The Jordan Wigner transformations and the fermionization of the XYZ spin Heisenberg chain. Algebra, geometry and physics?

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We generalize our previous results for the (anisotropic) fermion XYZ Heisenberg chain; we avoid the free-fermion restriction. The departing point in the definition of two different Jordan Wigner transformations, significant for the Hamiltonians. Afterwards, with 'their' square roots we define four different formulations of the local transition matrices with fermion operators. Due to our generalizations of the Yang-Baxter relation we observe a special role for the sign associated to the *modulus* k of the Jacobi elliptic functions. We define four Hamiltonians. We obtain the conserved quantities. Our construction and results suggest the importance of the geometry of the time and the space in various ways and a possible application to the elementary particles.

Keywords: time space, Jordan Wigner, fermion XYZ chain, non free fermion, Yang-Baxter, conserved quantities, quarks, leptons.

I. INTRODUCTION.

In this research, we are going to consider three related models:

first, the eight-vertex model, a two dimensional classical statistical model,
second, the one dimensional ferromagnetic Heisenberg chain,
represented by an anisotropic (XYZ) Hamiltonian in terms of spins, and
third, again a one dimensional anisotropic (XYZ) chain but now in terms of fermion operators.

The relationship of the first two and their solution was presented by R. J. Baxter fifty years ago culminating a process started forty years earlier.[1] [2] The situation with the third model, the one with fermions is different. Felderhof studied a particular case, the XY model, a free fermion model and he did so including some restrictions, specifically with cyclic boundaries he imposed an even number of fermion operators, and he wrote: "In order to recover the translational symmetry which is broken by the Jordan Wigner transformation" (pairs of fermions, related to a resulting boson structure?) [3]. We also mention studies with the XXZ fermion model and with their composition in the way of the Hubbard model. [4] [5] [6]

In another research, we work in the solving of the general anisotropic fermion model (XYZ) by defining the Hamiltonian [7]:

$$\mathcal{H}\,(\mathbf{J_z},\mathbf{J_x},\mathbf{J_y}) = \sum_{n=1}^{N}\mathcal{H}_{n+1,n} = -\sum_{n=1}^{N}\Big\{\mathbf{J_z}\,(\mathbf{n}_{n+1}\mathbf{n}_n+\tilde{\mathbf{n}}_{n+1}\tilde{\mathbf{n}}_n)+\frac{\mathbf{J_x}+\mathbf{J_y}}{2}\,(\mathbf{a}^{\dagger}_{n+1}\mathbf{a}_n-\mathbf{a}_{n+1}\mathbf{a}^{\dagger}_n)+\frac{\mathbf{J_x}-\mathbf{J_y}}{2}\,(\mathbf{a}_{n+1}\mathbf{a}_n-\mathbf{a}^{\dagger}_{n+1}\mathbf{a}^{\dagger}_n)\Big\}, \tag{1.1}$$

with the usual sheer algebraic method. The first main equation, which is a modified Yang-Baxter relation is:

$$\mathcal{R}\,(\mu-\nu,\ k)\ [\,\mathcal{L}_n\,(\mu,-k)\underset{s}{\otimes}\mathcal{L}_n(\nu,\ k)\,]\ =\ [\,\mathcal{L}_n(\nu,-k)\underset{s}{\otimes}\mathcal{L}_n(\mu,\ k)\,]\ \mathcal{R}\,(\mu-\nu,\ k)\,. \tag{1.2}$$

We point out two questions, as possible perspectives:

- the symmetric appearance of $\mathbf{n}$ and $\tilde{\mathbf{n}}=\mathbf{1}-\mathbf{n}=\mathbf{1}-\mathbf{a}^{\dagger}\,\mathbf{a}=\mathbf{a}\,\mathbf{a}^{\dagger}\overset{?}{=}\mathbf{b}^{\dagger}\,\mathbf{b}$, perhaps a possibility for antiparticles, [8] (Page 4; in Section V. Appendix: cites),
- the symmetries under the interchanges: $\mathbf{J}_x\longleftrightarrow\mathbf{J}_y$ ($-k\longleftrightarrow k$, k the modulus of the Jacobi elliptic functions) with $\{N\to 1\}$ instead of $\{1\to N\}$. Using (1.1) with the relation $\mathbf{J_x}-\mathbf{J_y}\ =\ -\ k\ 2\beta$ (see (3.12)) we obtain: $\mathcal{H}_{n+1,n}(\mathbf{J_z},\mathbf{J_x},\mathbf{J_y})=\mathcal{H}_{n+1,n}(-k)\ =\ \mathcal{H}_{n,n+1}(\,k)\,)=\mathcal{H}_{n,n+1}(\mathbf{J_z},\mathbf{J_y},\mathbf{J_x})$, perhaps a possibility for chiral particles.

In case these considerations are meaningful, the Jordan Wigner transformations would be interesting for the elementary particles (quarks and leptons), not just for the magnetic chains with our modeling Hamiltonian (1.1).

The matrices $\{\sigma^+,\,\sigma^-,\,\hat{\sigma}\}$ in (3.2) are "explicit matrix representations" of creation, annihilation and number operators (see Section V. Appendix: cites, and also the b_r , $b_r^{\dagger}$, N_r in [9]) acting in state vectors in $\mathbb{C}^2$. Jordan and Wigner in their research [10] specify that the ordering (<), connoted by the sub-index r (in their transformation), means: " " "comes before" (not, as usual, "less than") " ". The "invention" of Jordan and Wigner [8] (Page 7; in Section V. Appendix: cites), driving to the anti-commutator structure, has a formal algebraic expression in the direct product of sub-spaces in $\mathbb{C}^{2^N}$ (see [2] and Section III. Appendix. A). The previous r sub-indexes are not related with a placing in the time and the space.

Even though in the eight vertex model and in the Heisenberg chains there are orderings of the particles, with specific reference to directions and placements in the space; the $\{\,\mathbf{J_x}\,,\,\mathbf{J_y}\,,\,\mathbf{J_z}\,\}$ and the n sub-indexes in the chains.

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The matrices $\{\sigma^x,\ \sigma^y,\ \sigma^z\}$ in (3.1) are "explicit matrices": σ^z an algebraic tool in the Jordan Wigner transformation. And at once, $\{\sigma^x,\ \sigma^y,\ \sigma^z\}$ are deeply related with the geometry of the vectors in a three dimensional space. Pauli in [11] pointed out this in relation to quaternions (he thanks to Jordan who "brought" to his attention this point) and also in Cartan [12]. The rotations in the three dimensional space and the quaternions are intimately related. We assume the chain in (1.1) in the direction z with σ^z.

In this sense, we can assign to the algebraic form of the first set in the equations (2.1) a geometric interpretation: $\mathbb{1}$ and σ^z are vectors in a four dimensional geometry (see (3.1)); they are in the imaginary exponentials. We associate $\mathbb{1}$ with a time type dimension and σ^z with a space type dimension. $\pm(\mathbb{1} \pm \sigma^z)$ are null vectors in a Minkowskian space. Look also at their relative opposite signs, as it concern the space part with respect to the time part, and the appearance of the operators $\mathbf{n}$ and $\tilde{\mathbf{n}}$ (also last equations in (2.3)) as number operators for a particle and its antiparticle respectively. These aspects have to do essentially with our first question.

The possible interpretation of the second question is more subtle. We depart from our Hamiltonian (1.1). Let us look at the interactions ($\{\mathbf{J_z}, \mathbf{J_x}, \mathbf{J_y}\}$). We associate in $\mathcal{H}(\mathbf{J_z}, \mathbf{J_x}, \mathbf{J_y})$ the $\mathbf{J_z}$ with a z direction and similarly with the x and y directions taken as a right handed system (from $\mathbf{J_x}$ (x) and $\mathbf{J_y}$ (y) to $\mathbf{J_z}$ (z)). The fermion Yang-Baxter relation (1.2) suggests the definition of two different Hamiltonians: $\mathcal{H}(-k)$ and $\mathcal{H}(k)$. The equality $\mathbf{J_x} - \mathbf{J_y} = -2\,k\beta$ let us interchange the roles of the $-k$ and $+k$ and in correspondence the $\mathbf{J_x}$ and $\mathbf{J_y}$. Afterwards we establish the relation of the Hamiltonians: the previous $\mathcal{H}(\mathbf{J_z}, \mathbf{J_x}, \mathbf{J_y}) = \mathcal{H}(-k)$ on one side and on the other side, of $\mathcal{H}(\mathbf{J_z}, \mathbf{J_y}, \mathbf{J_x}) = \mathcal{H}(k)$ with a left handed system: from $\mathbf{J_x}$ (y) and $\mathbf{J_y}$ (x) to $\mathbf{J_z}$ (z). Finally, looking at the chain from the opposite direction, $-z$ $(-\sigma^z)$ (from $\{N \to 1\}$ instead of $\{1 \to N\}$), we consider it also as a right handed system (from $\mathbf{J_x}$ (y) and $\mathbf{J_y}$ (x)) for the Hamiltonian $\mathcal{H}_{N\to 1}(\mathbf{J_z}, \mathbf{J_y}, \mathbf{J_x})$, which is equal to $\mathcal{H}_{1\to N}(\mathbf{J_z}, \mathbf{J_x}, \mathbf{J_y})$. In the departing (spin) Hamiltonian (3.11) to make the change $\sigma^j \longrightarrow -\sigma^j$, $j \in \{x, y, z\}$ is not interesting (see after (3.15)).

Taking account of our two first points (particles, antiparticles and chirality), we can rewrite (1.1) in the form:

$$
\mathcal{H} = -\sum_{n=1}^{N} \left\{ \mathcal{H}^{\mathbf{a}^\dagger}_{n+1,n} + \mathcal{H}^{\mathbf{b}^\dagger}_{n+1,n} + \mathcal{H}^{\mathbf{a}^\dagger\mathbf{b}^\dagger}_{n+1,n} \right\} \qquad
\begin{cases}
\mathcal{H}^{\mathbf{a}^\dagger}_{n+1,n} = \dfrac{\mathbf{J_z}}{2}\left(\mathbf{n}_{n+1}\mathbf{n}_n + \mathbf{n}_n\mathbf{n}_{n*1}\right) + \dfrac{\mathbf{J_x}+\mathbf{J_y}}{4}\left(\mathbf{a}^\dagger_{n+1}\mathbf{a}_n + \mathbf{a}^\dagger_n\mathbf{a}_{n+1}\right) \\
\mathcal{H}^{\mathbf{b}^\dagger}_{n+1,n} = \dfrac{\mathbf{J_z}}{2}\left(\tilde{\mathbf{n}}_{n+1}\tilde{\mathbf{n}}_n + \tilde{\mathbf{n}}_n\tilde{\mathbf{n}}_{n+1}\right) - \dfrac{\mathbf{J_x}+\mathbf{J_y}}{4}\left(\mathbf{b}^\dagger_{n+1}\mathbf{b}_n + \mathbf{b}^\dagger_n\mathbf{b}_{n+1}\right) \\
\mathcal{H}^{\mathbf{a}^\dagger\mathbf{b}^\dagger}_{n+1,n} = \dfrac{\mathbf{J_x}-\mathbf{J_y}}{4}\left(\mathbf{b}^\dagger_{n+1}\mathbf{a}_n + \mathbf{a}^\dagger_n\mathbf{b}_{n+1}\right) + \dfrac{\mathbf{J_y}-\mathbf{J_x}}{4}\left(\mathbf{a}^\dagger_{n+1}\mathbf{b}_n + \mathbf{b}^\dagger_n\mathbf{a}_{n+1}\right)
\end{cases} . \tag{1.3}
$$

The algebraic formal exposition deserves attention. The starting point consist in the representation (definition) of the spin operators in a chain (action in $\mathbb{C}^{2^N}$) with the direct products ($N-1$ times) shown in Section III. Appendix. A. This is treated in more detail in [2]. The spin, the creation and the annihilation operators, and the commutators or the anti-commutators written in this form. In the Baxter's method we define 2×2 matrices, the **local** transition matrices; their matrix elements contain the previous operators (originally the spin matrices), ignoring their structures in relation to $\mathbb{C}^{2^N}$. But we sustain the usage of the commutators or the anti-commutators of the previous operators in the products of these matrices; this is the inheritance of the definitions and constructions in the Section III. Appendix. (A and B) and in the subsection II. A. We do the usual matrix product (we do not the direct product) of two consecutive local transition matrices that refer to two consecutive locations. In other stage we do the direct products of the local transition matrices at the same position of the chain, but depending on different parameters, obtaining 4×4 matrices. The *'Ultra locality'* property, mixing both products, specially remarkable.

In this research, we start defining the Jordan Wigner transformations (see (2.1)) significant for the Hamiltonians (with the ϵ in our notation). We add 'their' square roots (see (2.4)) which are relevant for the definition of different fermion local transition matrices. Afterwards we are able to relate the different generalized Hamiltonians with the various fermion local transition matrices and to obtain similar results to the ones of Baxter for the *XYZ* spin Heisenberg chain.

We need some more comments about the presentation of our results. The readers familiar with this field and looking exclusively for the fermionization of the XYZ spin chain can just read Section II and perhaps Section IV. Appendix: proofs. For them this introduction could be, at most or at least, controversial. The Section II follows closely and extends previous results in [7]. The Section III. Appendix: reminders, has a double purpose: to establish the notation and to serve as a helping guide for the novice in the field, of well known results. A final appendix with the program of studies containing this research.

Final annotations. This author found the relation (1.2) in the second semester in 1987, once finished a research in the Hubbard model under the guidance of the late Professor Miki Wadati. Misguided by the appearance of the two signs with the *modulus* k, this author tried to generalize the problem by introducing the γ matrices (thinking in a certain analogy with the Dirac equation), unsuccessfully. For a long period of time there were no more significant results and as a consequence the decision for its no communication. Surprisingly, without hope for obtaining new results, recently, this author has found the results presented in [7] and in this Study.

In the meantime (for the last ten years) the work in the Jordan Wigner transformation drove me along a very different pathway. Is it possible a more general Jordan Wigner transformation, which would consider discrete four dimensional vectors? Could this account for a geometrical description of the leptons and the quarks? [13] In some sense, the procedure for answering these questions is similar to the one discovered by Rodrigues and also by Hamilton with the quaternions in the middle of the ninetieth century for the description of the rotations in the three dimensional space, but this time in $\mathbb{C}^{4N}$.

II. THE JORDAN WIGNER TRANSFORMATIONS AND THE BAXTER'S METHOD.

A. The Jordan Wigner transformations. [10] Algebra and geometry. A non local transformation.

We assume the contents of Section III. Appendix. (A and B). In the usual way we define: $\mathbf{n}_k = \mathbf{a}_k^\dagger\, \mathbf{a}_k = \mathbf{n}_k^2$, and we denote: $\tilde{\mathbf{n}}_k = \mathbf{a}_k\, \mathbf{a}_k^\dagger = \tilde{\mathbf{n}}_k^2$, satisfying: $\mathbf{n}_k + \tilde{\mathbf{n}}_k = \mathbf{1}$.

