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From Ramond fermions to Lamé equations for orthogonal curvilinear coordinates

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Abstract

We show how Ramond free neutral Fermi fields lead to a τ -function theory of BKP type which describes iso-orthogonal deformations of systems of orthogonal curvilinear coordinates. We also provide a vertex operator representation for the classical Ribaucour transformation. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

In [5] the multicomponent BKP hierarchies of integrable systems were introduced by using free neutral Ramond fermions, the building blocks of the *fermionic quark model* for current algebras [12]. Some correlations functions of the theory, the so called τ -functions, satisfy a bilinear identity, defined in terms of vertex operators, which is equivalent to the BKP hierarchy of integrable nonlinear partial differential equations.

In this Letter we show that this approach can be applied to the theory of systems of orthogonal curvilinear coordinates, a classical subject in differential geometry [2,9,1], and its iso-orthogonal deformations. Let us remind the reader that since the last century distinguished geometers, among others Gauss, Dupin, Binet, Lamé, Darboux, Egorov and

Bianchi established the basis of this theory. In particular, Dardoux's book [3] is a standard reference for the Lamé equations [14], which describe these systems, see also [1] for the problem of triply orthogonal systems of surfaces. Geometers were also interested in those transformations (symmetries) [3,10] preserving the orthogonal character of systems of curvilinear coordinates and, in particular a fairly general transformation of this type was derived by Ribaucour [15]. Very recently the Lamé equations have been integrated by means of the inverse scattering technique [18] and algebra-geometric solutions in terms of theta functions have been constructed in [13]. Moreover, it has been found that the Lamé equations are relevant in the classification problem of 2D topological field theories. In particular, it was shown by Dubrovin [8] that the decomposable associativity equations of Witten–Dijkgraaf–Verlinde–Verlinde [17,7] can be described in terms of some particular subclass of Lamé systems, those of Egorov type, a particular class of flat diagonal metrics [4,11].

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Further, in [13], the partition function of the theory was found in terms of theta functions.

In this Letter we give:

1. A τ -function and bilinear equation representation of the Lamé equations.
2. A correspondence between the vertex operator of the theory and the classical Ribaucour transformation.

Our method is based on the Grassmannian formalism [16] as well as on same identities for Baker functions coming from Fay identities for τ -functions. For analogous results regarding the N -component KP hierarchy, charged fermions, conjugate nets and its transformations see [6].

The layout of the paper is as follows. Next, in §2, we present a brief account of the Lamé equations and the Ribaucour transformation. Then in §3 we remind the reader the quantum field theory of the BKP hierarchy in terms of τ -functions. In §4 we introduce the Baker matrix and show the relation between the BKP hierarchy and orthogonal curvilinear coordinates. Finally, in §5 we identify the action of the vertex operator with the classical Ribaucour transformation.

2. Lamé equations for orthogonal nets

Systems of curvilinear coordinates $\mathbf{u} := (u_1, \dots, u_N)$ in the Euclidean space \mathbb{R}^N are determined by diffeomorphisms $\mathbf{u} \rightarrow \mathbf{x}(\mathbf{u})$, with $\mathbf{x} = (x_1, \dots, x_N)$ being the Cartesian coordinates in \mathbb{R}^N .

Curvilinear coordinates such that the coordinates lines are orthogonal satisfy:

$$\frac{\partial \mathbf{x}}{\partial u_i} \cdot \frac{\partial \mathbf{x}}{\partial u_j} = 0, \quad i \neq j,$$

and the corresponding normalized tangent vectors are given by

$$\mathbf{g}_i = \frac{1}{H_i} \frac{\partial \mathbf{x}}{\partial u_i}, \quad i = 1, \dots, N.$$

where

$$H_i := \left\| \frac{\partial \mathbf{x}}{\partial u_i} \right\|, \quad i = 1, \dots, N,$$

are the so called Lamé coefficients. It turns out that the orthonormal frame $\{\mathbf{g}_j\}_{j=1}^N$ satisfies

$$\frac{\partial \mathbf{g}_i}{\partial u_j} - \beta_{ij} \mathbf{g}_j = 0, \quad i, j = 1, \dots, N, i \neq j, \quad (1)$$

