Discrete Levy Transformations and Casorati Determinant Solutions of Quadrilateral Lattices

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Abstract

Sequences of discrete Levy and adjoint Levy transformations for the multidimensional quadrilateral lattices are studied. After a suitable number of iterations we show how all the relevant geometrical features of the transformed quadrilateral lattice can be expressed in terms of multi-Casorati determinants. As an example we dress the Cartesian lattice.

*On leave of absence from Beijing Graduate School, CUMT, Beijing 100083, China
†Supported by *Beca para estancias temporales de doctores y tecnólogos extranjeros en España: SB95-A01722297*
‡Partially supported by CICYT: proyecto PB95–0401

1. Recently it has been shown that multidimensional quadrilateral lattices are integrable [6] and a number of results about this system have been obtained. Let us mention the reduction mechanism based on the $\bar{\partial}$ formalism [4], the multidimensional circular lattice [2], the relation with multi-component KP through the Miwa transformation and geometrical meaning of the τ-function [5] and Darboux and more general transformations [14, 7].

Quadrilateral lattice equations, as a discrete integrable system, appeared for the first time in [1], however no geometrical understanding of this system can be found there. We should mention that the quadrilateral lattice has a continuum limit to conjugate nets [3, 8]. Since last century [12] it has been known that there is a transformation, called Levy transformation, that preserves the conjugacy character of the net. This transformation was iterated, for the bidimensional case, in [10]; and recently [13] we have used standard techniques in Soliton Theory to obtain closed formulae, in terms of multi-Wroński determinants, for all the relevant geometrical objects. The analog of this transformation, say discrete Levy, can be found, for the points of the lattice, in [1] and a detailed geometrical exposition of it is contained in [7].

The aim of this paper is to obtain similar results for the quadrilateral lattice as we did with conjugate nets, namely to iterate the discrete Levy transformation and its adjoint [7] to get closed formulae in terms of multi-Casorati determinants for all the geometrical features of the transformed lattice. We should mention that in [14] it was obtained, for zero background, multi-Casorati determinant representations for quadrilateral lattices; however, the expressions for the tangent vectors and Lamé coefficients are much more involved than here and no closed expression is given for the points of the lattice. Notice that for the Hirota equation (discrete KP) Casorati determinant representations can be found in [15].

The layout of this letter is as follows. In the first section we remind the reader some basic facts of the quadrilateral lattices and the discrete Levy tarnsformation. Next, in §3 we give the main result of this letter that is extended to the adjoint case in §4. In §5 we briefly indicate how to write our formulae, expressions depending on discrete multi-Wroński determinants, in terms of multi-Casorati determinants. Finally, in §6 we analyze a simple example by applying our main result to the Cartesian lattice [14]. We conlude the letter with an Appendix contaning the proof of a lemma in §3.

2. A Multidimensional Quadrilateral Lattice (MQL) [6] is an N-dimensional

($N \geq 2$) lattice:

$$\boldsymbol{x} : \mathbb{Z}^N \to \mathbb{R}^D,\ M \geq N, \quad \boldsymbol{n} := (n_1, \ldots, n_N) \in \mathbb{Z}^N, \quad D \geq N$$

such that each elementary quadrilateral of it is planar. It can be shown that this condition can be rewritten as the following discrete Laplace equation

$$\Delta_i \Delta_j \boldsymbol{x} = (T_i A_{ij}) \Delta_i \boldsymbol{x} + (T_j A_{ji}) \Delta_j \boldsymbol{x} \ , \ \ i \neq j, \quad i, j = 1, \ldots, N, \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^D$ is an arbitrary point of the lattice, A_{ij} are $N(N-1)$ real functions of $\boldsymbol{n}$, T_j is the translation operator in the j-th variable:

$$T_j(f(n_1, \ldots, n_j, \ldots, n_N)) = f(n_1, \ldots, n_j + 1, \ldots, n_N)$$

and $\Delta_j = T_j - 1$ is the corresponding difference operator.

The following nonlinear constraints must hold as compatibility conditions in order to have planarity in each pair of directions:

$$\Delta_k A_{ij} = (T_j A_{jk}) A_{ij} + (T_k A_{kj}) A_{ik} - (T_k A_{ij}) A_{ik}, \ \ i \neq j \neq k \neq i, \tag{2}$$

which characterize completely all the MQL's.

