Associativeness versus recursiveness

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***Abstract*—** **Fuzzy connectives use to be assumed associative. In this way, key operational difficulties are solved by means of a single binary operator. In this paper we point out that the main property in order to assure operativeness should be *recursiveness*, which is weaker than associativity. If calculus can be obtained by means of a recursive application of a sequence of binary connectives, we still can develop operative models. It is then clearly seen that a *fuzzy rule* should be always understood as a family of fuzzy connectives. Associativity will appear when a fuzzy rule can be characterized by a single binary connective. Associativity assumption is therefore excluding from our model key rules in practice.**

Key words: Associativity, Fuzzy Connectives.

I. INTRODUCTION.

The notion of *aggregation* plays a key role in many real problems. We are continously ellaborating global opinions from chunks of partial information, by means of amalgamation processes.

For simplicity, mathematical models use to assume that each piece of input information and the final output itself are all of them of the same nature. Assuming a unique binary aggregation function being associative, we can get the final aggregated value by successively applying such a binary operator, no matter the number of items under consideration. If in addition this binary operator is commutative, then such a successive calculus does not depend upon any particular order of the items.

Obviously, such a mathematical approach to aggregation problems requiring associativity and commutativity is an oversimplification of reality. A particular problem does not need a general theory in order to be solved. If a particular decision-making problem is always described by means of only three criteria, we do not need to impose that our model is able to deal with four-criteria problems.

But since we frequently do not know in advance the dimension of the input information, we still need a model capable of dealing with arbitrary dimensions. An aggregation rule is therefore a family of aggregation operators consistently solving any aggregation problem, no matter the number of items introduced as input. Only such a family of aggregation rules can be said to be an *aggregation rule*, and it may require a previous re-arrangement of data (that is, an ordering rule on the items).

Anyway, such an *aggregation rule* should also be *operative*. In this paper we point out that such an operativeness can be assured by imposing a weaker condition than associativity: the existence of a *recursive* definition, by means of a sequence of binary operators.

II. CONNECTIVE RULES.

Standard fuzzy connective operators for conjunction and disjunction (t-norms and t-conorms; see, e.g., [7]) are assumed associative and commutative. The whole connective rule is fully characterized by means of a unique binary connective aggregation, a binary connective aggregating one item to another item (see [5] on the general concept of aggregation operator, as considered in this paper).

But we can find in the fuzzy literature some important connectives not being associative. OWA (*Ordered Weighted Averaging*) operators, for example. They were introduced by Yager [12] in order to fill the gap between *min* (which is the maximal t-norm) and *max* (which is the minimal t-conorm), on the basis of the natural (decreasing) ordering.

DEFINITION 1 *An OWA operator of dimension n is a connective operator*

$$\phi : [0,1]^n \to [0,1]$$

such that for any list $(a_1, \ldots, a_n)$ then

$$\phi(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i a_{[i]}.$$

for some associated list of weights $W = (w_1, \ldots, w_n)$ such that

1. *$w_i \geq 0$ for all $i \in \{1, \ldots, n\}$*

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2. $\sum_{i=1}^{n} w_i = 1$

OWA operators assume that data input are previously re-arranged according to the decreasing natural ordering on the real line. But each Yager's OWA operator does not really characterize any rule. OWA operators are not associative.

For example, the mean operator of n numbers can be introduced as the OWA operator with equal weights ($w_i = 1/n$ for all $i = 1, \ldots, n$). It is defined as a mapping

$$M_n : [0,1]^n \rightarrow [0,1]$$

such that

$$M_n(a_1, \ldots, a_n) = \frac{\sum_{i=1}^{n} a_i}{n}.$$

Such a mean M_n is just the mean of n numbers (it has been defined for a fixed n). It is just an operator, not a *rule*. When we refer to the *mean rule*, we mean the family of operators

$$\{M_n\}_{n=2}^{\infty}.$$

Such a *mean rule* tells us how to evaluate the mean of any arbitrary finite set of numbers. The *mean rule* is not a single mapping, but the above complete sequence of mappings.

