A solution to a slightly subcritical elliptic problem with non-power nonlinearity

Mónica Clapp*, Rosa Pardo†, Angela Pistoia ‡, and Alberto Saldaña§

June 30, 2020

Abstract

We consider a slightly subcritical Dirichlet problem with a non-power nonlinearity in a bounded smooth domain. For this problem, standard compact embeddings cannot be used to guarantee the existence of solutions as in the case of power-type nonlinearities. Instead, we use a Ljapunov-Schmidt reduction method to show that there is a positive solution which concentrates at a non-degenerate critical point of the Robin function. This is the first existence result for this type of generalized slightly subcritical problems.

Keywords: blow-up solutions, critical Sobolev exponent, Ljapunov-Schmidt reduction

MSC2020: 35B44, 35B33, 35J60.

1 Introduction

We consider the problem

$$\begin{cases} -\Delta u = f_\varepsilon(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, and

$$f_\varepsilon(u) := \frac{|u|^{2^*-2}u}{[\ln(\mathrm{e} + |u|)]^\varepsilon}, \quad \varepsilon \geq 0. \tag{1.2}$$

Here, $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent.

If $\varepsilon = 0$, then (1.1) is called the pure critical problem. In this case, the existence of solutions is strongly affected by the geometry of the domain. Indeed, Pohozaev's identity [20] ensures the non-existence of solutions in star-shaped domains, while the existence of a positive solution was established by Bahri and Coron [2] in a domain with non-trivial topology.

*M. Clapp was partially supported by UNAM-DGAPA-PAPIIT grant IN100718 and CONACYT grant A1-S-10457 (Mexico).

†R. Pardo was partially supported by Unión Iberoamericana de Universidades (UIU) ref. UCM-04-2019, by grants MTM2016-75465-P, and PID2019-103860GB-I00, MICINN, Spain, and by UCM-BSCH, Spain, GR58/08, Grupo 920894.

‡A. Pistoia was partially supported by Fondi di Ateneo "Sapienza" Universit'a di Roma (Italy) and project Vain-Hopes within the program VALERE: VAnviteLli pEr la RicErca.

§A. Saldaña was partially supported by the Alexander von Humboldt Foundation (Germany).

Most of the analysis of slightly subcritical problems has been focused on power-type nonlinearities ($|u|^{2^*-2-\varepsilon}u$ instead of $f_\varepsilon(u)$). However, if one considers problems with other types of subcritical behavior (such as (1.1) with (1.2)), then many of the techniques developed for the power nonlinearity cannot be applied anymore. For example, one cannot use directly the compactness of Sobolev embeddings to guarantee the convergence of Palais-Smale sequences associated to (1.1). Another well-known approach to find solutions of elliptic problems is to find a uniform a priori bound and establish an existence result using Leray-Schauder degree theory. However, if $\varepsilon > 0$, then the existence of a uniform a priori bound for the L^∞-norm of all positive solutions to the problem (1.1) is, in general, not known. The classical results of Gidas and Spruck [10] and de Figueiredo, Lions, and Nussbaum [8] do not apply to this problem. In this direction, some progress has been made recently. In [5], Castro and Pardo obtained a priori bounds for nonlinearities including (1.2) with $\varepsilon > \frac{2}{N-2}$, which are not covered by [8, 10]. The arguments rely on the moving plane method (providing uniform a priori bounds in a neighborhood of the boundary), the Pohozaev identity, $W^{1,q}$ regularity for $q > N$, and Morrey's theorem. Using the Kelvin transform, they extend the existence of uniform a priori bounds to non-convex domains, see [5, 6]. These results are, however, only available for $\varepsilon > \frac{2}{N-2}$, and do not include slightly subcritical problems.

We believe that the study of problems such as (1.1) with (1.2), for $\varepsilon > 0$ small, improves our understanding of more general subcritical problems and helps to develop more flexible and powerful tools in nonlinear analysis.

