Optical system design for orthosymplectic transformations in phase space

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On the basis of a matrix formalism, we analyze the paraxial optical systems composed by generalized lenses and fixed free-space intervals, suitable for orthosymplectic transformations in phase space. Flexible configurations to perform the attractive operations for optical information processing such as image rotation, separable fractional Fourier transformation, and twisting for different parameters are proposed. © 2006 Optical Society of America

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1. INTRODUCTION

Many interesting applications of first-order optical systems for information processing have been proposed in the past decade. Indeed, first-order optical systems are used for beam characterization, mode conversion, and typical signal processing operations such as filtering and correlation, etc. Several important basic transforms—scaling, quadratic phase modulation, fractional Fourier transformation (FT)—are performed by the systems constructed by spherical lenses separated by convenient free-space intervals. The list of operations significantly enlarges when cylindrical lenses are applied. It allows one to produce image rotation and nonsymmetrical scaling, which lead to affine image transformation, separable fractional FT, and the combination of all these operations responsible for canonical transformation in phase space.

Many particular systems that perform scaling, separable fractional FT, image rotation Hermite–Gaussian to Laguerre–Gaussian mode conversion, etc. are known.[1–18] Nevertheless, to the best of our knowledge no systematic analysis of the design of these systems has been done.

In this paper we consider first-order optical systems constructed only by generalized lenses and free-space intervals. We restrict ourselves to the design of systems associated with orthosymplectic matrices,[14,15,18] which describe such attractive transformations as image rotation, twisting, fractional FT, etc.[4–18] Our main objective is to find a minimal lens–free-space flexible system configuration to perform a given transform. For the case of one or two parametric transforms, as used with, for example, separable fractional FT, twisting, or image rotation, we are interested in systems that are easily adjustable for new parameter values. A setup with fixed free-space intervals and anamorphic lenses whose rotations are performed experimentally or by a spatial light modulator (SLM) is a promising candidate for these systems.

In Section 2 we introduce the matrix formalism and some transforms useful for information processing. In Section 3 we consider a general system composed of n generalized lenses and $n+1$ free-space intervals, and derive the equation that relates the parameters of the transfer matrix. In Section 4 symmetrical systems are studied and flexible configurations for the separable fractional FT and the twisting operation are proposed. The practical realization of the required lenses is discussed in Section 5. Section 6 is devoted to the design of the image rotator. The paper ends with concluding remarks.

2. FIRST-ORDER OPTICAL SYSTEMS: FUNDAMENTALS AND USEFUL EXAMPLES

First-order lossless optical systems are characterized by the well-known real 4×4 ray transformation matrix $\mathbf{T}$,[1,18] which relates the position $\mathbf{r}_i$ and direction $\mathbf{q}_i$ of an incoming ray to the position $\mathbf{r}_o$ and direction $\mathbf{q}_o$ of the outgoing ray:

$$\begin{bmatrix}\mathbf{r}_o\\ \mathbf{q}_o\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{r}_i\\ \mathbf{q}_i\end{bmatrix}=\mathbf{T}\begin{bmatrix}\mathbf{r}_i\\ \mathbf{q}_i\end{bmatrix}. \quad (1)$$

Notice that $\mathbf{r}^t=(x,y)$ and $\mathbf{q}^t=(q_x,q_y)$. Matrix $\mathbf{T}$ is symplectic, which means that $\mathbf{T}^t\mathbf{Q}\mathbf{T}=\mathbf{Q}$, where $\mathbf{T}^t$ denotes the transposition operation of $\mathbf{T}$ and $\mathbf{Q}$ is the 4×4 skew-symmetric matrix:

$$\mathbf{Q}=\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{I} & \mathbf{0}\end{bmatrix}, \quad (2)$$

where $\mathbf{I}$ is a unity 2×2 submatrix. Because of the symplecticity condition,

$$\mathbf{A}\mathbf{B}^t=\mathbf{B}\mathbf{A}^t,\quad \mathbf{C}\mathbf{D}^t=\mathbf{D}\mathbf{C}^t,\quad \mathbf{A}\mathbf{D}^t-\mathbf{B}\mathbf{C}^t=\mathbf{I},$$

$$\mathbf{A}^t\mathbf{C}=\mathbf{C}^t\mathbf{A},\quad \mathbf{B}^t\mathbf{D}=\mathbf{D}^t\mathbf{B},\quad \mathbf{A}^t\mathbf{D}-\mathbf{C}^t\mathbf{B}=\mathbf{I}, \quad (3)$$

the 4×4 ray transformation matrix is described by only ten free parameters. Below we reserve capital bold letters $\mathbf{T}$, $\mathbf{Q}$, $\mathbf{M}$, and $\mathbf{R}$ to indicate the 4×4 ray transformation matrix; other bold letters correspond to the 2×2 submatrices, which compose the entire 4×4 matrix. In the text we use dimensionless variables: $\mathbf{r}_{i,o}/\lambda\rightarrow\mathbf{r}_{i,o}$, $\mathbf{q}_{i,o}/\lambda\rightarrow\mathbf{q}_{i,o}$, $\mathbf{B}/\lambda\rightarrow\mathbf{B}$, $\mathbf{C}\lambda\rightarrow\mathbf{C}$, where λ is the wavelength.