With the expressions: $\mathrm{e}^{\mathrm{i}\,\alpha\,\sigma_k^z} = \cos\alpha\,\mathbb{1} + \mathrm{i}\,\sin\alpha\,\sigma_k^z$ and $\left\{\begin{array}{l} \mathrm{e}^{\mathrm{i}\,\alpha\,\mathbf{n}_k} = \mathbf{1} + (\mathrm{e}^{\mathrm{i}\alpha} - 1)\,\mathbf{n}_k = \tilde{\mathbf{n}}_k + \mathrm{e}^{\mathrm{i}\alpha}\,\mathbf{n}_k \\ \mathrm{e}^{\mathrm{i}\,\alpha\,\tilde{\mathbf{n}}_k} = \mathbf{1} + (\mathrm{e}^{\mathrm{i}\alpha} - 1)\,\tilde{\mathbf{n}}_k = \mathbf{n}_k + \mathrm{e}^{\mathrm{i}\alpha}\,\tilde{\mathbf{n}}_k \end{array}\right.$,

we define:

$$\left.\begin{array}{l} {v_k^+}^2 = {v_k^+}^{-2} = \mathrm{e}^{\mathrm{i}\,(\pm\pi)\,\frac{1}{2}(\ \mathbb{1}_k + \sigma_k^z)} = -\sigma_k^z \\ {v_k^-}^2 = {v_k^-}^{-2} = \mathrm{e}^{\mathrm{i}\,(\mp\pi)\,\frac{1}{2}(-\mathbb{1}_k + \sigma_k^z)} = \ \ \sigma_k^z \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} {u_k^+}^2 = {u_k^+}^{-2} = \mathrm{e}^{\mathrm{i}\,(\pm\pi)\,\mathbf{n}_k} = \tilde{\mathbf{n}}_k - \mathbf{n}_k \\ {u_k^-}^2 = {u_k^-}^{-2} = \mathrm{e}^{\mathrm{i}\,(\mp\pi)\,\tilde{\mathbf{n}}_k} = \mathbf{n}_k - \tilde{\mathbf{n}}_k \end{array}\right. . \tag{2.1}$$

In brief: ${v_k^\epsilon}^2 = {v_k^\epsilon}^{-2} = \mathrm{e}^{\mathrm{i}\,(\pm\pi)\,\frac{1}{2}(\epsilon\mathbb{1}_k + \sigma_k^z)} = -\epsilon\,\sigma_k^z \quad \longleftrightarrow \quad {u_k^\epsilon}^2 = {u_k^\epsilon}^{-2} = \epsilon\,\mathrm{e}^{\mathrm{i}\,(\pm\pi)\,\mathbf{n}_k} = \epsilon\,(\tilde{\mathbf{n}}_k - \mathbf{n}_k)$,

which satisfy: ${v_k^\epsilon}^4 = \mathbb{1} = {v_k^\epsilon}^2\,{v_k^\epsilon}^{-2}$, ${u_k^\epsilon}^4 = \mathbf{1} = {u_k^\epsilon}^2\,{u_k^\epsilon}^{-2}$, ${v_k^\epsilon}^2\,{v_k^{-\epsilon}}^{-2} = -\mathbb{1}$, ${u_k^\epsilon}^2\,{u_k^{-\epsilon}}^{-2} = -\mathbf{1}$, with ϵ either $+$ or $-$. The signs in the $(\pm\pi)$ in the exponents and of ϵ in the above definitions are not relevant for the achievement of the anti-commutators. The value of, or the sign represented by ϵ is relevant in the definitions of the fermion Heisenberg Hamiltonians.

The creation and annihilation operators via a Jordan Wigner transformation:

$$\mathbf{a}_{\mathcal{M}}^\dagger = v_{M-1}^2\ \sigma_m^+ = \sigma_m^+\ v_{M-1}^2\ , \qquad \mathbf{a}_{\mathcal{M}} = v_{M-1}^2\ \sigma_m^- = \sigma_m^-\ v_{M-1}^2\ , \tag{2.2}$$

with v_{M-1}^2 either one of the following two: ${v_{M-1}^\pm}^2 = \prod_{k=1}^{m-1} {v_k^\pm}^2 = (\mp\sigma^z)_{M-1}$. We write the relations:

$$\left.\begin{array}{l} \mathbf{n}_m = \mathbf{a}_m^\dagger\,\mathbf{a}_m = \mathbf{a}_{\mathcal{M}}^\dagger\,\mathbf{a}_{\mathcal{M}} \ = \sigma_m^+\sigma_m^- = \overset{\wedge}{\sigma}_m \\ \tilde{\mathbf{n}}_m = \mathbf{a}_m\,\mathbf{a}_m^\dagger = \mathbf{a}_{\mathcal{M}}\,\mathbf{a}_{\mathcal{M}}^\dagger \ = \sigma_m^-\sigma_m^+ = \overset{\vee}{\sigma}_m \end{array}\right\} \quad \begin{array}{l} 2\mathbf{n}_m - \mathbf{1}_m = \mathbf{n}_m - \tilde{\mathbf{n}}_m = \overset{\wedge}{\sigma}_m - \overset{\vee}{\sigma}_m = \sigma_m^z \\ \mathbf{n}_m = \frac{1}{2}\,(\ \mathbb{1} + \sigma^z)_m\ ,,\quad \tilde{\mathbf{n}}_m = \frac{1}{2}\,(-\mathbb{1} + \sigma^z)_m \end{array} , \tag{2.3}$$

Using this with (2.1), we have: ${v_k^+}^{\pm 2} = -{v_k^-}^{\pm 2} = {u_k^+}^{\pm 2} = -{u_k^-}^{\pm 2}$, although their significances are different, the first two deal with spins and the last two with fermions (with the same matrix representation).

We invert the formulas in (2.2) in the form: $\sigma_m^+ = \mathbf{a}_{\mathcal{M}}^\dagger\,u_{M-1}^2 = u_{M-1}^2\,\mathbf{a}_{\mathcal{M}}^\dagger$ and $\sigma_m^- = \mathbf{a}_{\mathcal{M}}\,u_{M-1}^2 = u_{M-1}^2\,\mathbf{a}_{\mathcal{M}}$.

In order to define the local transition matrices we use the following square roots (we ignore their negatives):

$$\begin{array}{lllll} \left\{\sqrt{{u_k^+}^2}\right\}: & u_k^+ = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\mathbf{n}_k} = \tilde{\mathbf{n}}_k + \mathrm{i}\,\mathbf{n}_k & \text{and} & {u_k^+}^\dagger = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\mathbf{n}_k} = \tilde{\mathbf{n}}_k - \mathrm{i}\,\mathbf{n}_k = {u_k^+}^{-1} \\ \left\{\sqrt{{u_k^-}^2}\right\}: & u_k^- = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\tilde{\mathbf{n}}_k} = \mathbf{n}_k + \mathrm{i}\,\tilde{\mathbf{n}}_k & \text{and} & {u_k^-}^\dagger = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\tilde{\mathbf{n}}_k} = \mathbf{n}_k - \mathrm{i}\,\tilde{\mathbf{n}}_k = {u_k^-}^{-1} \end{array} . \tag{2.4}$$

With (2.4) we define the matrices:

$$\mathbb{U}_k = \begin{pmatrix} u_k & \mathbb{0} \\ \mathbb{0} & u_k^\dagger \end{pmatrix} = \begin{pmatrix} u_k & \mathbb{0} \\ \mathbb{0} & u_k^{-1} \end{pmatrix}, \qquad \mathbb{U}_{M-1} = \prod_{k=1}^{m-1} \mathbb{U}_k\ , \qquad \text{with} \qquad u_k \in \left\{ u_k^+\ ,\ {u_k^+}^\dagger\ ,\ u_k^-\ ,\ {u_k^-}^\dagger \right\} . \tag{2.5}$$

And $\begin{pmatrix} u_k^\dagger & \mathbb{0} \\ \mathbb{0} & u_k \end{pmatrix} = \begin{pmatrix} u_k^{-1} & \mathbb{0} \\ \mathbb{0} & u_k \end{pmatrix} = \mathbb{U}_k^{-1}$.

What is a fermion? At this point, it is pertinent to mention two cites: a) "This invention (by Jordan and Wigner [3]), is very useful although its physical meaning appears to be obscure: the sign of an expression in the amplitudes . . . " (Appendix V),
b) in Professor Wen's book [14] (page 144) there are three notorious statements; explicitly, the second one tells us about the "**non local**" aspect of the fermions (through their construction) and of their "**weirdness**".

Most commonly, they can be taken as 'given' ($\mathbf{a}_{\mathcal{M}}^\dagger \longrightarrow \mathbf{a}_m^\dagger$ and $\mathbf{a}_{\mathcal{M}} \longrightarrow \mathbf{a}_m$ the primitive objects), with a mathematical structure similar to the one of the bosons (see Section III. Appendix A) but this time imposing anticommutators instead of commutators, and in this sense with a '**local**' character. For the fermion *XYZ* chain, the Jordan Wigner transformation is useful as a helping guide for obtaining some results for this chain which show a very close parallelism with known results with the spins. After all, we can ignore this tool (the Jordan Wigner transformation) and we can start with the definitions in (2.7).

B. The Yang-Baxter type relations (with fermion operators).

We assume the contents of the Section III. Appendix C.

In a first step we transform the 2×2 local transition matrix with spins $L_m(\lambda)=\begin{pmatrix}\alpha\,\hat{\sigma}+\gamma\,\check{\sigma} & \delta\,\sigma^{+}+\beta\,\sigma^{-}\\ \beta\,\sigma^{+}+\delta\,\sigma^{-} & \gamma\,\hat{\sigma}+\alpha\,\check{\sigma}\end{pmatrix}_m$:

$$L_m(\lambda)\;\longrightarrow\;\mathbb{U}_{M-1}^{-1}\begin{pmatrix}\alpha\,\mathbf{n}_m+\gamma\,\tilde{\mathbf{n}}_m & \delta\,\mathbf{a}_{\mathcal{M}}^{\dagger}+\beta\,\mathbf{a}_{\mathcal{M}}\\ \beta\,\mathbf{a}_{\mathcal{M}}^{\dagger}+\delta\,\mathbf{a}_{\mathcal{M}} & \gamma\,\mathbf{n}_m+\alpha\,\tilde{\mathbf{n}}_m\end{pmatrix}\mathbb{U}_{M-1}=\mathbb{U}_{M}^{-1}\,\mathbb{U}_m\begin{pmatrix}\alpha\,\mathbf{n}_m+\gamma\,\tilde{\mathbf{n}}_m & \delta\,\mathbf{a}_{\mathcal{M}}^{\dagger}+\beta\,\mathbf{a}_{\mathcal{M}}\\ \beta\,\mathbf{a}_{\mathcal{M}}^{\dagger}+\delta\,\mathbf{a}_{\mathcal{M}} & \gamma\,\mathbf{n}_m+\alpha\,\tilde{\mathbf{n}}_m\end{pmatrix}\mathbb{U}_{M-1}=\mathbb{U}_{M}^{-1}\,\mathcal{L}_m(\lambda)\,\mathbb{U}_{M-1}\,.$$

We have denoted with $\mathcal{L}_m(\lambda)$ the 2×2 local transition matrix with fermion operators, which we justify as follows.

In order to maintain the form of the monodromy matrix also for the fermions, $\overleftarrow{\mathcal{T}}_N(\lambda)$ (a 2×2 matrix), we do:

$$\overleftarrow{\mathbf{T}}_N(\lambda)=\overleftarrow{\prod_{m=1}^{N}}L_m(\lambda)\;\longrightarrow\;\mathbb{U}_N^{-1}\mathcal{L}_N(\lambda)\ldots\mathcal{L}_{m+1}(\lambda)\,\mathbb{U}_M\mathbb{U}_M^{-1}\mathcal{L}_m\ldots\mathcal{L}_1(\lambda)\mathbb{U}_0=\mathbb{U}_N^{-1}\Big(\overleftarrow{\prod_{m=1}^{N}}\mathcal{L}_m(\lambda)\Big)\,\mathbb{U}_0=\mathbb{U}_N^{-1}\,\overleftarrow{\mathcal{T}}_N(\lambda)\,\mathbb{U}_0\,.$$

This and the local character of the fermion operators as a local fermion particle, in the sense described in the last paragraph of the previous subsection, justify the writing of a sub-index m instead of $\mathcal{M}$, which is the sub-index appearing for the creation and annihilation operators in the Jordan Wigner construction. We could ignore $\mathbb{U}_0$ and $\mathbb{U}_N^{-1}$, or we could implement periodic boundary conditions in order to define $\mathbb{U}_0$ and also, later, for the transfer matrix. Therefore, in the fermion framework:

$$\begin{pmatrix}\alpha\,\mathbf{n}_m+\gamma\,\tilde{\mathbf{n}}_m & \delta\,\mathbf{a}_{\mathcal{M}}^{\dagger}+\beta\,\mathbf{a}_{\mathcal{M}}\\ \beta\,\mathbf{a}_{\mathcal{M}}^{\dagger}+\delta\,\mathbf{a}_{\mathcal{M}} & \gamma\,\mathbf{n}_m+\alpha\,\tilde{\mathbf{n}}_m\end{pmatrix}\;\longrightarrow\;\begin{pmatrix}\alpha\,\mathbf{n}+\gamma\,\tilde{\mathbf{n}} & \delta\,\mathbf{a}^{\dagger}+\beta\,\mathbf{a}\\ \beta\,\mathbf{a}^{\dagger}+\delta\,\mathbf{a} & \gamma\,\mathbf{n}+\alpha\,\tilde{\mathbf{n}}\end{pmatrix}_m=\mathbf{L}_m(\lambda)\;, \tag{2.6}$$

so that: $\mathcal{L}_m(\lambda)=\mathbb{U}_m\,\mathbf{L}_m(\lambda)$ are the 2×2 fermion **local** transition matrices.

We define four different *fermion local transition matrices* with the four possibilities in (2.5):

$\epsilon=+$, which we denote with the upper-indexes "a" and "b":

$$\begin{aligned}
&a)\ \text{for}\ u_n=u_n^{+}:\quad \mathcal{L}_n^{a}(\lambda)=\mathbb{U}_n^{a}\,\mathbf{L}_n(\lambda)=\begin{pmatrix}u_n^{+} & \mathbb{0}\\ \mathbb{0} & u_n^{+\dagger}\end{pmatrix}\mathbf{L}_n(\lambda)=\begin{pmatrix}\mathrm{i}\,\alpha\,\mathbf{n}+\gamma\,\tilde{\mathbf{n}} & \mathrm{i}\,\delta\,\mathbf{a}^{\dagger}+\beta\,\mathbf{a}\\ -\mathrm{i}\,\beta\,\mathbf{a}^{\dagger}+\delta\,\mathbf{a} & -\mathrm{i}\,\gamma\,\mathbf{n}+\alpha\,\tilde{\mathbf{n}}\end{pmatrix}_n\\
&b)\ \text{for}\ u_n=u_n^{+\dagger}:\quad \mathcal{L}_n^{b}(\lambda)=\mathbb{U}_n^{b}\,\mathbf{L}_n(\lambda)=\begin{pmatrix}u_n^{+\dagger} & \mathbb{0}\\ \mathbb{0} & u_n^{+}\end{pmatrix}\mathbf{L}_n(\lambda)=\begin{pmatrix}-\mathrm{i}\,\alpha\,\mathbf{n}+\gamma\,\tilde{\mathbf{n}} & -\mathrm{i}\,\delta\,\mathbf{a}^{\dagger}+\beta\,\mathbf{a}\\ \mathrm{i}\,\beta\,\mathbf{a}^{\dagger}+\delta\,\mathbf{a} & \mathrm{i}\,\gamma\,\mathbf{n}+\alpha\,\tilde{\mathbf{n}}\end{pmatrix}_n=\begin{pmatrix}1&0\\0&\mathrm{i}\end{pmatrix}\mathcal{L}_n^{a\dagger}(\lambda)\begin{pmatrix}1&0\\0&-\mathrm{i}\end{pmatrix}\\
&\epsilon=-,\ \text{which we denote with the upper-indexes "}c\text{" and "}d\text{":}\\
&c)\ \text{for}\ u_n=u_n^{-}\ :\quad \mathcal{L}_n^{c}(\lambda)=\mathbb{U}_n^{c}\,\mathbf{L}_n(\lambda)=\begin{pmatrix}u_n^{-} & \mathbb{0}\\ \mathbb{0} & u_n^{-\dagger}\end{pmatrix}\mathbf{L}_n(\lambda)=\begin{pmatrix}\alpha\,\mathbf{n}+\mathrm{i}\,\gamma\,\tilde{\mathbf{n}} & \delta\,\mathbf{a}^{\dagger}+\mathrm{i}\,\beta\,\mathbf{a}\\ \beta\,\mathbf{a}^{\dagger}-\mathrm{i}\,\delta\,\mathbf{a} & \gamma\,\mathbf{n}-\mathrm{i}\,\alpha\,\tilde{\mathbf{n}}\end{pmatrix}_n=\begin{pmatrix}\mathrm{i}&0\\0&1\end{pmatrix}\mathcal{L}_n^{a\dagger}(\lambda)\begin{pmatrix}1&0\\0&-\mathrm{i}\end{pmatrix}\\
&d)\ \text{for}\ u_n=u_n^{-\dagger}:\quad \mathcal{L}_n^{d}(\lambda)=\mathbb{U}_n^{d}\,\mathbf{L}_n(\lambda)=\begin{pmatrix}u_n^{-\dagger} & \mathbb{0}\\ \mathbb{0} & u_n^{-}\end{pmatrix}\mathbf{L}_n(\lambda)=\begin{pmatrix}\alpha\,\mathbf{n}-\mathrm{i}\,\gamma\,\tilde{\mathbf{n}} & \delta\,\mathbf{a}^{\dagger}-\mathrm{i}\,\beta\,\mathbf{a}\\ \beta\,\mathbf{a}^{\dagger}+\mathrm{i}\,\delta\,\mathbf{a} & \gamma\,\mathbf{n}+\mathrm{i}\,\alpha\,\tilde{\mathbf{n}}\end{pmatrix}_n=\begin{pmatrix}-\mathrm{i}&0\\0&\mathrm{i}\end{pmatrix}\mathcal{L}_n^{a}(\lambda)
\end{aligned}\;. \tag{2.7}$$

