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Injectiveness and Discontinuity of Multiplicative Convex Functions

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Abstract: In the present work we study the set of multiplicative convex functions. In particular, we focus on the properties of injectiveness and discontinuity. We will show that a non constant multiplicative convex function is at most 2-injective, and construct multiplicative convex functions which are discontinuous at infinitely many points.

Keywords: convexity; convex function; multiplicative convex

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1. Introduction and Background Research on the Set of Multiplicative Convex Functions

The class of multiplicative convex functions was introduced as a way of extending the property of classical convexity from the arithmetic mean to the geometric mean. With this idea, Niculescu proposed that a multiplicative convex function would verify the relation

$$f(x^{\lambda}y^{1-\lambda}) \leq f(x)^{\lambda}f(y)^{1-\lambda},$$

for every choice of $x, y > 0$ and $0 < \lambda < 1$ ([1]). On the other hand, in [2] the authors proposed a different way to define multiplicative convexity were the aim was to upgrade the classical definition from addition to multiplication. In this direction, the following definition was introduced:

Definition 1 ([3]). *A function $f : (0, \infty) \to [0, \infty)$ is said to be* **multiplicative convex** *(or f is mc) if, for every $\mu > 0$ and $x, y \geq 0$ we have*

$$f(x^{\mu}y^{1/\mu}) \leq f(x)^{\mu}f(y)^{1/\mu}.$$

If, besides, $f(1) = 1$ then we will say that f is **multiplicative convex 1** *(or mc1, for short).*

We will denote

$$\mathcal{MC} = \{f : (0, \infty) \to [0, \infty) \,:\, f \text{ is mc}\},$$
$$\mathcal{MC}1 = \{f : (0, \infty) \to [0, \infty) \,:\, f \text{ is mc}1\}$$

In [2] it was shown that the distinction between the sets $\mathcal{MC}$ and $\mathcal{MC}1$ (namely, $f(1) = 1$) was crucial. In fact, the condition $f(1) = 1$ suffices to completely describe the set $\mathcal{MC}1$:

Theorem 1 ([2]). *Let $f : (0, \infty) \to [0, \infty)$. Then, f is an mc1-function if and only if it can be written in the form*

$$f(x) = \begin{cases} b^{\log_a(x)} & \text{if } 0 < x \leq 1, \\ b'^{\log_{a'}(x)} & \text{if } x > 1, \end{cases} \tag{1}$$

where a, b, a' and b' satisfy the following conditions:

1. $0 < a < 1$ *and* $a' > 1$.
2. *If* $b < 1$, *then* $\log_b(b') \leq \log_a(a') < 0$ *(which, in particular, implies* $b' > 1$*).*
3. *If* $b > 1$, *then* $\log_b(b') \geq \log_a(a')$.

In particular, Theorem 1 implies that an $mc1$-function can show one (and only one) of the following behaviors: it is either monotone (increasing or decreasing) or it decreases over the interval $(0, 1)$ and increases over the interval $(1, \infty)$. Functions that behave like the latter ones (decrease over $(0, b)$ and increase over (b, ∞) for certain number $b > 0$) will be called decreasing–increasing functions.

On the other hand, requiring $f(1) > 1$ (for an *mc*-function cannot have $f(1) < 1$) implied a huge difference with respect to the previous situation, where everything was under control: for one thing (see [3]), $\mathcal{MC}$ is closed under addition, product, multiplication by a scalar no smaller than 1 and composition (if the first function acting in the composition is not decreasing), but it was also proved the existence of discontinuous *mc*-functions:

Theorem 2 ([3])**.** *Let $\alpha > 1$, $\alpha < \beta \leq \alpha^2$ and f be an $mc1$-function. Define the function*

$$g(x) = \begin{cases} \alpha f(x) & \text{if } 0 < x \leq 1, \\ \beta f(x) & \text{if } x > 1. \end{cases}$$

Then g is a discontinuous mc-function.

The following theorem can be used to prove the existence of an *mc*-function which is discontinuous at any given point. We include the constructive proof for the sake of completeness.

Proposition 1 ([3])**.** *Let $x_0 > 0$. Then, there exists an mc-function that is discontinuous at x_0*

Proof. Assume first $x_0 < 1$ and let f be an $mc1$-function and g be an increasing discontinuous function from Theorem 2. Then,

$$h_{x_0}(x) = g\left(\frac{1}{f(x_0)} f(x)\right)$$

is an *mc*-function which is discontinuous at x_0.

If now $x_0 > 1$, then we may just define

$$h^{(x_0)}(x) = h_{\frac{1}{x_0}}\left(\frac{1}{x}\right).$$

□

Given the aim of this paper, we shall introduce some notation to denote the set of points of discontinuity of functions.

Definition 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We shall denote*

$$\mathfrak{D}(f) = \{x \in \mathbb{R} : f \text{ is discontinuous on } x\}.$$

This set has some regular algebraic properties. For example, what follows is a standard calculus exercise:

Proposition 2. *Let f and g be two functions so that $\mathfrak{D}(f) \cap \mathfrak{D}(g) = \emptyset$. Then,*

$$\mathfrak{D}(f+g) = \mathfrak{D}(fg) = \mathfrak{D}(f) \cup \mathfrak{D}(g).$$

In this paper we have two main objectives: the first one is to complete a description of the set of general *mc*-functions in a similar way as how the set of *mc*1-functions was described in [2] (where, before succeeding in giving a complete characterization of the set, the authors proved that *mc*1-functions were continuous and either monotone or non increasing-non decreasing). The results in Section 3 will lead to the conclusions that a general *mc*-function is of one of the mentioned two behaviors (monotone or decreasing–increasing), which in particular implies that the set $\mathcal{MC}$ is of cardinality $\mathfrak{c}$ and that an *mc*-function is continuous everywhere except from possibly a countable set.

The second aim is to provide examples of *mc*-functions which are discontinuous at infinitely many points. The main result in Section 4 is held by some propositions and lemmas encompassed in Number Theory.

We will complete this paper by analyzing the set of discontinuous *mc*-functions from an algebraic point of view. Section 5 follows the line of action shown in [3] and focuses on the existence of certain algebraic structures whose non-zero elements fulfill some properties (see, e.g., [4–8]).

2. Other Fundamental Concepts and Classical Results

The following two theorems will be of great importance for the goals of this paper.