Using the Iwasawa decomposition, any ray transformation matrix[15,18] can be expressed as a product of an anamorphic lens, an anamorphic magnifier, and an orthosymplectic system:

$$\mathbf{T} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{L} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ -\mathbf{Y} & \mathbf{X} \end{bmatrix}, \tag{4}$$

where

$$\mathbf{S} = (\mathbf{AA}^t + \mathbf{BB}^t)^{1/2} = \mathbf{S}^t,$$

$$\mathbf{X} + i\mathbf{Y} = (\mathbf{AA}^t + \mathbf{BB}^t)^{-1/2}(\mathbf{A} + i\mathbf{B}),$$

$$\mathbf{L} = (\mathbf{CA}^t + \mathbf{DB}^t)(\mathbf{AA}^t + \mathbf{BB}^t)^{-1} = \mathbf{L}^t. \tag{5}$$

Here t stands for transposition operation. The first matrix in the decomposition performs an anamorphic quadratic phase modulation. Several flexible schemes based on two or more cylindrical and spherical lenses to modulate the phase have been proposed.[2,8] The scaling operation, described by the second matrix, is also well studied.[4,18]

The last orthosymplectic matrix ($\mathbf{M}^t = \mathbf{M}^{-1}$) is a general expression for a variety of attractive transforms.[14,15,18] Thus the submatrices

$$\mathbf{X}_{fr\mathrm{FT}} = \begin{bmatrix} \cos\gamma_x & 0 \\ 0 & \cos\gamma_y \end{bmatrix}, \quad \mathbf{Y}_{fr\mathrm{FT}} = \begin{bmatrix} \sin\gamma_x & 0 \\ 0 & \sin\gamma_y \end{bmatrix}, \tag{6}$$

are associated with the separable fractional Fourier transform,[9–11,14,15,18] which produces the rotations at planes (x, q_x) and (y, q_y) of the phase space. The separable fractional FT for angles $\gamma_x = 0$, $\gamma_y = \pi$ corresponds to image reflection.

The following submatrices

$$\mathbf{X}_{\mathrm{rot}} = \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix}, \quad \mathbf{Y}_{\mathrm{rot}} = \mathbf{0} \tag{7}$$

describe an image rotator operation,[7,14,18] related to the rotation in position and spatial-frequency planes (x, y) and (q_x, q_y).

The gyrator (twisting) operation associated with

$$\mathbf{X}_{\mathrm{gyr}} = \begin{bmatrix} \cos\alpha & 0 \\ 0 & \cos\alpha \end{bmatrix}, \quad \mathbf{Y}_{\mathrm{gyr}} = \begin{bmatrix} 0 & \sin\alpha \\ \sin\alpha & 0 \end{bmatrix} \tag{8}$$

describes the rotation at the twisted position and spatial-frequency planes (x, q_y) and (y, q_x).[14,15,18] In particular, the gyrator operation is responsible for the Hermite–Gaussian to Laguerre–Gaussian mode conversion.[14–18] The design of these three particular systems will be considered in detail. It was shown in Ref. 19 that any orthosymplectic system can be constructed from a fractional FT system embedded in two rotator systems. Then, in general, the third system (gyrator) can be obtained as a combination of the first ones. Nevertheless, we will consider it separately so as to simplify the system design. The conditions of Eqs. (3) for the case of orthosymplectic (orthogonal and symplectic) matrices reduce to[18]

$$\mathbf{XY}^t = \mathbf{YX}^t, \quad \mathbf{X}^t\mathbf{Y} = \mathbf{Y}^t\mathbf{X}, \quad \mathbf{XX}^t + \mathbf{YY}^t = \mathbf{I}. \tag{9}$$

3. SYSTEM DESIGN

The basic elements we have for the design of a given first-order optical system characterized by a matrix $\mathbf{T}$ are generalized lenses and intervals of free space. A generalized lens [see the first matrix in Eq. (4)] is a superposition of m cylindrical or spherical lenses and is represented by a submatrix $\mathbf{L}$[2,8]:

$$\mathbf{L} = \frac{1}{2} \begin{bmatrix} \sum_{i=1}^{m} p_i(1+\cos 2\varphi_i) & \sum_{i=1}^{m} p_i \sin 2\varphi_i \\ \sum_{i=1}^{m} p_i \sin 2\varphi_i & \sum_{i=1}^{m} p_i(1-\cos 2\varphi_i) \end{bmatrix}, \tag{10}$$

where p_i is the power (in dimensionless coordinates $p_i\lambda \to p_i$) and φ_i is a rotation angle for the ith lens. Note that $\mathbf{L} = \mathbf{L}^t$. The lenses might be constructed by assembling several cylindrical and spherical lenses, performed holographically, or implemented by a SLM.[12]

Let us consider a system (see Fig. 1) that contains n lenses with $n+1$ free-space intervals of length z_i ($z_i/\lambda \to z_i$) between themselves and the input–output planes. The matrix describing this system is given by

$$\mathbf{T}_n = \begin{bmatrix} \mathbf{A}_n & \mathbf{B}_n \\ \mathbf{C}_n & \mathbf{D}_n \end{bmatrix} = \begin{bmatrix} \mathbf{I} & z_{n+1}\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{L}_n & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & z_n\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \cdots \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{L}_1 & \mathbf{I} \end{bmatrix} \times \begin{bmatrix} \mathbf{I} & z_1\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{11}$$

One can easily derive the expression for the system with one lens $\mathbf{L}_1$:

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & z_2\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{L}_1 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & z_1\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_1 z_2 + \mathbf{I} & \mathbf{L}_1 z_1 z_2 + \mathbf{I}(z_1+z_2) \\ \mathbf{L}_1 & \mathbf{L}_1 z_1 + \mathbf{I} \end{bmatrix} \tag{12}$$

and establish the following relation among submatrices $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{C}_1$, and $\mathbf{D}_1$:

$$\mathbf{B}_1 = \mathbf{A}_1 z_1 - \mathbf{C}_1 z_1 z_2 + \mathbf{D}_1 z_2. \tag{13}$$