We can prove the following *Yang-Baxter type relationships* (see Section IV. Appendix: proofs):

$$\mathcal{R}\,(\mu-\nu,rk)\;[\mathcal{L}_n^{\{{a\atop d}\}}(\mu,-rk)\underset{s}{\otimes}\mathcal{L}_n^{\{{a\atop d}\}}(\nu,\;rk)]\;=\;[\mathcal{L}_n^{\{{a\atop d}\}}(\nu,-rk)\underset{s}{\otimes}\mathcal{L}_n^{\{{a\atop d}\}}(\mu,\;rk)]\;\mathcal{R}\,(\mu-\nu,rk)\,, \tag{2.8a}$$

$$\mathcal{R}^{t}(\mu-\nu,rk)\;[\mathcal{L}_n^{\{{b\atop c}\}}(\mu,-rk)\underset{s}{\otimes}\mathcal{L}_n^{\{{b\atop c}\}}(\nu,\;rk)]\;=\;[\mathcal{L}_n^{\{{b\atop c}\}}(\nu,-rk)\underset{s}{\otimes}\mathcal{L}_n^{\{{b\atop c}\}}(\mu,\;rk)]\;\mathcal{R}^{t}(\mu-\nu,rk)\,, \tag{2.8b}$$

with r either $+$ or $-$. Where:

$$\mathcal{R}(\mu-\nu,rk)=\begin{pmatrix}a&0&0&r\,\mathrm{i}\,d\\0&b&\mathrm{i}\,c&0\\0&-\mathrm{i}\,c&b&0\\-r\,\mathrm{i}\,d&0&0&a\end{pmatrix}\;,,\quad \mathcal{R}\,(\mu-\nu,rk)\,\mathcal{R}^{t}(\mu-\nu,rk)=\begin{pmatrix}a^2-d^2&0&0&0\\0&b^2-c^2&0&0\\0&0&b^2-c^2&0\\0&0&0&a^2-d^2\end{pmatrix}\;, \tag{2.9}$$

and $\det\mathcal{R}=\det\mathcal{R}^{t}=(a^2-d^2)(b^2-c^2)=a^2(1-k^2b^2c^2)(b^2-c^2)$. $\mathcal{R}$ and $\mathcal{R}^{t}$ have inverses except for a discrete set of values. The $\underset{s}{\otimes}$ is the symbol for a Grassmann direct product defined by $[\mathcal{A}\underset{s}{\otimes}\mathcal{B}]_{ik,jl}=(-1)^{[P(i)+P(j)]P(k)}\,\mathcal{A}_{ij}\,\mathcal{B}_{kl}$ with $P(1)=0$ and $P(2)=1$. [15] [16] In the resulting matrix the multiplying minus signs (in green color) in the indicated positions:

$$\mathcal{A}\underset{s}{\otimes}\mathcal{B}\;=\;\begin{pmatrix}*&*&*&*\\ *&*&-&-\\ *&*&*&*\\ -&-&*&*\end{pmatrix}. \tag{2.10}$$

The scalar equations obtained from (2.8,a) are the same ones as the already well known for the eight-vertex model:

$$\left\{\begin{array}{l} a(\mu-\nu,k)\ \alpha(\mu)\ \delta(\nu) + d(\mu-\nu,k)\ \beta\ \alpha(\nu) = -(-\delta(\nu))\,\gamma(\mu)\ c(\mu-\nu,k) + \alpha(\nu)\ \delta(\mu)\ \beta \\ a(\mu-\nu,k)\ \gamma(\mu)\ \beta - d(\mu-\nu,k)\,(-\delta(\mu))\,\gamma(\nu) = \beta\ \alpha(\mu)\ c(\mu-\nu,k) + \gamma(\nu)\ \beta\ \beta \\ -a(\mu-\nu,k)\,(-\delta(\mu))\,\gamma(\nu) + d(\mu-\nu,k)\ \gamma(\mu)\ \beta = \alpha(\nu)\ \delta(\mu)\ c(\mu-\nu,k) - (-\delta(\nu))\gamma(\nu)\ \beta \\ a(\mu-\nu,k)\ \beta\ \alpha(\nu) + d(\mu-\nu,k)\ \alpha(\mu)\ \delta(\nu) = \gamma(\nu)\ \beta\ c(\mu-\nu,k) + \beta\ \alpha(\mu)\ \beta \\ -a(\mu-\nu,k)\,(-\delta(\mu))\ \beta + d(\mu-\nu,k)\ \gamma(\mu)\ \gamma(\nu) = \alpha(\nu)\ \alpha(\mu)\ d(\mu-\nu,k) - (-\delta(\nu))\ \beta\ a(\mu-\nu,k) \\ \beta\ \alpha(\mu)\ \gamma(\nu) + c(\mu-\nu,k)\ \beta\ \beta = \alpha(\nu)\ \gamma(\mu)\ \beta - (-\delta(\nu))\ \delta(\mu)\ c(\mu-\nu,k) \end{array}\right. , \tag{2.11}$$

satisfied with the Jacobi elliptic functions via the Baxter method; they are the equations (3.9) in III. Appendix C (a second proof).

'Ultra locality' of the $\mathcal{L}_n^x$ operators. For the four definitions in (2.7) (x anyone of $\{a,b,c,d\}$), we demonstrate:

$$[\,\mathcal{L}_{n+1}^x(\mu,-rk) \underset{s}{\otimes} \mathcal{L}_{n+1}^x(\nu,rk)\,]\,[\,\mathcal{L}_n^x(\mu,-rk) \underset{s}{\otimes} \mathcal{L}_n^x(\nu,rk)\,] = [\,\mathcal{L}_{n+1}^x(\mu,-rk)\ \mathcal{L}_n^x(\mu,-rk)\,] \underset{s}{\otimes} [\,\mathcal{L}_{n+1}^x(\nu,rk)\ \mathcal{L}_n^x(\nu,rk)\,]\,.$$

Thanks to this property we obtain:

$$\prod_{n=1}^{\overleftarrow{N}} \Big[\,\mathcal{L}_n^x(\mu,-rk) \underset{s}{\otimes} \mathcal{L}_n^x(\nu,rk)\,\Big] = \Big[\prod_{n=1}^{\overleftarrow{N}} \mathcal{L}_n^x(\mu,-rk)\Big] \underset{s}{\otimes} \Big[\prod_{n=1}^{\overleftarrow{N}} \mathcal{L}_n^x(\nu,rk)\Big]\,. \tag{2.12}$$

Also with the Yang-Baxter type relations (2.8):

$$\mathcal{R}\,(\mu-\nu,rk) \prod_{n=1}^{\overleftarrow{N}} \Big[\,\mathcal{L}_n^{\left\{\substack{a\\d}\right\}}(\mu,-rk) \underset{s}{\otimes} \mathcal{L}_n^{\left\{\substack{a\\d}\right\}}(\nu,rk)\,\Big] = \prod_{n=1}^{\overleftarrow{N}} \Big[\,\mathcal{L}_n^{\left\{\substack{a\\d}\right\}}(\nu,-rk) \underset{s}{\otimes} \mathcal{L}_n^{\left\{\substack{a\\d}\right\}}(\mu,rk)\,\Big]\,\mathcal{R}\,(\mu-\nu,rk)\,, \tag{2.13a}$$

$$\mathcal{R}^t(\mu-\nu,rk) \prod_{n=1}^{\overleftarrow{N}} \Big[\,\mathcal{L}_n^{\left\{\substack{b\\c}\right\}}(\mu,-rk) \underset{s}{\otimes} \mathcal{L}_n^{\left\{\substack{b\\c}\right\}}(\nu,rk)\,\Big] = \prod_{n=1}^{\overleftarrow{N}} \Big[\,\mathcal{L}_n^{\left\{\substack{b\\c}\right\}}(\nu,-rk) \underset{s}{\otimes} \mathcal{L}_n^{\left\{\substack{b\\c}\right\}}(\mu,rk)\,\Big]\,\mathcal{R}^t(\mu-\nu,rk)\,, \tag{2.13b}$$

The definitions of the *fermion monodromy matrices*:

$$\overleftarrow{\mathcal{T}}_N^{\,x}(\lambda) = \prod_{n=1}^{\overleftarrow{N}} \mathcal{L}_n^x(\lambda) = \mathcal{L}_N^x(\lambda)\ \dots\ \mathcal{L}_n^x(\lambda)\ \dots\ \mathcal{L}_1^x(\lambda)\,, \tag{2.14}$$

with (2.13) and (2.12) drive to:

$$\overleftarrow{\mathcal{T}}_N^{\left\{\substack{a\\d}\right\}}(\mu,-rk) \underset{s}{\otimes} \overleftarrow{\mathcal{T}}_N^{\left\{\substack{a\\d}\right\}}(\nu,rk) = \mathcal{R}^{-1}(\mu-\nu,rk)\Big[\,\overleftarrow{\mathcal{T}}_N^{\left\{\substack{a\\d}\right\}}(\nu,-rk) \underset{s}{\otimes} \overleftarrow{\mathcal{T}}_N^{\left\{\substack{a\\d}\right\}}(\mu,rk)\,\Big]\,\mathcal{R}\,(\mu-\nu,rk)\,, \tag{2.15a}$$

$$\overleftarrow{\mathcal{T}}_N^{\left\{\substack{b\\c}\right\}}(\mu,-rk) \underset{s}{\otimes} \overleftarrow{\mathcal{T}}_N^{\left\{\substack{b\\c}\right\}}(\nu,rk) = \mathcal{R}^{t^{-1}}(\mu-\nu,rk)\Big[\,\overleftarrow{\mathcal{T}}_N^{\left\{\substack{b\\c}\right\}}(\nu,-rk) \underset{s}{\otimes} \overleftarrow{\mathcal{T}}_N^{\left\{\substack{b\\c}\right\}}(\mu,rk)\,\Big]\,\mathcal{R}^t(\mu-\nu,rk)\,. \tag{2.15b}$$

valid by analytic continuation for all the values of μ and ν, which include those ones for which the determinant of $\mathcal{R}$ is zero.

With the definition of a *'supertrace'* (for any of the four cases):

$$\mathrm{str}\,(\,\mathcal{L}\,) = \mathrm{tr}\,(\sigma^z\,\mathcal{L}\,)\,, \tag{2.16}$$

it is easy to prove (with all the cyclic possibilities):

$$\mathrm{str}\Big(\mathcal{L}_N^x\ \dots\ \mathcal{L}_n^x\ \dots\ \mathcal{L}_1^x\Big) = \mathrm{str}\Big(\mathcal{L}_{N-1}^x\ \dots\ \mathcal{L}_n^x\ \dots\ \mathcal{L}_1^x\mathcal{L}_N^x\Big) = \mathrm{str}\Big(\mathcal{L}_1^x\mathcal{L}_N^x\ \dots\ \mathcal{L}_n^x\ \dots\ \mathcal{L}_2^x\Big)\,, \tag{2.17}$$

The *fermion transfer matrices* :

$$\mathfrak{T}_N^x(\lambda,rk) = \mathrm{str}\,(\overleftarrow{\mathcal{T}}_N^{\,x}(\lambda,rk)) = \mathrm{str}\Big(\prod_{n=1}^{\overleftarrow{N}} \mathcal{L}_n^x(\lambda,rk)\Big) = \mathrm{str}\Big(\prod_{n=1}^{\overleftarrow{N}}\Big[\sum_{m_n=0}^{\infty} {}_{m_n}\mathcal{L}_n^x(rk)\,(\lambda-\eta)^{m_n}\Big]\Big)\,. \tag{2.18}$$

We also apply the supertrace to equations of the type (2.15):

$$\mathrm{str}\Big[\,\overleftarrow{\mathcal{T}} \underset{s}{\otimes} \overleftarrow{\mathcal{T}'}\,\Big] = \mathrm{tr}\Big[(\sigma^z\otimes\sigma^z)\Big(\,\overleftarrow{\mathcal{T}} \underset{s}{\otimes} \overleftarrow{\mathcal{T}'}\,\Big)\Big] = \mathrm{tr}\Big[\Big(\sigma^z\,\overleftarrow{\mathcal{T}}\,\Big)\otimes\Big(\sigma^z\,\overleftarrow{\mathcal{T}'}\,\Big)\Big] = \Big[\mathrm{str}\Big(\,\overleftarrow{\mathcal{T}}\,\Big)\Big]\Big[\mathrm{str}\Big(\,\overleftarrow{\mathcal{T}'}\,\Big)\Big]\,. \tag{2.19}$$

With the supertrace in equations (2.15) and using $[\,\sigma^z\otimes\sigma^z\,,\,\mathcal{R}\,] = [\,\sigma^z\otimes\sigma^z\,,\,\mathcal{R}^t\,] = 0$, we obtain for the four cases:

$$\mathfrak{T}_N^x(\mu,-rk)\ \mathfrak{T}_N^x(\nu,rk) = \mathfrak{T}_N^x(\nu,-rk)\ \mathfrak{T}_N^x(\mu,rk)\,, \tag{2.20}$$

In the process of obtaining conserved quantities we need a property similar to the simplifying property used for the equations (3.24)-(3.27) in the Section III. Appendix C:

$$\Big\{\mathrm{str}\Big[\Big(\,\overleftarrow{\mathcal{T}}_N^{\,x}(\eta)\Big)^{-1}\Big]\Big\}\ \mathfrak{T}_N^x(\eta) = \mathfrak{T}_N^x(\eta)\ \Big\{\mathrm{str}\Big[\Big(\,\overleftarrow{\mathcal{T}}_N^{\,x}(\eta)\Big)^{-1}\Big]\Big\} = \mathbf{1}\,,\qquad \text{with}\qquad \mathfrak{T}_N^x(\eta) = \mathrm{str}\Big(\overleftarrow{\mathcal{T}}_N^{\,x}(\eta)\Big)\,.$$

We introduce this results to obtain (2.37) after (2.36). Baxter, with the spin chain, used $\big\{T(\eta)\big\}^{-1}$.