$$\frac{\partial \mathbf{g}_i}{\partial u_i} + \sum_{\substack{k=1, \dots, N \\ k \neq i}} \beta_{ki} \mathbf{g}_k = 0, \quad i = 1, \dots, N, \quad (2)$$

where β_{ij} are the rotation coefficients:

$$\beta_{ij} := \frac{1}{H_i} \frac{\partial H_j}{\partial u_i}.$$

The compatibility of this system implies the so called Lamé equations for the rotation coefficients:

$$\frac{\partial \beta_{ij}}{\partial u_k} - \beta_{ik} \beta_{kj} = 0, \quad i, j, k = 1, \dots, N, \text{ with } i, j, k \text{ different}, \quad (3)$$

$$\frac{\partial \beta_{ij}}{\partial u_i} + \frac{\partial \beta_{ji}}{\partial u_j} + \sum_{\substack{k=1, \dots, N \\ k \neq i, j}} \beta_{ki} \beta_{kj} = 0,$$

$$i, j = 1, \dots, N, i \neq j. \quad (4)$$

The Cartesian coordinates (x_1, \dots, x_N) are recovered from the following Laplace equations:

$$\frac{\partial^2 \mathbf{x}}{\partial u_i \partial u_j} = \frac{\partial \ln H_i}{\partial u_j} \frac{\partial \mathbf{x}}{\partial u_i} + \frac{\partial \ln H_j}{\partial u_i} \frac{\partial \mathbf{x}}{\partial u_j},$$

$$i, j = 1, \dots, N, i \neq j,$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_i^2} + \frac{1}{2} \frac{\partial (H_i^2)}{\partial u_i} \sum_{k=1}^N \frac{1}{H_k^2} \frac{\partial \mathbf{x}}{\partial u_k} = 0, \quad i = 1, \dots, N,$$

once the Lamé coefficients are determined.

Given a system of orthogonal curvilinear coordinates it is of interest to derive transformations providing a new set of orthogonal curvilinear coordinates. A transformation of this type was found in the last century and is known as the Ribaucour transformation. It requires the introduction of a potential in the following manner: given functions ζ_i such that

$$\frac{\partial \zeta_i}{\partial u_j} = \beta_{ij} \zeta_j, \quad i, j = 1, \dots, N, i \neq j, \quad (5)$$

one can define a potential $\Omega(\zeta, H)$ through the compatible equations

$$\frac{\partial \Omega(\zeta, H)}{\partial u_i} = \zeta_i H_i \quad (6)$$

Ribaucour transformation. Given solutions ζ_i of (5), $i = 1, \dots, N$, new rotation coefficients $\mathcal{R}(\beta_{ij})$, orthonormal tangent vectors $\mathcal{R}(\mathbf{g}_i)$, Lamé coefficients $\mathcal{R}(H_i)$ and flat coordinates $\mathcal{R}(\mathbf{x})$ are given by

$$\begin{aligned} \mathcal{R}(\beta_{ij}) &= \beta_{ij} - 2 \frac{\zeta_i}{\sum_{k=1}^N \zeta_k^2} \left[\frac{\partial \zeta_j}{\partial u_j} + \sum_{\substack{k=1, \dots, N \\ k \neq j}} \beta_{kj} \zeta_k \right], \\ \mathcal{R}(\mathbf{g}_i) &= \mathbf{g}_i - 2 \frac{\zeta_i}{\sum_{k=1}^N \zeta_k^2} \sum_{k=1}^N \zeta_k \mathbf{g}_k, \\ \mathcal{R}(H_i) &= H_i - 2 \frac{1}{\sum_{k=1}^N \zeta_k^2} \\ &\quad \times \left[\frac{\partial \zeta_j}{\partial u_j} + \sum_{\substack{k=1, \dots, N \\ k \neq j}} \beta_{kj} \zeta_k \right] \Omega(\zeta, H), \\ \mathcal{R}(\mathbf{x}) &= \mathbf{x} - 2 \frac{1}{\sum_{k=1}^N \zeta_k^2} \Omega(\zeta, H) \sum_{k=1}^N \zeta_k \mathbf{g}_k. \end{aligned} \quad (7)$$

