DISTRIBUTION OF PRIMES AND APPROXIMATION ON WEIGHTED DIRICHLET SPACES

EVA A. GALLARDO-GUTIÉRREZ AND DANIEL SECO

Abstract. We study zero-free regions of the Riemann zeta function ζ related to an approximation problem in the weighted Dirichlet space D_{-2} which is known to be equivalent to the Riemann Hypothesis since the work of Báez-Duarte. We prove, indeed, that analogous approximation problems for the standard weighted Dirichlet spaces D_α when $\alpha \in (-3,-2)$ give conditions so that the half-plane $\{s \in \mathbb{C} : \Re(s) > -\frac{\alpha+1}{2}\}$ is also zero-free for ζ. Moreover, we extend such results to a large family of weighted spaces of analytic functions ℓ^p_α. As a particular instance, in the limit case $p = 1$ and $\alpha = -2$, we provide a new proof of the Prime Number Theorem.

1. Introduction and prelimnaries

The Riemann zeta function ζ is a very classical object in Mathematics and its links to various properties in Number Theory is well established nowadays. Given a complex number $s \in \mathbb{C}$ with real part $\Re(s) > 1$, the Riemann zeta function is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}.$$

For other values of $s \in \mathbb{C}\backslash\{1\}$, it admits a unique analytic continuation which is also commonly denoted by ζ.

As shown by Riemann, the localization of the zeros of ζ has deep connections with the distribution of the prime numbers. In particular, a distribution proposed by Gauss is satisfied if the famous Riemann Hypothesis (RH) holds true. Recall that (RH) asserts that those zeros of ζ that are not negative integers, namely the *nontrivial zeros*, lie on the *critical line*

$$\{s \in \mathbb{C} : \Re(s) = 1/2\}.$$

Although there are some positive evidences of this conjecture, the problem at large remains open.

It is well known that the nontrivial zeros are actually symmetric with respect to the critical line, and in terms of their real parts the only thing known to date is that they lie in the strip

$$\{s \in \mathbb{C} : \Re(s) \in (0,1)\}.$$

A function that is strongly associated with the distribution of the zeros of the Riemann zeta function is the so-called Möbius function μ. Recall that μ is the arithmetic function defined on the

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set of positive integers $\mathbb{N}$ taking values in $\{-1,0,1\}$, namely, $\mu : \mathbb{N} \to \{-1,0,1\}$, mapping each square-free factorization natural number k to $(-1)^{\omega(k)}$ where $\omega(k)$ is the number of prime factors of k, and taking each k divisible by a square prime, to 0. We refer to [4] for this and many other related classical topics in Analytic Number Theory.

Our main aim in this work is establishing sufficient conditions on the behavior of sums related to the Möbius function which guarantee that the Riemann zeta function has zero-free regions of the complex plane $\mathbb{C}$. In this regard, our work is strongly influenced by the classical ones in this context of Nyman [13], Beurling [5], Báez-Duarte [1, 2], Balazard-Saias [3] and the recent one by Waleed Noor [17]. In this framework, the classical theorem of Beurling [5] states that the property that the region

$$\Omega_t := \{s \in \mathbb{C} : \Re(s) > t\}$$

contains no zeros of ζ is equivalent to the fact that the constant function 1 lies in the closure of a particular family of functions in $L^{1/t}(0,1)$, where the interesting case is when $p = 1/t \in [1,2]$ (the case $p = 2$ was already considered in Nyman's thesis).

More concretely, if we define for $k \geq 2, n \in \mathbb{N}$, the functions

$$r_k(n) = k\left\{\frac{n}{k}\right\}, \tag{1}$$

where $\{\cdot\}$ denotes the fractional part, Báez-Duarte [1, 2] showed that Beurling's version of the problem could be translated into the problem of approximating arbitrarily close the function $\frac{1}{1-z}$ in the *weighted Dirichlet space* D_{-2} by linear combinations of the family of functions

$$R_k(z) = \sum_{n=1}^{\infty} r_k(n) z^{n-1},$$

with $k \geq 2$. Recall that the *weighted Dirichlet space* D_α, where $\alpha \in \mathbb{R}$, consists of the holomorphic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the unit disc $\mathbb{D}$ such that

$$\|f\|_\alpha^2 := \sum_{n=0}^{\infty} |a_n|^2 (n+1)^\alpha < \infty.$$

Weighted Dirichlet spaces are particular instances of the so-called weighted Hardy spaces $H^2(\beta)$ associated to a sequence $\beta = \{\beta_n\}_{n\geq 0}$ introduced by Shields in the seventies in order to study weighted shifts operators [15]. They are Hilbert spaces of analytic functions and play an important role in Operator Theory. Particular instances of α's yield classical spaces of analytic functions, namely, for $\alpha = -1$, D_α is the classical Bergman space A^2, $\alpha = 0$ yields the Hardy space H^2 and $\alpha = 1$, the Dirichlet space D. Note that the continuous inclusion $D_{\alpha_1} \subsetneq D_{\alpha_2}$ holds for all $\alpha_2 < \alpha_1$. Moreover, when $\alpha > 1$ the spaces D_α are continuously embedded in the disc algebra $\mathcal{A}$. We refer to [10, 9, 11] for more on these spaces, and to [2] for the connection mentioned here with ζ. In our work, a relevant role will be played by D_α when $\alpha \in [-3,-2]$.

In 2004, Balazard and Saias [3] showed that the Báez-Duarte criterion was connected with estimating sums defined in terms of the Möbius function μ. In particular, they provided specific sums in terms of the Möbius function whose value was directly connected with the (RH). A key fact in their work was that, if $1/(1-z)$ is approximated by $\frac{1}{z}\sum_{k=2}^{n} c_{k,n} R_k$, then for each $k \geq 2$, the limit as n tends to ∞ of $c_{k,n}$ must be a specific value, namely,

$$\lim_{n\to\infty} c_{k,n} = c_{k,\infty} = -\mu(k)/k.$$

More recently, Waleed Noor [17] has transformed further the problem by means of isometries turning Báez-Duarte criterion into a question about approximating the constant function 1 in the Hardy space H^2 by means of linear combinations of the sequence of functions $\{h_k\}_{k=2}^{\infty}$ given by

$$h_k(z) = \frac{1}{1-z} \log\left(\frac{1+z+...+z^{k-1}}{k}\right), \quad (z \in \mathbb{D}). \tag{2}$$

In this setting, our main contribution is providing conditions on sums related to the Möbius function that guarantee the lack of zeros of the Riemann zeta function ζ in the regions Ω_t. Our approach is based on finding new ways of expressing the Beurling criterion in other spaces of analytic functions following the spirit of Báez-Duarte. Nevertheless, the criteria we exhibit are of a radically different nature to other known criteria associated to the Möbius function and, in particular, it will allow us to provide a new proof of the Prime Number Theorem and establish, likewise, an extension of Waleed Noor's Theorem to the setting of weighted Dirichlet spaces.

The rest of the manuscript is organized as follows. We close this introductory section with some preliminaries regarding the spaces of analytic functions in the disc which will play a key role in order to provide the desired estimates. In Section 2, we prove Theorem 2.1 regarding zero-free regions of the Riemann zeta function and convergence in weighted Dirichlet spaces. As an application, we provide another proof of the Prime Number Theorem (see Subsection 2.1). In Section 3 we establish a sufficient condition on the behavior of sums related to the Möbius function guaranteeing that the Riemann zeta function has zero-free regions within the critical strip (Theorem 3.1). A key tool in this context is a classical theorem of S. Selberg [14]. In Section 4 we consider different approximations of the constant sequence $\mathbb{1}$ in $\operatorname{span}\{r_k : k \geq 2\}$ *à la* Balazard and Saias. Finally, in Section 5, we present a generalization of Waleed Noor's techniques, by introducing a family of conditions, each of which guarantees a zero-free region for the Riemann ζ function.