In the same way a similar expression can be obtained for the systems with zero (free-space interval only), two, three, and four lenses. The corresponding equations are summarized as

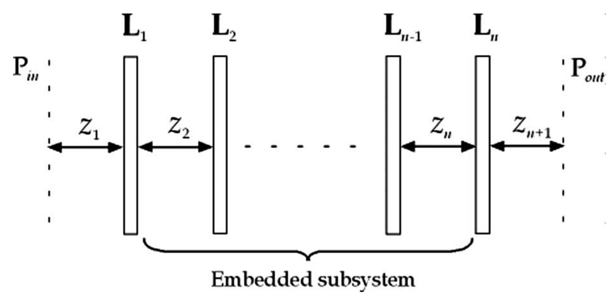

Fig. 1. Optical setup. P_{in} and P_{out} are the input and output planes, respectively. Generalized lenses $\mathbf{L}_m$ and free-space intervals z_m are also displayed.

$$\mathbf{B}_0 = \mathbf{A}_0 z_1,$$

$$\mathbf{B}_1 = \mathbf{A}_1 z_1 - \mathbf{C}_1 z_1 z_2 + \mathbf{D}_1 z_2,$$

$$\mathbf{B}_2 = \mathbf{A}_2 z_1 - \mathbf{C}_2 z_1 z_3 + \mathbf{D}_2 z_3 + \mathbf{I} z_2,$$

$$\mathbf{B}_3 = \mathbf{A}_3 z_1 - \mathbf{C}_3 z_1 z_4 + \mathbf{D}_3 z_4 + \mathbf{L}_2 z_2 z_3 + \mathbf{I}(z_2 + z_3),$$

$$\mathbf{B}_4 = \mathbf{A}_4 z_1 - \mathbf{C}_4 z_1 z_5 + \mathbf{D}_4 z_5 + \mathbf{L}_2 z_2 (z_3 + z_4) + \mathbf{L}_3 z_4 (z_2 + z_3) + \mathbf{L}_3 \mathbf{L}_2 z_2 z_3 z_4 + \mathbf{I}(z_2 + z_3 + z_4). \quad (14)$$

We observe that the submatrix $\mathbf{B}$ is a sum of submatrix $\mathbf{A}$ multiplied by the distance of the first free-space interval, submatrix $\mathbf{D}$ multiplied by the distance of the last free-space interval, submatrix $\mathbf{C}$ with opposite sign multiplied by the distance of the first and the last free-space intervals, and a term related only to the subsystem embedded between the first and the last lenses excluding them (see Fig. 1). Analyzing Eqs. (14) we have found that this additional term corresponds to the submatrix $\mathbf{B}$ of the embedded system. Then the general expression that relates the submatrices of the matrix $\mathbf{T}_n$ $(n \geq 2)$ can be formulated as

$$\mathbf{B}_n = \mathbf{A}_n z_1 - \mathbf{C}_n z_1 z_{n+1} + \mathbf{D}_n z_{n+1} + \mathbf{B}^e_{n-2}, \quad (15)$$

where the submatrix $\mathbf{B}^e_{n-2}$ is related to the embedded subsystem, which contains $n-2$ lenses and $n-1$ free-space intervals. Equation (15) significantly simplifies the system design as will be seen in the next section.

4. SYMMETRIC SYSTEMS

Regarding the one-dimensional case, it can be shown that there is only one orthosymplectic matrix that corresponds to the one-dimensional fractional FT.[18] The known optical systems implementing a one-dimensional fractional FT and a two-dimensional fractional FT for the same angle in both orthogonal directions[6] are symmetric. This means that $z_{n+2-k} = z_k$ and $\mathbf{L}_{n+1-k} = \mathbf{L}_k$ for $k = 1, \ldots, n/2$. In the case of odd n, the central lens can be represented as a cascade of two identical ones. For example, symmetric systems with one lens $\mathbf{L}$ can be represented by the matrix $\mathbf{T} = \mathbf{M}_z \mathbf{M}_L \mathbf{M}_z = \mathbf{M}_z \mathbf{M}_{L/2} \mathbf{M}_{L/2} \mathbf{M}_z$. Correspondingly, the symmetric system with two lenses can be described by $\mathbf{T} = \mathbf{M}_{z_1} \mathbf{M}_{L_1} \mathbf{M}_{z_2} \mathbf{M}_{L_1} \mathbf{M}_{z_1} = \mathbf{M}_{z_1} \mathbf{M}_{L_1} \mathbf{M}_{z_2/2} \mathbf{M}_{z_2/2} \mathbf{M}_{L_1} \mathbf{M}_{z_1}$. In general the symmetric system of n lenses can be represented by the following transformation matrix:

$$\mathbf{T} = \widetilde{\mathbf{M}_1} \widetilde{\mathbf{M}_2}, \ldots, \widetilde{\mathbf{M}_m} \mathbf{M}_m, \ldots, \mathbf{M}_2 \mathbf{M}_1 = \tilde{\mathbf{R}} \mathbf{R}, \quad (16)$$

where $\mathbf{M}_k = \mathbf{M}_{L_k} \mathbf{M}_{z_k}$, $\widetilde{\mathbf{M}_k} = \mathbf{M}_{z_k} \mathbf{M}_{L_k}$, $m = n/2$ if n is an even number, and $m = (n+1)/2$ if n is an odd number. In the case of odd n, the matrix $\mathbf{M}_m$ is composed by the lens $\mathbf{L}_{(n+1)/2}$ with half of the focal distance. Note that if

