On the finite dimension of attractors of parabolic problems in $I\!R^N$ with general potentials*

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1 Introduction

In this paper we study the finite dimensionality of attractors of nonlinear reaction diffusion equations of the type

$$\begin{cases} \partial_t u - \Delta u = f(x,u), & x \in I\!R^N \\ u(0) = u_0. \end{cases} \tag{1.1}$$

Since the equation is set in the unbounded domain $I\!R^N$ even the existence of the attractor has not been successfully understood until recently. This contrast with the case of bounded domain for which both, existence of the attractor and its finite dimensionality, have been addressed several years ago and is nowadays well understood, see for instance [13, 20, 18] and reference therein. Note that, for the case of unbounded domains, most of the difficulties to analyze the asymptotic behavior of solutions of (1.1) comes from the lack of compactness of the Sobolev emebeddings. To overcome this difficulty and in order to analyze the dissipativity mechanisms of (1.1), to measure the compactness effects of the nonlinear flow and ultimately to show the existence of an attractor, several approaches have been followed in the literature. These includes the use of weighted Sobolev spaces, locally uniform spaces, spaces of bounded and uniformly continuous functions etc. See [2] for an exhaustive discussion on this matter.

In all of these cases, at the end, a global attractor $\mathcal{A}$ is obtained and in most cases, this attractor is a bounded set in $L^2(I\!R^N) \cap L^\infty(I\!R^N)$.

Actually, this is our starting point in this paper. We will assume that we already have an attractor $\mathcal{A}$ or a compact invariant set of (1.1), which lies in a bounded set of $L^2(I\!R^N) \cap L^\infty(I\!R^N)$ and we address the question of the finite dimensionality of the attractor in the space $L^2(I\!R^N)$. Some of the examples that have been our guidelines are the following.

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• Babin and Vishik in their pioneering work in this field, [6], consider nonlinear terms of the form

$$f(x,u) := -\lambda_0 u + f_0(u) + g(x), \tag{1.2}$$

with $\lambda_0 > 0$, $g \in L^2(I\!R^N)$ and suitable growth and sign assumptions that guarantee that (1.1) can be solved with initial data $u_0 \in L^2(I\!R^N)$. Under further sign assumptions on f_0 they show the attractor exist in suitable weighted spaces. The case of an $L^2(I\!R^N)$ attractor was proved later in [22].

Then they prove that if $|f_0'(u)| \leq C|u|^{\alpha_0}C_1(u)$, for some $\alpha_0 > 0$ and some continuous function $C_1(u)$, then the Hausdorff dimension of the attractor is finite.

• Efendiev and Zelik in [10] assume that

$$f(x,u) := -\lambda_0 u + f_0(u, \nabla u) + g(x) \tag{1.3}$$

with $\lambda_0 > 0$, the dissipative condition $f_0(u, \nabla u).u \leq 0$ and the growth assumption

$$|\frac{\partial f_0}{\partial u}(u, \nabla u)| \leq C(1 + |u|^p)(1 + |\nabla u|)^r,$$
$$|\frac{\partial f_0}{\partial \xi}(u, \nabla u)| \leq C(1 + |u|^{p+1})(1 + |\nabla u|^{r-1}).$$

for some $r < 2$. They show then that the Hausdorff dimension of the attractor is finite.

• In another pioneering work, F. Abergel in [1], considers a nonlinear term of the form

$$f(x,u) := -\lambda_0 u - u\big(l(x)\nabla u\big) + g(x) \tag{1.4}$$

with $\lambda_0 > 0$, in an exterior unbounded domain in $I\!R^2$, where $l(x)$ is a smooth divergence free vector field, that is $Div\ l(x) = 0$. He also proves that the Hausdorff dimension of the attractor is finite.

In all these examples, one of the crucial assumptions is the fact that the first order linear term in (1.2), (1.3) and (1.4) is of the form $-\lambda_0 u$ with λ_0 a positive constant. This implies that the linear elliptic operator $A := -\Delta + \lambda_0$ has a positive bottom spectrum, actually given by λ_0, which implies exponential decay of the corresponding linear semigroup. Moreover, the fact that λ_0 is constant is essential in the method of proof of the finite dimensionality of the attractors in the papers mentioned above.

In this paper, following the approach in [2], we assume more flexible structure conditions on the linear and nonlinear terms in (1.1). More precisely, we assume that in (1.1) we have

$$f(x,u) := m(x)u + f_0(x,u) + g(x), \tag{1.5}$$

with

$$f_0(x,0) = 0, \quad \frac{\partial f_0}{\partial u}(x,0) = 0. \tag{1.6}$$

The linear potential m is assumed to be an element of the locally uniform space $L^{\sigma}_{U}(I\!R^N)$, for some $\sigma > N/2$, and in particular the potential m may change sign and it does not need to be constant. Moreover, we do not prescribe any behavior of the potential m as $|x| \to \infty$ and m may also have local singularities. The space $L^{\sigma}_{U}(I\!R^N)$ (see [2, 3]) is defined as the set of $\phi \in L^{\sigma}_{loc}(I\!R^N)$ such that

$$\sup_{x \in \mathbb{R}^N} \int_{B(x,1)} |\phi(y)|^{\sigma} \, dy < \infty \tag{1.7}$$

with norm

$$\|\phi\|_{L^{\sigma}_{U}(\mathbb{R}^N)} = \sup_{x \in \mathbb{R}^N} \|\phi\|_{L^{\sigma}(B(x,1))}.$$

With these hypotheses and some mild growth assumptions on f_0, local existence for (1.1) can be achieved in suitable function spaces of initial data. See [2] for the case of standard Lebesgue spaces, [3, 4] for the case of locally uniform spaces and [17] for the case of weighted spaces.

Concerning global existence the main assumptions in the references above is

$$f(x,s)s \le C(x)|s|^2 + D(x)|s|, \quad \text{for all} \quad s \in I\!R, \ x \in I\!R^N, \tag{1.8}$$

where $C \in L^{\sigma}_{U}(I\!R^N)$ for $\sigma > N/2$ and suitable integrability assumptions on D.

Also, uniform estimates on solutions and the existence of an attractor are obtained provided that the operator $\Delta + C(x)$ has negative exponential type; see Definition 3.1 below.

We will show in this paper that, regardless of the space of initial data for (1.1), if we assume that the attractor or the compact invariant set, satisfies

$$\mathcal{A} \subset L^{\infty}(I\!R^N) \cap L^{p}(I\!R^N) \cap L^{2}(I\!R^N) \quad \text{is bounded.}$$

for some $p > N/2$, then it has a finite Hausdorff and fractal dimensions in the metric of $L^2(I\!R^N)$, see Theorem 4.3. For this, we will make use of general techniques developed, e.g. in Temam [20], Babin and Vishik [5], Robinson [18], Hale [13] or Chepyzhov and Vishik [8], but we will have to analyze in detail the effect of a nonconstant and, possibly, sign changing singular potential m in the equation.