C. The fermion XYZ Heisenberg Hamiltonians.

We do a Jordan Wigner transformation of the XYZ Heisenberg Hamiltonian (in (3.11)), with $v_k^{+\,2} = -\sigma_k^z \longleftrightarrow u_k^{+\,2} = \tilde{\mathbf{n}}_k - \mathbf{n}_k$. We drop the term with $\frac{\mathbf{J_z}}{2}\sum_{n=1}^{N}\mathbf{1}$, and we write the fermion Heisenberg Hamiltonians ($\epsilon = +$):

$$\mathcal{H}^{+}(\mathbf{J_z}, \mathbf{J_x}, \mathbf{J_y}) = -\sum_{n=1}^{N}\Big\{ \mathbf{J_z}\,(\mathbf{n}_{n+1}\mathbf{n}_n + \tilde{\mathbf{n}}_{n+1}\tilde{\mathbf{n}}_n) + \frac{\mathbf{J_x}+\mathbf{J_y}}{2}\,(\mathbf{a}^{\dagger}_{n+1}\mathbf{a}_n - \mathbf{a}_{n+1}\mathbf{a}^{\dagger}_n) + \frac{\mathbf{J_x}-\mathbf{J_y}}{2}\,(\mathbf{a}_{n+1}\mathbf{a}_n - \mathbf{a}^{\dagger}_{n+1}\mathbf{a}^{\dagger}_n)\Big\}\,, \tag{2.21}$$

$$\begin{aligned}\mathcal{H}^{+}(\alpha_1,\beta,\ \delta_1) &= \frac{1}{\beta}\sum_{n=1}^{N}\Big\{\ \alpha_1(\mathbf{n}_{n+1}\ \mathbf{n}_n + \tilde{\mathbf{n}}_{n+1}\tilde{\mathbf{n}}_n)\ +\ (\mathbf{a}^{\dagger}_{n+1}\ \mathbf{a}_n - \mathbf{a}_{n+1}\ \mathbf{a}^{\dagger}_n)\ +\ \delta_1\,(\mathbf{a}_{n+1}\ \mathbf{a}_n - \mathbf{a}^{\dagger}_{n+1}\ \mathbf{a}^{\dagger}_n)\Big\} = \\ &= \frac{1}{\beta}\sum_{n=1}^{N}\Big\{\ \alpha_1(\mathbf{n}_{n+1}\ \mathbf{n}_n + \tilde{\mathbf{n}}_{n+1}\tilde{\mathbf{n}}_n)\ +\ (\mathbf{a}^{\dagger}_{n+1}\ +\delta_1\,\mathbf{a}_{n+1})\,\mathbf{a}_n\ -\ (\mathbf{a}_{n+1}\ +\delta_1\,\mathbf{a}^{\dagger}_{n+1})\,\mathbf{a}^{\dagger}_n\ \Big\}\ .\end{aligned} \tag{2.22}$$

If we use for the Jordan Wigner transformation, instead of the previous one, this other one: $v_k^{-\,2} = \sigma_k^z \longleftrightarrow u_k^{-\,2} = \mathbf{n}_k - \tilde{\mathbf{n}}_k$, we get ($\epsilon = -$):

$$\mathcal{H}^{-}(\alpha_1,\beta,\ \delta_1) = \frac{1}{\beta}\sum_{n=1}^{N}\Big\{\ \alpha_1(\mathbf{n}_{n+1}\ \mathbf{n}_n + \tilde{\mathbf{n}}_{n+1}\tilde{\mathbf{n}}_n)\ -\ (\mathbf{a}^{\dagger}_{n+1}\ +\delta_1\,\mathbf{a}_{n+1})\,\mathbf{a}_n\ +\ (\mathbf{a}_{n+1}\ +\delta_1\,\mathbf{a}^{\dagger}_{n+1})\,\mathbf{a}^{\dagger}_n\ \Big\}\ . \tag{2.23}$$

Also, in order to relate them with the results: in equations (2.8) (see Section III. Appendix C, equations (3.11) - (3.15)) and with equations of the type (3.18), we define:

$$\mathcal{H}^{+}(\mathbf{J_z},\ \mathbf{J_y}, \mathbf{J_x}) = -\sum_{n=1}^{N}\Big\{ \mathbf{J_z}\,(\mathbf{n}_{n+1}\mathbf{n}_n + \tilde{\mathbf{n}}_{n+1}\tilde{\mathbf{n}}_n) + \frac{\mathbf{J_x}+\mathbf{J_y}}{2}\,(\mathbf{a}^{\dagger}_{n+1}\mathbf{a}_n - \mathbf{a}_{n+1}\mathbf{a}^{\dagger}_n) - \frac{\mathbf{J_x}-\mathbf{J_y}}{2}\,(\mathbf{a}_{n+1}\mathbf{a}_n - \mathbf{a}^{\dagger}_{n+1}\mathbf{a}^{\dagger}_n)\Big\}\,, \tag{2.24}$$

$$\mathcal{H}^{+}(\alpha_1, \beta, -\delta_1) = \frac{1}{\beta}\sum_{n=1}^{N}\Big\{\ \alpha_1(\mathbf{n}_{n+1}\ \mathbf{n}_n + \tilde{\mathbf{n}}_{n+1}\tilde{\mathbf{n}}_n)\ +\ (\mathbf{a}^{\dagger}_{n+1} - \delta_1\,\mathbf{a}_{n+1})\,\mathbf{a}_n\ -\ (\mathbf{a}_{n+1} - \delta_1\,\mathbf{a}^{\dagger}_{n+1})\mathbf{a}^{\dagger}_n\ \Big\}. \tag{2.25}$$

We write in a compact way four different forms of the fermion Heisenberg Hamiltonians:

$$\begin{aligned}\mathcal{H}^{\epsilon}(\alpha_1, \beta, r\,\delta_1) &= \mathcal{H}^{r\,\epsilon} = \sum_{n=1}^{N}\mathcal{H}^{\epsilon}_{n+1,n}(\alpha_1, \beta, r\,\delta_1) = \sum_{n=1}^{N}\mathcal{H}^{r\,\epsilon}_{n+1,n} = \\ &= \frac{1}{\beta}\sum_{n=1}^{N}\Big\{\alpha_1\,(\mathbf{n}_{n+1}\ \mathbf{n}_n + \tilde{\mathbf{n}}_{n+1}\ \tilde{\mathbf{n}}_n) + \epsilon\,[\,(\mathbf{a}^{\dagger}_{n+1}\mathbf{a}_n - \mathbf{a}_{n+1}\mathbf{a}^{\dagger}_n) + r\,\delta_1\,(\mathbf{a}_{n+1}\mathbf{a}_n - \mathbf{a}^{\dagger}_{n+1}\mathbf{a}^{\dagger}_n)\,]\Big\}\,,\end{aligned} \tag{2.26}$$

with r and ϵ either $+$ or $-$.

And, accordingly we write the generating functions, using the series development for the α, γ and δ functions (see (3.10)):

$$\begin{aligned}\widetilde{\mathcal{H}}^{r\,\epsilon} &= \widetilde{\mathcal{H}}^{\epsilon}(\alpha, \gamma, \beta, r\,\delta) = \widetilde{\mathcal{H}}^{r\,\epsilon}_{N+1,N} + \cdots + \widetilde{\mathcal{H}}^{r\,\epsilon}_{n+1,n} + \cdots + \widetilde{\mathcal{H}}^{r\,\epsilon}_{2,1} = \sum_{n=1}^{N}\widetilde{\mathcal{H}}^{\epsilon}_{n+1,n}(\alpha, \gamma, \beta, r\,\delta) = \sum_{n=1}^{N}\widetilde{\mathcal{H}}^{r\,\epsilon}_{n+1,n} = \\ &= \frac{1}{\beta}\sum_{n=1}^{N}\Big\{(\alpha\mathbf{n}_{n+1} + \beta\,\tilde{\mathbf{n}}_{n+1})\,\mathbf{n}_n + (\alpha\tilde{\mathbf{n}}_{n+1} + \beta\,\mathbf{n}_{n+1})\,\tilde{\mathbf{n}}_n\ +\ \epsilon\,[\,(\gamma\,\mathbf{a}^{\dagger}_{n+1} + r\,\delta\,\mathbf{a}_{n+1})\,\mathbf{a}_n\ -\ (\gamma\,\mathbf{a}_{n+1} + r\,\delta\,\mathbf{a}^{\dagger}_{n+1})\,\mathbf{a}^{\dagger}_n\,]\Big\} = \\ &= \sum_{n=1}^{N}\Big\{\mathbf{1}\Big\}\ +\ \frac{1}{\beta}\sum_{n=1}^{N}\Big\{\alpha_1\,(\mathbf{n}_{n+1}\,\mathbf{n}_n + \tilde{\mathbf{n}}_{n+1}\,\tilde{\mathbf{n}}_n)\ +\ \epsilon\,[\,(\mathbf{a}^{\dagger}_{n+1} + r\,\delta_1\,\mathbf{a}_{n+1})\,\mathbf{a}_n\ -\ (\mathbf{a}_{n+1} + r\,\delta_1\,\mathbf{a}^{\dagger}_{n+1})\,\mathbf{a}^{\dagger}_n\,]\Big\}(\lambda-\eta)\ + \\ &\quad +\ \frac{1}{\beta}\sum_{n=1}^{N}\Big\{\,\alpha_2\,(\mathbf{n}_{n+1}\,\mathbf{n}_n + \tilde{\mathbf{n}}_{n+1}\,\tilde{\mathbf{n}}_n)\ +\ \epsilon\,r\,\delta_2\,(\mathbf{a}_{n+1}\ \mathbf{a}_n\ -\ \mathbf{a}^{\dagger}_{n+1}\ \mathbf{a}^{\dagger}_n\,)\Big\}\,(\lambda-\eta)^2\ +\ \ldots\ .\end{aligned} \tag{2.27}$$

They satisfy:

$$\widetilde{\mathcal{H}}^{r\,\epsilon}_{n+1,n} = \widetilde{\mathcal{H}}^{-r\,\epsilon}_{n,n+1} \tag{2.28}$$

Concerning the notation in (2.27) (see (3.11)-(3.15)): the functions α, β and γ have positive and negative values depending in the values of their variables (μ, ν, η, k), so that it is not essential the writing of the relative signs represented by r and ϵ. Even so, we have introduced the notation with the changing in the signs represented by the r and ϵ to emphasize the appearance of relative opposite signs in various of the equations that follows after the relations (2.8), last example of this in the equation (2.34).

There is a correspondence between the first two summing terms in (2.27) (see (3.11)-(3.12)) and the ones appearing in Baxter and in Takhtadzhan and Faddeev (their equations (2.11)-(2.13)) [2] up to the factor $\beta = sn(2\eta) = -\frac{1}{\mathbf{J}}$ (see their equation (2.11)).

D. The conserved quantities.

For the cases established in (2.7), with $\left(\mathcal{L}^x_n(\eta) = {}_0\mathcal{L}^x_n \right)$:

$$\mathcal{L}^a_n(\eta) = \beta \begin{pmatrix} \mathrm{i}\,\mathbf{n} & \mathbf{a} \\ -\mathrm{i}\,\mathbf{a}^\dagger & \tilde{\mathbf{n}} \end{pmatrix}_n , \quad \mathcal{L}^b_n(\eta) = \beta \begin{pmatrix} -\mathrm{i}\,\mathbf{n} & \mathbf{a} \\ \mathrm{i}\,\mathbf{a}^\dagger & \tilde{\mathbf{n}} \end{pmatrix}_n , \quad \mathcal{L}^c_n(\eta) = \beta \begin{pmatrix} \mathbf{n} & \mathrm{i}\mathbf{a} \\ \mathbf{a}^\dagger & -\mathrm{i}\,\tilde{\mathbf{n}} \end{pmatrix}_n , \quad \mathcal{L}^d_n(\eta) = \beta \begin{pmatrix} \mathbf{n} & -\mathrm{i}\mathbf{a} \\ \mathbf{a}^\dagger & \mathrm{i}\,\tilde{\mathbf{n}} \end{pmatrix}_n , \tag{2.29}$$

we obtain:

$$\begin{cases} \left[\mathcal{L}^x_{n+1}(\lambda, rk)\ \mathcal{L}^x_n(\eta)\right] = \left[\mathcal{L}^x_{n+1}(\eta)\ \mathcal{L}^x_n(\eta)\right]\ \widetilde{\mathcal{H}}^{\,r\,\epsilon}_{\,n+1,n} \\ \left[\mathcal{L}^x_n(\eta)\ \mathcal{L}^x_{n-1}(\lambda, rk)\right] = \widetilde{\mathcal{H}}^{\,(-r)\epsilon}_{\,n,n-1}\ \left[\mathcal{L}^x_n(\eta)\ \mathcal{L}^x_{n-1}(\eta)\right] \end{cases} , \qquad \begin{cases} \epsilon = + \quad \text{with} \quad x \in \{a\,,\, b\} \\ \epsilon = - \quad \text{with} \quad x \in \{c\,,\, d\} \end{cases} . \tag{2.30}$$

By imposing: one of the fermion local transition matrices defined in $\lambda = \eta$ (in the n^{th} position of the chain) we obtain:

$$\left[\mathcal{L}^x_N(\lambda)\ \ldots\ \mathcal{L}^x_{n+1}(\lambda)\ \mathcal{L}^x_n(\eta)\ \mathcal{L}^x_{n-1}(\lambda)\ \ldots\ \mathcal{L}^x_1(\lambda)\right](rk) = \left\{ \overleftarrow{\prod_{j=2}^{n}} \widetilde{\mathcal{H}}^{\,(-r)\epsilon}_{\,j,j-1} \right\} \left[\, \overleftarrow{\mathcal{T}}^{\,x}_{N}\ (\eta) \right] \left\{ \overleftarrow{\prod_{i=n}^{N-1}} \widetilde{\mathcal{H}}^{\,r\,\epsilon}_{\,i+1,i} \right\} . \tag{2.31}$$

In particular:

$$\begin{cases} \left[\mathcal{L}^x_N(\lambda)\ \ldots\ \mathcal{L}^x_n(\lambda)\ \ldots\ \mathcal{L}^x_2(\lambda)\ \mathcal{L}^x_1(\eta)\right](\ rk) = \left[\overleftarrow{\mathcal{T}}^{\,x}_{N}\ (\eta)\right]\left(\overleftarrow{\prod_{i=1}^{N-1}} \widetilde{\mathcal{H}}^{\,r\,\epsilon}_{\,i+1,i} \right) \\ \left[\mathcal{L}^x_N(\eta)\ \mathcal{L}^x_{N-1}(\lambda)\ \ldots\ \mathcal{L}^x_n(\lambda)\ \ldots\ \mathcal{L}^x_1(\lambda)\right](-rk) = \left(\overleftarrow{\prod_{j=2}^{N}} \widetilde{\mathcal{H}}^{\,r\epsilon}_{\,j,j-1} \right)\left[\overleftarrow{\mathcal{T}}^{\,x}_{N}\ (\eta)\right] \end{cases} . \tag{2.32}$$

For the periodic boundary case: $\mathbf{a}^\dagger_{N+1} = \mathbf{a}^\dagger_1$, $\mathbf{a}_{N+1} = \mathbf{a}_1$. We also use $\left(\mathcal{L}^x_{N-1}\ \ldots\ \mathcal{L}^x_n\ \ldots\ \mathcal{L}^x_1\ \mathcal{L}^x_N\right)$ and the other cyclic possibilities, which will be included later for obtaining (2.34) in relation to (2.17).

With the series developments (for the four different cases):

$$\mathcal{L}^x_n(\lambda) = \sum_{m_n=0}^{\infty} {}_{m_n}\mathcal{L}^x_n\ (\lambda-\eta)^{m_n} \qquad \text{and} \qquad \widetilde{\mathcal{H}}^{\,r\,\epsilon} = \sum_{n=1}^{N} \sum_{m_n=0}^{\infty} {}_{m_n}\widetilde{\mathcal{H}}^{\,r\,\epsilon}_{\,n+1,n}\ (\lambda-\eta)^{m_n} , \tag{2.33}$$

we write the fermion transfer matrices (2.18) in a series development, up to the order $N-1$ (we use the symbol $\eqcirc$), by substituting (2.33) in (2.32):

$$\begin{matrix} \mathfrak{T}^{\,x}_N(\lambda,\ \ rk) \eqcirc \mathfrak{T}^{\,x}_N(\eta)\ \left(\sum_{l=0}^{N-1} C^x_l\ (\eta, rk)\ (\lambda-\eta)^l \right) \\ \mathfrak{T}^{\,x}_N(\lambda, -rk) \eqcirc \left(\sum_{l=0}^{N-1} C^x_l(\eta, rk)\ (\lambda-\eta)^l \right)\ \mathfrak{T}^{\,x}_N(\eta) \end{matrix} , \qquad \mathfrak{T}^{\,x}_N(\eta) = \mathrm{str}\left[\, \overleftarrow{\mathcal{T}}^{\,x}_{N}\ (\eta) \right] . \tag{2.34}$$

where the C^x_l are sums of products of various ${}_{m_n}\widetilde{\mathcal{H}}^{\,r\,\epsilon}_{\,n+1,n}$, the ϵ and the ϵrk in the way they appear in (2.27) and (2.30). We have to take care of the orderings in the products. The first three and the last one are:

$$\begin{aligned} C^x_0\ (\eta,\ \ k) &= \mathbf{1} \\ C^x_1\ (\eta, rk) &= {}_1\widetilde{\mathcal{H}}^{\,\epsilon}(\alpha_1\,,\, \gamma = 1\,,\, \beta\,,\, r\,\delta_1\,) = \mathcal{H}^{\,\epsilon}(\alpha_1\,,\, \beta\,,\, r\,\delta_1\,) \\ C^x_2\ (\eta, rk) &= \Big\{ \sum_{n=1}^{N} {}_2\widetilde{\mathcal{H}}_{n,n+1} + \Big[\sum_{m=2}^{N-1} {}_1\widetilde{\mathcal{H}}_{m+1,m}\ \big(\sum_{n=1}^{m-1} {}_1\widetilde{\mathcal{H}}_{n+1,n}\ \big) + {}_1\widetilde{\mathcal{H}}_{1,N} \big(\sum_{n=2}^{N-1} {}_1\widetilde{\mathcal{H}}_{\,n+1,n} \big) + {}_1\widetilde{\mathcal{H}}_{2,1}\ {}_1\widetilde{\mathcal{H}}_{\,1,N} \Big] \Big\}^{r\,\epsilon}(\eta, k) \end{aligned} , \tag{2.35}$$

$$\bullet\ \bullet\ \bullet$$

$$\begin{cases} C^x_{N-1}(\eta, rk) = \Big\{ \cdots + \big({}_1\widetilde{\mathcal{H}}_{N,N-1} \ldots {}_1\widetilde{\mathcal{H}}_{2,1} \big) + \cdots + \big({}_1\widetilde{\mathcal{H}}_{n-1,n-2} \ldots {}_1\widetilde{\mathcal{H}}_{1,N} \ldots {}_1\widetilde{\mathcal{H}}_{n+1,n} \big) + \cdots + \big({}_1\widetilde{\mathcal{H}}_{N-1,N-2} \ldots {}_1\widetilde{\mathcal{H}}_{1,N} \big) \Big\}^{r\,\epsilon}(\eta, k) \\ (\ \text{we skip:} \qquad\qquad \text{no}\ \ {}_1\widetilde{\mathcal{H}}_{1,N} \qquad\qquad\qquad\qquad \text{no}\ \ {}_1\widetilde{\mathcal{H}}_{n,n-1} \qquad\qquad\qquad\qquad \text{no}\ \ {}_1\widetilde{\mathcal{H}}_{N,N-1}\ ) \end{cases}$$

where we have used the periodicity condition: $\widetilde{\mathcal{H}}^{\,r\,\epsilon}_{\,N+1,N} = \widetilde{\mathcal{H}}^{\,r\,\epsilon}_{\,1,N}$, and also: $\widetilde{\mathcal{H}}^{\,r\,\epsilon}_{\,n+1,n}(\alpha_{m_n}, \gamma_{m_n}, \beta, \delta_{m_n}) = {}_{m_n}\widetilde{\mathcal{H}}^{\,r\,\epsilon}_{\,n+1,n}$.