Theorem 3 ([9], Darboux-Froda's Theorem)**.** *The set of points of discontinuity of a monotone function is at most countable.*

Theorem 4 ([10])**.** *The cardinality of the set of monotone functions is $\mathfrak{c}$.*

Besides this, we will also make use of Steiner's problem, which we include bellow:

Lemma 1 ([11])**.** *The maximum of the function $f(x) = x^{1/x}$ is attained at $x = e$. In fact, the function f is increasing on $(0, e)$ and decreasing on (e, ∞).*

With respect to the algebraic properties of the set $\mathcal{MC}$, we remark that in general we cannot ensure that, given an *mc*-function f, αf is also an *mc*-function for $\alpha \geq 0$ but for $\alpha \geq 1$. Because of this, we have to consider the following definition.

Definition 3 ([3])**.** *Let X be a vector space and $M \subseteq X$. We say that M is an infinite dimensional* **truncated cone** *if M fulfills the following properties:*

1. *M is closed under addition.*
2. *M is closed under multiplication by scalars greater than or equal to 1.*
3. *M contains a set of infinite cardinality of linearly independent elements.*

The maximal possible cardinality of a set as in property (3) is the **dimension** *of the truncated cone.*

If, furthermore, we can define a multiplication on X (satisfying the usual properties on itself and with respect to addition) and M is closed under multiplication, then we say that M is an **algebraic truncated cone***. In that case,*

1. *the* **linear dimension** *of the truncated cone will be the maximal possible cardinality so that there exists a subset of such cardinality and consisting on linearly independent elements.*
2. *The* **algebraic dimension** *will be the maximal possible cardinality so that there exists a subset of such cardinality and consisting on algebraic independent elements (that is, so that the only polynomial vanishing on them is the null polynomial).*

Trivially, we obtain that the algebraic dimension is not greater than the linear dimension.

3. Study of the Injectiveness of an *mc*-Function

In order to conclude that an *mc*-function f is either monotone or decreasing–increasing, we will first show that we have that $f_{|f^{-1}(0,1)}$ is injective and that $f_{|f^{-1}[1,\infty)}$ is at most 2-injective.

Theorem 5. *Let f be an mc-function and define $A = f^{-1}(0,1)$. Then $f_{|A}$ is injective.*

Proof. Assume otherwise, that is, we can find $0 < \theta_1 < \theta_2$ so that $f(\theta_1) = f(\theta_2) = c < 1$. Assume furthermore that $1 < \theta_1, \theta_2$. We claim that we must have $\theta_2 < \theta_1^4$. Indeed, if $\theta_2 \geq \theta_1^4$, then

$$1 = \log_{\theta_2}(\theta_2) \geq 4\log_{\theta_2}(\theta_1) \quad \text{so}$$
$$0 \leq 1 - 4\log_{\theta_2}(\theta_1)$$

and therefore we can define

$$\mu = \frac{1 + \sqrt{1 - 4\log_{\theta_2}(\theta_1)}}{2} > 0,$$

which is a solution to $\theta_1^{1/\mu}\theta_2^{\mu} = \theta_2$. This allows us to conlcude

$$f(\theta_2) = f(\theta_1^{1/\mu}\theta_2^{\mu}) \leq f(\theta_1)^{1/\mu} f(\theta_2)^{\mu} = f(\theta_2)^{\mu + 1/\mu},$$

finding a contradiction to the condition $f(\theta_2) < 1$ and proving our claim.

Define next

$$g(x) = f\Big(\frac{\theta_1^4}{\theta_2}x\Big).$$

Then,

$$g\Big(\frac{\theta_1\theta_2}{\theta_1^4}\Big) = g\Big(\frac{\theta_2^2}{\theta_1^4}\Big),$$

so by our previous claim,

$$\frac{\theta_2^2}{\theta_1^4} < \Big(\frac{\theta_1\theta_2}{\theta_1^4}\Big)^4 = \frac{\theta_1^4\theta_2^4}{\theta_1^{16}} \quad \text{from which}$$
$$\theta_1^8 < \theta_2^2,$$

arriving at a contradiction.

If $0 < \theta_1 < 1 < \theta_2$, then $\log_{\theta_2}(\theta_1) < 0$ and therefore

$$\mu = \frac{1 + \sqrt{1 - 4\log_{\theta_2}(\theta_1)}}{2}$$

is always well-defined and positive, so we can again arrive at the contradiction $f(\theta_2) \leq f(\theta_2)^{\mu + 1/\mu}$.

Any other situation may be reduced to one of the previous two via the auxiliary function $h(x) = f(1/x)$. □

Theorem 6. *There is no mc-function, f, so that we can find $1 < \theta_1 < \theta_2$ such that* $\max\{f(1), f(\theta_2)\} < f(\theta_1)$.

Proof. If we choose

$$\mu = \log_{\theta_2}(\theta_1) < 1$$

then it must be

$$f(\theta_1) \leq f(\theta_2)^{\mu} f(1)^{1/\mu}.$$

Let us choose $0 < a, b < 1$, $a', b' > 1$, and $\alpha > 1$ so that

$$\begin{aligned} \log_b(b') &= \log_a(a'), \\ \alpha b'^{\log_{a'}(\theta_1)} f(\theta_1) &> f(1), \\ b'^{\log_{a'}\left(\frac{\theta_1}{\theta_2}\right)} f(\theta_1) &> f(\theta_2), \\ \alpha^{2-(2\mu+1/\mu)} &> \frac{f(\theta_2)^{\mu} f(1)^{1/\mu}}{f(\theta_1)}. \end{aligned} \tag{2}$$

Then, if we define the function

$$g(x) = \begin{cases} \alpha b^{\log_a(x)} & \text{if } 0 < x \leq 1, \\ \alpha^2 b'^{\log a'(x)} & \text{if } x > 1, \end{cases}$$

we can apply Theorems 1 and 2 to conclude that g is a (discontinuous) *mc*-function. Hence, $h(x) = f(x)g(x)$ would also be an *mc*-function. Now, we can deduce the following chain of equivalent inequalities:

$$\begin{aligned} \alpha^{2-\left(2\log_{\theta_2}(\theta_1)+1/\log_{\theta_2}(\theta_1)\right)} &> \frac{f(\theta_2)^{\log_{\theta_2}(\theta_1)} f(1)^{1/\log_{\theta_2}(\theta_1)}}{f(\theta_1)}, \\ \alpha^2 f(\theta_1) &> \left[\alpha^2 f(\theta_2)\right]^{\log_{\theta_2}(\theta_1)} [\alpha f(1)]^{1/\log_{\theta_2}(\theta_1)}, \\ \alpha^2 b^{\log_a(\theta_1)} f(\theta_1) &> \left[\alpha^2 f(\theta_2)\right]^{\log_{\theta_2}(\theta_1)} b^{\log_a(\theta_1)} [\alpha f(1)]^{1/\log_{\theta_2}(\theta_1)}, \\ \alpha^2 b^{\log_a(a')\log_{a'}(\theta_1)} f(\theta_1) &> \left[\alpha^2 f(\theta_2)\right]^{\log_{\theta_2}(\theta_1)} b^{\log_a(a')\log_{a'}(\theta_2)\log_{\theta_2}(\theta_1)} \\ &\quad \cdot [\alpha f(1)]^{1/\log_{\theta_2}(\theta_1)}, \\ \alpha^2 b^{\log_b(b')\log_{a'}(\theta_1)} f(\theta_1) &> \left[\alpha^2 b^{\log_b(b')\log_{a'}(\theta_2)} f(\theta_2)\right]^{\log_{\theta_2}(\theta_1)} \\ &\quad \cdot [\alpha f(1)]^{1/\log_{\theta_2}(\theta_1)}, \\ \alpha^2 b'^{\log_{a'}(\theta_1)} f(\theta_1) &> \left[\alpha^2 b'^{\log_{a'}(\theta_2)} f(\theta_2)\right]^{\log_{\theta_2}(\theta_1)} [\alpha f(1)]^{1/\log_{\theta_2}(\theta_1)}, \\ h(\theta_1) &> h(\theta_2)^{\mu} h(1)^{1/\mu}. \end{aligned}$$

On the other hand, applying the requisites over a, a', b, b' and α from Equation (2),

$$\begin{aligned} h(\theta_1) = \alpha^2 b'^{\log_{a'}(\theta_1)} f(\theta_1) &> \max\{\alpha^2 b'^{\log_{a'}(\theta_2)} f(\theta_2), \alpha f(1)\} \\ &= \max\{f(\theta_2), f(1)\}, \end{aligned}$$

arriving at a contradiction.

□

Corollary 1. *Let f be an mc-function and $\theta_1 < \theta_2 < \theta_3$. Then, it cannot be* $\max\{f(\theta_1), f(\theta_3)\} < f(\theta_2)$.

Proof. Assume otherwise. If $\theta_1 = 1$, then this corollary is just Theorem 6. For $1 < \theta_1$, consider $g_1(x) = f(\theta_1 x)$. For $\theta_3 > 1$, consider $g_2(x) = f\left(\frac{\theta_3}{x}\right)$.

For $\theta_3 < 1$, consider first $\tilde{g}_3(x) = f\left(\frac{1}{x}\right)$ and then $g_3(x) = \tilde{g}_3\left(\frac{1}{\theta_3}x\right)$.

In any case, g_i would be an *mc*-function which contradicts Theorem 6. □

Corollary 2. *Let f be an mc-convex function which is not locally constant (that is, for every real number x and interval I with x in I we can find $x_1 \neq x_2$ also in I so that $f(x_1) \neq f(x_2)$).*

Then f is at most 2-injective (meaning that for every real number y $\#f^{-1}(y) \leq 2$).

Proof. Let y be a real number and x_1, x_2, x_3 be so that $f(x_1) = f(x_2) = f(x_3)$. Because f is not locally constant, we can find $x_1 < x^{(1)} < x_2 < x^{(2)} < x_3$ so that $f(x^{(1)}) \neq f(x_2) \neq f(x^{(2)})$ and this is a contradiction with Theorem 6. □

The following Corollary generalizes Theorem 2.1 from [2]:

Corollary 3. *Let f be an mc-function (continuous or not). Then, f is either monotone or decreasing–increasing.*

Proof. This Corollary is easily proved if we consider the situation where f is not monotone and we apply Theorem 6 repeatedly. □

Corollary 3, in combination with Theorem 4, implies that the set $\mathcal{MC}$ must be of cardinality less than $\mathfrak{c}$. On the other hand, since the function $f(x) = b$ is an *mc*-function for $b \geq 1$, we conclude the following Theorem, which answers several questions posted in [3]:

Theorem 7. *The set $\mathcal{MC}$ has cardinality $\mathfrak{c}$.*

In particular, this implies that the algebraic structures considered in [3] are of the greatest possible dimension.

4. On the Set of Points of Discontinuity of an *mc*-Function

It is obvious that, if $A = \{x_1, x_2, \ldots, x_n\}$ is any finite set and, for $1 \leq k \leq n$, f_{x_k} is an *mc*-function that is discontinuous at x_k, then $g(x) = f_{x_1}(x) + f_{x_2}(x) + \ldots + f_{x_n}(x)$ is discontinuous over the set A. The question we will focus on now is whether it is posible to find an *mc*-function which is discontinuous over an infinite set. We remark that, taking Theorem 4 and Corollary 3 into account, the set A must be countable.

The main result of this section is as follows:

Theorem 8. *Let $\mathcal{X} = \{x_n\}_{n=1}^{\infty} \subseteq (0,1)$ be a decreasing sequence. Then, there exists an mc-function which is discontinuous on $\mathcal{X}$.*

Before giving the proof, we will need some preliminary lemmas and definitions.

Lemma 2. *Let Γ be any index set, $\{f_\alpha\}_{\alpha\in\Gamma} \subseteq \mathcal{MC}$ and assume that, for $x > 0$,*

$$\{f_\alpha(x) \,:\, \alpha \in \Gamma\}$$

is a bounded set.

Then,

$$g(x) = \sup_{\alpha\in\Gamma}\{f_\alpha(x)\}$$

is an mc-function.

Proof. We shall use the following fact, if $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are bounded sequences, then

$$\lim_{n\to\infty} \sup_{k\geq n} y_k z_k \leq \Big(\lim_{n\to\infty} \sup_{k\geq n} y_k\Big)\Big(\lim_{n\to\infty} \sup_{k\geq n} z_k\Big).$$

Let now $y, z, \mu > 0$. Then, we can find a sequence $\{\alpha_n\}_{n=1}^{\infty} \subseteq \Gamma$ so that $g(y^\mu z^{1/\mu}) = \lim_{n\to\infty} f_{\alpha_n}(y^\mu z^{1/\mu})$. Now,

$$\begin{aligned} g(y^\mu z^{1/\mu}) &= \lim_{n\to\infty} f_{\alpha_n}(y^\mu z^{1/\mu}) \\ &\leq \lim_{n\to\infty} \sup_{k\geq n} f_{\alpha_k}(y^\mu z^{1/\mu}) \\ &\leq \lim_{n\to\infty} \sup_{k\geq n} \Big[f_{\alpha_k}(y)^\mu f_{\alpha_k}(z)^{1/\mu}\Big] \\ &\leq \Big[\lim_{n\to\infty} \sup_{k\geq n} f_{\alpha_k}(y)^\mu\Big]\Big[\lim_{n\to\infty} \sup_{k\geq n} f_{\alpha_k}(z)^{1/\mu}\Big] \\ &\leq g(y)^\mu g(z)^{1/\mu}. \end{aligned}$$

□

Definition 4. *Given a decreasing sequence* $\{x_n\}_{n=1}^{\infty} \subset (0,1)$ *and* $n \geq 1$*, we will define the following elements:*

$$\begin{aligned} a_n &= \left(\frac{x_n}{x_{n+1}}\right)^{x_{n+1}}, \\ \alpha_n &= \prod_{k\geq n} a_k, \\ \beta_n &= \prod_{k\geq n} a_k^2. \end{aligned}$$

Lemma 3. *For every* $n \geq 1$*,* α_n *and* β_n *are well-defined.*

Proof. We will just show that β_1 is well-defined.

