An elliptic–parabolic equation with a nonlocal term for the transient regime of a plasma in a Stellarator.

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Abstract. We prove the existence and the regularity of a suitable weak solution of a nonlocal 2D free-boundary problem involving the notions of relative rearrangement and monotone rearrangement. The non local elliptic-parabolic equation that we are concerned with arises in the study of the transitory regime for a magnetically confined fusion plasma in a Stellarator device. The model is obtained from ideal 3D MHD system by applying some averaging arguments and by taking into account some arguments on the characteristic times for the involved phenomena, implying that the plasma region satisfies an equilibrium equation.

Key words: *plasma physics, elliptic-parabolic problem, relative rearrangement, free-boundary.*

1 Introduction.

This paper deals with the mathematical treatment of a two-dimensional transient model associated to a family of equilibrium states for a fusion plasma magnetically confined in a Stellarator device. The model under consideration is obtained from 3-D MHD system through the averaging results by Hender & Carreras [18], by applying some arguments on the characteristic times for the involved phenomena (we shall work at the resistive diffusion timescale). Using a similar approach to the already followed for the equilibrium regime (see Díaz & Rakotoson [12]), the model can be formulated as a free-boundary problem in the following terms: Let

Ω be a subset of $\mathbb{R}^N$ $(N=2)$ and a positive time $T>0$, given $\lambda>0$, $F_v>0$, a, $b\in L^\infty(\Omega)$ with $a\not\equiv 0$ and $b>0$ a. e. in Ω, $u_0\in H^1(\Omega)$ and $\gamma<0$, find

$$u:[0,T]\times\Omega\longrightarrow\mathbb{R} \text{ and } F:\mathbb{R}\longrightarrow\mathbb{R},$$

such that $F(s)=F_v$ for any $s\leq 0$ and (u,F) satisfying the *inverse problem*

$$(\mathcal{P}_I)\begin{cases} \dfrac{\partial}{\partial t}\beta(u)-\Delta u=aF(u)+F(u)F'(u)+\lambda bu_+ & \text{in} \qquad]0,T[\times\Omega=Q, \\[2mm] u(t,x)=\gamma & \text{on} \qquad]0,T[\times\partial\Omega, \\[2mm] \beta(u(0,x))=\beta(u_0(x)) & x\in\Omega, \\[2mm] \displaystyle\int_{\{u(t)\geq\theta\}}[F(u(t))F'(u(t))+\lambda bu_+]\,dx=0 & \text{for any}\quad \theta\in[\inf u,\sup u], \end{cases}$$

with $r_+:=\max(r,0)$, and $\beta(r):=\min(0,r)=-r_-$ for $r\in\mathbb{R}$ and for the time dependent functions, we have used the following notations: for a measurable function $u:]0,T[\times\Omega\to\mathbb{R}$ and for a fixed $t\in]0,T[$, we set $u(t):\Omega\to\mathbb{R}$, $u(t)(x)=u(t,x)$.

In order to determine the unknown function F, we can reformulate the above problem using the notion of *relative rearrangement*, as it was done in Díaz [7] (see also [12]) for the stationary regime. In this way, if (u,F) is a solution of $(\mathcal{P}_I)$ such that $u(t)\in\mathcal{U}\subset\mathcal{C}^0(\Omega)$ a. e. $t\in]0,T[$, where

$$\mathcal{U}=\left\{v\in W^{2,p}(\Omega)\text{ for any }1\leq p<\infty\text{ and } meas\{x\in\Omega:\nabla v(x)=\mathbf{0}\}=0\right\},$$

then u satisfies the following uncoupled *nonlocal problem*

$$(\mathcal{P})\begin{cases} \dfrac{\partial}{\partial t}\beta(u)-\Delta u=aG(u)+J(u) & \text{in} \quad]0,T[\times\Omega, \\[2mm] u(t,x)=\gamma & \text{on} \quad \Sigma_T, \\[2mm] \beta(u(0,x))=\beta(u_0(x)) & x\in\Omega, \end{cases}$$

where G, $J:L^2(0,T;H^1(\Omega))\longrightarrow\mathbb{R}$ are given by

$$G(u)(t,x)=a\left[F_v^2-\lambda\int_{|u(t)>0|}^{|u(t)>u_+(t,x)|}[u_+(t)]'_*(\sigma)b_{*u}(t,\sigma)d\sigma\right]_+^{1/2}, \qquad (1)$$

$$J(u)(t,x)=\lambda u_+(t,x)\left[b(x)-b_{*u(t)}\left(|u(t)>u(t,x)|\right)\right] \qquad (2)$$

and, thus, $F(u)=G(u)$ and $F(u)F'(u)+\lambda bu_+=J(u)$. As in [12], it is not difficult to show that any solution u of $(\mathcal{P})$ without any flat level set is also a solution of $(\mathcal{P}_I)$. Here, u_* denotes the decreasing rearrangement of u, b_{*u} is

the relative rearrangement of b with respect to u (see Section 3 below for the definitions) and $|E|$ denotes the Lebesgue measure of a set E. We recall that the notion of the relative rearrangement was introduced first by Mossino & Temam [23] and that it is closely related to the concept of *averaging over a magnetic surface* largely used in the plasma confinement literature (see, e.g., [15], [17]).

In the present paper we shall study existence and regularity of solutions of $(\mathcal{P})$. Let us point out that there exists a large number of mathematical works concerning the study of existence for similar elliptic-parabolic problems, most of them appearing in the context of partially saturated flows in porous media (see, e.g., [3] or [31] and the references therein). We shall also mention the important work of Alt & Luckhaus [2] where a general method for elliptic-parabolic systems is developed. The main differences between the model we are interested in and the references above lie on the fact of β not being strictly increasing, as well as on the nonlocal character of our nonlinearities G and J. In particular, we shall see that these functions ask for a high regularity hypothesis in order to be continuous.

In Section 2 we shall give some indications on the modeling. Section 3 will be devoted to recall the notions of relative and monotone rearrangements as well as some useful properties. Due to the nonlocal term $b_{*u(t)}$ in $(\mathcal{P})$ we shall also introduce two different notions of weak solutions of $(\mathcal{P})$, which it will be also done in this section. Some general results of these classes of solutions are collected in Section 4. Finally, in Section 5 we present the main results of the paper. As we have stated before, we shall prove the existence and regularity of a global weak (in the sense above) solution of $(\mathcal{P})$. As a matter of fact, we consider a more general family of problems $(\mathcal{P}_\alpha)$, where we have replaced β by $\beta_\alpha(r) := -r_- + \alpha r_+$, for countably many α satisfying $0 \le \alpha \le 1$. We shall first study the case $\alpha > 0$. For the proof we use a Galerkin argument for a uniformly parabolic approached problem obtained by replacing β_α by β_ε with β_ε a C^1 such that $\beta_\varepsilon(0) = 0$, $\alpha < \varepsilon \le \beta'_\varepsilon \le 2$ and $\beta_\varepsilon \to \beta_\alpha$ as $\varepsilon \to 0$. We use some properties of the relative rearrangement ([12], [21], [23], [25], [27] and [28]), as well as some *a priori* estimates, in order to pass to the limit as $\varepsilon \to 0$, thanks to . Finally we pass to the limit $\alpha \to 0$, under some additional assumptions, thanks to a recent compactness result due to Rakotoson & Temam [26], finding, in this way, a solution of $(\mathcal{P})$.

2   Modeling

We assume the plasma as an ideal fluid and so we use the ideal incompressible MHD model which provides a single–fluid description of the macroscopic plasma

behavior. The equations of MHD are given by

$$\frac{D\mathbf{v}}{Dt} = \mathbf{J} \times \mathbf{B} - \nabla p, \qquad \text{(conservation of momentum)}, \tag{3}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{\mu}\mathbf{J}, \qquad \text{(Ohm's Law)}, \tag{4}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \text{(Faraday's Law)}, \tag{5}$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \text{(Conservation of } \mathbf{B}), \tag{6}$$

$$\nabla \times \mathbf{B} = \mathbf{J}, \qquad \text{(Ampère's Law)}, \tag{7}$$

In these equations the electromagnetic variables are the electric field $\mathbf{E}$, the magnetic field $\mathbf{B}$ and the current density $\mathbf{J}$. We denote by $\mathbf{v}$ the fluid velocity, p the fluid pressure and $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$ the convective derivative. The parameter μ represents the electric conductivity. In the plasma region we shall assume $\mu = 0$, i.e., the plasma is a perfect conductor and so equations (3)–(7) becomes the system of ideal MHD. In the vacuum region we shall take $\mu = 1$.

We are interested in studying quasi–stationary processes, that is, the processes occurring on a slow time–scale, the resistive diffusion time–scale. In this time–scale, plasma would evolve through a series of states each of which would be very nearly in equilibrium, i.e., at each instant t, plasma can be regarded as being in MHD equilibrium ([16]). Following [4, Chapter IV] (see also [16]) we define some characteristic time constants of the plasma and introduce some plasma physics phenomena in order to neglect some quantities in the above system. To analyze slow phenomena we need to retain only the principal terms in the relation (3). So, we can neglect $\frac{D\mathbf{v}}{Dt}$ in plasma region at time t, $\Omega_p(t)$. The time dependence comes from the fact that we shall not neglect the term $\frac{\partial \mathbf{B}}{\partial t}$. Thus,

$$\nabla p = \mathbf{J} \times \mathbf{B} \tag{8}$$

is satisfied at each instant t in $\Omega_p(t)$. From (8) it follows that, for any instant t,

$$\mathbf{B}(t) \cdot \nabla p(t) = 0 \tag{9}$$

$$\mathbf{J}(t) \cdot \nabla p(t) = 0. \tag{10}$$

Then the pressure is constant on each magnetic surface; these surface are nested toroids (see [16], [14]). Thus, it is useful to introduce a set of new toroidal coordinates $(\widetilde{\rho}, \widetilde{\theta}, \widetilde{\phi})$, such that: $\widetilde{\rho} = \widetilde{\rho}(x, y, z)$ is an arbitrary function which is constant on each nested toroid and $\widetilde{\theta} = \widetilde{\theta}(x, y, z)$ is the poloidal angle which is constant on any toroidal circuit but changes by 2π over a poloidal circuit (here by a toroidal circuit we mean any closed loop that encircles the axis of the torus once, and by a poloidal circuit a closed loop that encircles the minor axis once). The

toroidal angle $\widetilde{\phi}$ is defined analogously but interchanging the words poloidal by toroidal. Among the special choices of $(\widetilde{\rho}, \widetilde{\theta}, \widetilde{\phi})$, we shall take the *Boozer vacuum coordinates system* ([5]) which are very useful for Stellarators since magnetic field lines becomes "straights" in the $(\widetilde{\theta}, \widetilde{\phi})$–plane. In what follows, for the sake of simplicity in the notation, we shall denote this set of coordinates by (ρ, θ, ϕ).

For a vacuum configuration (i.e. without any plasma) the magnetic field $\mathbf{B}_v$ may be written in contravariant form as

$$\mathbf{B}_v = B_0 \rho \nabla \rho \times \nabla(\theta - t_v(\rho)\phi)$$

where $t_v(\rho)$ is the so called *vacuum rotational transform* and B_0 is a positive constant. The covariant form of $\mathbf{B}_v$ is

$$\mathbf{B}_v = F_v \nabla \phi \qquad (11)$$

where F_v is a constant (which customary is taken as positive). In practice, it is used the quasi–cylindrical–like Boozer set of coordinates $(\rho, \rho\theta, \phi)$ which have the usual near–axis behavior of the field components commonly used.

