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Multivariate Toda hierarchies and biorthogonal polynomials

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ABSTRACT

A new multivariate Toda hierarchy of nonlinear partial differential equations adapted to multivariate biorthogonal polynomials is discussed. This integrable hierarchy is associated with non-standard multivariate biorthogonality. Wave and Baker functions, linear equations, Lax and Zakharov–Shabat equations, KP type equations, appropriate reductions, Darboux or linear spectral transformations, and bilinear equations involving linear spectral transformations are presented.

1. Introduction

In this paper an extension of the multivariate Toda hierarchy introduced in [1], associated with multivariate orthogonal polynomials (MVOP), is given to general multivariate biorthogonal polynomials (MVBOP). For this new integrable hierarchy, which has the MVOP as a particular reduction, we find the multivariate linear spectral transformations.

1.1. Historical background and state of the art

Sato [2,3] and Date, Jimbo, Kashiwara and Miwa [4–6] introduced geometrical tools, like the infinite-dimensional Grassmannian and infinite dimensional Lie groups an Lie algebras, which have become essential in the description of integrable hierarchies. In Mulase's seminal paper [7] the factorization problem, dressing procedure, and linear systems were shown to be the relevant keys for integrability. Multicomponent versions of the integrable Toda equations [8–10] played a prominent role in the connection with orthogonal polynomials and differential geometry. In [11–15] multicomponent versions of the KP hierarchy were analyzed, while in [16,17] we can find a study of the multi-component Toda lattice hierarchy, block Hankel/Toeplitz reductions, discrete flows, additional symmetries and dispersionless limits. In [18,19] the relation of the multicomponent KP–Toda with mixed multiple orthogonal polynomials was discussed.

Adler and van Moerbeke showed the prominent role played by the Gauss–Borel factorization problem for understanding the strong bonds between orthogonal polynomials and integrable systems. In particular, their studies on the 2D Toda hierarchy – what they called the discrete KP hierarchy – neatly established the deep connection among standard orthogonality of polynomials and integrability of nonlinear equations of Toda type, see [20–24] and also [25]. Let us also mention that multicomponent Toda systems or non-Abelian versions of Toda equations with matrix orthogonal polynomials were studied, for example, in [26,27] (on the real line) and in [28,29] (on the unit circle).

Darboux transformations for multivariate orthogonal polynomials and Toda hierarchies were first studied in [1,30]. These transformations are multidimensional extensions of the Christoffel transformations. In [30] a multivariate extension of the classical 1D Christoffel

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formula in terms of quasi-determinants [31–33] and poised sets [30,33] was given. In this general multidimensional framework in the paper [34] a Christoffel formula for multivariate Laurent polynomials orthogonal with respect to a measure supported in the unit torus was found. In [35] linear relations between two families of multivariate orthogonal polynomials were studied. Despite that [35] does not deal with Geronimus formulas it deals with linear connections among two families of orthogonal polynomials. This is a first step towards a connection formulas for the multivariate Geronimus transformation. Elwin Christoffel, when discussing Gauss quadrature rules in [36], found explicit formulas relating sequences of orthogonal polynomials corresponding to two measures $\mathrm{d}\,x$ and $p(x)\,\mathrm{d}\,x$, with $p(x) = (x - q_1)\cdots(x - q_N)$. The so called Christoffel formula is a classical result which can be found in a number of orthogonal polynomials textbooks, see for example [37–39].

Within a linear functional approach to the theory of orthogonal polynomials, see [40–44], given a linear functional $u \in (\mathbb{R}[x])'$ its canonical or elementary Christoffel transformation is a new moment functional given by $\hat{u} = (x-a)u$, $a \in \mathbb{R}$, [38,45,46]. Its right inverse is called the Geronimus transformation, i.e., the elementary or canonical Geronimus transformation is a new moment linear functional $\check{u}$ such that $(x - a)\check{u} = u$. In this case we can write $\check{u} = (x - a)^{-1}u + \xi\delta(x - a)$, where $\xi \in \mathbb{R}$ is a free parameter and $\delta(x)$ is the Dirac functional supported at the point $x = a$ [47,48]. Multiple Geronimus transformations [49] appear when one studies general inner products $\langle\cdot,\cdot\rangle$ such that the multiplication by a polynomial operator h is symmetric and satisfies $\langle h(x)p(x), q(x)\rangle = \int p(x)q(x)\,\mathrm{d}\,\mu(x)$ for a nontrivial probability measure μ.

In [50] Vasily Uvarov considered the multiplication of the measure by a rational function with prescribed zeros and poles, and got determinantal formulas – in terms of the original orthogonal polynomials and its Cauchy transformations – for the perturbed polynomials. That is, he worked out in [50, §1] the linear spectral transformation without masses. Moreover, he also introduced in [50, §2] the so called canonical Uvarov transformation the moment linear functional u is transformed into $\check{u} = u + \xi\delta(x - a)$ with $\xi \in \mathbb{R}$, and presented a determinantal formulas – in terms of kernel polynomials – for several masses of this type, $\check{u} = u+\xi_1\delta(x-a_1)+\cdots+\xi_N\delta(x-a_N)$. The Stieltjes function $F(x) := \sum_{n=0}^{\infty}\frac{\langle u,x^n\rangle}{x^{n+1}}$ of a linear functional $u \in (\mathbb{R}[x])'$ is relevant in the theory of orthogonal polynomials for several reasons, is in particular remarkable on account of its close relation with Padé approximation theory, see [51,52]. Alexei Zhedanov studied in [53] the following rational spectral transformations of the Stieltjes function

$$F(x) \mapsto \tilde{F}(x) = \frac{A(x)F(x) + B(x)}{C(x)F(x) + D(x)},$$

as a natural extension of the above mentioned three canonical transformations. Here $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are polynomials such that $\tilde{F}(x) = \sum_{n=0}^{\infty}\frac{\langle\tilde{u},x^n\rangle}{x^{n+1}}$ is a new Stieltjes function. Linear spectral transformations correspond to the choice $c(x) = 0$, of which particular cases are the canonical Christoffel transformations $\tilde{F}(x) = (x-a)F(x)-F_0$ and the canonical Geronimus transformation of $\tilde{F}(x) = \frac{F(x)+\tilde{F}_0}{x-a}$. Every linear spectral transformation of a moment functional is given as a composition of Christoffel and Geronimus transformations [53].

Regarding orthogonal polynomials in several variables we refer the reader to the excellent monographs [54,55]. Milch [56] and Karlin and McGregor [57] considered multivariate Hahn and Krawtchouk polynomials in relation with growth birth and death processes. Since 1975 substantial developments have been achieved, let us mention the spectral properties of these multivariate Hahn and Krawtchouk polynomials, see [58]. Orthogonal polynomials and cubature formulæon the unit ball, the standard simplex, and the unit sphere were studied in [59] finding a strong connections between both themes. The common zeros of multivariate orthogonal polynomials were discussed in [60] where relations with higher dimensional quadrature problems were found. A description of orthogonal polynomials on the bicircle and polycircle and their relation to bounded analytic functions on the polydisk is given in [61], here a Christoffel–Darboux like formula, related in this bivariate case with stable polynomials, and Bernstein–Szegő measures are used, allowing for a new proof of Ando theorem in operator theory. Bivariate orthogonal polynomials linked to a moment functional satisfying the two-variable Pearson type differential equation and an extension of some of the characterizations of the classical orthogonal polynomials in one variable was discussed in [62]; in the paper [63] an analysis of a bilinear form obtained by adding a Dirac mass to a positive definite moment functional in several variables is given.

The approach to linear spectral transformations and Toda hierarchies used in this paper, which is based on the Gauss–Borel factorization problem, has been used before in different contexts. We have connected integrable systems with orthogonal polynomials of diverse types:

(i) As already mentioned, mixed multiple orthogonal polynomials and multicomponent Toda was analyzed in [19,64]. This technique has applications to random walks beyond birth and death [65,66].
(ii) Matrix orthogonal Laurent polynomials on the circle and CMV orderings were considered [67]. The Christoffel transformation has been recently discussed for matrix orthogonal polynomials in the real line [68–70].
(iii) Applications to semiclassical discrete orthogonal polynomials and Laguerre–Freud equations can be found in [71–73].

1.2. Results and layout of the paper

Previously, see [1], we have considered semi-infinite matrices G of multivariate bimoments and discussed the corresponding symmetries, which are of generalized Hankel type. In this paper we will extend this setting. In Section 2 we explore different scenarios by assuming that the matrix G is the Gram matrix of a bilinear form, which has a Gauss–Borel factorization and is related to multivariate biorthogonal polynomials. We first give the general setting for this integrable hierarchy, finding the corresponding Lax and Zakharov–Shabat equations and the role played by the Baker and adjoint Baker functions. Some reductions, like the multi-Hankel that leads to MVOP, and extensions of it are presented. We also consider the action of the multivariate linear spectral transformations and find the Christoffel–Geronimus–Uvarov formula in this broader situation. Finally, we find generalized bilinear equations that involve linear spectral transformations.

1.3. Preliminary material

Following [30], a brief account of complex multivariate orthogonal polynomials in a D-dimensional real space (MVOP) is given. Cholesky factorization of a semi-infinite moment matrix will be keystone to build such objects. Consider D independent real variables $\boldsymbol{x}=(x_1,x_2,\dots,x_D)^\top\in\mathbb{R}^D$, and the corresponding ring of complex multivariate polynomials $\mathbb{C}[\boldsymbol{x}]\equiv\mathbb{C}[x_1,\dots,x_D]$. Given a multi-index $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_D)^\top\in\mathbb{Z}_+^D$ of non-negative integers write $\boldsymbol{x}^{\boldsymbol{\alpha}}=x_1^{\alpha_1}\cdots x_D^{\alpha_D}$ and say that the length of $\boldsymbol{\alpha}$ is $|\boldsymbol{\alpha}|:=\sum_{a=1}^D\alpha_a$. This length induces a total ordering of monomials: $\boldsymbol{x}^{\boldsymbol{\alpha}}<\boldsymbol{x}^{\boldsymbol{\alpha}'}\Leftrightarrow|\boldsymbol{\alpha}|<|\boldsymbol{\alpha}'|$. For each non-negative integer $k\in\mathbb{Z}_+$ introduce the set

$$[k]:=\{\boldsymbol{\alpha}\in\mathbb{Z}_+^D:|\boldsymbol{\alpha}|=k\},$$

build up with those vectors in the lattice $\mathbb{Z}_+^D$ with a given length k. The graded lexicographic order for $\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2\in[k]$ is

$$\boldsymbol{\alpha}_1<\boldsymbol{\alpha}_2\Leftrightarrow\exists p\{0,1,\dots,D-1\}\text{ such that }\alpha_{1,1}=\alpha_{2,1},\dots,\alpha_{1,p}=\alpha_{2,p}\text{ and }\alpha_{1,p+1}<\alpha_{2,p+1},$$

and if $\boldsymbol{\alpha}^{(k)}\in[k]$ and $\boldsymbol{\alpha}^{(l)}\in[l]$, with $k<l$ then $\boldsymbol{\alpha}^{(k)}<\boldsymbol{\alpha}^{(l)}$. Given the set of integer vectors of length k use the lexicographic order and write

$$[k]=\left\{\boldsymbol{\alpha}_1^{(k)},\boldsymbol{\alpha}_2^{(k)},\dots,\boldsymbol{\alpha}_{|[k]|}^{(k)}\right\}\text{ with }\boldsymbol{\alpha}_a^{(k)}<\boldsymbol{\alpha}_{a+1}^{(k)}.$$

Here $|[k]|$ is the cardinality of the set $[k]$. This is the dimension of the linear space of homogeneous multivariate polynomials of total degree k. Either counting weak compositions or multi-sets one obtains the multi-choose number, $|[k]|=\left(\!\binom{D}{k}\!\right)=\binom{D+k-1}{k}$. The dimension of the linear space $\mathbb{C}_k[x_1,\dots,x_D]$ of multivariate polynomials of degree less or equal to k is

$$N_k=1+|[2]|+\cdots+|[k]|=\binom{D+k}{D}.$$

The vector of monomials

$$\chi:=\begin{bmatrix}\chi_{[0]}\\\chi_{[1]}\\\vdots\end{bmatrix}\quad\text{where}\quad\chi_{[k]}:=\begin{bmatrix}\boldsymbol{x}^{\boldsymbol{\alpha}_1}\\\boldsymbol{x}^{\boldsymbol{\alpha}_2}\\\vdots\\\boldsymbol{x}^{\boldsymbol{\alpha}_{|[k]|}}\end{bmatrix},\qquad\chi^*:=\Big(\prod_{a=1}^Dx_a^{-1}\Big)\chi(x_1^{-1},\dots,x_D^{-1}).$$

will be useful. Observe that for $k=1$ we have that the vectors $\boldsymbol{\alpha}_a^{(1)}=\boldsymbol{e}_a$ for $a\in\{1,\dots,D\}$ form the canonical basis of $\mathbb{R}^D$, and for any $\boldsymbol{\alpha}_j\in[k]$ we have $\boldsymbol{\alpha}_j=\sum_{a=1}^D\alpha_j^a\boldsymbol{e}_a$. For the sake of simplicity unless needed we will drop off the super-index and write $\boldsymbol{\alpha}_j$ instead of $\boldsymbol{\alpha}_j^{(k)}$, as it is understood that $|\boldsymbol{\alpha}_j|=k$.

