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APPLICATIONS TO RISK THEORY OF A MONTECARLO MULTIPLE INTEGRATION METHOD

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ABSTRACT The evaluation of multiple integrals is a commonly encountered problem in risk theory, specially in ruin probability. Using Monte Carlo simulation we will obtain an unbiased and consistent point estimator, and also confidence intervals as approximations of a special case of multiple integral frequently used in risk theory. The variance reduction achieved compared to straight simulation and some specific properties make this approach interesting when approximating ruin probabilities.

1. INTRODUCTION

Let us define the discrete stochastic process $\{S_t\}$:

$$S_t = \sum_{i=1}^{t} z_i \qquad z_i \geq 0$$

where $z_i \quad i = 1, ..., t$ are independent random variables with p.d.fs. $g_i(x)$, c.d.fs. $G_i(x)$ and $E_{g_i}\{z_i\} = \mu_i < \infty$.

Let us define the following function:

$$\begin{aligned}\int_{\mathcal{R}(\mathbf{X})} \varphi(\mathbf{S}) &= H^{*t}(\mathbf{X}) = P\{S_j \leq x_j \quad j = 1, ..., t\} = \\ &= \int_0^{x_1}\int_{S_1}^{x_2}\cdots\int_{S_{t-2}}^{x_{t-1}} G_t(x_t - S_{t-1})\, g_{t-1}(S_{t-1} - S_{t-2})\ldots \\ &\quad \cdots g_2(S_2 - S_1)\, g_1(S_1)\, dS_1 \cdots dS_{t-1} \qquad (1.1)\end{aligned}$$

where $\mathbf{X} = (x_1, x_2, \ldots, x_t)$ and $\mathbf{S} = (S_1, S_2, \ldots, S_{t-1})$

Expression (1.1) is considering the probability that the paths of stochastic process $\{S_t\}$ will be bounded by vector $\mathbf{X}$.

Evaluating multiple integrals is one of the classical problems of numerical analysis, principal among the methods design for this purpose we highlight quadratures formulas, equidistributed sequences and MonteCarlo [6, Fishman(1996).Chapter2]:

1.1. quadrature formulas

The most commonly encountered multivariable quadrature formulas are direct extensions of quadrature formulas for the one-dimensional case. The number of evaluations of the function to integrate is n^t where n is the number of evaluation points for one-dimensional case and t is the dimension of the multiple integral, making this method very messy when t is not small because the number of evaluations is exponentially increased. Under certain conditions ([6], [1, Bahvalov(1959)] and [7, Haber(1970)]) the absolute error of the approximation is $O\left(n^{-\frac{j}{t}}\right)$, for some $j \geq 1$.

1.2. Equidistributed points

In this case the absolute error is $O\left(\delta n^{-1} (\ln(n))^t\right)$ under certain conditions ([6], [7] and [8, Niederreiter(1978,92)]) .

1.3. Monte Carlo methods

The convergence of the approximation is $O\left(n^{-\frac{1}{2}}\right)$ provided that $\int_{\Re} \varphi^2(\mathrm{X}) < \infty$.

Each approach has advantages and limitations. The convergence of deterministic methods seems clearly better (specially if the dimension of the integral t is not large) when they can be applied. Nevertheless, the applicability matters make Monte Carlo techniques competitive because the verification of the conditions we cited for deterministic methods becomes very difficult as t increases or very restrictive for function $\varphi(\mathrm{X})$.

Reproducing [6, Fishman(1996).Chapter2], we can argument that Monte Carlo methods can be applied considerably more broadly to functions that merely satisfy $\int_{\Re} \varphi^2(\mathrm{X}) < \infty$. Also, the Monte Carlo error depends on φ only through this integral, and in no way on the continuity and variational properties of φ. Another interesting fact is that Monte Carlo methods allow one to estimate error from generated data, whereas one needs to rely in considerably more global measures of error when employing deterministic techniques. Finally, the Monte Carlo convergence is always $O\left(n^{-\frac{1}{2}}\right)$ regardless the dimension of the integral t, this is an interesting aspect when t is large because in the deterministic methods this convergence worsen as t increases.

2. Simple Unbiased estimator

We will introduce the estimator:

$$H^{*t}(\mathbf{X}) \simeq \mathcal{H}^{*t}(\mathbf{X}) = G_t(x_t - S_{t-1})\, G_{t-1}(x_{t-1} - S_{t-2}) \ldots G_2(x_2 - S_1)\, G_1(x_1) \tag{2.1}$$

where z_i $i = 1, ..., t$ are random numbers generated using the following density functions:

$$S_1 \longrightarrow \mathfrak{D}_1(S_1) = \frac{g_1(S_1)}{G_1(x_1)} \qquad S_1 \in [0, x_1]$$

$$S_j \longrightarrow \mathfrak{D}_j(S_j) = \frac{g_j(S_j - S_{j-1})}{G_j(x_j - S_{j-1})} \qquad S_j \in [S_{j-1}, x_j] \qquad j > 1 \tag{2.2}$$

