

On the cosmological variation of the fine structure constant

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Abstract

A phenomenological model is proposed to explain the recent observed cosmological variation of the fine structure constant as an effect of the quantum vacuum, assuming a flat universe with cosmological constant Λ in the cases $(\Omega_M, \Omega_\Lambda)$ equal to $(0.3, 0.7)$ and $(1, 0)$. Because of the fourth Heisenberg relation, the lifetime of the virtual pairs of the zero-point radiation must depend on the gravitational potential Φ , so that the quantum vacuum changes its density and acquires a relative permittivity different from one. Since the matter was more concentrated in the past, the gravitational potential of all the universe was stronger and the optical density of the vacuum higher, the electron charge being therefore more renormalized and smaller than now. The model is based on a first order Newtonian approximation that is valid for the range of the observations, but not for very high redshift, the prediction being that $\Delta\alpha/\alpha$ is proportional to $\{\Omega_M[a(t)^{-1} - 1] - 2\Omega_\Lambda[a(t)^2 - 1]\}$, $a(t)$ being the scale factor. This agrees with the observations.

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Introduction. The observations of the absorption lines of distant quasars by Webb *et al* seem to indicate that the fine structure constant was smaller in the past [1]. The idea that the fundamental constants are actually changing goes back to the “large number hypothesis” by Dirac [2], what stimulated a number of proposals to formulate the variation of the constants, see for instance ref. [3] in which α is written in terms of a scalar field, or other theories in which the variation of α is induced by a variation of the light velocity [4] (see also [5]).

This letter proposes a way to understand this phenomenon as an effect of the quantum vacuum. More precisely, it is argued here that, because of the fourth Heisenberg relation, the density of the sea of virtual particles in the quantum vacuum must change in a gravitational field, with a corresponding variation of its permittivity and permeability. As a consequence the quantum vacuum was optically denser in the past, when the universe was more contracted than now and the gravitational potential due to all the universe was stronger. The observed electron charge e is the result of the renormalization of its bare charge $e_{\text{bare}}(> e)$ by the quantum vacuum [6], so that a denser vacuum implies a smaller observed charge. It would be desirable to study this question in the frame of a rigorous quantum field theory, but the quantum vacuum is not understood well enough today in order to do that (for instance it is not known why its energy density seems to be so small). The only available alternative (other than waiting for a future theory) is to try a phenomenological approach, as is done here.

In the model here proposed, the quantum vacuum is treated as a transparent optical medium characterized by its permittivity and permeability, the change of α being a consequence of the fourth Heisenberg relation applied to the gravitational interaction of the virtual pairs in the zero point radiation with all the universe. The analysis is nonrelativistic (although with the inclusion of the rest-mass energy mc^2) and only weak Newtonian gravitational potentials Φ are considered for which $|\Phi|/c^2 \ll 1$.

The fourth Heisenberg relation and the vacuum density. Traditionally, quantum physics has stated that the sea of virtual pairs that are created and destroyed constantly in the quantum vacuum, *i. e.* the zero-point energy, has infinite density, as follows from the simple application of its basic principles. However, there is now evidence that this density may be finite. On the average and phenomenologically, a virtual pair created with energy E (including rest-mass energy, kinetic energy and electromagnetic energy) will live during a time $\tau_0 = \hbar/E$, according to the fourth Heisenberg

relation. A constant number density of pairs is established in this way at a certain equilibrium between the number of particles created and destroyed per unit time. This has an important consequence: virtual pairs must live longer in the gravitational field created by a mass distribution, because they have an extra negative potential energy $E\Phi/c^2$ there (Φ being the Newtonian potential). Indeed their lifetime must be

$$\tau_\Phi = \hbar/(E + E\Phi/c^2) = \tau_0/(1 + \Phi/c^2). \quad (1)$$

The consequence is clear if unexpected: the number density of pairs \mathcal{N} depends on the gravitational potential Φ as

