1}^k a_i = 1\}.$$

Furthermore, for $1 \leq p' \leq p$, and $a_i \geq 0$ with $\sum_{i=1}^k a_i = 1$ it always holds that

$$\left(\sum_{i=1}^k a_i|Tw_i|^{p'}\right)^{\frac{1}{p'}} \leq \left(\sum_{i=1}^k a_i|Tw_i|^p\right)^{\frac{1}{p}}.$$

Hence the unit ball of $W_{T,p'}$ is contained in that of $W_{T,p}$.

REMARK 8.1.6. The operator $\varphi : W_{T,p} \to E$ constructed in the proof is an injective, interval preserving lattice homomorphism. Moreover, it satisfies that the image of the closed unit ball $\varphi(B_{W_{T,p}})$ is a closed set in E. This allows us to introduce a class $\mathcal{C}$ consisting of operators $T : E \to F$ between Banach lattices which are injective, interval preserving lattice homomorphisms, such that the image of the closed unit ball $T(B_E)$ is closed in F. The importance of this class will become clear next.

As for Proposition 8.1.3, the construction of Theorem 8.1.4 is in a sense minimal.

PROPOSITION 8.1.7. *Let $T : X \to E$ be a p-convex operator, such that there exist a p-convex Banach lattice $\widetilde{W}$ and operators $A : X \to \widetilde{W}$ and $B : \widetilde{W} \to E$ with $T = BA$ and B belonging to the class $\mathcal{C}$. Then there exists an operator $v : W_{T,p} \to \widetilde{W}$ such that $vR = A$ and $Bv = \varphi$.*

$$\begin{tikzcd}
X \arrow[rr, "T"] \arrow[dr, "A"] \arrow[ddr, "R"'] & & E \\
& \widetilde{W} \arrow[ur, "B"] & \\
& W_{T,p} \arrow[u, dashed, "v"'] \arrow[uur, "\varphi"'] &
\end{tikzcd}$$

PROOF. Let us define v. Let $y \in W_{T,p}$ with $\|y\|_{W_{T,p}} \leq 1$. By definition, there exists a sequence (y_n) in E such that $y_n \to \varphi(y)$ in E, and for each $n \in \mathbb{N}$,

$$|y_n| \leq \Big(\sum_{i=1}^{k_n} |Tx_i^n|^p\Big)^{\frac{1}{p}}$$

with $\sum_{i=1}^{k_n} \|x_i^n\|_X^p \leq 1$. Notice that since B is a lattice homomorphism

$$\Big(\sum_{i=1}^{k_n} |Tx_i^n|^p\Big)^{\frac{1}{p}} = \Big(\sum_{i=1}^{k_n} |BAx_i^n|^p\Big)^{\frac{1}{p}} = B\Big(\Big(\sum_{i=1}^{k_n} |Ax_i^n|^p\Big)^{\frac{1}{p}}\Big),$$

where $\Big(\sum_{i=1}^{k_n} |Ax_i^n|^p\Big)^{\frac{1}{p}}$ belongs to $\widetilde{W}$. Hence, since B is interval preserving there exists $w_n \in \widetilde{W}$ with $|w_n| \leq \Big(\sum_{i=1}^{k_n} |Ax_i^n|^p\Big)^{\frac{1}{p}}$ such that $B(w_n) = y_n$. Notice that since $\widetilde{W}$ is p-convex and $\sum_{i=1}^{k_n} \|x_i\|_X^p \leq 1$, for every n we have

$$\|w_n\|_{\widetilde{W}} \leq \Big\|\Big(\sum_{i=1}^{k_n} |Ax_i^n|^p\Big)^{\frac{1}{p}}\Big\|_{\widetilde{W}} \leq M^{(p)}(I_{\widetilde{W}})\Big(\sum_{i=1}^{k_n} \|Ax_i^n\|^p\Big)^{\frac{1}{p}} \leq M^{(p)}(I_{\widetilde{W}})\|A\|.$$

Now, since the image of the unit ball of $\widetilde{W}$ under B is closed, and $B(w_n) = y_n$ converge to $\varphi(y) \in E$, there exists $w \in \widetilde{W}$ with $\|w\|_{\widetilde{W}} \leq M^{(p)}(I_{\widetilde{W}})\|A\|$ such that $B(w) = \varphi(y)$. Moreover, this element is unique because B is injective. Let us define $v(y) = w$. It is clear, because of the injectivity of B, that $v : W_{T,p} \to \widetilde{W}$ is linear. Moreover, by the previous argument v is a bounded operator of norm less than or equal to $M^{(p)}(I_{\widetilde{W}})\|A\|$.

It is clear by construction that $B \circ v = \varphi$. Moreover, since $B \circ A = T = \varphi \circ R = B \circ v \circ R$ and B is injective, we also get that $A = v \circ R$ as desired. This finishes the proof. $\square$

2. Duality relations

In this section we show the relation between the Banach lattices constructed in the proofs of Theorems 8.1.1 and 8.1.4. We need some preliminary lemmas first.

LEMMA 8.2.1. *Let E be a Banach lattice, $x, y \in E_+$, $x \wedge y = 0$, and $z^* \in E^*_+$. There exist u^*, v^* in E^*_+ such that $z^* = u^* + v^*$, $u^* \wedge v^* = 0$ and*

$$\begin{cases} \langle z^*, x\rangle = \langle u^*, x\rangle \\ \langle z^*, y\rangle = \langle v^*, y\rangle \end{cases}$$

PROOF. By [**94**, Lemma 1.4.3], there exist $z^*(x)$ and $z^*(y)$ in E^*_+ such that

$$\begin{cases} \langle z^*(x), u\rangle = \langle z^*, u\rangle & \text{for all } u \in E_x \\ \langle z^*(x), u\rangle = 0 & \text{for all } u \in \{x\}^\perp \end{cases}$$

$$\begin{cases} \langle z^*(y), u\rangle = \langle z^*, u\rangle & \text{for all } u \in E_y \\ \langle z^*(y), u\rangle = 0 & \text{for all } u \in \{y\}^\perp \end{cases}$$

where E_x denotes the principal ideal generated by x in E, and $\{x\}^\perp$ denotes the orthogonal complement of x (i.e. $\{x\}^\perp = \{u \in E : u \wedge x = 0\}$). Moreover, without loss of generality we can assume that $z^*(x), z^*(y) \le z^*$ (simply consider $z^*(x) \wedge z^*$ and $z^*(y) \wedge z^*$).

Let now

$$\begin{cases} u_1^* = z^*(x) - z^*(x) \wedge z^*(y), \\ v_1^* = z^*(y) - z^*(x) \wedge z^*(y). \end{cases}$$

Clearly, $u_1^*, v_1^* \in E^*_+$, $u_1^* \wedge v_1^* = 0$ and

$$\begin{cases} \langle u_1^*, u\rangle = \langle z^*(x), u\rangle & \text{for all } u \in \{x\}^\perp + \{y\}^\perp, \\ \langle v_1^*, u\rangle = \langle z^*(y), u\rangle & \text{for all } u \in \{x\}^\perp + \{y\}^\perp. \end{cases}$$

Since E^* is Dedekind complete, we can consider P the band projection onto the band generated by u_1^*, and Q the band projection onto the band generated by v_1^*. Let us denote

$$u_2^* = P(z^*) \qquad\qquad v_2^* = Q(z^*).$$

Thus, we have

$$u_1^* \le u_2^* \le z^*,$$
$$v_1^* \le v_2^* \le z^*.$$

Therefore, for every positive element $u \in E_x$ we have

$$\langle u_2^*, u\rangle \le \langle z^*, u\rangle = \langle u_1^*, u\rangle \le \langle u_2^*, u\rangle.$$

Hence, $\langle u_2^*, u\rangle = \langle z^*, u\rangle$ for every $u \in E_x$. Analogously, $\langle v_2^*, u\rangle = \langle z^*, u\rangle$ for every $u \in E_y$.

Since P and Q are projections onto disjoint bands, we can write $z^* = u_2^* + v_2^* + w^*$, where w^* is disjoint with u_2^* and v_2^*. Moreover, since z^* and u_2^* coincide on E_x, then v_2^* and w^* are null on E_x. Analogously, u_2^* and w^* are null on E_y. Therefore, we can take

$$\begin{cases} u^* = u_2^*, \\ v^* = v_2^* + w^*, \end{cases}$$

which satisfy the required properties. □

LEMMA 8.2.2. *Let E be a Banach lattice. For any $z^* \in E_+^*$, and $x_1, \ldots, x_n \in E_+$, there exist $x_1^*, \ldots, x_n^*$ in E_+^*, such that $z^* = \sum_{i=1}^{n} x_i^*$, $x_i^* \wedge x_j^* = 0$ for $i \neq j$, and*

$$\langle z^*, \bigvee_{i=1}^{n} x_i \rangle = \sum_{i=1}^{n} \langle x_i^*, x_i \rangle.$$

PROOF. Recall that for $x, y \in E_+$ it holds

$$x \vee y = (x - x \wedge y) + (y - x \wedge y) + x \wedge y.$$

Hence, if we let $u = x - x \wedge y$ and $v = y - x \wedge y$, then clearly $u \wedge v = 0$, and by Lemma 8.2.1, there exist u^*, v^* in E_+^* such that $z^* = u^* + v^*$, $u^* \wedge v^* = 0$ and

$$\begin{cases} \langle z^*, u \rangle = \langle u^*, u \rangle \\ \langle z^*, v \rangle = \langle v^*, v \rangle. \end{cases}$$

Therefore,

$$\begin{aligned} \langle z^*, x \vee y \rangle &= \langle u^*, u \rangle + \langle v^*, v \rangle + \langle u^* + v^*, x \wedge y \rangle \\ &= \langle u^*, u + x \wedge y \rangle + \langle v^*, v + x \wedge y \rangle \\ &= \langle u^*, x \rangle + \langle v^*, y \rangle. \end{aligned}$$

By induction on n, the claimed result follows. □

Recall that given a set A in a Banach space X, the *polar* of A is the set $A^0 = \{x^* \in X^* : |\langle x^*, x \rangle| \leq 1 \, \forall x \in A\}$. Similarly, for a set B in X^*, the dual of a Banach space, the *prepolar* of B is the set $B_0 = \{x \in X : |\langle x^*, x \rangle| \leq 1 \, \forall x^* \in B\}$.

LEMMA 8.2.3. *Let $T : X \to E$ be p-convex, and let*

$$S := \{y \in E : |y| \leq \left(\sum_{i=1}^{k} |Tx_i|^p \right)^{\frac{1}{p}}, \text{ with } \sum_{i=1}^{k} \|x_i\|^p \leq 1\}.$$

Since $T^* : E^* \to X^*$ is q-concave (with $\frac{1}{p} + \frac{1}{q} = 1$), we can consider ρ_{T^*}, the seminorm which induces the norm on $V_{T^*,q}$. Hence, we can also consider the convex set $U := \{y^* \in E^* : \rho_{T^*}(y^*) \leq 1\}$. Then

$$\overline{S}^0 = U,$$

where $\overline{S}$ denotes the closure of S in E.

PROOF. First of all, we claim that $\overline{S} \subset U_0$.

Indeed, let $y \in E$ be such that $|y| \leq \left(\sum_{i=1}^{n}|Tx_i|^p\right)^{\frac{1}{p}}$ with $\sum_{i=1}^{n}\|x_i\|^p \leq 1$. For every $y^* \in E^*$ such that $\rho_{T^*}(y^*) \leq 1$, we have:

$$\begin{aligned} |\langle y^*, y\rangle| &\leq \langle |y^*|, \left(\sum_{i=1}^{n}|Tx_i|^p\right)^{\frac{1}{p}}\rangle \\ &= \langle |y^*|, \sup\left\{\sum_{i=1}^{n}a_iTx_i : \sum_{i=1}^{n}|a_i|^q \leq 1\right\}\rangle \\ &= \sup\left\{\left\langle |y^*|, \bigvee_{m=1}^{N}\left(\sum_{i=1}^{n}a_i^mTx_i\right)\right\rangle : \sum_{i=1}^{n}|a_i^m|^q \leq 1,\, m=1,\dots,N,\, N\in\mathbb{N}\right\}. \end{aligned}$$

Where we have made use of [**94**, Cor. 1.3.4.ii)] in the last step.

Now, by Lemma 8.2.2, there exist $(y_m^*)_{m=1}^N$ pairwise disjoint elements of E_+^* such that $|y^*| = \sum_{m=1}^N y_m^*$ and

$$\left\langle |y^*|, \bigvee_{m=1}^{N}\left(\sum_{i=1}^{n}a_i^mTx_i\right)\right\rangle = \sum_{m=1}^{N}\langle y_m^*, \sum_{i=1}^{n}a_i^mTx_i\rangle.$$

Therefore,

$$\begin{aligned} \left\langle |y^*|, \bigvee_{m=1}^{N}\left(\sum_{i=1}^{n}a_i^mTx_i\right)\right\rangle &= \sum_{m=1}^{N}\langle y_m^*, \sum_{i=1}^{n}a_i^mTx_i\rangle \\ &= \sum_{m=1}^{N}\sum_{i=1}^{n}a_i^m\langle T^*(y_m^*), x_i\rangle \\ &= \sum_{i=1}^{n}\langle T^*\left(\sum_{m=1}^{N}a_i^my_m^*\right), x_i\rangle \\ &\leq \left(\sum_{i=1}^{n}\|T^*z_i^*\|^q\right)^{\frac{1}{q}}\left(\sum_{i=1}^{n}\|x_i\|^p\right)^{\frac{1}{p}}, \end{aligned}$$

where $z_i^* = \sum_{m=1}^{N}a_i^my_m^*$.