1.1. **Preliminaries.** Throughout the rest of the manuscript, $\mathbb{D}$ will denote the unit disc of the complex plane $\mathbb{C}$. The following weighted ℓ^p spaces of analytic functions on $\mathbb{D}$ will be essential regarding estimates of sums related to the Möbius function.

Definition 1.1. Let $1 \leq p \leq 2$ and $\alpha \in \mathbb{R}$ be fixed. The space X_α^p consists of holomorphic functions $g(z) = \sum_{k=0}^{\infty} a_k z^k$ in $\mathbb{D}$ such that the norm

$$\|g\|_{p,\alpha} := \left(\sum_{k=0}^{\infty} |a_k|^p (k+1)^\alpha\right)^{1/p}$$

is finite.

Note that X_α^p are Banach spaces of analytic functions which comprise the introduced weighted Dirichlet spaces when $p = 2$, namely, $D_\alpha = X_\alpha^2$, or the Wiener Algebra $\mathcal{W}$, that is, $\mathcal{W} = X_0^1$. In particular, X_α^p are examples of *functional Banach spaces* (see [7, Definition 1.1]).

To each function g in X_α^p we associate the sequence of Taylor coefficients of $zg(z)$, and define the sequence space ℓ_α^p for $1 \leq p \leq 2$ and $\alpha \in \mathbb{R}$ as

$$\ell_\alpha^p := \{u = \{u(k)\}_{k \geq 1} : g(z) := \sum_{k=1}^{\infty} u(k) z^{k-1} \in X_\alpha^p\}. \tag{3}$$

We endow ℓ_α^p with the X_α^p-norm:

$$\|\{u(k)\}_{k\geq 1}\|_{p,\alpha} = \left\|\sum_{k=1}^{\infty} u(k) z^{k-1}\right\|_{p,\alpha},$$

which turns ℓ^p_α into a Banach space. Denoting the sequence with constant value 1 by $\mathbb{1}$, that is, $\mathbb{1}(n) = 1$ for all $n \geq 1$, it is clear that $\mathbb{1} \in \ell^p_\alpha$ for all $\alpha < -1$. Indeed, $\mathbb{1}$ corresponds to the function $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$.

2. Zero-free regions of the Riemann zeta function

For $k \geq 2$, recall the r_k function defined on $\mathbb{N}$ introduced in (1) by

$$r_k(n) = k\left\{\frac{n}{k}\right\}, \quad (n \in \mathbb{N}).$$

The main goal of this section is proving the following theorem:

Theorem 2.1. *Let $1 < p \leq 2$ and $\alpha \in (-1-p, -1-\frac{p}{2}]$. Assume that there exists a sequence of linear combinations $\{S_N = \mathbb{1} + \sum_{k=2}^{N} c_{k,N}\, r_k\}_{N=1}^{\infty}$ converging to 0 in the X^p_α-norm. Then the Riemann zeta function ζ has no zeros in the half-plane $\Omega_{p,\alpha} := \{z \in \mathbb{C} : \Re(z) > -\frac{\alpha+1}{p}\}$. Moreover, if the same holds for $p = 1$ and $\alpha \in [-2, -3/2]$, then ζ has no zeros lying on the* closed *half-plane $\Omega_{1,\alpha} := \{z \in \mathbb{C} : \Re(z) \geq -(\alpha+1)\}$.*

Remark 2.2. In particular, the previous theorem addresses a way to prove the Prime Number Theorem: we will see it is enough to show the convergence condition for the case $p = 1$, $\alpha = -2$. On the other hand, the case $p = 1$, $\alpha = -3/2$ cannot happen: if there was such a convergence condition there would be no non-trivial zeros of the zeta function at all, which we know to be false.

Before proving Theorem (2.1), we observe the following:

Let $m \in \mathbb{N}$ be fixed. From the work in [3], a natural candidate for approximating $\mathbb{1}$ in span$\{r_k : k \geq 2\}$ in X^2_{-2} norm appears to be

$$\sum_{k=2}^{m} r_k \frac{\mu(k)}{k}.$$

By modifying this approach slightly, we study the following attempt at approximating $\mathbb{1}$:

$$F_m := \mathbb{1} + \left(\sum_{k=2}^{m} r_k \frac{\mu(k)}{k} - \sum_{k=1}^{m} r_m \frac{\mu(k)}{k}\right). \tag{4}$$

Accordingly, in order to apply this result, we will make appropriate choices of m's belonging to an infinite subset $M \subset \mathbb{N}$ of values to obtain accurate estimates in the required approximation. Indeed, as we will show, in order to provide a new proof of the Prime Number Theorem, we may take $M = \mathbb{N}$. Nevertheless, for more advanced estimates, it will be necessary to determine a suitable choice of the subset M.

In order to prove Theorem 2.1, we introduce a set of real functions defined on $(0,1)$ by

$$\mathcal{C} = \left\{ f(x) = \sum_{k=1}^{N} c_k \left\{\frac{1}{kx}\right\} : x \in (0,1),\ \sum_{k=1}^{N} \frac{c_k}{k} = 0 \right\}. \tag{5}$$

Note that $\mathcal{C} \subset L^p(0,1)$ for $1 \leq p \leq \infty$. Moreover, each function in $\mathcal{C}$ is constant on intervals of the form $\left(\frac{1}{n+1}, \frac{1}{n}\right]$ for $n \geq 1$: the value of c_1 is determined by the remaining values so that a basis of $\mathcal{C}$ is given by the functions of the form $g_k(x) = k\left\{\frac{1}{kx}\right\} - \left\{\frac{1}{x}\right\}$, for $k \geq 2$. On the interval $\left(\frac{1}{n+1}, \frac{1}{n}\right]$, these functions have constant values $g_k(x) = r_k(n)$.

We will also need the following lemma auxiliary to the main result of Beurling in [5] (see formula (2) there):

Lemma 2.3. *If* $f \in \mathcal{C}$ *with* $f(x) = \sum_{k=1}^{N} c_k \left\{\frac{1}{kx}\right\}$, $x \in (0,1)$ *and* $\Re(z) > 0$ *then*

$$\int_0^1 f(x)x^{z-1}dx = -\frac{\zeta(z)\sum_{k=2}^{N} c_k k^{-z}}{z}.$$

From Lemma 2.3, adding $1/z$ on both sides makes the following expression valid on any z with $\Re(z) > 0$:

$$\int_0^1 (1+f(x))x^{z-1}dx = \frac{1}{z}\left(1-\zeta(z)\sum_{k=2}^{N} c_k k^{-z}\right). \tag{6}$$

Suppose, for the moment, we can find functions in $\mathcal{C}$, namely $\{f_N\}_{N=1}^{\infty}$, making the absolute value of the integral in the left-hand side arbitrarily small, and that z is a zero of ζ with $\Re(z) > 0$. Then it must happen that

$$0 = \frac{1}{z},$$

which can't happen.

This is to be read as a possibility for showing that a fixed value z is not a zero of ζ: we only need to show that the integral on the left hand side of (6) can be made arbitrarily small (by choosing $f \in \mathcal{C}$). Therefore our aim in order to prove Theorem 2.1 is to find a sequence converging to the function 1 by means of functions $\{f_N\}_{N=2}^{\infty}$ in terms of the integral.