3. Ramond fermions and BKP hierarchies

The N -component BKP hierarchy can be defined in terms of Ramond neutral free Fermi fields as follows [5]. First, we introduce a set of anticommuting quantum fields $\Phi_i(z)$, $i = 1, \dots, N$, satisfying

$$\{\Phi_i(z), \Phi_j(z')\} = \delta_{ij} \delta(z + z'),$$

where $\delta(z)$ denotes the Dirac distribution on the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. These quantum fields have a Laurent expansion in z :

$$\Phi_i(z) = \sum_{n \in \mathbb{Z}} z^n \Phi_{i,n},$$

with

$$\{\Phi_{i,n}, \Phi_{j,m}\} = \delta_{ij} \delta_{n,-m}, \quad i, j = 1, \dots, N, \quad n, m \in \mathbb{Z}.$$

The vacuum $|0\rangle$ and antivacuum $\langle 0|$ are defined by the relations:

$$\begin{aligned} \Phi_{i,n} |0\rangle &= 0, \quad n < 0, \\ \langle 0| \Phi_{i,n} &= 0, \quad n > 0, \\ \langle 0| \Phi_{i,0} \Phi_{j,0} |0\rangle &= 0. \end{aligned}$$

Now, we consider an infinite number of time labels:

$$\begin{aligned} \mathbf{t} &= (t_1, \dots, t_N) \in \mathbb{C}^{N \cdot \infty}, \\ \mathbf{t}_i &:= (t_{i,1}, t_{i,3}, t_{i,5}, \dots) \in \mathbb{C}^\infty, \end{aligned}$$

in terms of which we construct the operator:

$$H(\mathbf{t}) := \frac{1}{2} \sum_{\substack{i=1, \dots, N \\ n \in \mathbb{Z} \\ l \geq 0}} (-1)^{n+1} t_{j,2l+1} \Phi_{j,n} \Phi_{j,n-2l+1}.$$

The quadratic products $\Phi_{i,n}, \Phi_{j,m}$ when exponentiated generate a Lie group G . Given an element $g \in G$, there is a set of associated τ -functions given by the following expectation values

$$\begin{aligned} \tau(\mathbf{t}) &:= \langle 0 | \exp(H(\mathbf{t})) g | 0 \rangle, \\ \tau_{ij}(\mathbf{t}) &:= \langle 0 | \exp(H(\mathbf{t})) g \Phi_{i,0} \Phi_{j,0} | 0 \rangle, \\ i, j &= 1, \dots, N. \end{aligned}$$

These correlations satisfy

$$\begin{aligned} \tau_{ij} + \tau_{ji} &= 0, \quad i, j = 1, \dots, N, \quad i \neq j, \\ 2\tau_{ii} &= \tau, \quad i = 1, \dots, N. \end{aligned}$$

and the next bilinear equation – which defines the N -component BKP hierarchy –

$$\begin{aligned} \sum_{k=1}^N \int_{S^1} \frac{dz}{2\pi i z} [X_k(z) \tau_{ik}(\mathbf{t})] [X_k(-z) \tau_{jk}(\mathbf{t}')] \\ = \sum_{k=1}^N \tau_{ki}(\mathbf{t}) \tau_{kj}(\mathbf{t}'), \quad i, j = 1, \dots, N, \end{aligned} \quad (8)$$

for suitable \mathbf{t} and \mathbf{t}' . Here we are using the following vertex operators

$$X_i(z) := \exp(\xi(z, \mathbf{t}_i)) \mathbb{V}_i(-z),$$

$$\xi(z, \mathbf{t}_i) := \sum_{n=1}^{\infty} z^{2n+1} t_{i,2n+1},$$

$$\mathbb{V}_i(z) f(\mathbf{t}) := f(\mathbf{t} + [1/z] \mathbf{e}_i),$$

$$[1/z] := 2 \left(\frac{1}{z}, \frac{1}{3z^3}, \frac{1}{5z^5}, \dots \right),$$

with $\{e_i\}_{i=1}^N$ being the canonical generators of \mathbb{C}^N , so that the bilinear Eq. (8) can be written as

$$\begin{aligned} & \sum_{k=1}^N \int_{S^1} \frac{dz}{2\pi i z} \exp(\xi(z, t_k) - \xi(z, t'_k)) \\ & \times \tau_{ik}(t - [1/z]e_k) \tau_{jk}(t' + [1/z]e_k) \\ & = \sum_{k=1}^N \tau_{ki}(t) \tau_{kj}(t'), \quad i, j = 1, \dots, N. \end{aligned} \quad (9)$$