Note that, since $(y_m^*)_{m=1}^N$ are pairwise disjoint we have that

$$\begin{aligned} \left(\sum_{i=1}^{n}|z_i^*|^q\right)^{\frac{1}{q}} &= \left(\sum_{i=1}^{n}\sum_{m=1}^{N}|a_i^my_m^*|^q\right)^{\frac{1}{q}} \\ &= \left(\sum_{m=1}^{N}|y_m^*|^q\sum_{i=1}^{n}|a_i^m|^q\right)^{\frac{1}{q}} \\ &\leq \sum_{m=1}^{N}y_m^* = |y^*|. \end{aligned}$$

This implies that $\left(\sum_{i=1}^{n}\|T^*z_i^*\|^q\right)^{\frac{1}{q}} \leq 1$, since $\rho_{T^*}(y^*) \leq 1$. Therefore, for any y^* with $\rho_{T^*}(y^*) \leq 1$,

$$\begin{aligned} |\langle y^*, y\rangle| &\leq \sup\left\{\left(\sum_{i=1}^{n}\left\|T^*\left(\sum_{m=1}^{N}a_i^my_m^*\right)\right\|^q\right)^{\frac{1}{q}}\left(\sum_{i=1}^{n}\|x_i\|^p\right)^{\frac{1}{p}} : \sum_{i=1}^{n}|a_i^m|^q \leq 1,\, m=1,\dots,N,\, N\in\mathbb{N}\right\} \\ &\leq 1. \end{aligned}$$

This means that $y \in U_0$. Since U_0 is closed, this proves that $\overline{S} \subseteq U_0$ as claimed.

Therefore, it follows that $(U_0)^0 \subseteq \overline{S}^0$. So in particular, $U \subseteq \overline{S}^0$.

Let us prove now the converse inclusion ($\overline{S}^0 \subseteq U$). Given $y^* \in \overline{S}^0$, we want to show that $\rho_{T^*}(y^*) \leq 1$. To this end, let $y_1^*, \ldots, y_k^*$ be elements in E^*, such that

$$\Big(\sum_{i=1}^k |y_i^*|^q\Big)^{\frac{1}{q}} \leq |y^*|.$$

Notice that since S is solid, then so is $\overline{S}^0$. In particular, $|y^*| \in \overline{S}^0$ whenever $y^* \in \overline{S}^0$.

Now, for every $\varepsilon > 0$ there exist $x_1, \ldots, x_k$ in X, such that $\sum_{i=1}^k \|x_i\|^p \leq 1$, and

$$\Big(\sum_{i=1}^k \|T^* y_i^*\|^q\Big)^{\frac{1}{q}} \leq |\sum_{i=1}^k \langle T^* y_i^*, x_i\rangle| + \varepsilon.$$

Moreover, by [**87**, Prop. 1.d.2] we have

$$\begin{aligned} |\sum_{i=1}^k \langle T^* y_i^*, x_i\rangle| = |\sum_{i=1}^k \langle y_i^*, T x_i\rangle| &\leq \sum_{i=1}^k \langle |y_i^*|, T|x_i|\rangle \\ &\leq \langle \Big(\sum_{i=1}^k |y_i^*|^q\Big)^{\frac{1}{q}}, \Big(\sum_{i=1}^k |T x_i|^p\Big)^{\frac{1}{p}}\rangle \\ &\leq \langle |y^*|, \Big(\sum_{i=1}^k |T x_i|^p\Big)^{\frac{1}{p}}\rangle \leq 1 \end{aligned}$$

because $|y^*| \in \overline{S}^0$. Therefore, $\rho_{T^*}(y^*) \leq 1$ for every $y^* \in \overline{S}^0$. This finishes the proof.

$\square$

REMARK 8.2.4. Note that the equality $\overline{S}^0 = U$ proved above, yields in particular that U is weak*-closed. Hence, by the bipolar theorem it also holds that $\overline{S} = U_0$.

The main result of this section is the following.

THEOREM 8.2.5. *Let $T : X \to E$ be p-convex. By Theorem 8.1.4, T can be factored through $W_{T,p}$; moreover, since $T^* : E^* \to X^*$ is q-concave for $\frac{1}{p} + \frac{1}{q} = 1$, T^* can also be factored through $V_{T^*,q}$. It holds that:*

(1) *$V_{T^*,q}$ is lattice isometric to a sublattice of $(W_{T,p})^*$,*
(2) *$W_{T,p}$ is lattice isometric to a sublattice of $(V_{T^*,q})^*$.*

Moreover, under this identifications $V_{T^*,q}$ is always an ideal in $(W_{T,p})^*$, and if E is order continuous, then $W_{T,p}$ is an ideal of $(V_{T^*,q})^*$.

PROOF. We stick to the notation of Theorems 8.1.1 and 8.1.4. Let us consider the inclusion $\varphi : W_{T,p} \hookrightarrow E$. Hence, we also have $\varphi^* : E^* \to (W_{T,p})^*$. This allows us to define

$$\begin{array}{rccc} A: & E^*/\rho_{T^*}^{-1}(0) & \longrightarrow & (W_{T,p})^* \\ & y^* + \rho_{T^*}^{-1}(0) & \longmapsto & \varphi^*(y^*) \end{array}$$

Notice that for every $y^* \in E^*$ we have

$$\begin{aligned}\|\varphi^*(y^*)\|_{(W_{T,p})^*} &= \sup\{\langle \varphi^*(y^*), y\rangle : \|y\|_{W_{T,p}} \leq 1\}\\ &= \sup\{\langle y^*, \varphi(y)\rangle : y \in \overline{S}\}\\ &= \inf\{\lambda > 0 : y^* \in \lambda \overline{S}^0\}\\ &= \rho_{T^*}(y^*),\end{aligned}$$

by Lemma 8.2.3.

The equality $\|\varphi^*(y^*)\|_{(W_{T,p})^*} = \rho_{T^*}(y^*)$ implies that A is well defined. Moreover, it can be extended to an isometry between $V_{T^*,q}$ and $(W_{T,p})^*$.

Furthermore, since the unit ball of $W_{T,p}$ is a solid subset of E, then φ is interval preserving (i.e. $\varphi([0,x]) = [0, \varphi(x)]$ for $x \in W^+_{T,p}$). Thus, φ^* is a lattice homomorphism (see Theorem 1.5.11). Now, for $v \in V_{T^*,q}$, we can consider a sequence (y^*_n) in E^* such that $\lim\limits_n y^*_n + \rho^{-1}_{T^*}(0) = v$ in $V_{T^*,q}$. Hence, we have

$$A(|v|) = \lim_n A(|y^*_n| + \rho^{-1}_{T^*}(0)) = \lim_n \varphi^*(|y^*_n|) = \lim_n |\varphi^*(y^*_n)| = \lim_n |A(y^*_n + \rho^{-1}_{T^*}(0))| = |A(v)|.$$

Therefore, A is a lattice homomorphism, which implies that $V_{T^*,q}$ is lattice isometric to a sublattice of $(W_{T,p})^*$.

In order to see that $V_{T^*,q}$ is in fact an ideal of $(W_{T,p})^*$, let $y \in (W_{T,p})^*$ with $0 \leq y \leq A(x)$ for some $x \in V_{T^*,q}$. Notice that $x = \lim \phi(x^*_n)$ in $V_{T^*,q}$, where (x^*_n) belong to E^*. Thus,

$$A(x) = \lim A(\phi(x^*_n)) = \lim \varphi^*(x_n).$$

If we denote $y_n = y \wedge \varphi^*(x_n)$, then we clearly have that y_n tends to y in $(W_{T,p})^*$. Moreover, since φ is a lattice homomorphism, by Theorem 1.5.11, it follows that φ^* is interval preserving. Hence, since $0 \leq y_n \leq \varphi^*(x^*_n)$, for every $n \in \mathbb{N}$, there exists $u^*_n \in [0, x^*_n]$, such that $y_n = \varphi^*(u^*_n)$. Notice that $\varphi^*(u^*_n)$ tends to y in $(W_{T,p})^*$. In particular, we have $\rho_{T^*}(u^*_n - u^*_m) = \varphi^*(u^*_n - u^*_m) \to 0$ when $n, m \to \infty$, which yields that $\phi(u^*_n)$ tends to u^* in $V_{T^*,q}$. By construction, we obtain that $A(u^*) = y$, which implies that A is interval preserving. This shows that $V_{T^*,q}$ is an ideal of $(W_{T,p})^*$, as claimed.

On the other hand, we can also define a mapping $B : W_{T,p} \to (V_{T^*,q})^*$. Indeed, given $y \in \overline{S}$ and $y^* \in E^*$, since $\overline{S} = U_0$, we have

$$\langle y^*, \varphi(y)\rangle \leq \rho_{T^*}(y^*).$$

Therefore, for every $y \in W_{T,p}$ and $y^* \in E^*$ we have

$$\langle y^*, \varphi(y)\rangle \leq \rho_{T^*}(y^*)\|y\|_{W_{T,p}}.$$

Hence, there exists a unique element $B(y) \in (E^*/\rho^{-1}_{T^*}(0))^*$ such that

$$\langle y^* + \rho^{-1}_{T^*}(0), B(y)\rangle = \langle y^*, \varphi(y)\rangle$$

for every $y^* \in E^*$. Clearly, $B(y)$ is a linear functional which is continuous for the norm in $V_{T^*,q}$, thus, it can be extended to an element of $(V_{T^*,q})^*$, with $\|B(y)\|_{(V_{T^*,q})^*} \leq \|y\|_{W_{T,p}}$. Hence,

$B : W_{T,p} \to (V_{T^*,q})^*$ is a linear mapping which is bounded of norm ≤ 1. Moreover, for $y \in W_{T,p}$ we have

$$\begin{aligned}\|B(y)\|_{(V_{T^*,q})^*} &= \sup\{\langle v, B(y)\rangle : \|v\|_{V_{T^*,q}} \leq 1\} \\ &= \sup\{\langle y^*, \varphi(y)\rangle : \rho_{T^*}(y^*) \leq 1\}\end{aligned}$$

which is the value of the Minkowski functional of $U_0 = \overline{S}$ at $\varphi(y)$. Hence,

$$\|B(y)\|_{(V_{T^*,q})^*} = \inf\{\lambda \geq 0 : \varphi(y) \in \lambda\overline{S}\} = \|y\|_{W_{T,p}}.$$

This means that B is an isometry.

Moreover, for $y^* \in E^*_+$ and every $y \in W_{T,p}$ we have

$$\begin{aligned}\langle y^* + \rho_{T^*}^{-1}(0), |B(y)|\rangle &= \sup\{|\langle x^* + \rho_{T^*}^{-1}(0), B(y)\rangle| : |x^*| \leq y^*\} \\ &= \sup\{|\langle x^*, \varphi(y)\rangle| : |x^*| \leq y^*\} \\ &= \langle y^*, |\varphi(y)|\rangle,\end{aligned}$$

and since φ is a lattice homomorphism we have

$$\langle y^* + \rho_{T^*}^{-1}(0), |B(y)|\rangle = \langle y^*, \varphi(|y|)\rangle = \langle y^* + \rho_{T^*}^{-1}(0), B(|y|)\rangle.$$

Since this holds for every $y^* \in E^*_+$, we have that $|B(y)| = B(|y|)$. Therefore, B is a lattice homomorphism and the claimed result follows.

To prove the last statement, let $u \in (V_{T^*,q})^*$ such that $0 \leq u \leq B(y)$ for some $y \in W_{T,p}$. We consider $\phi : E^* \to V_{T^*,q}$ the operator in Theorem 8.1.1 induced by the quotient map. Since ϕ is positive, so is $\phi^* : (V_{T^*,q})^* \to E^{**}$. It holds that

$$\phi^*(u) \leq \phi^*(B(y)) = \varphi(y),$$

where $\varphi : W_{T,p} \to E$ is the operator given in Theorem 8.1.4. Indeed, for every $y^* \in E^*$ we have

$$\langle \phi^*(B(y)), y^*\rangle = \langle B(y), \phi^*(y^*)\rangle = \langle \varphi(y), y^*\rangle.$$

Hence, $\phi^*(u) \in [0, \varphi(y)]$ in E^{**}. However, if E is order continuous and $\varphi(y)$ belongs to E, then, by Theorem 1.2.2, we have $[0, \varphi(y)] \subset E$. Moreover, since φ is interval preserving, there exists $x \in [0, y]$ in $W_{T,p}$, such that $\phi^*(u) = \varphi(x)$. This implies that $u = B(x)$, which means that $W_{T,p}$ is an ideal in $(V_{T^*,q})^*$. $\square$

Notice that the isometries A and B given in the proof of Theorem 8.2.5 may not be surjective, as the following examples show.

EXAMPLE 8.2.6. Let $T : L_\infty(0,1) \hookrightarrow L_1(0,1)$ denote the formal inclusion. Clearly, for every $1 \leq p \leq \infty$, T is p-convex. First, notice that the set

$$S = \{f \in L_1(0,1) : |f| \leq \Big(\sum_{i=1}^n |Tf_i|^p\Big)^{\frac{1}{p}}, \sum_{i=1}^n \|f_i\|^p_{L_\infty} \leq 1\},$$

satisfies that $\overline{S} = \{f \in L_1(0,1) : \|f\|_{L_\infty} \leq 1\}$. This implies that

$$W_{T,p} = L_\infty(0,1).$$

On the other hand, if we consider the adjoint operator $T^* : L_1(0,1)^* \to L_\infty(0,1)^*$, which is p'-concave (for $\frac{1}{p} + \frac{1}{p'} = 1$), then for $f \in L_\infty(0,1) = L_1(0,1)^*$ we have

$$\begin{aligned} \|T^* f\|_{L_\infty^*} &= \sup\{\langle T^* f, g\rangle : \|g\|_{L_\infty} \leq 1\} \\ &= \sup\{\langle f, Tg\rangle : \|g\|_{L_\infty} \leq 1\} \\ &= \|f\|_{L_1}. \end{aligned}$$

From here, it follows that the expression

$$\rho_{T^*,p'}(f) = \sup\left\{\Big(\sum_{i=1}^n \|T^* f_i\|_{L_\infty^*}^{p'}\Big)^{\frac{1}{p'}} : \Big(\sum_{i=1}^n |f_i|^{p'}\Big)^{\frac{1}{p'}} \leq |f|\right\},$$

satisfies trivially $\|f\|_{L_1} \leq \rho_{T^*,p'}$. While on the other hand, for $f \in L_\infty(0,1)$ and $(f_i)_{i=1}^n$ with $(\sum_{i=1}^n |f_i|^{p'})^{1/p'} \leq |f|$ we have

$$\Big(\sum_{i=1}^n \|T^* f_i\|_{L_\infty^*}^{p'}\Big)^{\frac{1}{p'}} = \Big(\sum_{i=1}^n \|f_i\|_{L_1}^{p'}\Big)^{\frac{1}{p'}} \leq \Big\|\Big(\sum_{i=1}^n |f_i|^{p'}\Big)^{\frac{1}{p'}}\Big\|_{L_1} \leq \|f\|_{L_1}.$$

Thus, $\rho_{T^*,p'}(f) = \|f\|_{L_1}$, which implies that

$$V_{T^*,p'} = L_1(0,1).$$

Hence, the isometry $A : V_{T^*,p'} \to (W_{T,p})^*$ given in Theorem 8.2.5 is not surjective.