$$\mathcal{N}_\Phi = \mathcal{N}_0/(1 + \Phi/c^2). \quad (2)$$

In other words, the density of the quantum vacuum depends on the gravitational potential because it must have there an extra number density of pairs depending on Φ/c^2 . If $\Phi < 0$, the quantum vacuum becomes denser, if $\Phi > 0$, it becomes thinner. These two cases correspond to the gravity created by the mass and the cosmological constant, respectively. However, this change of the number density can produce no gravitational effects, since the gravitational mass of the virtual pairs in a volume depends on Φ through the product of their mass E/c^2 by the number density per unit volume and unit energy $\mathcal{N}(E)$. This product does not change since the second factor acquires a factor $(1 + \Phi/c^2)^{-1}$ in a gravitational field, the first a factor $(1 + \Phi/c^2)$, as we have seen. The effect that we are considering is optical and dielectric, not gravitational. There is no contradiction with the usual Lorentz invariance of the quantum vacuum, since the presence of a mass or any non uniform gravitational field breaks the Lorentz symmetry. This variation of the virtual pairs lifetime in a gravitational field is not an *ad hoc* hypothesis, but an unavoidable consequence of the fourth Heisenberg relation. Note also that the virtual particles considered here are not created by gravity: they are just the usual zero point particles that fill the space everywhere, according to elementary quantum physics: they live a bit longer (or shorter), that is all.

In the following we will deal with the variation of the observed values of certain quantities between a spacetime point of the universe with potential Φ and a terrestrial observatory. As they will be expressed at first order in the potential, $(\Phi - \Phi_\oplus)$ will be written instead of Φ , Φ_\oplus being the present gravitational potential of all the universe here at Earth.

Vacuum permittivity and permeability in a gravitational potential. Following the previous considerations, we will admit as a phenomenological hypothesis that the quantum vacuum can be considered as a substratum, similar to an ordinary transparent optical medium and characterized by a permittivity and a permeability that depend on Φ . As Φ becomes more negative, its density increases and the bare electron charge e_{bare} is renormalized further or, otherwise stated, the observed charge must become smaller. In such a view the permittivity and the permeability of the quantum vacuum must change to $\epsilon_r \epsilon_0$ and $\mu_r \mu_0$, the first factor expressing the effect of its thickening (or lightening). As we assume a weak field, we can express the relative permittivity and permeability at first order in the potential. Since both are equal to 1 now at Earth, their value at a spacetime point with potential Φ can be expressed as:

$$\epsilon_r = 1 - \beta(\Phi - \Phi_{\oplus})/c^2, \quad \mu_r = 1 - \gamma(\Phi - \Phi_{\oplus})/c^2, \quad (3)$$

β and γ being certain coefficients, which must be positive since the quantum vacuum is dielectric but paramagnetic (its effect on the magnetic field is due to the magnetic moments of the virtual pairs). It must be stressed that eqs. (3) are a plausible hypothesis, consequence of the fourth Heisenberg relation.

Because of (3) the observed values of the electron charge and light velocity at potential Φ must be equal to $e/\epsilon_r = e(1 + \beta(\Phi - \Phi_{\oplus})/c^2)$ and $c/\sqrt{\epsilon_r \mu_r} = c[1 + (\beta + \gamma)(\Phi - \Phi_{\oplus})/2c^2]$, at first order (the more negative is Φ , the smaller the charge and the slower the light). Consequently $\alpha = e^2/4\pi\hbar c\epsilon_0$ must change to $\alpha\sqrt{\mu_r/\epsilon_r^3}$. This means that the change in the observed value of α , as computed at Earth from the light absorbed or emitted at a spacetime point with potential Φ , must be

$$\Delta\alpha/\alpha = \xi(\Phi - \Phi_{\oplus})/c^2, \quad (4)$$

where $\xi = (3\beta - \gamma)/2$ and Φ_{\oplus} is the potential at Earth.