The paper is organized as follows. In Section 2 we present some of the general tools to estimate the dimension of comapct invariant sets. In Section 3 we prove that, under suitable assumptions on the attractor, the semigroup defined by (1.1) is uniformly differentable in $L^2(I\!R^N)$, which is a technical condition required to apply the results in Section 2. In Section 4 we prove the main theorem quoted above about the finite dimensionality of the attractor, see Theorem 4.3 and give some other extensions of our technique. The result is particularized for the prototype problem (1.1) with bistable nonlinear term

$$f(x,s) = m(x)s - n(x)s^3,$$

with $0 \le n \in L^{\infty}(I\!R^N)$. Also, in Theorem 4.6 we generalized the result in [6] for nonlinearities of the form

$$f(x,u) = m(x)u + f_0(u) + g(x), \quad f_0 \in C^1(I\!R) \quad f_0'(0) = 0 = f_0(0).$$

Then, in Section 5, we treat some examples in which the nonlinear term depends on the gradient in the spirit of the examples above. More precisely in Theorem 5.1 we generalize the result in [10] for nonlinear terms of the form

$$f(x, u, \nabla u) = m(x)u + f_0(u, \nabla u) + g(x).$$

On the other hand in Theorem 5.2 we generalize the result in [1] for

$$f(x, u, \nabla u) = m(x)u + u(l(x)\nabla u) + g(x).$$

2 A general technique for the dimension of a compact invariant set.

In this section we present some well known basic results concerning Hausdorff and fractal dimensions of compact invariant sets. A detailed exposition can be found for example in Temam [20], Babin and Vishik [5], Robinson [18], Hale [13], Chepyzhov and Vishik [8]. Assume X is a metric space and $A \subset X$ is a compact set. The Hausdorff and fractal dimensions are defined by the following procedure which approximates the d-dimensional volume of A by a covering with balls of a certain radius. More precisely, we have

Definition 2.1. *Let $d \in I\!R^+$ and $\epsilon > 0$. We define*

$$\mu_H(A, d, \epsilon) = \inf\{\sum_{i\in I} r_i^d : A \subset \bigcup_{i\in I} B(x_i, r_i),\ r_i \leq \epsilon\}$$

where the infimum is taken over all the finite coverings of A with balls of radius $r_i \leq \epsilon$.

Observe that $\mu_H(A, d, \epsilon)$ is a decreasing function of ϵ and d.

Definition 2.2. *The d-dimensional Hausdorff measure of A, $\mu_H(A, d)$ is defined as*

$$\mu_H(A, d) := \lim_{\epsilon\to 0} \mu_H(A, d, \epsilon) = \sup_{\epsilon>0} \mu_H(A, d, \epsilon).$$

Observe that $\mu_H(A, d) \in [0, \infty]$ and if $\mu_H(A, d) < \infty$ then $\mu_H(A, \tilde{d}) = 0$ for all $\tilde{d} > d$. Moreover if $\mu_H(A, d) > 0$ then $\mu_H(A, \hat{d}) = \infty$ for all $\hat{d} < d$.

Definition 2.3. *The Hausdorff dimension of A is the smallest d for which $\mu_H(A, d)$ is finite, that is*

$$d_H(A) = \inf_{d>0}\{d : \mu_H(A, d) = 0\}.$$

We define now the Fractal dimension of A.

Definition 2.4. *For a given* $\epsilon > 0$*, let* $n_A(\epsilon)$ *denote the minimum number of balls* $B_X(x_i, \epsilon)$ *or radius* ϵ *which is needed to cover* A*. Then,*
i) The fractal (d, ϵ)*-dimensional measure of* A *is given by*

$$\mu_F(A, d, \epsilon) := \epsilon^d n_A(\epsilon).$$

ii) The fractal $d-$*dimensional measure of* A *is given by*

$$\mu_F(A, d) := \limsup_{\epsilon \to 0} \mu_F(A, d, \epsilon).$$

iii) The fractal dimension of A*, also, known as capacity, is the number*

$$d_F(A) = \limsup_{\epsilon \to 0} \frac{\log\ n_A(\epsilon)}{\log\ (\frac{1}{\epsilon})}.$$

It is then clear that $\mu_H(A, d, \epsilon) \leq \mu_F(A, d, \epsilon)$, which implies $\mu_H(A, d) \leq \mu_F(A, d)$, hence $d_H(A) \leq d_F(A)$. There are know examples for which $d_H(A) < d_F(A)$, see [20].

When the set A is a compact invariant set for a nonlinear semigroup, then the dimension of A can be estimated by using some properties of the flow. We refer to [16] for a pioneer work in this idea and for [11, 15] for other important contributions.

In this case, for (1.1) we take $X = L^2(I\!R^N)$ and assume A is a compact invariant set for the nonlinear semigroup $\{T(t)\}_{t\geq 0}$ defined by (1.1), that is

$$T(t)A = A, \quad \text{for } t \geq 0. \tag{2.1}$$

Then we assume that $\{T(t)\}_{t\geq 0}$ is uniformly differentiable on A, that is, for each $u_0 \in A$ there exists a linear bounded operator $L = L(t, u_0) \in \mathcal{L}(X, X)$ such that for every $t \geq 0$ we have

$$\sup_{\{u_0, v_0 \in A,\ 0 < \|u_0 - v_0\|_X \leq \epsilon\}} \frac{\|T(t)v_0 - T(t)u_0 - L(t, u_0)(v_0 - u_0)\|_X}{\|v_0 - u_0\|_X} \xrightarrow{\epsilon \to 0} 0 \tag{2.2}$$

and

$$\sup_{u_0 \in A} \|L(t, u_0)\|_{\mathcal{L}(X,X)} < +\infty. \tag{2.3}$$

The geometric idea behind the estimate of the dimension is to analyze the evolution of a d-dimensional volume under the action of the semigroup on the invariant set $\mathcal{A}$. Then one searches for the smallest d for which any d-dimensional volume contracts asymptotically as $t \to \infty$.

Then we take $u_0 \in A$ and consider d orthogonal functions in $L^2(I\!R^N)$ and we denote by $V_d(0)$ the d-dimensional volume delimited by them. Then these vectors and volume evolve by the flow of the equation (1.1) linearized along the trajectory $u(t, u_0)$. Hence the volume

$V_d(t)$ is given by the initial volume times the factor $\exp\Big(\int_0^t Tr\big(A_1(s,u_0)\circ Q_d(s)\big)\ ds\Big)$ where $A_1(t)$ is the linearized operator from (1.1) long $u(t,u_0)$, that is

$$\partial_t U = \Delta U + \frac{\partial f}{\partial u}(x, u(t,u_0))U := A_1(t)U \tag{2.4}$$

and Q_d is a suitable orthogonal projection of rank d. Hence, to obtain the exponential decay of $V_d(t)$ it is enough to show that for all such projections Q_d, we have

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t Tr\Big(A_1(s,u_0)\circ Q_d(s)\Big)\ ds < 0.$$

Definition 2.5. *For each $d \in \mathbb{N}$ we define the d-dimensional trace of $A_1(t)$ as,*

$$Tr_d(A_1(t)) := \sup_{E_d} Tr_d(A_1(t), E_d) := \sup_{E_d} \sum_{i=1}^{d} (A_1(t)\varphi_i, \varphi_i)_{L^2(\mathbb{R}^N)} \tag{2.5}$$

where E_d is a d-dimensional subspace of $L^2(\mathbb{R}^N)$, and $\varphi_i \in E_d$, $i = 1,\dots,d$ is an orthonormal basis in $L^2(\mathbb{R}^N)$ of E_d.

From the above, it is sufficient to verify that for some $d \in \mathbb{N}$ one has

$$\limsup_{t\to\infty} \sup_{u_0\in A} \frac{1}{t}\int_0^t Tr_d(A_1(\tau))d\tau < 0 \tag{2.6}$$

and this implies that $dim_H A \le dim_F A < \infty$, see [18, 20, 8].

3 Uniform differentiability on the attractor

In what follows we check the uniform differentiability conditions (2.2), (2.3) for equation (1.1). Note that for this, it is not really required below that the set $\mathcal{A}$ is an attractor. We just need that it is positively invariant.

In what follows we will make use of the following property of the Schrödinger operator $\Delta + m(x)I$, with $m \in L^\sigma_U(\mathbb{R}^N)$, $\sigma > N/2$: for some $\mu \in \mathbb{R}$ we have

$$-\mu \int_{\mathbb{R}^N} |z|^2\, dx \le \inf\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla z|^2 - \int_{\mathbb{R}^N} m(x)|z|^2\, dx, \quad z \in H^1(\mathbb{R}^N)\}, \tag{3.1}$$

see [19].

Definition 3.1. *We say that the Schrödinger operator $\Delta + m(x)I$, with $m \in L^\sigma_U(\mathbb{R}^N)$, $\sigma > N/2$ has a negative exponential type if we can take $\mu < 0$ in (3.1).*

Note that this is equivalent to saying that the analytic semigroup generated by $\Delta + m(x)I$, that we denote by $S_m(t)$, decays expoentially.