Obviously, M_n is not associative. But the *mean rule* can be applied to any finite family of real numbers without taking into account the particular ordering in which they have been arranged. In particular, any mean M_n can be recursively obtained from the previous means $M_2, \ldots, M_{n-1}$, just taking into account that

$$M_n(a_1, \ldots, a_n) = \frac{(n-1)M_{n-1}(a_1, \ldots, a_{n-1}) + a_n}{n}$$

where $M_2(a, b) = (a + b)/2$. Alternatively,

$$M_n(a_1, \ldots, a_n) = \frac{a_1 + (n-1)M_{n-1}(a_2, \ldots, a_n)}{n}$$

(see [10,11], where this property is exploided within the context of group decision making).

It is not so simple to define what an *OWA rule* should be. Although several interesting families of OWA operators have been introduced in the past (see, e.g., [13,14,15]), these families are just showing the great flexibility in the choice of types of OWA operators. They can not be properly understood as *OWA rules*.

In a previous paper [1], the authors developed the concept of *OWA rule* on the basis of the existence of a consistent representation in terms of families of binary OWA operators (see also [2,3,4]). Such a representation enables OWA rules being operative, and capable of solving any problem of arbitrary dimension. Now we propose to extend this approach in order to get the general notion of *connective rule*.

III. RECURSIVE CONNECTIVE RULES.

A connective rule should allow an aggregated value for any possible dimension of the list of items to be aggregated. That is, a connective rule should be a sequence of connective operators

$$\{\phi_n : [0,1]^n \rightarrow [0,1]\}_{n>1}$$

allowing the aggregation of any finite number of items. Obviously, in order to be considered as a *rule*, some *consistency* assumption has to be imposed on the family of connectives. Not every family of connectives defines a connective rule. Not every family of connective operators can be considered as *consistent*.

We shall focus our attention here on those connective rules allowing the aggregation of arbitrary lists in a recursive manner. In particular, we shall consider those families of connective operators that can be defined by means of a left or a right recursive application of binary operators, once an appropriate re-arrangement of the items to be aggregated has been previously realized. In this way, a connective rule should be understood as a family of connective operators, allowing such a recursive evaluation. Since not every family of connective operators will allow its recursive definition, such a property of *recursiveness* plays a first consistency role.

First of all we distinguish between left and right recursiveness, but we shall check later that either both exist or none of them does.

DEFINITION 2 *A left recursive connective rule is a family of connective operators*

$$\{\phi_n : [0,1]^n \rightarrow [0,1]\}_{n>1}$$

such that there exists a sequence of binary operators

$$\{L_n : [0,1]^2 \rightarrow [0,1]\}_{n>1}$$

such that

$$\phi_n(a_1, \ldots, a_n) = L_n(\ldots L_2(\pi_1(a_1), \pi_2(a_2)) \ldots, \pi_n(a_n))$$

for some sequence of permutations

$$\pi_n : \{a_1, a_2, \ldots, a_n\} \rightarrow \{a_1, a_2, \ldots, a_n\}$$

such that

$$\pi_n(a) < \pi_n(b) \Rightarrow \pi_{n+1}(a) < \pi_{n+1}(b).$$

Obviously, $\phi_2 = L_2$.

The above condition on the sequence of permutations is needed for consistency, in order to assure that the relative position of values is kept all throughout the process. In this way we are assuming the existence of an unique underlying linear *ordering rule* in the real unit interval, to be applied previously to each aggregation aggregator. Such an ordering rule tells us the exact position each new element has to be placed in any previously given ordered set of numbers.

Of course, it may be the case that such an ordering rule is keeping positions as presented, that is,

$$\pi(a_i) < \pi(a_j) \Leftrightarrow i < j.$$

We talk then about the *identity* ordering rule.

The natural decreasing order in another interesting choice. It is the underlying ordering rule for OWA operators, frequently named *sorting permutation*. Given a list of n numbers $(a_1, \ldots, a_n)$, its sorting permutation is

$$\sigma(a_1, \ldots, a_n) = (a_{[1]}, \ldots, a_{[n]})$$

where $a_{[i]} \in \{a_1, \ldots, a_n\}$ for all i and $a_{[i]} \geq a_{[j]}$ for all $i \leq j$.