4. From BKP hierarchy to Lamé equations

The above τ -function formulation of the bilinear equation of the N -component BKP hierarchy allows for a useful Baker-function description. For this aim we introduce a non-normalized wave-function $\varphi(z, t)$

$$\varphi_{ij}(z, t) := \tau_{ij}(t - [1/z]e_j) \exp(\xi(z, t_j)).$$

The bilinear Eq. (8) becomes

$$\int_{S^1} \frac{dz}{2\pi i z} \varphi(z, t) \varphi(-z, t') = \mathcal{T}'(t) \mathcal{T}(t'),$$

with

$$\mathcal{T}(t) := (\tau_{ij}(t)).$$

Observe that

$$\varphi(z, t) = \rho(z, t) \psi_0(z, t),$$

where $\psi_0(z, t) := \text{diag}(\exp(\xi(z, t_1)), \dots, \exp(\xi(z, t_N)))$, and ρ has the following asymptotic expansion

$$\rho(z) \sim \mathcal{T} + \alpha z^{-1} + \mathcal{O}(z^{-2}), \quad z \rightarrow \infty.$$

In order to get a Baker function one normalizes the above wave function by its dominant behavior at $z = \infty$; i.e., the Baker function is

$$\psi(z, t) := \mathcal{T}^{-1}(t) \varphi(z, t),$$

and the bilinear Eq. (8) becomes

$$\begin{aligned} & \int_{S^1} \frac{dz}{2\pi i z} \psi(z, t) \psi^t(-z, t') = G(t) G^t(t'), \\ & G(t) := \mathcal{T}^{-1}(t) \mathcal{T}'(t). \end{aligned} \quad (10)$$

The asymptotic behavior of the Baker function is, by construction,

$$\psi(z, t) = \chi(z, t) \psi_0(z, t),$$

where

$$\chi(z, t) \sim 1 + \beta(t) z^{-1} + \mathcal{O}(z^{-2}), \quad z \rightarrow \infty.$$

A direct consequence of (10), when $t = t'$, is that the matrix $G(t)$ is orthogonal. Moreover, assuming that the Baker function is meromorphic inside the unit circle with poles at $\{z_1, \dots, z_n\}$, we see that the integrand in (10) has poles at $\{0\} \cup \{z_1, \dots, z_n\} \cup \{-z_1, \dots, -z_n\}$, and, therefore, by evaluating the corresponding residues, we get:

$$\begin{aligned} & \psi(0, t) \psi^t(0, t') \\ & + \sum_{i=1}^n \frac{1}{z_i} \left[\text{Res}_{z_i}(\psi(z, t)) \psi^t(-z_i, t') \right. \\ & \left. - \psi(-z_i, t) \text{Res}_{z_i}(\psi^t(z, t')) \right] \\ & = G(t) G^t(t'). \end{aligned}$$

Transposing this relation we conclude that

$$\begin{aligned} & \sum_{i=1}^n \frac{1}{z_i} \left[\text{Res}_{z_i}(\psi(z, t)) \psi^t(-z_i, t') \right. \\ & \left. - \psi(-z_i, t) \text{Res}_{z_i}(\psi^t(z, t')) \right] = 0; \end{aligned}$$

hence, it follows that

$$\psi(0, t) \psi^t(0, t') = G(t) G^t(t').$$

Therefore

$$G^{-1}(t) \psi(0, t) = G^t(t') (\psi^t)^{-1}(0, t'),$$

is a constant orthogonal matrix. We notice that (10) determines G up to $G \rightarrow GC, C \in O_N$, so that we can set:

$$G(t) = \psi(0, t).$$

In order to derive linear systems for the Baker functions we will use a Grassmannian like approach [16]. Observe that in terms of the matrix function

$$\Phi(z, t) := G^{-1}(t) \psi(z, t)$$

the bilinear relation reads

$$\int_{S^1} \frac{dz}{2\pi i z} \Phi(z, t) \Phi^t(-z, t') = 1.$$

Consider now the affine space W_{affine} of functions $w(z)$ such that

$$\int_{S^1} \frac{dz}{2\pi i z} w(z) \Phi^t(-z, t') = 1,$$

for all suitable t' , and the linear space W of functions $w(z)$ such that

$$\int_{S^1} \frac{dz}{2\pi i z} w(z) \Phi^t(-z, t') = 0,$$

for all suitable t' .