Example 8.2.7. Let $T : \ell_1 \hookrightarrow \ell_\infty$ denote the formal inclusion. Clearly, T is ∞-convex. Moreover, it is easy to see that the set

$$S = \{y \in \ell_\infty : |y| \leq \bigvee_{i=1}^n |y_i|, \bigvee_{i=1}^n \|y_i\|_{\ell_1} \leq 1\},$$

satisfies $\overline{S} = B_{c_0}$. Hence,

$$W_{T,\infty} = c_0.$$

On the other hand, let $T^* : \ell_\infty^* \to \ell_1^*$ be the adjoint operator, which is 1-convex. It is well known that $\ell_\infty^* = \ell_1^{**}$ can be decomposed as

$$\ell_1^{**} = J(\ell_1) \oplus J(\ell_1)^\perp,$$

where $J(\ell_1)$ denotes the canonical image of ℓ_1 in its bidual, and $J(\ell_1)^\perp$ its disjoint complement.

Notice that every $y \in J(\ell_1)^\perp$, viewed as an element of ℓ_∞^*, satisfies $y|_{c_0} = 0$. Indeed, for every $n \in \mathbb{N}$, let e_n denote the sequence formed by zeros with 1 in the n^{th} entry. For $y \in J(\ell_1)^\perp$, by disjointness we have

$$\begin{aligned} 0 = \langle |y| \wedge J(e_n), e_n\rangle &= \inf\{\langle |y|, x\rangle + \langle e_n, z\rangle : x, z \in \ell_\infty^+, x + z = e_n\} \\ &= \inf\{\lambda\langle |y|, e_n\rangle + 1 - \lambda : \lambda \in [0,1]\} \\ &= \langle |y|, e_n\rangle, \end{aligned}$$

for every $n \in \mathbb{N}$, which clearly implies $y|_{c_0} = 0$. In particular, for $y \in J(\ell_1)^\perp$ we have

$$\begin{aligned} \|T^*(y)\| &= \sup\{\langle T^*(y), x\rangle : x \in \ell_1, \|x\|_{\ell_1} \le 1\} \\ &= \sup\{\langle y, Tx\rangle : x \in \ell_1, \|x\|_{\ell_1} \le 1\} \\ &= 0, \end{aligned}$$

since $Tx \in c_0 \subset \ell_\infty$ for every $x \in \ell_1$. Therefore, for $y \in J(\ell_1)^\perp$, since $J(\ell_1)^\perp$ is solid, we have

$$\rho_{T^*,1}(y) = \sup\left\{\sum_{i=1}^n \|T^* y_i\|_{\ell_\infty} : \sum_{i=1}^n |y_i| \le |y|\right\} = 0.$$

While for $y \in J(\ell_1)$ we have

$$\rho_{T^*}(y) = \sup\left\{\sum_{i=1}^n \|T^* y_i\|_{\ell_\infty} : \sum_{i=1}^n |y_i| \le |y|\right\} = \|y\|_{\ell_1}.$$

Hence,

$$V_{T^*,1} = \ell_1.$$

This implies that the isometry $B : W_{T,\infty} \to (V_{T^*,1})^*$ is not surjective, since c_0 is clearly a proper subspace of ℓ_∞.

EXAMPLE 8.2.8. Let $T : \ell_1 \to c$ be the operator defined by

$$T(x_1, x_2, \ldots, x_n, \ldots) = (x_1, x_1 + x_2, \ldots, \sum_{k=1}^n x_k, \ldots),$$

where c denotes the space of convergent sequences of real numbers. Clearly, T is positive and p-convex for every $1 \le p \le \infty$. Notice that the set

$$S = \{y \in c : |y| \le \Big(\sum_{i=1}^n |Ty_i|^p\Big)^{\frac{1}{p}}, \sum_{i=1}^n \|y_i\|_{\ell_1}^p \le 1\},$$

contains the constant sequence equal to one, so since S is solid, $\overline{S}$ coincides with the closed unit ball of c. Hence,

$$W_{T,p} = c.$$

Now, we can consider the adjoint operator $T^* : c^* \to \ell_1^*$, which is clearly q-concave for every $1 \le q \le \infty$. Recall that c^* can be identified with the space $\ell_1(\mathbb{N})$ in the following way: for an element $x = (x_0, x_1, \ldots)$ in $\ell_1(\mathbb{N})$ and another element $y = (y_1, y_2, \ldots)$ in c, we set

$$\langle x, y\rangle = x_0 \lim y_n + \sum_{n=1}^\infty x_n y_n.$$

Therefore, for a positive element $x \in c^*$ we have

$$\begin{aligned} \|T^* x\|_{\ell_1^*} &= \sup\{\langle T^* x, y\rangle : \|y\|_{\ell_1} \le 1\} \\ &= \sup\{\langle x, Ty\rangle : \|y\|_{\ell_1} \le 1\} \\ &\ge \langle x, Te_1\rangle = \textstyle\sum_{n=0}^\infty x_n = \|x\|_{c^*}, \end{aligned}$$

where e_1 denotes the sequence formed by 1 in the first position followed by zeros. Since $\|T\| \leq 1$, it holds that $\|T^*x\|_{\ell_1^*} = \|x\|_{c^*}$ for every positive $x \in c^*$. This implies that

$$\rho_{T^*,q}(x) = \sup\left\{ \Big(\sum_{i=1}^n \|T^*x_i\|_{\ell_1^*}^q \Big)^{\frac{1}{q}} : \Big(\sum_{i=1}^n |x_i|^q \Big)^{\frac{1}{q}} \leq |x| \right\} = \|x\|_{c^*},$$

which yields that

$$V_{T^*,q} = c^*.$$

Notice that, in particular, the operator $\varphi : c \hookrightarrow c$ defined in Theorem 8.1.4 coincides with the identity on c, and the operator $\phi : c^* \to c^*$ defined in Theorem 8.1.1 coincides as well with the identity on c^*. Now, by the definition of the operator $B : W_{T,p} \to (V_{T^*,q})^*$ in Theorem 8.2.5, it follows that for every $y \in c$ and $y^* \in c^*$ we have

$$\langle B(y), y^* \rangle = \langle B(y), \phi(y^*) \rangle = \langle \varphi(y), \phi(y^*) \rangle = \langle y, y^* \rangle.$$

Hence, $B = J$, where $J : c \to c^{**}$ denotes the canonical inclusion of c into its bidual. Now since c is not order continuous, it follows that $B(c)$ is not an ideal in $(V_{T^*,q})^*$, and this shows that the last statement of Theorem 8.2.5 does not hold without the assumption of order continuity on E.

3. Complex interpolation of Banach lattices

So far, we have been implicitly considering real Banach lattices. Throughout this section we will be using the complex interpolation method for Banach lattices, hence we need to consider complex Banach lattices. However, our final results, given in the next section, remain true for real Banach lattices by means of complexifying and taking real parts after the interpolation constructions.

Recall that a compatible pair of Banach spaces (X_0, X_1) is a pair of Banach spaces X_0, X_1 which are continuously included in a topological vector space X. In the context of Banach lattices, we will say that two Banach lattices X_0, X_1 form a *compatible pair of Banach lattices* (X_0, X_1) if there exists a Riesz space X, and inclusions $i_j : X_j \hookrightarrow X$ which are continuous interval preserving lattice homomorphisms, for $j = 0, 1$. In this way, X_0 and X_1 can be considered as ideals of X.

Given a compatible pair of Banach lattices, (X_0, X_1), for each $\theta \in [0, 1]$ we will consider three different constructions:

(1) $X_0^{1-\theta} X_1^{\theta}$ denotes the space of elements $x \in X_0 + X_1$ such that

$$|x| \leq \lambda |x_0|^{1-\theta} |x_1|^{\theta},$$

for some $\lambda > 0$, $x_0 \in X_0$ and $x_1 \in X_1$, with $\|x_0\|_{X_0} \leq 1$, $\|x_1\|_{X_1} \leq 1$. The norm in this space is given by

$$\|x\|_{X_0^{1-\theta} X_1^{\theta}} = \inf\{\lambda > 0 : |x| \leq \lambda |x_0|^{1-\theta} |x_1|^{\theta} \text{ for some } \|x_0\|_{X_0} \leq 1,\ \|x_1\|_{X_1} \leq 1\}.$$

(2) $[X_0, X_1]_\theta$ denotes the space of elements $x \in X_0 + X_1$ which can be represented as $x = f(\theta)$ for some $f \in \mathcal{F}(X_0, X_1)$. Recall that $\mathcal{F}(X_0, X_1)$ denotes the linear space of functions $f(z)$ defined in the strip $\Pi = \{z \in \mathbb{C} : 0 \leq \Re z \leq 1\}$, with values in the space $X_0 + X_1$, such that
   - $f(z)$ is continuous and bounded for the norm of $X_0 + X_1$ in Π,
   - $f(z)$ is analytic for the norm of $X_0 + X_1$ in the interior of Π,
   - $f(it)$ assumes values in X_0 and is continuous and bounded for the norm of X_0, while $f(1+it)$ assumes values in X_1 and is continuous and bounded for the norm of X_1.

   In $\mathcal{F}(X_0, X_1)$ we can consider the norm $\|f\|_{\mathcal{F}(X_0,X_1)} = \max\{\sup_t \|f(it)\|_{X_0}, \sup_t \|f(1+it)\|_{X_1}\}$. The norm in $[X_0, X_1]_\theta$ is given by
$$\|x\|_{[X_0,X_1]_\theta} = \inf\{\|f\|_{\mathcal{F}(X_0,X_1)} : f(\theta) = x\}.$$

(3) $[X_0, X_1]^\theta$ denotes the space of elements $x \in X_0 + X_1$ which can be represented as $x = f'(\theta)$ for some $f \in \overline{\mathcal{F}}(X_0, X_1)$. Now $\overline{\mathcal{F}}(X_0, X_1)$ denotes the linear space of functions $f(z)$ defined in the strip $\Pi = \{z \in \mathbb{C} : 0 \leq \Re z \leq 1\}$, with values in the space $X_0 + X_1$, such that
   - $\|f(z)\|_{X_0+X_1} \leq c(1 + |z|)$ for some constant $c > 0$ and for every $z \in \Pi$,
   - $f(z)$ is continuous in Π and analytic in the interior of Π for the norm of $X_0 + X_1$,
   - $f(it_1) - f(it_2)$ has values in X_0 and $f(1 + it_1) - f(1 + it_2)$ in X_1 for any $-\infty < t_1 < t_2 < \infty$ and
$$\|f\|_{\overline{\mathcal{F}}(X_0,X_1)} = \max\left\{\sup_{t_1,t_2}\left\|\frac{f(it_2) - f(it_1)}{t_2 - t_1}\right\|_{X_0}, \sup_{t_1,t_2}\left\|\frac{f(1 + it_2) - f(1 + it_1)}{t_2 - t_1}\right\|_{X_1}\right\} < \infty.$$

   The norm in $[X_0, X_1]^\theta$ is given by
$$\|x\|_{[X_0,X_1]^\theta} = \inf\{\|f\|_{\overline{\mathcal{F}}(X_0,X_1)} : f'(\theta) = x\}.$$

These spaces are Banach lattices provided that (X_0, X_1) is a compatible pair of Banach lattices. Moreover, $[X_0, X_1]_\theta$ and $[X_0, X_1]^\theta$ are always interpolation spaces, while $X_0^{1-\theta}X_1^\theta$ is an intermediate space between X_0 and X_1 which is an interpolation space under certain extra assumptions. We refer to [**28**], [**80**], [**90**], and [**91**] for more information on these spaces.

Next theorem, extends a result of V. A. Sestakov [**126**], which was originally proved for the case of Banach lattices of measurable functions, showing how these constructions are related to each other.

THEOREM 8.3.1. *Let X_0, X_1 be a compatible pair of Banach lattices. For every $\theta \in (0, 1)$ it holds that*
$$[X_0, X_1]_\theta = \overline{X_0 \cap X_1}^{[X_0,X_1]_\theta} = \overline{X_0 \cap X_1}^{X_0^{1-\theta}X_1^\theta},$$
with equality of norms.

Before the proof of Theorem 8.3.1 we need the following.