Consider semi-infinite matrices A with a block or partitioned structure induced by the graded reversed lexicographic order

$$A=\begin{bmatrix}A_{[0],[0]}&A_{[0],[1]}\cdots\cdots\\A_{[1],[0]}&A_{[1],[1]}\cdots\cdots\\\vdots&\vdots\end{bmatrix},\qquad A_{[k],[\ell]}=\begin{bmatrix}A_{\boldsymbol{\alpha}_1^{(k)},\boldsymbol{\alpha}_1^{(\ell)}}&\cdots\cdots&A_{\boldsymbol{\alpha}_1^{(k)},\boldsymbol{\alpha}_{|[l]|}^{(\ell)}}\\\vdots&&\vdots\\A_{\boldsymbol{\alpha}_{|[k]|}^{(k)},\boldsymbol{\alpha}_1^{(\ell)}}&\cdots\cdots&A_{\boldsymbol{\alpha}_{|[k]|}^{(k)},\boldsymbol{\alpha}_{|[l]|}^{(\ell)}}\end{bmatrix}\in\mathbb{C}^{|[k]|\times|[l]|}.$$

Use the notation $0_{[k],[\ell]}\in\mathbb{C}^{|[k]|\times|[l]|}$ for the rectangular zero matrix, $0_{[k]}\in\mathbb{C}^{|[k]|}$ for the zero vector, and $\mathbb{I}_{[k]}\in\mathbb{C}^{|[k]|\times|[k]|}$ for the identity matrix. For the sake of simplicity just write 0 or $\mathbb{I}$ for the zero or identity matrices, and assume that the sizes of these matrices are the ones indicated by their position in the partitioned matrix.

The vector space of complex multivariate polynomials $\mathbb{C}_k[\boldsymbol{x}]$ in D real variables of degree less or equal to k with the norm $\left\|\sum_{|\boldsymbol{\alpha}|\le k}P_{\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}}\right\|_n:=\sum_{|\boldsymbol{\alpha}|\le k}|P_{\boldsymbol{\alpha}}|$, gives a nesting of Banach spaces $\mathbb{C}_n[\boldsymbol{x}]\subset\mathbb{C}_{n+1}[\boldsymbol{x}]$ whose inductive limit gives a topology to the space $\mathbb{C}[\boldsymbol{x}]$. The elements of the algebraic dual $u\in(\mathbb{C}[\boldsymbol{x}])^*$, which are called linear functionals, are linear maps $u:\mathbb{C}[\boldsymbol{x}]\to\mathbb{C}$; the notation $P(\boldsymbol{x})\overset{u}{\mapsto}\langle u,P(\boldsymbol{x})\rangle$ will be used. Two polynomials $P(\boldsymbol{x}),Q(\boldsymbol{x})\in\mathbb{C}[\boldsymbol{x}]$ are said to be orthogonal with respect to u if $\langle u,P(\boldsymbol{x})Q(\boldsymbol{x})\rangle=0$. The topological dual $(\mathbb{C}[\boldsymbol{x}])'$ has the dual weak topology characterized by the semi-norms $\left\{\|\cdot\|_P\right\}_{P(\boldsymbol{x})\in\mathbb{C}[\boldsymbol{x}]}$, $\|u\|_P:=|\langle u,P(\boldsymbol{x})\rangle|$. This family of seminorms is equivalent to the family of seminorms given by $\|u\|^{(k)}:=\sup_{|\boldsymbol{\alpha}|=k}|\langle u,\boldsymbol{x}^{\boldsymbol{\alpha}}\rangle|$. Moreover, the topological dual $(\mathbb{C}[\boldsymbol{x}])'$ is a Fréchet space and $(\mathbb{C}[\boldsymbol{x}])'=(\mathbb{C}[\boldsymbol{x}])^*$ and every linear functional is continuous. Linear functionals can be multiplied by polynomials $\langle Qu,P(\boldsymbol{x})\rangle:=\langle u,Q(\boldsymbol{x})P(\boldsymbol{x})\rangle$, $\forall P(\boldsymbol{x})\in\mathbb{C}[\boldsymbol{x}]$. It can be also shown, that in this case the space of generalized functions $(\mathbb{C}[\boldsymbol{x}])'$ coincide with the space of formal series $\mathbb{C}[\![\boldsymbol{x}]\!]$. For more information regarding linear functional's approach to orthogonal polynomials see [40,41,43,44].

Definition 1. Associated with the linear functional $u\in(\mathbb{C}[x])'$ define the following moment matrix

$$G:=\langle u,\chi(\boldsymbol{x})\big(\chi(\boldsymbol{x})\big)^\top\rangle.$$

In block form can be written as

$$G=\begin{bmatrix}G_{[0],[0]}&G_{[0],[1]}\cdots\cdots\\G_{[1],[0]}&G_{[1],[1]}\cdots\cdots\\\vdots&\vdots\end{bmatrix}.$$

Truncated moment matrices are given by

$$G^{[l]}:=\begin{bmatrix}G_{[0],[0]}&\cdots\cdots\cdots&G_{[0],[l-1]}\\\vdots&&\vdots\\G_{[l-1],[0]}&\cdots\cdots\cdots&G_{[l-1],[l-1]}\end{bmatrix}.$$

Notice that from the above definition we know that

Proposition 1. *The moment matrix is a symmetric matrix,* $G = G^\top$.

This result implies that a Gauss–Borel factorization of it, in terms of lower unitriangular and upper triangular matrices, is a Cholesky factorization.

In terms of quasi-determinants, see [1,30,33,34,74], we have

Proposition 2. *Let us assume that the last quasi-determinants* $\Theta_*(G^{[k+1]})$, $k \in \{0, 1, \dots\}$, *of the truncated moment matrices are not singular. Then, the Cholesky factorization*

$$G = S^{-1} H S^{-\top}, \tag{1}$$

with

$$S^{-1} = \begin{bmatrix} I & 0 & \cdots \\ (S^{-1})_{[1],[0]} & \ddots & \ddots \\ (S^{-1})_{[2],[0]} & (S^{-1})_{[2],[1]} & \ddots \\ \vdots & \vdots & \ddots \end{bmatrix}, \qquad H = \begin{bmatrix} H_{[0]} & 0 & \cdots \\ 0 & H_{[1]} & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix},$$

and symmetric matrices $H_{[k]} = (H_{[k]})^\top$, *can be performed. Moreover, the rectangular blocks can be expressed in terms of last quasi-determinants of truncations of the moment matrix*

$$H_{[k]} = \Theta_*(G^{[k+1]}), (S^{-1})_{[k],[l]} = \Theta_*(G_k^{[l+1]})\Theta_*(G^{[l+1]})^{-1}.$$

Definition 2. The monic MVOP associated to the linear functional u are

$$P(\boldsymbol{x}) = S\chi(\boldsymbol{x}) = \begin{bmatrix} P_{[0]}(\boldsymbol{x}) \\ P_{[1]}(\boldsymbol{x}) \\ \vdots \end{bmatrix}, \quad P_{[k]}(\boldsymbol{x}) = \sum_{\ell=0}^{k} S_{[k],[l]}\chi_{[l]}(\boldsymbol{x}) = \begin{bmatrix} P_{\boldsymbol{\alpha}_1^{(k)}}(\boldsymbol{x}) \\ \vdots \\ P_{\boldsymbol{\alpha}_{|[k]|}^{(k)}}(\boldsymbol{x}) \end{bmatrix}, \quad P_{\boldsymbol{\alpha}_i^{(k)}} = \sum_{l=0}^{k}\sum_{j=1}^{|[l]|} S_{\boldsymbol{\alpha}_i^{(k)},\boldsymbol{\alpha}_j^{(l)}} \boldsymbol{x}^{\boldsymbol{\alpha}_j^{(l)}}. \tag{2}$$

Observe that $P_{[k]}(\boldsymbol{x}) = \chi_{[k]}(\boldsymbol{x}) + \beta_{[k]}\chi_{[k-1]}(\boldsymbol{x}) + \cdots$ is a vector constructed with the polynomials $P_{\boldsymbol{\alpha}_i}(\boldsymbol{x})$ of degree k, each of which has only one monomial of degree k; i.e., we can write $P_{\boldsymbol{\alpha}_i}(\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{\alpha}_i} + Q_{\boldsymbol{\alpha}_i}(\boldsymbol{x})$, with $\deg Q_{\boldsymbol{\alpha}_i} < k$. Here β is th semi-infinite matrix with all its elements being zero but for its first subdiagonal $\beta = \operatorname{subdiag}_1(\beta_{[1]}, \beta_{[2]}, \dots)$ with coefficients given by $\beta_{[k]} := S_{[k],[k-1]}$.

Proposition 3 (*Orthogonality Relations*)**.** *The MVOP satisfy* $\left\langle u, P_{[k]}(\boldsymbol{x})\big(P_{[l]}(\boldsymbol{x})\big)^\top\right\rangle = \delta_{k,l}H_{[k]}$, *or, equivalently, the orthogonality relations*

$$\left\langle u, P_{[k]}(x)\big(P_{[l]}(\boldsymbol{x})\big)^\top\right\rangle = \left\langle u, P_{[k]}(\boldsymbol{x})\big(\chi_{[l]}(\boldsymbol{x})\big)^\top\right\rangle = 0, \qquad l = 0, 1, \dots, k-1, \tag{3}$$

$$\left\langle u, P_{[k]}(\boldsymbol{x})\big(P_{[k]}(\boldsymbol{x})\big)^\top\right\rangle = \left\langle u, P_{[k]}(\boldsymbol{x})\big(\chi_{[k]}(\boldsymbol{x})\big)^\top\right\rangle = H_{[k]}. \tag{4}$$

Therefore, the following orthogonality conditions

$$\langle u, P_{\boldsymbol{\alpha}_i^{(k)}}(\boldsymbol{x}) P_{\boldsymbol{\alpha}_j^{(l)}}(\boldsymbol{x})\rangle = \langle u, P_{\boldsymbol{\alpha}_i^{(k)}}(\boldsymbol{x})\boldsymbol{x}^{\boldsymbol{\alpha}_j^{(l)}}\rangle = 0,$$

are fulfilled for $l \in \{0, 1, \dots, k-1\}$, $i \in \{1, \dots, |[k]|\}$ and $j \in \{1, \dots, |[l]|\}$, with the normalization conditions

$$\langle u, P_{\boldsymbol{\alpha}_i}(\boldsymbol{x}) P_{\boldsymbol{\alpha}_j}(\boldsymbol{x})\rangle = \langle u, P_{\boldsymbol{\alpha}_i}(\boldsymbol{x})\boldsymbol{x}^{\boldsymbol{\alpha}_j}\rangle = H_{\boldsymbol{\alpha}_i,\boldsymbol{\alpha}_j}, \quad i, j \in \{1, \dots, |[k]|\}.$$

Let us introduce the matrices Λ_a that represent the multiplication by x_a in the lexicographically ordered basis of monomials.

Definition 3. The spectral matrices are given by

$$\Lambda_a = \begin{bmatrix} 0 & (\Lambda_a)_{[0],[1]} & 0 & \cdots \\ \vdots & \ddots & (\Lambda_a)_{[1],[2]} & \ddots \\ & & \ddots & (\Lambda_a)_{[2],[3]} \\ \vdots & & & \ddots \end{bmatrix}, \qquad a \in \{1, \dots, D\},$$

where the entries in the first block superdiagonal are

$$(\Lambda_a)_{\boldsymbol{\alpha}_i^{(k)},\boldsymbol{\alpha}_j^{(k+1)}} = \delta_{\boldsymbol{\alpha}_i^{(k)}+\boldsymbol{e}_a,\boldsymbol{\alpha}_j^{(k+1)}}, \quad a \in \{1, \dots, D\},\ i \in \{1, \dots, |[k]|\},\ j \in \{1, \dots, |[k+1]|\},$$

and the associated vector $\boldsymbol{\Lambda} := (\Lambda_1, \dots, \Lambda_D)^\top$. Finally, we introduce the Jacobi matrices

$$J_a := S\Lambda_a S^{-1}, \quad a \in \{1, \dots, D\}, \tag{5}$$

and the Jacobi vector

$$\boldsymbol{J} = (J_1, \dots, J_D)^\top.$$

Proposition 4.

(i) *The spectral matrices commute among them*

$$\Lambda_a\Lambda_b=\Lambda_b\Lambda_a,\quad a,b\in\{1,\dots,D\}.$$

(ii) *The spectral properties*

$$\Lambda_a\chi(\boldsymbol{x})=x_a\chi(\boldsymbol{x}),\quad a\in\{1,\dots,D\}\tag{6}$$

hold.