We introduce time evolutions of these spaces by $W_{\text{affine}} \mapsto W_{\text{affine}}(t) = W_{\text{affine}} \psi_0(z, t)$ and $W \mapsto W(t) = W \psi_0(z, t)$.

The main properties of these spaces are

Proposition 1.

1. The only element in $W_{\text{affine}}(t)$ with asymptotic expansion of the form

$$M(t) + \mathcal{O}(z^{-1}), \quad \text{when } z \rightarrow \infty$$

$$\text{is } \Phi(z, t) \psi_0(z, t).$$

2. The linear space $W(t)$ has no elements with asymptotic expansion

$$R(t) z^{-n} + \mathcal{O}(z^{-n-1}), \quad R \neq 0, n \geq 0,$$

$$\text{when } z \rightarrow \infty.$$

With these at hand we can show that the BKP hierarchy constitutes a set of iso-orthogonal deformations of orthogonal nets. We shall denote by ψ_i and g_i the i -th row of the matrices ψ and G , and use the notation $u_i = t_{i,1}$ as well as E_{kl} for the matrix $(\delta_{ik} \delta_{lj})$, and $P_i = E_{ii}$.

Theorem 1.

1. The vectors $\psi_i(z, t)$ satisfy:

$$\frac{\partial \psi_i}{\partial u_j} - \beta_{ij} \psi_j = 0, \quad i, j = 1, \dots, N, i \neq j.$$

2. The vectors $g_i(z, t)$ satisfy

$$\frac{\partial g_i}{\partial u_j} - \beta_{ij} g_j = 0, \quad i, j = 1, \dots, N, i \neq j,$$

$$\frac{\partial g_i}{\partial u_i} + \sum_{\substack{k=1, \dots, N \\ k \neq i}} \beta_{ki} g_k = 0, \quad i = 1, \dots, N.$$

3. The coefficients β_{ij} are solutions of the Lamé equations:

$$\frac{\partial \beta_{ij}}{\partial u_k} - \beta_{ik} \beta_{kj} = 0, \quad i, j, k = 1, \dots, N,$$

with i, j, k different,

$$\frac{\partial \beta_{ij}}{\partial u_i} + \frac{\partial \beta_{ji}}{\partial u_j} + \sum_{\substack{k=1, \dots, N \\ k \neq i, j}} \beta_{ki} \beta_{kj} = 0,$$

$$i, j = 1, \dots, N, i \neq j.$$

Proof.

1. From (10) we easily deduce that

$$P_j \left(\frac{\partial \psi}{\partial u_i} - \frac{\partial G}{\partial u_i} G^{-1} \psi \right) \in W, \quad i \neq j.$$

Then, because

$$\frac{\partial \psi}{\partial u_i} - \frac{\partial G}{\partial u_i} G^{-1} \psi \sim (P_i z + \mathcal{O}(1)) \psi_0(z, t)$$

from statement 2 of Proposition 1 we conclude the identity

$$P_j \frac{\partial \psi}{\partial u_i} = P_j \frac{\partial G}{\partial u_i} G^{-1} \psi, \quad i, j = 1, \dots, N, i \neq j.$$

On the other hand:

$$\int_{S^1} \frac{dz}{2\pi i z} \frac{\partial \psi}{\partial u_i} (z, t) \psi^t(-z, t) = \frac{\partial G}{\partial u_i} (t) G^{-1} (t)$$

and evaluating the residue of the integrand at $z = \infty$ we get

$$\beta_{ji}(t) E_{ji} = P_j \frac{\partial G}{\partial u_i} (t) G^{-1} (t),$$

$$i, j = 1, \dots, N, i \neq j,$$

which gives the stated result.