Lemma 8.3.2. *Let $F : \Pi \to X_0 + X_1$ be a function in $\mathcal{F}(X_0, X_1)$ of the form*

$$F(z) = e^{\delta z^2} \sum_{j=1}^{N} x_j e^{\lambda_j z},$$

where $\delta > 0$, the λ_j are real, and $x_j \in X_0 \cap X_1$. It holds that

$$\|F(\theta)\|_{X_0^{1-\theta} X_1^{\theta}} \leq \|F\|_{\mathcal{F}(X_0,X_1)}.$$

Proof. Let $F : \Pi \to X_0 \cap X_1$ be a function in $\mathcal{F}(X_0, X_1)$ of the form

$$F(z) = e^{\delta z^2} \sum_{j=1}^{N} x_j e^{\lambda_j z},$$

where $\delta > 0$, the λ_j are real, and $x_j \in X_0 \cap X_1$. Let $x = \sum_{j=1}^{N} |x_j|$. We can consider the principal (non closed) ideal in $X_0 \cap X_1$ generated by x, equipped with the norm that makes it isomorphic to a $C(K)$ space for some compact K (i.e. $\|y\| = \inf\{\lambda > 0 : |y| \leq \lambda x\}$. We clearly have the inclusions

$$C(K) \hookrightarrow X_0 \cap X_1 \hookrightarrow X_0 + X_1,$$

which are bounded. Moreover, since $|x_j| \leq x$, we have $x_j \in C(K)$, so we can consider

$$F(\omega, z) = e^{\delta z^2} \sum_{j=1}^{N} x_j(\omega) e^{\lambda_j z},$$

as a function of $\omega \in K$, and $z \in \Pi$. For each $z \in \Pi$, $F(\cdot, z)$ belongs to $C(K)$. Hence, applying [**28**, 9.4, ii)], for any $\omega \in K$ we have

$$|F(\omega,\theta)| \leq \left[\frac{1}{1-\theta} \int_{-\infty}^{+\infty} |F(\omega, it)| \mu_0(\theta,t) dt\right]^{1-\theta} \left[\frac{1}{\theta} \int_{-\infty}^{+\infty} |F(\omega, 1+it)| \mu_1(\theta,t) dt\right]^{\theta},$$

where μ_0 and μ_1 are the Poisson kernels for the strip Π, given by

$$\mu_0(\theta,t) = \frac{e^{-\pi t} \sin \pi\theta}{\sin^2 \pi\theta + [\cos \pi\theta - e^{-\pi t}]^2} \qquad \mu_1(\theta,t) = \frac{e^{-\pi t} \sin \pi\theta}{\sin^2 \pi\theta + [\cos \pi\theta + e^{-\pi t}]^2},$$

(see [**28**]). Hence, setting

$$g(\omega) = \frac{1}{1-\theta} \int_{-\infty}^{+\infty} |F(\omega, it)| \mu_0(\theta,t) dt,$$

and

$$h(\omega) = \frac{1}{\theta} \int_{-\infty}^{+\infty} |F(\omega, 1+it)| \mu_1(\theta,t) dt,$$

we find that g and h belong to $C(K)$. Indeed, for any ω_1, ω_2 in K, we have

$$\begin{aligned}
|g(\omega_1) - g(\omega_2)| &\leq \tfrac{1}{1-\theta} \int_{-\infty}^{+\infty} |F(\omega_1, it) - F(\omega_2, it)| \mu_0(\theta, t) dt \\
&\leq \tfrac{1}{1-\theta} \int_{-\infty}^{+\infty} \left| \sum_{j=1}^{N} (x_j(\omega_1) - x_j(\omega_2)) e^{i\lambda_j t} \right| e^{-\delta t^2} \mu_0(\theta, t) dt \\
&\leq \tfrac{1}{1-\theta} \int_{-\infty}^{+\infty} \sum_{j=1}^{N} |x_j(\omega_1) - x_j(\omega_2)| e^{-\delta t^2} \mu_0(\theta, t) dt \\
&= \sum_{j=1}^{N} |x_j(\omega_1) - x_j(\omega_2)|.
\end{aligned}$$

This inequality together with the fact that x_j belongs to $C(K)$ for $j = 1, \ldots, N$, proves that $g \in C(K)$. The proof for h is identical.

Moreover,

$$\begin{aligned}
\|g\|_{X_0} &= \left\| \tfrac{1}{1-\theta} \int_{-\infty}^{+\infty} |F(\omega, it)| \mu_0(\theta, t) dt \right\|_{X_0} \\
&\leq \tfrac{1}{1-\theta} \int_{-\infty}^{+\infty} \|F(\omega, it)\|_{X_0} \mu_0(\theta, t) dt \\
&\leq \|F\|_{\mathcal{F}(X_0, X_1)} \tfrac{1}{1-\theta} \int_{-\infty}^{+\infty} \mu_0(\theta, t) dt \\
&= \|F\|_{\mathcal{F}(X_0, X_1)},
\end{aligned}$$

and similarly

$$\|h\|_{X_1} \leq \|F\|_{\mathcal{F}(X_0, X_1)}.$$

Therefore,

$$\|F(\theta)\|_{X_0^{1-\theta} X_1^{\theta}} \leq \|F\|_{\mathcal{F}(X_0, X_1)}.$$

And the proof is finished. □

Proof of Theorem 8.3.1. If x is an element in $\overline{X_0 \cap X_1}^{[X_0, X_1]_\theta}$, by the definition of the norm in $[X_0, X_1]_\theta$, for every $\varepsilon > 0$, we can take F in $\mathcal{F}(X_0, X_1)$, such that $F(\theta) = x$ and

$$\|F\|_{\mathcal{F}(X_0, X_1)} \leq \|x\|_{[X_0, X_1]_\theta} + \varepsilon.$$

By [**80**, Chapter IV, Thm. 1.1], we can consider a sequence (F_n) in $\mathcal{F}(X_0, X_1)$, of elements of the form

$$e^{\delta z^2} \sum_{j=1}^{N} x_j e^{\lambda_j z},$$

where $x_j \in X_0 \cap X_1$ and $\lambda_j \in \mathbb{R}$, such that $\|F - F_n\|_{\mathcal{F}(X_0, X_1)} \to 0$. Then we have

$$\|F_n(\theta) - x\|_{[X_0, X_1]_\theta} = \|F_n(\theta) - F(\theta)\|_{[X_0, X_1]_\theta} \leq \|F_n - F\|_{\mathcal{F}(X_0, X_1)} \to 0.$$

By Lemma 8.3.2, for $n, m \in \mathbb{N}$ we have

$$\|F_n(\theta) - F_m(\theta)\|_{X_0^{1-\theta} X_1^{\theta}} \leq \|F_n - F_m\|_{\mathcal{F}(X_0, X_1)} \to 0,$$

when $n, m \to \infty$, and

$$\|F_n(\theta)\|_{X_0^{1-\theta} X_1^{\theta}} \leq \|F_n\|_{\mathcal{F}(X_0, X_1)} \to \|F\|_{\mathcal{F}(X_0, X_1)} \leq \|x\|_{[X_0, X_1]_\theta} + \varepsilon.$$

Therefore, $F_n(\theta)$ also converges to a limit in $X_0^{1-\theta}X_1^\theta$ of norm not exceeding $\|x\|_{[X_0,X_1]_\theta}+\varepsilon$. However, since $X_0^{1-\theta}X_1^\theta$ and $[X_0,X_1]_\theta$ are both continuously embedded in X_0+X_1, it follows that x is also the limit of $F_n(\theta)$ for the norm of $X_0^{1-\theta}X_1^\theta$. Hence, $x\in X_0^{1-\theta}X_1^\theta$ and $\|x\|_{X_0^{1-\theta}X_1^\theta}\leq \|x\|_{[X_0,X_1]_\theta}+\varepsilon$. Since this is true for all $\varepsilon>0$, we have

$$\|x\|_{X_0^{1-\theta}X_1^\theta}\leq \|x\|_{[X_0,X_1]_\theta}.$$

We will show now that for $x\in X_0^{1-\theta}X_1^\theta$,

$$\|x\|_{[X_0,X_1]^\theta}\leq \|x\|_{X_0^{1-\theta}X_1^\theta},$$

that is, the inclusion mapping

$$X_0^{1-\theta}X_1^\theta\hookrightarrow [X_0,X_1]^\theta$$

is bounded with norm smaller than or equal to one. Indeed, let $x\in X_0^{1-\theta}X_1^\theta$ be such that $\|x\|_{X_0^{1-\theta}X_1^\theta}\leq 1$. Then for every $\varepsilon>0$ we have $g\in X_0^+$, and $h\in X_1^+$ such that $\|g\|_{X_0}\leq 1$, $\|h\|_{X_1}\leq 1$, and $|x|\leq (1+\varepsilon)g^{1-\theta}h^\theta$ in X_0+X_1.

Now, if $\mathcal{I}$ denotes the (non closed) order ideal generated by $g\vee h$ in X_0+X_1, then $\mathcal{I}$ can be viewed as a space $C(K)$ over some compact Hausdorff space K. Since $|x|\leq (1+\varepsilon)g^{1-\theta}h^\theta$ in X_0+X_1, we can consider

$$f(t)=\frac{x(t)}{g^{1-\theta}(t)h^\theta(t)},$$

which is well defined for all $t\in K$ such that $g(t)h(t)\neq 0$. This allows us to define

$$F(t,z)=\begin{cases} f(t)g(t)^{1-z}h(t)^z & \text{if } g(t)h(t)\neq 0,\\ 0 & \text{in any other case.}\end{cases}$$

Note that, since $g,h\leq g\vee h$, we have $\|g\|_{C(K)},\|h\|_{C(K)}\leq 1$; hence, for every $z\in\Pi$,

$$\sup_{t\in K}|F(t,z)|\leq 1+\varepsilon.$$

Clearly, for $z\in\mathring{\Pi}$ we can consider $\phi(z)\in C(K)$ defined by $\phi(z)(t)=F(t,z)$. We claim that

$$\phi:\mathring{\Pi}\to C(K)$$

is analytic. Indeed, note that for every $t\in K$ fixed, $F(t,\cdot)$ is analytic on $\mathring{\Pi}$. Hence,

$$\phi(z)(t)=F(t,z)=\frac{1}{2\pi i}\int_\gamma \frac{F(t,\xi)}{\xi-z}d\xi$$

for every $t\in K$, and for any circumference γ of center z contained in $\mathring{\Pi}$. Since this identity is valid for every $t\in K$, we get

$$\phi(z)=\frac{1}{2\pi i}\int_\gamma\frac{\phi(\xi)}{\xi-z}d\xi.$$

This means that $\phi:\mathring{\Pi}\to C(K)$ is analytic.

Now, let us define

$$F_1(t,z)=\int_{\gamma_z}F(t,\xi)d\xi,$$

for $t \in K$ and $z \in \Pi$, where γ_z is any path joining $\frac{1}{2}$ and z, with all its points except possibly z inside $\overset{\circ}{\Pi}$. Note that since F is analytic in $\overset{\circ}{\Pi}$ and $\sup_{t \in K} |F(t,z)| \leq 1+\varepsilon$, for all $z \in \Pi$, F_1 is independent of the path γ_z, so it is well defined. Therefore, we can define $\phi_1 : \Pi \to B(K)$, where $B(K)$ denotes the bounded measurable functions on K, by

$$\phi_1(z) = F_1(\cdot, z) = \int_{\gamma_z} \phi(\xi) d\xi,$$

for $z \in \Pi$. Since $\phi : \overset{\circ}{\Pi} \to C(K)$ is analytic, so is ϕ_1 on $\overset{\circ}{\Pi}$, and clearly $\phi_1(\overset{\circ}{\Pi}) \subseteq C(K)$. Moreover,

$$\|\phi_1(z) - \phi_1(z')\|_{C(K)} \leq (1+\varepsilon)|z - z'|$$

for $z, z' \in \overset{\circ}{\Pi}$. Hence, for any z in the border of Π, let $z_n \in \overset{\circ}{\Pi}$ be such that $z_n \to z$. Since $\|\phi_1(z_n) - \phi_1(z_m)\|_{C(K)} \leq (1+\varepsilon)|z_n - z_m|$, we get that $\phi_1(z_n)$ is a Cauchy sequence in $C(K)$, hence convergent to some $\psi \in C(K)$. In particular, for every $t \in K$, $\phi_1(z_n)(t) \to \psi(t)$ and since $\phi_1(z_n)(t) = \int_{\gamma_{z_n}} F(t,\xi) d\xi$ we get that $\psi(t) = \int_{\gamma_z} F(t,\xi) d\xi$. This implies that $\phi_1(\Pi) \subseteq C(K)$, and

$$\|\phi_1(z) - \phi_1(z')\|_{C(K)} \leq (1+\varepsilon)|z - z'|$$

for $z, z' \in \Pi$. Thus $\phi_1 : \Pi \to C(K)$ is continuous.

Now, for $u, v \in \mathbb{R}$, and for every $\alpha \in (0,1)$ let γ_α be the path formed by the rectilinear segments $[iu, \alpha + iu]$, $[\alpha + iu, \alpha + iv]$ and $[\alpha + iv, iv]$. Hence, for every $\alpha \in (0,1)$ and $t \in K$ such that $g(t)h(t) \neq 0$

$$\begin{aligned}
|F_1(t,iu) - F_1(t,iv)| &\leq \textstyle\int_{\gamma_\alpha} |F(t,\xi)| d\xi \\
&= \textstyle\int_{[iu,\alpha+iu]} |F(t,\xi)| d\xi + \int_{[\alpha+iu,\alpha+iv]} |F(t,\xi)| d\xi + \int_{[\alpha+iv,iv]} |F(t,\xi)| d\xi \\
&\leq \alpha(1+\varepsilon) + (1+\varepsilon) g(t)^{1-\alpha} h(t)^\alpha |u - v| + \alpha(1+\varepsilon) \\
&\leq (g(t)^{1-\alpha} h(t)^\alpha |u-v| + 2\alpha)(1+\varepsilon).
\end{aligned}$$

Thus, letting $\alpha \to 0^+$, we get

$$\frac{|F_1(t,iu) - F_1(t,iv)|}{|u-v|} \leq (1+\varepsilon) g(t)$$

for $t \in K$ with $g(t)h(t) \neq 0$. Since the same inequality holds trivially if $g(t) = 0$, we have that

$$\frac{|\phi_1(iu) - \phi_1(iv)|}{|u-v|} \leq (1+\varepsilon) g$$

in X_0. Analogously we have

$$\frac{|\phi_1(1+iu) - \phi_1(1+iv)|}{|u-v|} \leq (1+\varepsilon) h$$

in X_1. Since X_0 and X_1 are order ideals, it clearly follows that $\frac{|\phi_1(iu) - \phi_1(iv)|}{|u-v|} \in X_0$ and $\frac{|\phi_1(1+iu) - \phi_1(1+iv)|}{|u-v|} \in X_1$.

Therefore, since

$$\left.\frac{d\phi_1}{dz}\right|_{z=\theta} = \phi(\theta) = x,$$

we get that

$$\begin{aligned}\|x\|_{[X_0,X_1]^\theta} &\leq \|\phi_1\|_{\overline{\mathcal{F}}} \\ &= \max\left[\sup_{u,v}\left\|\tfrac{|\phi_1(iu)-\phi_1(iv)|}{|u-v|}\right\|_{X_0}, \sup_{u,v}\left\|\tfrac{|\phi_1(1+iu)-\phi_1(1+iv)|}{|u-v|}\right\|_{X_1}\right] \\ &\leq 1+\varepsilon.\end{aligned}$$

Since this holds for every $\varepsilon > 0$, we have proved that

$$X_0^{1-\theta}X_1^\theta \hookrightarrow [X_0, X_1]^\theta$$

is continuous with norm smaller than or equal to one.