$$\mathbf{M}_k = \mathbf{M}_{L_k} \mathbf{M}_{z_k} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & z_k \mathbf{I} \\ \mathbf{L}_k & z_k \mathbf{L}_k + \mathbf{I} \end{bmatrix}, \quad (17)$$

then

$$\widetilde{\mathbf{M}_k} = \mathbf{M}_{z_k} \mathbf{M}_{L_k} = \begin{bmatrix} \mathbf{d} & \mathbf{b} \\ \mathbf{c} & \mathbf{a} \end{bmatrix} = \begin{bmatrix} z_k \mathbf{L}_k + \mathbf{I} & z_k \mathbf{I} \\ \mathbf{L}_k & \mathbf{I} \end{bmatrix}, \quad (18)$$

$$\widetilde{\mathbf{M}_k}^{-1} = \begin{bmatrix} \mathbf{d} & \mathbf{b} \\ \mathbf{c} & \mathbf{a} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{a}^t & -\mathbf{b}^t \\ -\mathbf{c}^t & \mathbf{d}^t \end{bmatrix} = \begin{bmatrix} \mathbf{a} & -\mathbf{b} \\ -\mathbf{c} & \mathbf{d} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -z_k \mathbf{I} \\ -\mathbf{L}_k & z_k \mathbf{L}_k + \mathbf{I} \end{bmatrix}. \quad (19)$$

We observe that $\widetilde{\mathbf{M}_k}^{-1}(\mathbf{L}_k, z_k) = \mathbf{M}_k(-\mathbf{L}_k, -z_k)$. This result is also valid for a subsystem constructed by a single lens or by a free-space interval. It is easy to see that if the transformation matrix of the subsystem is

$$\mathbf{R} = \begin{bmatrix} \mathbf{A}_r & \mathbf{B}_r \\ \mathbf{C}_r & \mathbf{D}_r \end{bmatrix}, \quad (20)$$

then

$$\tilde{\mathbf{R}} = \begin{bmatrix} \mathbf{D}_r^t & \mathbf{B}_r^t \\ \mathbf{C}_r^t & \mathbf{A}_r^t \end{bmatrix}. \quad (21)$$

Finally the transformation matrix for a symmetric system can be written as

$$\mathbf{T} = \tilde{\mathbf{R}} \mathbf{R} = \begin{bmatrix} \mathbf{D}_r^t \mathbf{A}_r + \mathbf{B}_r^t \mathbf{C}_r & \mathbf{D}_r^t \mathbf{B}_r + \mathbf{B}_r^t \mathbf{D}_r \\ \mathbf{C}_r^t \mathbf{A}_r + \mathbf{A}_r^t \mathbf{C}_r & \mathbf{C}_r^t \mathbf{B}_r + \mathbf{A}_r^t \mathbf{D}_r \end{bmatrix}, \quad (22)$$

or using the symplecticity conditions of Eqs. (3) for the matrix $\mathbf{R}$

$$\mathbf{T} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_r^t \mathbf{A}_r + \mathbf{B}_r^t \mathbf{C}_r & 2\mathbf{D}_r^t \mathbf{B}_r \\ 2\mathbf{C}_r^t \mathbf{A}_r & (\mathbf{D}_r^t \mathbf{A}_r + \mathbf{B}_r^t \mathbf{C}_r)^t \end{bmatrix}. \quad (23)$$

This means that $\mathbf{A} = \mathbf{D}^t$, $\mathbf{B} = \mathbf{B}^t$, and $\mathbf{C} = \mathbf{C}^t$.

Correspondingly, an orthosymplectic system can be implemented by a symmetrical system only if its submatrices $\mathbf{X}$ and $\mathbf{Y}$ are equal to its transpose:

$$\mathbf{X} = \mathbf{X}^t,$$

$$\mathbf{Y} = \mathbf{Y}^t. \quad (24)$$

From this requirement we can conclude that image rotation cannot be performed by a symmetrical system (except for the rotation at π) since in this case $\mathbf{X} \neq \mathbf{X}^t$, as can be seen from Eqs. (7).

If an orthosymplectic matrix satisfies Eqs. (24), then according to Eq. (9) we obtain the following restrictions:

$$\mathbf{XY} = \mathbf{YX}, \quad \mathbf{X}^2 + \mathbf{Y}^2 = \mathbf{I}. \quad (25)$$

Note that this relation holds for the separable fractional FT and the gyrator matrices [see Eqs. (6) and (8)].

Let us now design the symmetrical systems able to perform the fractional FT and gyrator operations. On the basis of Eqs. (3) and (23) we derive that

$$\mathbf{X} = 2\mathbf{D}_r^t \mathbf{A}_r - \mathbf{I},$$

$$\mathbf{Y} = 2\mathbf{D}_r^t \mathbf{B}_r = -2\mathbf{C}_r^t \mathbf{A}_r, \quad (26)$$

which means that the following relation has to hold:

$$\mathbf{D}_r^t\mathbf{B}_r + \mathbf{C}_r^t\mathbf{A}_r = \mathbf{0}. \tag{27}$$

There are two well-known symmetrical optical configurations containing one or two lenses correspondingly, which perform the fractional FT for the same angle in both orthogonal directions.[5,6] Several other schemes to perform separable fractional FT have been proposed.[9–11] However, they are not flexible; thus, to change the angle of the fractional FT, the distances between the lens and input–output planes and the lens powers have to be changed. Recently, an optical system based on two quadrupoles[8] to perform a separable fractional FT has been proposed.[13] This configuration permits one to change the fractional angles by a corresponding rotation of the quadrupoles, but the resulting fractional FT has a scaling depending on the angle values.