Proof of Theorem 2.1. Note that approximating the function 1 with linear combinations of functions of the form $\left\{\frac{1}{kx}\right\}$ is equivalent to approximating the sequence of its values over intervals of the form $\left(\frac{1}{n+1}, \frac{1}{n}\right)$, by the locally constant values of functions in $\mathcal{C}$. Hence, if $f \in \mathcal{C}$ with $f(x) = \sum_{k=1}^{N} c_k \left\{\frac{1}{kx}\right\}$, $x \in (0,1)$, we deduce that for complex numbers z with $\Re(z) > 0$,

$$\begin{aligned}\int_0^1 (1+f(x))x^{z-1}dx &= \sum_{n=1}^{\infty}\int_{\frac{1}{n+1}}^{\frac{1}{n}} (1+f(x))x^{z-1}dx \\ &= \sum_{n=1}^{\infty}\left(1+f\left(\frac{1}{n}\right)\right)\int_{\frac{1}{n+1}}^{\frac{1}{n}} x^{z-1}dx \\ &= \sum_{n=1}^{\infty}\left(1+\sum_{k=2}^{N} c_k r_k(n)\right)\int_{\frac{1}{n+1}}^{\frac{1}{n}} x^{z-1}dx. \end{aligned} \tag{7}$$

In the last step here, we used that $r_k(n) = k\left\{\frac{n}{k}\right\} = k\left\{\frac{1}{kx}\right\} - \left\{\frac{1}{x}\right\}$ for all $x \in (\frac{1}{n+1}, \frac{1}{n}]$, and that $f \in \mathcal{C}$ and thus $c_1 = -\sum_{k=2}^{N} \frac{c_k}{k}$.

Taking absolute values and writing the last expression in terms of the sequence S_N whose existence is guaranteed by hypothesis, (7) turns into

$$\left|\sum_{n=1}^{\infty} S_N(n)\int_{\frac{1}{n+1}}^{\frac{1}{n}} x^{z-1}dx\right| \le \frac{1}{|z|}\sum_{n=1}^{\infty}|S_N(n)|\left|\frac{1}{n^{\Re(z)}} - \frac{1}{(n+1)^{\Re(z)}}\right|. \tag{8}$$

Assume first that $p = 1$, $\alpha \in [-2, -3/2]$, and $\Re(z) \geq -(\alpha+1)$. Then the right hand side (8) is bounded by $2\sum_{n=1}^{\infty} |S_N(n)|\,(n+1)^{\alpha}$ which converges to 0 by assumption.

On the other hand, if $1 < p \leq 2$ and $\alpha \in (-1-p, -1-\frac{p}{2}]$, bearing in mind that $n^{-\Re(z)} - (n+1)^{-\Re(z)} \approx n^{-1-\Re(z)}$, turns (8) into

$$\begin{aligned} \left|\sum_{n=1}^{\infty} S_N(n) \int_{\frac{1}{n+1}}^{\frac{1}{n}} x^{z-1}dx\right| &\leq \sum_{n=1}^{\infty} |S_N(n)|\, n^{-1-\Re(z)} \\ &\leq \left(\sum_{n=1}^{\infty} |S_N(n)|^p\, n^{\alpha}\right)^{1/p} \left(\sum_{n=1}^{\infty} n^{-q(1+\Re(z)+\frac{\alpha}{p})}\right)^{1/q} \end{aligned} \tag{9}$$

where last inequality follows upon applying Hölder inequality. Here q is the reciprocal of p, namely $\frac{1}{p} + \frac{1}{q} = 1$.

The first factor on the right hand side of (9) converges to 0 by hypothesis while the second factor is a convergent sum because the exponent of n in the sum is smaller than -1, which is true precisely because $\Re(z) > -\frac{\alpha+1}{p}$. This concludes the proof of Theorem 2.1. □

Remark 2.4. Applying Hölder inequality to the absolute value of the left-hand side is at the core of Beurling's criterion for establishing regions free of ζ zeros. In obtaining the Báez-Duarte criterion, this last step is done *after* applying Hölder inequality to the integral in the spirit of Beurling, but here we decidedly want to perform this step prior to the introduction of estimates.

2.1. **An application of Theorem 2.1: The Prime Number Theorem.** Our aim in this subsection is providing another proof of the famous Prime Number Theorem: *If $\pi(x)$ is the number of primes less than or equal to x, then $x^{-1}\pi(x)\log x \to 1$ as $x \to \infty$.* That is, $\pi(x)$ is asymptotically equal to $x/\log x$ as $x \to \infty$. It is well-known that the Prime Number Theorem is equivalent to the following statement:

Prime Number Theorem. Let $z \in \mathbb{C}$ with $\Re(z) = 1$. Then $\zeta(z) \neq 0$.

With the definition of F_m given in (4) and Theorem 2.1 at hand, the proof of the Prime Number Theorem will be reduced to case $p = 1$ of the following result:

Proposition 2.5. *Let $1 \leq p \leq 2$. If $\alpha = -p-1$, then there exists a constant C such that for all $m \in \mathbb{N}$,*

$$\|F_m\|_{p,\alpha} \leq C.$$

Moreover, if $\alpha < -p-1$, then

$$\|F_m\|_{p,\alpha} \to 0 \quad \text{as} \quad m \to \infty.$$

Suppose, for the moment, that Proposition 2.5 is proved.

Proof of Prime Number Theorem. Let $p = 1$ and $\alpha = -2$ in Proposition 2.5. Then $\{F_m\}_{m\geq 1}$ is a bounded sequence in X^1_{-2} norm. In addition, for $\alpha < -2$ the sequence of functions $\{F_m\}_{m\geq 1}$ converges to zero in X^1_{α}-norm so $\{F_m\}$ converges pointwise to zero. Now, X^1_{-2} is a functional Banach space such that the span of the point evaluation linear functionals are dense in its dual space. Hence, $\{F_m\}_{m\geq 1}$ is a weakly convergent sequence (to zero) in X^1_{-2} (see [7, Proposition 1.2]). A classical result in ℓ^1 spaces [6, Proposition 5.2] guarantees that this sequence is convergent to 0 in X^1_{-2}. Upon applying Theorem 2.1, the Prime Number Theorem follows. □

Remark 2.6. In spaces with a weighted ℓ^2 norm, one can still apply a similar idea: rather than the subsequence of F_m being norm convergent, it is possible to establish that it is weakly convergent and then, passing to a further subsequence, Mazur's Theorem (see [8, Theorem 1, Chapter II], for instance), yields the existence of a sequence of convex combinations $\{S_N\}_{N\in\mathbb{N}}$, that is actually convergent to zero in the space norm.

At this point, it is worthy to note that those α's satisfying the last statement of Proposition 2.5 must have a maximum threshold. From applying Theorem 2.1 to the sequence F_m, larger zero-free regions of the Riemann zeta function would be deduced. Indeed, if such an estimate was available for $\|F_m\|_{1,-3/2}$ or $\|F_m\|_{p,c}$ for $p \in (1,2]$ and $c > -1 - \frac{p}{2}$, it would mean the Riemann zeta function would have no non-trivial zeros at all, but the existence of some such zeros were even known to Riemann.

We are left with the task of proving Proposition 2.5 and for this end, we require to show some properties of the approximation errors F_m.

2.1.1. *Basic properties of F_m.* Let $\mu(t)$ be the Möbius function evaluated at $t \in \mathbb{N}$. It is standard that

$$\sum_{k=1}^{n} \mu(k) \left\lfloor \frac{n}{k} \right\rfloor = 1,$$

where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$.