Equations (1) can be written as first order systems [6], for this we introduce functions H_i, $i = 1, \ldots, N$, defined by

$$\Delta_j H_i = A_{ij} H_i. \tag{3}$$

Then (1) reads

$$\Delta_j \boldsymbol{X}_i = (T_j Q_{ij}) \boldsymbol{X}_j, \quad i \neq j, \tag{4}$$

where the scalar functions Q_{ij} and the D-dimensional vectors $\boldsymbol{X}_i$ are defined by the equations

$$\Delta_i H_j = Q_{ij} T_i H_i, \ \ i \neq j, \tag{5}$$

$$\Delta_i \boldsymbol{x} = (T_i H_i) \boldsymbol{X}_i, \tag{6}$$

whose compatibility gives the equations

$$\Delta_j Q_{ik} = (T_j Q_{ijk}) Q_{jk}, \ \ i \neq j \neq k \neq i. \tag{7}$$

Equations (2) (or (7)) are the multidimensional quadrilateral lattice equations.

Given a solution ξ_j of

$$\Delta_k \xi_j = (T_k Q_{jk})\xi_k,$$

for each of the N possible directions of the lattice there is a corresponding discrete Levy transformation that reads for the i-th case:

$$\boldsymbol{x}[1] = \boldsymbol{x} - \frac{\Omega(\xi, H)}{\xi_i}\boldsymbol{X}_i,$$

$$\begin{cases} \boldsymbol{X}_i[1] = \dfrac{\xi_i \Delta_i \boldsymbol{X}_i - (\Delta_i \xi_i)\boldsymbol{X}_i}{\xi_i}, \\ \boldsymbol{X}_k[1] = \dfrac{\xi_i \boldsymbol{X}_k - \xi_k \boldsymbol{X}_i}{\xi_i}, \end{cases}$$

$$\begin{cases} H_i[1] = -\dfrac{\Omega(\xi, H)}{\xi_i}, \\ H_k[1] = H_k - Q_{ik}\dfrac{\Omega(\xi, H)}{\xi_i}, \end{cases}$$

$$\begin{cases} Q_{ik}[1] = -\dfrac{Q_{ik}(\Delta_i \xi_i) - \xi_i \Delta_i Q_{ik}}{\xi_i}, \\ Q_{ki}[1] = -\dfrac{\xi_k}{\xi_i}, \\ Q_{kl}[1] = -\dfrac{\xi_k Q_{il} - \xi_i Q_{kl}}{\xi_i}, \end{cases}$$

where $k, l = 1, \ldots, N$ with $k \neq l \neq i$. Here we have introduced the potential $\Omega(\xi, H)$ defined by

$$\Delta_k \Omega(\xi, H) = \xi_k T_k H_k, \quad k = 1, \ldots, N,$$

which are compatible equations by means of the equations satisfied by ξ_k and H_k.

3. As in the continuum case, i. e. conjugate nets, we have N different elementary Levy's transformations. Here we are going to iterate these transformations to get closed formulae for the transformed lattice. For simplicity we shall assume that in the iteration process at least one discrete Levy transformation has been made per direction in the lattice.

To present our main result, we introduce some convenient notations. Given any set of functions $\{\xi_j^i\}_{\substack{i=1,\dots,M\\ j=1\dots,N}}$ we denote by $W_j[n]$ the following discrete Wroński matrix

$$W_j[n] := W_j(\xi_j^1,\dots,\xi_j^M) := \begin{pmatrix} \xi_j^1 & \xi_j^2 & \dots & \xi_j^M \\ \Delta_j\xi_j^1 & \Delta_j\xi_j^2 & \dots & \Delta_j\xi_j^M \\ \vdots & \vdots & & \vdots \\ \Delta_j^{n-1}\xi_j^1 & \Delta^{n-1}\xi_j^2 & \dots & \Delta_j^{n-1}\xi_j^M \end{pmatrix}.$$