In this paper, we establish the existence of a solution to (1.1) which blows-up at a point in Ω when $\varepsilon \to 0$. First, let us introduce the so-called standard bubbles

$$U(y) := \alpha_N \frac{1}{(1+|y|^2)^{\frac{N-2}{2}}}, \qquad U_{\delta,\xi}(x) = \delta^{-\frac{N-2}{2}} U\left(\frac{x-\xi}{\delta}\right), \quad \delta > 0,\ \xi \in \mathbb{R}^N, \tag{1.3}$$

where $\alpha_N = [N(N-2)]^{\frac{N-2}{4}}$. Next, let G be the Green function of $-\Delta$ in Ω with Dirichlet boundary condition, and let H be its regular part, *i.e.*,

$$G(x,y) = c_N \left(\frac{1}{|x-y|^{N-2}} - H(x,y)\right), \qquad x, y \in \Omega,$$

where $c_N = \frac{1}{(N-2)\omega_N}$ and ω_N denotes the surface area of the unit sphere in $\mathbb{R}^N$. The function $\varrho : \Omega \to \mathbb{R}$ given by

$$\varrho(x) := H(x,x)$$

is called the Robin function. Our main result is the following one.

Theorem 1.1. *Let $\xi^* \in \Omega$ be a non-degenerate critical point of the Robin function. Then, there exists a solution to (1.1) which blows up at ξ^* as $\varepsilon \to 0$. More precisely, there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, there is a solution $u_\varepsilon \in H_0^1(\Omega)$ of (1.1) of the form*

$$u_\varepsilon = U_{\delta(\varepsilon),\xi(\varepsilon)} + \Phi_\varepsilon, \quad \textit{where} \quad \int_\Omega |\nabla \Phi_\varepsilon|^2 = \mathcal{O}\left(\frac{\varepsilon}{|\ln \varepsilon|}\right) \tag{1.4}$$

and with $U_{\delta(\varepsilon),\xi(\varepsilon)}$ as in (1.3). The concentration parameter $\delta(\varepsilon)$ and the blow-up point $\xi(\varepsilon)$ satisfy

$$\delta(\varepsilon)\left(\frac{|\ln \varepsilon|}{\varepsilon}\right)^{\frac{1}{N-2}} \to d > 0 \quad \textit{and} \quad \xi(\varepsilon) \to \xi^* \quad \textit{as } \varepsilon \to 0.$$

This seems to be the first existence result for problem (1.1) when $\varepsilon > 0$ is arbitrarily small. We point out that in any domain Ω the Robin function has at least one critical point which is a minimum point (since it tends to infinity at $\partial\Omega$) and also that the minimum value is strictly positive. Moreover, Micheletti and Pistoia in [14, Theorem 1.1] proved that, for almost every domain Ω, the Robin function is a Morse function, i.e., all its critical points are non-degenerate. It is also known that the origin is a non-degenerate critical point of the Robin function of a smooth bounded domain of $\mathbb{R}^N$ which is symmetric with respect to the origin and convex in any direction $x_1, \dots, x_N$, as proved by Grossi in [11].

Theorem 1.1 is shown using the Ljapunov-Schmidt reduction method. One of the advantages of this approach is that we obtain explicit information about the behavior of the solution. In particular, it is interesting to compare the blow-up rate of the solution u_ε given by Theorem 1.1 ($\|u_\varepsilon\|_\infty \sim (|\ln\varepsilon|\varepsilon^{-1})^{\frac{1}{2}}$) with the blow-up rate $\varepsilon^{-\frac{1}{2}}$ associated with the usual power nonlinearity, as shown by Bahri and Rey [3].

Theorem 1.1 is a first step towards establishing existence and multiplicity of positive and/or sign-changing solutions to problem (1.1) which blow up and/or blow down at different points in Ω as $\varepsilon \to 0$. This is motivated by a series of results which have been obtained in the last decades in the subcritical regime with power-type nonlinearities, namely, when the nonlinear term $f_\varepsilon(u)$ is replaced by $|u|^{2^*-2-\varepsilon}u$ with $\varepsilon > 0$. In this case the compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*-\varepsilon}(\Omega)$ yields the existence of a least-energy solution to

$$\begin{cases} -\Delta u = |u|^{2^*-2-\varepsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

by standard variational methods. Han in [12] proved that as $\varepsilon \to 0$ this solution blows up at a point $\xi_0 \in \Omega$ and its limit profile is a rescaling of the standard bubble (1.3). Flucher and Wei in [9] proved that ξ_0 is the minimum of the Robin function. The existence of positive solutions of (1.5) which blow up at different points in Ω was studied by Bahri, Li, and Rey in [3] using a finite dimensional reduction procedure. A similar argument was used by Bartsch, Micheletti, and Pistoia in [4] to prove the existence of sign-changing solutions of (1.5) which blow up or blow down at different points in Ω. In both cases the location of the blow-up and blow-down point is given in terms of a reduced energy which involves the Green and the Robin function. Finally, we also recall that sign-changing solutions of (1.5), which blow up and down at the same point (sometimes called the nodal towering point) have been found by Pistoia and Weth in [18]. In particular, Musso and Pistoia in [16] proved that any non-degenerate critical point of the Robin function is a nodal towering point.