Likewise, the property that

$$\left| \sum_{k=1}^{m} \frac{\mu(k)}{k} \right| \leq 1$$

can be proved without recurring to the Prime Number Theorem (which is indeed equivalent to this sum vanishing in the limit, see Lemma 3.2, Remark 3.3, and Theorems 5.9 and 5.11 in [4]). However, if $n \in \mathbb{N}$ is different from m, we define the 2-variable functions

$$G(n,m) = \sum_{k=1}^{m} \mu(k) \left\lfloor \frac{n}{k} \right\rfloor,$$

and

$$H(n,m) = \sum_{k=2}^{m} \mu(k) \left\{ \frac{n}{k} \right\}.$$

Having in mind the expression (4) for F_m, we notice the term on $k = m$ is 0. In addition, F_m can be easily evaluated at any $n \in \mathbb{N}$:

$$F_m(n) = 1 + H(n,m) - r_m(n) \sum_{k=1}^{m} \frac{\mu(k)}{k}, \tag{10}$$

which can be simplified by decomposing $r_t(q) = t\{q/t\} = q - t\lfloor q/t \rfloor$, giving

$$
\begin{aligned}
F_m(n) &= 1 + n\sum_{k=2}^{m}\frac{\mu(k)}{k} - \sum_{k=2}^{m}\mu(k)\left\lfloor\frac{n}{k}\right\rfloor - n\sum_{k=1}^{m}\frac{\mu(k)}{k} + m\left\lfloor\frac{n}{m}\right\rfloor\sum_{k=1}^{m}\frac{\mu(k)}{k}\\
&= 1 - \sum_{k=2}^{m}\mu(k)\left\lfloor\frac{n}{k}\right\rfloor - n + m\left\lfloor\frac{n}{m}\right\rfloor\sum_{k=1}^{m}\frac{\mu(k)}{k}.
\end{aligned}
$$

Hence,

$$
F_m(n) = 1 - G(n,m) + m\left\lfloor\frac{n}{m}\right\rfloor\sum_{k=1}^{m}\frac{\mu(k)}{k}. \tag{11}
$$

In particular, when $n < m$, then $\left\lfloor\frac{n}{m}\right\rfloor = 0$ and therefore

$$
\sum_{k=1}^{m}\mu(k)\left\lfloor\frac{n}{k}\right\rfloor = \sum_{k=1}^{n}\mu(k)\left\lfloor\frac{n}{k}\right\rfloor = 1.
$$

This means that

$$
F_m(n) = 0, \quad n \le m. \tag{12}
$$

Likewise, from (10) it follows that

$$
|F_m(n)| \le 1 + \sum_{k=2}^{m} 1 + r_m(n) < 2m, \quad n, m \in \mathbb{N}. \tag{13}
$$

With (12) and (13) at hands, the proof of Proposition 2.5 goes as follow.

Proof of Proposition 2.5. Suppose $\alpha \le -2$. Upon applying (12) and (13) we deduce

$$
\|F_m\|_{p,\alpha}^p = \sum_{n=m}^{\infty} |F_m(n)|^p (n+1)^\alpha \lesssim \frac{m^p}{|\alpha+1|}(m+1)^{\alpha+1},
$$

which converges to 0 if $\alpha < -p-1$ and remains bounded if $\alpha = -p-1$ as claimed. □

3. Möbius estimates and zero-free regions of the Riemann zeta function

As mentioned in the introduction, the distribution of the zeros of the Riemann zeta function is strongly connected to the behavior of the Möbius function μ. The main goal of this section is to establish a sufficient condition on the behavior of sums related to the Möbius function which guarantees that the Riemann zeta function has some zero-free regions within the critical strip, namely, the strip consisting of those complex numbers with imaginary part in $(0,1)$. More precisely, we prove the following:

Theorem 3.1. *Let $s \in [\frac{1}{2}, 1)$ and $\omega : [0,+\infty) \to [0,+\infty]$ be a non-decreasing function satisfying $\omega\left(\frac{1}{2}\right) < \infty$. Suppose that for $n \in (m, m^{1/s})$, we have*

$$
|G(n,m)| \le \left(m^s + \frac{n^s}{\sqrt{\log(n)}} + \left(\frac{n}{m}\right)^{\frac{s}{1-s}}\right) \cdot \omega\left(m\left|\sum_{k=1}^{m}\frac{\mu(k)}{k}\right|\right). \tag{14}
$$

Then the Riemann zeta function has no zeros in the region

$$
\Omega_s = \{z \in \mathbb{C} : \Re(z) > s\}.
$$

Before proceeding with the proof, a few remarks are in order.

Remark 3.2.

(a) Instead of assuming (14), the same result follows if G is replaced by H: Since $\frac{s}{1-s} \geq 1$, and $\lfloor \frac{n}{k} \rfloor = \frac{n}{k} - \{\frac{n}{k}\}$ the change of G by H only affects the choice of function $\omega(t)$ by adding (at most) a term $t+1$ to it.
(b) For H, the hypotheses are trivial in the limit $s = 1 \equiv \omega$ since $|H(n,m)| \leq m$, even changing the term $\left(\frac{n}{m}\right)^{\frac{s}{1-s}}$ by the smaller $\frac{n}{m}$. In that case, for G, (14) also holds in the limit $s = 1$ with $1+t$.
(c) When $n \notin (m, m^{1/s})$, (14) does hold automatically with $\omega(t) = 2 + t$. Indeed:
  (c1) If $n \geq m^{1/s}$, then

$$|H(n,m)| \leq m = m^{(\frac{1}{s}-1)(\frac{s}{1-s})} = \left(\frac{m^{1/s}}{m}\right)^{\frac{s}{1-s}} \leq \left(\frac{n}{m}\right)^{\frac{s}{1-s}}.$$

  (c2) If $n \leq m$, then $n/k < 1$ for $n < k \leq m$ and thus

$$|G(n,m)| = |G(n,n)| = 1.$$

(d) For $s = 1/2$ and $\omega(t) = 1 + t$, the assumption (14) does hold at least for $n \leq 360.000$.
(e) When $\left|\sum_{k=1}^{m} \frac{\mu(k)}{k}\right| \geq \frac{1}{\sqrt{m}}$, there is nothing to be shown. This means our condition goes in a direction that is completely different from other known conditions for Riemann Hypothesis based on μ: The values of m for which $\left|\sum_{k=1}^{m} \frac{\mu(k)}{k}\right| \geq 1/\sqrt{m}$ are not an obstruction at all. In particular, by S. Selberg's theorem [14], there are infinitely many values of $m \in \mathbb{N}$ such that

$$m\left|\sum_{k=1}^{m} \frac{\mu(k)}{k}\right| \leq \frac{1}{2}.$$

The proof of Theorem 3.1 is based on the existence of these values.
(f) Depending on the value of n, the leading term of the first factor of the right-hand side of (14) is,

$$\begin{aligned} m^s, &\quad \text{when} \quad n \in \left(m, m(\log m)^{\frac{1}{2s}}\right], \\ \frac{n^s}{\sqrt{\log(n)}}, &\quad \text{when} \quad n \in \left(m(\log m)^{\frac{1}{2s}}, \frac{m^{1/s}}{(\log m)^{\frac{1-s}{2s^2}}}\right], \\ \left(\frac{n}{m}\right)^{\frac{s}{1-s}}, &\quad \text{when} \quad n \in \left(\frac{m^{1/s}}{(\log m)^{\frac{1-s}{2s^2}}}, m^{1/s}\right]. \end{aligned}$$

Proof. Let us apply Theorem 2.1 with $p = 2$ and $\alpha = -(1+2s)$. In Hilbert spaces, we can apply Mazur's Theorem (see Remark 2.6), which tells us that the weak closure of a linear space is equal to its closure. Thus, we just need to prove that under our hypotheses F_m is weakly convergent in the X^2_α norm (or equivalently, in the D_α norm). Since we know that pointwise convergence holds from Proposition 2.5, then we can procede as in the proof of the Prime Number Theorem: it is enough to check that $\|F_m\|_{p,\alpha}$ stays bounded when $m \in M$ for some (increasing) subsequence $M = \{m_r\}_{r\in\mathbb{N}} \subset \mathbb{N}$.