For any partition of $M = m_1 + m_2 + \dots + m_N$, we construct a discrete multi-Wroński matrix

$$\mathcal{W} := \begin{pmatrix} W_1[m_1] \\ \vdots \\ W_N[m_N] \end{pmatrix}.$$

Before going to our main result we need the following technical lemma, whose proof is given in the appendix:

Lemma. *We have the relations*

$$\Delta_k|\mathcal{W}| = T_k|\tilde{\mathcal{W}}|, \tag{8}$$

$$|\mathcal{W}| = T_k|\bar{\mathcal{W}}|, \tag{9}$$

where $\tilde{\mathcal{W}}$ and $\bar{\mathcal{W}}$ are obtained from $\mathcal{W}$ by replacing the last row of the k-th block by $T_k^{-1}\Delta_k^{m_k}\boldsymbol{\xi}_k$ and $T_k^{-1}\Delta_k^{m_k-1}\boldsymbol{\xi}_k$, respectively.

With this at hand we have the following:

Theorem. *Given M functions $\{\xi_i^j\}_{\substack{i=1,\dots,N\\ j=1,\dots,M}}$ and $\boldsymbol{X}_i = (X_i^1,\dots,X_i^D)^{\mathrm{t}}$, $i = 1,\dots,N$, all of them solutions of (4) and H_i, $i = 1,\dots,N$, solutions of (5), for given Q_{ij}, then new solutions $\boldsymbol{X}_i[M], H_i[M]$ and $Q_{ij}[M]$ are defined by:*

$$X_i^\ell[M] = \frac{|\mathbb{X}_i^\ell|}{|\mathcal{W}|}, \quad H_i[M] = -\frac{|\mathbb{H}_i|}{|\mathcal{W}|}, \quad Q_{ij}[M] = -\frac{|\mathcal{W}_{ij}|}{|\mathcal{W}|},$$

where

$$\mathbb{X}_i^\ell = \begin{pmatrix} \mathcal{W} & \boldsymbol{v}^\ell \\ \Delta_i^{m_i}\boldsymbol{\xi}_i & \Delta_i^{m_i}X_i^\ell \end{pmatrix},$$

with

$$\boldsymbol{v}^\ell := (\boldsymbol{v}_1^\ell,\dots,\boldsymbol{v}_N^\ell)^{\mathrm{t}},\ \textit{being}\ \boldsymbol{v}_k^\ell := (X_k^\ell,\Delta_k X_k^\ell,\dots,\Delta_k^{m_k-1}X_k^\ell),$$
$$\boldsymbol{\xi}_i := (\xi_i^1,\dots,\xi_i^M),$$

$\mathbb{H}_i$ *is obtained from* $\mathcal{W}$ *by replacing the last row of the* i*-th block by* $\Omega(\boldsymbol{\xi},H)$ *and* $\mathcal{W}_{ij}$ *by replacing the last row of the* j*-th block by* $\Delta_i^{m_i}\boldsymbol{\xi}_i$*. In the partition* $M=m_1+m_2+\cdots+m_N$ *we need* $m_i\in\mathbb{N}$*.*

Moreover, for the new quadrilateral lattice we have the parametrization

$$\boldsymbol{x}[M]=\frac{1}{|\mathcal{W}|}\left(\begin{vmatrix}\mathcal{W} & \boldsymbol{v}^1\\ \Omega(\boldsymbol{\xi},H) & x^1\end{vmatrix},\dots,\begin{vmatrix}\mathcal{W} & \boldsymbol{v}^D\\ \Omega(\boldsymbol{\xi},H) & x^D\end{vmatrix}\right)^{\mathrm{t}}.$$

Proof. We first need to show that

$$\Delta_k X_i^\ell[M]=(T_kQ_{ik}[M])X_k^\ell[M],$$

or equivalently that the following bilinear equation holds

$$|\mathcal{W}|\,\Delta_k\left|\mathbb{X}_i^\ell\right|-\left|\mathbb{X}_i^\ell\right|\Delta_k\,|\mathcal{W}|+\left|\mathbb{X}_k^\ell\right|T_k\,|\mathcal{W}_{ik}|=0.$$