We introduce these results in the relations (2.20) $\left(\mathfrak{T}^{\,x}_N(\mu, rk)\ \mathfrak{T}^{\,x}_N(\nu, -rk) = \mathfrak{T}^{\,x}_N(\nu, rk)\ \mathfrak{T}^{\,x}_N(\mu, -rk) \right)$:

$$\begin{aligned} &\mathfrak{T}^{\,x}_N(\eta)\ \left(\sum_{l=0}^{N-1} C^x_l\ (\eta,\ rk)\ (\mu-\eta)^l \right) \left(\sum_{ll=0}^{N-1} C^x_{ll}\ (\eta,\ rk)\ (\nu-\eta)^{ll} \right) \mathfrak{T}^{\,x}_N(\eta) = \\ &= \mathfrak{T}^{\,x}_N(\eta)\ \left(\sum_{n=0}^{N-1} C^x_n\ (\eta,\ rk)\ (\nu-\eta)^n \right) \left(\sum_{nn=0}^{N-1} C^x_{nn}(\eta,\ rk)\ (\mu-\eta)^{nn} \right) \mathfrak{T}^{\,x}_N(\eta) \end{aligned} . \tag{2.36}$$

As k can be positive and negative, we can eliminate the sign r in these expressions. Finally:

$$\left[\, C^x_m\ (\eta,\ k)\ \ ,\ \ C^x_n\ (\eta,\ k) \,\right] = \mathbf{0}\,, \qquad \{\, m\,,\, n\,\} \in \{\, 0,\ \ldots\ , N-1\,\}\ . \tag{2.37}$$

The $C^x_m(\eta,\ k)$ contain among them the fermionic *XYZ* Heisenberg Hamiltonians. They are conserved quantities,

Conclusion. Further work.

The two main points are: the fermionization of the Heisenberg chain and the meaning of the Jordan Wigner transformation.

For the first one: we have presented results for several fermion Hamiltonians, without the free-fermion restriction. For them, in the form (1.1) and extended to the forms (2.26) we still can look for interpretations in condensed matter, particularly for the superconductivity. But, in the form (1.3): is it possible to develop a related quantum inverse problem with fermion operators?

With the second one: are fermions local or non-local objects? Let us recall one more historical note. Both, the concept of the spin and the Jordan Wigner transformation appeared a hundred years ago in relation to atomic physics, energies were behind, but there were not the time and the space in their definition. For both, the Pauli matrices (spin matrices) are the relevant **algebraic tools**. The spin in order to explain the Stern Gerlach experiment and the position of some spectral lines under the action of magnetic fields. The spin was viewed as an intrinsic rotation of an electron (fermion) in an **internal space** (something we do not know about); besides, if the electrons would be rotating (internally?), classically they would rotate at faster velocities than the velocity of the light. Physicist ignore that interpretation. [17] The Jordan Wigner transformation, in its origin also without reference to the space. Later on, its application to the fermionization of the XYZ model introduces the space in one or another way.

We suggest that the fermions (leptons and quarks) are local objects (point-like?) as it concern the actual experiments at present time. That is also its usage in this research (the m sub-index instead of the $\mathcal{M}$). The non-local aspect is clearly exposed in [14] with the terms $\hat{t}_{ij}(\{\sigma^z_{i'}\})\,\sigma^+_i\sigma^-_j$ in an XYZ Hamiltonian, which in the process of fermionization (thanks to the Jordan Wigner transformation) end up by having the form $t_{ij}\,c^\dagger_i c_j$. Professor Wen proposes an interpretation of the fermions as "ends of condensed strings" in a string-net theory. But, where do these condensed strings 'live'?, in an **internal space**?

This author proposes something different: *a WHOLE chain as it appears in the Jordan Wigner transformation, but with a generalization using discrete four dimensional complex vectors ($\mathbb{C}^4$) provide a geometrical representation of the creation and annihilation operators of a LEPTON or a QUARK*, obviously non-local. [13] The values of their electric charges emerge in a simple way. Is $\mathbb{C}^{4N}$ containing an **internal space**? [18] [19]

III. APPENDIX: REMINDERS.

A. For the notation.

The Pauli spin matrices σ^j with the identity constitute a basis for the linear space of $2x2$ matrices over the complex field, the **Pauli basis**. It is also a basis for the vectors in $\mathbb{R}^4$. It has the form:

$$\left\{ \; \mathbb{1} = \begin{pmatrix}1&0\\0&1\end{pmatrix}, \quad \sigma^z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \quad \sigma^x = \begin{pmatrix}0&1\\1&0\end{pmatrix}, \quad \sigma^y = \begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix} \; \right\}. \tag{3.1}$$

With the simple change: $\sigma^\pm = \frac{1}{2}(\sigma^x \pm \mathrm{i}\,\sigma^y)$, $\overset{\wedge}{\sigma} = \frac{1}{2}(\mathbb{1}+\sigma^z)$, $\overset{\vee}{\sigma} = \frac{1}{2}(\mathbb{1}-\sigma^z)$, we write the **canonical set** :

$$\left\{ \; \sigma^+ = \begin{pmatrix}0&1\\0&0\end{pmatrix}, \quad \sigma^- = \begin{pmatrix}0&0\\1&0\end{pmatrix}, \quad \overset{\wedge}{\sigma} = \sigma^+\sigma^- = \begin{pmatrix}1&0\\0&0\end{pmatrix}, \quad \overset{\vee}{\sigma} = \sigma^-\sigma^+ = \begin{pmatrix}0&0\\0&1\end{pmatrix} \; \right\}. \tag{3.2}$$

We consider an ordered set with N positions. We assign to every position in the set any of these spin matrices and we denote them with a sub-index, so that we write $\mathbf{A}_k$ for σ^j in the k^{th} position of the set with $j = \{z, x, y\}$. Similarly for the matrices in the canonical set, definition (3.2) with $j = \{+, -, \wedge, \vee\}$. With a particular $\pm\sigma^z$ for $\mathbf{A}$, we write a capital sub-index $M-1$ to denote the same $\mathbf{A}$ matrix for all the sites, from the 1 to the $m-1$ sites: $\mathbf{A}_{M-1} = \prod_{k=1}^{m-1}\mathbf{A}_k$ $(2 \le m \le N)$. We use a *Mathcal* sub-index $\mathcal{M}$ in the following way: $\mathbf{U}_{\mathcal{M}} = \mathbf{A}_{M-1}\Sigma_m$, with $\mathbf{A}$ either $+\sigma^z$ or $-\sigma^z$ and Σ either σ^+ or σ^-. We also implement these notations with the fermion operators. Schematically:

$$\begin{array}{lclcccccccccccccc}
\mathbb{1}_N & = & \mathbb{1} & = & \mathbb{1} & \otimes & \dots & \otimes & \mathbb{1} & \otimes & \mathbb{1} & \otimes & \mathbb{1} & \otimes & \dots & \otimes & \mathbb{1} \\
\lambda\,\Sigma_m & = & (\lambda\,\Sigma)_m & \equiv & \mathbb{1} & \otimes & \dots & \otimes & \mathbb{1} & \otimes & \lambda\Sigma & \otimes & \mathbb{1} & \otimes & \dots & \otimes & \mathbb{1} \\
\mathbf{A}_{M-1} & \equiv & \prod_{k=1}^{m-1}\mathbf{A}_k & = & \mathbf{A} & \otimes & \dots & \otimes & \mathbf{A} & \otimes & \mathbb{1} & \otimes & \mathbb{1} & \otimes & \dots & \otimes & \mathbb{1} \\
\mathbf{U}_{\mathcal{M}} & \equiv & \mathbf{A}_{M-1}\Sigma_m & = & \mathbf{A} & \otimes & \dots & \otimes & \mathbf{A} & \otimes & \Sigma & \otimes & \mathbb{1} & \otimes & \dots & \otimes & \mathbb{1} \\
 & & & & \uparrow & & & & \uparrow & & \uparrow & & & & & & \uparrow \\
 & & & & 1 & & & & m-1 & & m & & & & & & N
\end{array},$$

We see this algebraic presentation in Takhtadzhan and Faddeev, with the explicit treatment of the matrix direct products and the distinction of operating over $\mathbb{C}^{2^N}$ or over $\mathbb{C}^2$ [2]. And also in Wen's book [14].

The Pauli spin matrices ($\{j,k,l\} = \{x,y,z\}$) satisfy:

$$\sigma^j_m\,\sigma^k_m = \delta^{jk}\,\mathbb{1} + \mathrm{i}\,\epsilon^{jkl}\sigma^l_m, \qquad \text{so that:} \qquad \begin{cases} [\,\sigma^j_m\,,\sigma^k_m\,] = \sigma^j_m\,\sigma^k_m - \sigma^k_m\,\sigma^j_m = \mathrm{i}\,2\epsilon^{jkl}\,\sigma^l_m \\ \{\,\sigma^j_m\,,\sigma^k_m\,\} = \sigma^j_m\,\sigma^k_m + \sigma^k_m\,\sigma^j_m = 2\,\delta^{jk}\,\mathbb{1} \end{cases}.$$

and, for $m_1 \neq m_2$ $\quad [\sigma^j_{m_1}, \sigma^k_{m_2}] = \sigma^j_{m_1}\sigma^k_{m_2} - \sigma^k_{m_2}\sigma^j_{m_1} = \mathbb{0}$

B. For the Jordan-Wigner transformation. [10] , [8] (page 7).

In order to obtain operators obeying the Pauli's exclusion principle, the spin matrices σ^+_m and σ^-_m are transformed in the fermion creation and annihilation operators $a^\dagger_{\mathcal{M}}$ and $a_{\mathcal{M}}$ (without spin), by means of a **Jordan Wigner transformation**, in the following way:

$$\begin{array}{l} \mathbf{a}^\dagger_{\mathcal{M}} = \mathbf{v}^{\epsilon\;2}_{M-1}\;\sigma^+_m = \sigma^+_m\;\mathbf{v}^{\epsilon\;2}_{M-1} \\ \mathbf{a}_{\mathcal{M}} = \mathbf{v}^{\epsilon\;2}_{M-1}\;\sigma^-_m = \sigma^-_m\;\mathbf{v}^{\epsilon\;2}_{M-1} \end{array} \qquad \text{with} \qquad \mathbf{v}^{\pm\;2}_{M-1} = (\mp\sigma^z)_{M-1} = \prod_{k=1}^{m-1}\left(\mathrm{e}^{b\mathrm{i}\frac{\pi}{2}(\pm\mathbb{1}+\sigma^z)}\right)_k , \tag{3.3}$$

$b \in \{+,-\}$ unessential here. $\mathbf{v}^{\epsilon\;2}_{M-1}$ either $\mathbf{v}^{+\;2}_{M-1}$ or $\mathbf{v}^{-\;2}_{M-1}$ ($\epsilon \in \{+\,,-\}$). Also, it is $\mathbf{v}^{\pm\;4}_{M-1} = \prod_{k=1}^{m-1}(\mp\sigma^z)^2_k = \mathbb{1}$, and $b\frac{1}{2}(\pm\mathbb{1}+\sigma^z) \in \{\pm\overset{\wedge}{\sigma},\ \pm\overset{\vee}{\sigma}\}$ related to the number operators.

The essence of the method, the algebraic root for the transformation, consists in:

$$\sigma^z_m\;\sigma^\pm_m\;\sigma^z_m = -\sigma^\pm_m \qquad \text{or} \qquad \sigma^z_m\;\sigma^\pm_m = -\,\sigma^\pm_m\;\sigma^z_m = \pm\,\sigma^\pm_m\;. \tag{3.4}$$

With these elements we write the following commutation relations and the desired anticommutation relations (with (3.3)):

$$\begin{array}{c} SPIN \\ \left.\begin{array}{c} [\sigma^+_{m_1},\sigma^+_{m_2}] = [\sigma^-_{m_1},\sigma^-_{m_2}] = \mathbb{0} \\ [\sigma^+_{m_1},\sigma^-_{m_2}] = \delta_{m_1m_2}\;\sigma^z_{m_1} \\ \{\sigma^+_m,\sigma^-_m\} = \mathbb{1} \end{array}\right\} \end{array} \overset{J.W.}{\longleftrightarrow} \begin{array}{c} FERMION \\ \left\{\begin{array}{l} \{\mathbf{a}^\dagger_{m_1},\mathbf{a}^\dagger_{m_2}\} = \{\mathbf{a}_{m_1},\mathbf{a}_{m_2}\} = \mathbf{0} \\ \{\mathbf{a}^\dagger_{m_1},\mathbf{a}_{m_2}\} = \delta_{m_1m_2}\;\mathbf{1} \end{array}\right. \end{array} . \tag{3.5}$$

C. For the eight-vertex model and the XYZ spin ferromagnetic chain of Heisenberg.

For the eight-vertex model. The **local transition matrix**, from now on the 2×2 matrix:

$$\begin{aligned} L_n(\lambda) &= \mathrm{w}_4\,\mathbb{1}\otimes\mathbb{1}_n + \mathrm{w}_3\,\sigma^z\otimes\sigma^z_n + \mathrm{w}_1\,\sigma^x\otimes\sigma^x_n + \mathrm{w}_2\,\sigma^y\otimes\sigma^y_n = \\ &= \begin{pmatrix} \mathrm{w}_4\,\mathbb{1}_n + \mathrm{w}_3\sigma^z_n & \mathrm{w}_1\sigma^x_n - \mathrm{i}\,\mathrm{w}_2\sigma^y_n \\ \mathrm{w}_1\sigma^x_n + \mathrm{i}\,\mathrm{w}_2\sigma^y_n & \mathrm{w}_4\,\mathbb{1}_n - \mathrm{w}_3\sigma^z_n \end{pmatrix} = \begin{pmatrix} \alpha\,\overset{\wedge}{\sigma}_n + \gamma\,\overset{\vee}{\sigma}_n & \delta\,\sigma^+_n + \beta\,\sigma^-_n \\ \beta\,\sigma^+_n + \delta\,\sigma^-_n & \gamma\,\overset{\wedge}{\sigma}_n + \alpha\,\overset{\vee}{\sigma}_n \end{pmatrix} . \end{aligned} \tag{3.6}$$

With the functions $\alpha,\ \beta,\ \gamma$ and δ parametrized in the following way:

$$\left\{\begin{array}{lll} \alpha(\lambda,\eta) & = \mathrm{w}_4(\lambda)+\mathrm{w}_3(\lambda) & = sn(\lambda+\eta\,,\,k) \\ \gamma(\lambda,\eta) & = \mathrm{w}_4(\lambda)-\mathrm{w}_3(\lambda) & = sn(\lambda-\eta\,,\,k) \\ \beta(\eta) & = \mathrm{w}_1(\lambda)+\mathrm{w}_2(\lambda) & = sn(\;2\eta\;,\;k\;) \\ \delta(\lambda,\eta) & = \mathrm{w}_1(\lambda)-\mathrm{w}_2(\lambda) & = k\,\beta\,\alpha(\lambda,\eta)\,\gamma(\lambda,\eta) \end{array}\right. \xrightarrow{\lambda\,=\,\lambda'+\eta} \left\{\begin{array}{lcl} \alpha(\lambda',\eta) & = & sn(\lambda'+2\eta\,,\,k) \\ \gamma(\lambda') & = & sn(\;\lambda'\,,\,k\,) \\ \beta(\,\eta\,) & = & sn(\;2\eta\,,\,k\;) \\ \delta(\lambda',\eta) & = & k\,\beta(\eta)\,\alpha(\lambda',\eta)\,\gamma(\lambda') \end{array}\right. . \tag{3.7}$$