In order to evaluate the X^2_α-norm of F_m, firstly, we apply (12) to get rid of terms where $n \leq m$. Then, by means of the estimate (13) applied to the tail of the sum, we obtain:

$$\begin{aligned} \|F_m\|^2_{2,s} &= \sum_{n=m}^{m^{1/s}} |F_m(n)|^2 \left(\frac{1}{n^{2s}} - \frac{1}{(n+1)^{2s}} \right) \\ &\quad + \sum_{n=m^{1/s}+1}^{\infty} |F_m(n)|^2 \left(\frac{1}{n^{2s}} - \frac{1}{(n+1)^{2s}} \right) \\ &\leq \sum_{n=m}^{m^{1/s}} |F_m(n)|^2 \left(\frac{1}{n^{2s}} - \frac{1}{(n+1)^{2s}} \right) + 4m^2 m^{-2}. \end{aligned}$$

The only thing remaining is to show that under the hypotheses of Theorem 3.1, there is a $C > 0$ and an (increasing) subsequence $M = \{m_r\}_{r\in\mathbb{N}} \subset \mathbb{N}$ such that for $m \in M$,

$$\sum_{n=m}^{m^{1/s}} |F_m(n)|^2 \left(\frac{1}{n^{2s}} - \frac{1}{(n+1)^{2s}} \right) \leq C. \tag{15}$$

By a theorem of S. Selberg [14], there are infinitely many values of $m \in \mathbb{N}$ for which the quantity $\sum_{k=1}^{m} \frac{\mu(k)}{k}$ changes sign. If m is such that

$$\left(\sum_{k=1}^{m-1} \frac{\mu(k)}{k} \right) \cdot \left(\sum_{k=1}^{m} \frac{\mu(k)}{k} \right) < 0,$$

then we have both

$$\left| \sum_{k=1}^{m-1} \frac{\mu(k)}{k} \right| \leq \frac{1}{m} \quad \text{and} \quad \left| \sum_{k=1}^{m} \frac{\mu(k)}{k} \right| \leq \frac{1}{m}.$$

In fact, we can then choose either m or $m-1$ and there are infinitely many values of $m \in \mathbb{N}$ such that

$$m \left| \sum_{k=1}^{m} \frac{\mu(k)}{k} \right| \leq \frac{1}{2}. \tag{16}$$

We restrict to those values of m, i.e., M is formed by the values $m \in \mathbb{N}$ with the property (16) and from now on we asssume that $m \in M$. Bearing that in mind, we will denote $t = m \left| \sum_{k=1}^{m} \frac{\mu(k)}{k} \right|$.

Then (11) leads to

$$|F_m(n)| \leq 1 + t(n/m) + \left| \sum_{k=1}^{m} \mu(k) \lfloor \frac{n}{k} \rfloor \right|.$$

Notice that $1 \leq \frac{s}{1-s}$ and therefore, if (14) holds, F_m satisfies

$$|F_m(n)| \leq \left(m^s + \frac{n^s}{\sqrt{\log n}} + \left(\frac{n}{m} \right)^{\frac{s}{1-s}} \right) \omega'(t),$$

where $\omega'(s) := \omega(s) + 1 + s$.

Let us finally check that (15) holds. By our last considerations, F_m can be controlled with ω' yielding

$$\begin{aligned}
&\sum_{n=m}^{m^{1/s}} |F_m(n)|^2 \left(\frac{1}{n^{2s}} - \frac{1}{(n+1)^{2s}}\right) \\
&\quad \lesssim (\omega'(t))^2 \cdot \left(\sum_{n=m}^{m^{1/s}} \left(m^{2s} + \frac{n^{2s}}{\log n} + \left(\frac{n}{m}\right)^{\frac{2s}{1-s}}\right) \cdot \left(\frac{1}{n^{2s}} - \frac{1}{(n+1)^{2s}}\right)\right) \\
&\quad \lesssim \omega'(1/2)^2 \cdot \left(1 + \sum_{n=m}^{m^{1/s}} \frac{1}{n \log n} + m^{\frac{-2s}{1-s}} \cdot \left(\sum_{n=m}^{m^{1/s}} n^{\frac{2s}{1-s}-1-2s}\right)\right) \\
&\quad := \omega'(1/2)^2 \cdot (1 + \mathrm{Sum}_1 + \mathrm{Sum}_2)\,.
\end{aligned}$$

where we used that $\left(\frac{1}{n^{2s}} - \frac{1}{(n+1)^{2s}}\right) \approx n^{-1-2s}$.

Since $2s(\frac{1}{1-s} - 1) - 1 > -1$, we have a bound for Sum_2:

$$\mathrm{Sum}_2 \lesssim m^{\frac{-2s}{1-s} + \frac{1}{s} \cdot \frac{2s^2}{1-s}} = 1.$$

On the other hand,

$$\mathrm{Sum}_1 \lesssim [\log \log x]_m^{m^{1/s}} = \log(1/s)$$

which is a constant. This concludes the proof. □

Remark 3.3. A similar proof is possible, based on a $p \neq 2$ version of the previous, but unfortunately, a different value of p would have required that the denominator in the term $\frac{n^s}{\sqrt{\log n}}$ from (14) be replaced with $\frac{n^s}{(\log n)^{1/p}}$, which is a stronger assumption for $p < 2$.

3.1. **Further remarks on Theorem 3.1: towards the needed estimates on Möbius function sums.**

Clearly, assumption (14) plays a key role in the proof of Theorem 3.1. In order to provide some insights into (14), we show estimates which involve the classical *Mertens function*.

Recall that the Mertens function $M : \mathbb{N} \to \mathbb{N}$ is defined by

$$M(t) = \sum_{n=0}^{t} \mu(n),$$

and many authors have studied possible relations with the Riemann zeta function, some of which are present in [4].

Proposition 3.4. *For $n, m \in \mathbb{N}$ we have*

$$G(n,m) = 1 + \sum_{t=1}^{\left\lfloor \frac{n}{m} \right\rfloor} \left(M(m) - M(n/t)\right).$$

Proof. Note that

$$\begin{aligned}
\sum_{k=1}^{m}\mu(k)\left\lfloor\frac{n}{k}\right\rfloor &= \sum_{t=\lfloor\frac{n}{m}\rfloor}^{n} t \sum_{\substack{1\le k\le m\\ \lfloor\frac{n}{k}\rfloor=t}}\mu(k)\\
&= \left\lfloor\frac{n}{m}\right\rfloor\left(M(m)-M(\frac{n}{\lfloor\frac{n}{m}\rfloor+1})\right)+\sum_{t=\lfloor\frac{n}{m}\rfloor+1}^{n} t\left(M(\frac{n}{t})-M(\frac{n}{t+1})\right)\\
&= \left\lfloor\frac{n}{m}\right\rfloor M(m)+\sum_{t=\lfloor\frac{n}{m}\rfloor+1}^{n} M(\frac{n}{t}).
\end{aligned}$$

Then, using the fact that we just proved for $m=n$ we have

$$1=\sum_{k=1}^{n}\mu(k)\left\lfloor\frac{n}{k}\right\rfloor=\sum_{t=1}^{n}M(n/t).$$

This can be substituted above to give

$$\sum_{k=1}^{m}\mu(k)\left\lfloor\frac{n}{k}\right\rfloor=\left\lfloor\frac{n}{m}\right\rfloor M(m)+1-\sum_{t=1}^{\lfloor\frac{n}{m}\rfloor}M(\frac{n}{t})=1+\sum_{t=1}^{\lfloor\frac{n}{m}\rfloor}\left(M(m)-M(n/t)\right),$$

as claimed. □

We can take advantage of this proposition to show that two values of n that are close to each other satisfy (14) with similar functions ω, and thus we might only need to prove (14) for a *dense enough* grid of values m,n.

Proposition 3.5. *Let $n=qm+d$ and $n'=qm+d'$ with $d,d'\in\{0,...,m-1\}, q\in\{1,...,m^{\frac{1-s}{s}}\}$. Then*

$$|G(n,m)-G(n',m)|\lesssim q+|d-d'|\log r,$$

where $r=e+\min(|d-d'|,q)$.

Notice $q\le n/m$ and so this principle works when $d-d'$ is small compared to the right-hand side in (14).