Indeed, the product that defines β_1 converges if and only if $\sum_{k=1}^{\infty} \log(a_k^2)$ converges. Now,

$$\begin{aligned} \sum_{k=1}^{\infty} \log(a_k^2) &= 2\sum_{k=1}^{\infty} \log\left[\left(\frac{x_k}{x_{k+1}}\right)^{x_{k+1}}\right] = 2\sum_{k=1}^{\infty} x_{k+1} \log\left(\frac{x_k}{x_{k+1}}\right) \\ &= 2\sum_{k=1}^{\infty} x_{k+1} \log\left(1 + \frac{x_k - x_{k+1}}{x_{k+1}}\right) \leq 2\sum_{k=1}^{\infty} x_{k+1} \frac{x_k - x_{k+1}}{x_{k+1}} \\ &= 2\sum_{k=1}^{\infty} (x_k - x_{k+1}) \leq 4x_1. \end{aligned}$$

□

Lemma 4. *Let* $n \geq 3$*,* $1 \leq m \leq n-2$ *and* $\{x_k\}_{k=1}^{\infty}$*,* $\{a_k\}_{k=1}^{\infty}$ *be as in Definition 4. Then,*

$$\left(\frac{x_n}{x_m}\right)^{x_m} \prod_{k=m}^{n-1} a_k < \left(\frac{x_n}{x_{m+1}}\right)^{x_{m+1}} \prod_{k=m+1}^{n-1} a_k$$

Proof. Just notice that

$$\begin{aligned}\left(\frac{x_n}{x_m}\right)^{x_m}\prod_{k=m}^{n-1}a_k &= \left(\frac{x_n}{x_m}\right)^{x_m}a_m\prod_{k=m+1}^{n-1}a_k \\ &= \left(\frac{x_n}{x_m}\right)^{x_m}\left(\frac{x_m}{x_{m+1}}\right)^{x_{m+1}}\prod_{k=m+1}^{n-1}a_k \\ &= \left(\frac{x_n}{x_m}\right)^{x_m}\left(\frac{x_m}{x_{m+1}}\right)^{x_{m+1}}\left(\frac{x_{m+1}}{x_n}\right)^{x_{m+1}}\left(\frac{x_n}{x_{m+1}}\right)^{x_{m+1}}\prod_{k=m+1}^{n-1}a_k \\ &= \left(\frac{x_n}{x_m}\right)^{x_m-x_{m+1}}\left(\frac{x_n}{x_{m+1}}\right)^{x_{m+1}}\prod_{k=m+1}^{n-1}a_k \\ &< \left(\frac{x_n}{x_{m+1}}\right)^{x_{m+1}}\prod_{k=m+1}^{n-1}a_k,\end{aligned}$$

based on the fact that $\{x_k\}_{k=1}^{\infty}$ is a decreasing sequence. □

Lemma 5. *Let $\{x_n\}_{n=1}^{\infty} \subset (0,1)$ be a decreasing sequence. Then, for every $n \geq 1$ and $m \geq n+2$,*

$$\left(\frac{x_n}{x_{n+1}}\right)^{x_{n+1}}\left(\frac{x_{n+1}}{x_{n+2}}\right)^{x_{n+2}}\cdot\ldots\cdot\left(\frac{x_{m-1}}{x_m}\right)^{x_m} > \left(\frac{x_n}{x_m}\right)^{x_m}$$

Proof. We will proceed by induction on m. If $m = n+2$, then

$$\begin{aligned}\left(\frac{x_n}{x_{n+1}}\right)^{x_{n+1}}\left(\frac{x_{n+1}}{x_{n+2}}\right)^{x_{n+2}} &= \left(\frac{x_n}{x_{n+1}}\right)^{x_{n+1}-x_{n+2}}\left(\frac{x_n}{x_{n+1}}\right)^{x_{n+2}}\left(\frac{x_{n+1}}{x_{n+2}}\right)^{x_{n+2}} \\ &= \left(\frac{x_n}{x_{n+1}}\right)^{x_{n+1}-x_{n+2}}\left(\frac{x_n}{x_{n+2}}\right)^{x_{n+2}} \\ &> \left(\frac{x_n}{x_{n+2}}\right)^{x_{n+2}}.\end{aligned}$$

Assuming the result is true for m, then

$$\begin{aligned}&\left(\frac{x_n}{x_{n+1}}\right)^{x_{n+1}}\left(\frac{x_{n+1}}{x_{n+2}}\right)^{x_{n+2}}\cdot\ldots\cdot\left(\frac{x_{m-1}}{x_m}\right)^{x_m}\left(\frac{x_m}{x_{m+1}}\right)^{x_{m+1}} \\ &\quad > \left(\frac{x_n}{x_m}\right)^{x_m}\left(\frac{x_m}{x_{m+1}}\right)^{x_{m+1}} \\ &\quad = \left(\frac{x_n}{x_m}\right)^{x_m-x_{m+1}}\left(\frac{x_n}{x_m}\right)^{x_{m+1}}\left(\frac{x_m}{x_{m+1}}\right)^{x_{m+1}} \\ &\quad > \left(\frac{x_n}{x_{m+1}}\right)^{x_{m+1}},\end{aligned}$$

as desired. □

Corollary 4. *For every $n \geq 1$ and $m \geq n+2$,*

$$\alpha_m\left(\frac{x_n}{x_m}\right)^{x_m} < \alpha_{n+1}\left(\frac{x_n}{x_{n+1}}\right)^{x_{n+1}} \quad \textit{and}$$
$$\beta_m\left(\frac{x_n}{x_m}\right)^{x_m} < \beta_{n+1}\left(\frac{x_n}{x_{n+1}}\right)^{x_{n+1}},$$

where x_k, α_k and β_k are as in Definition 4.