Theorem 1. *The estimator $\mathcal{H}^{*t}(\mathbf{X})$ is unbiased.*

Proof:

The expected value of the estimator $\mathcal{H}^{*t}(\mathbf{X})$ can be expressed:

$E\{\mathcal{H}^{*t}(\mathbf{X})\} = \int_0^{x_1} \int_{S_1}^{x_2} \cdots \int_{S_{t-2}}^{x_{t-1}} G_t(x_t - S_{t-1})\, G_{t-1}(x_{t-1} - S_{t-2})$

$\ldots G_2(x_2 - S_1)\, G_1(x_1) \frac{g_1(S_1)}{G_1(x_1)} \frac{g_2(S_2-S_1)}{G_2(x_2-S_1)} \cdots \frac{g_{t-1}(S_{t-1}-S_{t-2})}{G_{t-1}(x_{t-1}-S_{t-2})} dS_1 \cdots dS_{t-1} =$

$\int_0^{x_1} \int_{S_1}^{x_2} \cdots \int_{S_{t-2}}^{x_{t-1}} G_t(x_t - S_{t-1})\, g_{t-1}(S_{t-1} - S_{t-2}) \ldots g_2(S_2 - S_1)\, g_1(S_1)$

$dS_1 \cdots dS_{t-1} = H^{*t}(\mathbf{X})$ *Q.E.D.*

3. Variance of the simple estimator

The Variance of the estimator will be studied using the following two theorems:

Theorem 2. *The variance of the estimator $\mathcal{H}^{*t}(\mathbf{X})$ has an upper bound:*

$$Var\{\mathcal{H}^{*t}(\mathbf{X})\} \leq \prod_{i=1}^{t} G_i(x_i)\, H^{*t}(\mathbf{X}) - \left[H^{*t}(\mathbf{X})\right]^2 \tag{3.1}$$

Proof:

Due to the fact that $\mathcal{H}^{*t}(\mathbf{X})$ is an unbiased estimator:

$$Var\{\mathcal{H}^{*t}(\mathbf{X})\} = E\left\{\left(\mathcal{H}^{*t}(\mathbf{X})\right)^2\right\} - \left[H^{*t}(\mathbf{X})\right]^2 \tag{3.2}$$

Let us study the former expected value:

$E\left\{\left(\mathcal{H}^{*t}(\mathbf{X})\right)^2\right\} = \int_0^{x_1} \int_{S_1}^{x_2} \cdots \int_{S_{t-2}}^{x_{t-1}} G_t^2(x_t - S_{t-1})\, G_{t-1}^2(x_{t-1} - S_{t-2})$

$$\ldots G_2^2(x_2-S_1)\,G_1^2(x_1)\,\tfrac{g_1(S_1)}{G_1(x_1)}\tfrac{g_2(S_2-S_1)}{G_2(x_2-S_1)}\cdots\tfrac{g_{t-1}(S_{t-1}-S_{t-2})}{G_{t-1}(x_{t-1}-S_{t-2})}dS_1\cdots dS_{t-1}\leq$$
$$G_t(x_t)\,G_{t-1}(x_{t-1})\ldots G_2(x_2)\,G_1(x_1)\int_0^{x_1}\int_{S_1}^{x_2}\cdots$$
$$\cdots\int_{S_{t-2}}^{x_{t-1}}G_t(x_t-S_{t-1})\,G_{t-1}(x_{t-1}-S_{t-2})\ldots G_2(x_2-S_1)\,G_1(x_1)$$

$$\frac{g_1(S_1)}{G_1(x_1)}\frac{g_2(S_2-S_1)}{G_2(x_2-S_1)}\cdots\frac{g_{t-1}(S_{t-1}-S_{t-2})}{G_{t-1}(x_{t-1}-S_{t-2})}dS_1\cdots dS_{t-1} \qquad (3.3)$$

because:

$$Max\left[G_t(x_t-S_{t-1})\,G_{t-1}(x_{t-1}-S_{t-2})\ldots G_2(x_2-S_1)\,G_1(x_1)\right]=$$

$$=G_t(x_t)\,G_{t-1}(x_{t-1})\ldots G_2(x_2)\,G_1(x_1)$$

and $G_i(y)$ $i=1,\ldots,t$ *are distribution functions and then non-decreasing .*
Substituting (3.3) into (3.2) we get the statement of the theorem.
Q.E.D.