(iii) *The moment matrix G satisfies*

$$\Lambda_aG=G\big(\Lambda_a\big)^\top,\quad a\in\{1,\dots,D\}.\tag{7}$$

(iv) *The Jacobi matrices J_a are block tridiagonal, with blocks of increasing sizes, and satisfy*

$$J_aH=HJ_a^\top,\quad a\in\{1,\dots,D\}.$$

Definition 4. The Christoffel–Darboux kernel is

$$K_n(\boldsymbol{x},\boldsymbol{y}):=\sum_{m=0}^{n}\big(P_{[m]}(\boldsymbol{x})\big)^\top(H_{[m]})^{-1}P_{[m]}(\boldsymbol{y})$$

In terms of the Christoffel–Darboux kernel and a linear functional u we define the operator

$$S_n(f)(\boldsymbol{x}):=\langle u,f(\boldsymbol{y})K_n(\boldsymbol{y},\boldsymbol{x})\rangle\,.$$

Proposition 5.

(i) *If $P(\boldsymbol{x})=\sum_{j\geq0}c_{[j]}P_{[j]}(\boldsymbol{x})\in\mathbb{C}[\boldsymbol{x}]$ is an arbitrary multivariate polynomial, we have*

$$S_n(P)(\boldsymbol{x})=\sum_{m=0}^{n}c_{[m]}P_{[m]}(\boldsymbol{x}).\tag{8}$$

(ii) *For any vector $\boldsymbol{n}\in\mathbb{C}^D$, the following Christoffel–Darboux formula is fulfilled*

$$\begin{aligned}&\big(\boldsymbol{n}\cdot(\boldsymbol{x}-\boldsymbol{y})\big)K_n(\boldsymbol{x},\boldsymbol{y})\\&\quad=\big(P_{[n+1]}(\boldsymbol{x})\big)^\dagger\Big((\boldsymbol{n}\cdot\boldsymbol{\Lambda})_{[n],[n+1]}\Big)^\top(H_{[n]})^{-1}P_{[n]}(\boldsymbol{y})-\big(P_{[n]}(\boldsymbol{x})\big)^\dagger(H_{[n]})^{-1}(\boldsymbol{n}\cdot\boldsymbol{\Lambda})_{[n],[n+1]}P_{[n+1]}(\boldsymbol{y}).\end{aligned}$$

2. Multivariate 2D Toda lattice

We explore the situation described in Section 1.3 but not specifically with multivariate polynomials in mind. The block structure of the semi-infinite matrices has been described there. In [1] we considered a semi-infinite matrix G such that $\Lambda_aG=G(\Lambda_a)^\top$, $a\in\{1,\dots,D\}$, a Cholesky factorization

$$G=S^{-1}H(S)^{-\top},$$

and flows preserving this structure. In that manner we obtained nonlinear equations for which the MVOP provided solutions. Then, in [30] we derived a quasi-determinantal Christoffel formula for the multivariate Christoffel transformations for MVOP. A similar development could be performed here with the more general linear spectral transformations, but we will follow a more general approach.

The Toda type flows discussed in [1] for multivariate moment matrices can be extended further. The integrable hierarchy has the MVOP as solutions, but this is only a part of its space of solutions, as the MVOP sector corresponds to a particular choice of G. In this paper we will analyze this Toda hierarchy, that we name as multivariate 2D Toda hierarchy, on its own, associated as we will see to non standard orthogonality. Therefore, we now consider any possible block Gauss–Borel factorizable semi-infinite matrix

$$G=(S_1)^{-1}H(S_2)^{-\top}$$

where, S_1,S_2 are lower unitriangular block semi-infinite matrices, and H is a diagonal block semi-infinite matrix. Recall that the blocks have increasing size.

2.1. Non-standard multivariate biorthogonality. Bilinear forms

Definition 5. In the linear space of multivariate polynomials $\mathbb{C}[\boldsymbol{x}]$ we consider a bilinear form $\langle\cdot,\cdot\rangle$ whose Gram semi-infinite matrix is G, i.e.

$$\langle P(\boldsymbol{x}),Q(\boldsymbol{x})\rangle=\sum_{\substack{|\boldsymbol{\alpha}|\leq\deg P\\|\boldsymbol{\beta}|\leq\deg Q}}P_{\boldsymbol{\alpha}}G_{\boldsymbol{\alpha},\boldsymbol{\beta}}Q_{\boldsymbol{\beta}},\quad G_{\boldsymbol{\alpha},\boldsymbol{\beta}}=\big\langle\boldsymbol{x}^{\boldsymbol{\alpha}},\boldsymbol{x}^{\boldsymbol{\beta}}\big\rangle.\tag{9}$$

Whenever the sum $\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{Z}_+^D}P_{\boldsymbol{\alpha}}G_{\boldsymbol{\alpha},\boldsymbol{\beta}}Q_{\boldsymbol{\beta}}$ converges in some sense, the corresponding extension of this bilinear form to the linear space of power series $\mathbb{C}[\![\boldsymbol{x}]\!]$ can be considered.

In general, the semi-infinite matrix G has no further structure and, consequently, we do not expect it to be symmetric or to be related to a linear functional, for example. We say that we are dealing with a non standard bilinear form. The bilinear form (9) induces another bilinear form which is a bilinear map from semi-infinite vectors of polynomials (or power series when possible) into the semi-infinite matrices.

Definition 6. Given two semi-infinite vectors of polynomials $v(\boldsymbol{x})=(v_{\boldsymbol\alpha}(\boldsymbol{x}))_{\boldsymbol\alpha\in\mathbb{Z}_+^D}$ and $w(\boldsymbol{x})=(w_{\boldsymbol\alpha}(\boldsymbol{x}))_{\boldsymbol\alpha\in\mathbb{Z}_+^D}$, with $v_{\boldsymbol\alpha},w_{\boldsymbol\alpha}\in\mathbb{C}[\boldsymbol{x}]$ (or $\mathbb{C}[\![\boldsymbol{x}]\!]$ when possible) we consider the following semi-infinite matrix

$$\big\langle v(\boldsymbol{x}),\big(w(\boldsymbol{x})\big)^\top\big\rangle=\Big(\big\langle v(\boldsymbol{x}),\big(w(\boldsymbol{x})\big)^\top\big\rangle_{\boldsymbol\alpha,\boldsymbol\beta}\Big),\big\langle v(\boldsymbol{x}),\big(w(\boldsymbol{x})\big)^\top\big\rangle_{\boldsymbol\alpha,\boldsymbol\beta}:=\big\langle v_{\boldsymbol\alpha}(\boldsymbol{x}),w_{\boldsymbol\beta}(\boldsymbol{x})\big\rangle,\boldsymbol\alpha,\boldsymbol\beta\in\mathbb{Z}_+^D.$$

A similar definition holds for a polynomial $p(\boldsymbol{x})\in\mathbb{C}[\boldsymbol{x}]$, i.e.,

$$\langle v(\boldsymbol{x}),p(\boldsymbol{x})\rangle:=\Big(\big\langle v_{\boldsymbol\alpha}(\boldsymbol{x}),p(\boldsymbol{x})\big\rangle\Big)_{\boldsymbol\alpha\in\mathbb{Z}_+^D},\quad \langle p(\boldsymbol{x}),(v(\boldsymbol{x}))^\top\rangle:=\bigg(\Big(\big\langle p(\boldsymbol{x}),v_{\boldsymbol\alpha}(\boldsymbol{x})\big\rangle\Big)_{\boldsymbol\alpha\in\mathbb{Z}_+^D}\bigg)^\top.$$

Proposition 6. *Given three semi-infinite vectors* $v^{(i)}(\boldsymbol{x})=\big(v^{(i)}_{\boldsymbol\alpha}(\boldsymbol{x})\big)_{\boldsymbol\alpha\in\mathbb{Z}_+^D}$, $i\in\{1,2,3\}$, *the formulæ*

$$\begin{aligned}\big\langle v^{(1)}(\boldsymbol{x}),\big(v^{(2)}(\boldsymbol{x})\big)^\top\big\rangle v^{(3)}(\boldsymbol{z})&=\big\langle v^{(1)}(\boldsymbol{x}),\big(v^{(2)}(\boldsymbol{x})\big)^\top v^{(3)}(\boldsymbol{z})\big\rangle,\\ \big(v^{(3)}(\boldsymbol{z})\big)^\top\big\langle v^{(1)}(\boldsymbol{z}),\big(v^{(2)}(\boldsymbol{x})\big)^\top\big\rangle&=\big\langle\big(v^{(3)}(\boldsymbol{z})\big)^\top v^{(1)}(\boldsymbol{x}),v^{(2)}(\boldsymbol{x})\big\rangle\end{aligned}\tag{10}$$

hold.

Using this non standard bilinear form we have the corresponding Gram matrix

$$G=\big\langle\chi(\boldsymbol{x}),\big(\chi(\boldsymbol{x})\big)^\top\big\rangle.\tag{11}$$

When there is a linear form $u\in\big(\mathbb{C}[\boldsymbol{x}]\big)'$ such that $\langle P(\boldsymbol{x}),Q(\boldsymbol{x})\rangle=\langle u,P(\boldsymbol{x})Q(\boldsymbol{x})\rangle$ we find that $G=\big\langle u,\chi(\boldsymbol{x})\big(\chi(\boldsymbol{x})\big)^\top\big\rangle$ is the corresponding moment matrix.

Proposition 7. *For any polynomial* $\mathcal{Q}(\boldsymbol{x})\in\mathbb{C}[\boldsymbol{x}]$ *we have*

$$\mathcal{Q}(\boldsymbol{\Lambda})G=\big\langle\mathcal{Q}(\boldsymbol{x})\chi(\boldsymbol{x}),\big(\chi(\boldsymbol{x})\big)^\top\big\rangle,\quad G\big(\mathcal{Q}(\boldsymbol{\Lambda})\big)^\top=\big\langle\chi(\boldsymbol{x}),\big(\chi(\boldsymbol{x})\big)^\top\mathcal{Q}(\boldsymbol{x})\big\rangle.$$

Proof. Use (6). □

2.2. A multivariate 2D Toda hierarchy

In terms of the continuous time parameters sequences $t=\{t_1,t_2\}\subset\mathbb{R}$ given by

$$t_i:=\{t_{i,\boldsymbol\alpha}\}_{\boldsymbol\alpha\in\mathbb{Z}_+^D},\quad i\in\{1,2\},$$

we consider the formal time power series

$$t_i(\boldsymbol{x}):=\sum_{\boldsymbol\alpha\in\mathbb{Z}_+^D}t_{i,\boldsymbol\alpha}\boldsymbol{x}^{\boldsymbol\alpha},\quad i\in\{1,2\},$$

the following vacuum wave semi-infinite matrices

$$W_i^{(0)}(t_i)=\exp\Big(\sum_{\boldsymbol\alpha\in\mathbb{Z}_+^D}t_{i,\boldsymbol\alpha}\boldsymbol{\Lambda}^{\boldsymbol\alpha}\Big),\quad i\in\{1,2\},$$

and the perturbed semi-infinite Gram matrix

$$G(t)=W_1^{(0)}(t_1)G\Big(W_2^{(0)}(t_2)\Big)^{-\top}.\tag{12}$$

Notice that these flows do respect the multi-Hankel condition, if initially we have $\Lambda_aG=G(\Lambda_a)^\top$, $a\in\{1,\dots,D\}$, then, for any further time, we will have $\Lambda_aG(t)=G(t)\big(\Lambda_a\big)^\top$, $a\in\{1,\dots,D\}$. Hence, $G(t)=G(t_1-t_2)$.

We will assume that the block Gauss–Borel factorization do exist, at least for an open subset of times containing $t=0$

$$G(t)=\big(S_1(t)\big)^{-1}H(t)\big(S_2(t)\big)^{-\top}.\tag{13}$$

Then, we consider the semi-infinite vectors of polynomials

$$P_1(t,\boldsymbol{x}):=S_1(t)\chi(\boldsymbol{x}),\quad P_2(t,\boldsymbol{x}):=S_2(t)\chi(\boldsymbol{x}),\tag{14}$$

being its component $P_{i,\boldsymbol\alpha}(t,\boldsymbol{x})$, $i\in\{1,2\}$, $\boldsymbol\alpha\in\mathbb{Z}_+^D$, a t-dependent monic multivariate polynomial in $\boldsymbol{x}$ of degree $|\boldsymbol\alpha|$.

Then, the Gauss–Borel factorization (13) implies the bi-orthogonality condition

$$\big\langle P_{1,[k]}(t,\boldsymbol{x}),P_{2,[l]}(t,\boldsymbol{x})\big\rangle=\delta_{k,l}H_{[k]}(t).$$

Here we used the bilinear form $\langle\cdot,\cdot\rangle$ with Gram matrix $G(t)$. We also consider the wave matrices

$$W_1(t):=S_1(t)W_1^{(0)}(t_1),\quad W_2(t):=\tilde{S}_2(t)\big(W_2^{(0)}(t_2)\big)^\top, \tag{15}$$

where $\tilde{S}_2:=H(t)\big(S_2(t)\big)^{-\top}$.