The **Yang-Baxter relation**: $R(\mu-\nu,\eta,k)\;[\,L_n(\mu,\eta,k)\,\otimes\,L_n(\nu,\eta,k)\,] = [\,L_n(\nu,\eta,k)\,\otimes\,L_n(\mu,\eta,k)\,]\;R(\mu-\nu,\eta,k)$,

$$R(\mu-\nu) = \begin{pmatrix} a(\mu-\nu) & 0 & 0 & d(\mu-\nu) \\ 0 & b & c(\mu-\nu) & 0 \\ 0 & c(\mu-\nu) & b & 0 \\ d(\mu-\nu) & 0 & 0 & a(\mu-\nu) \end{pmatrix} \quad \text{with} \quad \left\{\begin{array}{l} a(\mu-\nu) = sn(\mu-\nu+2\eta\,,k) \\ c(\mu-\nu) = sn(\mu-\nu\,,k) \\ \quad b \;\; = sn(2\eta\,,k) \\ d(\mu-\nu) = k\;b\;a\;c \end{array}\right. \underset{\mu-\nu=\mu'-\nu'}{\overset{\substack{\mu=\mu'+\eta \\ \nu=\nu'+\eta}}{\longrightarrow}} \left\{\begin{array}{lcl} a(\mu'-\nu') & = & a(\mu-\nu) \\ c(\mu'-\nu') & = & c(\mu-\nu) \\ \quad b & = & sn(2\eta,\;k) \\ d(\mu'-\nu') & = & d(\mu-\nu) \end{array}\right. . \tag{3.8}$$

The Yang-Baxter relation drives to the following six scalar relations of the Jacobi elliptic functions:

$$\left.\begin{array}{l} a(\mu-\nu)\;\alpha(\mu)\;\delta(\nu) + d(\mu-\nu)\;\beta\;\alpha(\nu) = \delta(\nu)\;\gamma(\mu)\;c(\mu-\nu) + \alpha(\nu)\;\delta(\mu)\;\beta \\ a(\mu-\nu)\;\gamma(\mu)\;\beta + d(\mu-\nu)\;\delta(\mu)\;\gamma(\nu) = \beta\;\alpha(\mu)\;c(\mu-\nu) + \gamma(\nu)\;\beta\;\beta \\ a(\mu-\nu)\;\delta(\mu)\;\gamma(\nu) + d(\mu-\nu)\;\gamma(\mu)\;\beta = \alpha(\nu)\;\delta(\mu)\;c(\mu-\nu) + \delta(\nu)\;\gamma(\nu)\;\beta \\ a(\mu-\nu)\;\beta\;\alpha(\nu) + d(\mu-\nu)\;\alpha(\mu)\;\delta(\nu) = \gamma(\nu)\;\beta\;c(\mu-\nu) + \beta\;\alpha(\mu)\;\beta \\ a(\mu-\nu)\;\delta(\mu)\;\beta + d(\mu-\nu)\;\gamma(\mu)\;\gamma(\nu) = \alpha(\nu)\;\alpha(\mu)\;d(\mu-\nu) + \delta(\nu)\;\beta\;a(\mu-\nu) \\ \beta\;\alpha(\mu)\;\gamma(\nu) + c(\mu-\nu)\;\beta\;\beta = \alpha(\nu)\;\gamma(\mu)\;\beta + \delta(\nu)\,\delta(\mu)\;c(\mu-\nu) \end{array}\right\}(\eta,k)\;\;. \tag{3.9}$$

The series development of the Jacobi elliptic function $sn(\lambda,k)$: [20]

$$sn(\lambda,k) \;=\; \lambda - \tfrac{1+k^2}{3!}\,\lambda^3 + \tfrac{1+14k^2+k^4}{5!}\,\lambda^5 - \tfrac{1+135k^2+135k^4+k^6}{7!}\,\lambda^7 + \tfrac{1+1228k^2+5478k^4+1228k^6+k^8}{9!}\,\lambda^9 - \ldots\ldots\;, \qquad [\,|\lambda| < |K'|\,]$$

In our expressions we take λ equal to $\mu\pm\eta$, $\nu\pm\eta$, 2η, also to $\mu-\nu$ and $\mu-\nu+2\eta$. The functions β, α and γ verify: $\beta(-\eta,k) = -\beta(\eta,k)$, $\alpha(\lambda,\eta,k) = \gamma(\lambda,-\eta,k)$, $\gamma(-\lambda,-\eta,k) = -\gamma(\lambda,\eta,k)$; and $\delta(\lambda,\eta,rk) = rk\;\beta(\eta,k)\;\alpha(\lambda,\eta,k)\;\gamma(\lambda,\eta,k)$. α, β and γ are even in the parameter k: $\alpha(k) = \alpha(-k)$, $\beta(k) = \beta(-k)$ and $\gamma(k) = \gamma(-k)$; but the function δ is odd: $\delta(k) = -\delta(-k)$. Taking their series developments in powers of $(\lambda-\eta)$:

$$\Big\{\alpha(\lambda,\eta,k) = \sum_{m=0}^{\infty}\alpha_m(\eta,k)(\lambda-\eta)^m\Big\},\quad \Big\{\gamma(\lambda,\eta,k) = \sum_{m=1}^{\infty}\gamma_m(k)(\lambda-\eta)^m,\;\; (m \text{ odd})\Big\} \;\text{ and }\; \Big\{\delta(\lambda,\eta,k) = k\,\beta\sum_{m=1}^{\infty}\sum_{l=0}^{m}\alpha_l\gamma_{m-l}(\lambda-\eta)^m\Big\},$$

$$\left\{\begin{array}{llll}
\alpha_0 \;=\; \beta\,; & \alpha_1 \;=\; cn(2\eta,\;k)\;dn(2\eta,\;k)\,; & \alpha_2 \;=\; \ldots\,; & \alpha_3 \;=\; \ldots\,;\\
\gamma_0 \;=\; 0\,; & \gamma_1 \;=\; 1\,; & \gamma_2 \;=\; 0\,; & \gamma_3 \;=\; \ldots\,;\\
\beta_0 \;=\; \beta = sn(2\eta,k); & \beta_1 \;=\; 0\,; & \beta_2 \;=\; 0\,; & \beta_3 \;=\; 0\,;\\
\delta_0 \;=\; 0\,; & \delta_1 \;=\; k\,\beta^2\,; & \delta_2 \;=\; k\,\beta\,\alpha_1\,; & \delta_3 \;=\; \ldots\,.
\end{array}\right. \tag{3.10}$$

The **Heisenberg Hamiltonian**:

$$\begin{aligned} H(\mathbf{J_z},\mathbf{J_x},\mathbf{J_y}) \;&=\; \sum_{n=1}^{N} H_{n+1,n} \;=\; -\frac{1}{2}\sum_{n=1}^{N}\Big\{\mathbf{J_z}\,\sigma^z_{n+1}\,\sigma^z_n + \mathbf{J_x}\,\sigma^x_{n+1}\,\sigma^x_n + \mathbf{J_y}\,\sigma^y_{n+1}\,\sigma^y_n\Big\} \;=\\ &=-\frac{1}{2}\sum_{n=1}^{N}\Big\{\mathbf{J_z}\,\sigma^z_{n+1}\sigma^z_n \;+\; (\mathbf{J_x}+\mathbf{J_y})\,(\sigma^+_{n+1}\sigma^-_n+\sigma^-_{n+1}\sigma^+_n) \;+\; (\mathbf{J_x}-\mathbf{J_y})\,(\sigma^+_{n+1}\sigma^+_n+\sigma^-_{n+1}\sigma^-_n)\Big\}\,. \end{aligned} \tag{3.11}$$

We relate it with the eight-vertex model:

$$\left\{\begin{array}{lllll}
\mathbf{J_z} &=& \mathbf{J}\;cn(2\eta,\;k)\;dn(2\eta,\;k) &=& \mathbf{J}\;\;\alpha_1\\
\mathbf{J_x} &=& \mathbf{J}\,(\,1\;+\;k\;sn^2(2\eta,\;k)) &=& \mathbf{J}\,(1\;+\;k\,\beta^2)\\
\mathbf{J_y} &=& \mathbf{J}\,(\,1\;-\;k\;sn^2(2\eta,\;k)) &=& \mathbf{J}\,(1\;-\;k\,\beta^2)\\
 & & \mathbf{J} &=& -\beta^{-1}
\end{array}\right. ,\qquad
\left\{\begin{array}{lllll}
\alpha_1 &=& \mathbf{J_z}/\mathbf{J} &=& cn(2\eta,k)\;\;dn(2\eta,k)\\
\gamma_1 &=& (\mathbf{J_x}+\mathbf{J_y})/(2\mathbf{J}) &=& 1\\
\delta_1 &=& (\mathbf{J_x}-\mathbf{J_y})/(2\mathbf{J}) &=& k\,\beta^2 \;=\; k\;sn^2(2\eta,k)
\end{array}\right. \quad : \tag{3.12}$$

rewriting it in the form:

$$H(\,\alpha_1,\beta,\;\;\delta_1\,) \;=\; \frac{1}{\beta}\sum_{n=1}^{N}\Big\{-\frac{\alpha_1}{2}\mathbb{1} + \alpha_1\,(\hat{\sigma}_{n+1}\hat{\sigma}_n + \check{\sigma}_{n+1}\check{\sigma}_n) \;+\; (\sigma^+_{n+1}\sigma^-_n+\sigma^-_{n+1}\sigma^+_n) \;+\; k\,\beta^2\,(\sigma^+_{n+1}\sigma^+_n+\sigma^-_{n+1}\sigma^-_n)\Big\}\,. \tag{3.13}$$

Also:

$$H(\,\alpha_1,\beta,-\delta_1\,) \;=\; \frac{1}{\beta}\sum_{n=1}^{N}\Big\{-\frac{\alpha_1}{2}\mathbb{1} + \alpha_1\,(\hat{\sigma}_{n+1}\hat{\sigma}_n + \check{\sigma}_{n+1}\check{\sigma}_n) \;+\; (\sigma^+_{n+1}\sigma^-_n+\sigma^-_{n+1}\sigma^+_n) \;-\; k\,\beta^2\,(\sigma^+_{n+1}\sigma^+_n+\sigma^-_{n+1}\sigma^-_n)\Big\}\,, \tag{3.14}$$

in correspondence to:

$$H(\mathbf{J_z}\,,\,\mathbf{J_y}\,,\,\mathbf{J_x}) \;=\; -\frac{1}{2}\sum_{n=1}^{N}\Big\{\mathbf{J_z}\,\sigma^z_{n+1}\,\sigma^z_n + \mathbf{J_y}\,\sigma^x_{n+1}\,\sigma^x_n + \mathbf{J_x}\,\sigma^y_{n+1}\,\sigma^y_n\Big\}\,. \tag{3.15}$$

With (3.12) it is: $\mathbf{J_x}-\mathbf{J_y} = -2\,k\beta$. These Hamiltonians satisfy: $H_{\,n+1,n}(\mp k) \;=\; H_{\,n,n+1}(\mp k)$ but we have:

$$H_{\,n+1,n}(\mathbf{J_z},\mathbf{J_x},\mathbf{J_y}) = H_{\,n+1,n}(-k) \;\neq\; H_{\,n+1,n}(\;k) = H_{\,n+1,n}(\mathbf{J_z}\,,\,\mathbf{J_y}\,,\,\mathbf{J_x})\,.$$

We define:

$$\begin{aligned} \widetilde{H}(\alpha,\gamma,\beta,\delta) &= \sum_{n=1}^{N-1}\widetilde{H}_{n+1,n} = \frac{1}{\beta}\sum_{n=1}^{N}\Big\{\frac{1}{2}(\alpha+\beta)\,\mathbb{1} + \frac{1}{2}(\alpha-\beta)\,\sigma^z_{n+1}\sigma^z_n \;+\; \gamma\,(\sigma^+_{n+1}\sigma^-_n+\sigma^-_{n+1}\sigma^+_n) \;+\; \delta\,(\sigma^+_{n+1}\sigma^+_n+\sigma^-_{n+1}\sigma^-_n)\Big\} =\\ &= \frac{1}{\beta}\sum_{n=1}^{N}\Big\{(\alpha\,\hat{\sigma}_{n+1} + \beta\,\check{\sigma}_{n+1})\;\hat{\sigma}_n + (\alpha\,\check{\sigma}_{n+1} + \beta\,\hat{\sigma}_{n+1})\;\check{\sigma}_n + (\gamma\,\sigma^+_{n+1} + \delta\,\sigma^-_{n+1})\,\sigma^-_n \;+\; (\gamma\,\sigma^-_{n+1} + \delta\,\sigma^+_{n+1})\,\sigma^+_n\Big\}\,. \end{aligned} \tag{3.16}$$

With

$$\widetilde{H}\Big(\alpha(\lambda),\gamma(\lambda),\beta,\delta(\lambda)\Big) = \sum_{n=1}^{N}\sum_{m_n=0}^{\infty}{}_{m_n}\widetilde{H}_{n+1,n}\,(\lambda-\eta)^{m_n}:\qquad {}_0\widetilde{H} = \sum_{n=1}^{N}\Big\{\mathbb{1}\Big\} = (N)\,\mathbb{1},\qquad {}_1\widetilde{H} = \frac{\alpha_1}{2\beta}\,(N)\,\mathbb{1} + H(\,\alpha_1,\beta,\,\delta_1\,). \tag{3.17}$$

The structure of $\widetilde{H}_{n+1,n}$: $\dfrac{1}{\beta}\begin{pmatrix}\alpha & . & . & \delta\\ . & \beta & \gamma & .\\ . & \gamma & \beta & .\\ \delta & . & . & \alpha\end{pmatrix}$ is similar to the form of the matrix R.