Theorem 3. *If* $H^{*t}(\mathbf{X})<1$ *(non trivial case) then the variance of the estimator* $\mathcal{H}^{*t}(\mathbf{X})$ *is less than the variance of the direct simulation of the value of* $H^{*t}(\mathbf{X})$ *:*

$$Var\left\{\mathcal{H}^{*t}(\mathbf{X})\right\} \;<\; H^{*t}(\mathbf{X})-\left[H^{*t}(\mathbf{X})\right]^2$$

Proof:

$$H^{*t}(\mathbf{X})=\int_0^{x_1}\int_{S_1}^{x_2}\cdots\int_{S_{t-2}}^{x_{t-1}}G_t(x_t-S_{t-1})\,G_{t-1}(x_{t-1}-S_{t-2})$$

$$\ldots G_2(x_2-S_1)\,G_1(x_1)\,\tfrac{g_1(S_1)}{G_1(x_1)}\tfrac{g_2(S_2-S_1)}{G_2(x_2-S_1)}\cdots\tfrac{g_{t-1}(S_{t-1}-S_{t-2})}{G_{t-1}(x_{t-1}-S_{t-2})}dS_1\cdots dS_{t-1}$$

It is obvious that:

$$\left[G_t(x_t-S_{t-1})\,G_{t-1}\left(x_{t-1}-S_{t-2}\right)\ldots G_2(x_2-S_1)\,G_1(x_1)\right]^2\leq$$

$$\left[G_t(x_t-S_{t-1})\,G_{t-1}\left(x_{t-1}-S_{t-2}\right)\ldots G_2(x_2-S_1)\,G_1(x_1)\right]$$

because $G_i(x)$ *are distribution functions.*

If $H^{*t}(\mathbf{X})<1$ *then we always can find at least one vector* $\left(S_1^*,S_2^*,\ldots,S_{t-1}^*\right)$ *for which:*

$$G_t(x_t-S^*{}_{t-1})\,G_{t-1}\left(x_{t-1}-S_{t-2}^*\right)\ldots G_2(x_2-S_1^*)\,G_1(x_1)<1$$

and:

$$\int_0^{x_1}\int_{S_1}^{x_2}\cdots\int_{S_{t-2}}^{x_{t-1}}\tfrac{g_1(S_1)}{G_1(x_1)}\tfrac{g_2(S_2-S_1)}{G_2(x_2-S_1)}\cdots\tfrac{g_{t-1}(S_{t-1}-S_{t-2})}{G_{t-1}(x_{t-1}-S_{t-2})}$$

$$dS_1\cdots dS_{t-1}=1$$

then at least for $\left(S_1^*, S_2^*, \ldots, S_{t-1}^*\right)$:

$$\left[G_t\left(x_t - S^*{}_{t-1}\right) G_{t-1}\left(x_{t-1} - S_{t-2}^*\right) \ldots G_2\left(x_2 - S_1^*\right) G_1\left(x_1\right)\right]^2 <$$

$$\left[G_t\left(x_t - S^*{}_{t-1}\right) G_{t-1}\left(x_{t-1} - S_{t-2}^*\right) \ldots G_2\left(x_2 - S_1^*\right) G_1\left(x_1\right)\right]$$

finally:

$$E\left\{\left(\mathcal{H}^{*t}(\mathbf{X})\right)^2\right\} = \int_0^{x_1} \int_{S_1}^{x_2} \cdots \int_{S_{t-2}}^{x_{t-1}} G_t^2\left(x_t - S_{t-1}\right) G_{t-1}^2\left(x_{t-1} - S_{t-2}\right)$$

$$\ldots G_2^2\left(x_2 - S_1\right) G_1^2\left(x_1\right) \tfrac{g_1(S_1)}{G_1(x_1)} \tfrac{g_2(S_2 - S_1)}{G_2(x_2 - S_1)} \cdots \tfrac{g_{t-1}(S_{t-1} - S_{t-2})}{G_{t-1}(x_{t-1} - S_{t-2})} dS_1 \cdots dS_{t-1} <$$
$$\int_0^{x_1} \int_{S_1}^{x_2} \cdots \int_{S_{t-2}}^{x_{t-1}} G_t\left(x_t - S_{t-1}\right) G_{t-1}\left(x_{t-1} - S_{t-2}\right) \ldots G_2\left(x_2 - S_1\right) G_1\left(x_1\right)$$
$$\tfrac{g_1(S_1)}{G_1(x_1)} \tfrac{g_2(S_2 - S_1)}{G_2(x_2 - S_1)} \cdots\cdots \tfrac{g_{t-1}(S_{t-1} - S_{t-2})}{G_{t-1}(x_{t-1} - S_{t-2})} dS_1 \cdots dS_{t-1} = H^{*t}(\mathbf{X})$$

and:

$$Var\left\{\mathcal{H}^{*t}(\mathbf{X})\right\} = E\left\{\left(\mathcal{H}^{*t}(\mathbf{X})\right)^2\right\} - \left[H^{*t}(\mathbf{X})\right]^2 < H^{*t}(\mathbf{X}) - \left[H^{*t}(\mathbf{X})\right]^2$$

Q.E.D.