Proof. First, upon applying the previous Proposition, and having in mind that

$$|M(x)-M(y)|\le|x-y|+1,$$

we obtain

$$\begin{aligned}
\left|\sum_{k=1}^{m}\mu(k)\left(\left\lfloor\frac{n}{k}\right\rfloor-\left\lfloor\frac{n'}{k}\right\rfloor\right)\right| &= \left|\sum_{t=1}^{q}\left(M(n/t)-M(n'/t)\right)\right|\\
&\le\sum_{t=1}^{q}\left(\left\lfloor\frac{|d-d'|}{t}\right\rfloor+1\right)\le q+|d-d'|\sum_{t=1}^{r}\frac{1}{t}.
\end{aligned}$$

□

4. Balazard and Saias approach: a further approximation of $\mathbb{1}$ in span$\{r_k : k \geq 2\}$

As we discussed in Section 2, the work of Balazard and Saias [3] suggests natural candidates for approximating $\mathbb{1}$ in span$\{r_k : k \geq 2\}$ in X^2_{-2} norm, namely,

$$\sum_{k=2}^{m} r_k \frac{\mu(k)}{k}.$$

Previously, we modified such approach and considered the sequences F_m given by (4), that is,

$$F_m = \mathbb{1} + \left(\sum_{k=2}^{m} r_k \frac{\mu(k)}{k} - \sum_{k=1}^{m} r_m \frac{\mu(k)}{k} \right),$$

as an attempt at approximating $\mathbb{1}$ in the scheme of Theorem 2.1.

The aim of this section is approximating $\mathbb{1}$ by a linear combination of a *selection* of $\{r_k : 2 \leq k \leq n\}$ which provides a different insight in order to determine zero-free regions of ζ. In particular, for each $m \geq 2$ we will consider approximations with linear combinations of the r_k's such that k divides m, that is, span$\{r_k : 2 \leq k | m\}$. More precisely, instead of F_m we will consider the sequences

$$F'_m := \mathbb{1} + \left(\sum_{\substack{k=2 \\ k|m}}^{m} r_k \frac{\mu(k)}{k} - \sum_{\substack{k=1 \\ k|m}}^{m} r_m \frac{\mu(k)}{k} \right). \tag{17}$$

In addition, we will restrict ourselves to the subsequence of positive integers $\{m_t\}$ defined as the product of the first t prime numbers, that is, $m_t = \prod_{i=1}^{t} p_i$ is the prime decomposition of m_t. For the sake of simplicity, in what follows we will denote F'_{m_t} by F'_m.

This choice of approximation also satisfies the assumptions in Theorem 2.1 with the same constant c_1 since, for fixed $n \in \mathbb{N}$ and as $t \to \infty$, each coefficient $F'_m(n)$ converges towards the corresponding $F_m(n)$. However, as we will show, the conclusions we can obtain are different in nature. The point of establishing a different selection of the set of r_k here is that the necessary estimates on the Möbius function sums may have a better chance of being proved, since the arithmetic properties of the Möbius function with respect to the divisors of a number are well known.

A standard property of the Möbius function that will be useful for us is the following (see [4, Theorem 2.16]):

Lemma 4.1. *Let $d \in \mathbb{N}$. Then*

$$\sum_{\substack{k=1 \\ k|d}}^{d} \mu(k) = \delta(d).$$

Analogously as in Section 2, we introduce two double indexed functions

$$G'(n,m) = \sum_{\substack{k=1 \\ k|m}}^{m} \mu(k) \left\lfloor \frac{n}{k} \right\rfloor.$$

$$H'(n,m) = \sum_{\substack{k=2 \\ k|m}}^{m} \mu(k) \left\{ \frac{n}{k} \right\},$$

and deduce:

$$F'_m(n) = 1 + H'(n,m) - r_m(n) \sum_{\substack{k=1 \\ k|m}}^{m} \frac{\mu(k)}{k}, \tag{18}$$

and

$$F'_m(n) = 1 - G'(n,m) + m \left\lfloor \frac{n}{m} \right\rfloor \sum_{\substack{k=1 \\ k|m}}^{m} \frac{\mu(k)}{k}. \tag{19}$$

Consider $t \in \mathbb{N}$ and $m \in \mathbb{N}$, with $m = \prod_{i=1}^{t} p_i$ both fixed. Now, from the definition of F'_m, and for $n \le p_t$, notice that $r_k(n) - r_m(n) = 0$ for all $k > p_t$, and thus

$$F'_m(n) = F_{p_t}(n) = 0, \quad n \le p_t. \tag{20}$$

Apart from this, we remark a key distinction of F'_m with respect to F_m: *F'_m is m-periodic* since r_k is m-periodic whenever $k|m$.

In D_α spaces for $\alpha \in [-3,-2]$, any m-periodic sequence (of Taylor coefficients) defines a function f that has a norm $\|f\|_\alpha$ between $\|q_m(f)\|_\alpha$ and $2\|q_m(f)\|_\alpha$, where $q_m(f)$ is the Taylor polynomial of f of degree less or equal to m. Therefore, in order to show boundedness of $\|F'_m\|_\alpha$ we just need to show boundedness of the norm contributed by the coefficients of order $p_t < n < m$ (when $n = m$ then $F'_m(n) = 1$).

By making use of the standard notation $\#E$ for the cardinal of a set E and (u,v) for the greatest common divisor of u and v, the first basic result towards understanding the remaining case is the following:

Theorem 4.2. *Let $1 \le n < m$. Then*

$$F'_m(n) = -\#\{1 < k \le n : (k,m) = 1\}.$$

Proof. If $n \le p_t$ then we already know that the result is true from (20). If $n > p_t$, we just need to show that when $(n,m) = 1$ we have

$$F'_m(n) = -1 + F'_m(n-1), \tag{21}$$

whereas otherwise we have

$$F'_m(n) = F'_m(n-1). \tag{22}$$

Let us first assume $(n,m) = 1$. Then $r_k(n) - r_k(n-1) = 1$ for all $k|m$. Thus,

$$F'_m(n) - F'_m(n-1) = \left(\sum_{\substack{k=2 \\ k|m}}^{m} \frac{\mu(k)}{k} - \sum_{\substack{k=1 \\ k|m}}^{m} \frac{\mu(k)}{k} \right) = -1.$$

On the other hand, if $(n,m) = D \neq 1$, then $r_k(n) - r_k(n-1) = 1$ for $k|m$ with $k \nmid n$ while $r_k(n) - r_k(n-1) = 1 - k$ when $k|D$. Hence, the only difference with the previous is contributed by k dividing D and in this case we have

$$F'_m(n) - F'_m(n-1) = -1 - \left(\sum_{\substack{k=2 \\ k|D}}^{D} \mu(k) - \sum_{\substack{k=1 \\ k|D}}^{D} \mu(k) \right) = -1 + 1 = 0.$$

Then, for any $n > p_t$,

$$F'_m(n) = -\sum_{\substack{k=p_t+1 \\ (k,m)=1}}^{n} 1,$$

which is the claimed value. □

The game changes then, from approximating the function $\mathbb{1}$ to approximating $-F'_m$ by means of linear combinations of r_k. The following seems like a starting point on how to approximate $-F'_m$. We use the standard notation φ for the Euler totient function. Define s_t as:

$$s_t = \frac{\varphi(m)-1}{m-p_t}(r_{p_t} - r_m). \tag{23}$$

Notice s_t is a reasonable guess firstly in the sense that it is still m-periodic and its first p_t values are equal to 0; moreover, $s_t(n)$ grows with n at the adequate speed, meaning that $r_{p_t}(n) - r_m(n)$ jumps by p_t at the multiples of p_t and remains constant elsewhere, implying an approximately linear pace of increase for s_t up to

$$s_t(m-1) \approx \varphi(m) - 1 = -F'_m(m-1).$$

So taking this approximation is a bet on the distribution of numbers relatively prime with m, between $p_t + 1$ and $m - 1$, being equidistributed to some degree.