With (3.6), (3.16) and $L_n(\eta) \;=\; {}_0L_n \;=\; \beta\begin{pmatrix}\hat{\sigma}_n & \sigma^-_n\\ \sigma^+_n & \check{\sigma}_n\end{pmatrix}$, we obtain (products of 2×2 matrices):

$$\left\{\begin{array}{lll}
[\,L_{n+1}(\lambda)\;L_n(\eta)\,] &=& [\,L_{n+1}(\eta)\;L_n(\eta)\,]\;\;\widetilde{H}_{n+1,n}(\lambda,\eta)\\
[\,L_n(\eta)\;L_{n-1}(\lambda)\,] &=& \widetilde{H}_{n,n-1}(\lambda,\eta)\;\;[\,L_n(\eta)\;L_{n-1}(\eta)\,]
\end{array}\right. . \tag{3.18}$$

The **monodromy matrix** (or transition matrix) (a 2×2 matrix);

$$\overleftarrow{\mathbf{T}}_N(\lambda) \;=\; L_N(\lambda)\ldots L_n(\lambda)\ldots L_1(\lambda) \;=\; \overleftarrow{\prod_{n=1}^{N}} L_n(\lambda) \;=\; \overleftarrow{\prod_{n=1}^{N}}\Big[\sum_{m=0}^{\infty} {}_mL_n(\lambda-\eta)^m\Big]. \tag{3.19}$$

Thanks to a commutative property:

$$\overleftarrow{\prod_{n=1}^{N}}\Big[L_n(\mu,k)\otimes L_n(\nu,k)\Big] \;=\; \Big[\overleftarrow{\prod_{n=1}^{N}} L_n(\mu,k)\Big]\otimes\Big[\overleftarrow{\prod_{n=1}^{N}} L_n(\nu,k)\Big], \tag{3.20}$$

the Yang-Baxter relation for them is:

$$[\,\overleftarrow{\mathbf{T}}_N(\mu,k)\otimes\overleftarrow{\mathbf{T}}_N(\nu,k)\,] \;=\; R(\mu-\nu)^{-1}\,[\,\overleftarrow{\mathbf{T}}_N(\nu,k)\otimes\overleftarrow{\mathbf{T}}_N(\mu,k)\,]\;R(\mu-\nu)\,. \tag{3.21}$$

The **transfer matrix** (an expression with operators, like $\widetilde{H}$) :

$$T_N(\lambda) \;=\; \mathrm{tr}\Big(\overleftarrow{\mathbf{T}}_N(\lambda)\Big) \;=\; \mathrm{tr}\Big(\overleftarrow{\prod_{n=1}^{N}} L_n(\lambda)\Big) \;=\; \mathrm{tr}\Big(\overleftarrow{\prod_{n=1}^{N}}\Big[\sum_{m=0}^{\infty} {}_mL_n(\eta,k)\,(\lambda-\eta)^m\Big]\Big)\,. \tag{3.22}$$

It is: $T_N(\eta) = \mathrm{tr}(\overleftarrow{\mathbf{T}}_N(\eta)) = \mathrm{tr}\big(L_N(\eta)\ldots L_n(\eta)\ldots L_1(\eta)\big) = \mathrm{tr}\big(L_{N-1}(\eta)\ldots L_n(\eta)\ldots L_1(\eta)\,L_N(\eta)\big)$. With $\beta = sn(2\eta)$:

$$T_N(\eta) \;=\; \mathrm{tr}\,(\,\overleftarrow{\mathbf{T}}_N(\eta)\,) \;=\; \beta^N\,\mathrm{tr}\Big(\overleftarrow{\prod_{n=1}^{N}}\begin{pmatrix}\overset{\wedge}{\sigma}_n & \sigma^-_n\\ \sigma^+_n & \overset{\vee}{\sigma}_n\end{pmatrix}\Big)\,. \tag{3.23}$$

Taking the trace in the Yang-Baxter relation (3.21) we have:

$$[\,T_N(\mu,k)\,,\,T_N(\nu,k)\,] \;=\; \mathbb{0}\,. \tag{3.24}$$

There were considered two cases: a) an “infinite chain” ($N\to\infty$), and b) a periodic boundary ($\sigma^j_{N+1}=\sigma^j_1$). The original procedure followed by Baxter and also exposed in [2] (their equations (2.10) - (2.21)) take the transfer matrices $T_N(\lambda)$ as generating functions of a family of commuting quantities. In particular Baxter found the following relation with the Hamiltonian of the XYZ spin Heisenberg chain: $\mathbf{H} = -sn(2\eta)\,\frac{d}{d\lambda}\log T(\lambda)\Big|_{\lambda=\eta} + \frac{1}{2}\mathbf{J}_z N I_N$, where it is used $\big\{T(\eta)\big\}^{-1}$. By doing higher derivatives it can be continued the procedure.

The method developed here is different. We use as a departing point the equations in (3.18), which obliges to have at least one ${}_0L_n = L_n(\eta)$ in the power development of the transfer matrices. This has a very important consequence: for obtaining our results we have to restrict the validity of the series development of the transfer matrix up to the order $N-1$. We denote these conditions with the symbol $\bumpeq$. Let us show the main steps:

$$\left\{\begin{array}{ll} L_N(\lambda)\;\ldots\;L_2(\lambda)\,L_1(\eta) \;=\; \overleftarrow{\prod_{n=1}^{N}} L_n(\eta)\,\overleftarrow{\prod_{n=1}^{N-1}}\widetilde{H}_{n+1,n} \;=\; \overleftarrow{\mathbf{T}}_N(\eta)\;\Big(\overleftarrow{\prod_{n=1}^{N-1}}\;\widetilde{H}_{n+1,n}\Big) & \\ L_{N-1}(\lambda)\;\ldots\;L_1(\lambda)\;L_N(\eta) \;=\; \overleftarrow{\mathbf{T}}_N(\eta)\;\Big(\Big(\overleftarrow{\prod_{n=1}^{N-2}}\widetilde{H}_{n+1,n}\Big)\,\widetilde{H}_{1,N}\Big) & \text{(periodicity condition: } \widetilde{H}_{1,N}=\widetilde{H}_{N+1,N}) \end{array}\right. . \tag{3.25}$$

And also with $\overleftarrow{\mathbf{T}}_N(\eta)$ in the position to the right instead of the position to the left. Which, with (3.22) and (3.23) give us:

$$T_N(\lambda) = \mathrm{tr}\Big(\Big[\sum_{m_N=0}^{\infty} {}_{m_N}L_N\,(\lambda-\eta)^{m_N}\Big]\ldots\Big[\sum_{m_n=0}^{\infty} {}_{m_n}L_n\,(\lambda-\eta)^{m_n}\Big]\ldots\Big[\sum_{m_1=0}^{\infty} {}_{m_1}L_1\,(\lambda-\eta)^{m_1}\Big]\Big) \bumpeq T_N(\eta)\Big\{\sum_{m=0}^{N-1} C_m\,(\lambda-\eta)^m\Big\}\,, \tag{3.26}$$

with $\quad {}_0L_n = L_n(\eta) = \beta\begin{pmatrix}\overset{\wedge}{\sigma} & \sigma^-\\ \sigma^+ & \overset{\vee}{\sigma}\end{pmatrix}_n, \qquad {}_{m_n}L_n(\lambda) = \begin{pmatrix}\alpha_{m_n}\overset{\wedge}{\sigma}+\gamma_{m_n}\overset{\vee}{\sigma} & \delta_{m_n}\,\sigma^+\\ \delta_{m_n}\,\sigma^- & \gamma_{m_n}\overset{\wedge}{\sigma}+\alpha_{m_n}\overset{\vee}{\sigma}\end{pmatrix}_n$

and $\quad \widetilde{H}_{n+1,n}(\alpha_{m_n},\gamma_{m_n},\beta,\delta_{m_n}) = {}_{m_n}\widetilde{H}_{n+1,n}$. These substitutions let us obtain the C_m.

The $C_m = C_m(\eta,k)$ are sums of products of various ${}_l\widetilde{H}_{n+1,n}$ ($l\in\{0,\ldots,N-1\}$). C_0 and C_1 related with the expressions in (3.17). The Heisenberg Hamiltonian is in C_1. The C_m are conserved quantities. With (3.23)-(3.26):

$$[\,C_m(\eta,k)\,,\,C_{m'}(\eta,k)\,] \;=\; \mathbb{0}\,, \qquad\qquad \{m\,,\,m'\}\in\{0,\;\ldots\;,N-1\} \tag{3.27}$$

IV. APPENDIX: PROOFS.

A. Proof of the Yang-Baxter type relation in the fermion formulation.

$$\left.\begin{array}{l} \quad u_m\,\mathbf{n}_m\,u_m^{-1} = u_m^{-1}\,\mathbf{n}_m\,u_m = \mathbf{n}_m\,, \qquad u_m\,\tilde{\mathbf{n}}_m\,u_m^{-1} = u_m^{-1}\,\tilde{\mathbf{n}}_m\,u_m = \tilde{\mathbf{n}}_m \\ \left\{\begin{array}{l} u_m^{+}\,\mathbf{a}_m^{\dagger}\,u_m^{+-1} = -\,u_m^{+-1}\,\mathbf{a}_m^{\dagger}\,u_m^{+} = u_m^{--1}\,\mathbf{a}_m^{\dagger}\,u_m^{-} = -\,u_m^{-}\,\mathbf{a}_m^{\dagger}\,u_m^{--1} = \mathrm{i}\,\mathbf{a}_m^{\dagger} \\ -\,u_m^{+}\,\mathbf{a}_m\,u_m^{+-1} = u_m^{+-1}\,\mathbf{a}_m\,u_m^{+} = -\,u_m^{--1}\,\mathbf{a}_m\,u_m^{-} = u_m^{-}\,\mathbf{a}_m\,u_m^{--1} = \mathrm{i}\,\mathbf{a}_m \end{array}\right. \end{array}\right\} \quad \text{and} \quad \mathcal{L}_m(\lambda) = \begin{pmatrix} e & o \\ \hat{o} & \hat{e} \end{pmatrix}_m .$$

$$L_m(\mu,k)\otimes L_m(\nu,k) \longrightarrow [\,\mathbb{U}_{\mathcal{M}-1}^{-1}\,\mathbb{U}_m^{-1}\,\mathcal{L}_m(\mu,k)\,\mathbb{U}_{\mathcal{M}-1}\,]\otimes[\,\mathbb{U}_{\mathcal{M}-1}^{-1}\,\mathbb{U}_m^{-1}\,\mathcal{L}_m(\nu,k)\,\mathbb{U}_{\mathcal{M}-1}\,] =$$

$$= [\,\mathbb{U}_{\mathcal{M}-1}^{-1}\otimes\mathbb{U}_{\mathcal{M}-1}^{-1}\,]\,[\,\{\mathbb{U}_m^{-1}\,\mathcal{L}_m(\mu,k)\}\otimes\{\mathbb{U}_m^{-1}\,\mathcal{L}_m(\nu,k)\}\,]\,[\,\mathbb{U}_{\mathcal{M}-1}\otimes\mathbb{U}_{\mathcal{M}-1}\,] =$$

$$= [\,\mathbb{U}_{\mathcal{M}-1}\otimes\mathbb{U}_{\mathcal{M}-1}\,]^{-1}\begin{pmatrix} u^{-1}e\;u^{-1}e' & u^{-1}e\;u^{-1}o' & u^{-1}o\;u^{-1}e' & u^{-1}o\;u^{-1}o' \\ u^{-1}e\;u\hat{o}' & u^{-1}e\;u\hat{e}' & u^{-1}o\;u\hat{o}' & u^{-1}o\;u\hat{e}' \\ u\hat{o}\;u^{-1}e' & u\hat{o}\;u^{-1}o' & u\hat{e}\;u^{-1}e' & u\hat{e}\;u^{-1}o' \\ u\hat{o}\;u\hat{o}' & u\hat{o}\;u\hat{e}' & u\hat{e}\;u\hat{o}' & u\hat{e}\;u\hat{e}' \end{pmatrix}_m [\,\mathbb{U}_{\mathcal{M}-1}\otimes\mathbb{U}_{\mathcal{M}-1}\,] =$$

$$= [\,\mathbb{U}_{\mathcal{M}}\otimes\mathbb{U}_{\mathcal{M}}\,]^{-1}\begin{pmatrix} u\,e\,u^{-1}\;e' & u\,e\,u^{-1}\;o' & u\,o\,u^{-1}\;e' & u\,o\,u^{-1}\;o' \\ u^{-1}e\,u\;\hat{o}' & u^{-1}e\,u\;\hat{e}' & u^{-1}o\,u\;\hat{o}' & u^{-1}o\,u\;\hat{e}' \\ u\,\hat{o}\,u^{-1}\;e' & u\,\hat{o}\,u^{-1}\;o' & u\,\hat{e}\,u^{-1}\;e' & u\,\hat{e}\,u^{-1}\;o' \\ u^{-1}\hat{o}\,u\;\hat{o}' & u^{-1}\hat{o}\,u\;\hat{e}' & u^{-1}\hat{e}\,u\;\hat{o}' & u^{-1}\hat{e}\,u\;\hat{e}' \end{pmatrix}_m [\,\mathbb{U}_{\mathcal{M}-1}\otimes\mathbb{U}_{\mathcal{M}-1}\,] =$$

$$= [\,\mathbb{U}_{\mathcal{M}}\otimes\mathbb{U}_{\mathcal{M}}\,]^{-1}\begin{pmatrix} e\,e' & e\,o' & u\,o\,u^{-1}\;e' & u\,o\,u^{-1}\;o' \\ e\,\hat{o}' & e\,\hat{e}' & u^{-1}o\,u\;\hat{o}' & u^{-1}o\,u\;\hat{e}' \\ u\,\hat{o}\,u^{-1}\;e' & u\,\hat{o}\,u^{-1}\;o' & \hat{e}\,e' & \hat{e}\,o' \\ u^{-1}\hat{o}\,u\;\hat{o}' & u^{-1}\hat{o}\,u\;\hat{e}' & \hat{e}\,\hat{o}' & \hat{e}\,\hat{e}' \end{pmatrix}_m [\,\mathbb{U}_{\mathcal{M}-1}\otimes\mathbb{U}_{\mathcal{M}-1}\,] = \qquad *$$

a) for: $u_m = u_m^{+} = \mathrm{e}^{\,\mathrm{i}\frac{\pi}{2}\mathbf{n}_m} = \tilde{\mathbf{n}}_m + \mathrm{i}\,\mathbf{n}_m$ and $\mathcal{L}_m^a(\lambda) = \mathbb{U}_m^a\,\mathbf{L}_m(\lambda) = \begin{pmatrix} \mathrm{i}\,\alpha\,\mathbf{n}_m + \gamma\,\tilde{\mathbf{n}}_m & \mathrm{i}\,\delta\,\mathbf{a}_m^{\dagger} + \beta\,\mathbf{a}_m \\ -\mathrm{i}\,\beta\,\mathbf{a}_m^{\dagger} + \delta\,\mathbf{a}_m & -\mathrm{i}\,\gamma\,\mathbf{n}_m + \alpha\,\tilde{\mathbf{n}}_m \end{pmatrix} = \begin{pmatrix} e_m^a & o_m^a \\ \hat{o}_m^a & \hat{e}_m^a \end{pmatrix}$

$$\left\{\begin{array}{l} u_m^{+}\,o_m^a\,u_m^{+-1} = (\tilde{\mathbf{n}}_m + \mathrm{i}\,\mathbf{n}_m)\,(\mathrm{i}\,\delta\,\mathbf{a}_m^{\dagger} + \beta\,\mathbf{a}_m)\,(\tilde{\mathbf{n}}_m - \mathrm{i}\,\mathbf{n}_m) = (-\,\delta\,\mathbf{a}_m^{\dagger} - \mathrm{i}\beta\,\mathbf{a}_m) = (\mathrm{i}\,(-\delta)\,\mathbf{a}_m^{\dagger} + \beta\,\mathbf{a}_m)\,(-\mathrm{i}) = o_m^{-a}\,(-\mathrm{i}) \\ u_m^{+-1}\,o_m^a\,u_m^{+} = (\tilde{\mathbf{n}}_m - \mathrm{i}\,\mathbf{n}_m)\,(\mathrm{i}\,\delta\,\mathbf{a}_m^{\dagger} + \beta\,\mathbf{a}_m)\,(\tilde{\mathbf{n}}_m + \mathrm{i}\,\mathbf{n}_m) = (\delta\,\mathbf{a}_m^{\dagger} + \mathrm{i}\beta\,\mathbf{a}_m) = -(\mathrm{i}\,(-\delta)\,\mathbf{a}_m^{\dagger} + \beta\,\mathbf{a}_m)\,(-\mathrm{i}) = -\,o_m^{-a}\,(-\mathrm{i}) \\ u_m^{+}\,\hat{o}_m^a\,u_m^{+-1} = (\tilde{\mathbf{n}}_m + \mathrm{i}\,\mathbf{n}_m)\,(-\mathrm{i}\beta\,\mathbf{a}_m^{\dagger} + \delta\,\mathbf{a}_m)\,(\tilde{\mathbf{n}}_m - \mathrm{i}\,\mathbf{n}_m) = (\beta\,\mathbf{a}_m^{\dagger} - \mathrm{i}\,\delta\,\mathbf{a}_m) = (\mathrm{i})\,(-\mathrm{i}\beta\,\mathbf{a}_m^{\dagger} + (-\delta)\,\mathbf{a}_m) = \mathrm{i}\,\hat{o}_m^{-a} \\ u_m^{+-1}\,\hat{o}_m^a\,u_m^{+} = (\tilde{\mathbf{n}}_m - \mathrm{i}\,\mathbf{n}_m)\,(-\mathrm{i}\beta\,\mathbf{a}_m^{\dagger} + \delta\,\mathbf{a}_m)\,(\tilde{\mathbf{n}}_m + \mathrm{i}\,\mathbf{n}_m) = (-\beta\,\mathbf{a}_m^{\dagger} + \mathrm{i}\,\delta\,\mathbf{a}_m) = -\,(\mathrm{i})\,(-\mathrm{i}\beta\,\mathbf{a}_m^{\dagger} + (-\delta)\,\mathbf{a}_m) = -\mathrm{i}\,\hat{o}_m^{-a} \end{array}\right. .$$

$$* \quad = [\,\mathbb{U}_{\mathcal{M}}^a\otimes\mathbb{U}_{\mathcal{M}}^a\,]^{-1}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&\mathrm{i}&0\\0&0&0&\mathrm{i} \end{pmatrix}_m \begin{pmatrix} e\,e' & e\,o' & o^{-}\,e' & o^{-}\,o' \\ e\,\hat{o}' & e\,\hat{e}' & -o^{-}\,\hat{o}' & -o^{-}\,\hat{e}' \\ \hat{o}^{-}\,e' & \hat{o}^{-}\,o' & \hat{e}\,e' & \hat{e}\,o' \\ -\hat{o}^{-}\,\hat{o}' & -\hat{o}^{-}\,\hat{e}' & \hat{e}\,\hat{o}' & \hat{e}\,\hat{e}' \end{pmatrix}_m^a \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-\mathrm{i}&0\\0&0&0&-\mathrm{i} \end{pmatrix}_m [\,\mathbb{U}_{\mathcal{M}-1}^a\otimes\mathbb{U}_{\mathcal{M}-1}^a\,]$$