2. Since $G(t) = \psi(0, t)$, its rows satisfy the same linear system as the rows of the Baker function (see 1 of the Theorem). Moreover, as it is an orthogonal matrix its rows form an orthonormal frame $\{g_i\}_{i=1}^N$. Hence, we have:

$$\frac{\partial g_i}{\partial u_i} = \sum_{k=1}^N A_{ik} g_k$$

with

$$\frac{\partial \mathbf{g}_i}{\partial u_i} \cdot \mathbf{g}_k = A_{ik}.$$

But from $\mathbf{g}_i \cdot \mathbf{g}_k = \delta_{ik}$ it follows that

$$\frac{\partial \mathbf{g}_k}{\partial u_i} \cdot \mathbf{g}_i = -A_{ik}, \quad A_{ii} = 0;$$

and recalling that $\frac{\partial \mathbf{g}_k}{\partial u_i} = \beta_{ki} \mathbf{g}_i$ we get

$$\frac{\partial \mathbf{g}_i}{\partial u_i} + \sum_{\substack{k=1, \dots, N \\ k \neq i}} \beta_{ki} \mathbf{g}_k = 0.$$

3. The Lamé equations are the compatibility conditions for the linear system satisfied by the rows of G . \square

5. The Ribaucour transformation as a vertex operator

In this section we are going to identify the action of the vertex operator $\mathbb{V}_i(z)$ with the classical Ribaucour transformation. For this aim we shall use some identities for the Baker functions which in turn correspond to Fay identities for the underlying τ -functions. If we set $\mathbf{t} \mapsto \mathbf{t} + [1/p]\mathbf{e}_i$ and $\mathbf{t}' \mapsto \mathbf{t}$ in (10) then taking into account that

$$\begin{aligned} \mathbb{V}_i(p) \psi(z, \mathbf{t}) &= \chi(z, \mathbf{t} + [1/p]\mathbf{e}_i) \\ &\times \left[1 - \frac{2z}{z-p} P_i \right] \psi_0(z, \mathbf{t}), \end{aligned}$$

for $|p| > 1$, we obtain

$$\begin{aligned} 1 - 2P_i - \text{Res}_p(\mathbb{V}_i(p) \psi(z, \mathbf{t})) P_i \psi^t(-p, \mathbf{t}) \\ = (\mathbb{V}_i(p) G(\mathbf{t})) G(\mathbf{t})^t. \end{aligned} \quad (11)$$

Now, the orthogonal character of the right-hand side of this formula implies

$$\begin{aligned} (1 - 2P_i) \psi(-p, \mathbf{t}) P_i \text{Res}_p(\mathbb{V}_i(p) \psi^t(z, \mathbf{t})) \\ + \text{Res}_p(\mathbb{V}_i(p) \psi(z, \mathbf{t})) P_i \psi^t(-p, \mathbf{t}) (1 - 2P_i) \\ = \left[\sum_{k=1}^N \psi_{ki}^2(-p, \mathbf{t}) \right] \\ \times \text{Res}_p(\mathbb{V}_i(p) \psi(z, \mathbf{t})) \\ \times P_i \text{Res}_p(\mathbb{V}_i(p) \psi^t(z, \mathbf{t})), \end{aligned}$$

By multiplying thus expression on the left by P_j and on the right by P_i it follows that

$$\begin{aligned} (-1)^{\delta_{ij}} \psi_{ji}(-p, \mathbf{t}) \text{Res}_p(\mathbb{V}_i(p) \psi_{ii}(z, \mathbf{t})) \\ - \text{Res}_p(\mathbb{V}_i(p) \psi_{ji}(z, \mathbf{t})) \psi_{ii}(-p, \mathbf{t}) \\ = \left[\sum_{k=1}^N \psi_{ki}^2(-p, \mathbf{t}) \right] \\ \times \text{Res}_p(\mathbb{V}_i(p) \psi_{ii}(z, \mathbf{t})) \\ \times \text{Res}_p(\mathbb{V}_i(p) \psi_{ji}(z, \mathbf{t})), \end{aligned}$$

or equivalently

$$\text{Res}_p(\mathbb{V}_i(p) \psi_{ji}(z, \mathbf{t})) = (-1)^{\delta_{ij}} \frac{2\psi_{ji}(-p, \mathbf{t})}{\sum_{k=1}^N \psi_{ki}^2(-p, \mathbf{t})}. \quad (12)$$