We will make use of the following well known estimate, for which we give a proof due to Peter Humphries [12]:

Lemma 4.3. *Let $1 \le n < m = \prod_{i=1}^{t} p_i$. Then, as $t \to \infty$, we have*

$$\sum_{\substack{1\le j\le n \\ (j,m)=1}} 1 = n\frac{\varphi(m)}{m} + \mathcal{O}(2^t).$$

Proof. By Lemma 4.1, we can see that

$$\sum_{\substack{1\le j\le n \\ (j,m)=1}} 1 = \sum_{j=1}^{n} \sum_{d|(j,m)} \mu(d) = \sum_{d|m} \mu(d) \sum_{\substack{1\le j\le n \\ d|j}} 1 = \sum_{d|m} \mu(d) \left\lfloor \frac{n}{d} \right\rfloor =: G'(n,m).$$

Now, in the expression for G' we use the decomposition $\left\lfloor \frac{n}{d} \right\rfloor = \frac{n}{d} - \left\{ \frac{n}{d} \right\} = \frac{n}{d} + O(1)$ to obtain

$$G'(n,m) = n\sum_{d|m} \frac{\mu(d)}{d} - H'(n,m) = \frac{n\varphi(m)}{m} + O(2^t),$$

since 2^t is the number of divisors of m. □

Notice the relations we used on G' and H', that

$$G'(n,m) + H'(n,m) = \frac{n\varphi(m)}{m}$$

is valid for all n, m. This Lemma is relevant since the left-hand side in its statement is $1 - F'_m(n)$. We would like to be able to apply Lemma 4.3 together with the following:

Proposition 4.4. *For $n < m$,*

$$s_t(n) + n\frac{\varphi(m)}{m} = \mathcal{O}\left(\frac{p_t}{\log\log m}\right).$$

Proof. Since $n < m$ we know $r_m(n) = n$. Thus,

$$(r_{p_t} - r_m)(n) = p_t \left\{ \frac{n}{p_t} \right\} - n = -p_t \left\lfloor \frac{n}{p_t} \right\rfloor .$$

Substituting this value in $s_t(n)$ yields

$$s_t(n) = -\left\lfloor \frac{n}{p_t} \right\rfloor p_t \frac{\varphi(m) - 1}{m - p_t}.$$

But we can decompose

$$s_t(n) + n\frac{\varphi(m)}{m} = p_t \left\{ \frac{n}{p_t} \right\} \frac{\varphi(m) - 1}{m - p_t} + \frac{n}{m - p_t} - \frac{n\varphi(m)p_t}{m(m - p_t)}.$$

Since $m >> p_t \gtrsim \log m$ and $n < m$, the second term is a number between 0 and 2 and thus, clearly contributes $\mathcal{O}\left(\frac{p_t}{\log\log m}\right)$. From [4, Lemma 4.13 and Theorem 4.15], we see that $\varphi(m)/m \approx \frac{1}{\log\log m}$. As the fractional part of a number is bounded by 1, we get the desired estimate for the first term as well. A mixture of all the same arguments gives the same bound for the last term. □

As a closing remark of this section, note that Lemma 4.4 along with Proposition 4.3 yield that the contributions of some ranges of values of n ($2^{2t} \leq n < m$) to the norm of $s_t + F'_m$ are unimportant in order to show uniform boundedness. Since m has t different prime factors, when t is large it is to be expected that this unimportant range becomes most of the range of possible values, but good estimates at the first values of n are needed.

Indeed, pointwise convergence is easy to see since $\|F'_m\|_\alpha \to 0$ as $t \to \infty$ whenever $\alpha < -3$: just estimating $|F'_m(n)| < n - p_t$ when $n > p_t$ (and 0 otherwise) shows that $\|F'_m(n)\|^2_\alpha \lesssim (p_t)^{3+\alpha}$, which vanishes as $t \to \infty$. For large values of t, adding s_t to the mix does not affect this convergence (its norm is less than that of $r_{p_t} - r_m$, which for $\alpha < -3$ goes to 0 as $t \to \infty$).

The final observation is that one doesn't need to take r_{p_t} and in fact it makes sense to make use of all the divisors of m to approximate the function $\sum_{\substack{1\leq j\leq n \\ (j,m)=1}} 1$.

5. An extension of the work by Waleed Noor

The main aim of this final section is pushing further the recent approach of Waleed Noor [17] to the context of weighted Dirichlet spaces in order to provide zero-free regions for the Riemann ζ function.

Let us denote by Ψ the correspondence between D_α and ℓ^2_α introduced in Subsection 1.1, namely:

$$\begin{array}{cccc} \Psi: & \ell^2_\alpha & \longrightarrow & D_\alpha \\ & \{a_k\}_{k\geq 1} & \longrightarrow & \Psi\Big(\{a_k\}_{k\geq 1}\Big) = \displaystyle\sum_{k=1}^{\infty} a_k z^{k-1} \end{array}$$

We have shown in Theorem 2.1 that for $\alpha \in (-3, -2]$, the statement that the Riemann zeta function has no zeros on the band $\{z \in \mathbb{C} : \Re(z) > -\frac{\alpha+1}{2}\}$ can be derived provided we establish the existence of some linear combinations of $\{\Psi(r_k) : k \geq 2\}$ approximating

$$f_0(z) := \frac{1}{1 - z}, \quad (z \in \mathbb{D})$$

in the D_α norm.

Waleed Noor's work [17] addresses an analysis of the linear operator $T : D_{-2} \to D_0$ defined by

$$Tf(z) = \frac{((1-z)f(z))'}{1-z}, \quad (z \in \mathbb{D}).$$

In particular, he proves that such operator is bijective and bounded and identifies $T^{-1}(f_0)$ as well as $\{T^{-1}(\Psi(r_k)) : k \geq 2\}$.

In order to extend previous results to D_α, we introduce the operators T_a, $T_{a,b}$ and $T_{a,h}$ where $a, b \in \mathbb{R}$ and h is a function as follows:

$$T_a f(z) = \frac{((1-z)^a f(z))'}{(1-z)^a},$$

$$T_{a,b} f(z) = \frac{((1-z)^a f(z))'}{(1-z)^b},$$

$$T_{a,h} f(z) = \frac{((1-z)^a f(z))'}{(1-z)^a} \cdot h(z).$$

In particular, $T = T_1$; if $f_1 \equiv 1$, $T_a = T_{a,f_1} = T_{a,a}$; and $T_{a,b} = T_{a,(1-z)^{a-b}}$. We also have that $T_{a,h} = M_h \cdot T_a$ whenever such composition is defined (where M_g denotes the operator of multiplication by the function g). Recall that a function $h \in D_\alpha$ is said to be *cyclic* (in D_α) if $\{z^k h : k \in \mathbb{N}\}$ spans a dense subset of D_α. We refer to [11] for more on cyclic functions in D_α spaces.

Proposition 5.1. *Let $\alpha \in [-3, -2]$ and h be a bounded and cyclic function in D_α. Then $T_{a,h} : D_{\alpha+2} \to D_\alpha$ is bijective and bounded if and only if $a \geq -\frac{1+\alpha}{2}$.*

Remark 5.2. In the particular cases of T_a and $T_{a,b}$ the requirement that h is a cyclic bounded function is, respectively, automatically true and equivalent to $b \leq a$.

Proof. We start with the *boundedness*: we know from [17] that T is bounded from D_0 to D_{-2}, but the same proof yields boundedness $D_{\alpha+2} \to D_\alpha$. Now, $T_a = D - aM_{\frac{1}{1-z}}$, where D is the derivative. From the definition of the spaces D_α, given in terms of coefficients, it is easy to see that the derivative is a bounded operator from $D_{\alpha+2}$ to D_α. Thus, $M_{\frac{1}{1-z}}$ must be bounded as the difference between two bounded operators $D - T$. Hence, T_a is bounded. Now, $T_{a,h} = M_h T_a$ and the operator M_h is bounded for a bounded function even in $D_{\alpha+2} \to D_{\alpha+2}$: H^∞ coincides with the set of multipliers in all Dirichlet spaces larger than H^2 [16].