The Stellarators–type configurations are very complicated due to the fully three–dimensional nature of the device. To simplify the model to a two–dimensional problem different averaging methods were used for the study of stationary models: see Greene and Johnson [16], and Hender and Carreras [18]. Following the last reference we may decompose the magnetic field in terms of its toroidally averaged and rapidly varying parts. For a general function f this decomposition takes the form

$$f = \langle f \rangle + \widetilde{f}, \text{ with } \langle f \rangle := \frac{1}{2\pi} \int_0^{2\pi} f d\phi \ .$$

In our case, motivated by the set of coordinates $(\rho, \rho\theta, \phi)$, the natural way of doing that is

$$\frac{B^i}{D} = \left\langle \frac{B^i}{D} \right\rangle + \left(\frac{\widetilde{B}^i}{D} \right)$$

where B^i are the contravariant components of the vacuum magnetic field, $i = \rho, \ \theta, \ \phi$, and D is the Jacobian $D = (\nabla \rho \times \rho \nabla \theta) \cdot \nabla \phi$.

Using a suitable assumption (the Stellarator expansion hypothesis) Hender and Carreras [18] show that (6) leads to the equation

$$\frac{\partial}{\partial \rho} \left(\rho \left\langle \frac{B^\rho(t)}{D} \right\rangle \right) + \frac{\partial}{\partial \theta} \left(\left\langle \frac{B^\theta(t)}{D} \right\rangle \right) = 0,$$

and thus to the existence of the *averaged poloidal flux function* $\psi = \psi(t, \rho, \ \theta)$ defined at each instant t by

$$\left\langle \frac{B^\rho(t)}{D} \right\rangle = \frac{1}{\rho} \frac{\partial \psi(t)}{\partial \theta} \quad \text{ and } \left\langle \frac{B^\theta(t)}{D} \right\rangle = -\frac{\partial \psi(t)}{\partial \rho} \qquad . \qquad (12)$$

They also show that, when MHD equilibrium (8) exists, then $\langle B_\phi \rangle$ is a function ψ alone and the same for $\langle p \rangle$ (recall (9)). Thus, in our case, as equation (8) is satisfied in the plasma region, by introducing the usual notation

$$F(\psi) := \langle B_\phi \rangle \quad \text{and} \quad p(\psi) := \langle p \rangle , \tag{13}$$

and following Hender and Carreras [18], we obtain a Grad–Shafranov type equation for ψ in the plasma region $\Omega_p(t)$ at each $t \geq 0$:

$$-\mathcal{L}\psi = a(\rho,\theta)F(\psi) + F(\psi)F'(\psi) + b(\rho,\theta)p'(\psi). \tag{14}$$

That is, ψ satisfies, at each instant, an equilibrium equation, where

$$\mathcal{L}\psi := \frac{1}{\rho} \left\{ \frac{\partial}{\partial \rho}(a_{\rho\rho}\frac{\partial \psi}{\partial \rho}) + \frac{\partial}{\partial \rho}(a_{\rho\theta}\frac{\partial \psi}{\partial \theta}) + \frac{\partial}{\partial \theta}(a_{\theta\rho}\frac{\partial \psi}{\partial \rho}) + \frac{\partial}{\partial \theta}(a_{\theta\theta}\frac{\partial \psi}{\partial \theta}) \right\}$$

with

$$a_{\rho\rho}(\rho,\theta) := \rho \left\langle g^{\rho\rho} \right\rangle (\rho,\theta)$$
$$a_{\rho\theta}(\rho,\theta) = a_{\theta\rho}(\rho,\theta) := \left\langle g^{\rho\theta} \right\rangle (\rho,\theta)$$
$$a_{\theta\theta}(\rho,\theta) := \frac{1}{\rho} \left\langle g^{\theta\theta} \right\rangle (\rho,\theta)$$

and where $\langle g^{i,j} \rangle$, i, $j = \rho$, θ are the averaged components of the Riemannian metric associated to the vacuum coordinates system (all those coefficients are 2π–periodic functions of θ). The rest of the coefficients in (14) are given by

$$a(\rho,\theta) := \frac{B_0}{\rho F_v} \left[\frac{\partial}{\partial \rho} \left(\rho^2 t(\rho) \left\langle g^{\rho\rho} \right\rangle \right) + \frac{\partial}{\partial \theta} \left(\rho t(\rho) \left\langle g^{\rho\theta} \right\rangle \right) \right]$$

and

$$b(\rho,\theta) := \frac{F_v}{B_0} \left\langle \frac{1}{D} \right\rangle (\rho,\theta).$$

We remark that $b > 0$ and that usually function a does not have any singularity.

As we have already pointed out, equation (14) only holds on the (averaged) region occupied by the plasma. Following [4, Chapter IV] we analyze the vacuum region at time t and we obtain, by using (4), (5), (7), and relation (11), that the equation satisfied by ψ in Ω_v is

$$-\mathcal{L}\psi = aF_v - \frac{\partial \psi}{\partial t},$$

where $\Omega_v := \bigcup_{t \in [0,T]} \{t\} \times \Omega_v(t)$. In order to obtain a global formulation as a free boundary problem we remark that in the vacuum region $\nabla p = 0$. Besides, it is clear that the free boundary (separating the plasma and vacuum regions at time t) is a (toroidal) magnetic surface and, as $p = p(\psi)$, by normalizing, we can identify the free boundary as the level line $\{\psi(t) = 0\}$, the plasma region as $\{\psi(t) > 0\}$ (and thus $\{p > 0\}$) and the vacuum region by $\{\psi(t) < 0\}$ (and

$\{p = 0\}$). It is also well–known that the pressure cannot be obtained from the (MHD) system and some constitutive law must be assumed. Here, for simplicity, we shall assume a quadratic law (see, e.g., Temam [33])

$$p = \frac{\lambda}{2}[\psi_+]^2, \qquad \psi_+ = \max\{\psi, 0\} \tag{15}$$

which is compatible with the above normalization. In order to give an unified formulation for the present model, we extend the *unknown* function $F(\psi)$ for negative values of ψ. We use again (11) and so we must find $\psi(t, \rho, \theta)$ and $F : \mathbb{R} \to \mathbb{R}_+$ such that $F(s) = F_v$ for any $s \leq 0$, satisfying

$$-\frac{\partial \psi_-}{\partial t} - \mathcal{L}\psi = a(\rho, \theta)F(\psi) + F(\psi)F'(\psi) + \lambda b(\rho, \theta)\psi_+ \, . \tag{16}$$

The above equation is satisfied on any bidimensional open set (in the variables (ρ, θ)) associated to a physical three–dimensional domain Ω^3 (i.e. in the original cartesian variables (X, Y, Z)) containing in its interior the plasma region. If we take as Ω^3 the interior of a vacuum magnetic surface, the construction of the Boozer coordinates implies that the associated open set in the (ρ, θ) variables becomes $\Omega = \{(\rho, \theta) : \rho \in (0, R), \theta \in (0, 2\pi)\}$. The boundary of Ω^3 is assumed to be a *perfectly conducting wall* and thus the boundary conditions become (see Díaz [7])

$$\psi(t) = \gamma \text{ on }]0, T[\times \partial\Omega.$$

To complete the formulation of the problem under consideration we must add the Stellarator condition imposing a zero net current within each flux magnetic surface. According the averaging method by Hender and Carreras [18] this condition can be expressed (Díaz [7]) for a. e. $t \in]0, T[$ as

$$\int_{\{\psi(t) \geq \tau\}} [F(\psi)F'(\psi) + \lambda b\psi_+] \rho d\rho d\theta = 0 \text{ for any } \tau \in [\inf\psi, \sup\psi]. \tag{17}$$

Notice that in the case of Stellarator devices this condition comes from the design of the external conductors. This contrast with the usual condition of positive total current due to the inner toroidal current in the plasma for Tokamaks configurations (see, e.g., Temam [33] and Blum [4]). Any way, (17) is an ideal situation, and in practice some known current arises at the interior of each magnetic surface. An approach to the dynamic problem associated to this situation has been initiated in [9] (see [11] for the stationary regime).

Summarizing we arrive to the formulation given in $(\mathcal{P}_I)$ of Section 1, where for the sake of simplicity we have replaced the $\mathcal{L}$ operator by the Laplacian one, Δ.

3   Preliminary results and definitions.

In this section we recall briefly the notion of relative rearrangement and some useful properties of it. Let Ω be a bounded measurable set of $\mathbb{R}^N$, $N \geq 1$. For any measurable subset E of Ω we denote by $|E|$ its Lebesgue measure $\int_E dx$. The decreasing rearrangement of a measurable function u defined on Ω is given by

$$u_*(s) = \inf \{ \theta \in \mathbb{R} : |u > \theta| \leq s \},$$

where $s \in (0, |\Omega|)$. We will say that u has a *flat region at the level* θ if $|u = \theta|$ is strictly positive. We recall that given a measurable function u, there exists at most a countable family D of flat regions $P_u(\theta_i) := \{u = \theta_i\}$. We denote by $P(u) = \bigcup_{i \in D} P_u(\theta_i)$ the union of all the flat regions of u. Given $v \in L^1(\Omega)$, we define a function w on $[0, |\Omega|]$ by:

$$w(s) = \int_{\{u > u_*(s)\}} v(x)\, dx \; + \; \int_0^{s - |u > u_*(s)|} \left(v|_{P(s)} \right)_* (\sigma)\, d\sigma \ ,$$

where $P(s) = \{ x \in \Omega : u(x) = u_*(s) \}$ and $v|_{P(s)}$ is the restriction of v to $P(s)$. The following lemma was proved in [22] (see also [23]).

Lemma 3.1 *Let* $u \in L^1(\Omega)$ *and* $v \in L^p(\Omega)$ *for some* $1 \leq p \leq +\infty$. *Then* $w \in W^{1,p}(\Omega_*)$ *and* $\left\| \dfrac{dw}{ds} \right\|_{L^p(\Omega_*)} \leq \|v\|_{L^p(\Omega)}$, *where* $\Omega_* := (0, \ |\Omega|)$.

Definition 3.1 *The function* $\dfrac{dw}{ds}$ *is called the* relative rearrangement of v with *respect to the* u *and it is denoted by*

$$v_{*u} = \frac{dw}{ds}.$$

For more details of the notion of the relative rearrangement, one can consult [21], [23], [25], [27]. We will need some lemmas already proved in previous papers.

Lemma 3.2 ([12]) *Let* $(u, \ \varphi) \in L^1(\Omega)^2$. *Then,*

   i) $\varphi_{*(u+c)} = \varphi_{*u}$,

   ii) $(\varphi + c)_{*u} = \varphi_{*u} + c$ *for any constant* c.

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Lemma 3.3 ([12]) *Let u_n, $u \in W^{1,p}(\Omega)$, $1 \le p \le +\infty$, such that either meas$\{x \in \Omega : |\nabla u|(x) = 0\} = 0$ or $u \in W^{N,r}_{loc}(\Omega)$ for some $r > 1$. Then, if u_n converges to u in $W^{1,p}(\Omega)$ for some $p > N$, we get that u'_{n*} converges strongly to u'_* in $L^q(\Omega)$, for any $1 \le q < q_c := \dfrac{1}{1 - \frac{1}{N} + \frac{1}{p}}$.*

Lemma 3.4 ([12]) *Let u_n, u be in $L^1(\Omega)$ and assume that u_n converges to u in $L^1(\Omega)$. Then, for all $v \in L^p(\Omega)$ (for a given p, $1 < p \le +\infty$) we have*

$$\left(v\chi_{\Omega\backslash P(u)}\right)_{*u_n} \rightharpoonup \left(v\chi_{\Omega\backslash P(u)}\right)_{*u}$$

weakly in $L^p(\Omega_*)$ if $p < +\infty$, and weakly–start in $L^\infty(\Omega_*)$ if $p = +\infty$ (where χ_E denotes the characteristic function of the set E).