4. Sample mean estimator

Let us define now this new estimator as a sample mean of $\mathcal{H}^{*t}(\mathbf{X})$:

$$\aleph^{*t}(\mathbf{X}, n) = \frac{\sum_{i=1}^{n} \mathcal{H}_i^{*t}(\mathbf{X})}{n} = \frac{\sum_{i=1}^{n} G_t\left(x_t - S_{t-1}^i\right) G_{t-1}\left(x_{t-1} - S_{t-2}^i\right) \ldots G_2\left(x_2 - S_1^i\right) G_1\left(x_1\right)}{n} \tag{4.1}$$

using (2.1), where S_j^i are random numbers generated from the p.d.fs. (2.2) for $i = 1, \ldots, n$ and $j = 1, \ldots, t-1$:

$$S_1^i \longrightarrow \mathfrak{D}_1^i\left(S_1^i\right) = \frac{g_1\left(S_1^i\right)}{G_1\left(x_1\right)} \qquad S_1^i \in [0, x_1]$$

$$S_j^i \longrightarrow \mathfrak{D}_j^i\left(S_j\right) = \frac{g_j\left(S_j^i - S_{j-1}^i\right)}{G_j\left(x_j - S_{j-1}^i\right)} \qquad S_j^i \in \left[S_{j-1}^i, x_j\right] \qquad j > 1$$

and $\left\{\mathcal{H}_i^{*t}(\mathbf{X})\right\}_{i=1}^n$ is a sample of independent estimators $\mathcal{H}^{*t}(\mathbf{X})$

As a sample mean of an unbiased estimator, $\aleph^{*t}(\mathbf{X},n)$ is also unbiased and consistent with variance bounds (in the non trivial case):

$$Var\left\{\aleph^{*t}(\mathbf{X},n)\right\} = \frac{Var\left\{\mathcal{H}^{*t}(\mathbf{X})\right\}}{n} < \frac{H^{*t}(\mathbf{X}) - \left[H^{*t}(\mathbf{X})\right]^2}{n} \quad (4.2)$$

$$Var\left\{\aleph^{*t}(\mathbf{X},n)\right\} \leq \frac{\prod_{i=1}^{t} G_i(x_i)\, H^{*t}(\mathbf{X}) - \left[H^{*t}(\mathbf{X})\right]^2}{n}$$

under fairly general conditions that include $\int_{\Re} \varphi^4(\mathbf{X}) < \infty$ [6] :

$$\lim_{n\to\infty} \quad \aleph^{*t}(\mathbf{X},n) \longrightarrow N\left[H^{*t}(\mathbf{X}), \sqrt{\frac{Var\left\{\mathcal{H}^{*t}(\mathbf{X})\right\}}{n}}\right] \quad (4.3)$$

and the interval estimation with a confidence level 1-α is:

$$\left[\aleph^{*t}(\mathbf{X},n) \mp \phi(1-\alpha)\sqrt{\frac{Var\left\{\mathcal{H}^{*t}(\mathbf{X})\right\}}{n}}\right] \quad (4.4)$$

we can use an estimator of the variance of the $\mathcal{H}^{*t}(\mathbf{X})$:

$$Var\left\{\mathcal{H}^{*t}(\mathbf{X})\right\} \simeq k = \frac{1}{n-1}\left(\sum_{i=1}^{n}\left(\mathcal{H}_i^{*t}(\mathbf{X})\right)^2 - n\sum_{i=1}^{n}\left(\mathcal{H}_i^{*t}(\mathbf{X})\right)\right) \quad (4.5)$$

as recommended in [6, pg. 68], k is a strongly consistent estimator of $Var\left\{\mathcal{H}^{*t}(\mathbf{X})\right\}$.

then an asymptotically valid confidence interval can be:

$$\left[\aleph^{*t}(\mathbf{X},n) \mp \phi(1-\alpha)\sqrt{\frac{k}{n}}\right] \quad (4.6)$$

We can also avoid the use of an estimator for the variance substituting the result of Theorem 2 into (4.4) and get a broader confidence interval :

$$\left[\aleph^{*t}(\mathbf{X},n) \mp \phi(1-\alpha)\sqrt{\frac{\mathcal{T}}{n}}\right]$$

where:

$$\mathcal{T} = \prod_{i=1}^{t} G_i(x_i)\, \aleph^{*t}(\mathbf{X}) - \left[\aleph^{*t}(\mathbf{X})\right]^2$$

5. Some comments about the use of the estimator $\aleph^{*t}(\mathrm{X})$

The first aspect that should be considered is the true variance reduction, proved in theorem 3, that could be achieved using the simple estimator $\mathcal{H}^{*t}(\mathrm{X})$ compared with the straight simulation. The result of Theorem 2 (3.1) is certainly an upper bound for the variance of the estimator, but when the values of vector $X = (x_1, x_2, \ldots, x_t)$ are large the true reduction of the variance will stay hidden. For this last reason we consider more proper the use of the estimated variance (4.5) in order to assess the true variance reduction achieved.