From now on, we shall denote the underlying ordering rule just by π.

Right recursiveness can be analogously defined to left recursiveness.

DEFINITION 3 *A sequence of connective operators*

$$\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$$

is said a right recursive connective rule whenever

$$\phi_n(a_1, \ldots, a_n) = R_n(\pi(a_1), R_{n-1}(\pi(a_2), \ldots, R_2(\pi(a_{n-1}), \pi(a_n)) \ldots))$$

holds for some family of binary operators

$$\{R_n : [0,1]^2 \to [0,1]\}_{n>1}$$

and some ordering rule π.

Obviously, an operator $\phi_n : [0,1]^n \to [0,1]$ allows a right recursive definition if and only if it allows its left recursive definition: if for example we have

$$\phi_n(a_1, \ldots, a_n) = R_n(\pi(a_1), R_{n-1}(\pi(a_2), \ldots, R_2(\pi(a_{n-1}), \pi(a_n)) \ldots))$$

then we can define $\hat{\pi}$ such that $\hat{\pi}(i) < \hat{\pi}(j)$ if and only if $\pi(i) > \pi(j)$, and $L_k(a,b) = R_k(b,a)$ in such a way that

$$\phi_n(a_1, \ldots, a_n) = L_n(L_{n-1}(\ldots L_2(\hat{\pi}(a_1), \hat{\pi}(a_2)) \ldots, \hat{\pi}(a_{n-1})), \hat{\pi}(a_n)).$$

Such an ordering rule $\hat{\pi}$ is to be known as the *dual* ordering rule of π.

But the existence of a right (left) recursion representation of a given operator does not imply in general the existence of an equivalent left (right) recursion representation by means of the same underlying ordering rule π.

Example 1 *Let us consider the following two OWA operators:*

$$\phi_2(a_1, a_2) = \frac{1}{2}a_{[1]} + \frac{1}{2}a_{[2]}$$

and

$$\phi_3(a_1, a_2, a_3) = \frac{1}{4}a_{[1]} + \frac{1}{4}a_{[2]} + \frac{1}{2}a_{[3]}$$

They are consistent with left recursivenes, since

$$\phi_3(a_1, a_2, a_3) = \frac{1}{2}\left(\frac{1}{2}a_{[1]} + \frac{1}{2}a_{[2]}\right) + \frac{1}{2}a_{[3]}$$

But there is no function $h : [0,1]^2 \to [0,1]$ such that

$$\phi_3(a_1, a_2, a_3) = h(a_{[1]}, \frac{1}{2}a_{[2]} + \frac{1}{2}a_{[3]}).$$

Hence, ϕ_2 and ϕ_3 are not consistent with right recursiveness.

Moreover, some rules will allow no one-side recursive definition (see [1]). In particular, we have the following characterization for the OWA case.

THEOREM 1 *Let us consider a family of OWA operators*

$$\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}.$$

Then it can be defined by left recursiveness (i.e., it is LR consistent*) if and only if*

$$w_{i,k}w_{j,k+1} = w_{j,k}w_{i,k+1}$$

for all $i,j = 1,2,\ldots,k$ and every k. Analogously, such a family of OWA operators can be defined by right recursiveness (i.e., it is RR consistent*) if and only if*

$$w_{i,k}w_{j+1,k+1} = w_{j,k}w_{i+1,k+1}$$

for all $i,j = 1,2,\ldots,k$ and every k.

In some cases, such a recursive representation of a connective rule is fixed from the underlying ordering rule, as shown in the following result.

THEOREM 2 *Let us assume*

$$\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$$

a left recursive connective rule with respect to the ordering rule π, and such that $\phi_n(0,\ldots,0) = 0$ and $\phi_n(1,\ldots,1) = 1$, with ϕ_n continuous and strictly increasing in each coordenate, for all n. Then $\{L_n\}_{n>1}$ is unique with respect to π, in such a way that

$$\phi_n(a_1, \ldots, a_n) = L_n(L_{n-1}(\ldots L_2(\pi(a_1), \pi(a_2)) \ldots, \pi(a_{n-1})), \pi(a_n))$$

(Analogous result holds for right recursiveness).