Lemma 3.5 ([13], [28]) *Let $v \in L^p(\Omega)$ with $1 \le p < +\infty$ and u, $u_n \in W^{1,r}(\Omega)$, $1 < r \le +\infty$, such that meas$\{x \in \Omega : |\nabla u(x)| = 0\} = 0$ and meas$\{x \in \Omega : |\nabla u_n(x)| = 0\} = 0$. If u_n converges to u in $W^{1,r}(\Omega)$ for some $1 < r \le +\infty$, then $v_{*u_n} \longrightarrow v_{*u}$, when $n \longrightarrow \infty$ strongly in $L^p(\Omega_*)$. Furthermore, $v_{*u_n}(|u_n > u_n(\cdot)|)$ converges strongly to $v_{*u}(|u > u(\cdot)|)$ in $L^p(\Omega)$.*

As a direct consequence of this lemma, we have the following result

Lemma 3.6 ([28]) *Let $v \in L^p(\Omega)$ with $1 \le p < +\infty$ and let $(\lambda_k, \varphi_k)_{k \ge 1}$ be the sequence of eigenvalues and eigenfunctions of $-\Delta$ on Ω with Dirichlet conditions; i.e. $-\Delta\varphi_k = \lambda_k\varphi_k$, $\varphi_k \in H^1_0(\Omega)$. Consider the finite dimensional vector space $V_m = \{\varphi_1, ..., \varphi_m\}$. Then, the maps*

   i) $u \in V_m\backslash\{0\} \longrightarrow b_{*u} \in L^p(\Omega_*)$,

   ii) $u \in V_m\backslash\{0\} \longrightarrow b_{*u}(|u > u(\cdot)|) \in L^p(\Omega)$,

   are strongly continuous.

Given $\theta \in \mathbb{R}$, let $|u(t) > \theta|$ be the measure of the set $\{y \in \Omega : u(t)(y) > \theta\}$ and analogously let $|u(t) = \theta|$ be the measure of the set $\{y \in \Omega : u(t)(y) = \theta\}$. For a fixed $\sigma \in \Omega_* =]0, |\Omega|[$, the *monotone rearrangement* of $u(t)$ at σ is $u(t)_*(\sigma) = u_*(t, \sigma) = \inf\{\theta \in \mathbb{R}, |u(t) > \theta| \le \sigma\}$. Given a function $b \in L^1(Q)$ ($Q :=]0, T[\times\Omega$), we set for $\sigma \in]0, |\Omega|[$, $t \in]0, T[$

$$w(t, \sigma) = \int\limits_{\{x : u(t)(x) > u_*(t,\sigma)\}} b(t, x)\, dx \quad + \int\limits_0^{\sigma - |u(t) > u_*(t,\sigma)|} \left(b(t)|_{\{u(t) = u_*(t,\sigma)\}}\right)_* (s)\, ds$$

where $b(t)|_{\{u(t) = u_*(t,\sigma)\}}$ is the restriction of $b(t)$ to the set $\{x : u(t)(x) = u_*(t, \sigma)\}$. The *relative rearrangement of $b(t)$ with respect to $u(t)$* is the weak derivative

$\frac{\partial w}{\partial \sigma}(t, \sigma)$, we set $b(t)_{*u(t)}(\sigma) = b_{*u}(t, \sigma) = \frac{\partial w}{\partial \sigma}(t, \sigma)$. Properties on monotone rearrangement given above are valid for $u(t)_*$. On the other hand, from Lemma 3.1, we know that $w(t) \in W^{1,p}(\Omega_*)$, provided that $b(t) \in L^p(\Omega)$ and $\left| \frac{\partial w}{\partial \sigma}(t) \right|_{L^p(\Omega_*)} \leq |b(t)|_{L^p(\Omega)}$, $1 \leq p \leq +\infty$.

We recall a general result on functions $v \in H^1(0, T; L^1(\Omega))$ which can be proved by using rearrangement techniques.

Lemma 3.7 ([22]) *Assume* $v \in H^1(0, T; L^1(\Omega))$. *Then, for almost every* $t \in]0, T[$, *we have that* $\frac{\partial v(t, \cdot)}{\partial t}$ *is constant on any set where* $v(t, \cdot)$ *is constant.*

Due to the presence of nonlocal terms in $(\mathcal{P})$, two different notions of weak solutions can be introduced.

Definition 3.2 *We will say that a function* u *is a* first category weak solution *of* $(\mathcal{P})$ *if the function* $v = u - \gamma$ *satisfies:*

1. $v \in L^2(0, T; H_0^1(\Omega))$, $\frac{\partial}{\partial t}\beta(v + \gamma) \in L^2(0, T; H^{-1}(\Omega))$.

2. *For a. e.* $t \in]0, T[$, *the function* $u(t)$ *has not flat regions.*

3. $\frac{\partial}{\partial t}\beta(v + \gamma) - \Delta v = aG(v + \gamma) + J(v + \gamma)$ *in* $\mathcal{D}'(\Omega)$ *for a. e.* $t \in]0, T[$ *and* $\beta(v + \gamma)|_{t=0} = \beta(u_0)$.

Definition 3.3 *We will say that a function* u *is a* second category weak solution *of* $(\mathcal{P})$ *if the function* $v = u - \gamma$ *satisfies:*

1. $v \in L^2(0, T; H_0^1(\Omega))$, $\frac{\partial}{\partial t}\beta(v + \gamma) \in L^2(0, T; H^{-1}(\Omega))$.

2. (Relative rearrangement condition) *There exists a bounded function* $b^v \in L^\infty(Q)$, *satisfying for a. e.* $t \in]0, T[$, *for all* $\theta \in \mathbb{R}$ *and for all* $\varphi \in C(\mathbb{R})$, *with* $\varphi(v(t)) \in L^1(\Omega)$

$$\int\limits_{\{x : v(t,x) > \theta\}} b^v \varphi(v(t, x)) dx = \int\limits_{\{x : v(t,x) > \theta\}} b \varphi(v(t, x)) dx$$

and

$$ess \inf_{\Omega} b \leq b^v \leq ess \sup_{\Omega} b .$$

3. $\frac{\partial}{\partial t}\beta(v + \gamma) - \Delta v = aG(v + \gamma) + p'(v + \gamma)[b - b^v]$ *in* $\mathcal{D}'(\Omega)$ *for a. e.* t, *and* $\beta(v + \gamma)|_{t=0} = \beta(u_0)$.

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Remark 3.1 Notice that $\beta(v+\gamma) = -u_- = T^*_{-\gamma}(v) - \gamma$ were $T^*_{-\gamma}$ is the truncation at level $(-\gamma)$ function

$$T^*_{-\gamma}(r) = \left\{ \begin{array}{ll} r & if \;\; r < -\gamma \\ -\gamma & if \;\; r \geq -\gamma \end{array} \right. .$$

On the other hand, since $T^*_{-\gamma}(\cdot)$ is a Lipschitz function, we have that $T^*_{-\gamma}(v) \in L^2(0,T;H^1_0(\Omega))$. Then, using that condition 1 (in both cases) implies that $\frac{\partial}{\partial t}\left(T^*_{-\gamma}(v)\right) \in L^2(0,T;H^{-1}(\Omega))$, by well–known interpolation results (Simon [35]) we conclude that $T^*_{-\gamma}(v)$ and $\beta(v+\gamma) \in C([0,T];L^2(\Omega))$ and so the restriction $\beta(v+\gamma)|_{t=0}$ is well defined.

Lemma 3.8 *Let v be an $L^1(\Omega)$ without any flat zones, i.e.* $\mathrm{meas}(P(v)) = 0$, *and $b \in L^1(\Omega)$. For all $\phi \in C(\mathbb{R})$, we have*

$$\int_\Omega b_{*v}\left(|v > v(x)|\right)\phi(v)(x)dx = \int_\Omega b\phi(v)dx .$$

Proof From the equimeasurable property and the mean value operator property (see for example [23], [21]), we have:

$$\int_\Omega b_{*v}\left(|v > v(x)|\right)\phi(v(x))dx = \int_{\Omega_*} b_{*v}(\sigma)\phi(v_*(\sigma))d\sigma = \int_\Omega b\phi(v)dx.$$

Lemma 3.9 *Assume that $N = 2$ and let v^ε be a sequence in $W^{1,2+\delta}(\Omega)$, for some $\delta > 0$ and $v \in H^2(\Omega)$ such that v^ε converges to v in $W^{1,2+\delta}(\Omega)$ as $\varepsilon \to 0$. Then, $\lim\limits_{\varepsilon \searrow 0} G(v^\varepsilon + \gamma)(x) = G(v+\gamma)(x)$ for a. e. $x \in \Omega$.*

Proof Let θ be the characteristic function of the set $P(v_*)$. From the above convergence we get $\frac{dv^\varepsilon_*}{d\sigma} \to \frac{dv_*}{d\sigma}$ in $L^q(\Omega_*)$ as $\varepsilon \searrow 0$ with q given in Lemma 3.3 . Thus $\lim\limits_{\varepsilon \searrow 0} \theta \frac{dv^\varepsilon_*}{d\sigma} = 0$ in $L^q(\Omega_*)$. Arguing as in Rakotoson–Seoane [30], we conclude that

$$(1-\theta)b_{*v^\varepsilon} \rightharpoonup (1-\theta)b_{*v} \tag{18}$$

weakly–star in $L^\infty(\Omega_*)$.