On the gravitational potential of all the universe. As said before, this work considers only weak gravitational potentials that verify $|\Phi|/c^2 \ll 1$ and can be studied therefore in the Newtonian limit [7]. When that condition is no longer verified, the gravitational energy of a body is comparable to its rest energy and a relativistic theory must be used instead. This point, however, needs a clarification, since there are two independent different aspects in a Newtonian approximation. One is that the force on a particle must be weak and equal to $-\nabla\Phi$, the potential Φ being determined up to

an additive constant, other is the value of the potential energy of a particle. When studying a bounded system, as the Earth-Moon pair, a galaxy or a cluster, the effect of the faraway bodies can be neglected in the study of its motion, taking only the potential Φ caused by the nearby masses. The reason is clearly that the distant bodies contribute to Φ with a term that is almost independent of the space coordinates, at the scale of the system studied, and has therefore a negligible effect on the forces on the system. This is correct for all practical purposes concerning the motion. However, the gravitation being a long range force, one must be careful to include all the faraway matter that could have an effect on the problem *if something depends on its gravitational potential energy. For this reason, the potentials Φ and Φ_{\oplus} that appear in eqs. (1)-(4) will be taken in this work as being due to all the universe.* Note that this potential must include the contribution of matter, either ordinary or dark, as well as the effect of the cosmological constant.

Indeed the gravitational potential energy of a body is defined as the energy needed to bring it from the infinity to its actual position, without changing its kinetic energy or any other nongravitational energy. It could be argued that a body can not be brought actually from the infinity, since this is farther away than the horizon of the visible universe. However, it is clear that less energy is needed to create a virtual pair if Φ is negative than if it is zero, the difference playing the same role as the gravitational energy when bringing a body to a spacetime point at potential Φ . Therefore, the variation of the lifetime of the virtual pairs is due in this model to the gravitational interaction of the virtual pairs of the quantum vacuum with all the matter and radiation in the universe, and also with all the quantum vacuum itself.

It is assumed here that the universe is flat (*i.e.* $k = 0$) and consists in ordinary plus dark matter (with zero pressure) and dark energy due to the cosmological constant Λ . The densities over the critical density at present time are noted as usual Ω_M, Ω_Λ (with $\Omega_M + \Omega_\Lambda = 1$). Taking $p_\Lambda = -\rho_\Lambda c^2$ as the equation of state of the quantum vacuum, the source of its gravity in the Newtonian approximation is $\rho_\Lambda + 3p_\Lambda/c^2 = -2\rho_\Lambda$. Consequently, the space average potentials at present time due to the matter and to the quantum vacuum are $\Phi_{av,M} = \Omega_M \Phi_0$ and $\Phi_{av,\Lambda} = -2\Omega_\Lambda \Phi_0$, respectively, where $\Phi_0/c^2 = -\int_0^{R_U} G\rho_{cr}4\pi r dr/c^2 \simeq -0.3$ is the potential that would be created by a mass distribution with the critical density, $R_U = 3,000$ Mpc being taken as the visible universe radius. Most of these potentials is due to the larger distances. This means that the average gravitational potential

produced by the visible universe is now equal to $\Phi_{\text{av}} = \Phi_0(\Omega_M - 2\Omega_\Lambda)$. The effect of the inhomogeneities can be neglected. For instance, the contributions to the potential of the Sun, of Earth and of the Galaxy at a terrestrial laboratory are, respectively, $\simeq -10^{-8}c^2$, $\simeq -7 \times 10^{-10}c^2$ and $\simeq -6 \times 10^{-7}c^2$, which are much weaker than Φ_{av} .