First we prove the continuous dependence of solutions of (1.1) with respect of initial data.

Lemma 3.2. *Assume that f_0 in (1.5) satisfies*

$$\left|\frac{\partial f_0}{\partial s}(x,s)\right| \le C(R), \quad \textit{for} \quad |s| \le R, \quad x \in I\!R^N. \tag{3.2}$$

Moreover assume the attractor $\mathcal{A}$ exists in $L^2(I\!R^N)$ and it is bounded in $L^\infty(I\!R^N)$.

Then, if u and v are solutions of (1.1) with initial data $u_0, v_0 \in \mathcal{A}$ then

$$\|v(t) - u(t)\|_{L^2(\boldsymbol{R}^N)} \le \gamma(t)\|u_0 - v_0\|_{L^2(\boldsymbol{R}^N)} \tag{3.3}$$

with $\gamma(t) = e^{C_{\mathcal{A}} t}$ for some constant $C_{\mathcal{A}}$ depending on the attractor $\mathcal{A}$.

Proof. Note that $z = v - u$ satisfies

$$\left\{ \begin{array}{l} z_t - \Delta z = m(x)z + f_0(x,v) - f_0(x,u), \\ z(0) = v_0 - u_0. \end{array} \right. \tag{3.4}$$

Then multiplying by z and integrating in $I\!R^N$ we get

$$\frac{1}{2}\frac{d}{dt}\|z\|^2_{L^2(\boldsymbol{R}^N)} \quad + \quad \int_{\boldsymbol{R}^N} |\nabla z|^2 \, dx = \int_{\boldsymbol{R}^N} m(x) z^2 \, dx + \int_{\boldsymbol{R}^N} [f_0(x,v) - f_0(x,u)] z \, dx.$$

Since $\mathcal{A}$ is bounded in $L^\infty(I\!R^N)$ and from assumption (3.2) we have

$$|f_0(x,v) - f_0(x,u)| \le C_{\mathcal{A}} \, |z|, \quad x \in I\!R^N.$$

and then

$$\frac{1}{2}\frac{d}{dt}\|z\|^2_{L^2(\boldsymbol{R}^N)} + \int_{\boldsymbol{R}^N} (|\nabla z|^2 - m(x)z^2) \, dx \le C_{\mathcal{A}} \|z\|^2_{L^2(\boldsymbol{R}^N)}.$$

From (3.1) we have, for some $\mu \in I\!R$,

$$\frac{1}{2}\frac{d}{dt}\|z\|^2_{L^2(\boldsymbol{R}^N)} \le (C_{\mathcal{A}} + \mu)\|z\|^2_{L^2(\boldsymbol{R}^N)}.$$

Thus, Gronwall's lemma gives the result.$\square$

With this we can now prove

Theorem 3.3. *Assume the nonlinear term $f(x,s)$ in problem (1.1), satisfies*

$$|\frac{\partial^2 f}{\partial s^2}(x,s)| \le C(R), \quad |s| \le R, \quad x \in I\!R^N. \tag{3.5}$$

Assume furthermore that problem (1.1) has an attractor $\mathcal{A} \subset L^2(I\!R^N) \cap L^\infty(I\!R^N)$. Then, the semigroup associated to (1.1) is uniformly differentiable on $\mathcal{A}$, that is (2.2) and (2.3) are satisfied.

Proof. Assume $u(t)$ and $v(t)$ are solutions of (1.1) with initial data $u_0, v_0 \in \mathcal{A}$ respectively and let $U(t)$ be the solution of the linearized equation along $u(t)$, (2.4), with initial data $v_0 - u_0$, that is

$$\begin{cases} \partial_t U - \Delta U = \frac{\partial f}{\partial u}(x, u(t))U, \\ U(0) = v_0 - u_0. \end{cases} \tag{3.6}$$

We are going to show that $L(t, u_0)(v_0 - u_0) = U(t)$. Then we define $\theta(t) = v(t) - u(t) - U(t)$, which satisfies

$$\begin{cases} \partial_t \theta - \Delta \theta = \frac{\partial f}{\partial u}(x, u(t))\theta + G(x, t), \\ \theta(0) = 0, \end{cases} \tag{3.7}$$

with

$$G(x, t) := f(x, v(t, x)) - f(x, u(t, x)) - \frac{\partial f}{\partial u}(x, u(t, x))(v(t, x) - u(t, x)).$$

Using (3.5), we have

$$|G(x, t)| \le |\frac{\partial^2 f}{\partial u^2}(x, \eta)|[v(t, x) - u(t, x)]^2,$$

for some intermediate value η. Since $\mathcal{A}$ is bounded in $L^\infty(I\!\!R^N)$, we get

$$|G(x, t)| \le C_{\mathcal{A}}[v(t, x) - u(t, x)]^2. \tag{3.8}$$

We now estimate the $L^2(I\!\!R^N)$ norm of the solution of (3.7) and for this we rewrite the equation as

$$\partial_t \theta - \Delta \theta - m(x)\theta = \frac{\partial f}{\partial u}(x, u(t))\theta - \frac{\partial f}{\partial u}(x, 0)\theta + G(x, t)$$

where we have used that $\frac{\partial f}{\partial u}(x, 0) = m(x)$. Multiplying by θ and integrating in $I\!\!R^N$ we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(\boldsymbol{R}^N)} + \int_{\boldsymbol{R}^N} |\nabla \theta|^2 \, dx - \int_{\boldsymbol{R}^N} m(x)\theta^2 \, dx \;&=\; \int_{\boldsymbol{R}^N} \left(\frac{\partial f}{\partial u}(x, u(t)) - \frac{\partial f}{\partial u}(x, 0)\right)\theta^2 \, dx \\ &+ \int_{\boldsymbol{R}^N} G\theta \, dx. \end{aligned} \tag{3.9}$$

Again, since $\mathcal{A}$ is bounded in $L^\infty(I\!\!R^N)$ and using (3.5) we get

$$\left|\frac{\partial f}{\partial u}(x, u(t)) - \frac{\partial f}{\partial u}(x, 0)\right| \le C_{\mathcal{A}}.$$

Now, Hölder's inequality, Sobolev emdeddings and Young's inequality gives

$$\begin{aligned} \int_{\boldsymbol{R}^N} |G\theta| \;&\le\; \|\theta\|_{L^{\frac{2N}{N-2}}(\boldsymbol{R}^N)} \|G\|_{L^{\frac{2N}{N+2}}(\boldsymbol{R}^N)} \le C\|\theta\|_{H^1(\boldsymbol{R}^N)}\|G\|_{L^{\frac{2N}{N+2}}(\boldsymbol{R}^N)} \le \\ &\frac{1}{2}\int_{\boldsymbol{R}^N} |\nabla \theta|^2 + \frac{1}{2}\int_{\boldsymbol{R}^N} |\theta|^2 + C\|G\|^2_{L^{\frac{2N}{N+2}}(\boldsymbol{R}^N)} \end{aligned}$$

and then, from (3.9) we have

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\theta|^2\,dx - \int_{\mathbb{R}^N} m(x)\theta^2\,dx \le (C_{\mathcal{A}} + \frac{1}{2})\int_{\mathbb{R}^N} |\theta|^2\,dx + C\|G\|^2_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)}.$$

Now from (3.1) we get, for some $\mu \in \mathbb{R}$,

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(\mathbb{R}^N)} - \mu\|\theta\|^2_{L^2(\mathbb{R}^N)} \le C\|G\|^2_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)}. \tag{3.10}$$