Proposition 8. *The wave matrices satisfy*

$$\big(W_1(t)\big)^{-1}W_2(t)=G. \tag{16}$$

Proof. It follows from the Gauss–Borel factorization (13). □

Given a semi-infinite matrix A we have unique splitting $A=A_++A_-$ where A_+ is an upper triangular block matrix while is A_- a strictly lower triangular block matrix. The Gauss–Borel factorization (16) has the following differential consequences

Proposition 9. *The following equations hold*

$$\begin{aligned}
&\frac{\partial S_1}{\partial t_{1,\boldsymbol{\alpha}}}(S_1)^{-1}=-\Big(S_1\boldsymbol{\Lambda}^{\boldsymbol{\alpha}}(S_1)^{-1}\Big)_-, && \frac{\partial S_1}{\partial t_{2,\boldsymbol{\alpha}}}(S_1)^{-1}=\Big(\tilde{S}_2\big(\boldsymbol{\Lambda}^\top\big)^{\boldsymbol{\alpha}}(\tilde{S}_2)^{-1}\Big)_-,\\
&\frac{\partial \tilde{S}_2}{\partial t_{1,\boldsymbol{\alpha}}}(\tilde{S}_2)^{-1}=\Big(S_1\boldsymbol{\Lambda}^{\boldsymbol{\alpha}}(S_1)^{-1}\Big)_+, && \frac{\partial \tilde{S}_2}{\partial t_{2,\boldsymbol{\alpha}}}(\tilde{S}_2)^{-1}=-\Big(\tilde{S}_2\big(\boldsymbol{\Lambda}^\top\big)^{\boldsymbol{\alpha}}(\tilde{S}_2)^{-1}\Big)_+.
\end{aligned}$$

Proof. Taking right derivatives of (16) yields

$$\frac{\partial W_1}{\partial t_{i,\boldsymbol{\alpha}}}(W_1)^{-1}=\frac{\partial W_2}{\partial t_{i,\boldsymbol{\alpha}}}(W_2)^{-1}, i\in\{1,2\},\quad j\in\mathbb{Z}_+,$$

where

$$\begin{aligned}
&\frac{\partial W_1}{\partial t_{1,\boldsymbol{\alpha}}}(W_1)^{-1}=\frac{\partial S_1}{\partial t_{1,\boldsymbol{\alpha}}}(S_1)^{-1}+S_1\boldsymbol{\Lambda}^{\boldsymbol{\alpha}}(S_1)^{-1}, && \frac{\partial W_1}{\partial t_{2,\boldsymbol{\alpha}}}(W_1)^{-1}=\frac{\partial S_1}{\partial t_{2,\boldsymbol{\alpha}}}(S_1)^{-1},\\
&\frac{\partial W_2}{\partial t_{1,\boldsymbol{\alpha}}}(W_2)^{-1}=\frac{\partial \tilde{S}_2}{\partial t_{1,\boldsymbol{\alpha}}}(\tilde{S}_2)^{-1}, && \frac{\partial W_2}{\partial t_{2,\boldsymbol{\alpha}}}(W_2)^{-1}=\frac{\partial \tilde{S}_2}{\partial t_{2,\boldsymbol{\alpha}}}(\tilde{S}_2)^{-1}+\tilde{S}_2\big(\boldsymbol{\Lambda}^\top\big)^{\boldsymbol{\alpha}}(\tilde{S}_2)^{-1},
\end{aligned}$$

and the result follows immediately. □

As a consequence, we deduce

Proposition 10. *The multivariate 2D Toda lattice equations*

$$\begin{aligned}
&\frac{\partial}{\partial t_{2,\boldsymbol{e}_b}}\Big(\frac{\partial H_{[k]}}{\partial t_{1,\boldsymbol{e}_a}}(H_{[k]})^{-1}\Big)+(\Lambda_a)_{[k],[k+1]}H_{[k+1]}\Big((\Lambda_b)_{[k],[k+1]}\Big)^\top(H_{[k]})^{-1}\\
&\quad-H_{[k]}\Big((\Lambda_b)_{[k-1],[k]}\Big)^\top(H_{[k-1]})^{-1}(\Lambda_a)_{[k-1],[k]}=0
\end{aligned}$$

hold.

Proof. From Proposition 9 we get

$$\frac{\partial H_{[k]}}{\partial t_{1,\boldsymbol{e}_a}}(H_{[k]})^{-1}=\beta_{[k]}(\Lambda_a)_{[k-1],[k]}-(\Lambda_a)_{[k],[k+1]}\beta_{[k+1]},\ \frac{\partial \beta_{[k]}}{\partial t_{2,\boldsymbol{e}_b}}=H_{[k]}\Big((\Lambda_b)_{[k-1],[k]}\Big)^\top(H_{[k-1]})^{-1},$$

where $\beta_{[k]}\in\mathbb{C}^{|[k]|\times|[k-1]|}$, $k=1,2,\dots$, are the first subdiagonal coefficients in S_1. □

These equations are just the first members of an infinite set of nonlinear partial differential equations, an integrable hierarchy. Its elements are given by

Definition 7. The Lax and Zakharov–Shabat matrices are given by

$$\begin{aligned}
&L_{1,a}:=S_1\Lambda_a(S_1)^{-1}, && L_{2,a}:=\tilde{S}_2(\Lambda_a)^\top(\tilde{S}_2)^{-1},\\
&B_{1,\boldsymbol{\alpha}}:=\big((\boldsymbol{L}_1)^{\boldsymbol{\alpha}}\big)_+, && B_{2,\boldsymbol{\alpha}}:=\big((\boldsymbol{L}_2)^{\boldsymbol{\alpha}}\big)_-.
\end{aligned}$$

The Baker functions are defined as

$$\Psi_1(t,\boldsymbol{z}):=W_1(t)\chi(\boldsymbol{z}),\quad \Psi_2(t,\boldsymbol{z}):=W_2(t)\chi^*(\boldsymbol{z}),$$

and the adjoint Baker functions by

$$\Psi_1^*(t,\boldsymbol{z}):=(W_1(t))^{-\top}\chi^*(\boldsymbol{z}),\quad \Psi_2^*(t,\boldsymbol{z}):=(W_2(t))^{-\top}\chi(\boldsymbol{z}),$$

here we switch for $\boldsymbol{x}\in\mathbb{R}^D$ to $\boldsymbol{z}\in\mathbb{C}$. We also consider the multivariate Cauchy kernel

$$\mathcal{C}(\boldsymbol{z},\boldsymbol{x}):=\frac{1}{\prod_{i=1}^D(z_i-x_i)}.$$

Proposition 11. *The Lax matrices can be written as*

$$L_{1,a}(t)=W_1(t)\Lambda_a(W_1(t))^{-1},\quad L_{2,a}(t)=W_2(t)(\Lambda_a)^\top(W_2(t))^{-1}, \tag{17}$$

and satisfy commutativity properties

$$[L_{1,a}(t),L_{1,b}(t)]=0,\quad [L_{2,a}(t),L_{2,b}(t)]=0,\quad a,b\in\{1,\dots,D\},$$

and the spectral properties

$$L_{1,a}(t)\Psi_1(t,\boldsymbol{x})=x_a\Psi_1(t,\boldsymbol{x}),\quad (L_{2,a}(t))^\top\Psi_2^*(t,\boldsymbol{x})=x_a\Psi_2^*(t,\boldsymbol{x}),\quad a\in\{1,\dots,D\}.$$

The Cauchy kernel satisfies

$$\big(\chi(\boldsymbol{x})\big)^\top\chi^*(\boldsymbol{z})=\mathcal{C}(\boldsymbol{z},\boldsymbol{x}),\quad |z_i|>|x_i|,\quad i\in\{1,\dots,D\}. \tag{18}$$

Theorem 1. *The Baker functions can be expressed in terms of the biorthogonal polynomials, the multivariate Cauchy kernel and the bilinear form as follows*

$$\Psi_1(t,\boldsymbol{z})=\mathrm{e}^{t_1(\boldsymbol{x})}P_1(t,\boldsymbol{z}), \tag{19}$$

$$\Psi_2^*(t,\boldsymbol{z})=\mathrm{e}^{-t_2(\boldsymbol{z})}(H(t))^{-\top}P_2(t,\boldsymbol{z}), \tag{20}$$

$$\Psi_2(t,\boldsymbol{z})=\langle\Psi_1(t,\boldsymbol{x}),\mathcal{C}(\boldsymbol{z},\boldsymbol{x})\rangle,\qquad |z_i|>|x_i|,\qquad i\in\{1,\dots,D\}, \tag{21}$$

$$\big(\Psi_1^*(t,\boldsymbol{z})\big)^\top=\Big\langle\mathcal{C}(\boldsymbol{z},\boldsymbol{x}),\big(\Psi_2^*(t,\boldsymbol{x})\big)^\top\Big\rangle,\qquad |z_i|>|x_i|,\qquad i\in\{1,\dots,D\}, \tag{22}$$

Proof. Eq. (19) follows easily

$$\begin{aligned}
\Psi_1(t,\boldsymbol{x})&=W_1(t)\chi_1(\boldsymbol{x}), && \text{from Definition 7}\\
&=S_1(t)W_1^{(0)}(t_1)\chi(\boldsymbol{x}) && \text{see (15)}\\
&=\mathrm{e}^{t_1(x)}S_1(t)\chi_1(\boldsymbol{x}) && \text{consequence of (6)}\\
&=\mathrm{e}^{t_1(x)}P_1(t,\boldsymbol{x}) && \text{directly from (14).}
\end{aligned}$$

To get (20) we argue similarly

$$\begin{aligned}
\Psi_2^*(t,\boldsymbol{z})&=\big(W_2(t)\big)^{-\top}\chi(\boldsymbol{z}), && \text{from Definition 7,}\\
&=H^{-\dagger}S_2(t)\big(W_2^{(0)}(t_2)\big)^{-1}\chi(\boldsymbol{z}) && \text{see (15)}\\
&=\mathrm{e}^{-\bar t_2(\boldsymbol{z})}H^{-\top}S_2(t)\chi(\boldsymbol{z}) && \text{consequence of (6)}\\
&=\mathrm{e}^{-\bar t_2(\boldsymbol{z})}H^{-\top}P_2(t,\boldsymbol{z}) && \text{follows from (14).}
\end{aligned}$$

To show (21) we proceed as follows, assume that $|z_i|>|x_i|$, $i\in\{1,\dots,D\}$.

$$\begin{aligned}
\Psi_2(t,\boldsymbol{z})&=W_2(t)\chi^*(\boldsymbol{z}) && \text{from Definition 7}\\
&=W_1(t)G\chi^*(\boldsymbol{z}) && \text{use the factorization (16)}\\
&=W_1(t)\Big\langle\chi(\boldsymbol{x}),\big(\chi(\boldsymbol{x})\big)^\top\Big\rangle\chi^*(\boldsymbol{z}) && \text{introduce the bilinear form expression (11)}\\
&=\Big\langle W_1(t)\chi(\boldsymbol{x}),\big(\chi(\boldsymbol{x})\big)^\top\chi^*(\boldsymbol{z})\Big\rangle && \text{use properties (10)}\\
&=\langle\Psi_1(t,\boldsymbol{x}),\mathcal{C}(\boldsymbol{z},\boldsymbol{x})\rangle && \text{consequence of (18) and Definition 7.}
\end{aligned}$$

We now prove (22), for $|z_i|>|x_i|$, $i\in\{1,\dots,D\}$,

$$\begin{aligned}
\Psi_1^*(t,\boldsymbol{z})&=\big(W_1(t)\big)^{-\top}\chi^*(\boldsymbol{z}) && \text{from Definition 7}\\
&=\big(W_2(t)\big)^{-\top}G^\dagger\chi^*(\boldsymbol{z}) && \text{follows from factorization (16)}\\
&=\big(W_2(t)\big)^{-\top}\Big(\big(\chi^*(\boldsymbol{z})\big)^\top G\Big)^\top\\
&=\big(W_2(t)\big)^{-\top}\Big(\Big\langle\big(\chi^*(\boldsymbol{z})\big)^\top\chi(\boldsymbol{x}),\big(\chi(\boldsymbol{x})\big)^\top\Big\rangle\Big)^\top && \text{use the bilinear expression (11)}\\
&=\big(W_2(t)\big)^{-\top}\Big(\Big\langle\mathcal{C}(\boldsymbol{z},\boldsymbol{x}),\big(\chi(\boldsymbol{x})\big)^\top\Big\rangle\Big)^\top && \text{see (18)}\\
&=\Big(\Big\langle\mathcal{C}(\boldsymbol{z},\boldsymbol{x}),\Big(\big(W_2(t)\big)^{-\top}\chi(\boldsymbol{x})\Big)^\top\Big\rangle\Big)^\top\\
&=\Big(\Big\langle\mathcal{C}(\boldsymbol{z},\boldsymbol{x}),\big(\Psi_2^*(t,\boldsymbol{x})\big)^\top\Big\rangle\Big)^\top && \text{from Definition 7, again.}\quad\square
\end{aligned}$$