In particular, we have

$$\overline{X_0 \cap X_1}^{X_0^{1-\theta}X_1^\theta} \hookrightarrow \overline{X_0 \cap X_1}^{[X_0,X_1]^\theta}$$

with norm smaller than one. Now, by [**21**], we have

$$\overline{X_0 \cap X_1}^{[X_0,X_1]^\theta} = \overline{X_0 \cap X_1}^{[X_0,X_1]_\theta},$$

with equality of norms. This proves the theorem. □

4. Factorization for operators which are both p-convex and q-concave

In section 1, it was proved that every p-convex (resp. q-concave) operator factors in a nice way through a p-convex (resp. q-concave) Banach lattice. However, if the operator is both p-convex and q-concave, can this factorization be improved? It is well-known that if E is a q-concave Banach lattice and F a p-convex Banach lattice, then every operator $T : E \to F$ is both p-convex and q-concave. Moreover, if an operator $T : E \to F$ between Banach lattices, has a factorization of the following form

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "\phi"'] & F \\ E_1 \arrow[r, "R"] & F_1 \arrow[u, "\psi"'] \end{tikzcd}$$

where ϕ and ψ are positive, E_1 q-concave, and F_1 p-convex, then T is both p-convex and q-concave [**81**]. Hence, the following question is natural:

Can a p-convex and q-concave operator $T : E \to F$ factor always in this way?

According to Theorems 8.1.1 and 8.1.4, this is true if the operator $T : E \to F$ can be written as $T = T_1 \circ T_2$, where T_1 is p-convex, and T_2 is q-concave. In fact, it turns out that the previous question is equivalent to the following one.

If $T : E \to F$ is p-convex and q-concave, do there exist operators T_1 and T_2, such that $T = T_1 \circ T_2$, where T_1 is p-convex, and T_2 is q-concave?

In general the answer to this question is negative, as the following shows.

Proposition 8.4.1. *Let $T : E \to F$ be an operator from ∞-convex Banach lattice (an AM-space) E to a q-concave Banach lattice F ($q < \infty$). If T can be factored as $T = SR$, with R q-concave and S ∞-convex, then T is compact.*

PROOF. If $T : E \to F$ has such a factorization, then by Theorems 8.1.1 and 8.1.4 we must have

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "\phi"'] & F \\ V \arrow[r, "T_1"] & W \arrow[u, "\varphi"'] \end{tikzcd}$$

where V is q-concave, W an AM-space, and ϕ, φ lattice homomorphisms.

Since ϕ and φ are positive and take values in q-concave Banach lattices, by Theorem 1.5.10, they are q-concave operators. Moreover, since both operators are defined on AM-spaces, by [**87**, Theorem 1.d.10], ϕ and φ are q-absolutely summing.

Therefore, $T = \varphi \circ (T_1 \circ \phi)$ is a product of two q-absolutely summing operators, hence compact, because every q-absolutely summing operator is weakly compact and Dunford-Pettis (cf. [**6**, Cor. 8.2.15]). □

In particular, let $T : C(0,1) \hookrightarrow L_q(0,1)$ denote the formal inclusion. Since T is positive, by Theorem 1.5.10, T is q-concave and ∞-convex. If we could factor T as $T = T_1 \circ T_2$, where T_1 is ∞-convex, and T_2 is q-concave, then by Proposition 8.4.1, T would be compact, which is not the case.

By duality, Proposition 8.4.1 immediately yields the following.

COROLLARY 8.4.2. *Let $T : E \to F$ be an operator from a p-convex Banach lattice E to a* 1*-concave Banach lattice (an AL-space) F. If T can be factored as $T = SR$, with R* 1*-concave and S p-convex, then T is compact.*

Despite this fact, as an application of the results of section 3, we have the following factorization for operators which are both p-convex and q-concave.

THEOREM 8.4.3. *Let E and F be Banach lattices, and let $T : E \to F$ be both p-convex and q-concave. For every $\theta \in (0,1)$ we can factor T in the following way*

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "\phi_\theta"'] & F \\ E_\theta \arrow[r, "R_\theta"] & F_\theta \arrow[u, "\varphi_\theta"'] \end{tikzcd}$$

where ϕ_θ and φ_θ are interval preserving lattice homomorphisms, E_θ is $(\frac{q}{1-\theta})$-concave, and F_θ is $(\frac{p}{\theta+(1-\theta)p})$-convex.

Before the proof, we need some lemmas first. Recall, that given a Banach space X, and $1 \le p < \infty$, $\ell_p(X)$ denotes the space of sequences (x_n) of X such that $(\|x_n\|_X)$ belongs to ℓ_p. This is a Banach space with the norm

$$\|(x_n)\|_{\ell_p(X)} = \Big(\sum_{n=1}^{\infty} \|x_n\|_X^p\Big)^{\frac{1}{p}}.$$

In order to keep a unified notation, for $p=\infty$, $\ell_\infty(X)$ will denote the space of sequences (x_n) of X such that $(\|x_n\|_X)$ belongs to c_0, equipped with the norm

$$\|(x_n)\|_{\ell_\infty(X)} = \sup \|x_n\|_X.$$

Notice that this space is usually denoted $c_0(X)$ in the literature.

Analogously, given a Banach lattice E, and $1\le p\le\infty$, $E(\ell_p)$ denotes the completion of the space of eventually null sequences (x_n) of E under the norm

$$\|(x_n)\|_{E(\ell_p)} = \begin{cases} \sup\limits_n \Big\| \Big(\sum\limits_{i=1}^n |x_i|^p\Big)^{\frac{1}{p}} \Big\|_E & \text{if } 1\le p<\infty, \\ \sup\limits_n \Big\| \bigvee\limits_{i=1}^n |x_i| \Big\|_E & \text{if } p=\infty. \end{cases}$$

LEMMA 8.4.4. *Let (F,G) be a compatible pair of Banach lattices, let $r,s\in[1,+\infty]$ and $\theta\in(0,1)$. For $\frac{1}{t}=\frac{1-\theta}{r}+\frac{\theta}{s}$, we have:*

(1) $\ell_r(F)^{1-\theta}\ell_s(G)^\theta = \ell_t(F^{1-\theta}G^\theta)$, *with equality of norms.*

(2) $\overline{\ell_r(F)\cap\ell_s(G)}^{\ell_r(F)^{1-\theta}\ell_s(G)^\theta} = \ell_t(\overline{F\cap G}^{F^{1-\theta}G^\theta})$.

PROOF. We present the proof in the case $r,s<\infty$. In the remaining cases, the argument works as well replacing the expression $(\sum|\cdot|^\infty)^{1/\infty}$ by $\bigvee|\cdot|$.

(1) Let us proof first the "$\subseteq$" part. Let $x=(x_n)\in\ell_r(F)^{1-\theta}\ell_s(G)^\theta$ with $\|x\|_{\ell_r(F)^{1-\theta}\ell_s(G)^\theta}\le 1$. This means that for every $\lambda>1$ we have $|x|\le\lambda|y|^{1-\theta}|z|^\theta$ with $y=(y_n)$, $z=(z_n)$, such that

$$\sum_{n=1}^\infty \|y_n\|_F^r \le 1,$$

and

$$\sum_{n=1}^\infty \|z_n\|_G^s \le 1.$$

Since

$$|x_n| \le \lambda|y_n|^{1-\theta}|z_n|^\theta = \lambda\|y_n\|_F^{1-\theta}\|z_n\|_G^\theta \Big(\frac{|y_n|}{\|y_n\|_F}\Big)^{1-\theta}\Big(\frac{|z_n|}{\|z_n\|_G}\Big)^\theta,$$

by the definition of the norm in $F^{1-\theta}G^\theta$, and Hölder's inequality we get

$$\begin{aligned} \|x\|_{\ell_t(F^{1-\theta}G^\theta)} &\le \Big(\sum_n (\lambda\|y_n\|_F^{1-\theta}\|z_n\|_G^\theta)^t\Big)^{\frac{1}{t}} \\ &\le \lambda\Big(\sum_n\|y_n\|_F^r\Big)^{\frac{1-\theta}{r}}\Big(\sum_n\|z_n\|_F^s\Big)^{\frac{\theta}{s}} \le \lambda. \end{aligned}$$

Since this is true for every $\lambda>1$, we get that x belongs to $\ell_t(F^{1-\theta}G^\theta)$, with $\|x\|_{\ell_t(F^{1-\theta}G^\theta)}\le 1$.

Now, for the "$\supseteq$" part, notice first that if x belongs to $\ell_t(F^{1-\theta}G^\theta)$ and $\|x\|_{\ell_t(F^{1-\theta}G^\theta)}<1$, then we have $|x_n|\le\lambda_n|y_n|^{1-\theta}|z_n|^\theta$ with $\|y_n\|_F\le 1$, $\|z_n\|_G\le 1$ and $\Big(\sum_{n=1}^\infty \lambda_n^t\Big)^{\frac{1}{t}}$. It is known that $\ell_r^{1-\theta}\ell_s^\theta=\ell_t$, hence there exist sequences (a_n) and (b_n) such that $\lambda_n=a_n^{1-\theta}b_n^\theta$ with $\Big(\sum_{n=1}^\infty a_n^r\Big)^{\frac{1}{r}}\le 1$ and $\Big(\sum_{n=1}^\infty b_n^s\Big)^{\frac{1}{s}}\le 1$. Thus, we can write

$$|x_n| \le |a_ny_n|^{1-\theta}|b_nz_n|^\theta$$

with $\Big(\sum_{n=1}^\infty \|a_n y_n\|_F^r\Big)^{\frac{1}{r}} \leq 1$ and $\Big(\sum_{n=1}^\infty \|b_n z_n\|_G^s\Big)^{\frac{1}{s}} \leq 1$. This means that $x=(x_n)$ belongs to $\ell_r(F)^{1-\theta}\ell_s(G)^\theta$ and $\|x\|_{\ell_r(F)^{1-\theta}\ell_s(G)^\theta} \leq 1$. Therefore, the inclusion

$$\ell_t(F^{1-\theta}G^\theta) \hookrightarrow \ell_r(F)^{1-\theta}\ell_s(G)^\theta$$

is continuous with norm less than or equal to one.

(2) Now, if $x=(x_n)$ belongs to $\ell_r(F)\cap\ell_s(G)$, in particular we have that $x_n \in F\cap G$ for each $n\in\mathbb{N}$. So by part (1), $x \in \ell_t(\overline{F\cap G}^{F^{1-\theta}G^\theta})$. By continuity, we can extend this inclusion to

$$\overline{\ell_r(F)\cap\ell_s(G)}^{\ell_r(F)^{1-\theta}\ell_s(G)^\theta} \subseteq \ell_t(\overline{F\cap G}^{F^{1-\theta}G^\theta}),$$

as desired.

Finally, if $x=(x_n)\in \ell_t(\overline{F\cap G}^{F^{1-\theta}G^\theta})$, then for every $n\in\mathbb{N}$ there exists a sequence $(x_n^k)_{k=1}^\infty \in F\cap G$ such that $x_n^k \to x_n$ in $F^{1-\theta}G^\theta$ when $k\to\infty$. For every $k\in\mathbb{N}$ take n_k such that $\|x_n^m - x_n\|_{F^{1-\theta}G^\theta} \leq \frac{1}{k2^n}$ for $n=1,\dots,k$ and every $m\geq n_k$. Hence, if we define

$$y_k = (x_1^{n_k}, x_2^{n_k}, \dots, x_k^{n_k}, 0, \dots)$$

then $y_k \to x$ in $\ell_t(F^{1-\theta}G^\theta)$ when $k\to\infty$. Since the inclusion $\ell_t(F^{1-\theta}G^\theta) \hookrightarrow \ell_r(F)^{1-\theta}\ell_s(G)^\theta$ is continuous, we get that $y_k\to x$ in $\ell_r(F)^{1-\theta}\ell_s(G)^\theta$ as well, and since for every $k\in\mathbb{N}$, y_k belongs to $\ell_r(F)\cap\ell_s(G)$ we can conclude that $x\in\overline{\ell_r(F)\cap\ell_s(G)}^{\ell_r(F)^{1-\theta}\ell_s(G)^\theta}$. This finishes the proof. □

LEMMA 8.4.5. *Let (F,G) be a compatible pair of Banach lattices. Let $r,s\in[1,+\infty]$ and $\theta\in(0,1)$. For $\frac{1}{t}=\frac{1-\theta}{r}+\frac{\theta}{s}$, and for every $n\in\mathbb{N}$ the inclusion*

$$F(\ell_r^n)^{1-\theta}G(\ell_s^n)^\theta \hookrightarrow F^{1-\theta}G^\theta(\ell_t^n)$$

is bounded of norm less than or equal to one.

PROOF. Let $x=(x_1,\dots,x_n)\in F(\ell_r^n)^{1-\theta}G(\ell_s^n)^\theta$ with $\|x\|_{F(\ell_r^n)^{1-\theta}G(\ell_s^n)^\theta}<1$. Then, we have

$$|x| \leq |y|^{1-\theta}|z|^\theta,$$

with $\|y\|_{F(\ell_r^n)}\leq 1$ and $\|z\|_{g(\ell_s^n)}\leq 1$. This means that $y=(y_1,\dots,y_n)$, $z=(z_1,\dots,z_n)$ with $\|\Big(\sum_{i=1}^n |y_i|^r\Big)^{\frac{1}{r}}\|_F \leq 1$ and $\|\Big(\sum_{i=1}^n |z_i|^s\Big)^{\frac{1}{s}}\|_G\leq 1$.