Now, if we set $I(v^\varepsilon(x))$ the interval given by $[|v^\varepsilon > v^\varepsilon_+(x)|, \;\; |v^\varepsilon > 0|]$, then for all $\sigma \in \overline{\Omega}_*$, $\;\; \sigma \neq |v > v_+(x)|$, $\;\; \sigma \neq |v > 0|$, one has

$$\lim\limits_{\varepsilon \searrow 0}(1-\theta)\chi_{I(v^\varepsilon(x))}(\sigma) = (1-\theta)\chi_{I(v(x))}(\sigma) . \tag{19}$$

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From relation (18) and (19) and the fact $\lim_{\varepsilon} |\theta \frac{dv_*^\varepsilon}{ds}|_q = 0$, one finds (for almost every $x \in \Omega$ using the fact that $v_{+*}^\varepsilon(\sigma) \to v_{+*}$ in $L^r(\Omega_*), \quad r < \infty$) that

$$\lim_{\varepsilon \to 0} \int_{|v^\varepsilon > 0|}^{|v^\varepsilon > v_+^\varepsilon(x)|} \frac{dv_*^\varepsilon}{d\sigma}(\sigma) b_{*v^\varepsilon}(\sigma) p'(v_*^\varepsilon(\sigma)) d\sigma =$$

$$= \int_{|v > 0|}^{|v > v_+(x)|} \frac{dv_*}{d\sigma}(\sigma)(1 - \theta)^2 b_{*v}(\sigma) p'(v_*(\sigma)) d\sigma$$

$$= \int_{|v > 0|}^{|v > v_+(x)|} \frac{d}{d\sigma}(p(v_*(\sigma))) b_{*v}(\sigma) d\sigma.$$

The above limit implies the result. ■

4 General results on first and second category weak solutions.

Proposition 4.1 *Any first category solution u is a second category solution.*

Proof Let u be a first category solution, and let $v = u - \gamma$. We set $b^v(t, x) = b_{*u}(|u(t) > u(t, x)|)$, for $(t, x) \in Q$. By invariance of such quantity with respect to a translation by a constant one has:

$$b^v(t, x) = b_{*v}(|v(t) > v(t, x)|)$$

(recall $|v(t) > v(t, x)| = $ measure $\{y \in \Omega, \ v(t, y) > v(t, x)\}$). It remains to check the relative rearrangement condition. Indeed, using property 2 of the definition, the equimeasurability and the mean value theorem, on has for a. e. t, for all $\theta \in \mathbb{R}$

$$\int_{\{x:v(t,x)>\theta\}} b^v \varphi(v(t,x)) dx = \int_{\{\sigma:v_*(t,\sigma)>\theta\}} b_{*v} \varphi(v_*(t,\sigma)) d\sigma = \int_{\{x:v(t,x)>\theta\}} b \varphi(v(t,x)) dx .$$

■

Proposition 4.2 *If u is a second category solution satisfying that $u(t, \cdot)$ has not flat regions for a. e. $t \in]0, T[$ and that there exists a Borel map $g^u : \mathbb{R} \to \mathbb{R}$ such that*

$$g^u \circ u = b^v, \ v = u - \gamma .$$

Then u is a first category solution.

Proof It suffices to show that for a. e. $(t, x) \in Q$, $b^v(t, x) = b_{*u}(|u(t) > u(t, x)|)$. First, let us observe that as u is a second category solution, we deduce by an approximating argument that for any Borel function φ on $\mathbb{R}$ with $\varphi(v(t)) \in L^1(\Omega)$

$$\int_\Omega b^v \varphi(u(t, x)) dx = \int_\Omega b \varphi(u(t, x)) dx \tag{20}$$

and from the first hypothesis, by the mean value theorem and equimeasurability, we have:

$$\int_\Omega b \varphi(u(t, x)) dx = \int_\Omega b_{*u}(|u(t) > u(t, x)|) \varphi(u(t, x)) dx \ . \tag{21}$$

By the second assumption of Proposition 4.2, the function

$$\varphi(\sigma) = b_{*u}(|u(t) > \sigma|) - g^u(\sigma), \ \sigma \in \mathbb{R} \tag{22}$$

is a Borel function in $\mathbb{R}$ for almost every t. Thus from (20), (21) and (22) we deduce

$$\int_\Omega \left(b_{*u}(|u(t) > u(t, x)|) - g^u \circ u(t, x) \right)^2 dx = 0 \ .$$

$\blacksquare$

Let us now prove some properties of a second category solutions:

Theorem 4.1 *Any second category weak solution u satisfies*

$$|\beta(u(t)) - \beta(\gamma)|_{L^\infty(\Omega)} \leq |a|_\infty F_v t + |\beta(u_0) - \beta(\gamma)|_{L^\infty(\Omega)} \qquad \forall t \in [0, T] \ .$$

Proof For an integer $m \geq 2$, we set $g_m(\sigma) = |\sigma|^{m-2} \sigma$ and T_k the truncation operator given by

$$T_k(\sigma) = \begin{cases} \sigma & \text{if } |\sigma| \leq k, \\ k \operatorname{sign} \sigma & \text{if } |\sigma| > k, \end{cases}$$

so we get

$$w_{m,k} := g_m \circ T_k(\beta(u) - \beta(\gamma)) \in L^\infty(Q) \cap L^2(0, T; H_0^1(\Omega)).$$

By the relative rearrangement condition, one has:

$$< \frac{\partial}{\partial t} \beta(v+\gamma), w_{m,k}(t) > + \int_\Omega \nabla v(t, x) \cdot \nabla w_{m,k}(t, x) dx = \int_\Omega aG(v+\gamma) w_{m,k} dx \tag{23}$$

where $< \cdot, \cdot >$ denotes the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. By the *integration by parts formula* (see, e.g., Alt and Luckhaus [2]), we have

$$\frac{d}{dt} y_{m,k}(t) = < \frac{\partial}{\partial t} \beta(v(t) + \gamma), w_{m,k}(t) > \tag{24}$$

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where we set

$$y_{m,k}(t) = \int_\Omega dx \int_0^{\beta(v(t)+\gamma)+\gamma} g_m \circ T_k\,(\sigma)\,d\sigma \ .$$

Since $\displaystyle\int_\Omega \nabla v(t,x) \cdot \nabla w_{m,k}(t,x)dx \geq 0$ and $0 \leq G(v+\gamma) \leq F_v$, one gets from (23) and (24) (via Hölder inequality)

$$\frac{d}{dt}y_{mk}(t) \leq |a|_\infty F_v \int_\Omega |w_{m,k}(t,x)|dx \leq |a|_\infty F_v |\Omega|^{\frac{1}{m}} \left(\int_\Omega |w_{m,k}(t,x)|^{\frac{m}{m-1}} \right)^{1-\frac{1}{m}} .$$
(25)

We use the equality:

$$\int_\Omega |w_{m,k}(t,x)|^{\frac{m}{m-1}}dx = m y_{m,k}(t) - m k^{m-1} \int_\Omega \left(|\beta(v+\gamma) - \beta(\gamma)| - k\right)_+ dx, \quad (26)$$

which, together with (25), gives

$$y'_{m,k}(t) \leq m^{1-\frac{1}{m}}|\Omega|^{\frac{1}{m}}|a|_\infty F_v y_{m,k}(t)^{1-\frac{1}{m}}. \tag{27}$$

Thus, we can derive

$$y_{m,k}^{\frac{1}{m}}(t) \leq m^{-\frac{1}{m}}|\Omega|^{\frac{1}{m}}|a|_\infty F_v t + y_{m,k}^{\frac{1}{m}}(0).$$

Finally, using (26) and letting $m \to +\infty$ and $k \to +\infty$ we get the result. ∎

Remark 4.1 *Note that the above proof holds for any non decreasing Lipschitz function β and any G such that $0 \leq G \leq F_v$.*

In the special case of $\beta(\sigma) = \min(\sigma,\,0) + \alpha\sigma_+$, for countably many $\alpha \in [0,1]$, we have

Theorem 4.2 *Assume that $\beta(\sigma) = -\sigma_-$, $N = 2$. Then, for any second category weak solution u, one has for all $t \in [0,T]$*

$$|u_+(t)|_\infty \leq \frac{1}{4\pi}|a|_\infty F_v|\Omega| \ .$$

In particular $u \in L^\infty(Q)$.

If $\beta(\sigma) = -\sigma_- + \alpha\sigma_+$, $0 < \alpha \leq 1$, the above result hold provided that $\dfrac{\partial u}{\partial t} \in L^1(Q)$. Similar conclusions holds for any dimension $N \geq 3$.

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Proof Consider the case $\alpha = 0$. Since $\dfrac{\partial u_-}{\partial t} \in L^2\left(0, T; H^{-1}(\Omega)\right)$, one gets by approximation that for all $\theta > 0$, for a. e. $t \in]0, T[$

$$< \frac{\partial u_-}{\partial t}(t), (u_+(t) - \theta)_+ >= 0 \tag{28}$$

$(< \cdot, \ \cdot >$ denotes the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$). By the relative rearrangement condition, the equation gives:

$$\int\limits_{\{u_+(t) > \theta\}} |\nabla u_+(t)|^2 \, dx \ = \int_\Omega aG(u)(u_+(t) - \theta)_+ dx \ . \tag{29}$$

Deriving this last relation with respect to θ and using the fact that $0 \le G(u) \le F_v$, we get

$$-\frac{d}{d\theta} \int\limits_{\{u_+(t) > \theta\}} |\nabla u_+(t)|^2 \, dx \ \le |a|_\infty F_v |u_+(t) > \theta| \ .$$

Following Talenti's method (see, e.g., the exposition made in Mossino [21]), one derives for all $t \in [0, T]$

$$|u_+(t)|_\infty \le \frac{1}{4\pi} |a|_\infty F_v |\Omega| \ . \tag{30}$$

Combining Proposition 4.1 and this result, we get that $u \in L^\infty(Q)$.

Now, let $0 < \alpha \le 1$. We argue as in Mossino Rakotoson [22]. Using the relative rearrangement condition as in the relation (29), we derive for a. e. $\theta > 0$

$$\alpha \int\limits_{\{u_+(t) > \theta\}} \tfrac{\partial}{\partial t} u_+(t) \, dx \ -\frac{d}{d\theta} \int\limits_{\{u_+(t) > \theta\}} |\nabla u_+(t)|^2 \, dx \ = \int\limits_{\{u_+(t) > \theta\}} aG(u) \, dx \ . \tag{31}$$

From which we derive (see [10], for an analogous argument)

$$-4\pi s \frac{\partial}{\partial s} u_{+*}(t, s) \le |a| F_v s - \alpha \int_0^s \frac{\partial}{\partial t} u_{+*}(t, \sigma) d\sigma \tag{32}$$

for all $s \in (0, |\Omega|)$. If we introduce $K(t, s) = \int_0^s u_{+*}(t, \sigma) d\sigma$, then (32) gives the following partial differential inequality:

$$\begin{cases} \alpha \dfrac{\partial}{\partial t} K(t, s) - 4\pi s \dfrac{\partial^2}{\partial s^2} K(t, s) & \le |a|_\infty F_v s \\ K(t, 0) = 0, & \dfrac{\partial K}{\partial s}(t, |\Omega|) = 0 \end{cases}$$

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Introducing the function $\widehat{K}(s)$ satisfying:

$$|a|_\infty F_v s = -4\pi s \frac{d^2\widehat{K}}{ds^2}, \qquad \widehat{K}(0) = 0, \qquad \frac{d\widehat{K}}{ds}(|\Omega|) = 0$$

that is $\widehat{K}(s) = -\dfrac{|a|_\infty F_v}{4\pi}s^2 + \dfrac{|a|_\infty F_v}{4\pi}|\Omega|$. We deduce from comparison principle (see Díaz, Nagai and Rakotoson [10]) that $\widehat{K}(t,s) \leq \widehat{K}(s)$ for all $s \in [0,\ |\Omega|]$. In particular we deduce

$$|u_+(t)|_{L^\infty(\Omega)} \leq \left|\frac{d\widehat{K}}{ds}(0)\right| = \frac{|a|_\infty F_v |\Omega|}{4\pi}. \qquad \blacksquare$$

Theorem 4.3 *Any second category weak solution u satisfies the following estimate*

$$\int_0^t \int_\Omega |\nabla u(\sigma,x)|^2\, dxd\sigma + \int_\Omega dx \int_0^{u_0(x)} \beta(\sigma+\gamma)\, d\sigma \leq$$

$$\leq \int_\Omega u_0 \beta(u_0+\gamma) + |a|_\infty F_v \int_0^t d\sigma \int_\Omega |u-\gamma|(\sigma,x)\, dx$$

for all $t \in [0,T]$.