Proposition 12 (*The Integrable Hierarchy*)**.** *The wave matrices obey the evolutionary linear systems*

$$\frac{\partial W_1}{\partial t_{1,\alpha}}=B_{1,\alpha}W_1,\quad \frac{\partial W_1}{\partial t_{2,\alpha}}=B_{2,\alpha}W_1,\quad \frac{\partial W_2}{\partial t_{1,\alpha}}=B_{1,\alpha}W_2,\quad \frac{\partial W_2}{\partial t_{2,\alpha}}=B_{2,\alpha}W_2,$$

the Baker and adjoint Baker functions solve the following linear equations

$$\frac{\partial\Psi_1}{\partial t_{1,\boldsymbol\alpha}}=B_{1,\boldsymbol\alpha}\Psi_1,\qquad \frac{\partial\Psi_1}{\partial t_{2,\boldsymbol\alpha}}=B_{2,\boldsymbol\alpha}\Psi_1,\qquad \frac{\partial\Psi_2}{\partial t_{1,\boldsymbol\alpha}}=B_{1,\boldsymbol\alpha}\Psi_2,\qquad \frac{\partial\Psi_2}{\partial t_{2,\boldsymbol\alpha}}=B_{2,\boldsymbol\alpha}\Psi_2,$$
$$\frac{\partial\Psi_1^*}{\partial t_{1,\boldsymbol\alpha}}=-(B_{1,\boldsymbol\alpha})^\top\Psi_1,\qquad \frac{\partial\Psi_1}{\partial t_{2,\boldsymbol\alpha}}=-(B_{2,\boldsymbol\alpha})^\top\Psi_1^*,\qquad \frac{\partial\Psi_2^*}{\partial t_{1,\boldsymbol\alpha}}=-(B_{1,\boldsymbol\alpha})^\top\Psi_2^*,\qquad \frac{\partial\Psi_2^*}{\partial t_{2,\boldsymbol\alpha}}=-(B_{2,\boldsymbol\alpha})^\top\Psi_2^*,$$

the Lax matrices are subject to the following Lax equations

$$\frac{\partial L_{i,a}}{\partial t_{j,\boldsymbol\alpha}}=\left[B_{j,\boldsymbol\alpha},L_{i,a}\right],$$

and Zakharov–Sabat matrices fulfill the following Zakharov–Shabat equations

$$\frac{\partial B_{i',\boldsymbol\alpha'}}{\partial t_{i,\boldsymbol\alpha}}-\frac{\partial B_{i,\boldsymbol\alpha}}{\partial t_{i',\boldsymbol\alpha'}}+\left[B_{i,\boldsymbol\alpha},B_{i',\boldsymbol\alpha'}\right]=0.$$

Proof. Follows from Proposition 9. □

2.3. KP type hierarchies

In [1] it is shown that the KP type construction appears also in the MVOP context. Here we show that they admit an extension to this broader scenario not linked to MVOP of multispectral Toda hierarchies.

Definition 8. Given two semi-infinite matrices $Z_1(t)$ and $Z_2(t)$ we say that

- $Z_1(t)\in \mathfrak{l}W_1^{(0)}$ if $Z_1(t)\big(W_1^{(0)}(t_1)\big)^{-1}$ is a block strictly lower triangular matrix.
- $Z_2(t)\in \mathfrak{u}W_2^{(0)}$ if $Z_2(t)\big(W_2^{(0)}(t_2)\big)^{-\top}$ is a block upper triangular matrix.

Then, we can state the following *congruences*

Proposition 13. *Given two semi-infinite matrices $Z_1(t)$ and $Z_2(t)$ such that*

- $Z_1(t)\in \mathfrak{l}W_1^{(0)}$,
- $Z_2(t)\in \mathfrak{u}W_2^{(0)}$,
- $Z_1(t)G=Z_2(t)$.

then

$$Z_1(t)=0,Z_2(t)=0.$$

Proof. Observe that

$$Z_2(t)=Z_1(t)G=Z_1(t)\big(W_1(t)\big)^{-1}W_2(t),$$

where we have used (16). From here we get

$$Z_1(t)\big(W_1^{(0)}(t_1)\big)^{-1}\big(S_1(t)\big)^{-1}=Z_2(t)\big(W_2^{(0)}(t_2)\big)^{-\top}\big(\tilde S_2(t)\big)^{-1},$$

and, as in the LHS we have a strictly lower triangular block semi-infinite matrix while in the RHS we have an upper triangular block semi-infinite matrix, both sides must vanish and the result follows. □

Let us write $t_{i,(a_1,a_2,\dots,a_p)}$ to denote $t_{i,\boldsymbol\alpha}$ with $\boldsymbol\alpha=\boldsymbol e_{a_1}+\cdots+\boldsymbol e_{a_p}$ and introduce the diagonal block matrices $V_{a,b}=\operatorname{diag}((V_{a,b})_{[0]},(V_{a,b})_{[1]},(V_{a,b})_{[2]},\dots)$

$$V_{a,b}:=\frac{\partial\beta_1}{\partial t_{1,a}}\Lambda_b,\quad (V_{a,b})_{[k]}=\frac{\partial\beta_{1,[k]}}{\partial t_{1,a}}(\Lambda_b)_{[k-1],[k]},\quad U_{a,b}:=-V_{a,b}-V_{b,a},\tag{23}$$

in terms of the first block subdiagonal β_1 of S_1.

Proposition 14. *The Baker function Ψ_1 satisfies*

$$\frac{\partial\Psi_1}{\partial t_{1,(a,b)}}=\frac{\partial^2\Psi_1}{\partial t_{1,a}\partial t_{2,b}}+U_{a,b}\Psi_1.\tag{24}$$

Proof. On the one hand,

$$\frac{\partial W_1}{\partial t_{1,(a,b)}}=\Big(\frac{\partial S_1}{\partial t_{1,(a,b)}}+S_1\Lambda_a\Lambda_b\Big)W_1^{(0)}(t_1)$$
$$\frac{\partial^2 W_1}{\partial t_{1,a}\partial t_{1,b}}=\Big(\frac{\partial^2 S_1}{\partial t_{1,a}\partial t_{1,b}}+\frac{\partial S_1}{\partial t_{1,a}}\Lambda_b+\frac{\partial S_1}{\partial t_{1,b}}\Lambda_a+S_1\Lambda_a\Lambda_b\Big)W_1^{(0)}(t_1)$$

so that

$$\Big(\frac{\partial}{\partial t_{1,(a,b)}}-\frac{\partial^2}{\partial t_{1,a}\partial t_{1,b}}\Big)(W_1)=-\Big(\frac{\partial \beta_1}{\partial t_{1,a}}\Lambda_b+\frac{\partial \beta_1}{\partial t_{1,b}}\Lambda_a\Big)W_1^{(0)}(t_1)+\mathfrak{l}W_1^{(0)}$$

and, consequently,

$$\Big(\frac{\partial}{\partial t_{1,(a,b)}}-\frac{\partial^2}{\partial t_{1,a}\partial t_{1,b}}+\frac{\partial \beta_1}{\partial t_{1,a}}\Lambda_b+\frac{\partial \beta_1}{\partial t_{1,b}}\Lambda_a\Big)(W_1)=\mathfrak{l}W_1^{(0)}.$$

On the other hand,

$$\frac{\partial W_2}{\partial t_{1,(a,b)}}=\frac{\partial \tilde S_2}{\partial t_{1,(a,b)}}W_2^{(0)}(t_2),\ \frac{\partial^2 W_2}{\partial t_{1,a}\partial t_{1,b}}=\frac{\partial^2 \tilde S_2}{\partial t_{1,a}\partial t_{1,b}}W_2^{(0)}(t_2)$$

Now, we apply Proposition 13 with

$$Z_i=\Big(\frac{\partial}{\partial t_{1,(a,b)}}-\frac{\partial^2}{\partial t_{1,a}\partial t_{1,b}}-U_{a,b}\Big)(W_i),\quad i=1,2,$$

to get the result. □

Proceeding similarly we can reproduce the results of [1] for this more general case. The proofs are essentially as are there with slight modifications as just shown in the above developments. Associated with the third order times $t_{1,(a,b,c)}$ we introduce the following block diagonal matrices

$$V_{a,b,c}=\operatorname{diag}((V_{a,b,c})_{[0]},(V_{a,b,c})_{[1]},(V_{a,b,c})_{[2]},\dots)$$

with

$$\begin{aligned}V_{a,b,c}:=&\frac{\partial \beta_1^{(2)}}{\partial t_a}\Lambda_b\Lambda_c-\frac{\partial \beta_1}{\partial t_{1,a}}\Lambda_b\beta_1\Lambda_c,\\(V_{a,b,c})_{[k]}=&\Big(\frac{\partial \beta_{1,[k]}^{(2)}}{\partial t_{1,a}}(\Lambda_b)_{[k-2],[k-1]}-\frac{\partial \beta_{1,[k]}}{\partial t_a}(\Lambda_b)_{[k-1],[k]}\beta_{1,[k]}\Big)(\Lambda_c)_{[k-1],[k]},\end{aligned}$$

The Baker functions Ψ_1 satisfies the third order linear differential equations

$$\begin{aligned}\frac{\partial \Psi_1}{\partial t_{1,(a,b,c)}}=&\frac{\partial^3\Psi_1}{\partial t_{1,a}\partial t_{1,b}\partial t_{1,c}}-V_{a,b}\frac{\partial \Psi}{\partial t_c}-V_{c,a}\frac{\partial \Psi}{\partial t_{1,b}}-V_{b,c}\frac{\partial \Psi}{\partial t_{1,a}}\\&-\Big(\frac{\partial V_{a,b}}{\partial t_{1c}}+\frac{\partial V_{b,c}}{\partial t_{1,a}}+\frac{\partial V_{c,a}}{\partial t_{1,b}}+V_{a,b,c}+V_{b,c,a}+V_{c,b,a}\Big)\Psi_1,\end{aligned}$$

and a matrix type KP system of equations for $\beta_{1,[k]}$ and $\beta^{(2)}_{1,[k]}$ emerges [1]. For example, if we denote $t^{(3)}_{1,a}=t_{3,(a,a,a)}$ and $t^{(2)}_{1,a}=t_{1,(a,a)}$ we get the nonlinear partial differential system

$$\begin{aligned}0=&\frac{\partial}{\partial t_{1,a}}\left[\frac{\partial \beta_1}{\partial t_{1,a}}\Lambda_a\beta_1-\frac{\partial \beta_1^{(2)}}{\partial t_{1,a}}\Lambda_a-\frac{1}{2}\frac{\partial^2\beta_1}{\partial t_{1,a}^2}+\frac{1}{4}\frac{\partial \beta_1}{\partial t^{(2)}_{1,a}}\right],\\0=&3\frac{\partial^2}{\partial t_{1,a}^2}\left[\frac{1}{2}\frac{\partial \beta_1}{\partial t^{(2)}_{1,a}}-\frac{\partial^2\beta_1}{\partial t_{1,a}^2}+2\frac{\partial \beta_1}{\partial t_{1,a}}\Lambda_a\beta_1\right]\Lambda_a\\&+\frac{\partial}{\partial t_{1,a}}\left[2\frac{\partial^3\beta_1}{\partial t_{1,a}^3}-\frac{\partial \beta_1}{\partial t^{(3)}_{1,a}}+\left(\frac{\partial \beta}{\partial t_{1,a}}\Lambda_a\beta_1-\frac{\partial \beta_1^{(2)}}{\partial t_{1,a}}\Lambda_a\right)\Lambda_a\beta_1\right]\Lambda_a\\&+3\frac{\partial}{\partial t_{1,a}}\left[\left(2\frac{\partial \beta_1}{\partial t_{1,a}}\Lambda_a\beta_1^{(2)}+\frac{1}{2}\frac{\partial \beta_1^{(2)}}{\partial t^{(2)}_{1,a}}-\frac{\partial^2\beta_1^{(2)}}{\partial t_{1,a}^2}\right)\Lambda_a^2-2\frac{\partial \beta_1}{\partial t_{1,a}}\Lambda_a\frac{\partial \beta_1}{\partial t_{1,a}}\Lambda_a\right]\\&+3\frac{\partial \beta_1}{\partial t_{1,a}}\Lambda_a\left[\frac{\partial^2\beta_1}{\partial t_{1,a}^2}-2\frac{\partial \beta_1}{\partial t_{1,a}}\Lambda_a\beta_1-\frac{1}{2}\frac{\partial \beta_1}{\partial t_{1,a}}\right]\Lambda_a-6\frac{\partial^2\beta_1}{\partial t_{1,a}^2}\Lambda_a\beta_1^{(2)}(\Lambda_a)^2.\end{aligned}$$

2.4. Reductions

Let us explore some simple reductions.