Let us check the *surjectivity of* T_a. For this, let us find a polynomial function p_k such that

$$T_a p_k = z^k. \tag{24}$$

If this can be done for all $k \in \mathbb{N}$, since $\{z^k\}_{k\in\mathbb{N}}$ is an (orthogonal) basis of D_α, and thus span a dense subset, then the operator must be surjective.

Denote $p_k(z) = \sum_{i=0}^d a_i z^i$, for some d. Notice that $T_a = T + (1-a)M_{\frac{1}{1-z}}$. Thus, from the Taylor expansion of $1/(1-z)$ we can see that if p_k exists, solving (24), then p_k must satisfy

$$T_a p_k = \sum_{n=0}^\infty \left(\sum_{i=0}^n (i+1)(a_{i+1} - a_i) + (1-a) \sum_{i=0}^n a_i \right) z^n.$$

Now, the right hand side above is equal to

$$\sum_{n=0}^\infty \left(\sum_{i=0}^n ((i+1)a_{i+1} - (i+a)a_i) \right) z^n.$$

For this to be equal to z^k we must have

$$\sum_{i=0}^{k}((i+1)a_{i+1}-(i+a)a_i)=1, \tag{25}$$

and

$$\sum_{i=0}^{n}((i+1)a_{i+1}-(i+a)a_i)=0, \quad \forall n\neq k. \tag{26}$$

We can apply (26) with two consecutive values n and $n-1$ to get

$$a_{k+1}=\frac{1+a_k(k+a)}{k+1}, \quad a_{k+2}=\frac{-1+a_{k+1}(k+1+a)}{k+2},$$

and

$$a_{n+1}=a_n\left(\frac{n+a}{n+1}\right), \forall n\notin\{k,k+1\}.$$

Notice each a_i is obtained by multiplying by a positive number the previous one, except for $i=k+1$ or $i=k+2$. In particular, if p_k is a polynomial, it must be of degree k or $k+1$. If $a_{k+1}=0$ then a_{k+2} is not null, and thus, p_k can only be a polynomial if $a_{k+2}=0$ and then its degree is $k+1$. From now on let $b_n=a_0\prod_{t=0}^{n-1}\frac{t+a}{t+a}$. Altogether, this shows that

$$a_n=b_n, \quad \forall n\leq k,$$

$$a_n=0, \quad \forall n\geq k+2,$$

$$a_{k+1}=\frac{1}{k+1}+b_{k+1},$$

$$a_{k+2}=\frac{-1}{k+2}+\frac{k+1+a}{(k+2)(k+1)}+b_{k+2}.$$

Denote by B the classical beta function. The condition that $a_{k+2}=0$ determines the value of a_0 from that of a and k, and also allows us to express a_{k+1} more simply:

$$a_0=-B(k+1,a+1), a_{k+1}=\frac{1}{k+1+a}.$$

From this information, we can obtain the general form of a_i and, therefore, of p_k:

$$a_i=\frac{-aB(k+1,a)}{(i+a)(k+1+a)B(i+1,a)}, \quad 0\leq i\leq k,$$

$$p_k(z)=\frac{1}{k+1+a}\left(z^{k+1}-aB(k+1,a)\cdot\sum_{r=0}^{k}\frac{z^r}{(r+a)B(r+1,a)}\right). \tag{27}$$

Indeed, this choice of p_k does satisfy (24) and T_a is onto for any $a>0$ (which must hold if $a\geq\frac{-(1+\alpha)}{2}$). The same argument shows how to construct for all $k\in\mathbb{N}$, a polynomial p_k such that

$$T_{a,h}p_k(z)=z^k\cdot h(z). \tag{28}$$

Since h is cyclic, the set $\{z^kh\}_{k\in\mathbb{N}}$ spans a dense subset of D_α, and so, we have shown that $T_{a,h}$ *is onto.*

It remains to see that T_a *is one-to-one*: if it is, then $T_{a,h} = M_h T_a$ is too, since multiplying by a (not identically null) function is necessarily one-to-one. Now, consider two functions g_1 and g_2 in $D_{\alpha+2}$ such that

$$T_a g_1 = T_a g_2.$$

This means that $(1-z)^a(g_1 - g_2)$ must be a constant function $c \in \mathbb{C}$ and if $c \neq 0$ we must have $(1-z)^{-a} \in D_{\alpha+2}$, which happens exactly when $a < -\frac{1+\alpha}{2}$. This concludes the proof. □

Remark 5.3. Note that when $0 < a < -\frac{1+\alpha}{2}$, only the injectivity fails in the theorem.

For $k \geq 2$ let h_k be the function introduced in (2), that is,

$$h_k(z) = \frac{1}{1-z} \log\left(\frac{1+z+...+z^{k-1}}{k}\right), \quad (z \in \mathbb{D}).$$

Waleed Noor showed that

$$Th_k = \Psi(r_k). \tag{29}$$

In our setting, next results identify the inverse images of the special functions f_0 and $\Psi(r_k)$ under some of the introduced operators.

Lemma 5.4. *Let $b \neq 0$. Then*

$$T_{a,b} \frac{-(1-z)^{b-a}}{b} = f_0. \tag{30}$$

In the particular case of $b = 1 \leq a$, for $k \geq 2$, we also have

$$T_{a,1}\left(\frac{h_k}{(1-z)^{a-1}}\right) = \Psi(r_k). \tag{31}$$

Remark 5.5. Before proving the result, notice that $(1-z)^c \in D_\beta$ if and only if $c > \frac{\beta-1}{2}$. Thus for the functions we found $(1-z)^{b-a}$ to be in $D_{\alpha+2}$, there is an additional requirement that $b - a > \frac{\alpha+1}{2}$, in order to guarantee that the application of the operator is correct. With regards to $h_k(1-z)^{1-a}$ being elements of $D_{\alpha+2}$, notice $h_k \in D_{1-\varepsilon}$ for all $\varepsilon > 0$: Waleed Noor [17, Last line of proof of Lemma 7] showed that these functions have coefficients bounded by a multiple of k/n, and thus $h_k(1-z)^{1-a}$ is in $D_\alpha + 2$ precisely when $a > \frac{\alpha+3}{2}$. Since we are going to take $a \geq 1$ to apply the Lemma, a is in the correct range of values.

Proof. To see (30), call $f(z) = \frac{-(1-z)^{b-a}}{b}$. By direct computation we can see that

$$T_{a,b} f = \frac{\frac{b-a}{b}(1-z)^{b-a-1} \cdot (1-z) + \frac{a}{b}(1-z)^{b-a}}{(1-z)^{1-a+b}}.$$

The right hand side simplifies to f_0.

In order to check (31), we can use (29) to see that

$$T_{a,1}\left(\frac{h_k}{(1-z)^{a-1}}\right) = \frac{((1-z) \cdot h_k)'}{1-z} = Th_h = \Psi(r_k).$$

□

Gathering all the previous result, the following extension of Waleed Noor's criterion holds:

Theorem 5.6. *Let $\alpha \in [-3,-2]$, $a \in [1, \frac{1-\alpha}{2})$ and $\Omega_\alpha = \{z \in \mathbb{C} : \Re(z) > \frac{-1-\alpha}{2}\}$. The function $(1-z)^{1-a}$ is in the closure in $D_{\alpha+2}$ of* $\operatorname{span}\{(1-z)^{1-a}h_k : k \geq 2\}$ *if and only if Ω_α is a zero-free region of the Riemann zeta function.*

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Eva A. Gallardo-Gutiérrez
Departamento de Análisis Matemático y Matemática Aplicada,
Facultad de Matemáticas,
Universidad Complutense de Madrid,
Plaza de Ciencias Nº 3, 28040 Madrid, Spain
and Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM),
Madrid, Spain
Email address: `eva.gallardo@mat.ucm.es`

Daniel Seco
Departamento de Matemáticas,
Universidad Carlos III de Madrid
Avenida de la Universidad 30, 28911 Leganés (Madrid), Spain
and Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM),
Madrid, Spain.
Email address: `dseco@math.uc3m.es`