By Hölder's inequality we get that

$$\Big(\sum_{i=1}^n |x_i|^t\Big)^{\frac{1}{t}} \leq \Big(\sum_{i=1}^n (|y_i|^{1-\theta}|z_i|^\theta)^t\Big)^{\frac{1}{t}} \leq \Big(\sum_{i=1}^n |y_i|^r\Big)^{\frac{1-\theta}{r}}\Big(\sum_{i=1}^n |z_i|^s\Big)^{\frac{\theta}{s}}.$$

Since $\Big\|\Big(\sum_{i=1}^n |y_i|^r\Big)^{\frac{1}{r}}\Big\|_F \leq 1$ and $\Big\|\Big(\sum_{i=1}^n |z_i|^s\Big)^{\frac{1}{s}}\Big\|_G \leq 1$ it follows that $\Big\|\Big(\sum_{i=1}^n |x_i|^t\Big)^{\frac{1}{t}}\Big\|_{F^{1-\theta}G^\theta} \leq$ 1, as desired. □

LEMMA 8.4.6. *Let E be a Banach lattice, $r, s \in [1,\infty]$ and $\theta \in (0,1)$. For $\frac{1}{t} = \frac{1-\theta}{r} + \frac{\theta}{s}$ we have*

$$E(\ell_r)^{1-\theta} E(\ell_s)^\theta = E(\ell_t),$$

with equality of norms.

PROOF. Suppose first that $r, s < \infty$, and let $n \in \mathbb{N}$ be fixed. Let $x = (x_1, \dots, x_n)$ belong to $E(\ell_t^n)$ with

$$\Big\| \Big(\sum_{i=1}^n |x_i|^t \Big)^{\frac{1}{t}} \Big\|_E \le 1.$$

If we consider I the ideal of E generated by $(\sum_{i=1}^n |x_i|^t)^{1/t}$, then this can be identified with a space $C(K)$ for some compact Hausdorff space K. In this representation $(\sum_{i=1}^n |x_i|^t)^{1/t}$ corresponds to the function identically one on K. Since $|x_j| \le (\sum_{i=1}^n |x_i|^t)^{1/t}$, for every j there exists $f_j \in C(K)$ such that $(\sum_{i=1}^n |f_i(\omega)|^t)^{1/t} = 1$ for every $\omega \in K$.

Let us consider for $i = 1, \dots, n$ and each $\omega \in K$,

$$g_i(\omega) = |f_i(\omega)|^{\frac{t}{r}} \qquad \text{and} \qquad h_i(\omega) = |f_i(\omega)|^{\frac{t}{s}}.$$

Clearly, the functions g_i and h_i belong to $C(K)$ and satisfy

$$\sum_{i=1}^n |g_i(\omega)|^r = \sum_{i=1}^n |f_i(\omega)|^t = 1$$

and

$$\sum_{i=1}^n |h_i(\omega)|^s = \sum_{i=1}^n |f_i(\omega)|^t = 1,$$

for every $\omega \in K$. Moreover, we have

$$|g_i(\omega)|^{(1-\theta)} |h_i(\omega)|^\theta = |f_i(\omega)|^{\frac{t(1-\theta)}{r} + \frac{t\theta}{s}} = |f_i(\omega)|,$$

for each $\omega \in K$. Thus, if we consider the corresponding elements $(y_i)_{i=1}^n$ and $(z_i)_{i=1}^n$ in E, we have that $|x_i| = |y_i|^{(1-\theta)} |z_i|^\theta$, with

$$\Big\| \Big(\sum_{i=1}^n |y_i|^r \Big)^{\frac{1}{r}} \Big\|_E \le 1 \qquad \text{and} \qquad \Big\| \Big(\sum_{i=1}^n |z_i|^s \Big)^{\frac{1}{s}} \Big\|_E \le 1.$$

Therefore, $x = (x_1, \dots, x_n)$ belongs to $E(\ell_r^n)^{(1-\theta)} E(\ell_s^n)^\theta$ with norm less than or equal to one.

Now, in the remaining case, let us assume that r is infinite. Then we can consider for $i = 1, \dots, n$ and each $\omega \in K$ the functions

$$g_i(\omega) = 1 \qquad \text{and} \qquad h_i(\omega) = |f_i(\omega)|^{\frac{t}{s}},$$

which clearly belong to $C(K)$. Moreover, this functions satisfy

$$|f_i| = |g_i|^{(1-\theta)} |h_i|^\theta,$$

and for every $\omega \in K$,

$$\bigvee_{i=1}^n |g_i(\omega)| = 1$$

and

$$\sum_{i=1}^{n} |h_i(\omega)|^s = \sum_{i=1}^{n} |f_i(\omega)|^t = 1.$$

Therefore, considering the corresponding $(y_i)_{i=1}^n$ and $(z_i)_{i=1}^n$ in E, we have that $|x_i| = |y_i|^{(1-\theta)}|z_i|^{\theta}$, with

$$\Big\| \bigvee_{i=1}^{n} |y_i| \Big\|_E \leq 1 \qquad \text{and} \qquad \Big\| \Big(\sum_{i=1}^{n} |z_i|^s \Big)^{\frac{1}{s}} \Big\|_E \leq 1.$$

Therefore, $x = (x_1, \ldots, x_n)$ belongs to $E(\ell_\infty^n)^{(1-\theta)} E(\ell_s^n)^{\theta}$ and has norm less than or equal to one.

All this shows that $E(\ell_t^n) \hookrightarrow E(\ell_r^n)^{1-\theta} E(\ell_s^n)^{\theta}$ is continuous with norm less than or equal to one, for every $n \in \mathbb{N}$. Clearly, by Lemma 8.4.5, we get the converse inclusion and so, for every $n \in \mathbb{N}$,

$$E(\ell_r^n)^{1-\theta} E(\ell_s^n)^{\theta} = E(\ell_t^n)$$

holds with equality of norms. Since, by the definition of the space $E(\ell_p)$ we have

$$\|(x_n)\|_{E(\ell_p)} = \sup_n \|(x_k)_{k=1}^n\|_{E(\ell_p^n)},$$

the conclusion follows. $\square$

Proof of Theorem 8.4.3. Since T is p-convex, it can be factored through a p-convex Banach lattice Z as in Theorem 8.1.4:

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "R"'] & & F \\ & Z \arrow[ur, "\varphi"'] & \end{tikzcd}$$

where $\varphi : Z \to F$ is an injective interval preserving lattice homomorphism, and $Rx = Tx$ for all $x \in E$. Therefore, (Z, F) can be considered as a compatible interpolation pair of Banach lattices, and we can interpolate $T : E \to F$ and $R : E \to Z$ by the complex method of interpolation (see [**28**]) with parameter θ, and we get a Banach lattice

$$F_\theta = [(Z, F)]_\theta,$$

and an operator

$$T_\theta : E \to F_\theta.$$

Moreover, since φ is an inclusion, F_θ is also continuously included in F. Let us denote this inclusion by $\varphi_\theta : F_\theta \hookrightarrow F$.

We claim that F_θ is p_θ-convex, with $\frac{1}{p_\theta} = \frac{\theta}{p} + \frac{1-\theta}{1}$, that is $p_\theta = \frac{p}{\theta + (1-\theta)p}$. Indeed, first notice that if Z is p-convex then $F^{1-\theta} Z^{\theta}$ is p_θ-convex. This is because for positive operators it does hold that

$$T(|x_0|^{1-\theta}|x_1|^{\theta}) \leq (T|x_0|)^{1-\theta}(T|x_1|)^{\theta},$$

which implies that for positive operators $T : X_0 \to Y_0$, $T : X_1 \to Y_1$ the interpolated operator $T : X_0^{1-\theta} X_1^{\theta} \to Y_0^{1-\theta} Y_1^{\theta}$ is bounded. In our particular case, (see the discussion following [**87**, 1.d.3]) for every $n \in \mathbb{N}$, we have operators

$$\begin{array}{rcl} \hat{I}_n : \ell_1^n(F) & \longrightarrow & F(\ell_1^n) \\ (x_1, \ldots, x_n) & \longmapsto & (x_1, \ldots, x_n) \end{array} \qquad \begin{array}{rcl} \hat{I}_n : \ell_p^n(Z) & \longrightarrow & Z(\ell_p^n) \\ (x_1, \ldots, x_n) & \longmapsto & (x_1, \ldots, x_n) \end{array}$$

which are bounded uniformly on $n \in \mathbb{N}$. Since they are clearly positive, by the previous remark the following operators are also uniformly bounded

$$\begin{array}{rcl} I_n : \ell_1^n(F)^{1-\theta} \ell_p^n(Z)^{\theta} & \longrightarrow & F(\ell_1^n)^{1-\theta} Z(\ell_p^n)^{\theta} \\ (x_1, \ldots, x_n) & \longmapsto & (x_1, \ldots, x_n) \end{array}$$

Using Lemmas 8.4.4 and 8.4.5 we get that the operators

$$\begin{array}{ccccccc} \ell_{p_\theta^n}(F^{1-\theta} G^{\theta}) & \longrightarrow & \ell_1^n(F)^{1-\theta} \ell_p^n(Z)^{\theta} & \longrightarrow & F(\ell_1^n)^{1-\theta} Z(\ell_p^n)^{\theta} & \longrightarrow & F^{1-\theta} G^{\theta}(\ell_{p_\theta}^n \\ (x_1, \ldots, x_n) & \longmapsto & (x_1, \ldots, x_n) & \longmapsto & (x_1, \ldots, x_n) & \longmapsto & (x_1, \ldots, x_n) \end{array}$$

are also uniformly bounded on n. This means that $F^{1-\theta} G^{\theta}$ is p_θ-convex. Now, by Theorem 8.3.1, $F_\theta = \overline{F \cap Z}^{F^{1-\theta} Z^{\theta}}$, and since $F \cap Z$ is a sublattice of $F^{1-\theta} Z^{\theta}$, F_θ is also p_θ-convex.

Now we claim that T_θ is $(\frac{q}{1-\theta})$-concave. Indeed, since $T : E \to F$ is q-concave and $R : E \to Z$ is ∞-concave, the following maps are bounded:

$$\begin{array}{rcl} \check{T} : E(\ell_q) & \longrightarrow & \ell_q(F) \\ (x_1, x_2, \ldots) & \longmapsto & (Tx_1, Tx_2, \ldots) \end{array} \qquad \begin{array}{rcl} \check{R} : E(\ell_\infty) & \longrightarrow & \ell_\infty(Z) \\ (x_1, x_2, \ldots) & \longmapsto & (Rx_1, Rx_2, \ldots) \end{array}$$

Therefore, the interpolated map

$$\check{T}_\theta : [(E(\ell_q), E(\ell_\infty))]_\theta \to [(\ell_q(F), \ell_\infty(Z))]_\theta$$

is also bounded (cf. [**22**] or [**28**, §4]). Note that by Theorem 8.3.1 and Lemma 8.4.6, we have

$$[(E(\ell_q), E(\ell_\infty))]_\theta = \overline{E(\ell_q) \cap E(\ell_\infty)}^{E(\ell_q)^{1-\theta} E(\ell_\infty)^{\theta}} = \overline{E(\ell_q)}^{E(\ell_q)^{1-\theta} E(\ell_\infty)^{\theta}} = E(\ell_{q_\theta}),$$

where $\frac{1}{q_\theta} = \frac{\theta}{\infty} + \frac{1-\theta}{q}$ (see [**28**] and [**108**]). And by Lemma 8.4.4, we have the identity

$$[(\ell_q(F), \ell_\infty(Z))]_\theta = \overline{\ell_q(F) \cap \ell_\infty(Z)}^{\ell_q(F)^{1-\theta} \ell_\infty(Z)^{\theta}} = \ell_{q_\theta}(\overline{F \cap Z}^{F^{1-\theta} Z^{\theta}}) = \ell_{q_\theta}(F_\theta),$$

with equality of norms. Therefore, the map $\check{T}_\theta : E(\ell_{q_\theta}) \to \ell_{q_\theta}(F_\theta)$ is bounded, which means that T_θ is q_θ-concave ($q_\theta = \frac{q}{1-\theta}$).

Hence, we can now apply Theorem 8.1.1 to $T_\theta : E \to F_\theta$, and we get the factorization

$$\begin{tikzcd} E \arrow[rr, "T_\theta"] \arrow[dr, "\phi_\theta"'] & & F_\theta \\ & E_\theta \arrow[ur, "R_\theta"'] & \end{tikzcd}$$

through the q_θ-concave Banach lattice E_θ. Therefore, T can be factorized as claimed. $\square$

5. Other results

Recall the classical result proved by J. L. Krivine in [**81**]: If $T : X \to E$ is p-convex and $S : E \to Y$ is p-concave, then ST factors through some $L_p(\mu)$. In this direction, as another application of Theorem 8.3.1, we have the following result (see also [**117**]).

PROPOSITION 8.5.1. *Let $T : X \to E$ be p-convex and $S : E \to Y$ q-concave. For every $\theta \in (0,1)$ we can factor ST through a Banach lattice U_θ which is p_θ-convex and q_θ-concave (with as usual $p_\theta = \frac{p}{1-\theta}$ and $q_\theta = \frac{q}{q(1-\theta)+\theta}$).*

PROOF. By Theorem 8.1.4, we can factor T in the following way

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "\widetilde{T}"'] & & E \\ & W \arrow[ur, hook, "i"'] & \end{tikzcd}$$

Moreover, since $S \circ i : W \to Y$ is q-concave, by Theorem 8.1.1 we have the lattice seminorm $\rho_{S\circ i}$ which is continuous with respect to the norm in W ($\rho_{S\circ i}(x) \leq M_q(S \circ i)\|x\|_W$), and such that $W/\rho_{S\circ i}^{-1}(0)$ with the norm that $\rho_{S\circ i}$ induces becomes a q-concave Banach lattice, such that $S \circ i$ factors through it. But, since W is p-convex and $\rho_{S\circ i}^{-1}(0)$ is a closed ideal, it follows that $W/\rho_{S\circ i}^{-1}(0)$ with its quotient norm is also p-convex.

We can consider $X_0 = W/\rho_{S\circ i}^{-1}(0)$ with its quotient norm, and $X_1 = \widehat{W/\rho_{S\circ i}^{-1}(0)}$ (the completion under $\rho_{S\circ i}$) with the norm induced by $\rho_{S\circ i}$. Note that, for all y with $\rho_{S\circ i}(y) = 0$, we have that

$$\rho_{S\circ i}(x) = \rho_{S\circ i}(x + y) \leq M_{(q)}(S \circ i)\|x + y\|.$$

Thus,

$$\|x\|_{X_1} = \rho_{S\circ i}(x) \leq M_{(q)}(S \circ i) \inf\{\|x + y\| : \rho_{S\circ i}(y) = 0\} = \|x\|_{X_0},$$

which means that $X_0 \hookrightarrow X_1$ is bounded of norm less than or equal to $\leq M_{(q)}$.