Definition 9. Given two polynomials $\mathcal{Q}_1(\boldsymbol{x}),\mathcal{Q}_2(\boldsymbol{x})\in\mathbb{C}[\boldsymbol{x}]$ a semi-infinite matrix G is said to be $(\mathcal{Q}_1,\mathcal{Q}_2)$-invariant if

$$\mathcal{Q}_1(\boldsymbol{\Lambda})G=G\mathcal{Q}_2(\boldsymbol{\Lambda}^\top)\tag{25}$$

We will use the notation

$$\boldsymbol{L}_1:=(L_{1,1},\dots,L_{1,D})^\top,\quad \boldsymbol{L}_2:=(L_{2,1},\dots,L_{2,D})^\top.$$

Observe that according to Proposition 7 this reduction implies for the associated bilinear forms

$$\left\langle \mathcal{Q}_1(\boldsymbol{x})\chi(\boldsymbol{x}), \big(\chi(\boldsymbol{x})\big)^\top\right\rangle = \left\langle \chi(\boldsymbol{x}), \big(\chi(\boldsymbol{x})\big)^\top \mathcal{Q}_2(\boldsymbol{x})\right\rangle.$$

Proposition 15. *Given two polynomials $\mathcal{Q}_1(\boldsymbol{x}), \mathcal{Q}_2(\boldsymbol{x}) \in \mathbb{C}[\boldsymbol{x}]$, with powers written as*

$$(\mathcal{Q}_1(\boldsymbol{x}))^n = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}_+^D} \mathcal{Q}^n_{1,\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}}, \quad (\mathcal{Q}_2(\boldsymbol{x}))^n = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}_+^D} \mathcal{Q}^n_{2,\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}}$$

and a $(\mathcal{Q}_1, \mathcal{Q}_2)$-invariant initial condition G we find that

(i) The Lax semi-infinite matrices satisfy

$$\mathcal{Q}_1(\boldsymbol{L}_1) = \mathcal{Q}_2(\boldsymbol{L}_2). \tag{26}$$

(ii) For $n \in \{1, 2, \dots\}$ the wave matrices satisfy

$$\begin{aligned}
&\sum_{\boldsymbol{\alpha}\in\mathbb{Z}_+^D} \mathcal{Q}^n_{1,\boldsymbol{\alpha}}\frac{\partial W_1}{\partial t_{1,\boldsymbol{\alpha}}} + \sum_{\boldsymbol{\alpha}\in\mathbb{Z}_+^D} \mathcal{Q}^n_{2,\boldsymbol{\alpha}}\frac{\partial W_1}{\partial t_{2,\boldsymbol{\alpha}}} = W_1\big(\mathcal{Q}_1(\boldsymbol{\Lambda})\big)^n,\\
&\sum_{\boldsymbol{\alpha}\in\mathbb{Z}_+^D} \mathcal{Q}^n_{1,\boldsymbol{\alpha}}\frac{\partial W_2}{\partial t_{1,\boldsymbol{\alpha}}} + \sum_{\boldsymbol{\alpha}\in\mathbb{Z}_+^D} \mathcal{Q}^n_{2,\boldsymbol{\alpha}}\frac{\partial W_2}{\partial t_{2,\boldsymbol{\alpha}}} = W_2\big(\mathcal{Q}_2(\boldsymbol{\Lambda}^\top)\big)^n,
\end{aligned} \tag{27}$$

and the Lax matrices fulfill the invariance conditions

$$\begin{aligned}
&\sum_{\boldsymbol{\alpha}\in\mathbb{Z}_+^D} \mathcal{Q}^n_{1,\boldsymbol{\alpha}}\frac{\partial \boldsymbol{L}_1}{\partial t_{1,\boldsymbol{\alpha}}} + \sum_{\boldsymbol{\alpha}\in\mathbb{Z}_+^D} \mathcal{Q}^n_{2,\boldsymbol{\alpha}}\frac{\partial \boldsymbol{L}_1}{\partial t_{2,\boldsymbol{\alpha}}} = 0,\\
&\sum_{\boldsymbol{\alpha}\in\mathbb{Z}_+^D} \mathcal{Q}^n_{1,\boldsymbol{\alpha}}\frac{\partial \boldsymbol{L}_2}{\partial t_{1,\boldsymbol{\alpha}}} + \sum_{\boldsymbol{\alpha}\in\mathbb{Z}_+^D} \mathcal{Q}^n_{2,\boldsymbol{\alpha}}\frac{\partial \boldsymbol{L}_2}{\partial t_{2,\boldsymbol{\alpha}}} = 0.
\end{aligned} \tag{28}$$

Proof.

(i) Use (16), (17) and (25) for (26).

(ii) Observe that

$$\sum_{\boldsymbol{\alpha}\in\mathbb{Z}_+^D} \mathcal{Q}^n_{1,\boldsymbol{\alpha}}B_{1,\boldsymbol{\alpha}} = \Big(\big(\mathcal{Q}_1(\boldsymbol{L}_1)\big)^n\Big)_+, \sum_{\boldsymbol{\alpha}\in\mathbb{Z}_+^D} \mathcal{Q}^n_{2,\boldsymbol{\alpha}}B_{2,\boldsymbol{\alpha}} = \Big(\big(\mathcal{Q}_1(\boldsymbol{L}_2)\big)^n\Big)_-$$

and, consequently,

$$\sum_{\boldsymbol{\alpha}\in\mathbb{Z}_+^D} \mathcal{Q}^n_{1,\boldsymbol{\alpha}}B_{1,\boldsymbol{\alpha}} + \sum_{\boldsymbol{\alpha}\in\mathbb{Z}_+^D} \mathcal{Q}^n_{2,\boldsymbol{\alpha}}B_{2,\boldsymbol{\alpha}} = \big(\mathcal{Q}_1(\boldsymbol{L}_1)\big)^n = \big(\mathcal{Q}_2(\boldsymbol{L}_2)\big)^n,$$

and systems (27) and (28) follow from Proposition 12. □

An illustration of these type of reductions is the case studied in previous sections involving multivariate orthogonal polynomials, for a given linear functional $u \in (\mathbb{C}[x])'$, with $G = \langle u, \chi\chi^\top\rangle$. the matrix of bimoments. As we know this implies $\Lambda_a G = G(\Lambda_a)^\top$, $a \in \{1, \dots, D\}$, so that $L_{1,a} = S_1\Lambda S_1^{-1} = \tilde{S}_2\Lambda^\top\tilde{S}_2^{-1} = L_{2,a}$, $a \in \{1, \dots, D\}$. The Lax matrices $L_{1,a}$ and $L_{2,a}$ are lower and upper Hessenberg block matrices, respectively. Consequently, we have a tridiagonal block matrix form; i.e., a Jacobi block matrix

$$\boldsymbol{L}_1 = \boldsymbol{L}_2 = \boldsymbol{J}.$$

Moreover, these conditions imply an invariance property under the flows introduced, as we have that $G(t) = W_1^{(0)}(t_1 - t_2)G$, i.e., there are only one type of flows, or in differential form

$$(\partial_{1,\boldsymbol{\alpha}} + \partial_{2,\boldsymbol{\alpha}})W_1 = W_1\boldsymbol{\Lambda}^{\boldsymbol{\alpha}}, \qquad (\partial_{1,\boldsymbol{\alpha}} + \partial_{2,\boldsymbol{\alpha}})W_2 = W_2(\boldsymbol{\Lambda}^\top)^{\boldsymbol{\alpha}},$$

$$(\partial_{1,\boldsymbol{\alpha}} + \partial_{2,\boldsymbol{\alpha}})L_{1,a} = 0, \qquad (\partial_{1,\boldsymbol{\alpha}} + \partial_{2,\boldsymbol{\alpha}})L_{2,a} = 0.$$

2.5. The linear spectral transformation for the multivariate 2D Toda hierarchy

Following [53], we give the linear spectral transform for the multivariate Toda lattice just discussed. As a main result in Theorem 2 we get quasi-determinantal expressions for the transformed Baker function $(\hat{\Psi}_1)_{[k]}(t)$ and the quasi-tau matrices $\hat{H}_{[k]}(t)$.

Definition 10. Given two coprime polynomials $\mathcal{Q}_1(\boldsymbol{x})$ and $\mathcal{Q}_2(\boldsymbol{x})$, $\deg \mathcal{Q}_i = m_i$, we consider an initial condition G and a perturbed one $\hat{G}$ such that

$$\hat{G}\mathcal{Q}_2(\boldsymbol{\Lambda}^\top) = \mathcal{Q}_1(\boldsymbol{\Lambda})G. \tag{29}$$

We can achieve the perturbed semi-infinite matrix $\hat{G}$ in two steps, using an intermediate matrix $\check{G}$. First, we perform a Geronimus type transformation

$$\check{G}\mathcal{Q}_2(\boldsymbol{\Lambda}^\top) = G \tag{30}$$

and second, a Christoffel type transformation

$$\hat{G}=\mathcal{Q}_1(\boldsymbol{\Lambda})\check{G}. \tag{31}$$

Proposition 16. *Under the evolution prescribed in (12) if (29), (30)* and *(31) we have*

$$\hat{G}(t)\mathcal{Q}_2(\boldsymbol{\Lambda}^\top)=\mathcal{Q}_1(\boldsymbol{\Lambda})G(t),\quad \check{G}(t)\mathcal{Q}_2(\boldsymbol{\Lambda}^\top)=G(t),\quad \hat{G}(t)=\mathcal{Q}_1(\boldsymbol{\Lambda})\check{G}(t).$$

Proof. We just check the first as the others follow in an analogous manner:

$$\begin{aligned}\hat{G}(t)\mathcal{Q}_2(\boldsymbol{\Lambda}^\top)=&W_1^{(0)}(t_1)\hat{G}\big(W_2^{(0)}(t_2)\big)^{-\top}\mathcal{Q}_2(\boldsymbol{\Lambda}^\top)\\ =&W_1^{(0)}(t_1)\hat{G}\mathcal{Q}_2(\boldsymbol{\Lambda}^\top)\big(W_2^{(0)}(t_2)\big)^{-\top}\\ =&W_1^{(0)}(t_1)\mathcal{Q}_1(\boldsymbol{\Lambda})G\big(W_2^{(0)}(t_2)\big)^{-\top}\\ =&\mathcal{Q}_1(\boldsymbol{\Lambda})G(t).\quad\square\end{aligned}$$

In terms of bilinear forms (30) reads

$$\big\langle\chi(\boldsymbol{x}),\big(\chi(\boldsymbol{x})\big)^\top\mathcal{Q}_2(\boldsymbol{x})\big\rangle^{\check{}}=\big\langle\chi(\boldsymbol{x}),\big(\chi(\boldsymbol{x})\big)^\top\big\rangle$$

so that assuming we can divide by polynomials inside these bilinear forms a solution to (30) is

$$\check{G}=\Big\langle\chi(\boldsymbol{x}),\frac{\big(\chi(\boldsymbol{x})\big)^\top}{\mathcal{Q}_2(\boldsymbol{x})}\Big\rangle+\big\langle v,\chi(\boldsymbol{x})\big(\chi(\boldsymbol{x})\big)^\top\big\rangle, \tag{32}$$

where $v\in\big(\mathbb{C}[x]\big)'$ and $(\mathcal{Q}_2(\boldsymbol{x}))\subset\operatorname{Ker}(v)$. In fact, a more general case will be

$$\check{G}=\Big\langle\chi(\boldsymbol{x}),\frac{\big(\chi(\boldsymbol{x})\big)^\top}{\mathcal{Q}_2(\boldsymbol{x})}\Big\rangle+\big\langle, v,A\chi(\boldsymbol{x})\big(\chi(\boldsymbol{x})\big)^\top\big\rangle$$

where A is a semi-infinite matrix with rows having only a finite number of non vanishing coefficients.

Definition 11. We introduce the resolvent matrices

$$\omega_1(t):=\hat{S}_1(t)\mathcal{Q}_1(\boldsymbol{\Lambda})\big(S_1(t)\big)^{-1},\quad \omega_2(t):=\Big(S_2(t)\mathcal{Q}_2(\boldsymbol{\Lambda})\big(\hat{S}_2(t)\big)^{-1}\Big)^\top.$$

Proposition 17. *The resolvent matrices satisfy*

$$\hat{H}(t)\omega_2(t)=\omega_1(t)H(t). \tag{33}$$

Matrices $\omega_1(t),\omega_2(t)$ are block banded matrices, having as non-zero blocks only the first m_1 block superdiagonals and the first m_2 block subdiagonals. The blocks are of increasing size.

Proof. From the *LU* factorization we get

$$\big(\hat{S}_1(t)\big)^{-1}\hat{H}(t)\big(\hat{S}_2(t)\big)^{-\top}\mathcal{Q}_2(\boldsymbol{\Lambda}^\top)=\mathcal{Q}_1(\boldsymbol{\Lambda})\big(S_1(t)\big)^{-1}H(t)\big(S_2(t)\big)^{-\top},$$

so that

$$\hat{H}(t)\Big(S_2(t)\mathcal{Q}_2(\boldsymbol{\Lambda})\big(\hat{S}_2(t)\big)^{-1}\Big)^\top=\hat{S}_1(t)\mathcal{Q}_1(\boldsymbol{\Lambda})\big(S_1(t)\big)^{-1}H(t).\quad\square$$

We have the following important relation connecting the original multivariate orthogonal polynomials and the perturbed ones

Proposition 18 (*Connection Formulas*)**.** *We have*

$$\begin{aligned}\omega_1(t)P_1(t,\boldsymbol{x})&=\mathcal{Q}_1(\boldsymbol{x})\hat{P}_1(t,\boldsymbol{x}),\\ \big(\omega_2(t)\big)^\top\hat{P}_2(t,\boldsymbol{x})&=\mathcal{Q}_2(\boldsymbol{x})P_2(t,\boldsymbol{x}).\end{aligned}$$

Definition 12.