To this aim, using standard techniques [9], we consider the following $(2M+1)\times(2M+1)$ square matrix

$$\mathcal{A}_{ik}^\ell:=\begin{pmatrix} A_k & 0 & \Delta_k^{m_k-1}\boldsymbol{\xi}_k^{\mathrm{t}} & \Delta_k^{m_k}\boldsymbol{\xi}_k^{\mathrm{t}} & T_k\Delta_i^{m_i}\boldsymbol{\xi}_i^{\mathrm{t}}\\ 0 & A_k & \Delta_k^{m_k-1}\boldsymbol{\xi}_k^{\mathrm{t}} & \Delta_k^{m_k}\boldsymbol{\xi}_k^{\mathrm{t}} & T_k\Delta_i^{m_i}\boldsymbol{\xi}_i^{\mathrm{t}}\\ 0 & \boldsymbol{b}_k^\ell & \Delta_k^{m_k-1}X_k^\ell & \Delta_k^{m_k}X_k^\ell & T_k\Delta_i^{m_i}X_i^\ell\end{pmatrix},$$

where A_k is a $M\times(M-1)$ rectangular matrix

$$(A_k)^{\mathrm{t}}:=T_k\begin{pmatrix}W_1[m_1]\\ \vdots\\ \hat{W}_k[m_k]\\ \vdots\\ W_N[m_N]\end{pmatrix},$$

with $\hat{W}_k[m_k]$ obtained from $W_k[m_k]$ by deleting the last row, and

$$\boldsymbol{b}_k^\ell=(\boldsymbol{v}_1^\ell,\cdots,\hat{\boldsymbol{v}}_k^\ell,\dots,\boldsymbol{v}_N^\ell),$$

with $\hat{\boldsymbol{v}}_k^\ell$ obtained by deleting the last element in $\boldsymbol{v}_k^\ell$. Applying our lemma and the Laplace's general expansion theorem [16] to compute $\det \mathcal{A}_{ik}^\ell$, we obtain the desired bilinear relation.

Next, let us check the relation

$$\Delta_k H_i[M] = Q_{ki}[M] T_k H_k[M],$$

or equivalently that the following bilinear equation holds:

$$|\mathcal{W}|\Delta_k\,|\mathbb{H}_i| - |\mathbb{H}_i|\,\Delta_k\,|\mathcal{W}| + |\mathcal{W}_{ki}|\,T_k\,|\mathbb{H}_k| = 0.$$

This formula, as previously, follows from the Laplace's general expansion theorem when applied to the evaluation of the determinant of the $2M\times 2M$ matrix:

$$\mathcal{B}_{ik} := \begin{pmatrix} B_{ik} & 0 & \Delta_k^{m_k-1}\boldsymbol{\xi}_k^{\mathrm{t}} & \Delta_k^{m_k}\boldsymbol{\xi}_k^{\mathrm{t}} & T_k\Delta_i^{m_i-1}\boldsymbol{\xi}_i^{\mathrm{t}} & T_k\Omega(\boldsymbol{\xi},H)^{\mathrm{t}} \\ 0 & B_{ik} & \Delta_k^{m_k-1}\boldsymbol{\xi}_k^{\mathrm{t}} & \Delta_k^{m_k}\boldsymbol{\xi}_k^{\mathrm{t}} & T_k\Delta_i^{m_i-1}\boldsymbol{\xi}_i^{\mathrm{t}} & T_k\Omega(\boldsymbol{\xi},H)^{\mathrm{t}} \end{pmatrix},$$

where B_{ik} is a $M\times(M-2)$ rectangular matrix

$$(B_{ik})^{\mathrm{t}} := T_k \begin{pmatrix} W_1[m_1] \\ \vdots \\ \hat{W}_i[m_i] \\ \vdots \\ \hat{W}_k[m_k] \\ \vdots \\ W_N[m_N] \end{pmatrix}.$$