Therefore, we can interpolate X_0 and X_1. Since X_0 is p-convex and X_1 is q-concave, by [**108**] we get that $U_\theta = X_0^{1-\theta}X_1^\theta$ is p_θ-convex and q_θ-concave. The following diagram illustrates the situation:

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "\widetilde{T}"'] & & E \arrow[rr, "S"] & & Y \\ & W \arrow[ur, hook, "i"'] \arrow[rrr, "\phi"] \arrow[d] & & & X_1 \arrow[u, "S_1"'] \\ & X_0 \arrow[dr, hook] & & & X_1 \arrow[u, hook] \\ & & X_0^{1-\theta}X_1^\theta \arrow[urr, hook] & & \end{tikzcd}$$

□

REMARK 8.5.2. Proposition 8.5.1 was also proved in [**117**].

REMARK 8.5.3. Note that in the proof we could use [**108**] because $X_0 \hookrightarrow X_1$ and X_1 is q-concave, hence representable as a Banach lattice of functions over a measure space (Ω, Σ, μ).

COROLLARY 8.5.4. *If $T : E \to E$ is p-convex and q-concave, then T^2 factors through a p_θ-convex and q_θ-concave Banach lattice. In particular, it factors through a super reflexive Banach lattice.*

LEMMA 8.5.5. $\left(\overline{X_0 \cap X_1}^{X_0}\right)^{1-\theta}\left(\overline{X_0 \cap X_1}^{X_1}\right)^{\theta}$ *is dense in* $\overline{X_0 \cap X_1}^{X_0^{1-\theta}X_1^{\theta}}$.

PROOF. Let $x \in \overline{X_0 \cap X_1}^{X_0^{1-\theta}X_1^{\theta}}$, such that $\|x\|_{X_0^{1-\theta}X_1^{\theta}} \leq 1$, then for every $\varepsilon > 0$ we have $g \in X_0$, $h \in X_1$ such that $\|g\|_{X_0} \leq 1$, $\|h\|_{X_1} \leq 1$, and $|x| \leq (1+\varepsilon)|g|^{1-\theta}|h|^{\theta}$.

For $N \in \mathbb{N}$ let

$$g_N = g \wedge Nx,$$

and

$$h_N = h \wedge Nx.$$

Clearly g_N and h_N belong to $X_0 \cap X_1$, and $\|g_N\|_{X_0} \leq \|g\|_{X_0}$, $\|h_N\|_{X_1} \leq \|h\|_{X_1}$.

Let also

$$x_N = x \wedge (1+\varepsilon)g_N^{1-\theta}h_N^{\theta}.$$

Since $x = x \wedge (1+\varepsilon)g^{1-\theta}h^{\theta}$, it follows that

$$\|x_N - x\|_{X_0^{1-\theta}X_1^{\theta}} \leq (1+\varepsilon)\|g_N^{1-\theta}h_N^{\theta} - g^{1-\theta}h^{\theta}\|_{X_0^{1-\theta}X_1^{\theta}}.$$

Let us consider the ideal generated by $|g| \vee |h|$ in $X_0 + X_1$. As usual we can think of it as a $C(K)$ space. Now for those $k \in K$ such that $x_N(k) < x(k)$ (for the remaining we would have $x_N(k) = x(k)$), we have

$$(1+\varepsilon)g_N(k)^{1-\theta}h_N(k)^{\theta} < x(k).$$

Therefore, for those k's, if $g_N(k) = Nx(k)$ and $h_N(k) = Nx(k)$, then $(1+\varepsilon)Nx(k) < x(k)$ which is a contradiction for N large enough. Now, if $g_N(k) = Nx(k)$ and $h_N(k) = h(k)$, then

$$(1+\varepsilon)(Nx(k))^{1-\theta}h(k)^{\theta} \leq x(k)$$

yields

$$h(k) \leq \frac{1}{N^{\frac{1-\theta}{\theta}}}x(k)^{\theta}.$$

Thus we have proved that $Nx(k) \leq g(k)$ implies $h(k) \leq \frac{1}{N^{1-\theta}}x(k)$.

Hence,

$$\|(g - Nx)_+^{1-\theta}h^{\theta}\|_{X_0^{1-\theta}X_1^{\theta}} \leq \frac{1}{N^{1-\theta}}\|g\|_{X_0}^{1-\theta}\|x\|_{X_1}^{\theta}.$$

Since $g = (g \wedge Nx) + (g - Nx)_+$, we have

$$g^{1-\theta}h^{\theta} \leq (g \wedge Nx)^{1-\theta}h^{\theta} + (g - Nx)_+^{1-\theta}h^{\theta}.$$

Therefore,

$$0 \le g^{1-\theta}h^{\theta} - (g \wedge Nx)^{1-\theta}h^{\theta} \le (g - Nx)_+^{1-\theta}h^{\theta} \le \frac{1}{N^{1-\theta}}(g - Nx)_+^{1-\theta}x^{\theta}.$$

This yields

$$\|g^{1-\theta}h^{\theta} - (g_N)^{1-\theta}h^{\theta}\|_{X_0^{1-\theta}X_1^{\theta}} \le \|(g - Nx)_+^{1-\theta}h^{\theta}\|_{X_0^{1-\theta}X_1^{\theta}},$$

which goes to zero as N tends to infinity.

Reasoning analogously with $h_N = h \wedge Nx$, we arrive at $g_N^{1-\theta}h_N^{\theta} \to g^{1-\theta}h^{\theta}$ in the norm of $X_0^{1-\theta}X_1^{\theta}$ for $N \to \infty$. This implies $x_N \to x$ in $X_0^{1-\theta}X_1^{\theta}$, as we wanted to prove. $\square$

Bibliography

[1] Y.A. Abramovich, C. D. Aliprantis *An invitation to operator theory.* Graduate Studies in Mathematics, 50. American Mathematical Society, Providence, RI, (2002).

[2] Y. A. Abramovich, C. D. Aliprantis *Positive operators.* Handbook of the geometry of Banach spaces, Vol. I, 85–122. North-Holland, Amsterdam, 2001.

[3] Y.A. Abramovich, C. D. Aliprantis, and O. Burkinshaw, *Invariant subspaces of operators on ℓ_p spaces.* J. Funct. Anal., **115** (1993), 418–424.

[4] Y.A. Abramovich, C. D. Aliprantis, and O. Burkinshaw, *Invariant subspaces for positive operators.* J. Funct. Anal., **124**, 95–111 (1994).

[5] Abramovich, Y. A.; Aliprantis, C. D.; Sirotkin, G.; Troitsky, V. G. *Some open problems and conjectures associated with the invariant subspace problem.* Positivity 9 (2005), no. 3, 273–286.

[6] F. Albiac, and N. J. Kalton, *Topics in Banach space theory,* Graduate Texts in Mathematics **233**, Springer, (2006).

[7] D. J. Aldous, *Subspaces of L^1, via random measures,* Trans. Amer. Math. Soc. **267** (1981), no. 2, 445–463.

[8] D. J. Aldous, D. H. Fremlin *Colacunary sequences in L-spaces.* Studia Math. **71** (1982), no. 3, 297–304.

[9] C. D. Aliprantis, O. Burkinshaw *Positive compact operators on Banach lattices.* Math. Z. **174**, (1980) 289-298.

[10] C. D. Aliprantis, O. Burkinshaw *On weakly compact operators on Banach lattices.* Proc. Amer. Math. Soc. **83**, (1981) 573-578.

[11] C. D. Aliprantis and O. Burkinshaw, *Factoring compact and weakly compact operators through reflexive Banach lattices*, Trans. Amer. Math. Soc. 283 (1984), no. 1, 369–381.

[12] C. D. Aliprantis, and O. Burkinshaw, *Positive Operators,* Springer, Dordrecht, (2006).

[13] G. Androulakis, P. Dodos, G. Sirotkin, V. G. Troitksy. *Classes of strictly singular operators and their products* (to appear in Isral J. Math.)

[14] R. Anisca, and V. G. Troitsky, *Minimal vectors of positive operators,* Indiana Univ. Math. J. **54** (2005), no. 3, 861–872.

[15] N. Aronszajn, and K. T. Smith, *Invariant subspaces of completely continuous operators.* Ann. of Math. 60 (1954), 345–350.

[16] J. Avron, I. Herbst, and B. Simon, *Schrödinger operators with magnetic fields, I. General interactions.* Duke Math. J. **45** (1978), 847–883.

[17] A. Baernstein *On reflexivity and summability* Studia Math. **42** (1972), 91–94.

[18] S. Banach and S. Saks *Sur la convergence forte dans les champs L^p.* Studia Math. **2** (1930), 51–57.

[19] B. Beauzamy *Propriété de Banach-Saks* Studia Math. **66** (1980), 227–235.

[20] B. Beauzamy *Operateurs uniformément convexifiants.* Studia Math. **57**, (1976) 103–139.

[21] J. Bergh, *On the relation between the two complex methods of interpolation.* Indiana Univ. Math. J. **28** (1979), no. 5, 775–778.

[22] J. Bergh, J. Löfström, *Interpolation spaces. An introduction.* Grundlehren der Mathematischen Wissenschaften, No. **223**. Springer-Verlag, Berlin-New York, 1976.

[23] J. Bourgain, *The Szlenk index and operators on $C(K)$-spaces.* Bull. Soc. Math. Belg. Sér. B **31** (1979), no. 1, 87–117.

[24] J. Bourgain, *Dunford-Pettis operators on L^1 and the Radon-Nikodym property,* Israel J. Math. **37** (1980), 34–47.

[25] J. Bourgain, *An averaging result for c_0-sequences,* Bull. Soc. Math. Belg. 30 (1978) 83–87.

[26] J. Bourgain, *An averaging result for ℓ_1-sequences and applications to weakly conditionally compact sets in L^1_X,* Israel J. Math. 32 (1979) 289–298.

[27] A. Burger, *Mutation-selection balance and coninuum-of-alleles models.* Math. Biosci. 91 (1988), 67–83.

[28] A. P. Calderón, *Intermediate spaces and interpolation, the complex method.* Studia Math. **24**, (1964) 113–190.

[29] N. L. Carothers, S. J. Dilworth, *Subspaces of $L_{p,q}$.* Proc. Amer. Math. Soc. 104 (1988), no. 2, 537–545.

[30] P.G. Casazza, and T.J. Shura, *Tsirelson's space* Lecture Notes in Mathematics **1363** (1989).

[31] P. Cembranos, J. Mendoza, *Banach Spaces of Vector-valued Functions,* Lecture Notes in Mathematics, 1676, Springer-Verlag, Berlin, 1997.

[32] Z. Chen, *Weak sequential precompactness in Banach lattices,* Chinese Ann. Math. Ser. A 20 (1999), no. 5, 567–574; translation in Chinese J. Contemp. Math. 20 (1999), no. 4, 477–486 (2000).

[33] J. Creekmore *Type and Cotype in Lorentz L_{pq} Spaces* Indagationes Mathematicae. Proceedings A 84 (2), June 20, (1981).

[34] W. J. Davis, T. Figiel, W. B. Johnson, A. Pełczyński *Factoring weakly compact operators* J. Funtional Analysis **17** (1974), 311-327.

[35] A. Dehici, A. Jeribi, K. Latrach, *Spectral analysis of a transport operator arising in growing cell populations.* Acta Appl. Math. **92** (2006), 37–62.

[36] J. Diestel, C. J. Seifert, *The Banach-Saks ideal. I. Operators acting on $C(\Omega)$.* Comment. Math. Special Issue **1** (1978), 109–118.

[37] P. G. Dodds, D. H. Fremlin *Compact operators in Banach lattices.* Israel J. Math. **34**, (1979) 287-320.

[38] P. N. Dowling, N. Randrianantoanina, *Asymptotically isometric copies of c_0 and l^1 in Bochner-spaces,* J. Math. Anal. Appl. 262 (2001) 419–434.

[39] L. Drewnowski, I. Labuda, *Topological vector spaces of Bochner measurable functions,* Illinois J. Math. 46 (2002) 287–318.

[40] N. Dunford, J.T. Schwartz *Linear Operators. Part I, General Theory.* Pure and applied Mathematics, vol. VII, Interscience, New York, (1958).

[41] G. Emmanuele, *Copies of l_∞ in Köthe spaces of vector valued functions,* Illinois J. Math. 36 (1992) 293–296.

[42] P. Enflo. *On the invariant subspace problem for Banach spaces.* Acta Math. 158 (1987), no. 3-4, 213–313.

[43] P. Enflo, H. P. Rosenthal, *Some results concerning $L^p(\mu)$-spaces.* J. Functional Analysis 14 (1973), 325–348.

[44] P. Erdös and M. Magidor, *A note on regular methods of summability and the Banach-Saks property,* Proc. Amer. Math. Soc. 59 (1976), no. 2, 232–234.

[45] N. R. Farnum, *The Banach-Saks theorem in $C(S)$.* Canad. J. Math **26** (1974), 91–97.

[46] T. Figiel, N. Ghoussoub, and W. B. Johnson, *On the structure of non-weakly compact operators on Banach Lattices,* Math. Ann. **257**, (1981) 317-334.

[47] T. Figiel, W. B. Johnson, and L. Tzafriri, *On Banach lattices and spaces having local unconditional structure with applications to Lorentz funtion spaces,* J. Approximation Theory **13**, (1975) 395-412.

[48] J. Flores, *Problemas de mayoración en clases de operadores entre retículos de Banach.* Tesis doctoral, Universidad Complutense de Madrid (2000).

[49] J. Flores, *Some remarks on disjointly strictly singular operators,* Positivity 9, (2005) 385-396.

[50] J. Flores, F. L. Hernández *Domination by positive disjointly strictly singular operators.* Proc. Amer. Math. Soc. **129**, (2001) 1979-1986.

[51] J. Flores, and F. L. Hernández, *Domination by positive strictly singular operators,* J. London Math. Soc. (2) **66**, (2002) 433-452.

[52] J. Flores, F. L. Hernandez, N. J. Kalton, P. Tradacete *A characerization of strictly singular operators on Banach lattices* (preprint)

[53] J. Flores, F. L. Hernández, Y. Raynaud *Super strictly singular and cosingular operators and related classes.* (Preprint).