$$R(\mu-\nu,k)\left[\mathbb{U}_{\mathcal{M}}^a\otimes\mathbb{U}_{\mathcal{M}}^a\right]^{-1}\left[\begin{pmatrix}1&0\\0&-\mathrm{i}\end{pmatrix}\otimes\mathbb{1}\right]_m\left[\mathcal{L}_m^a(\mu,-k)\underset{s}{\otimes}\mathcal{L}_m^a(\nu,k)\right]\left[\begin{pmatrix}1&0\\0&-\mathrm{i}\end{pmatrix}\otimes\mathbb{1}\right]_m\left[\mathbb{U}_{\mathcal{M}-1}^a\otimes\mathbb{U}_{\mathcal{M}-1}^a\right] =$$

$$= \left[\mathbb{U}_{\mathcal{M}}^a\otimes\mathbb{U}_{\mathcal{M}}^a\right]^{-1}\left[\begin{pmatrix}1&0\\0&-\mathrm{i}\end{pmatrix}\otimes\mathbb{1}\right]_m\left[\mathcal{L}_m^a(\nu,-k)\underset{s}{\otimes}\mathcal{L}_m^a(\mu,k)\right]\left[\begin{pmatrix}1&0\\0&-\mathrm{i}\end{pmatrix}\otimes\mathbb{1}\right]_m\left[\mathbb{U}_{\mathcal{M}-1}^a\otimes\mathbb{U}_{\mathcal{M}-1}^a\right] R(\mu-\nu,k) \quad .$$

$$\mathcal{R}(\mu-\nu,k) = \left[\begin{pmatrix}1&0\\0&-\mathrm{i}\end{pmatrix}\otimes\mathbb{1}\right]_m\left[\mathbb{U}^a_{\left\{\substack{\mathcal{M}\\ \mathcal{M}-1}\right.}\otimes\mathbb{U}^a_{\left\{\substack{\mathcal{M}\\ \mathcal{M}-1}\right.}\right] R(\mu-\nu,k)\left[\mathbb{U}^a_{\left\{\substack{\mathcal{M}\\ \mathcal{M}-1}\right.}\otimes\mathbb{U}^a_{\left\{\substack{\mathcal{M}\\ \mathcal{M}-1}\right.}\right]^{-1}\left[\begin{pmatrix}1&0\\0&-\mathrm{i}\end{pmatrix}\otimes\mathbb{1}\right]_m = \begin{pmatrix} a&0&0&\mathrm{i}\,d\\0&b&\mathrm{i}\,c&0\\0&-\mathrm{i}\,c&b&0\\-\mathrm{i}\,d&0&0&a \end{pmatrix} \quad .$$

$$\mathcal{R}(\mu-\nu,\;k)\;[\,\mathcal{L}_m^a(\mu,-k)\underset{s}{\otimes}\mathcal{L}_m^a(\nu,\;k)\,] = [\,\mathcal{L}_m^a(\nu,-k)\underset{s}{\otimes}\mathcal{L}_m^a(\mu,\;k)\,]\;\mathcal{R}(\mu-\nu,\;k) \quad ,$$

$$\mathcal{R}(\mu-\nu,-k)\;[\,\mathcal{L}_m^a(\mu,\;k)\underset{s}{\otimes}\mathcal{L}_m^a(\nu,-k)\,] = [\,\mathcal{L}_m^a(\nu,\;k)\underset{s}{\otimes}\mathcal{L}_m^a(\mu,-k)\,]\;\mathcal{R}(\mu-\nu,-k) \quad .$$

Similarly for the case **d)** , and also for the cases **b)** and **c)**, but these last two with the transpose $\mathcal{R}^t$.

B. Proof of (2.30), an step (case a).

i) $$\mathcal{L}^a_{n+1}(0)\,\mathcal{L}^a_n(0)\,\widetilde{H}_{n+1,n}(\lambda,\,k) \;=\; \beta^2 \begin{pmatrix} \mathrm{i}\,\mathbf{n}_{n+1} & \mathbf{a}_{n+1} \\ -\mathrm{i}\,\mathbf{a}^\dagger_{n+1} & \tilde{\mathbf{n}}_{n+1}\end{pmatrix}\begin{pmatrix} \mathrm{i}\,\mathbf{n}_{n} & \mathbf{a}_{n} \\ -\mathrm{i}\,\mathbf{a}^\dagger_{n} & \tilde{\mathbf{n}}_{n}\end{pmatrix}\widetilde{H}_{n+1,n}(\lambda,\,k) \;=$$

$$= \mathcal{L}^a_{n+1}(0)\begin{pmatrix} \mathrm{i}\,(\alpha\mathbf{n}_{n+1}+\beta\,\tilde{\mathbf{n}}_{n+1})\mathbf{n}_n - \mathrm{i}\,(\gamma\,\mathbf{a}_{n+1}+\delta\,\mathbf{a}^\dagger_{n+1})\,\mathbf{a}^\dagger_n & (\alpha\mathbf{n}_{n+1}+\beta\,\tilde{\mathbf{n}}_{n+1})\,\mathbf{a}_n + (\gamma\,\mathbf{a}_{n+1}+\delta\,\mathbf{a}^\dagger_{n+1})\,\tilde{\mathbf{n}}_n \\ -\mathrm{i}\,(\alpha\,\tilde{\mathbf{n}}_{n+1}+\beta\,\mathbf{n}_{n+1})\,\mathbf{a}^\dagger_n + \mathrm{i}\,(\gamma\,\mathbf{a}^\dagger_{n+1}+\delta\,\mathbf{a}_{n+1})\mathbf{n}_n & (\alpha\,\tilde{\mathbf{n}}_{n+1}+\beta\mathbf{n}_{n+1})\,\tilde{\mathbf{n}}_n + (\gamma\,\mathbf{a}^\dagger_{n+1}+\delta\,\mathbf{a}_{n+1})\,\mathbf{a}_n\end{pmatrix} =$$

$$= \beta\begin{pmatrix} \mathrm{i}\,\mathbf{n}_{n+1} & \mathbf{a}_{n+1} \\ -\mathrm{i}\,\mathbf{a}^\dagger_{n+1} & \tilde{\mathbf{n}}_{n+1}\end{pmatrix}\begin{pmatrix} \alpha\mathbf{n}_{n+1}+\beta\,\tilde{\mathbf{n}}_{n+1} & \gamma\,\mathbf{a}_{n+1}+\delta\,\mathbf{a}^\dagger_{n+1} \\ \gamma\,\mathbf{a}^\dagger_{n+1}+\delta\,\mathbf{a}_{n+1} & \alpha\,\tilde{\mathbf{n}}_{n+1}+\beta\mathbf{n}_{n+1}\end{pmatrix}\begin{pmatrix} \mathrm{i}\,\mathbf{n}_{n} & \mathbf{a}_{n} \\ -\mathrm{i}\,\mathbf{a}^\dagger_{n} & \tilde{\mathbf{n}}_{n}\end{pmatrix} =$$

$$= \begin{pmatrix} \mathrm{i}\,\alpha\,\mathbf{n}_{n+1}+\gamma\,\tilde{\mathbf{n}}_{n+1} & \mathrm{i}\,\delta\,\mathbf{a}^\dagger_{n+1}+\beta\,\mathbf{a}_{n+1} \\ -\mathrm{i}\beta\,\mathbf{a}^\dagger_{n+1}+\delta\,\mathbf{a}_{n+1} & -\mathrm{i}\,\gamma\,\mathbf{n}_{n+1}+\alpha\,\tilde{\mathbf{n}}_{n+1}\end{pmatrix}\beta\begin{pmatrix} \mathrm{i}\,\mathbf{n}_{n} & \mathbf{a}_{n} \\ -\mathrm{i}\,\mathbf{a}^\dagger_{n} & \tilde{\mathbf{n}}_{n}\end{pmatrix} = \mathcal{L}^a_{n+1}(\lambda,\,k)\,\mathcal{L}^a_n(0)\,.$$

ii) $$\widetilde{\mathcal{H}}_{n+1,n}(\lambda,\,k)\,\mathcal{L}^a_{n+1}(0)\,\mathcal{L}^a_n(0) \;=\; \beta^2\,\widetilde{\mathcal{H}}_{n+1,n}(\lambda,\,k)\begin{pmatrix} \mathrm{i}\,\mathbf{n}_{n+1} & \mathbf{a}_{n+1} \\ -\mathrm{i}\,\mathbf{a}^\dagger_{n+1} & \tilde{\mathbf{n}}_{n+1}\end{pmatrix}\begin{pmatrix} \mathrm{i}\,\mathbf{n}_{n} & \mathbf{a}_{n} \\ -\mathrm{i}\,\mathbf{a}^\dagger_{n} & \tilde{\mathbf{n}}_{n}\end{pmatrix} =$$

$$= \begin{pmatrix} \mathrm{i}\,\alpha\mathbf{n}_{n+1}\mathbf{n}_n + \mathrm{i}\beta\,\mathbf{n}_{n+1}\tilde{\mathbf{n}}_n - \mathrm{i}\,\gamma\,\mathbf{a}_{n+1}\mathbf{a}^\dagger_n + \mathrm{i}\delta\,\mathbf{a}_{n+1}\,\mathbf{a}_n & \beta\mathbf{a}_{n+1}\mathbf{n}_n + \alpha\,\mathbf{a}_{n+1}\tilde{\mathbf{n}}_n - \gamma\,\mathbf{n}_{n+1}\,\mathbf{a}_n + \delta\,\mathbf{n}_{n+1}\,\mathbf{a}^\dagger_n \\ -\mathrm{i}\beta\mathbf{a}^\dagger_{n+1}\tilde{\mathbf{n}}_n - \mathrm{i}\alpha\,\mathbf{a}^\dagger_{n+1}\mathbf{n}_n - \mathrm{i}\gamma\,\tilde{\mathbf{n}}_{n+1}\,\mathbf{a}^\dagger_n + \mathrm{i}\delta\,\tilde{\mathbf{n}}_{n+1}\,\mathbf{a}_n & \alpha\tilde{\mathbf{n}}_{n+1}\tilde{\mathbf{n}}_n + \beta\,\tilde{\mathbf{n}}_{n+1}\mathbf{n}_n + \gamma\,\mathbf{a}^\dagger_{n+1}\mathbf{a}_n - \delta\,\mathbf{a}^\dagger_{n+1}\,\mathbf{a}^\dagger_n\end{pmatrix}\mathcal{L}^a_n(0) =$$

$$= \beta\begin{pmatrix} \mathrm{i}\,\mathbf{n}_{n+1} & \mathbf{a}_{n+1} \\ -\mathrm{i}\,\mathbf{a}^\dagger_{n+1} & \tilde{\mathbf{n}}_{n+1}\end{pmatrix}\begin{pmatrix} \alpha\mathbf{n}_n+\beta\,\tilde{\mathbf{n}}_n & \mathrm{i}\gamma\,\mathbf{a}_n - \mathrm{i}\delta\,\mathbf{a}^\dagger_n \\ \mathrm{i}\delta\,\mathbf{a}_n - \mathrm{i}\gamma\,\mathbf{a}^\dagger_n & \alpha\,\tilde{\mathbf{n}}_n+\beta\mathbf{n}_n\end{pmatrix}\begin{pmatrix} \mathrm{i}\,\mathbf{n}_{n} & \mathbf{a}_{n} \\ -\mathrm{i}\,\mathbf{a}^\dagger_{n} & \tilde{\mathbf{n}}_{n}\end{pmatrix} =$$

$$= \beta\begin{pmatrix} \mathrm{i}\,\mathbf{n}_{n+1} & \mathbf{a}_{n+1} \\ -\mathrm{i}\,\mathbf{a}^\dagger_{n+1} & \tilde{\mathbf{n}}_{n+1}\end{pmatrix}\begin{pmatrix} \mathrm{i}\,\alpha\,\mathbf{n}_n+\gamma\,\tilde{\mathbf{n}}_n & \mathrm{i}(-\delta)\,\mathbf{a}^\dagger_n+\beta\,\mathbf{a}_n \\ -\mathrm{i}\beta\,\mathbf{a}^\dagger_n+(-\delta)\,\mathbf{a}_n & -\mathrm{i}\,\gamma\,\mathbf{n}_n+\alpha\,\tilde{\mathbf{n}}_n\end{pmatrix} = \mathcal{L}^a_{n+1}(0)\,\mathcal{L}^a_n(\lambda,-k)\,.$$

V. APPENDIX: CITES.

About the creation and annihilation operators. Pauli's "Selected Topics in Field Quantization". (**1950-51**) (page 4) [8].

> " Formally, if one introduces
>
> $$\{\mathbf{a},\mathbf{a}^*\} \equiv \mathbf{a}\mathbf{a}^* + \mathbf{a}^*\mathbf{a} = \mathbf{1}, \quad \mathbf{a}^2 = 0, \quad \mathbf{a}^{*2} = 0\,, \qquad [2.10]$$
>
> then one obtains as solution
>
> $$\mathbf{a} = \begin{pmatrix}0&1\\0&0\end{pmatrix}, \qquad \mathbf{a}^* = \begin{pmatrix}0&0\\1&0\end{pmatrix}. \qquad [2.11]$$
>
> Furthermore, if we set
>
> $$\mathbf{N} \equiv \mathbf{a}^*\mathbf{a} = \begin{pmatrix}0&0\\0&1\end{pmatrix}, \qquad [2.12]$$
>
> then
>
> $$\mathbf{1}-\mathbf{N} \equiv \mathbf{a}\mathbf{a}^* = \begin{pmatrix}1&0\\0&0\end{pmatrix},$$
>
> $$\mathbf{N}(\mathbf{1}-\mathbf{N}) = \mathbf{0}. \qquad [2.13]$$
>
> This corresponds exactly to the exclusion principle. Note that here, in contrast with Bose-Einstein statistics , complete symmetry exists between **a** and $\mathbf{a}^*$, and **N** and $\mathbf{1}-\mathbf{N}$, respectively. "

About the Jordan Wigner transformation. Pauli's "Selected Topics in Field Quantization". (**1950-51**) (page 7) [8].

> " We will now quantize the amplitudes and, indeed, since we know empirically that electrons satisfy the exclusion principle (there is also a theoretical basis for this [2]), we will follow the scheme given at the end of Section 2. This invention (by Jordan and Wigner [3]), is very useful although its physical meaning appears to be obscure: the sign of an expression in the amplitudes becomes dependent upon the numbering of the normal modes. "

VI. APPENDIX: PROGRAM OF THE *STUDIES* CONTAINING THIS RESEARCH.

On the fermionization of the XYZ spin Heisenberg chain (algebra).
(2022) https://eprints.ucm.es/id/eprint/72882/ *Study -2)*

The JordanWigner transformations and the fermionization of the XYZ spin Heisenberg chain. Algebra, geometry and physics? (This study). (2022) https://eprints.ucm.es/id/eprint/74550/ Study -1)

A tentative model of creation and annihilation operators for neutrinos.
(2021) https://eprints.ucm.es/id/eprint/65151/ *Study 0)*

Expression of the 3- and 4-dimensional vectors in total polar exponential form.
(2021) https://eprints.ucm.es/id/eprint/65825/ *Study I,1)*

Vectors. Dimensions 4 and 8. (2023) https://eprints.ucm.es/id/eprint/76327/ *Study I,2)*

Geometry of the time and the space. *Study I)*

Geometry of the symmetries in dimension 4=(1+[1)+“2”], and general Time-Space-Spin vectors.
(2023) https://eprints.ucm.es/id/eprint/76328/ *Study II)*

Geometry and Physics of the Elementary Fermions. (On pride of Jordan Wigner Pauli Weyl Dirac). **1**.
(2021) https://eprints.ucm.es/id/eprint/69295/ *Study III)*

Geometry and Physics of the Elementary Fermions. **2**. *Study III)*

Axial vector magnetic charge and magnetic moment. Maxwell’s equations and Lorentz force law.
(2021) https://eprints.ucm.es/id/eprint/69294/ *Study IV)*

Addenda. *Study V)*

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