On the other hand, from (3.8) and since $u(t), v(t) \in \mathcal{A} \subset L^\infty(\mathbb{R}^N)$ and is bounded, we get, with $q = \frac{2N}{N+2}$

$$\begin{aligned} \|G\|^q_{L^q(\mathbb{R}^N)} &\le C_{\mathcal{A}}\int_{\mathbb{R}^N} |v(t)-u(t)|^{2q}\,dx = C_{\mathcal{A}}\int_{\mathbb{R}^N} |v(t)-u(t)|^{2q-2}|v(t)-u(t)|^2\,dx \\ &\le \hat{C}_{\mathcal{A}}\|v(t)-u(t)\|^2_{L^2(\mathbb{R}^N)} \end{aligned}$$

hence

$$\|G\|_{L^q(\mathbb{R}^N)} \le \hat{C}_{\mathcal{A}}\|u(t)-v(t)\|^{\frac{2}{q}}_{L^2(\mathbb{R}^N)} = \hat{C}_{\mathcal{A}}\|u(t)-v(t)\|^{1+\frac{2}{N}}_{L^2(\mathbb{R}^N)}. \tag{3.11}$$

Now, since (3.5) implies (3.2), using the continuous dependence on initial data in $L^2(\mathbb{R}^N)$, see Lemma 3.2 above, from (3.11) and (3.3), denoting $\alpha = \frac{2}{N}$ we get

$$\|G\|^2_{L^q(\mathbb{R}^N)} \le \hat{C}_{\mathcal{A}}\gamma(t)^{2(1+\alpha)}\|u_0 - v_0\|^{2(1+\alpha)}_{L^2(\mathbb{R}^N)}$$

with $\gamma(t) = e^{C_{\mathcal{A}}t}$. Substituting in (3.10) we get

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(\mathbb{R}^N)} - \mu\|\theta\|^2_{L^2(\mathbb{R}^N)} \le \frac{1}{2}\hat{C}_{\mathcal{A}}\gamma(t)^{2(1+\alpha)}\|u_0 - v_0\|^{2(1+\alpha)}_{L^2(\mathbb{R}^N)}.$$

Now Gronwall inequality gives

$$\|\theta(t)\|_{L^2(\mathbb{R}^N)} \le \Gamma(t)^{\frac{1}{2}}\|u_0 - v_0\|^{1+\alpha}_{L^2(\mathbb{R}^N)}$$

with $\Gamma(t) := \frac{1}{2}C_{\mathcal{A}}e^{2\mu t}[\int_0^t e^{-2\mu\tau}\gamma(\tau)^{2(1+\alpha)}\,d\tau]$. Then we get for $t > 0$

$$\sup_{\{u_0,v_0\in\mathcal{A},\,0\le\|u_0-v_0\|_{L^2(\mathbb{R}^N)}\le\epsilon\}} \frac{\|\theta(t)\|_{L^2(\mathbb{R}^N)}}{\|v_0 - u_0\|_{L^2(\mathbb{R}^N)}} \le \Gamma(t)^{\frac{1}{2}}\epsilon^\alpha \xrightarrow{\epsilon\to 0} 0$$

which proves (2.2).

Now we will prove (2.3). For this we write (3.6) as

$$\partial_t U - \Delta U - m(x)U = \frac{\partial f}{\partial u}(x, u(t))U - \frac{\partial f}{\partial u}(x, 0)U$$

and multiply by U and integrate in $I\!R^N$ to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^N} U^2\,dx + \int_{\mathbf{R}^N} |\nabla U|^2\,dx - \int_{\mathbf{R}^N} m(x)U^2\,dx \\ &= \int_{\mathbf{R}^N}\left[\frac{\partial f}{\partial u}(x,u(t)) - \frac{\partial f}{\partial u}(x,0)\right]U^2\,dx \le C_{\mathcal{A}}\int_{\mathbf{R}^N} U^2\,dx.\end{aligned} \tag{3.12}$$

Using again (3.1), we get from (3.12) that for some $\mu \in I\!R$

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|^2_{L^2(\mathbf{R}^N)} \le (C_{\mathcal{A}} + \mu)\|U(t)\|^2_{L^2(\mathbf{R}^N)}$$

and then Gronwall inequality implies

$$\|U(t)\|_{L^2(\mathbf{R}^N)} \le \|U_0\|_{L^2(\mathbf{R}^N)} e^{(C_{\mathcal{A}}+\mu)t}$$

and since $U(t) = L(t,u_0)(v_0 - u_0)$ we get the result.□

4 Estimate on the dimension of the attractor in unbounded domains.

In this section we apply the general technique sketched in Section 2 to problem (1.1). For this we will make use of the following results.

Lemma 4.1. *Assume the Schrödinger operator $\Delta + m(x)I$, with $m \in L^\sigma_U(I\!R^N)$, $\sigma > N/2$ has a negative exponential type, that is, we can take $\mu < 0$ in (3.1).*

Then for every $d \in I\!N$, we have

$$Tr_d(\Delta + m(x)I) \le \mu d$$

Proof. Note that for any orthonormal set φ_i, $i = 1,\dots,d$ in $L^2(I\!R^N)$ we have, from (3.1),

$$\sum_{i=1}^{d}((\Delta + m(x)I)\varphi_i,\varphi_i)_{L^2(\mathbf{R}^N)} = -\sum_{i=1}^{d}\left(\int_{\mathbf{R}^N}|\nabla\varphi_i|^2 - \int_{\mathbf{R}^N} m(x)\varphi_i^2\right) \le \mu d$$

which according to (2.5) gives the result.□

The next result, known as the Lieb–Thirring inequality will also be of great help below, see [12], [20].

Lemma 4.2. *Assume $\{\varphi_1,\dots,\varphi_d\} \subset H^1(I\!R^N)$ is an orthonormal set in $L^2(I\!R^N)$ and denote $\rho(x) := \sum_{i=1}^d(\varphi_i(x))^2$.*

Then for any p such that $\max\{1,\frac{N}{2}\} < p \le 1 + \frac{N}{2}$, there exists a constant $K = K(N,p) > 0$ independent of d and of the set $\{\varphi_1,\dots,\varphi_d\}$ such that

$$K\left(\int_{\mathbf{R}^N}\rho(x)^{\frac{p}{p-1}}\,dx\right)^{\frac{2(p-1)}{N}} \le \sum_{j=1}^{d}\int_{\mathbf{R}^N}|\nabla\varphi_j|^2\,dx.$$

For the case where the nonlinear term depends on the spatial variable, we have

Theorem 4.3. *Assume the nonlinear term in (1.1) is such that*

$$f(x,s)s \leq C(x)|s|^2 + D(x)|s|, \quad \textit{for all} \quad s \in I\!R,\ x \in I\!R^N, \tag{4.1}$$

$$|\frac{\partial^2 f}{\partial s^2}(x,s)| \leq C(R), \quad \textit{for} \quad |s| \leq R, \quad x \in I\!R^N. \tag{4.2}$$

for some $C \in L^\sigma_U(I\!R^N), \sigma > \frac{N}{2}$, $\sigma > 2$ *and* $0 \leq D \in L^p(I\!R^N)$, $p > \frac{N}{2}$.

Assume

$$\mathcal{A} \subset L^\infty(I\!R^N) \cap L^p(I\!R^N) \cap L^2(I\!R^N) \quad \textit{is bounded.}$$

Assume furthermore that the operator $\Delta + C(x)I$ *has a negative exponential type.*

Then the Hausdorff and fractal dimensions of $\mathcal{A}$ *are finite.*

Proof Since the linearization along any trajectory on the attractor is given by (2.4), we can now write $A_1(t) := \Delta + \frac{\partial f}{\partial u}(x, u(t,x))I$, as

$$A_1(t) := \Delta + \frac{\partial f}{\partial u}(x,\varphi(x)) + \frac{\partial f}{\partial u}(x,u(t,x)) - \frac{\partial f}{\partial u}(x,\varphi(x)) \tag{4.3}$$

where we construct $\varphi(x)$ below in such a way that $\varphi \in L^\infty(I\!R^N) \cap L^p(I\!R^N)$ for $p > \frac{N}{2}$ and the operator $\Delta + \frac{\partial f}{\partial u}(x,\varphi(x))$ has a negative exponential type.