Theorem 9. *Let $y_n \to y_0$ and $\{f_k\}_{k=1}^{\infty} \subseteq \mathbb{R}^{\mathbb{R}}$. Then,*

$$\sup_{k\in\mathbb{N}} \lim_{n\to\infty} \left\{ \sup_{m\geq n} f_k(y_m) \right\} \leq \lim_{n\to\infty} \sup_{m\geq n} \left\{ \sup_{k\in\mathbb{N}} f_k(y_m) \right\}.$$

Proof. Let $l, n \in \mathbb{N}$. We notice that

$$f_l(y_n) \leq \sup_{k\in\mathbb{N}} f_k(y_n).$$

Therefore,

$$\sup_{m\geq n} f_l(y_m) \leq \sup_{m\geq n} \left\{ \sup_{k\in\mathbb{N}} f_k(y_m) \right\}$$

and hence

$$\lim_{n\to\infty} \left\{ \sup_{m\geq n} f_l(y_m) \right\} \leq \lim_{n\to\infty} \sup_{m\geq n} \left\{ \sup_{k\in\mathbb{N}} f_k(y_m) \right\}$$

for every $l \in \mathbb{N}$. Finally,

$$\sup_{k\in\mathbb{N}} \lim_{n\to\infty} \left\{ \sup_{m\geq n} f_k(y_m) \right\} \leq \lim_{n\to\infty} \sup_{m\geq n} \left\{ \sup_{k\in\mathbb{N}} f_k(y_m) \right\}.$$

□

Corollary 5. *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of functions so that, for every $x \in \mathbb{R}$, $\{f_k(x) : k \geq 1\}$ is a bounded set and $\lim_{x\to x_0} f_k(x)$ exists for every $k \geq 1$. Define the function $g(x) = \sup_{k\geq 1} f_k(x)$. Then, if $\lim_{x\to x_0} g(x)$ also exists,*

$$\sup_{k\geq 1} \lim_{x\to x_0} f_k(x) \leq \lim_{x\to x_0} g(x) = \lim_{x\to x_0} \sup_{k\geq 1} f_k(x). \tag{3}$$

Remark 1. *Equality in (3) cannot generally be attained: as a counterexample, we choose the functions*

$$f_k(x) = 1 - \left(\frac{1}{1+x}\right)^k, \; x > -1, \, k \geq 0.$$

Then, $\lim_{x\to 0} f_k(x) = 0$ for every $k \geq 1$, so that $\sup_{k\geq 1} \lim_{x\to 0} f_k(x)$. On the other hand, $\sup_{k\geq 1} f_k(x) = 1$ for every $k \geq 1$, $x > -1$, so that $\lim_{x\to 0} \sup_{k\geq 1} f_k(x) = 1$.

Proof of Theorem 8. Define, given $m \in \mathbb{N}$, the following function:

$$f_m(t) = \begin{cases} \alpha_m \left(\frac{t}{x_m}\right)^{x_m} & \text{if } 0 < t \leq x_m, \\ \beta_m \left(\frac{t}{x_m}\right)^{x_m} & \text{if } t > x_m, \end{cases}$$

where x_m, α_m and β_m are as in Definition 4.

The function f_m is multiplicative convex and it is discontinuous at $t = x_m$, since it has been defined following the contruction from Proposition 1.

We will show that the function $g(t) = \sup\{f_n(t) : n \in \mathbb{N}\}$ is discontinuous on the set $\mathcal{X} = \{x_n\}_{n=1}^{\infty}$. First of all, for $t > 0$ the set $\{f_n(t) : n \in \mathbb{N}\}$ is bounded since

$$\alpha_n, \beta_n \leq \beta_1 \leq e^{4x_1},$$
$$t^{x_n} \leq \begin{cases} 1 & \text{if } 0 < t \leq 1, \\ t & \text{if } t > 1, \end{cases} \quad \text{for every } n \geq 1 \text{ and}$$
$$x_n^{-x_n} \leq e^{1/e}$$

because of Lemma 1.

Let $n, m \in \mathbb{N}$. Then,

$$f_m\Big(x_n\Big) = \begin{cases} \alpha_m \Big(\frac{x_n}{x_m}\Big)^{x_m} & \text{if } 1 \le m \le n, \\ \beta_m \Big(\frac{x_n}{x_m}\Big)^{x_m} & \text{if } m \ge n+1. \end{cases}$$

Notice that, if $m < n$

$$\begin{aligned} \alpha_m \left(\frac{x_n}{x_m}\right)^{x_m} &= \left(\frac{x_n}{x_m}\right)^{x_m} \prod_{k \ge m} a_k \\ &= \left(\frac{x_n}{x_m}\right)^{x_m} \left(\frac{x_m}{x_{m+1}}\right)^{x_{m+1}} \left(\frac{x_{m+1}}{x_{m+2}}\right)^{x_{m+2}} \cdot \ldots \\ &\quad \ldots \cdot \left(\frac{x_{n-1}}{x_n}\right)^{x_n} \prod_{k \ge n} \left(\frac{x_k}{x_{k+1}}\right)^{x_{k+1}} \\ &= \left(\frac{x_n}{x_m}\right)^{x_m} \left(\frac{x_m}{x_{m+1}}\right)^{x_{m+1}} \left(\frac{x_{m+1}}{x_{m+2}}\right)^{x_{m+2}} \cdot \ldots \cdot \left(\frac{x_{n-1}}{x_n}\right)^{x_n} \alpha_n \\ &< \left(\frac{x_n}{x_{n-1}}\right)^{x_{n-1}} \left(\frac{x_{n-1}}{x_n}\right)^{x_n} \alpha_n \\ &< \alpha_n, \end{aligned}$$

by Lemma 4.

If $m \ge n+2$, and using Corollary 4, we can prove that

$$\begin{aligned} \beta_n &= \left(\frac{x_n}{x_{n+1}}\right)^{2x_{n+1}} \beta_{n+1} \\ &> \left(\frac{x_n}{x_{n+1}}\right)^{x_{n+1}} \beta_{n+1} \\ &> \left(\frac{x_n}{x_m}\right)^{x_m} \beta_m. \end{aligned}$$

In conclusion, we can say that $g(x_n) < \beta_n$.
On the other hand,

$$\lim_{x \to x_n^+} f_m(x) = \begin{cases} \alpha_m \Big(\frac{x_n}{x_m}\Big)^{x_m} & \text{if } 1 \le m \le n-1, \\ \beta_m \Big(\frac{x_n}{x_m}\Big)^{x_m} & \text{if } m \ge n. \end{cases}$$

With a similar argument as before, $\sup\left\{ \lim_{x \to x_n^+} f_m(x) \,:\, m \ge 1 \right\} = \beta_n$.