Finally, we prove the formula for the points in the lattice $x^\ell[M] = \Omega(X^\ell[M], H[M])$ (see (6)). For that aim we consider the following $(2M+1)\times(2M+1)$ matrix

$$\mathcal{C}_k^\ell := \begin{pmatrix} A_k & 0 & \Delta_k^{m_k-1}\boldsymbol{\xi}_k^{\mathrm{t}} & \Delta_k^{m_k}\boldsymbol{\xi}_k^{\mathrm{t}} & T_k\Omega(\boldsymbol{\xi},H)^{\mathrm{t}} \\ 0 & A_k & \Delta_k^{m_k-1}\boldsymbol{\xi}_k^{\mathrm{t}} & \Delta_k^{m_k}\boldsymbol{\xi}_k^{\mathrm{t}} & T_k\Omega(\boldsymbol{\xi},H)^{\mathrm{t}} \\ 0 & \boldsymbol{b}_k^\ell & \Delta_k^{m_k-1}X_k^\ell & \Delta_k^{m_k}X_k^\ell & T_k\Omega(X^\ell,H) \end{pmatrix},$$

and use that $x^\ell = \Omega(X^\ell, H)$ and compute the $\det \mathcal{C}_k^\ell$ according to the Laplace's general expansion theorem. □

4. The discrete adjoint Levy transformation [7]:

$$\boldsymbol{x}[1]=\boldsymbol{x}-\frac{\Omega(\boldsymbol{X},\zeta)}{\zeta_i}H_i,$$

$$\begin{cases}\boldsymbol{X}_i[1]=-\dfrac{\Omega(\boldsymbol{X},\zeta)}{\zeta_i},\\ \boldsymbol{X}_k[1]=\boldsymbol{X}_k-Q_{ki}\dfrac{\Omega(\boldsymbol{X},\zeta)}{\zeta_i},\end{cases}$$

$$\begin{cases}H_i[1]=\dfrac{\zeta_i\Delta_i H_i-(\Delta_i\zeta_i)H_i}{\zeta_i},\\ H_k[1]=\dfrac{\zeta_i H_k-\zeta_k H_i}{\zeta_i},\end{cases}$$

$$\begin{cases}Q_{ki}[1]=-\dfrac{Q_{ki}\Delta_i\zeta_i-(\Delta_i Q_{ki})\zeta_i}{\zeta_i},\\ Q_{ik}[1]=-\dfrac{\zeta_k}{\zeta_i},\\ Q_{lk}[1]=-\dfrac{\zeta_k Q_{li}-\zeta_i Q_{lk}}{\zeta_i},\end{cases}$$

where $k,l=1,\dots,N$ with $k\neq l\neq i$, and ζ_k solves (5).

By similar considerations as in §**3** we get:

Proposition. *Given M functions $\{\zeta_i^j\}_{\substack{i=1,\dots,N\\ j=1,\dots,M}}$ and H_i, $i=1,\dots,N$, , all of them solutions of* (5)*, $\boldsymbol{X}_i=(X_i^1,\dots,X_i^D)^{\mathrm{t}}$, $i=1,\dots,N$ solutions of* (4) *for given Q_{ij}, then new solutions $\boldsymbol{X}_i[M],H_i[M]$ and $Q_{ij}[M]$ are defined by:*

$$X_i^\ell[M]=-\frac{|\tilde{\mathbb{X}}_i^\ell|}{|\mathcal{W}|},\quad H_i[M]=\frac{|\tilde{\mathbb{H}}_i|}{|\mathcal{W}|},\quad Q_{ij}[M]=-\frac{|\mathcal{W}_{ji}|}{|\mathcal{W}|},$$

where

$$\tilde{\mathbb{H}}_i=\begin{pmatrix}\mathcal{W} & \tilde{\boldsymbol{v}}\\ \Delta_i^{m_i}\boldsymbol{\zeta}_i & \Delta_i^{m_i}H_i\end{pmatrix},$$

with

$$\begin{aligned}&\tilde{\boldsymbol{v}}:=(\tilde{\boldsymbol{v}}_1,\dots,\tilde{\boldsymbol{v}}_N)^{\mathrm{t}},\ \textit{being}\ \tilde{\boldsymbol{v}}_k:=(H_k,\Delta_k H_k,\dots,\Delta_k^{m_k-1}H_k),\\ &\boldsymbol{\zeta}_i:=(\zeta_i^1,\dots,\zeta_i^M),\end{aligned}$$

$\mathbb{X}_i^\ell$ is obtained from $\mathcal{W}$ by replacing the last row of the i-th block by $\Omega(X^\ell,\boldsymbol{\zeta})$. In the partition $M=m_1+m_2+\cdots+m_N$ we need $m_i\in\mathbb{N}$.