(i) We introduce the semi-infinite matrix

$$R(t):=S_1(t)\check{G}(t). \tag{34}$$

(ii) If $k>m_2$, take an ordered set of multi-indices

$$\mathcal{M}_k:=\big\{\boldsymbol{\beta}_i\in(\mathbb{Z}_+)^D:|\boldsymbol{\beta}_i|<k\big\}_{i=1}^{r_{k,m_2}},$$

with cardinal given by

$$r_{k,m_2}:=|\mathcal{M}_k|=N_{k-1}-N_{k-m_2-1}=|[k-m_2]|+\cdots+|[k-1]|.$$

(iii) Associated with this set consider the truncations

$$R^{[\mathcal{M}_k]} := \begin{pmatrix} R_{[k-m_2],\boldsymbol{\beta}_1} & \dots & R_{[k-m_2],\boldsymbol{\beta}_{r_{k,m_2}}} \\ \vdots & & \vdots \\ R_{[k-1],\boldsymbol{\beta}_1} & \dots & R_{[k-1],\boldsymbol{\beta}_{r_{k,m_2}}} \end{pmatrix},$$
$$R_{\mathcal{M}_k} := (R_{[k],\boldsymbol{\beta}_1}, \dots, R_{[k],\boldsymbol{\beta}_{r_{k,m_2}}})$$

(iv) Then, the set $\mathcal{M}_k$ is said to be poised if the corresponding truncation is not singular

$$\begin{vmatrix} R_{[k-m_2],\boldsymbol{\beta}_1} & \dots & R_{[k-m_2],\boldsymbol{\beta}_{r_{k,m_2}}} \\ \vdots & & \vdots \\ R_{[k-1],\boldsymbol{\beta}_1} & \dots & R_{[k-1],\boldsymbol{\beta}_{r_{k,m_2}}} \end{vmatrix} \neq 0.$$

Proposition 19. *The matrix $R(t)$ can be expressed as follows*

$$R(t) = \left\langle P_1(t,\boldsymbol{x}), \frac{\big(\chi(\boldsymbol{x})\big)^\top}{\mathcal{Q}_2(\boldsymbol{x})}\right\rangle + \big\langle v, P_1(t,\boldsymbol{x})\big(\chi(\boldsymbol{x})\big)^\top\big\rangle. \tag{35}$$

Proof. Recall (32) and (34). □

Proposition 20. *We have the following relations*

$$(\omega_1(t)R(t))_{[k],[l]} = 0, \qquad l = 0, 1, \dots, k-1,$$
$$(\omega_1(t)R(t))_{[k],[k]} = \hat{H}_{[k]}(t).$$

Proof. Just follow the following chain of equalities

$$\begin{aligned} \omega_1(t)R(t) &= \hat{S}_1(t)\mathcal{Q}_1(\boldsymbol{\Lambda})(S_1(t))^{-1}S_1(t)\check{G}(t) \\ &= \hat{S}_1(t)\mathcal{Q}_1(\boldsymbol{\Lambda})\check{G}(t) \\ &= \hat{S}_1(t)\hat{G}(t) && \text{from (16)} \\ &= \hat{H}(t)(\hat{S}_2(t))^{-\top}, \end{aligned} \tag{36}$$

and the matrix $\omega_1 R$ is an upper triangular block matrix with $\hat{H}$ as its block diagonal. □

For $k < m_2$ we can write

$$\begin{aligned} &\big((\omega_1)_{[k],[0]}(t), \dots, (\omega_1)_{[k],[k+m_1-1]}(t)\big) = \\ &-\big(\mathcal{Q}_1(\boldsymbol{\Lambda})\big)_{[k],[k+m_1]}\big(R_{[k+m_1],[0]}(t), \dots, R_{[k+m_1],[k-1]}(t), \Psi_{1,[k+m_1]}(t,\boldsymbol{p}_1), \dots, \Psi_{[k+m_1]}(t,\boldsymbol{p}_{r_{1|k,m_1}})\big) \\ &\times \begin{pmatrix} R_{[k],[0]}(t) & \dots & R_{[0],[k-1]}(t) & \Psi_{1,[k]}(t,\boldsymbol{p}_1) & \dots & \Psi_{1,[k]}(t,\boldsymbol{p}_{r_{1|k,m_1}}) \\ \vdots & & \vdots & \vdots & & \vdots \\ R_{[k+m_1-1],[0]}(t) & \dots & R_{[k+m_1-1],[k-1]}(t) & \Psi_{1,[k+m_1-1]}(t,\boldsymbol{p}_1) & \dots & \Psi_{1,[k+m_1-1]}(t,\boldsymbol{p}_{r_{1|k,m_1}}) \end{pmatrix}^{-1}, \end{aligned}$$

while for $k \geq m_2$

$$\begin{aligned} &\big((\omega_1)_{[k],[k-m_2]}(t), \dots, (\omega_1)_{[k],[k+m_1-1]}(t)\big) = \\ &-\big(\mathcal{Q}_1(\boldsymbol{\Lambda})\big)_{[k],[k+m_1]}\big(R_{[k+m_1],\boldsymbol{\beta}_1}(t), \dots, R_{[k+m_1],\boldsymbol{\beta}_{r_{2|k,m_2}}}(t), \Psi_{1,[k+m_1]}(t,\boldsymbol{p}_1), \dots, \Psi_{1,[k+m_1]}(t,\boldsymbol{p}_{r_{1|k,m_1}})\big) \\ &\times \begin{pmatrix} R_{[k-m_2],\boldsymbol{\beta}_1}(t) & \dots & R_{[k-m_2],\boldsymbol{\beta}_{r_{2|k,m_2}}}(t) & \Psi_{1,[k-m_2]}(t,\boldsymbol{p}_1) & \dots & \Psi_{1,[k-m_2]}(t,\boldsymbol{p}_{r_{1|k,m_1}}) \\ \vdots & & \vdots & \vdots & & \vdots \\ R_{[k+m_1-1],\boldsymbol{\beta}_1}(t) & \dots & R_{[k+m_1-1],\boldsymbol{\beta}_{r_{2|k,m_2}}}(t) & \Psi_{1,[k+m_1-1]}(t,\boldsymbol{p}_1) & \dots & \Psi_{1,[k+m_1-1]}(t,\boldsymbol{p}_{r_{1|k,m_1}}) \end{pmatrix}^{-1}. \end{aligned}$$

We also have

$$(\omega_1(t))_{[k],[k+m_1]} = \big(\mathcal{Q}_1(\boldsymbol{\Lambda})\big)_{[k],[k+m_1]}.$$

Now we give the quasideterminantal expressions for the perturbed Baker–Akhiezer functions in terms of the original ones and the R factors:

Theorem 2 (*Christoffel–Geronimus–Uvarov Formula for Multispectral Toda Hierarchy*)**.** *A linear spectral transformation, as in (29), for the multispectral Toda hierarchy has the following effects on the Baker function $\Psi_{1,[k]}(t)$ and the quasi-tau matrices $H_{[k]}(t)$. Given a poised set*

$\mathcal{S}_k$, of multi-indices and nodes, we have a perturbed Baker function

$$\hat{\Psi}_{1,[k]}(t,\boldsymbol{x})=\frac{\big(\mathcal{Q}_1(\boldsymbol{\Lambda})\big)_{[k],[k+m_1]}}{\mathcal{Q}_1(\boldsymbol{x})}\times\Theta_*\begin{pmatrix}R_{[0],[0]}(t)&\dots&R_{[0],[k-1]}(t)&\Psi_{1,[0]}(t,\boldsymbol{p}_1)&\dots&\Psi_{1,[0]}(t,\boldsymbol{p}_{r_{1|k,m_1}})&\Psi_{1,[0]}(t,\boldsymbol{x})\\\vdots&&\vdots&\vdots&&\vdots&\vdots\\R_{[k+m_1],[k-1]}(t)&\dots&R_{[k+m_1],[k-1]}(t)&\Psi_{1,[k+m_1]}(t,\boldsymbol{p}_1)&\dots&\Psi_{1,[k+m_1]}(t,\boldsymbol{p}_{r_{1|k,m_1}})&\Psi_{1,[k+m_1]}(t,\boldsymbol{x})\end{pmatrix},$$

and a perturbed quasi-tau matrix

$$\hat{H}_{[k]}(t)=\big(\mathcal{Q}_1(\boldsymbol{\Lambda})\big)_{[k],[k+m_1]}\times\Theta_*\begin{pmatrix}R_{[0],[0]}(t)&\dots&R_{[0],[k-1]}(t)&\Psi_{1,[0]}(t,\boldsymbol{p}_1)&\dots&\Psi_{1,[0]}(t,\boldsymbol{p}_{r_{1|k,m_1}})&R_{[0],[k]}(t)\\\vdots&&\vdots&\vdots&&\vdots&\vdots\\R_{[k+m_1],[k-1]}(t)&\dots&R_{[k+m_1],[k-1]}(t)&\Psi_{1,[k+m_1]}(t,\boldsymbol{p}_1)&\dots&\Psi_{1,[k+m_1]}(t,\boldsymbol{p}_{r_{1|k,m_1}})&R_{[k+m_1],[k]}(t)\end{pmatrix}.$$

When $k\geq m_2$ we have the shorter alternative expressions

$$\hat{\Psi}_{1,[k]}(t,\boldsymbol{x})=\frac{\big(\mathcal{Q}_1(\boldsymbol{\Lambda})\big)_{[k],[k+m_1]}}{\mathcal{Q}_1(\boldsymbol{x})}\times\Theta_*\begin{pmatrix}R_{[k-m_2],\boldsymbol{\beta}_1}(t)&\dots&R_{[k-m_2],\boldsymbol{\beta}_{r_{2|k,m_2}}}(t)&\Psi_{1,[k-m_2]}(t,\boldsymbol{p}_1)&\dots&\Psi_{1,[k-m_2]}(t,\boldsymbol{p}_{r_{1|k,m_1}})&\Psi_{1,[k-m_2]}(t,\boldsymbol{x})\\\vdots&&\vdots&\vdots&&\vdots&\vdots\\R_{[k+m_1],\boldsymbol{\beta}_1}(t)&\dots&R_{[k+m_1],\boldsymbol{\beta}_{r_{2|k,m_2}}}(t)&\Psi_{1,[k+m_1]}(t,\boldsymbol{p}_1)&\dots&\Psi_{1,[k+m_1]}(t,\boldsymbol{p}_{r_{1|k,m_1}})&\Psi_{1,[k+m_1]}(t,\boldsymbol{x})\end{pmatrix},$$

$$\hat{H}_{[k]}(t)=\big(\mathcal{Q}_1(\boldsymbol{\Lambda})\big)_{[k],[k+m_1]}\times\Theta_*\begin{pmatrix}R_{[k-m_2],\boldsymbol{\beta}_1}(t)&\dots&R_{[k-m_2],\boldsymbol{\beta}_{r_{2|k,m_2}}}(t)&\Psi_{1,[k-m_2]}(t,\boldsymbol{p}_1)&\dots&\Psi_{1,[k-m_2]}(t,\boldsymbol{p}_{r_{1|k,m_1}})&R_{[k-m_2],[k]}(t)\\\vdots&&\vdots&\vdots&&\vdots&\vdots\\R_{[k+m_1],\boldsymbol{\beta}_1}(t)&\dots&R_{[k+m_1],\boldsymbol{\beta}_{r_{2|k,m_2}}}(t)&\Psi_{1,[k+m_1]}(t,\boldsymbol{p}_1)&\dots&\Psi_{1,[k+m_1]}(t,\boldsymbol{p}_{r_{1|k,m_1}})&R_{[k+m_1],[k]}(t)\end{pmatrix}.$$

and

$$\hat{H}_{[k]}(t)\Big(\big(\mathcal{Q}_2(\boldsymbol{\Lambda})\big)_{[k-m_2],[k]}\Big)^\top=\big(\mathcal{Q}_1(\boldsymbol{\Lambda})\big)_{[k],[k+m_1]}\times\Theta_*\begin{pmatrix}R_{[k-m_2],\boldsymbol{\beta}_1}(t)&\dots&R_{[k-m_2],\boldsymbol{\beta}_{r_{2|k,m_2}}}(t)&\Psi_{1,[k-m_2]}(t,\boldsymbol{p}_1)&\dots&\Psi_{1,[k-m_2]}(t,\boldsymbol{p}_{r_{1|k,m_1}})&H_{[k-m_2]}(t)\\R_{[k-m_2],\boldsymbol{\beta}_1}(t)&\dots&R_{[k-m_2],\boldsymbol{\beta}_{r_{2|k,m_2}}}(t)&\Psi_{1,[k-m_2]}(t,\boldsymbol{p}_1)&\dots&\Psi_{1,[k-m_2]}(t,\boldsymbol{p}_{r_{1|k,m_1}})&0\\\vdots&&\vdots&\vdots&&\vdots&\vdots\\R_{[k+m_1],\boldsymbol{\beta}_1}(t)&\dots&R_{[k+m_1],\boldsymbol{\beta}_{r_{2|k,m_2}}}(t)&\Psi_{1,[k+m_1]}(t,\boldsymbol{p}_1)&\dots&\Psi_{1,[k+m_1]}(t,\boldsymbol{p}_{r_{1|k,m_1}})&0\end{pmatrix}.$$

Regarding the Baker function Ψ_2 and its behavior under a general linear spectral transformation, using (21), we have for each component

$$\hat{\Psi}_{2,[k]}(t,\boldsymbol{z})=\langle\hat{\Psi}_{1,[k]}(t,\boldsymbol{x}),\mathcal{C}(\boldsymbol{z},\boldsymbol{x})\rangle,$$

and consequently Theorem 2 provides quasi-determinantal expression for $\hat{\Psi}_{2,[k]}$ performing the following replacements

$$\Psi_{1,[l]}(t,\boldsymbol{x})\to\Big\langle\frac{\Psi_{1,[l]}(t,\boldsymbol{x})}{\mathcal{Q}_1(\boldsymbol{x})},\mathcal{C}(\boldsymbol{z},\boldsymbol{x})\Big\rangle,\qquad l\in\{k-m_2,\dots,k+m_1\}.$$