Proof By the relative rearrangement condition and the *integration by parts formula* one has (multiplying by $v(t)$ the equation)

$$\frac{d}{dt} \int_\Omega \psi^*(\beta(v(t)+\gamma))\, dx + \int_\Omega |\nabla v(t,x)|^2 dx = \int_\Omega aG(v(t)+\gamma)v(t)dx \qquad (33)$$

where

$$\int_\Omega \psi^*(\beta(v(t)+\gamma))\, dx = \int_\Omega v(t)\beta(v(t)+\gamma)\, dx - \int_\Omega dx \int_0^{v(t,x)} \beta(\sigma+\gamma)\, d\sigma$$

Integrating (33) with respect to t, dropping some nonnegative term, and using the relations $0 \leq G(v+\gamma) \leq F_v$ we derive the result. $\qquad \blacksquare$

Making use of the Sobolev Poincaré inequality and Schwartz inequality, we easily have the

Corollary 4.1 *For all $t \in [0,T]$*

$$\int_0^t \int_\Omega |\nabla u(\sigma,x)|^2 dxd\sigma + 2\int_\Omega \int_0^{u_0(x)} \beta(\sigma+\gamma)\, d\sigma \leq 2\int_\Omega u_0\beta(u_0+\gamma)dx + \frac{\|a\|_\infty^2 F_v^2 |\Omega|}{\lambda_1}t,$$

where λ_1 is the first eigenvalue of the homogeneous Dirichlet problem associated to the Laplace operator.

5   Existence theorems.

In this section, we prove the existence of at least one second category solution u for $0 \le \alpha \le 1$. Our first result concerns the existence of a global solution for $\alpha > 0$:

Theorem 5.1 *Assume that $N = 2$, $u_0 \in H^1(\Omega)$, $\beta(\sigma) = \min(0, \sigma) + \alpha\sigma_+$ with $0 < \alpha \le 1$ and $\beta(u_0) \in L^\infty(\Omega)$. Then there exists at least a solution $u_\alpha \in L^2(0, T; H^2(\Omega))$ of the second category of $(\mathcal{P})$. Moreover, if $u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ then $u_\alpha \in L^2(0, T; H^2(\Omega)) \cap L^\infty(Q)$.*

The proof of Theorem 5.1 will consist in three steps and is available for more general β and for $N \ge 3$. Let $0 < \varepsilon < 1,$ and $\beta_\varepsilon \in C^\infty(\mathbb{R})$, satisfying

i) $\alpha \le \beta'_\varepsilon \le 1 + \alpha, \quad \beta_\varepsilon(0) = 0, \quad \beta_\varepsilon \to \beta \in H^1_{loc}(\mathbb{R})$, and

ii) for any $\sigma \in \mathbb{R}, \quad |\beta_\varepsilon(\sigma) - \beta(\sigma)| \le 2\varepsilon$.

We shall solve first the following approximate problem:

Theorem 5.2 *Assume $u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$. Then there exist $(w^\varepsilon, \widehat{b^\varepsilon}) \in L^\infty(Q)^2$ satisfying the following problem $(\mathcal{P}_\varepsilon)$*

i) $w^\varepsilon \in L^2(0, T; H^1_0(\Omega) \cap H^2(\Omega)), \quad \dfrac{\partial w^\varepsilon}{\partial t} \in L^2(Q)$

ii) $\dfrac{\partial}{\partial t}\beta_\varepsilon(w^\varepsilon + \gamma) - \Delta w^\varepsilon = aG(w^\varepsilon + \gamma) + p'(w^\varepsilon + \gamma)[b - \widehat{b^\varepsilon}]$

$\beta_\varepsilon(w^\varepsilon)|_{t=0} = \beta_\varepsilon(u_0 - \gamma)$ *(or equivalently $w^\varepsilon|_{t=0} = u_0 - \gamma$)*

iii) $\forall\phi : \mathbb{R} \to \mathbb{R}$ *continuous,* $\forall\theta \in \mathbb{R}$

$$\int\limits_{\{x:w^\varepsilon(t)(x)>\theta\}} \widehat{b^\varepsilon}\,\phi\,(w^\varepsilon)\,dx = \int\limits_{\{x:w^\varepsilon(t)(x)>\theta\}} b\phi\,(w^\varepsilon)\,dx$$

for a. e. $t \in]0, T[$ and $\operatorname*{essinf}_{\Omega} b \le \widehat{b^\varepsilon} \le \operatorname*{esssup}_{\Omega} b$.

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5.1 The Galerkin method: Existence of solution for a family of finite dimensional problems $(\mathcal{P}_{\varepsilon,m})$.

Let $(\lambda_k, \varphi_k)_{k\geq 1}$ be the eigenvalues and eigenfunctions associated to $-\Delta$ on Ω with boundary conditions, i.e.

$$-\Delta\varphi_k = \lambda_k\varphi_k, \quad \varphi_k \in H_0^1(\Omega).$$

We denote by V_m the vector space spanned by $\{\varphi_1, \cdots, \varphi_m\}$. For all $v \in V_m$, $v = \sum_{i=1}^m v^i\varphi_i$. We consider the following approximate problem: To find

$$w_m \in L^1(0,T;V_m), \ w_m(t) = \sum_{i=1}^m w_m^i(t)\varphi_i,$$

satisfying

$$\int_\Omega \left(\frac{\partial}{\partial t}\beta_\varepsilon(w_m(t) + \gamma)\right)\varphi_k dx + \int_\Omega \nabla w_m(t) \cdot \nabla\varphi_k dx =$$
$$= \int_\Omega a(x)G(w_m + \gamma)\varphi_k dx + \int_\Omega J(w_m + \gamma)\varphi_k dx, \quad k = 1, ..., m, \qquad ((\mathcal{P}_\varepsilon, m))$$

and the initial condition

$$w_m(0) = P_m(u_0 - \gamma).$$

where P_m is the orthogonal projection operator from $L^2(\Omega)$ onto V_m.

Theorem 5.3 *There exists w_m solution of problem $(\mathcal{P}_{\varepsilon,m})$. Furthermore, if $a \not\equiv 0$ then there exists k_0 such that $w_m \not\equiv 0$ for $m\geq k_0$.*

Proof The above problem can be written as a nonlinear differential system for the functions $w_m^1(t)$, ..., $w_m^m(t)$, $w_m(t) = \sum_{i=1}^m w_m^i(t)\varphi_i$. Indeed, $w_m^i(t)$ with $i = 1, ..., m$ verify

$$\sum_{i=1}^m a_{ik}(w_m(t))w_m^{i\prime}(t) + \sum_{i=1}^m b_{ik}w_m^i(t) = \widehat{\mathcal{I}}_k(w_m(t)), \quad k = 1, ..., m, \qquad (34)$$
$$w_m^i(0) = \text{the } i^{th} \text{ component of } P_m(u_0 - \gamma),$$

where, for $i, k = 1, ..., m$,

$$a_{ik}(w_m(t)) := \int_\Omega \beta_\varepsilon'(w_m + \gamma)\varphi_i\varphi_k dx,$$

$$b_{ik} := \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_k dx,$$

$$\widehat{\mathcal{I}}_k(w_m(t)) := \int_\Omega a(x)G(w_m(t) + \gamma)\varphi_k dx + \int_\Omega J(w_m(t) + \gamma)\varphi_k dx,$$

To prove the existence of a solution of the above initial value problem we shall need the following lemma:

Lemma 5.1 *The function* $\widehat{\mathcal{I}}_k : V_m \longrightarrow \mathbb{R}$ *is continuous,* $k = 1, ..., m$.

Proof Let $v \in V_m$. Then

$$\widehat{\mathcal{J}}_k(v) = \int_\Omega a(x)G(v+\gamma)\varphi_k dx + \int_\Omega J(v+\gamma)\varphi_k dx,$$

with G and J given by (1) and (2). Indeed, since from lemma (3.6) the map $v \in V_m\backslash\{0\} \longrightarrow b_{*v}(|v > v(\cdot)|) \in L^p(\Omega)$ is strongly continuous for any finite p and $|b_{*v}|_{L^\infty} \leq |b|_\infty$ (see [12]) we deduce that the map $v \in V_m \longrightarrow J(v+\gamma) \in L^p(\Omega)$ is strongly continuous.

Analogously, by lemmas (3.3) and (3.6) $v \in V_m\backslash\{0\} \longrightarrow b_{*v} \in L^p(\Omega_*)$ and $v \in V_m\backslash\{0\} \longrightarrow [p(v_* + \gamma)]' \in L^q(\Omega_*)$ are strongly continuous for any finite p and $q \in [1,2)$, and since $p'(\gamma) = 0$ $(\gamma \leq 0)$, we obtain that $v \in V_m \longrightarrow [p(v_* + \gamma)]' b_{*v} \in L^q(\Omega_*)$ is also continuous. We denote by $I(v,x)$ the interval: $I(v,x) = \left[\left| v + \gamma > (v+\gamma)_+ (x) \right|, |v+\gamma > 0| \right]$, $x \in \overline{\Omega}$. Let $(v_j)_{j\geq 1}$ be a sequence of V_m converging to v, if $v \neq 0$, the characteristic function $\chi_{I(v_j,x)}$ converges to $\chi_{I(v,x)}$ in $L^r(\Omega_*)$ for every r finite and every $x \in \overline{\Omega}$. So, using again that $p'(\gamma) = 0$ we deduce that for every $x \in \overline{\Omega}$:

$$\lim_{j \longrightarrow +\infty} \int_{\Omega_*} \chi_{I(v_j,x)}(\sigma) [p(v_{j*} + \gamma)]'(\sigma) b_{*v_j}(\sigma) d\sigma =$$

$$= \int_{\Omega_*} \chi_{I(v,x)}(\sigma) [p(v_* + \gamma)]'(\sigma) b_{*v}(\sigma) d\sigma.$$

Noting that $(v+\gamma)_* = v_* + \gamma$, $b_{*(v+\gamma)} = b_{*v}$, we find:

$$G(v_j + \gamma)(x) \overrightarrow{j\to\infty} G(v+\gamma)(x), \text{ a.e. } x \in \Omega.$$

Now, as $0 \leq G(v_j + \gamma) \leq F_v$, from the Lebesgue dominate convergence we get that:

$$v \in V_m \longrightarrow \int_\Omega a(x)G(v+\gamma)\varphi_k dx \text{ is continuous.} \quad \blacksquare$$

Proof of Theorem 5.3. Since $\{\varphi_1, ..., \varphi_m\}$ is free and β'_ε verifies $\beta'_\varepsilon \in C^1(\mathbb{R})$, $0 < \alpha \leq \beta'_\varepsilon < 2$, the matrix of coefficients $a_{ik}(w_m(t))$ is invertible. So, by Cauchy–Peano theorem, the nonlinear differential system (34) has a maximal solution defined on some interval $[0, T_m]$. The *a priori* estimates on w_m, we shall prove later, show that in fact $T_m = T$, $\forall m \geq 1$. Finally, if $a \not\equiv 0$, then $\int_\Omega a(x)\varphi_{k_0}(x)dx \neq 0$ for some k_0, so, if we assume that $w_m \equiv 0$ for some $m \geq k_0$, then we would arrive to $\int_\Omega a(x)\varphi_{k_0}(x)dx = 0$. $\quad \blacksquare$