To construct $\varphi(x)$ we proceed as follows. Let $R > 0$ to be determined later on. Then we define

$$0 \leq s_0(x) := \begin{cases} R(D(x) + |g(x)|), & \text{if} \quad D(x) + |g(x)| \leq 1 \\ R, & \text{if} \quad D(x) + |g(x)| \geq 1. \end{cases}$$

Now, for fixed $x \in I\!R^N$, the mean value theorem gives that there exists $\varphi(x) \in [0, s_0(x)]$ such that

$$\frac{\partial f}{\partial u}(x,\varphi(x)) = \frac{f(x,s_0(x)) - f(x,0)}{s_0(x)} \leq C(x) + \frac{D(x) + |g(x)|}{s_0(x)} = C(x) + P(x) \tag{4.4}$$

where $g(x) := f(x,0)$ and we have used (4.1). Therefore, we have

$$P(x) := \begin{cases} \frac{1}{R}, & \text{if} \quad D(x) + |g(x)| \leq 1, \\ \frac{D(x)+|g(x)|}{R}, & \text{if} \quad D(x) + |g(x)| \geq 1. \end{cases}$$

It is then clear that $0 \leq P(x) \leq \frac{1+D(x)+|g(x)|}{R}$ and for each $z \in I\!R^N$

$$\|P\|_{L^p(B(z,1))} \leq \frac{1}{R}\|\ 1 + D + |g|\ \|_{L^p(B(z,1))} \leq \frac{1}{R}\Big(|B(z,1)|^{\frac{1}{p}} + \|\ D + |g|\ \|_{L^p(\boldsymbol{R}^N)}\Big)$$

and then

$$\|P\|_{L^p_U(\boldsymbol{R}^N)} = \sup_{z \in \boldsymbol{R}^N} \|P\|_{L^p(B(z,1))} \leq \frac{1}{R}\Big(|B(0,1)|^{\frac{1}{p}} + \|\ D + |g|\ \|_{L^p(\boldsymbol{R}^N)}\Big).$$

Moreover, since $0 \leq \varphi(x) \leq s_0(x)$, we have $\varphi \in L^\infty(I\!R^N) \cap L^p(I\!R^N)$, for $p > \frac{N}{2}$.

Now we show that

$$\frac{\partial f}{\partial u}(\cdot, \varphi(\cdot)) \in L^\sigma_U(I\!R^N), \ \sigma > \frac{N}{2}.$$

In fact, from (1.5) and $m \in L^\sigma_U(I\!R^N)$, it will be enough to show that $\frac{\partial f_0}{\partial u}(x, \varphi(x)) \in L^\sigma_U(I\!R^N)$. For this, just note that from (4.2), implies $|\frac{\partial f_0}{\partial u}(x,u)| \leq C(R)|u|$, for $|u| \leq R$, and since $\varphi \in L^\infty(I\!R^N)$ we get $|\frac{\partial f_0}{\partial u}(\cdot, \varphi(\cdot))| \in L^\infty(I\!R^N) \subset L^\sigma_U(I\!R^N)$, $\sigma > \frac{N}{2}$.

Now we make use of the following. Since, by hypothesis, the operator $\Delta + C(x)I$ has a negative exponential type, from Lemma 2.2 in [2], there exists $C(\mu)$ such that if $P \in L^p_U(I\!R^N)$, $p > \frac{N}{2}$, with positive part such that $\|P^+\|_{L^p_U(\boldsymbol{R}^N)} \leq C(\mu)$, then the operator $\Delta + (C+P)I$ has a negative exponential type.

In particular, taking R sufficiently large above we have that the operator $\Delta + (C+P)I$ has a negative exponential type. But then from (4.4) we have $C(x) + P(x) - \frac{\partial f}{\partial u}(x, \varphi(x)) \geq 0$ and then

$$\frac{\partial f}{\partial u}(x, \varphi(x)) = C(x) + P(x) - \big(C(x) + P(x) - \frac{\partial f}{\partial u}(x, \varphi(x))\big)$$

and again Lemma 2.2 in [2] gives that the operator $\Delta + \frac{\partial f}{\partial u}(x, \varphi(x))$ has a negative exponential type.

Now we write the linearized operator $A_1(t)$ in (4.3) as

$$A_1(t) = (1-\delta)[\Delta + \frac{1}{1-\delta}\frac{\partial f}{\partial u}(x, \varphi(x))] + \delta\Delta + \frac{\partial f}{\partial u}(x, u(t,x)) - \frac{\partial f}{\partial u}(x, \varphi(x)) \qquad (4.5)$$

and we chose $\delta \in (0,1)$ sufficiently small such that the operator $\Delta + \frac{1}{1-\delta}\frac{\partial f}{\partial u}(x, \varphi(x))$ has a negative exponential type that we still denote $\mu < 0$. Hence from Lemma 4.1, $Tr_d(\Delta + \frac{1}{1-\delta}\frac{\partial f}{\partial u}(x, \varphi(x))) \leq \mu d$ and then for any choice of $\varphi_1, \varphi_2, \ldots, \varphi_d \in H^1(I\!R^N)$, which are orthonormal in $L^2(I\!R^N)$ and span a subspace E_d, we get

$$Tr_d(A_1(t), E_d) \leq (1-\delta)\mu d - \delta \sum_{i=1}^d \int_{\boldsymbol{R}^N} |\nabla \varphi_i|^2 \, dx + \sum_{i=1}^d \int_{\boldsymbol{R}^N} \Big(\frac{\partial f}{\partial u}(x, u(t,x)) - \frac{\partial f}{\partial u}(x, \varphi(x))\Big)\varphi_i^2 \, dx.$$

Denoting $\rho(x) := \sum_{i=1}^d \varphi_i(x)^2$ and applying the Lieb-Thirring inequality, Lemma 4.2, we get

$$\begin{aligned} Tr_d(A_1(t), E_d) \ &\leq \ (1-\delta)\mu d - \delta K \int_{\boldsymbol{R}^N} \rho(x)^{1+\frac{2}{N}} \, dx \\ &+ \ \int_{\boldsymbol{R}^N} \Big(\frac{\partial f}{\partial u}(x, u(t,x)) - \frac{\partial f}{\partial u}(x, \varphi(x))\Big) \rho(x) \, dx. \end{aligned} \qquad (4.6)$$

and then

$$\begin{aligned} Tr_d(A_1(t), E_d) \ &\leq \ \frac{(1-\delta)\mu d}{2} - \delta K \int_{\boldsymbol{R}^N} \rho(x)^{1+\frac{2}{N}} \, dx \\ &+ \ \int_{\boldsymbol{R}^N} \Big([\frac{\partial f}{\partial u}(x, u(t,x)) - \frac{\partial f}{\partial u}(x, \varphi(x))] + \frac{\mu(1-\delta)}{2}\Big)\rho(x) dx. \end{aligned} \qquad (4.7)$$

since $\int_{\boldsymbol{R}^N} \rho(x)\ dx = d$.

Then we define

$$J(t,x) := \max\{0,\ [\frac{\partial f}{\partial u}(x,u(t,x)) - \frac{\partial f}{\partial u}(x,\varphi(x))] + \frac{\mu(1-\delta)}{2}\} \tag{4.8}$$

and then we have

$$Tr_d(A_1(t),E_d) \le \frac{(1-\delta)\mu d}{2} - \delta K \int_{\boldsymbol{R}^N} \rho(x)^{1+\frac{2}{N}}\ dx + \int_{\boldsymbol{R}^N} J(t,x)\rho(x)dx.$$