Cosmological variation of α . Webb *et al* measured lines absorbed by distant gas clouds at high redshift, so that Φ in (4) must be the potential at the absorption time t , which can be taken to be approximately uniform, except for small scale inhomogeneities. In the past, when the universe was more compact and dense, the distances varied as the scale factor $a(t)$, the mass density as $a(t)^{-1}$ while the density of the quantum vacuum was constant (except for its variation due to eq. (2) that would give a second order effect). Consequently, the space average potential at time t was $\Phi_{\text{av}}(t) = \Phi_0(\Omega_M/a(t) - 2\Omega_\Lambda a^2(t))$. Introducing this expression in eq. (4) and neglecting the local inhomogeneities, it turns out that $(\alpha_z - \alpha)/\alpha$, can be expressed as

$$\frac{\Delta\alpha}{\alpha} = \xi \frac{\Phi_0}{c^2} \left[\Omega_M \left(\frac{1}{a(t)} - 1 \right) - 2\Omega_\Lambda (a^2(t) - 1) \right]. \quad (5)$$

Assuming a flat dust universe, the time evolution of the scale factor is

$$a(t) = \left(\frac{\Omega_M}{\Omega_\Lambda} \right)^{1/3} \sinh^{2/3} \left[\frac{(3\Lambda)^{1/2} t}{2} \right], \quad (6)$$

with $\Lambda = 8\pi G\Omega_\Lambda\rho_{\text{cr}}$ (eq. (6) reduces to $a(t) = (t/t_0)^{2/3}$ if $\Lambda = 0$, t_0 being the age of the universe). Equations (5)-(6) give the main result of this work: the time dependence of $\Delta\alpha/\alpha$. Note that, since $\xi\Phi_0 < 0$, $\Delta\alpha < 0$ for $t < 0$, so that the quantum vacuum must have been optically denser in the past (the electron charge was more renormalized than now since $\Phi_{\text{av}}(t) - \Phi_{\text{av}}(t_0) < 0$).

The thick line in Figure 1 shows the relative change of the fine structure constant given by eqs. (5)-(6) in the case ($\Omega_M = 0.3, \Omega_\Lambda = 0.7$) as compared with the observations by Webb *et al* [1] (in units of 10^{-5}) versus the look back-time (in units of the age of the universe), with $\xi = 1.3 \times 10^{-5}$. This is the value that gives the best fit (remind that $\Phi_0/c^2 \simeq -0.3$); it was obtained by minimizing χ^2 , the minimum value being 0.63 per point. The thin line shows the same result for ($\Omega_M = 1, \Omega_\Lambda = 0$). The best fit was obtained here for $\xi = 1.9 \times 10^{-5}$, the value of χ^2 being of 0.56 per point. Although the latter gives a slightly better fit, both curves fit well the observational points taking

into account the large error bars, their difference being small. As explained before, the light velocity must be affected also by the quantum vacuum and be time dependent. A similar argument shows that the refractive index at time t is $n(t) = c/c(t) = 1 - (\beta + \gamma)\Phi_0[\Omega_m(1/a - 1) - 2\Omega_\Lambda(a^2 - 1)]/2c^2 (> 1)$, so that the light velocity is an increasing function of time.

Validity of the model. Being based on a first order approximation in Φ , this model is valid only if $\Omega_M|\Phi_0|/a(t)c^2 \ll 1$, what means that the gravitational potential energy of a particle is much smaller than its rest energy. This condition is verified at present time, approximately at least, but loses progressively its validity when $a(t)$ decreases so that the model can not be extended arbitrarily towards the past (in its present nonrelativistic formulation).

It must be stressed that the singularity of (5) at $t = 0$ has no physical meaning since it is outside of the applicability of the model. Neither it can be said that α is predicted to change too rapidly or to vanish near time zero. One can not apply these ideas, therefore, to the recombination era, which happened at a very high redshift. But the interval of validity does cover the range of the observations by Webb *et al*, at least as a good approximation. To extend this model farther away in the past, it must be reformulated in relativistic terms.

This is not to say that the Newtonian approximation can not be used farther away in the past, for higher redshift. It can certainly be used whenever it is not necessary to consider the gravitational interaction of a system with all the universe.