For the new quadrilateral lattice we have the parametrization

$$\boldsymbol{x}[M]=\frac{1}{|\mathcal{W}|}\left(\begin{vmatrix}\mathcal{W} & \tilde{\boldsymbol{v}}^1\\ \Omega(X^1,\boldsymbol{\xi}) & x^1\end{vmatrix},\dots,\begin{vmatrix}\mathcal{W} & \tilde{\boldsymbol{v}}^D\\ \Omega(X^P,\boldsymbol{\xi}) & x^D\end{vmatrix}\right)^{\mathrm{t}}.$$

5. The result of our Theorem and Proposition are expressed in terms of discrete multi-Wroński matrices, however it is easy to write the formulae in terms of multi-Casorati determinants. For this aim it is only necessary to replace in $\mathcal{W},\mathbb{X}_i^\ell,\tilde{\mathbb{H}}_i,\boldsymbol{v},\tilde{\boldsymbol{v}}$ and $\mathcal{W}_{ij}$ the difference operator Δ by the shift operator T.

6. We now consider the dressing of the Cartesian background [14]. When $Q_{ij}=0$ the quantities $\boldsymbol{X}_i$ and H_i are arbitrary vector functions of n_i only, while the points in the lattice can be represented as

$$\boldsymbol{x}=\sum_{k=1}^{D}\boldsymbol{x}_k(n_k)+\boldsymbol{c},$$

where $\boldsymbol{x}_k(0)=\boldsymbol{0}$ and $\boldsymbol{c}$ is a constant vector, which characterize this parallelogram lattice [14]. Among them, the Cartesian lattice is obtained choosing $D=N$ and

$$\begin{aligned}&\boldsymbol{X}_i=\boldsymbol{e}_i,\ H_i=1,\\&\boldsymbol{x}=\boldsymbol{n},\end{aligned}$$

where $\{\boldsymbol{e}_i\}_{i=1}^N$ is the canonical basis of $\mathbb{R}^N$.

Next, in order to get the iterated Levy transformed lattice we apply our Theorem to these data. First, notice that $\boldsymbol{\xi}_i(\boldsymbol{n})=\boldsymbol{\xi}_i(n_i)$, $i=1,\dots,N$. Second, we have $\boldsymbol{v}_k^\ell=(\delta_k^\ell,0,\dots,0)$, where δ_k^ℓ is the Kronecker symbol. Observe also that $\boldsymbol{\Omega}:=\Omega(\boldsymbol{\xi},H)$ is defined by $\Delta_i\boldsymbol{\Omega}=\boldsymbol{\xi}_i$, and we can write $\boldsymbol{\Omega}(\boldsymbol{n})=\boldsymbol{\Omega}_1(n_1)+\cdots+\boldsymbol{\Omega}_N(n_N)$ and $\boldsymbol{\xi}_i=\Delta_i\boldsymbol{\Omega}_i$. Thus, we have that $\mathbb{X}_i^\ell$ is just the matrix obtained from $\mathcal{W}$ by replacing the first row of the ℓ-th block, W_ℓ, by $\Delta_i^{m_i}\boldsymbol{\xi}_i$. To get $\mathbb{H}_i$ we subsistute the last row of the i-th block of $\mathcal{W}$ by $-\boldsymbol{\Omega}$. Finally, the points of the lattice are

$$\boldsymbol{x}(\boldsymbol{n})=\boldsymbol{n}-\delta\boldsymbol{x}(\boldsymbol{n}),\quad \delta x_i:=\frac{|\mathcal{W}_i|}{|\mathcal{W}|},$$

where $\mathcal{W}_i$ is the matrix obtained from $\mathcal{W}$ by replacing the first row of the i-th block, W_i, by $\mathbf{\Omega}$.

The result of the iterated adjoint Levy transformation on the Cartesian lattice, as one can easily compute from our Proposition, gives the same transformed lattice as above.