Lemma 5.2 *If* w_m *is a solution of* $(\mathcal{P}_{\varepsilon,m})$ *then*

i) $\forall \phi : \mathbb{R} \to \mathbb{R}$ *Borelian with* $\phi(w_m(t)) \in L^1(\Omega)$ *we have*

$$\int_\Omega J(w_m(t) + \gamma)\phi(w_m(t))dx = 0$$

ii) w_m *remains in a bounded set of* $L^2(0, T; H_0^1(\Omega))$ *as* $m \to +\infty$ *and satisfies the following estimates, for all* $t \in [0, T]$

$$\int_0^t \int_\Omega |\nabla w_m(\sigma, x)|^2 dx d\sigma + 2 \int_\Omega dx \int_0^{w_m(0)} \beta_\varepsilon (\sigma + \gamma) d\sigma \leq$$

$$\leq 2 \int_\Omega w_m(0)\beta_\varepsilon(w_m(0) + \gamma)dx + \frac{|a|_\infty^2 F_v^2 |\Omega| t}{2\lambda_1}$$

Proof Since $w_m \not\equiv 0$, then for all t, $\mathrm{meas}\{x : |\nabla w_m(t, x)| = 0\} = 0$. Thus, using Lemma 3.8

$$\int_\Omega b_{*w_m} \left(|w_m(t) > w_m(t, x)|\right) \phi\left(w_m(t, x)\right) = \int_\Omega b\phi\left(w_m(t, x)\right) dx$$

which gives *i)*. Taking $w_m(t)$ as test function in the equation $(\mathcal{P}_{\varepsilon,m})$ and using the above property:

$$\int_\Omega \frac{\partial}{\partial t}\beta_\varepsilon \left(w_m(t) + \gamma\right) w_m(t)dx + \int_\Omega |\nabla w_m(t, x)|^2 dx = \int_\Omega aG(w_m + \gamma)w_m(t) .$$

Then, the proof follows exactly the same idea as for the proof of Corollary 4.1. ■

Lemma 5.3 *The sequence* $\frac{\partial w_m}{\partial t}$ *remains in a bounded set of* $L^2(Q)$ *as* $m \to \infty$, *and thus,* w_m *also remains in a bounded set of* $L^2(0, T; H^2(\Omega))$ *as* $m \to +\infty$. *Moreover, the sequence* $(w_m)_{m \geq 1}$ *remains in a bounded subset of* $H^1(0, T; L^2(\Omega))$ $\cap L^2(0, T; H^2(\Omega)) \cap \mathcal{C}([0, T]; \bar{H}_0^1(\Omega))$. *Furthermore,*

$$\int_0^T |w_m'(t)|_2^2 dt \leq \frac{1}{\alpha} |\nabla u_0|_2^2 + \frac{2}{\alpha^2} \left(\|a\|_\infty^2 F_v^2 |\Omega| T + \lambda^2 \left(\underset{\Omega}{\mathrm{osc}\,} b \right)^2 \|w_m\|_{L^2(Q)}^2 \right),$$

where $\underset{\Omega}{\mathrm{osc}\,} b$ *denotes the oscillation of* b *in* Ω.

Proof Multiplying $(\mathcal{P}_{\varepsilon,m})$ by $\frac{dw_m^j(t)}{dt}$ and adding these equations for $j = 1, ..., m$, we get

$$\int_\Omega \beta_\varepsilon'(w_m(t) + \gamma)) \left|w_m'(t)\right|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w_m(t)|^2 dx =$$

$$= \int_\Omega a(x)G(w_m + \gamma)w_m'(t)dx + \int_\Omega J(w_m + \gamma)w_m'dx.$$

Using the first assumption on β_ε and the estimates $0 \le G \le F_v$, $|J(w_m(t) + \gamma)| \le \lambda(w_m(t) + \gamma) \operatorname{osc}_\Omega b$ we obtain

$$\alpha \left|w_m'(t)\right|_2^2 + \frac{1}{2}\frac{d}{dt}\left|\nabla w_m(t)\right|_2^2 \le$$

$$\le |a|_\infty F_v \left|w_m'(t)\right|_1 + \lambda \operatorname{osc}_\Omega b \int_\Omega (w_m + \gamma)_+ w_m'(t)dx$$

$$\le |a|_\infty F_v |\Omega|^{1/2} \left|w_m'(t)\right|_2 + \lambda \operatorname{osc}_\Omega b \left|w_m(t)\right|_2 \left|w_m'(t)\right|_2,$$

where we used Holder's inequality and the fact that $\gamma < 0$. By applying Young's inequality, we get

$$\alpha \left|w_m'(t)\right|_2^2 + \frac{1}{2}\frac{d}{dt}\left|\nabla w_m(t)\right|_2^2$$

$$\le \delta C_0 \left|w_m'(t)\right|_2^2 + \frac{1}{4\delta}\left(|a|_\infty F_v |\Omega|^{1/2} + \lambda \operatorname*{osc}_\Omega b \left|w_m(t)\right|_2^2\right).$$

From the choice of δ and integrating in $]0, t[$, $t \le T$, we have:

$$\int_0^t \left|w_m'(s)\right|_2^2 ds + \frac{1}{\alpha}\left|\nabla w_m(t)\right|_2^2$$

$$\le \frac{1}{\alpha}\left|\nabla w_m(0)\right|_2^2 + \frac{1}{\alpha^2}C_0\left(|a|_\infty F_v |\Omega|_T^{1/2} + \lambda \operatorname{osc}_\Omega b \int_0^t \left|w_m(\sigma)\right|_2^2 d\sigma\right),$$

which leads to the boundedness in $H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega))$ by applying Lemma 5.2.

In order to show that w_m remains in a bounded set of $L^2(0, T; H^2(\Omega))$, we consider the orthogonal projection of $L^2(\Omega)$ onto V_m. The equation satisfied by w_m is equivalent to:

$$\begin{cases} P_m(\frac{\partial}{\partial t}\beta_\varepsilon(w_m(t) + \gamma))) - \Delta w_m = P_m(aG(w_m(t) + \gamma) + J(w_m(t) + \gamma), \\ w_m(t) \in V_m, \text{ for a.e. } t \in (0, T). \end{cases} \tag{35}$$

Lemma 5.2 and the estimate $0 \le G \le F_v$ ensure that $aG(w_m(t) + \gamma) + J(w_m(t) + \gamma)$ remains in a bounded set of $L^2(Q)$. Since $0 < \beta_\varepsilon' \le 2$, Lemma 5.3 implies that $\frac{\partial}{\partial t}\beta_\varepsilon(w_m(t) + \gamma))$ is bounded in $L^2(Q)$. From the equation in (35), we infer that Δw_m remains in a bounded set of $L^2(Q)$, and thus w_m remains in a bounded set of $L^2(0, T; H^2(\Omega))$. Finally, by using standard results (see, e.g., [20] Chap. I):

$$Y := H^1\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right) \hookrightarrow \mathcal{C}([0, T]; H_0^1(\Omega)),$$

we obtain the remaining boundedness.   ∎

5.2 Solvability of the approximated problem. Passing to the limit $m \to \infty$: Existence of solution for $(\mathcal{P}_\varepsilon)$.

By the above estimates, there exists a subsequence of $(w_m)_{m \geq 1}$, which we also denote by $(w_m)_{m \geq 1}$, and $w^\varepsilon \in Y$ such that

$$w_m \rightharpoonup w^\varepsilon \text{ weakly in } Y,$$

and so, by compactness results (see [19]) we get

$$w_m \to w^\varepsilon \text{ strongly in } L^2(0, T; W_0^{1,p}(\Omega)), \text{ with } p \in [2, +\infty). \tag{36}$$

Due to the uniform boundedness

$$\underset{\Omega}{\operatorname{essinf}} b \leq b_{*(w_m+\gamma)}(|w_m(t) > w_m(t;)|) \leq \underset{\Omega}{\operatorname{esssup}} b, \tag{37}$$

we get the existence of $\widehat{b}^\varepsilon \in L^\infty(Q)$ such that

$$b_{*(w_m+\gamma)}\left(|w_m(t) > w_m(t, \cdot)|\right) \rightharpoonup \widehat{b}^\varepsilon \text{ weakly–star in } L^\infty(Q).$$

Analogously, as $|G(w_m(t, x) + \gamma)| \leq F_v$ a. e. in Q, there exists $G_\infty^\varepsilon \in L^\infty(Q)$ such that

$$G(w_m + \gamma) \rightharpoonup G_\infty^\varepsilon \text{ in } L^\infty(Q) \text{ weakly star.}$$

Thus, w^ε is a solution of the following limit problem:

$$\begin{cases} \dfrac{\partial}{\partial t} \beta_\varepsilon(w^\varepsilon + \gamma) - \Delta w^\varepsilon = a G_\infty^\varepsilon + \lambda \, (w^\varepsilon + \gamma)_+ \left[b - \widehat{b}^\varepsilon \right], \\ w(0) = u_0 - \gamma, \\ w \in Y \cap L^2(0, T; H^2(\Omega)). \end{cases} \tag{38}$$

Applying Lemma 3.9, we derive the following conclusion

Lemma 5.4 $G_\infty^\varepsilon = G(w^\varepsilon + \gamma)$.

Proof Indeed from (36), we can deduce that there exists a subsequence of w_m, which we will denote also by w_m, such that $w_m(t) \to w^\varepsilon(t)$ in $W^{1,p}(\Omega)$ strong for $p \in [2, +\infty)$, a. e. t, and $w^\varepsilon(t) \in H^2(\Omega)$; thus, we may appeal to Lemma 3.9 to conclude.

Lemma 5.5 *For any $\phi \in C(\mathbb{R})$ and $\forall \theta \in \mathbb{R}$*

$$\int\limits_{\{x:w^\varepsilon(t,x)>\theta\}} \widehat{b}^\varepsilon \phi\,(w^\varepsilon)\, dx = \int\limits_{\{x:w^\varepsilon(t,x)>\theta\}} b \phi\,(w^\varepsilon)\, dx \quad \text{for a.e. } t \in]0, T[,$$

and

$$\underset{\Omega}{\operatorname{essinf}} b \leq \widehat{b}^\varepsilon(t, x) \leq \underset{\Omega}{\operatorname{esssup}} b \text{ a.e. on } Q.$$

Proof For fixed t, it suffices to prove the equality for θ such that $|w^\varepsilon(t) = \theta| = 0$. Let $b^\varepsilon_m(t,x) = b_{*w_m}\left(|w_m(t) > w_m(t,x)|\right)$. By Lemma 3.8, we know that

$$\int\limits_{\{x:w_m(t,x)>\theta\}} b^\varepsilon_m(t,x)\,\phi\left(w_m(t,x)\right)dx = \int\limits_{\{x:w_m(t,x)>\theta\}} b\phi\left(w_m(t,x)\right)dx \tag{39}$$

Thus, since $\phi(w_m(t)) \to \phi(w^\varepsilon(t)) \in L^2(\Omega)$ and $\lim \chi_{\{x:w_m(t)>\theta\}}(x) = \chi_{\{x:w^\varepsilon(t)>\theta\}}(x)$ for a. e. $x \in \Omega$ a. e. $t \in]0,T[$, we deduce the result from (39). ∎

End of the proof of Theorem 5.2. It suffices to collect Lemma 5.3, Lemma 5.4 and eq. (38). ∎

5.3 The full solvability of the model when $\alpha > 0$.

Collecting the uniform estimates with respect to ε of the Theorem 4.1, Theorem 4.2 and Theorem 4.3 one has:

Lemma 5.6 *There exist a constant $c > 0$, independent of ε and α, such that*

i) $\displaystyle\int_0^T \int_\Omega |\nabla w^\varepsilon(\sigma,x)|^2 d\sigma dx \le c,$

ii) $0 \le G(w^\varepsilon + \gamma) \le F_v,$

iii) $\displaystyle\operatorname*{essinf}_\Omega b \le \widehat{b}^\varepsilon(t,x)| \le \operatorname*{esssup}_\Omega b,$

iv) $|\beta_\varepsilon(w^\varepsilon + \gamma)|_{L^\infty(Q)} \le c, \quad \left|\dfrac{\partial}{\partial t}\beta_\varepsilon(w^\varepsilon + \gamma)\right|_{L^2(Q))} \le c\ .$

From the Lemma above, we may assume that there exists w_α, G_α, $\widehat{b}_\alpha$ such that $w^\varepsilon \rightharpoonup w_\alpha$ weakly in $L^2(0,T;H^1_0(\Omega))$, $G^\varepsilon \rightharpoonup G_\alpha$ weakly star in $L^\infty(Q)$, and $\widehat{b}^\varepsilon \rightharpoonup \widehat{b}_\alpha$ weakly star in $L^\infty(Q)$ as $\varepsilon \to 0$. Then we have the following lemma (on the strong convergence in which the estimates depend on $\frac{1}{\alpha}$).