On the Oklo and other data. Damour and Dyson [8] analyzed the data from the natural reactor which operated 1.8 billion years ago at Oklo (Gabon) and concluded that the relative change of α from then to now is in the interval $(-0.9 \times 10^{-7}, 1.2 \times 10^{-7})$ (assuming that other constants like the Fermi constant do not vary). Equation (5) gives for that time the relative change $\Delta\alpha/\alpha \simeq -1.3 \times 10^{-6}$ (resp. $\simeq -5.4 \times 10^{-7}$) if $\Omega_M = 0.3, \Omega_\Lambda = 0.7$ (resp. $\Omega_M = 1, \Omega_\Lambda = 0$), values which are outside but not far from that interval.

The Oklo study is thought to give the most powerful method to determine the variation of α with geochemical data but, according to Uzan [9] “one has to understand and to model carefully the correlations of the variations of α_w and g_s as well as the effect of μ ($= m_e/m_p$). This difficult but necessary task remains to be done”. There are several others studies which set bounds on the variation of $|\Delta\alpha|$, using a number of different data [9]. The results of the

present model are compatible with these other bounds, except for the one set by Olive *et al* using data of β -decay in meteorites [10], $|\Delta\alpha/\alpha| < 3 \times 10^{-7}$ during the past 4.6 Gyr (redshift about 0.45), while this model gives $\simeq 3 \times 10^{-6}$ (resp. $\simeq 1.7 \times 10^{-6}$) if $\Omega_M = 0.3, \Omega_\Lambda = 0.7$ (resp. $\Omega_M = 1, \Omega_\Lambda = 0$). However, this bound by Olive *et al* “can also be altered if the neutrinos are massive”, according to Uzan [9].

In any case, this model gives a fair account of the observations by Webb *et al*. Taking everything into account, it seems worth of consideration to explore its consequences.

Comparison with the gravitational redshift. The effect described here produces a frequency shift which may seem similar to the gravitational redshift $\Delta\omega/\omega = \Delta\Phi/c^2$. The two effects are different, however. The one observed by Webb *et al* is due to the variation of the fine structure constant. The principal part of the frequencies emitted by an atom is proportional to $(e^2/4\pi\hbar\epsilon_0)^2$, not exactly to α^2 while the relative separation between lines in the relativistic fine structure splitting does depend on $\alpha^2 = (e^2/4\pi\hbar c\epsilon_0)^2$. On the other hand, the gravitational shift affects equally to all the lines in a multiplet. In experiments that disregard the width of the multiplets, the contribution of the change of α to the frequency shift is obtained from (4), but with 2β instead of β , *i.e.* $\Delta\omega/\omega = 4\beta\Delta\Phi/c^2$. In that case, the observed shift would be $\Delta\omega/\omega = (1 + 4\beta)\Delta\Phi/c^2$, *i.e.* the addition of the two effects. The best confirmations of the gravitational redshift, those by Pound, Rebka and Snider, agree with the prediction of General Relativity up to about 1 % [11], but they refer to nuclear levels in which the electromagnetism plays only a part. This means that the results of this work are certainly compatible with the experiments on the gravitational redshift if $\beta \leq 2.5 \times 10^{-3}$, an inequality that this work assumes to be satisfied. However, there is a problem: as $\xi < 3\beta/2$, a bound on β is also a bound on ξ but the converse is not true. In any case, a necessary condition for the compatibility of the results of this work and the gravitational redshift experiments is that $\xi < 4 \times 10^{-3}$, while this model predicts that ξ is equal to about 1.3×10^{-5} and 1.9×10^{-5} in the two cases considered.

Two final comments. First, note that the dressed electron charge and the fine structure constant are not treated here as universal constants but as the result of the interaction of point charges with the quantum vacuum, their variations being an indirect consequence of the universal expansion, through the modification of their renormalization.