Here we will consider two symmetrical systems with three generalized lenses, which are able to perform a separable fractional FT for different angles in the x and y directions, and gyrator operation, without adjusting the distances when the angles are changed.

An orthosymplectic symmetric system with three lenses can be represented by the transformation matrix $\mathbf{T}=\mathbf{M}_{z_1}\mathbf{M}_{L_1}\mathbf{M}_{z_2}\mathbf{M}_{L_2/2}\mathbf{M}_{L_2/2}\mathbf{M}_{z_2}\mathbf{M}_{L_1}\mathbf{M}_{z_1}$. Let us consider the case $z_1=0$. Then the submatrices $\mathbf{A}_r$, $\mathbf{B}_r$, $\mathbf{C}_r$, and $\mathbf{D}_r$ are given by

$$\mathbf{A}_r = \mathbf{L}_1 z_2 + \mathbf{I},$$

$$\mathbf{B}_r = \mathbf{I} z_2,$$

$$\mathbf{C}_r = \mathbf{L}_1 + \frac{1}{2}\mathbf{L}_2 + \frac{1}{2}\mathbf{L}_2\mathbf{L}_1 z_2,$$

$$\mathbf{D}_r = \frac{1}{2}\mathbf{L}_2 z_2 + \mathbf{I}. \tag{28}$$

Then, based on Eqs. (26), we obtain

$$\mathbf{X} = z_2(2\mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_1\mathbf{L}_2 z_2) + \mathbf{I},$$

$$\mathbf{Y} = z_2(\mathbf{L}_2 z_2 + 2\mathbf{I}). \tag{29}$$

From $\mathbf{XY}=\mathbf{YX}$ one can see that $\mathbf{L}_1\mathbf{L}_2=\mathbf{L}_2\mathbf{L}_1$ and therefore Eq. (27) holds. In the nonsingular case $\det \mathbf{Y} \neq 0$,[19] the generalized lenses $\mathbf{L}_1$ and $\mathbf{L}_2$ can be written as a combination of the $\mathbf{X}$ and $\mathbf{Y}$ matrices as

$$\mathbf{L}_1 = (\mathbf{X} + \mathbf{I})\mathbf{Y}^{-1} - \frac{1}{z_2}\mathbf{I}, \tag{30a}$$

$$\mathbf{L}_2 = \frac{1}{z_2^2}(\mathbf{Y} - 2z_2\mathbf{I}). \tag{30b}$$

The separable fractional FT [see Eqs. (6)] is obtained when

$$\mathbf{L}_1(\gamma_x,\gamma_y) = \begin{bmatrix} \cot\left(\frac{\gamma_x}{2}\right) - \frac{1}{z_2} & 0 \\ 0 & \cot\left(\frac{\gamma_y}{2}\right) - \frac{1}{z_2} \end{bmatrix}, \tag{31a}$$

$$\mathbf{L}_2(\gamma_x,\gamma_y) = \frac{1}{z_2^2}\begin{bmatrix} \sin\gamma_x - 2z_2 & 0 \\ 0 & \sin\gamma_y - 2z_2 \end{bmatrix}. \tag{31b}$$

Furthermore, the gyrator operation, Eqs. (8), is obtained from Eqs. (30a) and (30b) when

$$\mathbf{L}_1(\alpha) = \begin{bmatrix} -\frac{1}{z_2} & \cot\left(\frac{\alpha}{2}\right) \\ \cot\left(\frac{\alpha}{2}\right) & -\frac{1}{z_2} \end{bmatrix},$$

$$\mathbf{L}_2(\alpha) = \frac{1}{z_2^2}\begin{bmatrix} -2z_2 & \sin\alpha \\ \sin\alpha & -2z_2 \end{bmatrix}. \tag{32}$$

Equations (31), (34), and (32) satisfy the form of the generalized lens [Eq. (10)]. The varying lenses used to perform the separable fractional FT and the gyrator operation for different angles can be implemented holographically or by a programmable SLM. In the last case all range of angles, limited only by the SLM resolution and the condition $\det \mathbf{Y} \neq 0$, can be reached without changing the distances or lens rotations. Moreover, both separable fractional FT or gyrator operations are performed by the same setup. In the case of the implementations of ordinary lenses, only some sets of angle parameters γ_x, γ_y or α are available without changing the lens power. The optical design of $\mathbf{L}_1$ and $\mathbf{L}_2$ for this case is discussed in Section 5.

Note that for a singular case, $\det \mathbf{Y}=0$, Eq. (30a) cannot be applied. Nevertheless, the generalized lens $\mathbf{L}_1$ is obtained from the following equation:

$$\mathbf{X} = \left(\mathbf{L}_1 + \frac{1}{z_2}\mathbf{I}\right)\mathbf{Y} - \mathbf{I}. \tag{33}$$

It is easy to see that if $\mathbf{Y}=\mathbf{0}$, then $\mathbf{X}=-\mathbf{I}$, which coincides with the separable fractional FT for angles $\gamma_x=\gamma_y=\pi$, gyrator operation for $\alpha=\pi$, and image rotation at angle π. This operation is realizable by the proposed setup for arbitrary $\mathbf{L}_1$ and $\mathbf{L}_2=-2z_2^{-1}\mathbf{I}$.