Hence, by Corollary 5

$$\begin{aligned} \lim_{x \to x_n^+} g(x) &= \lim_{x \to x_n^+} \sup\{f_m(x) \,:\, m \ge 1\} \\ &\ge \sup\left\{ \lim_{x \to x_n^+} f_m(x) \,:\, m \ge 1 \right\} \\ &= \beta_n > g(x_n). \end{aligned}$$

As a consequence, g is not continuous at x_n. □

The following result complements Theorem 8 and shows that $\mathfrak{D}(g) = \mathcal{X}$:

Proposition 3. *The function g considered in the proof of Theorem 8 is continuous on $(0, \infty) \setminus \mathcal{X}$.*

Proof. Let f_m be the functions defined in the proof of Theorem 8, $m_0 \geq 1$ and $x_{m_0} < x < x_{m_0-1}$. Then,

$$f_m(x) = \begin{cases} \alpha_m \left(\frac{x}{x_m}\right)^{x_m} & \text{if } m \leq m_0 - 1, \\ \beta_m \left(\frac{x}{x_m}\right)^{x_m} & \text{if } m \geq m_0. \end{cases}$$

If $m \leq m_0 - 1$, then $x < x_m$ so that

$$\begin{aligned} & x_m^{x_{m+1}-x_m} \leq x^{x_{m+1}-x_m}, \quad \text{from which} \\ & \left(\frac{x_m}{x_{m+1}}\right)^{x_{m+1}} \left(\frac{x}{x_m}\right)^{x_m} \leq \left(\frac{x}{x_{m+1}}\right)^{x_{m+1}} \quad \text{and therefore} \\ & \alpha_m \left(\frac{x}{x_m}\right)^{x_m} \leq \alpha_{m+1} \left(\frac{x}{x_{m+1}}\right)^{x_{m+1}}, \end{aligned}$$

leading us to conclude that

$$\sup\{f_m(x) \, : \, 1 \leq m \leq m_0 - 1\} = \alpha_{m_0-1} \left(\frac{x}{x_{m_0-1}}\right)^{x_{m_0-1}}.$$

If now $m \geq m_0$, then $x_m < x$, which implies that $x^{x_{m+1}-x_m} < x_m^{x_{m+1}-x_m}$. On the other hand, from $x_{m+1} < x_m$ we can deduce that

$$x_m^{x_{m+1}-x_m} < \frac{x_m^{2x_{m+1}-x_m}}{x_{m+1}^{x_{m+1}}},$$

which allows us to deduce that

$$x^{x_{m+1}-x_m} < \frac{x_m^{2x_{m+1}-x_m}}{x_{m+1}^{x_{m+1}}}.$$

This last inequality is equivalent to

$$\left(\frac{x}{x_{m+1}}\right)^{x_{m+1}} < \left(\frac{x_m}{x_{m+1}}\right)^{2x_{m+1}} \left(\frac{x}{x_m}\right)^{x_m},$$

that is,

$$\beta_{m+1} \left(\frac{x}{x_{m+1}}\right)^{x_{m+1}} < \beta_m \left(\frac{x}{x_m}\right)^{x_m}.$$

As a consequence,

$$\sup\{f_m(x) \, : \, m \geq m_0\} = \beta_{m_0} \left(\frac{x}{x_{m_0}}\right)^{x_{m_0}}$$

and therefore

$$\begin{aligned} g(x) &= \sup\{f_m(x) \, : \, m \geq 1\} \\ &= \max\left\{\alpha_{m_0-1} \left(\frac{x}{x_{m_0-1}}\right)^{x_{m_0-1}}, \beta_{m_0} \left(\frac{x}{x_{m_0}}\right)^{x_{m_0}}\right\}. \end{aligned}$$

Hence, on $(x_m.x_{m-1})$, g can be expressed as the maximum of two continuous functions and in conclusion it is continuous over that interval.

If next $0 < x < x_0 = \lim_{n\to\infty} x_n$ (which in particular implies that $x_0 \neq 0$), then $f_m(x) = \alpha_m \left(\frac{x}{x_m}\right)^{x_m}$ and, as above, we would have

$$\alpha_m \left(\frac{x}{x_m}\right)^{x_m} \leq \alpha_{m+1} \left(\frac{x}{x_{m+1}}\right)^{x_{m+1}}.$$

Therefore,

$$g(x) = \lim_{m\to\infty} \alpha_m \left(\frac{x}{x_m}\right)^{x_m} = \left(\frac{x}{x_0}\right)^{x_0},$$

which again is a continuous function.

For the case $x = x_0$ we notice that $g(x_0) = 1$. Assume $\{y_n\}_{n=1}^{\infty}$ is such that $y_n \to x_0$ as $n \to \infty$. If $y_n < x_0$ then $g(y_n) = \left(\frac{y_n}{x_0}\right)^{x_0}$. On the other hand, if $y_n > x_0$ then we can find $\{m_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ so that $x_{m_n} \le y_n < x_{m_n-1}$. As before,

$$g(y_n) = \max\left\{\alpha_{m_n-1}\left(\frac{y_n}{x_{m_n-1}}\right)^{x_{m_n-1}}, \beta_{m_n}\left(\frac{y_n}{x_{m_n}}\right)^{x_{m_n}}\right\}.$$

Since

$$\lim_{n\to\infty}\left(\frac{y_n}{x_0}\right)^{x_{m_0}} = \lim_{n\to\infty} \alpha_{m_n-1}\left(\frac{y_n}{x_{m_n-1}}\right)^{x_{m_n-1}} = \lim_{n\to\infty} \beta_{m_n}\left(\frac{y_n}{x_{m_n}}\right)^{x_{m_n}} = 1,$$

we are able to conclude that $\lim_{n\to\infty} g(y_n) = g(x_0)$.

Finally, for the case $x > x_1$ we would obtain that $f_m(x) = \beta_m\left(\frac{x}{x_m}\right)^{x_m}$ and hence (again, similarly as before)

$$g(x) = \beta_1\left(\frac{x}{x_1}\right)^{x_1},$$

so g is continuous on x. □

Remark 2. *If $\mathcal{X} \subseteq (0,1)$ is an increasing sequence, then there exists an mc-function so that $\mathfrak{D}(f) = \mathcal{X}$.*

Indeed, we only need to change the definition of the elements a_n in Definition 4 as follows:

$$a_n = \left(\frac{x_{n+1}}{x_n}\right)^{x_n}.$$

Corollary 6. *If $\mathcal{X}$ is a monotone sequence, then we can find an mc-function which is discontinuous on $\mathcal{X}$.*

Proof. Let $\mathcal{X}_1 = \mathcal{X} \cap (0,1)$ and $\mathcal{X}_2 = \mathcal{X} \cap [1,\infty)$. We can find two functions f_1 and f_2 so that f_1 is discontinuous only on $\mathcal{X}_1$ and f_2 is discontinuous only on the set $\left\{\frac{1}{x} : x \in \mathcal{X}_2\right\}$.