We conjecture that similar results as those obtained in [4, 18] can be extended to problem (1.1) with (1.2), however the proof requires some careful estimates to overcome the essential technical difficulties due to the strong nonlinearity (1.2).

To close this introduction, we mention that in [7], Damascelli and Pardo found a priori bounds for the p-Laplacian version of (1.1). Furthermore, the existence of uniform a priori bounds and, thus, of a positive solution to the Hamiltonian elliptic system

$$-\Delta u = v^p/[\ln(e+v)]^{\varepsilon_1}, \quad -\Delta v = u^q/[\ln(e+u)]^{\varepsilon_2}, \quad \text{in } \Omega, \qquad u = v = 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

with $\min\{\varepsilon_1, \varepsilon_2\} > 2/(N-2)$, and p, q lying in the critical Sobolev hyperbola $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$, was treated by Mavinga and Pardo in [13].

The paper is organized as follows. In Section 2 we include some notation and well-known results regarding the Ljapunov-Schmidt method. In Section 3 we structure the finite dimensional reduction for the problem, some of the results in this section are well known, but we include a proof for clarity and completeness. Finally, in Section 4, we find a critical point of the reduced problem. We close the paper with an appendix containing some useful estimates associated to the nonlinearity (1.2).

2 Preliminaries

Consider the Hilbert space $D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N, \mathbb{R}^N)\}$ with its usual inner product and norm,

$$\langle u, v\rangle := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v, \qquad \|u\| := \left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{1/2}.$$

It is well known that the standard bubbles

$$U(y) := \alpha_N \frac{1}{(1+|y|^2)^{\frac{N-2}{2}}}, \qquad U_{\delta,\xi}(x) = \delta^{-\frac{N-2}{2}} U\left(\frac{x-\xi}{\delta}\right), \quad \delta > 0,\ \xi \in \mathbb{R}^N, \tag{2.1}$$

are the only positive solutions of the problem

$$-\Delta u = |u|^{2^*-2}u, \qquad u \in D^{1,2}(\mathbb{R}^N), \tag{2.2}$$

where $\alpha_N = [N(N-2)]^{\frac{N-2}{4}}$. They satisfy

$$\|U_{\delta,\xi}\|^2 = |U_{\delta,\xi}|_{2^*}^{2^*} = S^{\frac{N}{2}},$$

where S is the best constant for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and $|\cdot|_p$ is the standard L^p-norm.

Set $p := 2^* - 1$. It is well known that the kernel of the linearized operator $-\Delta - pU^{p-1}\mathtt{I}$, *i.e.*, the space of solutions to the problem

$$-\Delta\psi = pU^{p-1}\psi, \qquad \psi \in D^{1,2}(\mathbb{R}^N), \tag{2.3}$$

is generated by the $N+1$ functions

$$\begin{aligned} \psi^0(y) &:= \frac{N-2}{2}\alpha_N \frac{|y|^2-1}{(1+|y|^2)^{N/2}}, \\ \psi^j(y) &:= (N-2)\alpha_N \frac{y_j}{(1+|y|^2)^{N/2}}, \quad j = 1, \dots, N. \end{aligned} \tag{2.4}$$

Set

$$\begin{aligned} \psi^0_{\delta,\xi}(x) &:= \delta^{-\frac{N-2}{2}}\psi^0\left(\frac{x-\xi}{\delta}\right) = \frac{N-2}{2}\alpha_N\, \delta^{\frac{N-2}{2}} \frac{|x-\xi|^2-\delta^2}{(\delta^2+|x-\xi|^2)^{N/2}}, \\ \psi^j_{\delta,\xi}(x) &:= \delta^{-\frac{N-2}{2}}\psi^j\left(\frac{x-\xi}{\delta}\right) = (N-2)\alpha_N\, \delta^{\frac{N}{2}} \frac{x_j-\xi_j}{(\delta^2+|x-\xi|^2)^{N/2}}, \quad j = 1, \dots, N. \end{aligned} \tag{2.5}$$