In general, the singular case for the separable fractional FT arises when at least one of the angles γ_x or γ_y equals $m\pi$, where m is an integer. Analyzing Eq. (33), we conclude that only the fractional FT operations for γ_x and/or γ_y equal to π can be obtained by the proposed scheme. In particular, in the case $\gamma_y=\pi$, the generalized lenses $\mathbf{L}_1$ and $\mathbf{L}_2$ are given by

$$\mathbf{L}_1(\gamma_x,\pi) = \begin{bmatrix} \cot\left(\frac{\gamma_x}{2}\right) - \frac{1}{z_2} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{L}_2(\gamma_x,\pi) = \frac{1}{z_2^2}\begin{bmatrix} \sin\gamma_x - 2z_2 & 0 \\ 0 & -2z_2 \end{bmatrix}. \tag{34}$$

The singular cases corresponding to the fractional FT at angles γ_x and/or γ_y equal 0, and gyrator operation for $\alpha=0$ (corresponding to the identity transform) cannot be realized by the proposed setup.

5. L_1 AND L_2 OPTICAL DESIGN FOR SEPARABLE FRACTIONAL FOURIER TRANSFORMATION AND GYRATOR OPERATOR IMPLEMENTATION

A useful generalized lens is obtained when three particular lenses are assembled, $\mathcal{L}=\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3$ (see Fig. 2), where $\mathcal{L}_1$ and $\mathcal{L}_2$ are convergent and divergent cylindrical lenses, respectively, and $\mathcal{L}_3$ is a convergent spherical lens.[8] The cylindrical lenses $\mathcal{L}_1$ and $\mathcal{L}_2$ can be rotated while $\mathcal{L}_3$ remains fixed. Therefore $\mathcal{L}$ can be written as a function of $\mathcal{L}_1(\varphi_1)$, $\mathcal{L}_2(\varphi_2)$, and $\mathcal{L}_3$ as follows:

$$\mathcal{L}=p\sin 2\omega\begin{bmatrix}-\sin 2\Omega & \cos 2\Omega\\ \cos 2\Omega & \sin 2\Omega\end{bmatrix}+p_3\mathbf{I}, \tag{35}$$

where $\varphi_1=\Omega+\omega$, $\varphi_2=\Omega-\omega$, and $p=p_1=-p_2$ and p_3 are the powers for the lenses $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$, correspondingly.

For the separable fractional FT [Eqs. (31)] for the angles $\gamma_x=-\gamma_y=\alpha$, $\mathbf{L}_2$ can be implemented by using lens $\mathcal{L}$ with the following parameters: $\Omega=-(\pi/4)$, $2\omega=\alpha=2\varphi_1+(\pi/2)$, $p=-(1/z_2^2)$, and $p_3=-(2/z_2)$. Notice that different values of angle α for $\mathbf{L}_2(\alpha,-\alpha)$ are obtained by only rotation of cylindrical lens $\mathcal{L}_1$, and the angle $\varphi_1+\varphi_2=-(\pi/2)$ is fixed.

The lens $\mathbf{L}_1(\alpha,-\alpha)$,

$$\mathbf{L}_1(\alpha,-\alpha)=\begin{bmatrix}\cot\left(\dfrac{\alpha}{2}\right)-\dfrac{1}{z_2} & 0\\ 0 & -\cot\left(\dfrac{\alpha}{2}\right)-\dfrac{1}{z_2}\end{bmatrix}, \tag{36}$$

can also be implemented by means of the $\mathcal{L}$ setup of Eq. (35), but only for certain values of α such that $\cot(\alpha/2)=p\sin 2\omega$. In this case, $\Omega=-(\pi/4)$, $2\omega=2\varphi_1+(\pi/2)$, and $p_3=-(1/z_2)$.

Analogically, the lens $\mathbf{L}_1(\alpha)$ for the gyrator operation of Eqs. (32) for the same angular sequence α as in the case of the fractional FT $\cot(\alpha/2)=p\sin 2\omega$ can be obtained by using $\mathcal{L}$ with the following parameters: $\Omega=0$, $\omega=\varphi_1$, and $p_3=-(1/z_2)$.

The lens $\mathbf{L}_2(\alpha)$ [Eqs. (32)] is obtained for $\Omega=0$, $2\omega=\alpha=2\varphi_1$, $p=(1/z_2^2)$, and $p_3=-(2/z_2)$. The variation of the angle α is reached by the lens rotation.

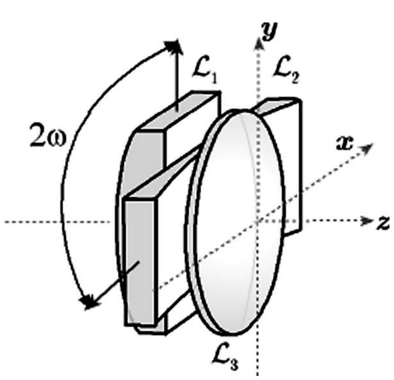

Fig. 2. Optical system associated with $\mathcal{L}=\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3$ generalized lens.

6. SYSTEMS FOR IMAGE ROTATION

Several schemes for image rotation by lens setups have been proposed.[4,7,14,18] Some of them do not relate to pure rotation since they introduce an additional phase modulation of the rotated image. We will call them imperfect rotators. Others are reflectors with rotation,[4] which also belong to the class of orthosymplectic systems, and are described by the submatrices

$$\mathbf{X}_{\mathrm{ref}}(\theta)=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}, \quad \mathbf{Y}=\mathbf{0}. \tag{37}$$

Note that a cascade of two reflectors with rotation angles θ_1 and θ_2 corresponds to a rotator with angle $\theta_1-\theta_2$,[7,14,18] since

$$\begin{aligned}\mathbf{X}_{\mathrm{rot}}(\theta_1-\theta_2)&=\mathbf{X}_{\mathrm{ref}}(\theta_2)\mathbf{X}_{\mathrm{ref}}(\theta_1)\\ &=\begin{bmatrix}\cos\theta_2 & \sin\theta_2\\ \sin\theta_2 & -\cos\theta_2\end{bmatrix}\begin{bmatrix}\cos\theta_1 & \sin\theta_1\\ \sin\theta_1 & -\cos\theta_1\end{bmatrix}\\ &=\begin{bmatrix}\cos(\theta_1-\theta_2) & \sin(\theta_1-\theta_2)\\ -\sin(\theta_1-\theta_2) & \cos(\theta_1-\theta_2)\end{bmatrix}.\end{aligned} \tag{38}$$

Our strategy will be to find a minimal configuration to perform an imperfect rotator and to correct it by an additional lens.