The required function is then $f(x) = f_1(x) + f_2(\frac{1}{x})$. □

5. Algebraic Structure on $\mathcal{MC} \setminus C(0,\infty)$

Theorem 10. *There exists an algebraic truncated cone of algebraic dimension $\mathfrak{c}$ every non-trivial algebraic combination of which is an mc-function which is discontinuous at infinitely many points.*

Proof. Let us consider a $\mathbb{Q}$-linearly independent set of cardinality $\mathfrak{c}$, $\{a_\zeta : \zeta < \mathfrak{c}\} \subseteq (4,6)$. Then, for every $\zeta < \mathfrak{c}$ we consider a decreasing sequence, $\mathcal{X}_\alpha = \{x_{k,\zeta}\}_{k=1}^{\infty}$ converging to $\frac{1}{a_\zeta}$ and so that $x_{1,\zeta} = \frac{1}{4}$.

Using Theorem 8 for every $\zeta < \mathfrak{c}$ there exists an *mc*-function $\tilde{f}_\zeta$ which is discontinuous on $\mathcal{X}_\zeta$.

In particular, taking a look at the proof of Theorem 8, the function $\tilde{f}_\zeta$ is defined as $\tilde{f}_\zeta(x) = \sup\{f_{k,\zeta}(x) : k \in \mathbb{N}\}$, where

$$f_{k,\zeta}(x) = \begin{cases} \alpha_{k,\zeta}\left(\frac{x}{x_{k,\zeta}}\right)^{x_{k,\zeta}} & \text{if } 0 < x \le x_{k,\zeta}, \\ \beta_{k,\zeta}\left(\frac{x}{x_{k,\zeta}}\right)^{x_{k,\zeta}} & \text{if } x > x_{k,\zeta} \end{cases},$$

$$\alpha_{k,\zeta} = \prod_{l \ge n}\left(\frac{x_{l,\zeta}}{x_{l+1,\zeta}}\right),$$

$$\beta_{k,\zeta} = \prod_{l \ge n}\left(\frac{x_{l,\zeta}}{x_{l+1,\zeta}}\right)^2.$$

In particular, if $x > x_{1,\zeta}$ then

$$\tilde{f}_\zeta(x) = \sup\left\{\beta_{k,\zeta}\left(\frac{x}{x_{k,\zeta}}\right)^{x_{k,\zeta}} \,:\, k \in \mathbb{N}\right\}.$$

Because of Lemma 1 and since $\left\{\frac{1}{x_{k,\zeta}}\right\}_{k=1}^{\infty}$ is an increasing sequence with $\frac{1}{x_{1,\zeta}} = 4 > e$, we obtain that if $x > x_{1,\zeta} = \frac{1}{4}$ then $\tilde{f}_\zeta(x) = \beta_{1,\zeta}(4x)^{1/4}$.

Define, for $\zeta < \mathfrak{c}$, the auxiliary function

$$g_\zeta(x) = \frac{4}{4^{1/4}\beta_{1,\zeta}} x^{a_\zeta - 1/4}$$

From the proof of Lemma 3, $\beta_{1,\zeta} < e^{4x_{1,\zeta}} = e$ and therefore

$$1 < \frac{4}{4^{1/4}\beta_{1,\zeta}},$$

so g_ζ is a (continuous) *mc*-function.

Define then the function

$$f_\zeta(x) = \tilde{f}_\zeta(x) g_\zeta(x).$$

Then f_ζ is an *mc*-function (because it is the product of two *mc*-functions) which is discontinuous on the set $\mathcal{X}_\zeta$.

Furthermore, if $\frac{1}{4} < x$,

$$f_\zeta(x) = \tilde{f}_\zeta(x) g_\zeta(x) = \beta_{1,\zeta}(4x)^{1/4}\frac{4}{4^{1/4}\beta_{1,\zeta}} x^{a_\zeta - 1/4} = 4x^{a_\zeta}.$$

Let us show that $B = \{f_\zeta \,:\, \zeta < \mathfrak{c}\}$ is an algebraically independent set.

Indeed, let $\zeta_1, \zeta_2, \ldots, \zeta_n < \mathfrak{c}$, $\lambda_1, \ldots, \lambda_m$ be non-zero numbers and $N = (n_{i,j})_{i,j=1}^{n,m}$ be a matrix consisting of natural numbers as entries and without two equal columns.

Assume that

$$f = \sum_{j=1}^{m} \lambda_j \prod_{i=1}^{n} f_{\zeta_i}^{n_{i,j}} = 0.$$

Then, we notice that for $\frac{1}{4} < x$ it must be

$$\begin{aligned} 0 = f(x) &= \sum_{j=1}^{m} \lambda_j \prod_{i=1}^{n} f_{\zeta_i}^{n_{i,j}}(x) \\ &= \sum_{j=1}^{m} \lambda_j \prod_{i=1}^{n} \left(4x^{a_{\zeta_i}}\right)^{n_{i,j}} \\ &= \sum_{j=1}^{m} \lambda_j \left(4^{\sum_{i=1}^{n} n_{i,j}}\right) x^{\sum_{i=1}^{n} a_{\zeta_i} n_{i,j}}. \end{aligned}$$

Since the elements $a_{\zeta_1}, a_{\zeta_2}, \ldots, a_{\zeta_n}$ are $\mathbb{Q}$-linearly independent and the columns of $N = \{n_{i,j}\}_{i,j=1}^{n,m}$ are pairwise different, we can conclude that the exponents

$$\sum_{i=1}^{n} a_{\zeta_i} n_{i,j}$$

are all different from each other.

Therefore, $f_{|(1/4,\infty)}$ is an identically null extended polynomial (with positive exponents). Hence, all its coefficients must be zero and, in conclusion, if $1 \le j \le m$

$$\lambda_j 4^{\sum_{i=1}^{n} n_{i,j}} = 0, \text{ which implies that } \lambda_j = 0$$

and in conclusion B is algebraically independent.

To finish with, let us choose an element f in the truncated cone generated by B. Then, we can find $\zeta_1, \zeta_2, \ldots, \zeta_n < \mathfrak{c}$, $\lambda_1, \ldots, \lambda_m \ge 1$ and a matrix consisting of natural numbers as entries and without two equal columns $N = \{n_{i,j}\}_{i,j=1}^{n,m}$ so that

$$f = \sum_{j=1}^{m} \lambda_j \prod_{i=1}^{n} f_{\zeta_i}^{n_{i,j}}.$$

Without loss of generality we may assume that $a_{\zeta_i} < a_{\zeta_{i+1}}$ for every $1 \le i \le n-1$. Then, f_{ζ_i} is continuous on $(0, a_{\zeta_2})$ for every $i \ge 2$.