One of the main features of this method is the fact that if we increase the dimension t, we do not have to start again the simulation process as should be done in direct simulation. When we get the estimator from (5) and store these pairs of values:

$$\left(\mathcal{H}_i^{*t}(\mathrm{X}), S_{t-1}^i\right) \qquad i = 1, \ldots, n$$

then:

$$\aleph^{*t+1}(\mathrm{X},n) = \frac{\sum_{i=1}^{n} \mathcal{H}_i^{*t+1}(\mathrm{X})}{n} =$$

$$= \frac{\sum_{i=1}^{n} G_{t+1}\left(x_{t+1}-S_t^i\right)G_t\left(x_t-S_{t-1}^i\right)G_{t-1}\left(x_{t-1}-S_{t-2}^i\right)\ldots G_2\left(x_2-S_1^i\right)G_1(x_1)}{n} =$$

$$= \frac{\sum_{i=1}^{n} G_{t+1}\left(x_{t+1}-S_t^i\right)\mathcal{H}_i^{*t}(\mathrm{X})}{n}$$

where using (2.2) :

$$S_t^i \longrightarrow \mathfrak{D}_t^i(S_t) = \frac{g_t\left(S_t^i - S_{t-1}^i\right)}{G_t\left(x_t - S_{t-1}^i\right)} \qquad S_t^i \in \left[S_{t-1}^i, x_t\right] \qquad i = 1, \ldots, n$$

and $\mathrm{X} = (x_1, x_2, \ldots, x_t, x_{t+1})$.

This last result means that increasing one unit the dimension of the multiple integral only imply generating n random numbers more and the total amount of random numbers required is n$(t-1)$, where t is the dimension considered. The save of number of steps – random numbers in our case – become even more obvious when we need to evaluate the integral (1.1) $H^{*t}(\mathrm{X})$ for $t = 1, 2, \ldots, k$, one by one up to a certain integer k, as the convolutions in the solution of a renewal equation, evaluation of compound processes, solving Fredholm equations of the second kind using Neumann series, in these cases the total amount of steps still remains $n(t-1)$.

There is another fact that could make this method appealing, if we want to approximate $H^{*t}(\mathbf{X})$ for different values of the last component of the vector $\mathbf{X}$ $(\mathbf{x}_t)$, it is not necessary to start another simulation again. We only need to evaluate again the values $G_t\left(x_t - S_{t-1}^i\right) \quad i = 1, \ldots, n$ for the new x_t.

6. Applications to Risk Theory

The integral of expression 1.1 is frequently found in Risk Theory. Two examples could be the n-fold convolutions of compound processes that model the Total Claims or the Ultimate Non-ruin Probability and discrete time non-ruin probability.

Confidence intervals were obtained using 4.6 with significance level $\alpha = 0.01$ and n=5,000.

6.1. Compound processes.

Compound processes are a very appropriate example to implement the advantanges of the estimator $\aleph^{*t}(\mathbf{X},n)$ described in section 5.

The infinte sums were calculated up to a certain number of terms (lim) for which the rest of the terms of the sums were smaller than 10^{-10}. Then the total amount of random number used to get all the convolutions up to *lim* will be n(lim-1), as it was stated in section 5.

The advantage compared with straight simulation is outstanding for two reasons. First the save of random numbers that in the case of straight simulation would be $\frac{(Lim-1)Lim}{2}n$ and second the reduction in the variance guaranteed by Theorem 3. In the cases studied, this approach could be considered reliable, even compared with deterministic numerical methods, because the number of steps–random numbers in our case–is not very large (see tables I, II and III).

Ultimate non-ruin probability in the classical case

The probability of ultimate survival could be written as a compound process:

$$\Phi(U) = \sum_{t=0}^{\infty} \frac{\theta}{1+\theta} \left(\frac{1}{1+\theta} \right)^t G^{*t}(U) \qquad (6.1)$$

(see for example [10, Theorem 11.4.5]) where $G^{*t}(U)$ is the t-fold convolution of a random variable with p.d.f.:

$$g(x) = \frac{1 - F(x)}{p_1}$$

and $F(x)$ is the c.d.f. of the claim size , $E_f\{x\} = p_1$ and θ the security loading.

The t-fold convolution $G^{*t}(U)$ could be expressed in terms of (1.1) and approximated using (5) and (4.6) :

$$G^{*t}(U) = H^{*t}(\mathbf{X}) \simeq \frac{\sum_{i=1}^{n} G\left(U - S_{t-1}^i\right) G\left(U - S_{t-2}^i\right) \dots G\left(U - S_1^i\right) G(U)}{n}$$

where $\mathbf{X} = \{U, U, \dots, U\}$ and $G_i(x) = G(x) \quad i = 1, 2, \dots, n.$

In Tables I and II this approach was tested respectively for exponential($p_1 = 1$) and Pareto $\left(F(x) = 1 - \left(\frac{\mu}{\mu+x} \right)^{\mu+1} \quad \mu = 1 \right)$ claim sizes and different values of θ and the initial reserves. The random numbers were generated by the inverse method from (2.2).