Hence, if a recursive connective rule contains a continuous strictly increasing operator of dimension n, then consistent operators of lower dimension can be obtained according the above result.

Consistent upper dimension operators can neither be freely chosen. In general, we have shown that in order to allow left recursiveness, operators in a connective rule should allow for all n the definition of a binary operator L_n such that

$$\phi_n(\pi(a_1),\ldots,\pi(a_{n-1}),\pi(a_n)) = L_n(\phi_{n-1}(\pi(a_1),\ldots,\pi(a_{n-1})),\pi(a_n)).$$

Analogously, right recursiveness holds if and only if we can define a sequence $\{R_n\}_{n>1}$ such that

$$\phi_n(\pi(a_1),\pi(a_2),\ldots,\pi(a_n)) = R_n(\pi(a_1),\phi_{n-1}(\pi(a_2),\ldots,\pi(a_n))).$$

Anyway, an interesting case to be analyzed will be that one in which both left and right recursions can share the same underlying ordering rule. That is, when

$$\phi_n(a_1,\ldots,a_n) = R_n(\pi(a_1),R_{n-1}(\pi(a_2),\ldots,R_2(\pi(a_{n-1}),\pi(a_n))\ldots)) = L_n(L_{n-1}(\ldots L_2(\pi(a_1),\pi(a_2))\ldots,\pi(a_{n-1})),\pi(a_n))$$

holds for some ordering rule π.

DEFINITION 4 *If both left and right recursiveness hold for the same ordering rule, we then talk about it as a* recursive rule.

In this way, recursiveness generalizes the concept of associativity, in the sense that recursive rules are the ones that can be evaluated iteratively (both sides), after an appropriate pre-arrangement of data. This ability of being iteratively evaluated was in fact the deep reason for associativity in practice.

Operational calculus usually implies an iterative reckoning. But this iterative calculus does not necessarily requieres a unique binary operator. As shown above, the *mean rule* allows both left and right recursive definitions, although it is not associative.

The *mean rule* verifies an additional property: both left and right recursive definitions do not depend on the permutation, i.e., they are the same no matter the particular ordering rule being chosen. Left and right recursion hold for any possible ordering rule. If such a condition holds, we can talk about *commutative* recursive rules. Commutative recursive rules will be those connective rules not depending on any particular ordering rule.

THEOREM 3 *Let us consider a connective rule $\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$ allowing left or right recursive definition (and therefore both). Then the following three properties are equivalent:*

(a) for some ordering rule π and its dual $\hat{\pi}$ we have

$$\phi_n(a_1,\ldots,a_n) = L_n(\ldots L_2(\pi(a_1),\pi(a_2))\ldots,\pi(a_n)) = R_n(\hat{\pi}(a_1),\ldots,R_2(\hat{\pi}(a_{n-1}),\hat{\pi}(a_n))\ldots))$$

for any given n;

(b) we can choose $\{L_n : [0,1]^2 \to [0,1]\}_{n>1}$ and $\{R_n : [0,1]^2 \to [0,1]\}_{n>1}$ such that $L_n(a,b) = R_n(b,a)$ for all n;

(c) for any ordering rule π we have

$$\phi_n(a_1,\ldots,a_n) = L_n(\ldots L_2(\pi(a_1),\pi(a_2))\ldots,\pi(a_n)) = R_n(\pi(a_1),\ldots,R_2(\pi(a_{n-1}),\pi(a_n))\ldots)).$$

Associativity appears when π can be taken as the identity rule and $L_n = L_2 = F = R_2 = R_n$ for all n (that is, the whole recursive connective rule is characterized by a unique associative binary connective F, with no pre-arrangement of data). This is the case when dealing with t-norms and t-conorms, for example (they are commutative, so pre-arrangement of data is neither needed). In this case, each connective rule $\{\phi_2,\ldots,\phi_n\ldots\}$ is defined by means of a unique commutative and associative binary operator $\phi : [0,1]^2 \to [0,1]$ such that

$$\phi_n(a_1,\ldots,a_n) = \phi(\ldots,\phi(\phi(b_1,b_2),b_3),\ldots,b_n) = \phi(b_1,\ldots,(b_{n-3},\phi(b_{n-2},\phi(b_{n-1},b_n)))\ldots)$$

for $(b_1,\ldots,b_n)$ any permutation of $(a_1,\ldots,a_n)$.