Second, because of the conservation of the charge conjugation, the pairs

are created with $L = S = 0$, so that their energy in a magnetic field is $E - 2\mu_B B$ with probability $1/2$, E being the non-magnetic energy. For magnetic field close to $B_0 = mc^2/\mu_B \simeq 8.8 \times 10^{13}$ gauss, some pairs would have zero total energy, their lifetime being infinite according to the fourth Heisenberg relation). If B approaches B_0 from below, the creation of the pairs would begin to be energetically free. There would be a threshold to some peculiar phenomenon, although it is not clear what this can be (note that the inequality $B \ll B_0$ is a necessary condition for the validity of the Euler-Heisenberg Lagrangian to study in QED the interactions of the quantum vacuum with a magnetic field [12]). Intriguingly, it turns out that B_0 is close to the highest magnetic field measured in magnetars, see Kouveliotou *et al* [13]. Commenting ref. [13], Kulkarni and Thompson [14] say, “our failure to detect radio pulsars with magnetic fields greater than 2×10^{13} gauss is because, at such high fields, quantum electrodynamic effects help to damp the radio emission” this being “direct evidence of magnetic fields strong enough to perturb the very structure of the vacuum”. This gives support to the present work: the upper limit for the magnetic field of magnetars could be just a manifestation of the fourth Heisenberg relation similar to the one considered here.

Summary and conclusions. In the phenomenological model presented here, the cosmological variation of the fine structure constant is due to the combined effect of the fourth Heisenberg relation and the gravitational interaction of the virtual pairs in the zero-point radiation with all the universe. The problem is studied in the Newtonian approximation. The quantum vacuum is treated as an optical medium characterized by its relative permittivity and permeability that depend on the average gravitational potential of the universe. The model predicts that $\Delta\alpha/\alpha$ is proportional to $\{\Omega_M[a(t)^{-1} - 1] - 2\Omega_\Lambda[a(t)^2 - 1]\}$ ($a(t)$ being the scale factor). The results for the cases $\Omega_M = 0.3, \Omega_\Lambda = 0.7$ and $\Omega_M = 1, \Omega_\Lambda = 0$ agree with the observations by Webb *et al* [1] on the cosmological variation of the fine structure constant, as is seen in figure 1. The argument goes as follows.

(i) Because of the fourth Heisenberg relation, the lifetime of the virtual pairs of the zero-point radiation depends on the gravitational potential Φ . This causes the permittivity and the permeability of the quantum vacuum, the observable electron charge and the light velocity to depend also on Φ . The consequent change of the observed fine structure constant is expressed at first order in terms of the average gravitational potential of the universe, and of a parameter ξ related to the renormalization effects of the quantum

vacuum.

(ii) As the universe was more compact in the past, its average gravitational potential was more negative (or less positive). Consequently, the lines of the spectra of distant quasars were absorbed with a more renormalized value of the electron charge than now. As a result, $\Delta\alpha/\alpha$ is given by eq. (5), which is plotted in fig. 1. The agreement seems good. Note that, in this model, the optical density of the quantum vacuum increases towards the past and decreases along the universe history.

(iii) The model can not be extended arbitrarily towards the past, since it is based on a first order Newtonian approximation that is valid for the recent past, including the range of the observations by Webb *et al*, but is no longer applicable to higher redshift. In particular, it can never be applied to the recombination era. A relativistic approach should be followed in order to extend the model to the remote past. Note finally that, according to this model, the light is also affected by the gravitational potential so that it was slower in the past.

The conclusion of this paper is that the combined effect of the gravitation of all the universe and of the fourth Heisenberg relation on the density of the zero-point radiation and the corresponding cosmological variations of α , e and c proposed here should be further investigated.

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Figure 1: $\Delta\alpha/\alpha$ (times 10^{-5}) vs fractional look-back time, predicted by this work in the cases $\Omega_M = 0.3, \Omega_\Lambda = 0.7$ (eqs. (5)-(6), thick line) and $\Omega_M = 1, \Omega_\Lambda = 0$ (thin line), as compared with the data by Webb *et al* [1] (explanation in the text).