From here, Hölder's inequality gives

$$\begin{aligned} Tr_d(A_1(t),E_d) &\le \frac{(1-\delta)\mu d}{2} - \delta K \int_{\boldsymbol{R}^N} \rho(x)^{1+\frac{2}{N}}\ dx \\ &+ \left(\int_{\boldsymbol{R}^N} |J(t,x)|^{1+\frac{N}{2}}\ dx\right)^{\frac{2}{N+2}} \left(\int_{\boldsymbol{R}^N} \rho(x)^{1+\frac{2}{N}} dx\right)^{\frac{N}{N+2}} \end{aligned}$$

and setting $V = V(t) = \left(\int_{\boldsymbol{R}^N} |J(t,x)|^{1+\frac{N}{2}}\ dx\right)^{\frac{2}{N+2}}$ and $y = \left(\int_{\boldsymbol{R}^N} \rho(x)^{1+\frac{2}{N}}\ dx\right)^{\frac{N}{N+2}}$ we get

$$Tr_d(A_1(t),E_d) \le \frac{(1-\delta)\mu d}{2} - Cy^{\frac{N+2}{N}} + V(t)y.$$

Now, Young's inequality gives, for every $\epsilon > 0$, $V(t)y \le \epsilon y^{\frac{N+2}{N}} + C_\epsilon V(t)^{\frac{N+2}{2}}$ and taking $\epsilon = \frac{C}{2}$ and the sup in all subspaces E_d we get

$$Tr_d(A_1(t)) \le \frac{(1-\delta)\mu d}{2} + C_1 V(t)^{\frac{N+2}{2}}. \tag{4.9}$$

Hence condition (2.6) is satisfied provided

$$\frac{2C_1}{(1-\delta)|\mu|} \limsup_{t\to\infty} \sup_{u_0\in\mathcal{A}} \frac{1}{t}\int_0^t V(\tau)^{\frac{N}{2}+1}\ d\tau < d. \tag{4.10}$$

Now for any trajectory in the attractor and $\hat{\delta} > 0$ we split

$$V(t)^{\frac{N}{2}+1} = \int_{\{x\in\boldsymbol{R}^N,\, |u(t,x)-\varphi(x)|\le\hat{\delta}\}} |J(t,x)|^{\frac{N}{2}+1}\ dx + \int_{\{x\in\boldsymbol{R}^N,\, |u(t,x)-\varphi(x)|>\hat{\delta}\}} |J(t,x)|^{\frac{N}{2}+1}\ dx. \tag{4.11}$$

From (4.2), the mean vaue theorem and using that $\|u\|_{L^\infty(\boldsymbol{R}^N)} \le R$ for all $u \in \mathcal{A}$ and $\|\varphi\|_{L^\infty(\boldsymbol{R}^N)} \le R$, we get

$$|\frac{\partial f}{\partial u}(x,\varphi(x)) - \frac{\partial f}{\partial u}(x,u(t,x))| \le C(R)|\varphi(x) - u(t,x)|, \quad x \in I\!R^N. \tag{4.12}$$

Hence, we chose $\hat{\delta}$ such that if $|u-\varphi| < \hat{\delta}$, then $C(R)\hat{\delta} < \frac{\mu(\delta-1)}{2}$. Thus from (4.8) we have

$$\int_{\{x\in\boldsymbol{R}^N,\, |u(t,x)-\varphi(x)|\le\hat{\delta}\}} |J(t,x)|^{\frac{N+2}{2}} dx = 0.$$

Now we deal with the second term in (4.11). Since $\mathcal{A}$ is bounded in $L^\infty(\mathbb{R}^N)$ and $\varphi \in L^\infty(\mathbb{R}^N)$, from (4.8) we get

$$|J(t,x)| \leq C_{\mathcal{A},\varphi} + \frac{|\mu|(1-\delta)}{2} := K_1$$

and thus

$$\int_{\{x\in \boldsymbol{R}^N,\, |u(t,x)-\varphi(x)|>\hat{\delta}\}} |J(t,x)|^{\frac{N+2}{2}}\, dx \leq K_1^{\frac{N+2}{2}} |\{x \in \mathbb{R}^N : |u(t,x)-\varphi| > \hat{\delta}\}|. \quad (4.13)$$

Using now that $\mathcal{A}$ is bounded in $L^p(\mathbb{R}^N)$ and $\varphi \in L^p(\mathbb{R}^N)$ we get that for all $u \in \mathcal{A}$

$$\hat{\delta}^p |\{x \in \mathbb{R}^N,\, |u(x)-\varphi(x)| > \hat{\delta}\}| \leq \int_{\boldsymbol{R}^N} |u(x)-\varphi(x)|^p\, dx \leq C = C(\mathcal{A}, \|\varphi\|^p_{L^p(\boldsymbol{R}^N)}),$$

and then substituting in (4.13) we get

$$\int_{\{x\in \boldsymbol{R}^N,\, |u(t,x)-\varphi(x)|>\hat{\delta}\}} |J(t,x)|^{\frac{N+2}{2}}\, dx \leq K_1^{\frac{N+2}{2}} \frac{C(\mathcal{A}, \|\varphi\|^p_{L^p(\boldsymbol{R}^N)})}{\hat{\delta}^p}.$$

From (4.10) we get the result.□

Remark 4.4. *Note that from the results in [2], the assumptions on the boundedness of the attractor in Theorem 4.3 are satisfied provided*

$$C \in L^\sigma_U(\mathbb{R}^N) \quad \textit{for some} \quad \sigma > N/2,\ \sigma > 2$$

and

$$D \in L^p(\mathbb{R}^N) \cap L^s(\mathbb{R}^N) \quad \textit{with} \quad p > N/2 \quad \textit{and} \quad 2 \geq s > \frac{2N}{N+4}.$$

Now we illustrate the scope of the result above with the following example.

Example 4.5. *Consider a prototype problem (1.1) with bistable nonlinear term*

$$f(x,s) = m(x)s - n(x)s^3.$$

Note that as soon as $0 \leq n \in L^\infty(\mathbb{R}^N)$ then (3.2), (3.5) and (4.2) are satisfied.

Assume there exists a decomposition

$$m(x) = m_0(x) - m_1(x), \quad \textit{with} \quad m_0, m_1 \in L^\sigma_U(\mathbb{R}^N), \quad \sigma > N/2,\ \sigma > 2,$$

such that the operator $\Delta - m_1(x)$ has negative exponential type.

Hence, using Young's inequality we have that (4.1) is satisfied with

$$C(x) = -m_1(x), \quad D(x) = A\frac{|m_0|^{3/2}(x)}{n^{1/2}(x)}$$

for some constant A.

Moreover, it was proved in [2] that the attractor $\mathcal{A}$ of (1.1) satisfies the remaining assumptions in Theorem 4.3, that is

$$\mathcal{A} \subset L^\infty(I\!R^N) \cap L^p(I\!R^N) \cap L^2(I\!R^N) \quad \text{is bounded}$$

provided

$$\frac{|m_0|^{3/2}}{n^{1/2}} \in L^r(I\!R^N) \cap L^p(I\!R^N) \cap L^s(I\!R^N)$$

for some $r > N/3$, $p > \frac{N}{2}$ and $2 \geq s > \frac{2N}{N+4}$.

Note that a source term $g(x) = f(x, 0)$ can also be considered as long as

$$g \in L^\infty(I\!R^N) \cap L^p(I\!R^N) \cap L^2(I\!R^N);$$

see Theorem 5.2 in [2] for sharper assumptions on g.

The next result considers the case in which f_0 does not depend on x and improves conditions in Theorem 3.3 in [6] with respect to the dimension of the attractor.

Theorem 4.6. *Consider the reaction diffusion equation*

$$\begin{cases} \partial_t u - \Delta u = m(x)u + f_0(u) + g(x), & x \in I\!R^N,\ t > 0 \\ u(0) = u_0 \end{cases}$$

with $g \in L^2(I\!R^N)$, $m \in L^\sigma_U(I\!R^N)$, $\sigma > \frac{N}{2}$, and

$$f_0 \in C^2(I\!R), \quad \text{with} \quad f_0(0) = 0 = f_0'(0). \tag{4.14}$$

Assume the attractor of this problem, $\mathcal{A}$, exists and satisfies

$$\mathcal{A} \subset L^\infty(I\!R^N) \cap L^2(I\!R^N), \quad \text{is bounded.}$$

Finally assume the operator $\Delta + m(x)I$ has a negative exponential type.

Then the Hausdorff and fractal dimensions of $\mathcal{A}$ are finite.