When we refer to a t-norm or a t-conorm as a *connective rule* we really mean the family of *connective operators* in such a way univocally defined (only one binary connective not depending on the ordering rule). The whole family of connective operators is fully characterized by its first connective operator of dimension 2, and no pre-arrangement of data is needed.

In some way we could say that a connective rule $\{\phi_n\}_{n>1}$ is recursive if and only if a set of general associativity equations (in the sense of Mak [9]) hold for each n, once the items have been properly ordered. In fact, recursiveness holds whenever

$$\phi_n(a_1,\ldots,a_n) = \phi_n(\pi(a_1),\ldots,\pi(a_n)) = R_n(\pi(a_1),\phi_{n-1}(\pi(a_2),\ldots,\pi(a_n))) = L_n(\phi_{n-1}(\pi(a_1),\pi(a_{n-1})),\pi(a_n))$$

for all n and some ordering rule π. If each one of these binary connective L_n, R_n can be assumed to be defined in the cartesian product of two nontrivial compact intervals on the real line, being continuous strictly increasing in each coordenate, then it can be shown (see [8]) that they are commutative and basically additive, in such a way that

$$\phi_n(a_1, \ldots, a_n) = \psi_0^{-1}(\psi_1(a_1) + \ldots + \psi_n(a_n))$$

for some $\psi_0, \psi_1, \ldots \psi_n$ striclty increasing continuous functions. This result can allow a particular representation of theorem 2. If we take, for example, the natural decreasing order as the underlying ordering, then each L_j is defined on a simplex $a_{j-1} \geq a_j$. Assuming the above conditions in a proper extended cartesian product of two nontrivial compact intervals, plus continutity and strict continuity, would assure such an additive solution.

IV. FINAL COMMENTS.

This paper explores a generalization of results previously obtained just for OWA operators [1]. A general approach to non-associative connective rules allowing an *operational* definition has been proposed, where *operational* we means the ability of a recursive one-by-one evaluation, on the basis of some previous re-arrangement of the data set.

As a consequence, it has been stressed the fact that a connective rule -in order to be properly considered as a rule- should be able to deal with any arbitrary number of items. An OWA operator is just an operator as the mean of n numbers is. None of them is a *connective rule* by themself, but only single connectives. Considerably many real life decision processes require at different times the aggregation of (possibly very large) lists of inputs of different dimensions. Connective rules have to be defined before knowing such a list. A connective rule is in general a rule allowing aggregation of any list, no matter its dimension.

Connective rules have been conceived here as consistent families of connective operators. They should at least allow a representation in terms of right and left recursion of binary connective operators, on the basis of a unique pre-arrangement of data. Associativity is just an easy way of assuring such an operational representation.

There are obviously families of OWA operators that represent *rules* in the sense that they allow the evaluation of any arbitrary number of items, not allowing the recursive approach as developed in this paper, but being *consistent* in some other alternative sense. This is the case, for example, of the *Binomial* OWA rule $\{\phi_n\}_{n>1}$ where each ϕ_n is an OWA operator of dimension n with weights

$$w_{i,n} = \binom{n}{i} a^i (1-a)^{n-i} \quad \forall i = 1, \ldots, n$$

for some fixed $a \in (0,1)$. Each one of these operators can be recursively defined, but the family itself does not verify the recursive OWA rule condition given in definition 4. An operative description of this family of OWA operators, still by means of a sequence of binary OWA operators and the natural decreasing ordering, can be based upon the ordered linkage property of OWA operators (see [6]).

Acknowledgment: This research has been partially supported by Dirección General de Investigación Cientifica y Técnica (Spain).

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