It can be demonstrated that a system with one lens cannot perform image rotation except in the trivial case $\theta=\pi$. Let us consider a system with two lenses $z_1\mathbf{L}_1z_2\mathbf{L}_2z_3$. The submatrix parameters in this case are given by

$$\mathbf{A}_2=\mathbf{L}_1(z_2+z_3)+\mathbf{L}_2z_3+\mathbf{L}_2\mathbf{L}_1z_2z_3+\mathbf{I}, \tag{39a}$$

$$\mathbf{B}_2=\mathbf{L}_1(z_2+z_3)z_1+\mathbf{L}_2(z_2+z_1)z_3+\mathbf{L}_2\mathbf{L}_1z_2z_3z_1+\mathbf{I}(z_2+z_3+z_1), \tag{39b}$$

$$\mathbf{C}_2=\mathbf{L}_1+\mathbf{L}_2+\mathbf{L}_2\mathbf{L}_1z_2, \tag{39c}$$

$$\mathbf{D}_2=\mathbf{L}_1z_1+\mathbf{L}_2(z_1+z_2)+\mathbf{L}_2\mathbf{L}_1z_2z_1+\mathbf{I}, \tag{39d}$$

and Eq. (15) for $n=2$,

$$\mathbf{B}_2=\mathbf{A}_2z_1-\mathbf{C}_2z_1z_3+\mathbf{D}_2z_3+\mathbf{I}z_2. \tag{40}$$

If we apply that $\mathbf{A}_2=\mathbf{D}_2=\mathbf{X}$, $\mathbf{B}_2=\mathbf{0}$ reduces to

$$\mathbf{C}_2z_1z_3-\mathbf{I}z_2=\mathbf{X}(z_1+z_3). \tag{41}$$

It is easy to see that the requirement $\mathbf{C}_2=\mathbf{0}$ leads to only the trivial solution $\theta=\pi$. Therefore we will first design a system with transfer matrix

$$\mathbf{T}=\begin{bmatrix}\mathbf{X} & \mathbf{0}\\ \mathbf{C}_2 & \mathbf{X}\end{bmatrix} \tag{42}$$

with $\mathbf{X}=(\mathbf{C}_2z_1z_3-\mathbf{I}z_2)/(z_1+z_3)$. From Eq. (39c) we derive that the imperfect rotator cannot be constructed by a two-lens configuration, since $\mathbf{C}_2$ is a symmetric matrix. Nevertheless, this configuration can be used for the imperfect reflector with rotation. Using Eq. (37) for $\mathbf{X}_{\mathrm{ref}}(\theta)$ and Eq. (41), we find that

$$\mathbf{C}_2 = \frac{z_1+z_3}{z_1z_3}\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\frac{z_2}{z_1z_3} = \mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_2\mathbf{L}_1 z_2. \quad (43)$$

Applying cylindrical and spherical lenses given by $\mathbf{L}_1$ and $\mathbf{L}_2$,

$$\mathbf{L}_1(\theta) = -\frac{z_3}{z_1z_2}\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix} - \frac{z_1+z_2}{z_1z_2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

$$\mathbf{L}_2(\theta) = -\frac{z_1}{z_2z_3}\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix} - \frac{z_2+z_3}{z_2z_3}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad (44)$$

we obtain the required configuration for an imperfect reflector with rotation for angle θ. Note that $\mathbf{D}_2 - z_1\mathbf{C}_2 = \mathbf{L}_2 z_2 + \mathbf{I}$. To compensate the undesirable phase modulation, the additional lens $\mathbf{L}_3 = -\mathbf{C}_2\mathbf{X}_{\text{ref}}^{-1}(\theta) = -\mathbf{C}_2\mathbf{X}_{\text{ref}}(\theta)$ has to be added. This lens is a combination of spherical and cylindrical lenses:

$$\mathbf{L}_3(\theta) = -\frac{z_2}{z_1z_3}\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix} - \frac{z_1+z_3}{z_1z_3}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \quad (45)$$

Therefore the combination of three lenses performs as a pure reflector with rotation angle θ. The configuration is flexible: The angle is changed by lens rotation.

Since two consecutive reflectors with rotation produce a pure rotation, then the cascade of these systems consisting of six lenses performs a rotator operation.

In a similar way it is possible to demonstrate that a cascade of two imperfect reflectors with rotation [see Eqs. (44)], constructed from a four-lens system, and an additional lens to compensate the phase shift also lead to the pure rotator operation.