Proof. Note that we cannot apply directly Theorem 4.3 since we are not assuming that the structure conditions (4.1) hold for this case. Nevertheless it is not difficult to see that we can follow basically the same argument as in the proof of Theorem 4.3.

Note that the linearization around a trajectory $u(t)$ on the attractor is given by

$$U_t = \Delta U + m(x)U + f_0'(u(t))U := A_1(t)U.$$

Then we write

$$A_1(t) := (1-\delta)\left[\Delta + \frac{m(x)I}{1-\delta}\right] + \delta\Delta + f_0'(u(t,x))I \tag{4.15}$$

and we chose $\delta \in (0,1)$ sufficiently small such that the operator $\Delta + \frac{m(x)I}{1-\delta}$ has a negative exponential type $\mu < 0$. This is equivalent to take $\varphi(x) \equiv 0$ in the proof of Theorem 4.3. In particular, (4.15) is the equivalent to (4.5) with $\varphi \equiv 0$. The rest of the proof follows exactly the same lines as the rest of the proof of Theorem 4.3 after statement (4.5). $\square$

5 Non linear terms depending on the gradients.

In this section we illustrate how the technique developed in the previous section can be applied to other two examples in which the nonlinear terms depend also on the gradients. Therefore we assume that the proofs of the existence of the attractor and the uniform differentiablity have been carried out.

We start with the following example which is similar to the case considered in [10] but allowing for a suitable potential $m(x)$ with no sign instead of a negative constant, see (1.3).

Hence, we consider the following problem

$$\begin{cases} \partial_t u - \Delta u = m(x)u + f_0(u, \nabla u) + g(x) \\ u(0) = u_0 \end{cases} \tag{5.1}$$

Theorem 5.1. *Assume $m \in L^\sigma_U(I\!R^N)$, $\sigma > \frac{N}{2}$, $g \in L^p(I\!R^N)$, $p > \frac{N}{2}$ and the following dissipativity assumption*

$$f_0(u, \xi)u \leq 0, \quad \textit{for} \quad u \in I\!R,\ \xi \in I\!R^N. \tag{5.2}$$

We also assume that for some $r < 2$ we have

$$\begin{aligned} |\frac{\partial f_0}{\partial u}(u, \nabla u)| &\leq C(1 + |u|^p)(1 + |\nabla u|^r), \\ |\frac{\partial f_0}{\partial \xi}(u, \nabla u)| &\leq C(1 + |u|^{p+1})(1 + |\nabla u|^{r-1}). \end{aligned} \tag{5.3}$$

Assume furthermore that the attractor, $\mathcal{A}$, exists and it is such that

$$\mathcal{A} \subset H^1(I\!R^N) \cap W^{1, r(1+\frac{N}{2})}(I\!R^N), \quad \textit{is bounded.} \tag{5.4}$$

Then if the operator $\Delta + m(x)I$ has a negative exponential type then $\mathcal{A}$ has finite Hausdorff and fractal dimensions.

Proof. Note that now the linearization along any trajectoy on $\mathcal{A}$ is given by

$$U_t = \Delta U + m(x)U + \frac{\partial f_0}{\partial u}(u(t,x), \nabla u(t,x))U + \frac{\partial f_0}{\partial \xi}(u(t,x), \nabla u(t,x))\nabla U := A_1(t)U.$$

Then we can write $A_1(t)$ as

$$A_1(t)U := (1-\delta)[\Delta U + \frac{1}{1-\delta} m(x)U] + \delta \Delta U + \frac{\partial f_0}{\partial u}(u, \nabla u)U + \frac{\partial f_0}{\partial \xi}(u, \nabla u)\nabla U$$

and we chose $\delta \in (0,1)$ sufficiently small such that the opearor $\Delta + \frac{1}{1-\delta} m(x)I$ has a negative exponential type $\mu < 0$.

Then from Lemma 4.1, for any choice of $\varphi_1, \varphi_2, \ldots, \varphi_d \in H^1(I\!R^N)$, which are orthonormal in $L^2(I\!R^N)$ and span a subspace E_d, we get

$$\begin{aligned} Tr_d(A_1(t), E_d) &\leq (1-\delta)\,\mu d - \delta \sum_{i=1}^{d} \int_{\boldsymbol{R}^N} |\nabla \varphi_i|^2 \, dx \\ &+ \sum_{i=1}^{d} \int_{\boldsymbol{R}^N} \frac{\partial f_0}{\partial u}(u, \nabla u) \varphi_i^2(x) \, dx + \sum_{i=1}^{d} \int_{\boldsymbol{R}^N} \frac{\partial f_0}{\partial \xi}(u, \nabla u) \nabla \varphi_i \varphi_i \, dx \quad (5.5) \end{aligned}$$

Now Young's inequality in the fourth term gives

$$\begin{aligned} Tr_d(A_1(t), E_d) &\leq (1-\delta)\mu d - \frac{\delta}{2} \sum_{i=1}^{d} \int_{\boldsymbol{R}^N} |\nabla \varphi_i|^2 \, dx + \int_{\boldsymbol{R}^N} \frac{\partial f_0}{\partial u}(u, \nabla u) (\sum_{i=1}^{d} \varphi_i^2) \, dx \\ &+ \frac{1}{2\delta} \int_{\boldsymbol{R}^N} |\frac{\partial f_0}{\partial \xi}(u, \nabla u)|^2 (\sum_{i=1}^{d} \varphi_i^2) \, dx. \end{aligned}$$

Now, denoting $\rho(x) = \sum_{i=1}^{d} \varphi_i^2(x)$ and using the Lieb-Thirring inequality, Lemma 4.2, we get

$$Tr_d(A_1(t), E_d) \leq (1-\delta)\mu d - \frac{\delta K}{2} \int_{\boldsymbol{R}^N} \rho(x)^{1+\frac{2}{N}} \, dx + \int_{\boldsymbol{R}^N} M(t,x) \rho(x) \, dx \qquad (5.6)$$

where we have set $M(t,x) := \frac{\partial f_0}{\partial u}(u, \nabla u) + \frac{1}{2\delta} |\frac{\partial f_0}{\partial \xi}(u, \nabla u)|^2$.

Note that assumption (5.2) implies $f_0(0, \xi) = 0$ and $\frac{\partial f_0}{\partial u}(0, \xi) \leq 0$, for $\xi \in I\!R^N$ and, in particular, $\frac{\partial f}{\partial \xi}(0,0) = 0$. Since $f_0 \in C^1_{u,\xi}(I\!R^{N+1})$, there exist $\beta > 0$ such that

$$\frac{\partial f_0}{\partial u}(u, v) + \frac{1}{2\delta} |\frac{\partial f_0}{\partial \xi}(u, v)|^2 \leq \frac{|\mu|(1-\delta)}{2} \quad \text{for } |u| \leq \beta \text{ and } |v| \leq \beta.$$

For such β fix and $u \in \mathcal{A}$, we consider the set

$$\Omega_\beta := \{x \in I\!R^N : |u(x)| \leq \beta, \ |\nabla u(x)| \leq \beta\},$$

and decompose

$$\int_{\boldsymbol{R}^N} M(t,x) \rho(x) \, dx = \int_{\Omega_\beta} M(t,x) \rho(x) \, dx + \int_{\boldsymbol{R}^N \setminus \Omega_\beta} M(t,x) \rho(x) \, dx$$

so we get

$$\int_{\Omega_\beta} M(t,x) \rho(x) \, dx \leq \int_{\Omega_\beta} \frac{|\mu|(1-\delta)}{2} \rho(x) \, dx \leq \frac{|\mu|(1-\delta)}{2} \int_{\boldsymbol{R}^N} \rho(x) \, dx = \frac{|\mu|(1-\delta)}{2} d$$