Appendix: In this appendix we give the proof of our lemma.

Proof of the Lemma. For the sake of simplicity we shift the k-th block to the end of the matrix and use the discrete version of the Leibnitz rule, $\Delta(ab) = (\Delta a)b + (Ta)\delta b$. Then, we can write

$$\Delta_k|\mathcal{W}| = \mathcal{F}_1 + \cdots + \mathcal{F}_N$$

with

$$\mathcal{F}_j := \mathcal{F}_{j1} + \cdots + \mathcal{F}_{jm_j},\ j \neq k,\ \mathcal{F}_{ji} := \begin{vmatrix} T_k W_1[m_1] \\ \vdots \\ T_k W_{j-1}[m_{j-1}] \\ \tilde{W}_{ji}[m_j] \\ W_{j+1}[m_{j+1}] \\ \vdots \\ W_k[m_k] \end{vmatrix},\ \tilde{W}_{ji}[m_j] := \begin{pmatrix} T_k\boldsymbol{\xi}_j \\ T_k\Delta_j\boldsymbol{\xi}_j \\ \vdots \\ T_k\Delta_j^{i-1}\boldsymbol{\xi}_j \\ \Delta_k\Delta_j^{i}\boldsymbol{\xi}_j \\ \Delta_j^{i+1}\boldsymbol{\xi}_j \\ \vdots \\ \Delta_j^{m_j-1}\boldsymbol{\xi}_j \end{pmatrix}.$$

We can remove the T_k, namely

$$\mathcal{F}_{ji} = \begin{vmatrix} W_1[m_1] \\ \vdots \\ W_{j-1}[m_{j-1}] \\ \breve{W}_{ji}[m_j] \\ W_{j+1}[m_{j+1}] \\ \vdots \\ W_k[m_k] \end{vmatrix},\ \breve{W}_{ji}[m_j] := \begin{pmatrix} \boldsymbol{\xi}_j \\ \Delta_j\boldsymbol{\xi}_j \\ \vdots \\ \Delta_j^{i-1}\boldsymbol{\xi}_j \\ \Delta_k\Delta_j^{i}\boldsymbol{\xi}_j \\ \Delta_j^{i+1}\boldsymbol{\xi}_j \\ \vdots \\ \Delta_j^{m_j-1}\boldsymbol{\xi}_j \end{pmatrix},$$

because $T_k\Delta_i^p\boldsymbol{\xi}_i$ can be expressed, using (4), as a linear combination:

$$T_k\Delta_i^p\boldsymbol{\xi}_i = \Delta_i^p\boldsymbol{\xi}_i + c_{k,i,p-1}\Delta_i^{p-1}\boldsymbol{\xi}_i + \cdots + c_{k,i,1}\Delta_i\boldsymbol{\xi}_i + c_{k,i,0}\boldsymbol{\xi}_i + c_{k,i}\boldsymbol{\xi}_k$$

for some scalar coefficients c's. Using this formula again we see that $\mathcal{F}_{ji} = 0$. So, we have

$$\Delta_k |\mathcal{W}| = \mathcal{F}_k$$

where

$$\mathcal{F}_k := \mathcal{F}_{k1} + \ldots + \mathcal{F}_{km_k}, \ \mathcal{F}_{kp} := \begin{vmatrix} T_k W_1[m_1] \\ \vdots \\ T_k W_N[m_N] \\ \breve{W}_{kp}[m_k] \end{vmatrix}, \ \breve{W}_{kp}[m_k] := \begin{pmatrix} T_k \boldsymbol{\xi}_k \\ T_k \Delta_k \boldsymbol{\xi}_k \\ \vdots \\ T_k \Delta_k^{p-1} \boldsymbol{\xi}_k \\ \Delta_k \Delta_k^p \boldsymbol{\xi}_k \\ \Delta_k^{p+1} \boldsymbol{\xi}_k \\ \vdots \\ \Delta_p^{m_p-1} \boldsymbol{\xi}_p \end{pmatrix}.$$

Now, it is obvious that

$$\mathcal{F}_k = \mathcal{F}_{km_k},$$

which gives the (8). For (9) one proceeds in a similar way. □

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