Lemma 5.7 *There exists a constant $c_\alpha > 0$ such that $|\frac{\partial w^\varepsilon}{\partial t}|_{L^2(Q)} \le c_\alpha$. In particular, w^ε remains in a bounded set of $L^2(0,T;H^2(\Omega))$ as ε go to zero and $w_\alpha \in L^2(0,T;H^2(\Omega))$, $w^\varepsilon \to w_\alpha$ in $L^2(0,T;H^1_0(\Omega))$.*

Proof Lemma 5.3 provides the estimate for $\dfrac{\partial w^\varepsilon}{\partial t}$ in $L^2(Q)$. From the equation satisfied by w^ε, we deduce that Δw^ε remains in a bounded set of $L^2(Q)$. Thus, w^ε remains in a bounded set of $L^2(0,T;H^2(\Omega))$ and the weak convergence in $L^2(0,T;H^1_0(\Omega))$ implies the weak convergence in $L^2(0,T;H^2(\Omega))$ to the same $w_\alpha \in L^2(0,T;H^2(\Omega))$. Therefore, the strong convergence of w^ε to w_α in $L^2(0,T;H^1_0(\Omega))$ is a direct consequence of standard compactness results (see [35] or [34]). ∎

Corollary 5.1 $G_\alpha(t,x) = G(w_\alpha + \gamma)(t,x)$ *for a. e.* $(t,x) \in Q$.

Proof By the above lemma and a new application of the compactness results (see [35] or [34]) it follows that for a. e. $t \in]0,T[$, $w_\alpha(t) \in H^2(\Omega)$ and $w^\varepsilon(t)$ converges to $w_\alpha(t)$ in $W^{1,p}(\Omega)$ for $2 \leq p < +\infty$. So, we may appeal Lemma 3.9 to deduce the result. ∎

Proof of Theorem 5.1 The above estimates allow us to pass to the limit in (38), and so, by using the results collected in this section, we get that w_α is a second category solution for $(\mathcal{P}_\alpha)$. The regularity is also a consequence of the above estimates. ∎

Furthermore, we have proved that there exists a positive constant c, independent of α such that:

Corollary 5.2 *The following estimates holds*

i) $\displaystyle\int_Q |\nabla w_\alpha(t,x)|^2 dx dt \leq c,$

ii) $|w_{\alpha_-}|_{L^2(0,T;H_0^1(\Omega))} \leq c,$

iii) $|w_\alpha|_{L^\infty(Q)} \leq c.$

5.4 The case $\alpha = 0$.

In this last section we full solve problem $(\mathcal{P})$, that is the case $\alpha = 0$. For the sake of simplicity in the notation, we introduce the notation

$$-f(w_{\alpha+}) := aG(w_\alpha + \gamma) + (w_\alpha + \gamma)_+[b - \widehat{b}_\alpha].$$

We have proved that this function remains in a bounded set of $L^\infty(Q)$ as α goes to zero. We denote by $M \geq \operatorname*{esssup}_Q |f(w_{\alpha+}(t,x))|$. Then the equation satisfied by w_α can be written as:

$$(\mathcal{P}_\alpha) \begin{cases} \dfrac{\partial}{\partial t}(w_\alpha + \gamma)_- - \alpha\dfrac{\partial}{\partial t}(w_\alpha + \gamma)_+ + \Delta w_\alpha = f(w_{\alpha+}) \\ w_\alpha = 0 \qquad \text{on } \Sigma_+ =]0,T[\times\partial\Omega \\ w_\alpha(0,x) = w_0 \end{cases}$$

where
$$w_\alpha \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)), \quad \frac{\partial w_\alpha}{\partial t} \in L^2(Q).$$

Lemma 5.8 *The sequence* $(w_\alpha + \gamma)_-$ *remains in a bounded set of* $L^\infty(0,T;H_0^1(\Omega) \cap H^1(0,T;L^2(\Omega))$ *as* α *goes to zero. Furthermore,*

$$\int_0^t \left|\frac{\partial}{\partial t}(w_\alpha(\sigma) + \gamma)_-\right|^2_{L^2(\Omega)} d\sigma + \int_\Omega \left|\nabla(w_\alpha(t) + \gamma)_-\right|^2 dx$$
$$\leq \left|\nabla(w_0 + \gamma)_-\right|^2_{L^2(\Omega)} + T|\Omega|M^2.$$

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Proof The following *integration by parts* can be justified by using a smooth approximation (which is possible thanks to the above regularity):

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\nabla\left(w_\alpha(t)+\gamma\right)_-\right|^2 dx = \int_\Omega \frac{\partial}{\partial t}\left(w_\alpha(t)+\gamma\right)_- \Delta w_\alpha(t)dx.$$

Multiplying the equation that w_α satisfies by $\dfrac{\partial}{\partial t}(w_\alpha+\gamma)_-$ we find:

$$\left|\frac{\partial}{\partial t}\left(w_\alpha(t)+\gamma\right)_-\right|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\int_\Omega \left|\nabla\left(w_\alpha(t)+\gamma\right)_-\right|^2 dx \le M\left|\frac{\partial}{\partial t}\left(w_\alpha(t)+\gamma\right)_-\right|_{L^1(\Omega)}$$

From which we deduce after integration:

$$\int_0^t \left|\frac{\partial}{\partial t}\left(w_\alpha(\sigma)+\gamma\right)_-\right|^2_{L^2(\Omega)} d\sigma + \int_\Omega \left|\nabla\left(w_\alpha(t)+\gamma\right)_-\right|^2 dx$$
$$\le \left|\nabla\left(w_0+\gamma\right)_-\right|^2_{L^2(\Omega)} + T|\Omega|M^2 .$$

The uniform boundedness of $(w_\alpha+\gamma)_-$ follows from this last inequality. ∎

From the uniform boundedness of w_α (see Theorem 5.1), it follows the existence of $w\in L^2(0,T;H^1_0(\Omega))\cap L^\infty(Q)$ such that

$$w_\alpha \rightharpoonup w \text{ weakly in } L^2(0,T;H^1_0(\Omega)),$$
$$w_\alpha \rightharpoonup w \text{ weakly star in } L^\infty(Q).$$

Using a standard compactness result, we may assume that there exists $z\in L^2(Q)$ such that $(w_\alpha+\gamma)_-\to z$ in $L^2(Q)$ as α goes to zero. The following lemma allows as to identify z :

Lemma 5.9 *The following identity is verified:*

$$z = (w+\gamma)_- = -\beta\left(w+\gamma\right).$$

Proof The $\mathbb{R}^2$-graph β generates a maximal monotone operator A on $L^2\left(0,T;L^2\left(\Omega\right)\right)$ (see [6], Chap. II), defined as

$$Av = -(v+\gamma)_- \qquad \forall v\in L^2(0,T;L^2\left(\Omega\right)).$$

From the weak convergence of $\left(w_\alpha+\gamma\right)_{\alpha>0}$ in $L^2\left(0,T;H^1_0\left(\Omega\right)\right)$ and the strong convergence of $\left(\beta\left(w_\alpha+\gamma\right)\right)_{\alpha>0}$ in $L^2\left(0,T;L^2\left(\Omega\right)\right)$, i.e.,

$$\left(w_\alpha+\gamma\right)\rightharpoonup w+\gamma \text{ weakly in } L^2(Q),$$

$$\beta\left(w_\alpha+\gamma\right)\to -z \text{ strongly in } L^2(Q).$$

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Thus, by using the theory of maximal monotone operators (see [6]) we arrive to

$$((w_\alpha + \gamma), -z) \in A$$

and so $-z = \beta (w_\alpha + \gamma)$ which ends the proof of the lemma. ∎

Finally, if we assume that $f(w_{\alpha +})$ converge weakly star to a function h in $L^\infty(Q)$, the, w satisfies the following limit problem $(\mathcal{P}_0)$

$$\begin{cases} \dfrac{\partial}{\partial t}(w + \gamma)_- + \Delta w = h, \\ w \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q), \\ (w + \gamma)_- = u_{0-} . \end{cases}$$

Furthermore, from Lemma 5.8 and Agmon–Douglis–Nirenberg, we deduce from the equation above that

$$w \in L^2(0, T; H^2(\Omega)) . \tag{40}$$

It remains to identify the function h. For this purpose we need the strong convergence of $w_\alpha(t)$ in $W^{1,2+\delta}(\Omega)$ with $\delta > 0$ and a.e. $t \in (0, T)$.(see Lemma 3.9) We shall obtain this convergence, whenever $(w_\alpha(t))_{\alpha > 0}$ converges weakly in $L^2(\Omega)$ to $w(t)$ a.e. t, by using a recent compactness result due to Rakotoson & Temam [26]. Previously, we shall prove the following proposition

Proposition 5.1 *Let us set, for every $\phi \in L^2(\Omega)$ and $h > 0$,*

$$\phi_{h\alpha}(t) = \frac{1}{h} \int_t^{t+h} \int_\Omega w_\alpha(\sigma, x) \phi(x) \, dx d\sigma.$$

Then, $w_\alpha(t) \rightharpoonup w(t)$ in $L^2(\Omega)$ for a. e. $t \in (0, T)$ if and only if

$$\lim_{h \to 0} \lim_{\alpha \to 0} \phi_{h\alpha}(t) = \lim_{\alpha \to 0} \lim_{h \to 0} \phi_{h\alpha}(t) \text{ for a.e. } t \in (0, T) \text{ and } \forall \phi \in L^2(\Omega). \tag{41}$$

Proof: Assume (41) then, from the boundedness of $(w_\alpha)_{\alpha > 0}$ given in Theorem 5.1 it follows the existence of a subsequence, that we still denote by $(w_\alpha)_{\alpha > 0}$, such that

$$w_\alpha \rightharpoonup w \text{ in } L^2(0, T; L^2(\Omega)) \text{-weak.}$$

In particular,

$$\int_0^T \int_\Omega w_\alpha(t, x) \psi(t) \phi(x) \, dx dt \to \int_0^T \int_\Omega w(t, x) \psi(t) \phi(x) \, dx dt$$

$\forall \psi \in L^2(0, T)$ and $\forall \phi \in L^2(\Omega)$, when $\alpha \to 0$. Let us fix $t_0 \in (0, T)$, $h > 0$ small enough and we define the sequence $(\psi_h)_{h > 0} \subset L^2(0, T)$ as

$$\psi_h(t) = \frac{1}{h} \chi_{[t_0, t_0 + h]}(t),$$

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where $\chi_{[t_0, t_0+h]}$ denotes the characteristic function of $[t_0, t_0 + h]$. Then,

$$\int_0^T \int_\Omega w_\alpha(t, x) \psi_h(t) \phi(x) \, dx dt \to \int_0^T \int_\Omega w(t, x) \psi_h(t) \phi(x) \, dx dt$$

when $\alpha \to 0$ and by (41) this convergence is uniformly in h. Then, passing to the limit $h \to 0$ in the above expression and taking into account that for any integrable function the complementary of its Lebesgue points is a set of zero measure (see, e.g., [6] pp. 140) we get

$$\int_\Omega w_\alpha(t_0, x) \phi(x) \, dx \to \int_\Omega w(t_0, x) \phi(x) \, dx \text{ when } \alpha \to 0, \text{ a.e. } t_0 \in (0, T),$$

and thus $w_\alpha(t) \rightharpoonup w(t)$ in $L^2(\Omega)$-weak a.e. $t \in (0, T)$.