On the other hand, Hölder's inequality gives

$$\int_{\boldsymbol{R}^N \setminus \Omega_\beta} M(t,x) \rho(x) \, dx \leq \left(\int_{\boldsymbol{R}^N \setminus \Omega_\beta} |M(t,x)|^{1+\frac{N}{2}} \, dx \right)^{\frac{2}{N+2}} \left(\int_{\boldsymbol{R}^N \setminus \Omega_\beta} \rho(x)^{1+\frac{2}{N}} \, dx \right)^{\frac{N}{N+2}}$$

and Young's inequality with $\epsilon > 0$ such that $\epsilon < \frac{\delta K}{2}$, gives

$$\leq \epsilon \int_{\mathbb{R}^N \setminus \Omega_\beta} \rho(x)^{1+\frac{2}{N}} \, dx + C_\epsilon \int_{\mathbb{R}^N \setminus \Omega_\beta} |M(t,x)|^{1+\frac{N}{2}} \, dx, \tag{5.7}$$

and then in (5.6) we get, taking the sup in all subspaces E_d,

$$Tr_d(A_1(t)) \leq \frac{(1-\delta)\mu d}{2} + C \int_{\mathbb{R}^N \setminus \Omega_\beta} |M(t,x)|^{1+\frac{N}{2}} \, dx. \tag{5.8}$$

Note now that from (5.3) and since $\mathcal{A}$ is bounded in $L^\infty(\mathbb{R}^N)$, we get $|\frac{\partial f}{\partial u}(u, \nabla u)| + \frac{1}{2\delta}|\frac{\partial f}{\partial \xi}(u, \nabla u)|^2 \leq C(1 + |\nabla u|^r)$ for some $r < 2$, and then

$$\int_{\mathbb{R}^N \setminus \Omega_\beta} |M(t,x)|^{1+\frac{N}{2}} \, dx \leq C \int_{\mathbb{R}^N \setminus \Omega_\beta} (1 + |\nabla u|^r)^{1+\frac{N}{2}} \, dx = C|\mathbb{R}^N \setminus \Omega_\beta| + C\|u\|^{r(1+\frac{N}{2})}_{W^{1,r(1+\frac{N}{2})}(\mathbb{R}^N)} \tag{5.9}$$

Now we use that $\mathcal{A}$ is bounded in $H^1(\mathbb{R}^N)$ to get, for any $u \in \mathcal{A}$,

$$\beta^2 |\mathbb{R}^N \setminus \Omega_\beta| \leq \int_{\mathbb{R}^N \setminus \Omega_\beta} [u^2 + |\nabla u|^2] \, dx \leq C_{\mathcal{A}}.$$

Therefore from (5.4), (5.8) and (5.9), we get

$$\sup_{u_0 \in \mathcal{A}} Tr_d(A_1(t)) \leq \frac{(1-\delta)\mu d}{2} + C(\mathcal{A})$$

and then (2.6) is satisfied provided

$$d > \frac{2C(\mathcal{A})}{(1-\delta)|\mu|}$$

and the result is roved.$\square$

The next example is similar to the one considered in [1] but again allowing for a more general potential, see (1.4).

Theorem 5.2. *Consider the problem*

$$\begin{cases} \partial_t u - \Delta u = m(x)u + u(l(x)\nabla u) + g(x) \\ u(0) = u_0, \end{cases} \tag{5.10}$$

and assume that $g \in L^2(\mathbb{R}^N)$, $l(x)$ *is a smooth vector field and* $m \in L^\sigma_U(\mathbb{R}^N)$, $\sigma > \frac{N}{2}$, *such that the operator* $\Delta + m(x)I$ *has a negative exponential type.*

Also, assume thet the attractor $\mathcal{A}$ *exists and satisfies*

$$\mathcal{A} \subset W^{1,1+\frac{N}{2}}(\mathbb{R}^N), \quad \textit{is bounded.} \tag{5.11}$$

Then $\mathcal{A}$ *has finite fractal and Hausdorff dimensions.*

Proof. Now the linearization of (5.10) along any solution on $\mathcal{A}$ is given by

$$U_t = \Delta U + m(x)U + u(t,x)(l(x)\nabla U) + l(x)\nabla u(t,x)U = A_1(t)U$$

and then we write

$$A_1(t) := (1-\delta)[\Delta + \frac{1}{1-\delta}m(x)I] + \delta\Delta + ul(x)\nabla + l(x)\nabla u I$$

where $\delta \in (0,1)$ is small enough such that the semigroup generated by $\Delta + \frac{1}{1-\delta}m(x)I$ decays exponentially with an exponential type $\mu < 0$. Then from Lemma 4.1, for any choice of $\varphi_1, \varphi_2, \ldots, \varphi_d \in H^1(I\!R^N)$, which are orthonormal in $L^2(I\!R^N)$ and span a subspace E_d, we get

$$\begin{array}{l} Tr_d(A_1(t), E_d) \le (1-\delta)\mu d - \delta \sum_{i=1}^d \int_{\mathbf{R}^N} |\nabla\varphi_i|^2 \, dx \\ \quad + \sum_{i=1}^d \int_{\mathbf{R}^N} |u| \, |l||\nabla\varphi_i||\varphi_i| \, dx + \sum_{i=1}^d \int_{\mathbf{R}^N} |l||\nabla u||\varphi_i|^2 \, dx. \end{array}$$

Now, Young's inequality in the third term gives,

$$Tr_d(A_1(t), E_d) \le (1-\delta)\mu d - \frac{\delta}{2}\sum_{i=1}^d \int_{\mathbf{R}^N} |\nabla\varphi_i|^2 \, dx + C\int_{\mathbf{R}^N} \Big[|u|^2|l|^2 + |l||\nabla u|\Big]\rho(x) \, dx$$

where we have set $\rho(x) := \sum_{i=1}^d \varphi_i^2$.

Now, Hölder's and Lieb-Thirring inequalities give

$$\begin{array}{l} Tr_d(A_1(t), E_d) \le (1-\delta)\mu d - \dfrac{\delta K}{2}\displaystyle\int_{\mathbf{R}^N} \rho(x)^{1+\frac{2}{N}} \, dx \\ \quad + C_\delta \Big(\displaystyle\int_{\mathbf{R}^N} \Big|u^2|l|^2 + |l||\nabla u|\Big|^{1+\frac{N}{2}} \, dx\Big)^{\frac{2}{N+2}} \Big(\displaystyle\int_{\mathbf{R}^N} \rho(x)^{1+\frac{2}{N}} \, dx\Big)^{\frac{N}{N+2}}. \end{array} \tag{5.12}$$

Hence, setting $y := \Big(\int_{\mathbf{R}^N} \rho(x)^{1+\frac{2}{N}} \, dx\Big)^{\frac{N}{N+2}}$ and $V(t) := \Big(\int_{\mathbf{R}^N} |u^2|l|^2 + |l||\nabla u|^{\frac{2+N}{2}}\Big)^{\frac{2}{N+2}}$, inequality (5.12) reads

$$Tr_d(A_1(t), E_d) \le (1-\delta)\mu d - \frac{\delta K}{2} y^{\frac{N+2}{N}} + CV(t)y$$

and again Young's inequality gives, after taking the sup in all subspaces E_d,

$$Tr_d(A_1(t)) \le (1-\delta)\mu d + C|V(t)|^{1+\frac{N}{2}}.$$

Therefore, (2.6) is satisfied provided

$$\frac{C}{|\mu|(1-\delta)} \limsup_{t\to\infty} \sup_{u_0\in\mathcal{A}} \frac{1}{t}\int_0^t |V(\tau)|^{1+\frac{N}{2}} \, d\tau < d. \tag{5.13}$$

Finally note that

$$|V(t)|^{1+\frac{N}{2}} \quad \le \quad C_N \max\{\|l\|_\infty^{2+N}, \|l\|_\infty^{\frac{2+N}{2}}\}\Big[\int_{\mathbf{R}^N} |u(t)|^{2+N} + \int_{\mathbf{R}^N} |\nabla u(t)|^{\frac{2+N}{2}}\Big] \tag{5.14}$$

and since $\mathcal{A} \subset W^{1,1+\frac{N}{2}}(I\!R^N) \hookrightarrow L^{2+N}(I\!R^N)$, we get the result.$\square$

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