Let us now demonstrate that the minimum number of generalized lenses to perform a pure rotator operation is four. We start from the consideration of the system that contains three lenses to perform an imperfect rotator $z_1\mathbf{L}_1z_2\mathbf{L}_2z_3\mathbf{L}_3z_4$ and then transform it to a pure rotator by an additional lens. The submatrices for the three-lens configuration are related to each other as

$$\mathbf{A}_3 = \mathbf{C}_3 z_4 + \mathbf{B}_1^e\mathbf{L}_1 + \mathbf{L}_2 z_3 + \mathbf{I},$$

$$\mathbf{B}_3 = \mathbf{A}_3 z_1 - \mathbf{C}_3 z_1 z_4 + \mathbf{D}_3 z_4 + \mathbf{B}_1^e,$$

$$\mathbf{C}_3 = \mathbf{L}_3[\mathbf{L}_1(z_2+z_3) + \mathbf{L}_2 z_3 + \mathbf{L}_2\mathbf{L}_1 z_2 z_3 + \mathbf{I}] + [\mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_2\mathbf{L}_1 z_2],$$

$$\mathbf{D}_3 = \mathbf{C}_3 z_1 + \mathbf{L}_3\mathbf{B}_1^e + \mathbf{L}_1 z_1 + \mathbf{L}_2(z_1+z_2) + \mathbf{L}_2\mathbf{L}_1 z_2 z_1 + \mathbf{I}, \quad (46)$$

$$\mathbf{B}_1^e = \mathbf{L}_2 z_3 z_2 + \mathbf{I}(z_2+z_3), \quad (47)$$

where the symbol e is related to the submatrix of the embedded system of Eq. (15). Applying the conditions for the imperfect rotator, $\mathbf{A}_3 = \mathbf{D}_3 = \mathbf{X}_{\text{rot}}$ and $\mathbf{B}_3 = \mathbf{0}$, we obtain

$$\mathbf{X}_{\text{rot}} = \frac{1}{(z_1+z_4)}(\mathbf{C}_3 z_1 z_4 - \mathbf{B}_1^e). \quad (48)$$

From this we conclude that it is impossible to construct a pure rotator with three lenses. Indeed, if $\mathbf{C}_3 = \mathbf{0}$ we cannot satisfy Eq. (48), since lenses (in particular $\mathbf{L}_2$) are represented by symmetrical matrices [Eq. (10)], while $\mathbf{X}_{\text{rot}}$ is not symmetrical.

Equations (46)–(48) are significantly simplified if we suppose that $z_1 = z_2 = z_3 = z_4 = z$ and choose $\mathbf{L}_2$ in the form of a cylindrical lens operating in the y direction:

$$\mathbf{L}_2 = \begin{bmatrix}0 & 0\\ 0 & p\end{bmatrix}. \quad (49)$$

Then the embedded system, and in particular $\mathbf{B}_1^e$, is completely defined, and the rest of the lenses, including the lens $\mathbf{L}_4 = -\mathbf{C}_3\mathbf{X}_{\text{rot}}^{-1}$ added in to compensate the quadratic phase modulation, are given by

$$\mathbf{L}_1 = (\mathbf{B}_1^e)^{-1}(\mathbf{I} - \mathbf{X}_{\text{rot}}) - \frac{2}{z}\mathbf{I},$$

$$\mathbf{L}_3 = -\left[z(\mathbf{B}_1^e)^{-1}(\mathbf{I} - \mathbf{X}_{\text{rot}}) + 3\mathbf{X}_{\text{rot}} + \mathbf{I} + \frac{2}{z}\mathbf{B}_1^e\right][z\mathbf{X}_{\text{rot}} + \mathbf{B}_1^e]^{-1},$$

$$\mathbf{L}_4 = -\frac{1}{z^2}(2z\mathbf{I} + \mathbf{B}_1^e\mathbf{X}^{-1}). \quad (50)$$

In particular for the case $p = -(4/z)$, the following set of lenses performing a pure rotator operation is obtained:

$$\mathbf{L}_1 = -\frac{1}{2z}\begin{bmatrix}1+\cos\theta & \sin\theta\\ \sin\theta & 1-\cos\theta\end{bmatrix} - \frac{1}{z}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix},$$

$$\mathbf{L}_2 = -\frac{4}{z}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},$$

$$\mathbf{L}_3 = -\frac{1}{2z}\begin{bmatrix}1+\cos\theta & -\sin\theta\\ -\sin\theta & 1-\cos\theta\end{bmatrix} - \frac{1}{z}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix},$$

$$\mathbf{L}_4 = -\frac{2}{z}\begin{bmatrix}1+\cos\theta & -\sin\theta\\ -\sin\theta & 1-\cos\theta\end{bmatrix}. \quad (51)$$

Note that $\mathbf{L}_3(\theta) = \mathbf{L}_1(-\theta)$. Thus we have demonstrated that four is the minimum number of generalized lenses needed to perform the pure rotator. The proposed scheme is flexible and can be performed by the combination of conventional cylindrical lenses. The rotation angle θ is changed only by lens rotation, without varying the distances between the lenses and input–output planes.

7. CONCLUSIONS

We have considered optical systems containing only lenses and free-space intervals, which are able to perform orthosymplectic transformation in phase space. Several useful equations that connect the parameters of the transformation matrix for a general and symmetrical anamorphic first-order optical system have been derived. It has been shown that symmetrical anamorphic systems cannot perform a rotation operation except for the trivial one.

Flexible configurations for three principal transformations in phase space, image rotation, fractional FT, and gyrator (twisting) have been obtained. The first one contains four generalized lenses and can be performed by the combination of conventional cylindrical lenses. The separable fractional FT and gyrator operations can be performed by a symmetrical anamorphic setup that contains three generalized lenses. Some combinations of the transform angles can be obtained by applying a set of conventional cylindrical and spherical or holographically multiplexed lenses. To cover all angular combinations, except some singular cases, these lenses have to be implemented by an SLM.

Other systems performing canonical transformations in phase space can be constructed by a combination of a separable fractional FT system embedded in two rotator setups.[19] Experimental verification of the proposed optical schemes will be done in the near future.

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