If we multiply (11) by P_j and $G(\mathbf{t})$ on the left and right, respectively, we get

$$\begin{aligned} (-1)^{\delta_{ij}} \mathbf{g}_j(\mathbf{t}) \\ - \text{Res}_p(\mathbb{V}_i(p) \psi_{ji}(z, \mathbf{t})) \sum_{k=1}^N \psi_{ki}(-p, \mathbf{t}) \mathbf{g}_k(\mathbf{t}) \\ = \mathbb{V}_i(p) \mathbf{g}_j(\mathbf{t}). \end{aligned}$$

This equation together with (12) shows that

$$\begin{aligned} (-1)^{\delta_{ij}} \mathbb{V}_i(p) \mathbf{g}_j(\mathbf{t}) \\ = \mathbf{g}_j(\mathbf{t}) \\ - \frac{2\psi_{ji}(-p, \mathbf{t})}{\sum_{k=1}^N \psi_{ki}^2(-p, \mathbf{t})} \left[\sum_{k=1}^N \psi_{ki}(-p, \mathbf{t}) \mathbf{g}_k(\mathbf{t}) \right]. \end{aligned}$$

Hence, we conclude

Theorem 2. The Ribaucour transformation of the Lamé equations with transformation data $\zeta_k(\mathbf{t}) = \psi_{ki}(-p, \mathbf{t})$, $k = 1, \dots, N$ is given by

$$\mathcal{R}(\mathbf{g}_j) = (-1)^{\delta_{ij}} \mathbb{V}_i(p) \mathbf{g}_j.$$

References

- [1] L. Bianchi, *Lezioni di Geometria Differenziale*, 3-a ed., Zanichelli, Bologna, 1924.
- [2] G. Darboux, *Leçons sur la théorie générale des surfaces IV*,

- Gauthier-Villars, Paris, 1896. Reprinted by Chelsea Publishing Company, New York, 1972.
- [3] G. Darboux, *Leçons sur les systèmes orthogonaux et les coordonnées curvilignes* (deuxième édition), Gauthier-Villars, Paris, 1910 (the first edition was in 1897). Reprinted by Éditions Jacques Gabay, Sceaux, 1993.
 - [4] G. Darboux, *Ann. L'École Normale* 3 (1866) 97.
 - [5] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, *Physica D* 4 (1982) 343.
 - [6] A. Doliwa, M. Mañas, L. Martínez Alonso, E. Medina, P.M. Santini, *Charged Free Fermions, Vertex Operators and Classical Theory of Conjugate Nets*, 1998, *solv-int*/9803015.
 - [7] R. Dijkgraaf, E. Verlinde, H. Verlinde, *Nucl. Phys. B* 352 (1991) 59.
 - [8] B. Dubrovin, *Nucl. Phys. B* 379 (1992) 627.
 - [9] L.P. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces*, Ginn, Co., Boston, 1909.
 - [10] L.P. Eisenhart, *Transformations of Surfaces*, Princeton University Press, Princeton, 1923. Reprinted by Chelsea Publishing Company, New York, 1962.
 - [11] D.-Th. Egorov, *Comp. Rend. Acad. Sci. Paris* 131 (1900) 668; 132 (1901) 174.
 - [12] P. Goddard, D. Olive, *Int. J. Mod. Phys. A* 1 (1986) 303.
 - [13] I.M. Krichever, *Func. Anal. Appl.* 31 (1997) 25.
 - [14] G. Lamé, *Leçons sur la théorie des coordonnées curvilignes et leurs diverses applications*, Mallet-Bachelier, Paris, 1859.
 - [15] A. Ribaucour, *Comp. Rend. Acad. Sci. Paris* 74 (1872) 1489.
 - [16] G. Segal, G. Wilson, *Publ. Math. IHES* 61 (1985) 5.
 - [17] E. Witten, *Nucl. Phys. B* 340 (1990) 281.
 - [18] V.E. Zakharov, *On Integrability of the Equations Describing N -Orthogonal Curvilinear Coordinate Systems and Hamiltonian Integrable Systems of Hydrodynamic Type I: Integration of the Lamé Equations*, Preprint, 1996.