Total claims compound process

The total claims are frequently modeled using compound processes:

$$T(x) = \sum_{t=0}^{\infty} P_t F^{*t}(x) \qquad x \geq 0$$

where :

– P_t :Probability of finding exactly t claims in the period considered.

– $F^{*t}(x)$: t-fold convolution of the claim size distribution.

As the t-fold convolutions are multiple integrals of the type (1.1), the estimator (5) could be used as an approximation.

Table III shows some results for a Poisson Compound process and exponential claims with different values for the expected number of claims(λ) and the expected claim size(p_1).

6.2. Finite time survival probability.

We will consider now the survival probability for finite(t) and discrete time.

Let us define the discrete stochastic process of the discount accumulated claims in the present moment $\{S_t\}$:

$$S_t = \sum_{i=1}^{t} z_i = \sum_{i=1}^{t} d_i y_i$$

where:

– $y_i \geq 0 \quad i = 1, \ldots, t$ are the total claims of the i-th period with p.d.f. $f_i(x)$, c.d.f. $F_i(x)$ and $E_{f_i}\{x\} = \mu_i$, assuming they are uniformly distributed over the period considered , they will be paid in the mid-point of this period.

– $d_i = \left(\frac{1}{(1+I_1)}\right)^{-\frac{1}{2}} \prod_{j=2}^{i} \left(\frac{1}{(1+I_j)}\right) \quad j = 1, \ldots, i$ the discount factor to the present moment for the amounts of the i-th period and I_j the rate of interest of the j-th period.

The c.d.fs. $G_i(\omega)$ and p.d.fs. $g_i(\omega)$ could be expressed:

$$\begin{aligned} G_i(\omega) &= P\{d_i y_i \leq \omega\} = P\left\{y_i \leq \frac{\omega}{d_i}\right\} = F_i\left(\frac{\omega}{d_i}\right) \\ g_i(\omega) &= f_i\left(\frac{\omega}{d_i}\right)\frac{1}{d_i} \end{aligned} \tag{6.2}$$

and $E_{g_i}\{\omega\} = d_i \mu_i$

Then the probability of non-ruin for t-1 periods and that the accumulated discount claims up to t are less than x is:

$P\{S_i \leq R_0 + P_i \quad i = 1, ..., t-1; S_t \leq x\} =$

$$= \int_0^{x_1} \int_{S_1}^{x_{21}} \cdots \int_{S_{t-2}}^{x_{t-1}} G_t(x_t - S_{t-1})\, g_{t-1}(S_{t-1} - S_{t-2}) \ldots$$

$$\cdots g_2(S_2 - S_1)\, g_1(S_1)\, dS_1 \cdots dS_{t-1} \tag{6.3}$$

(see for example [?] pg. 137 for a similar expression for a non-financial model (I=0))

where:

– R_0 : Initial reserves.

– $P_i = \sum_{j=1}^{i} d_j (1+\theta_j)\mu_j$: Accumulated discount premiums upto the i-th period.

It is clear that (6.3) is a multiple integral of the kind of (1.1) where $\mathbf{X}=\{R_0+P_1,\ldots,R_0+P_{t-1},x\}$and will be approximated using (5):

$$P\{S_i \leq R_0+P_i \quad i=1,...,t-1; S_t \leq x\} \simeq \aleph^{*t}(\mathbf{X},n) =$$

$$= \frac{\sum_{i=1}^{n} G_t\left(x-S_{t-1}^i\right) G_{t-1}\left(R_0+P_{t-1}-S_{t-2}^i\right) \ldots G_2\left(R_0+P_2-S_1^i\right) G_1\left(R_0+P_1\right)}{n}$$

and from (2.2) and (6.2) :

$$S_1^i \longrightarrow \mathfrak{D}_1^i\left(S_1^i\right) = \frac{g_1\left(S_1^i\right)}{G_1\left(R_0+P_1\right)} = \frac{f_1\left(\frac{S_1^i}{d_1}\right)\frac{1}{d_1}}{F_1\left(\frac{R_0+P_1}{d_1}\right)} \quad S_1^i \in [0, R_0+P_1]$$

$$S_j^i \longrightarrow \mathfrak{D}_j^i\left(S_j\right) = \frac{g_j\left(S_j^i-S_{j-1}^i\right)}{G_j\left(x_j-S_{j-1}^i\right)} = \frac{f_j\left(\frac{S_j^i-S_{j-1}^i}{d_j}\right)\frac{1}{d_j}}{F_j\left(\frac{R_0+P_j-S_{j-1}^i}{d_j}\right)}$$

$$S_j^i \in \left[S_{j-1}^i, R_0+P_j\right] \quad j>1$$

One fequently used model in Risk Theory is a simplication of the former one where:

$$F_i(x)=F(x) \qquad \theta_i=\theta \quad I_i=I=0 \quad \forall i \tag{6.4}$$

as in the paper of De Vylder and Goovaerts(1988) [5]. This probability of non-ruin could be expressed with (1.1) $H^{*t}(\mathbf{X})$ where :

$$\mathbf{X}=(R_0+c, R_0+2c,\ldots,R_0+tc) \qquad c=\mu(1+\theta)$$

In Tables IV and V, we obtained confidence intervals for different initial reserves and securirty loadings.