Assume now that the $L^2(\Omega)$-weak convergence of $(w_\alpha(t))_{\alpha > 0}$ to $w(t)$ holds for a.e. $t \in (0, T)$. We always have that

$$\lim_{h \to 0} \lim_{\alpha \to 0} \phi_{h\alpha}(t) = \lim_{h \to 0} \left[\int_0^T \int_\Omega w(t, x) \psi_h(t) \phi(x) \, dx dt \right] =$$
$$= \int_\Omega w(t, x) \phi(x) \, dx \text{ a.e. } t \in (0, T). \tag{42}$$

Moreover, by the Lebesgue theorem,

$$\lim_{\alpha \to 0} \lim_{h \to 0} \int_0^T \int_\Omega w_\alpha(t, x) \psi_h(t) \phi(x) \, dx dt = \lim_{\alpha \to 0} \int_\Omega w_\alpha(t, x) \phi(x) \, dx =$$
$$= \int_\Omega w(t, x) \phi(x) \, dx \text{ a.e. } t \in (0, T), \tag{43}$$

and so (41) holds. ∎

We arrive then to our main result:

Theorem 5.4 *Let $(w_\alpha)_{\alpha > 0}$ be the sequence given in Theorem 5.1 and assume that (41) holds. Then $w_\alpha \to w$ in $L^2(0, T; H_0^1(\Omega))$-strong and $u = w + \gamma$ is a second category weak solution to problem $(\mathcal{P})$. Furthermore,*

$$u \in L^2\left(0, T; \mathcal{C}^1(\Omega)\right) \text{ and } \beta(u) \in \mathcal{C}\left([0, T]; L^2(\Omega)\right).$$

Proof From the hypothesis (41) it follows that $w_\alpha(t) \rightharpoonup w(t)$ in $L^2(\Omega)$ for a. e. $t \in (0, T)$. We can then appeal to the compactness result of Rakotoson and Temam [26] and we deduce that $w_\alpha \to w$ in $L^2(Q)$ and for a. e. $t \in]0, T[$. Then, from Lemma 5.8 and the boundedness of w_α in $L^\infty(Q)$ one has:

$$\lim_{\alpha \searrow 0} \int_Q w_\alpha \frac{\partial}{\partial t}(w_\alpha + \gamma)_- dx dt = \int_Q w \frac{\partial}{\partial t}(w + \gamma)_- dx dt \tag{44}$$

and

$$\lim_{\alpha \searrow 0} \alpha \int_Q w_\alpha \frac{\partial}{\partial t}(w_\alpha + \gamma)_+ dxdt = 0$$

Since $f(w_{\alpha_+})$ converges weakly to h in $L^2(Q)$

$$\lim_{\alpha \searrow 0} \int_Q w_\alpha f(w_{\alpha t}) = \int_Q wh.$$

Multiplying by w_α the first equation of $(\mathcal{P}_\alpha)$ one deduces from $(\mathcal{P}_0)$:

$$\lim_{\alpha \searrow 0} \int_Q |\nabla w_\alpha|^2 dxdt = \int_Q w \frac{\partial}{\partial t}(w + \gamma)_- dxdt - \int_Q hwdxdt = \int_Q |\nabla w|^2 dxdt \quad (45)$$

Thus, the weak convergence of w_α to w in $L^2(0,T;H_0^1(\Omega))$ and (45) implies that

$$w_\alpha(t) \underset{\alpha \searrow 0}{\longrightarrow} w(t) \text{ in } H_0^1(\Omega) \text{ for a.e. } t \in]0,T[.$$

In fact, as w_α remains in a bounded set of $L^2(0,T;H^2(\Omega))$, from the above convergence we deduce, by Gagliardo–Nirenberg interpolation, that

$$w_\alpha(t) \underset{\alpha \searrow 0}{\longrightarrow} w(t) \text{ in } W^{1,p}(\Omega) \text{ for a.e. } t \in]0,T[, \text{ and } 2 \le p \le 4.$$

Since $w(t) \in H^2(\Omega)$, we may appeal Lemma 3.9 to derive that:

$$\lim_{\alpha \searrow 0} G(w_\alpha(t) + \gamma)(x) = G(w(t) + \gamma)(x)$$

for a. e. $(t,x) \in Q$. From the boundedness of $\widehat{b}_\alpha$, there exists $\widehat{b}_0$ such that $\widehat{b}_\alpha \rightharpoonup \widehat{b}_0$ weakly star in $L^\infty(Q)$, $\underset{\Omega}{\text{essinf}} b \le \widehat{b}_0 \le \underset{\Omega}{\text{esssup}} b$ and

$$\int_{\{x:w(t)(x)>\theta\}} b\phi(w(t)) dx = \lim_{\alpha \searrow 0} \int_{\{x:w_\alpha(t)(x)>\theta\}} \widehat{b}_\alpha \phi(w_\alpha(t)) dx = \int_{\{x:w(t)(x)>\theta\}} \widehat{b}_0 \phi(w(t)) dx$$

for a. e. $t \in]0,T[$, for any $\phi \in C(\mathbb{R})$ and for all $\theta \in \mathbb{R}$. Then, passing to the limit in $\mathcal{D}'(Q)$

$$\lim_\alpha f(w_{\alpha+}) = -aG(w + \gamma) - (w + \gamma)_+[b - \widehat{b}_0]$$

which implies that: $h = -aG(w + \gamma) - (w + \gamma)_+[b - \widehat{b}_0]$. Using the equation in problem $(\mathcal{P}_0)$ we deduce that $w + \gamma$ is a second category weak solution of $(\mathcal{P})$.

By arguing as in Lemma 5.3, we deduce that $\beta(u) \in \mathcal{C}([0,T];L^2(\Omega))$ and $u \in L^2(0,T;H^2(\Omega))$, we can then use Sobolev embedding to conclude that $u \in L^2(0,T;\mathcal{C}^1(\Omega))$. ∎

Remark 5.1 *We conjecture that condition (41) holds under stronger regularity on the initial datum (and so on the approximating solutions w_α).*

Finally, we give two results indicating when the second category weak solution we have found is a solution of the first category to problem $(\mathcal{P})$.

Theorem 5.5 *Let u be the second category weak solution of $(\mathcal{P})$ given in the above theorem, and assume that $a(x)$ does not have any flat regions. Then, if*

$$2\lambda \|b\|_\infty < \left(\frac{1-\nu}{\nu}\right) \inf_\Omega a^2, \tag{46}$$

where $\nu := \dfrac{2\lambda \|b\|_\infty S |\Omega|}{F_v^2} < 1$ and $S := |u_+|_\infty$, it follows that $u(t)$ has not flat regions for a. e. $t \in]0, T[$.

Proof Suppose that there exists $I \subset]0, T$, $|I| > 0$, and a constant $c \in \mathbb{R}$ such that

$$meas\,\{x \in \Omega : u(t, x) = c\} > 0, \text{ for a.e. } t \in I.$$

Then, if $c \leq 0$, by Stampacchia's theorem we can deduce that $\Delta u(t) = 0$ a. e. on $S_c := \{x \in \Omega : u(t, x) = c\}$.

Let us denote by $v := u_- \in H^1(0, T; L^2(\Omega))$, then by Lemma 3.7, it follows that $\dfrac{\partial u_-(t)}{\partial t} = K_c$ constant a. e. on S_c. Thus, necessarily

$$K_c = a(x) F_v \text{ a.e. } x \in S_c, \text{ a.e. } t \in I,$$

and so a has a positively measured flat region ($\{x : a = 0\}$ if $K_c = 0$, and $\left\{x : a = \frac{F_v}{K_c}\right\}$ otherwise), which measure is at least $meas(S_c)$, which contradicts the assumption on a. Let us assume now that $c > 0$. In this case, necessarily

$$0 = aG(u) + \lambda u_+ \left(b - \widehat{b}_0\right) \text{ a.e. on } S_c, \text{ and a.e. } t \in I.$$

But then, using the estimates we have on $G(u)$, u_+ and $\widehat{b}_0$, we arrive to

$$(2\lambda \|b\|_\infty \|u_+\|_\infty)^2 \geq \inf_\Omega a^2 \left[F_v^2 - 2\lambda \|b\|_\infty \|u_+\|_\infty^2\right],$$

which is a contradiction with (46). ∎

It remains to state some conditions in order to identify $\widehat{b}_0$ as b_{*u}, and so to obtain a first category solution:

Corollary 5.3 *Let us assume that there exists a Borel map $g^u : \mathbb{R} \longrightarrow \mathbb{R}$ such that $g^u \circ u = \widehat{b}_0$. Then, under the hypothesis of Theorem 5.5, u is a first category solution of $(\mathcal{P})$.*

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Proof It is a direct consequence of Theorem 5.5 and Lemma 4.2. ■

Proposition 5.2 *If* $mes\left\{x \in \Omega : |\nabla w^\varepsilon (t, x)| = 0\right\} = meas\{x \in \Omega : |\nabla w_\alpha (t, x)| = 0\} = meas\left\{x \in \Omega : |\nabla w (t, x)| = 0\right\} = 0$ *for a. e. t, then* w^ε, w_α *and w are solutions of the first category of their respective problems* $(\mathcal{P}_\varepsilon)$, $(\mathcal{P}_\alpha)$ *and* $(\mathcal{P})$.

Proof Following Lemma 3.5, as the sequence w_m given in Theorem 5.3 satisfies that $mes\left\{x \in \Omega : |\nabla w_m (t, x)| = 0\right\} = 0$ for a. e. $t \in]0, T[$, then the condition $mes\left\{x \in \Omega : |\nabla w^\varepsilon (t, x)| = 0\right\} = 0$ implies that

$$b_{*w_m(t)}\left(|w_m(t) > w_m(t, \cdot)|\right) \to b_{*w^\varepsilon(t)}\left(|w^\varepsilon(t) > w^\varepsilon(t, \cdot)|\right) \text{ in } L^p(\Omega),$$

as $m \to +\infty$, for $1 < p < +\infty$. Thus, we obtain that $\widehat{b}_\varepsilon = b_{*w^\varepsilon(t)}\left(|w^\varepsilon(t) > w^\varepsilon(t, x)|\right)$. Repeating the same argument for w_α, we get $\widehat{b}_\alpha = b_{*w_\alpha(t)}\left(|w_\alpha(t) > w_\alpha(t, x)|\right)$ and $\widehat{b}_0 = b_{*w(t)}\left(|w(t) > w(t, x)|\right)$, and so the conclusion is reached. ■

6   Acknowledgment

References

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