[54] J. Flores, F. L. Hernández and P. Tradacete, *Powers of operators dominated by strictly singular operators*, Q. J. Math. 59 (2008), 321–334.

[55] J. Flores and C. Ruiz, *Domination by positive Banach-Saks operators*, Studia Math. 173 (2006), no. 2, 185–192.

[56] J. Flores, P. Tradacete *Factorization and domination of positive Banach-Saks operators* Studia Math. 189 (2008), 91-101.

[57] J. Flores, P. Tradacete, and V.G. Troitsky, *Invariant subspaces of positive strictly singular operators on Banach lattices,* J. Math. Anal. Appl. **343** (2008), 743–751.

[58] J. Flores, P. Tradacete, V. G. Troitsky *Disjointly homogeneous Banach lattices and compact products of operators* (preprint).

[59] H. Gessesse, *Invariant subspaces of super left-commutants.* To appear in Proc. Amer. Math. Soc.

[60] H. Gessesse, and V. G. Troitsky, *Invariant subspaces of positive quasinilpotent operators on Banach spaces with a closed generating cone,* to appear in Positivity.

[61] N. Ghoussoub, W. B. Johnson *Factoring operators through Banach lattices not containing $C(0,1)$*, Math. Z. **194**, (1987) 153-171.

[62] M. González, J. M. Gutiérrez, *Polynomials on Schreier's space.* Rocky Mountain J. Math. **30** (2000), no. 2, 571–585.

[63] S. Guerre-Delabrière, *Classical sequences in Banach spaces.* Monographs and Textbooks in Pure and Applied Mathematics, 166. Marcel Dekker, Inc., New York, 1992.

[64] R. Haydon, M. Levy, Y. Raynaud *Randomly normed spaces.* Travaux en Cours [Works in Progress], 41. Hermann, Paris, (1991).

[65] S. Heinrich *Ultraproducts in Banach space theory.* J. Reine Angewandte fur Math. **313**, 72-104 (1980).

[66] W. Hensgen, *Some properties of the vector-valued Banach ideal space $E(X)$ derived from those of E and X,* Collect. Math. 43 (1992) 1–13.

[67] F. L. Hernández, *Disjointly strictly singular operators in Banach Lattices,* Proc. 18 Winter School on Abstract Analysis (Srni). Acta Univ. Carolinae-Mathematica et Physica **31**, (1990) 35-40.

[68] F. L. Hernández, S. Ya. Novikov, and E. M. Semenov, *Strictly singular embeddings between rearrangement invariant spaces,* Positivity and its applications (Nijmegen, 2001). Positivity **7**, no. 1-2, (2003) 119-124.

[69] F. L. Hernández, V. Peirats, *Orlicz function spaces without complemented copies of l^p*. Israel J. Math. **56** (1986), no. 3, 355–360.

[70] F. L. Hernández, B. Rodríguez-Salinas *On l_p complemented copies in Orlicz spaces II.* Israel J. Math. **68**, 27-55 (1989).

[71] F. L. Hernández, V. M. Sánchez, and E. M. Semenov, *Strictly singular inclusions into $L_1 + L_\infty$,* Math. Z. **258**, (2008) 87–106.

[72] J. Hoffmann-Jørgensen *Sums of independent Banach space valued random variables,* Studia Math. 52, (1974) 150–186.

[73] W. B. Johnson, B. Maurey, G. Schechtman, and L. Tzafriri, *Symmetric structures in Banach spaces,* Mem. Amer. Math. Soc. **217**, (1979).

[74] W. B. Johnson, and G. Schechtman, *Multiplication operators on* $L(L_p)$ *and* ℓ_p*-strictly singular operators,* (preprint) arXiv:0708.0560v1

[75] W. B. Johnson, E. Odell, *Subspaces of* L_p *which embed into* l_p. Compositio Math. 28 (1974), 37–49.

[76] M. I. Kadeč, and A. Pełczyński, *Bases, lacunary sequences and complemented subspaces in the spaces* L_p*,* Studia Math. **21**, (1962) 161-176.

[77] N. Kalton, P. Saab *Ideal properties of regular operators between Banach lattices.* Illinois J. Math **29**, (1985) 382-400.

[78] T. Kato, *Perturbation theory for nullity deficiency and other quantities of linear operators,* J. Analyse Math. **6**, (1958) 273-322.

[79] M. A. Krasnoselskii, P. P. Zabreiko, E. I. Pustylnik, P. E. Sobolevskii, *Integral operators in spaces of summable functions.* Noordhoff International Publishing, 1976.

[80] S. G. Kreĭn, Yu. Ī. Petunīn, E. M. Semënov, *Interpolation of linear operators.* Translations of Mathematical Monographs, 54. American Mathematical Society, 1982.

[81] J. L. Krivine, *Théorèmes de factorisation dans les espaces réticulés.* (French) Séminaire Maurey-Schwartz 1973–1974: *Espaces* L^p*, applications radonifiantes et géométrie des espaces de Banach,* Exp. Nos. 22 et 23. Centre de Math., École Polytech., Paris, 1974.

[82] S. Kwapień, *On Banach spaces containing* c_0*. A supplement to the paper by J. Hoffmann-Jørgensen: "Sums of independent Banach space valued random variables",* Studia Math. 52 (1974) 187–188.

[83] C. C. A. Labuschagne, *A Dodds-Fremlin property for Asplund and Radon-Nikodým operators*, Positivity 10 (2006), no. 2, 391–407.

[84] D. H. Leung *Embedding* $l^{p,\infty}$ *into* $L^{p,\infty}[0,1]$ *complementably.* Bull. London Math. Soc. **23** (1991), no. 6, 583–586.

[85] J. Lindenstrauss, L. Tzafriri, *L. On Orlicz sequence spaces. III.* Israel J. Math. **14** (1973), 368–389.

[86] J. Lindenstrauss, and L. Tzafriri, *Classical Banach Spaces I: Sequence Spaces,* Springer-Verlag, (1977).

[87] J. Lindenstrauss, and L. Tzafriri, *Classical Banach Spaces II: Function Spaces,* Springer-Verlag, (1979).

[88] V. I. Lomonosov, *Invariant subspaces of the family of operators that commute with a completely continuous operator,* Funkcional. Anal. i Priložen. **7** (1973), no. 3, 55–56.

[89] G. G. Lorentz *On the theory of spaces* Λ. Pacific J. Math. **1**, 411-429 (1951).

[90] G.Ya. Lozanovskii, *Certain Banach lattices.* Sibirsk. Mat. Ž. **10**, (1969) pp. 584–599.

[91] G.Ya. Lozanovskii, *Certain Banach lattices IV.* Sibirsk. Mat. Ž. **14**, (1973), pp. 140–155.

[92] W. A. J. Luxemburg, A. C. Zaanen, *Compactness of integral operators in Banach function spaces.* Math. Ann. 149 (1963), 150–180.

[93] V. Mascioni *A restriction-extension property for operators on Banach spaces.* Rocky Mountain J. Math. **24**, no. 4, (1994) 1497–1507.

[94] P. Meyer-Nieberg, *Banach Lattices,* Springer-Verlag (1991).

[95] V. D. Milman, *Certain properties of strictly singular operators,* (Russian) Funkcional. Anal. i Priložen. **3**, (1969) no. 1, 93–94.

[96] V. D. Milman, *Operators of class* C_0 *and* C_0^*. Teor. Funkcii Funkcional. Anal. i Prilozen. **10**, 15-26 (1970).

[97] R. J. Nagel, U. Schlotterbeck, *Kompaktheit von integraloperatoren auf Banachverbänden.* Math. Ann. 202 (1973), 301–306.

[98] C. P. Niculescu *Weak compactness in Banach lattices.* J. Oper. Theory **6** (1981), 217-231.

[99] T. Nishiura, D. Waterman *Reflexivity and summability.* Studia Math. **23** (1963), 53-57.

[100] M. Nowak, *Weak compactness in Köthe Bochner spaces and regular methods of summability,* Bull. Polish Acad. Sci. Math. 50 (2002) 417–425.

[101] M. Nowak, *Conditional and relative weak compactness in vector-valued function spaces,* J. Convex Anal. 12 (2005) 447–463.

[102] M. Nowak, *Order-weakly compact operators from vector-valued function spaces to Banach spaces,* Proc. Amer. Math. Soc. 135 (2007) 2803–2809.

[103] B. de Pagter, and A. Schep, *Measures of noncompactness of operators in Banach lattices,* J. Funct. Anal. 78 (1988), 31–55.

[104] J. R. Partington *On the Banach-Saks property.* Math. Proc. Cambridge Philos. Soc. **82** (1977), 369-374.

[105] A. Pelczynski *On strictly singular and strictly cosingular operators.* Bull. Acad. Pol. Sci. XVIII, (1965) 31-41.

[106] A. Pełczyński, H. P. Rosenthal, *Localization techniques in L^p spaces.* Studia Math. 52 (1974/75), 263–289.

[107] A. Pietsch, *Operator ideals.* North-Holland Mathematical Library, **20**, (1980).

[108] G. Pisier *Some applications of the complex interpolation method to Banach lattices.* J. Analyse Math. **35**, (1979), 264–281.

[109] L. D. Pitt, *A compactness condition for linear operators of function spaces.* J. Operator Theory **1** (1979), 49–54.

[110] G. Pisier, *Une propriéte de stabilité de la classe des espaces ne contenant pas ℓ_1,* C. R. Acad. Sci. Paris Sér. A 86 (1978) 747–749.

[111] A. Plichko *Superstrictly singular and superstrictly cosingular operators.* Functional analysis and its applications, 239–255, North-Holland Math. Stud., 197, Elsevier, Amsterdam, (2004).

[112] H. Raubenheimer, *On regular Riesz operators.* Quaest. Math. **23** (2000), 179–186.

[113] F. Räbiger, *Lower and upper 2-estimates for order bounded sequences and Dunford-Pettis operators between certain class of Banach lattices,* Functional analysis (Austin, TX, 1987/1989), 159–170, *Lecture Notes in Math.*, 1470, Springer, Berlin, 1991.

[114] C. J. Read, *A solution to the invariant subspace problem.* Bull. London Math. Soc. 16 (1984), no. 4, 337–401.

[115] C. J. Read, *A solution to the invariant subspace problem on the space ℓ_1.* Bull. London Math. Soc. 17 (1985), no. 4, 305–317.

[116] C. J. Read, *Strictly singular operators and the invariant subspace problem,* Studia Math. **132** No. 3, (1999), 203–226.

[117] S. Reisner *Operators which factor through convex Banach lattices.* Canad. J. Math. **32** (1980), 1482–1500.

[118] A. Rhandi, and R. Schnaubelt, *Asymptotic behaviour of a non-autonomous popolation equation with diffusion in L^1.* Discrete Contin. Dynam. Systems 5 (1999), 663–683.

[119] H. P. Rosenthal, *A characterization of Banach spaces containing l^1,* Proc. Nat. Acad. Sci.U.S.A. **71** (1974), 2411–2413.

[120] H.P. Rosenthal, *Convolution by a biased coin, The Altgeld Book*, University of Illinois, Functional Analysis Seminar (1975/76).

[121] M. Sbihi, *A resolvent approach to the stability of essential and critical spectra of perturbed C_0-semigroups on Hilbert spaces with applications to transport theory.* J. Evol. Equ. **7** (2007), 35–58.

[122] H. H. Schaefer, *Banach lattices and positive operators.* Die Grundlehren der mathematischen Wissenschaften, Band **215**. Springer-Verlag 1974.

[123] J. Schreier *Ein Gegenbeispiel zur Theorie der schwachen Konvergenz.* Studia Math. **2** (1930), 58–62.

[124] C. J. Seifert, *The dual of Baernstein's space and the Banach-Saks property.* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **26** (1978), no. 3, 237–239.

[125] E. M. Semenov, A. M. Shteĭnberg, *Estimates of norms of operator blocks in Banach lattices.* Mat. Sb. 54 (1986), no. 2, 317–334.

[126] V. A. Sestakov *Complex interpolation in Banach spaces of measurable functions.* (Russian) Vestnik Leningrad. Univ. No. 19 Mat. Meh. Astronom. Vyp. 4 (1974), 64–68, 171.

[127] W. Szlenk, *Sur les suites faiblement convergentes dans l'espace L.* (French) Studia Math. **25** (1965) 337–341.

[128] M. Talagrand, *Weak Cauchy sequences in $L^1(E)$,* Amer. J. Math. 106 (1984) 703–724.

[129] M. Talagrand, *Some weakly compact operators between Banach lattices do not factor through reflexive Banach lattices*, Proc. Amer. Math. Soc. 96 (1986), no. 1, 95–102.

[130] B.S. Tsirelson, *It is impossible to imbed 1_p of c_0 into an arbitrary Banach space (Russian),* Funkcional. Anal. i Priložen. **8** (1974), no. 2, 57–60.

[131] L. Weis, *On the surjective (injective) envelope of strictly (co-) singular operators,* Studia Math. 54 (1975) 285–290.

[132] L. Weis, *On perturbations of Fredholm operators in $L_p(\mu)$-Spaces,* Proc. Amer. Math. Soc. **67**, (1977) 287-292.

[133] L. Weis, *Integral operators and changes of density,* Indiana Univ. Math. J. **31** (1982), no. 1, 83–96.

[134] L. Weis, *Banach lattices with the subsequence splitting property,* Proc. Amer. Math. Soc. **105**, (1989) 87-96.

[135] R. J. Whitley, *Strictly singular operators and their conjugates.* Trans. Amer. Math. Soc. **113**, 252–261 (1964).

[136] A. W. Wickstead *Extremal structure of cones of operators.* Quart. J. Math. Oxford Ser (2) **32**, (1981) 239-253.

[137] W. Wnuk *Banch lattices with properties of the Schur type - a survey.* Conferenze del Seminario di Matematica dell'Università di Bari **249**, (1993) 1-24.

[138] W. Wnuk *Banch lattices with order continuous norm.* Polish Scientific Publishers, 1999.

[139] P. Wojtaszczyk, *Banach spaces for analysts,* Cambridge studies in Advanced Mathematics, vol. **25**, Cambridge University Press, (1991).