The values of the variance reduction using this approach compared with the straight simulation are shown in the column V.R.P(Variance reduction percentage) of the tables:

$$V.R.P. = \frac{Direct\ simulation\ variance - method\ variance}{Direct\ simulation\ variance} \times 100$$

The amount of random numbers is 5,000(t-1)— $n=5,000$ — and were generated from (2.2) using the cutpoint method ([6], [2]).

In the context and the assumptions of the paper by DeVylder and Goovaerts 6.4, our approach cannot compete with the method designed by these two authors, but when the model is generalized and the deterministic methods (often based in restrictive assumptions as stationarity) are very hard to find, this approach could certainly be interesting to consider instead of the straight simulation.

As an illustration, Table VI was calculated introducing a constant rate of interest I=0.06 in the non-financial model 6.4 and time span 10.

It is important to state the significant percentage of reduction of the variance(53-98%) add to the fact that we can get the severity of the ruin without restarting the simulation again(as explained in the last paragraph of Section 5)

7. Conclusions.

The discrete time non-ruin probability and the distribution of the severity of ruin are very difficult to obtain when the model considered includes different rates of interes, annual distributions of the total claims and security loadings for each period. The direct simulation is sometimes the only method available to approximate these probabilities.

The variance reduction achieved with estimator $\aleph^{*t}(\mathrm{X},n)$ 5 compared with direct simulation was proved in Theorem 3 and tested with examples (Tables IV, V and VI). The significant percentage of reduction of the variance(53-98%) add to the fact that we can get the severity of the ruin without restarting the simulation again(as explained in the last paragraph of Section 5) make this method clearly better than the straight simulation.

Finally, in the context on discrete time, this method also allows increasing the number of periods considered without restarting the simulations again.

Due to the this last property of estimator $\aleph^{*t}(\mathrm{X},n)$ the method was also used in approximations of t-fold convolutions in the context of Compound processes (Tables I, II and III) obtaining narrow confidence intervals with significance level $\alpha = 0.01$.

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Table I.

Confidence intervals for survival probability. $\alpha = 0.01$

$$\Phi(U) = \sum_{t=0}^{\infty} \frac{\theta}{1+\theta} \left(\frac{1}{1+\theta}\right)^t G^{*t}(U)$$

Pareto Claim size $\quad F(x) = 1 - \left(\frac{\mu}{\mu+x}\right)^{\mu+1} \quad \mu = 1$

Security Loading	initial reserves(U)	lower limit	upper limit	number of terms of the sum
	20	0.4994	0.5058	38
	100	0.8305	0.8369	89
$\theta = 0.10$	500	0.9742	0.9756	189
	1000	0.9878	0.9889	226
	20	0.7532	0.7585	35
	100	0.9469	0.9492	70
$\theta = 0.25$	500	0.9913	0.9915	100
	1000	0.9956	0.9959	103
	20	0.8792	0.8823	30
	100	0.9766	0.9775	50
$\theta = 0.50$	500	0.9957	0.9959	57
	1000	0.9979	0.9979	57
	20	0.9230	0.9249	27
	100	0.9853	0.9857	39
$\theta = 0.75$	500	0.9972	0.9973	41
	1000	0.9986	0.998	42
	20	0.9448	0.9460	24
	100	0.9890	0.9979	32
$\theta = 1.00$	500	0.9979	0.9979	34
	1000	0.9989	0.9989	34

Table II

Confidence intervals for survival probability. $\alpha = 0.01$

$$\Phi(U) = \sum_{t=0}^{\infty} \tfrac{\theta}{1+\theta} \left(\tfrac{1}{1+\theta}\right)^t G^{*t}(U)$$

Exponential claim size. $F(x) = 1 - e^{-\frac{1}{p_1}x}$ $p_1 = 1$

Security Loading	initial reserves(U)	lower limit	upper limit	number of terms
	1	0.1694	0.1704	13
	5	0.4181	0.4251	21
$\theta = 0.10$	10	0.6304	0.6385	28
	80	0.9993	0.994	115
	1	0.3441	0.3457	12
	5	0.7019	0.7096	20
$\theta = 0.25$	10	0.8885	0.8942	30
	80	0.9999	0.9999	93
	1	0.5216	0.5232	11
	5	0.8703	0.8758	20
$\theta = 0.50$	10	0.9748	0.9772	27
	80	0.9999	0.9999	57
	1	0.6270	0.6284	11
	5	0.9323	0.9358	19
$\theta = 0.75$	10	0.9915	0.9926	25
	80	0.9999	0.9999	42
	1	0.6961	0.6974	11
	5	0.9575	0.9601	18
$\theta = 1.00$	10	0.9963	0.9969	23
	80	0.9999	0.9999	34