Alternative expressions are achieved if the relation (36) is recalled. Indeed, it implies

$$\hat{\Psi}_2(t,\boldsymbol{z})=\omega_1R\big(W_2^{(0)}(t_2)\big)^\top\chi^*(\boldsymbol{z}).\tag{37}$$

Then, using (35) we conclude that the replacements to perform in Theorem 2 to find a quasi-determinantal expression for $\hat{\Psi}_{2,[k]}$ are

$$\Psi_{1,[l]}(t,\boldsymbol{x})\to\Big\langle P_{1,[l]}(t,\boldsymbol{x}),\mathrm{e}^{t_2(\boldsymbol{x})}\frac{\mathcal{C}(\boldsymbol{z},\boldsymbol{x})}{\mathcal{Q}_2(\boldsymbol{x})}\Big\rangle+\big\langle v,\mathrm{e}^{t_2(\boldsymbol{x})}P_{1,[l]}(t,\boldsymbol{x})\mathcal{C}(\boldsymbol{z},\boldsymbol{x})\big\rangle,\qquad l\in\{k-m_2,\dots,k+m_1\}.$$

In this general setting G is not restricted by a Hankel type constraint, thus given a polynomial $\mathcal{Q}(\boldsymbol{x})\in\mathbb{R}[\boldsymbol{x}]$ we have $G\mathcal{Q}(\boldsymbol{\Lambda}^\top)\neq\mathcal{Q}(\boldsymbol{\Lambda})G$. For example, instead of (29) we may have considered $\mathcal{Q}_2(\boldsymbol{\Lambda})\hat{G}=G\mathcal{Q}_1(\boldsymbol{\Lambda}^\top)$. In this case a transposition formally gives $\hat{G}^\top\mathcal{Q}_2(\boldsymbol{\Lambda}^\top)=\mathcal{Q}_1(\boldsymbol{\Lambda})G^\top$, which can be gotten from (29) by the replacement $G\mapsto G^\top$ and $\hat{G}\mapsto\hat{G}^\top$; i.e., at the level of the Gauss–Borel factorization (13)

$$\begin{aligned}&S_1\mapsto S_2, && H\mapsto H^\top, && S_2\mapsto S_1,\\&\hat{S}_1\mapsto\hat{S}_2, && \hat{H}\mapsto\hat{H}^\top, && \hat{S}_2\mapsto\hat{S}_1.\end{aligned}$$

Thus, previous formulas hold when the replacing P_1 by P_2 and the transposition of the matrices $H_{[k]}$ and $\hat{H}_{[k]}$ is performed.

A quite general transformation, which we will not explore in this paper, corresponds to

$$\mathcal{Q}_2^L(\boldsymbol{\Lambda})\hat{G}\mathcal{Q}_2^R(\boldsymbol{\Lambda}^\top)=\mathcal{Q}_1^R(\boldsymbol{\Lambda})G\mathcal{Q}_1^L(\boldsymbol{\Lambda}^\top),$$

for polynomials $\mathcal{Q}_1^L(\boldsymbol{x}),\mathcal{Q}_1^R(\boldsymbol{x}),\mathcal{Q}_2^L(\boldsymbol{x}),\mathcal{Q}_2^R(\boldsymbol{x})\in\mathbb{R}[\boldsymbol{x}]$. This transformation is preserved by the integrable flows introduced above; i.e., $\mathcal{Q}_2^L(\boldsymbol{\Lambda})\hat{G}(t)\mathcal{Q}_2^R(\boldsymbol{\Lambda}^\top)=\mathcal{Q}_1^R(\boldsymbol{\Lambda})G(t)\mathcal{Q}_1^L(\boldsymbol{\Lambda}^\top)$. Notice that this transformation for a multi-Hankel reduction $\Lambda_aG=G(\Lambda_a)^\top$, $a\in\{1,\dots,D\}$, is just the one considered in previous sections.

2.6. Generalized bilinear equations and linear spectral transformations

We are ready to show that the Baker functions at different times and their linear spectral transforms satisfy a bilinear equation as in the KP theory, see [4–6]. In the standard formulation [4–6] discrete times appeared in the bilinear equation, which in this case are identified, see for example [75], with the linear spectral transformations. To deduce the bilinear equations we use a similar method as in [16,17,23].

We begin with the following observation

Proposition 21. *The wave matrices $W_i(t)$, $i\in\{1,2\}$ and their linear spectral transformed wave matrices $\hat{W}_i(t')$, $i\in\{1,2\}$, according to the coprime polynomials $\mathcal{Q}_1(\boldsymbol{x}),\mathcal{Q}_2(\boldsymbol{x})\in\mathbb{C}[\boldsymbol{x}]$, fulfill*

$$\hat{W}_1(t')\mathcal{Q}_1(\boldsymbol{\Lambda})\big(W_1(t)\big)^{-1}=\hat{W}_2(t')\mathcal{Q}_2(\boldsymbol{\Lambda}^\top)\big(W_2(t)\big)^{-1}.$$

Proof. We have $G=\big(W_1(t)\big)^{-1}W_2(t)$ and $\hat{G}=\big(\hat{W}_1(t')\big)^{-1}\hat{W}_2(t')$. Hence, using (29) we deduce

$$\mathcal{Q}_1(\boldsymbol{\Lambda})\big(W_1(t)\big)^{-1}W_2(t)=\big(\hat{W}_1(t')\big)^{-1}\hat{W}_2(t')\mathcal{Q}_2(\boldsymbol{\Lambda}^\top).\quad\square$$

Now, we need

Lemma 1. *Given two semi-infinite matrices U and V we have*

$$UV=\frac{1}{(2\pi\,\mathrm{i})^D}\oint_{\mathbb{T}^D(\boldsymbol{r})}U\chi(\boldsymbol{z})\big(V^T\chi^*(\boldsymbol{z})\big)^\top\mathrm{d}z_1\cdots\mathrm{d}z_D=\frac{1}{(2\pi\,\mathrm{i})^D}\oint_{\mathbb{T}^D(\boldsymbol{r})}U\chi^*(\boldsymbol{z})\big(V^T\chi(\boldsymbol{z})\big)^\top\mathrm{d}z_1\cdots\mathrm{d}z_D.$$

Proof. Observe that

$$\chi(\chi^*)^\top=\begin{pmatrix}Z_{[0],[0]}&Z_{[0],[1]}&\cdots\\Z_{[1],[0]}&Z_{[1],[1]}&\cdots\\\vdots&\vdots&\end{pmatrix},\quad Z_{[k],[\ell]}:=\frac{1}{z_1\cdots z_D}\begin{pmatrix}\boldsymbol{z}^{\boldsymbol{k}_1-\boldsymbol{\ell}_1}&\boldsymbol{z}^{\boldsymbol{k}_1-\boldsymbol{\ell}_2}&\dots&\boldsymbol{z}^{\boldsymbol{k}_1-\boldsymbol{\ell}_{|[\ell]|}}\\\boldsymbol{z}^{\boldsymbol{k}_2-\boldsymbol{\ell}_1}&\boldsymbol{z}^{\boldsymbol{k}_2-\boldsymbol{\ell}_2}&\dots&\boldsymbol{z}^{\boldsymbol{k}_2-\boldsymbol{\ell}_{|[\ell]|}}\\\vdots&\vdots&&\vdots\\\boldsymbol{z}^{\boldsymbol{k}_{|[k]|}-\boldsymbol{\ell}_1}&\boldsymbol{z}^{\boldsymbol{k}_{|[k]|}-\boldsymbol{\ell}_2}&\dots&\boldsymbol{z}^{\boldsymbol{k}_{|[k]|}-\boldsymbol{\ell}_{|[\ell]|}}\end{pmatrix}.$$

If we now integrate in the polydisk distinguished border $\mathbb{T}^D(\boldsymbol{r})$ using the Fubini theorem we factor each integral in a product of D factors, where the ith factor is an integral over z_i on the circle centered at origin of radius r_i. This is zero unless the integrand is z_i^{-1} which occurs only in the principal diagonal. Consequently, we have

$$\oint_{\mathbb{T}^D(\boldsymbol{r})}\chi(\boldsymbol{z})\chi^*(\boldsymbol{z})^\top\mathrm{d}z_1\cdots\mathrm{d}z_D=\oint_{\mathbb{T}^D(\boldsymbol{r})}\chi^*(\boldsymbol{z})\chi(\boldsymbol{z})^\top\mathrm{d}z_1\cdots\mathrm{d}z_D=(2\pi\,\mathrm{i})^D\mathbb{I},$$

and the result follows. $\square$

We notice that Ψ_1 and Ψ_2^* lead to the computation of finite sums, i.e., polynomials, but Ψ_1^* and Ψ_2 involve Laurent series. We will denote by $\mathscr{D}_{2,\boldsymbol{\alpha}}(t)$ and $\mathscr{D}^*_{1,\boldsymbol{\alpha}}(t)$ the domains of convergence of $\Psi_{2,\boldsymbol{\alpha}}(t,\boldsymbol{z})$ and $\Psi^*_{1,\boldsymbol{\alpha}}(t,\boldsymbol{z})$, respectively. Recall that these domains are Reinhardt domains; i.e., if $\mathscr{D}\subset\mathbb{C}^D$ is the domain of convergence then for any $\boldsymbol{c}=(c_1,\dots,c_D)^\top\in\mathscr{D}$ we have that $\mathbb{T}^D(|c_1|,\dots,|c_D|)\subset\mathscr{D}$.

Theorem 3 (*Generalized Bilinear Equations*)**.** *For any pair of times t and t', points $\boldsymbol{r}_1\in\mathscr{D}^*_{1,\boldsymbol{\alpha}}(t)$ and $\boldsymbol{r}_2\in\hat{\mathscr{D}}_{2,\boldsymbol{\alpha}}(t')$ in the respective Reinhardt domains and D-dimensional tori $\mathbb{T}^D(\boldsymbol{r}_1)$ and $\mathbb{T}^D(\boldsymbol{r}_2)$, and multi-indices $\boldsymbol{\alpha},\boldsymbol{\alpha}'\in\mathbb{Z}_+^D$, the Baker and adjoint Baker functions and their linear spectral transformations satisfy the following bilinear identity*

$$\oint_{\mathbb{T}^D(\boldsymbol{r}_1)}\hat{\Psi}_{1,\boldsymbol{\alpha}'}(t',\boldsymbol{z})\Psi^*_{1,\boldsymbol{\alpha}}(t,\boldsymbol{z})\mathcal{Q}_1(\boldsymbol{z})\,\mathrm{d}z_1\cdots\mathrm{d}z_D=\oint_{\mathbb{T}^D(\boldsymbol{r}_2)}\hat{\Psi}_{2,\boldsymbol{\alpha}'}(t',\boldsymbol{z})\Psi^*_{2,\boldsymbol{\alpha}}(t,\boldsymbol{z})\mathcal{Q}_2(\boldsymbol{z})\,\mathrm{d}z_1\cdots\mathrm{d}z_D.$$

Proof. From Definition 7 and Lemma 1, choosing $U=\hat{W}_1(t')\mathcal{Q}_1\boldsymbol{\Lambda})$ and $V=\big(W_1(t)\big)^{-1}$ we get

$$\hat{W}_1(t')\mathcal{Q}_1(\boldsymbol{\Lambda})\big(W_1(t)\big)^{-1}=\frac{1}{(2\pi\,\mathrm{i})^D}\oint_{\mathbb{T}^D(\boldsymbol{r}_1)}\hat{\Psi}_1(t',\boldsymbol{z})\Psi_1^*(t,\boldsymbol{z})\mathcal{Q}_1(\boldsymbol{z})\,\mathrm{d}z_1\cdots\mathrm{d}z_D,$$

and choosing $U=\hat{W}_2(t')$ and $V=\mathcal{Q}_2(\boldsymbol{\Lambda}^\top)\big(W_2(t)\big)^{-1}$ we get

$$\hat{W}_2(t')\mathcal{Q}_2(\boldsymbol{\Lambda}^\top)\big(W_2(t)\big)^{-1}=\frac{1}{(2\pi\,\mathrm{i})^D}\oint_{\mathbb{T}^D(\boldsymbol{r}_2)}\hat{\Psi}_2(t',\boldsymbol{z})\Psi_2^*(t,\boldsymbol{z})\mathcal{Q}_2(\boldsymbol{z})\,\mathrm{d}z_1\cdots\mathrm{d}z_D.$$

Then, Proposition 21 implies the result. $\square$

Conclusions and outlook

In this paper we have discussed a type of multicomponent Toda hierarchy that is connected to nonstandard multivariate orthogonality, and also presented corresponding Darboux type transformations. An interesting open point is the Riemann–Hilbert formulation of this multivariate biorthogonality, see [34] for a presentation of Plemej's formulas in this context. Another interesting issue is the possible role of an extended tau function, possibly a quasideterminant, for more on quasi-tau matrices see [1].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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