For any function $h : \mathbb{R} \to \mathbb{R}$ we denote $h^{a^-} = \lim_{x \to a^-} h(x)$ and $h^{a^+} = \lim_{x \to a^+} h(x)$. We can then write

$$f = f_{\zeta_1}^{n_{1,1}} g_1 + \ldots + f_{\zeta_1}^{n_{j,1}} g_j + \ldots + f_{\zeta_1}^{n_{m,1}} g_m,$$

where g_j is an *mc*-function continuous on $(0, a_{\zeta_2})$.

For every $\beta \in \mathcal{X}_{\zeta_i} \cap (0, a_{\zeta_2})$, we have that

$$f^{\beta^+} = (f_{\zeta_1}^{\beta^+})^{n_{1,1}} g_1(\beta) + \ldots + (f_{\zeta_1}^{\beta^+})^{n_{j,1}} g_j(\beta) + \ldots + (f_{\zeta_1}^{\beta^+})^{n_{m,1}} g_1(\beta)$$

and

$$f^{\beta^-} = (f_{\zeta_1}^{\beta^-})^{n_{1,1}} g_1(\beta) + \ldots + (f_{\zeta_1}^{\beta^-})^{n_{j,1}} g_j(\beta) + \ldots + (f_{\zeta_1}^{\beta^-})^{n_{m,1}} g_m(\beta).$$

Since the function f_{ζ_1} is not continuous at β we may assume without loss of generality that $f_{\zeta_1}^{\beta^-} < f_{\zeta_1}^{\beta^+}$ (the procedure would be analogous for the case $f_{\zeta_1}^{\beta^-} > f_{\zeta_1}^{\beta^+}$) and observe that if $f_{\zeta_1}^{\beta^-} < f_{\zeta_1}^{\beta^+}$ then $(f_{\zeta_1}^{\beta^-})^q < (f_{\zeta_1}^{\beta^+})^q$ for every $q \in \mathbb{N} \setminus \{0\}$. We can now write

$$\begin{aligned} f^{\beta^+} - f^{\beta^-} =& (f_{\zeta_1}^{\beta^+})^{n_{1,1}} g_1(\beta) + \ldots + (f_{\zeta_1}^{\beta^+})^{n_{m,1}} g_m(\beta) \\ & - \left[(f_{\zeta_1}^{\beta^-})^{n_{1,1}} g_1(\beta) + \ldots + (f_{\zeta_1}^{\beta^-})^{n_{m,1}} g_m(\beta) \right] \\ =& g_1(\beta) \left[(f_{\zeta_1}^{\beta^+})^{n_{1,1}} - (f_{\zeta_1}^{\beta^-})^{n_{1,1}} \right] + \ldots + g_j(\beta) \left[(f_{\zeta_1}^{\beta^+})^{n_{j,1}} - (f_{\zeta_1}^{\beta^-})^{n_{j,1}} \right] \\ & + \ldots + g_m(\beta) \left[(f_{\zeta_1}^{\beta^+})^{n_{m,1}} - (f_{\zeta_1}^{\beta^-})^{n_{m,1}} \right] \\ >& 0, \end{aligned}$$

so that f is discontinuous on $\mathcal{X}_{\zeta_1} \cap (0, a_{\zeta_2})$ and the proof is done. □

6. Conclusions

In [2] the authors focused on those *mc*-functions that attained the value 1 at $x = 1$. It was proved that such functions were continuous and that in fact they could only be monotone, or decreasing until $x = 1$ and increasing from $x = 1$. In [3] some discontinuous *mc*-functions were introduced. This paper completes both of them in setting the cardinality of the set of all *mc*-functions at $\mathfrak{c}$ (and therefore "there are not really many more *mc*-functions out of the cathegory of continuous functions") and showing that every of these functions show either monotonous or decreasing–increasing behavior.

It also continues the ideas shown in [3] when it constructs a trunkated cone consisting on *mc*-functions which this time have infinitely many points of discontinuity.

There are still some questions that remain open, namely whether the set of discontinuous *mc*-functions is a truncated cone itself, and what kind of properties can be concluded for the set of points of discontinuity of an *mc*-function.

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References

1. Niculescu, C.P. Convexity according to the geometric mean. *Math. Inequal. Appl.* **2000**, *2*, 155–167. [CrossRef]
2. Jiménez-Rodríguez, P.; Martínez-Gómez, M.E.; Muñoz-Fernández, G.A.; Seoane-Sepúlveda, J.B. Describing multiplicative convex functions. *J. Convex Anal.* **2020**, *27*, 935–942.
3. Jiménez-Rodríguez, P.; Martínez-Gómez, M.E.; Muñoz-Fernández, G.A.; Seoane-Sepúlveda, J.B. Generalizing multiplicative convex functions. *J. Convex Anal.* **2021**, *28*.
4. Aron, R.M.; Gurariy, V.I.; Seoane Sepúlveda, J.B. Lineability and spaceability of sets of functions on $\mathbb{R}$. *Proc. Am. Math. Soc.* **2005**, *133*, 795–803. [CrossRef]
5. Aron, R.M.; Bernal, González, L.; Pellegrino, D.M.; Seoane Sepúlveda, J.B. *Lineability: The Search for Linearity in Mathematics*; Monographs and Research Notes in Mathematics; CRC Press: Boca Raton, FL, USA, 2016; pp. xix+308, ISBN 978-1-4822-9909-0.
6. Bastin, F.; Conejero, J.A.; Esser, C.; Seoane-Sepúlveda, J.B. Algebrability and nowhere Gevrey differentiability. *Israel J. Math.* **2015**, *205*, 127–143. [CrossRef]
7. Bernal-González, L.; Pellegrino, D.; Seoane-Sepúlveda, J.B. Linear subsets of nonlinear sets in topological vector spaces. *Bull. Am. Math. Soc.* **2014**, *51*, 71–130. [CrossRef]
8. Ciesielski, K.; Seoane-Sepúlveda, J. Differentiability versus continuity: Restriction and extension theorems and monstrous examples. *Bull. Am. Math. Soc.* **2019**, *56*, 211–260. [CrossRef]
9. Froda, A. Sur la Distribution des Propriétés de Voisinage des Fonctions de Variables Réelles. Ph.D. Thesis, Paris Science Faculty, Paris France, 1929.
10. Moschovakis, Y. Notes on set theory. In *Undergraduate Texts in Mathematics*, 2nd ed.; Springer: New York, NY, USA, 2006; p. xii+276, ISBN 978-0387-28722-5.
11. Steiner, J. Über das gröβte Product der Theile oder Summanden jeder Zahl. *Journal für die Reine und Angewandte Mathematik* **1850**, *40*, 208–208.