Table III

Confidence intervals for total claims Poisson Compound Process. $\alpha = 0.01$

$$T(x) = \sum_{t=0}^{\infty} \frac{e^{-\lambda}\lambda^t}{t!} F^{*t}(x) \qquad x \geq 0$$

Exponential claim size. $F(x) = 1 - e^{-\frac{1}{p_1}x}$

Parameters	x	number of terms	lower limit	upper limit
	1	6	0.386040	0.386049
	10	9	0.529791	0.530424
	100	14	0.975472	0.977482
	150	14	0.995754	0.996594
	1	7	0.162362	0.162413
	10	12	0.392771	0.395827
	100	18	0.995111	0.996465
	150	19	0.999834	0.999437
	1	7	0.148860	0.148873
	10	10	0.268473	0.269497
	100	16	0.912056	0.918233
	150	17	0.978257	0.981244

Table IV

Confidence intervals for discrete time non-ruin probability. $\alpha = 0.01$

Time span t=10. Poisson number of claims($\lambda = 1$). Exponential claim size $p_1 = 1$

Variance reduction percentage

$$V.R.P. = \frac{Direct\ simulation\ variance - method\ variance}{Direct\ simulation\ variance} \times 100$$

Classical case I=0

Initial reserves	security loading	V.R.P.	lower limit	upper limit
	0.05	97	0.2841	0.2885
	0.10	97	0.3106	0.3150
	0.15	97	0.3374	0.3417
U=0	0.20	97	0.3624	0.3670
	0.25	97	0.3892	0.3938
	0.30	97	0.4141	0.4188
	1.00	98	0.6980	0.7013
	0.05	94	0.4312	0.4378
	0.10	94	0.4590	0.4657
	0.15	95	0.4884	0.4949
U=20	0.20	95	0.5177	0.5241
	0.25	95	0.5460	0.5524
	0.30	95	0.5688	0.5751
	1.00	97	0.8115	0.8152
	0.05	78	0.9687	0.9733
	0.10	73	0.9695	0.9742
	0.15	75	0.9738	0.9780
U=200	0.20	80	0.9792	0.9826
	0.25	77	0.9808	0.9843
	0.30	78	0.9831	0.9863
	1.00	86	0.9974	0.9982

Table V

Confidence intervals for discrete time non-ruin probability. $\alpha = 0.01$

Time span t=100. Poisson number of claims($\lambda = 1$). Exponential claim size $p_1 = 1$

Variance reduction percentage

$$V.R.P. = \frac{\text{Direct simulation variance} - \text{method variance}}{\text{Direct simulation variance}} \times 100$$

Classical case I=0

Ininitial reserves	security loading	V.R.P.	lower limit	upper limit
	0.05	90	0.1164	0.1221
	0.10	91	0.1572	0.1635
	0.15	92	0.2046	0.2112
U=0	0.20	92	0.2483	0.2553
	0.25	93	0.2912	0.2982
	0.30	93	0.3330	0.3399
	0.05	84	0.1882	0.1972
	0.10	84	0.2412	0.2509
	0.15	86	0.3012	0.3111
U=20	0.20	87	0.3649	0.3748
	0.25	88	0.4163	0.4261
	0.30	89	0.4656	0.4750
	0.05	55	0.6693	0.6871
	0.10	53	0.7490	0.7657
U=200	0.15	53	0.8164	0.8312
	0.20	53	0.8612	0.8741
	0.25	55	0.9068	0.9174

Table VI

Confidence intervals for discrete time non-ruin probability. $\alpha = 0.01$

Time span t=10. Poisson number of claims($\lambda = 1$). Exponential claim size $p_1 = 1$

Variance reduction percentage

$$V.R.P. = \frac{Direct\ simulation\ variance - method\ variance}{Direct\ simulation\ variance} \times 100$$

Rate of Interest I=0.06

Ininitial reserves	security loading	V.R.P.	lower limit	upper limit
U=0	0.05	97	0.3203	0.3249
	0.10	97	0.3468	0.3515
	0.15	97	0.3734	0.3782
	0.20	97	0.4014	0.4062
	0.25	97	0.4255	0.4302
	0.30	97	0.4515	0.4562
	1.00	98	0.7149	0.7180
U=20	0.05	94	0.4905	0.4976
	0.10	94	0.5181	0.5251
	0.15	94	0.5453	0.5522
	0.20	94	0.5744	0.5810
	0.25	94	0.5943	0.6009
	0.30	95	0.6159	0.6223
	1.00	97	0.8300	0.8335
U=200	0.05	70	0.9875	0.9907
	0.10	77	0.9910	0.9933
	0.15	75	0.9916	0.9940
	0.20	75	0.9926	0.9948
	0.25	82	0.9946	0.9962