In particular,

$$\psi^0_{\delta,\xi} = \delta \frac{\partial U_{\delta,\xi}}{\partial \delta}, \qquad \psi^j_{\delta,\xi} = \delta \frac{\partial U_{\delta,\xi}}{\partial \xi_j}, \quad j = 1, \dots, N.$$

Note that, as $U_{\delta,\xi}$ solves (2.2), any solution ψ to (2.3) satisfies

$$\int_{\mathbb{R}^N} U^p_{\delta,\xi}\psi = \int_{\mathbb{R}^N} \nabla U_{\delta,\xi} \cdot \nabla \psi = p \int_{\mathbb{R}^N} U^p_{\delta,\xi}\psi.$$

In particular,

$$\langle U_{\delta,\xi}, \psi^j_{\delta,\xi} \rangle = \int_{\mathbb{R}^N} U^p_{\delta,\xi}\psi^j_{\delta,\xi} = 0 \qquad \forall j = 0, \dots, N. \tag{2.6}$$

We denote by $P : D^{1,2}(\mathbb{R}^N) \to H^1_0(\Omega)$ the orthogonal projection, *i.e.*, PW is the unique solution to the problem

$$-\Delta(PW) = -\Delta W \quad \text{in } \Omega, \qquad PW = 0 \quad \text{on } \partial\Omega.$$

Next, we collect some well known estimates.

Lemma 2.1. *The following expansions hold true*

$$PU_{\delta,\xi} = U_{\delta,\xi} - \alpha_N \delta^{\frac{N-2}{2}} H(\,\cdot\,,\xi) + \mathcal{O}(\delta^{\frac{N+2}{2}}), \tag{2.7}$$

$$P\psi^0_{\delta,\xi} = \psi^0_{\delta,\xi} - \frac{N-2}{2}\alpha_N \delta^{\frac{N-2}{2}} H(\cdot,\xi) + \mathcal{O}(\delta^{\frac{N+4}{2}}), \tag{2.8}$$

$$P\psi^j_{\delta,\xi} = \psi^j_{\delta,\xi} - \alpha_N \delta^{\frac{N}{2}} \partial_{\xi_j} H(\cdot,\xi) + \mathcal{O}(\delta^{\frac{N+2}{2}}) \quad \textit{for } j = 1, \dots, N, \tag{2.9}$$

as $\delta \to 0$ uniformly with respect to ξ in compact subsets of Ω. Moreover,

$$|P\psi^j_{\delta,\xi} - \psi^j_{\delta,\xi}|_{\frac{2N}{N-2}} = \begin{cases} \mathcal{O}(\delta^{\frac{N-2}{2}}) & \textit{if } j = 0, \\ \mathcal{O}(\delta^{\frac{N}{2}}) & \textit{if } j = 1, \dots, N, \end{cases} \tag{2.10}$$

and

$$\langle P\psi^i_{\delta,\xi}, P\psi^j_{\delta,\xi} \rangle = \begin{cases} c_i(1 + o(1)) > 0 & \textit{if } i = j \\ o(1) & \textit{if } i \neq j. \end{cases} \tag{2.11}$$

Proof. For the proof of (2.7), (2.8), and (2.9), see [21, Proposition 1]. Then (2.10) follows from (2.8) and (2.9). For (2.11) we argue as in [17, Lemma 3.1]. Observe that, by (2.8) and (2.9),

$$\langle P\psi^i_{\delta,\xi}, P\psi^j_{\delta,\xi} \rangle = \int_\Omega f_0'(U_{\delta,\xi})\psi^i_{\delta,\xi} P\psi^j_{\delta,\xi} = \int_\Omega f_0'(U_{\delta,\xi})\psi^i_{\delta,\xi}\psi^j_{\delta,\xi} + o(1),$$

for $i, j = 0, \dots, N$ as $\delta \to 0$. Then, changing variables,

$$\int_\Omega f_0'(U_{\delta,\xi})\psi^i_{\delta,\xi}\psi^j_{\delta,\xi} = (2^* - 1) \int_{\mathbb{R}^N} |U|^{2^*-2}\psi^i\psi^j + o(1),$$

and (2.11) follows by oddness, because

$$\psi^j(y) = \begin{cases} \frac{N-2}{2}\alpha_N \frac{|y|^2-1}{(1+|y|^2)^{N/2}}, & j = 0, \\ (N-2)\alpha_N \frac{y_j}{(1+|y|^2)^{N/2}}, & j = 1, \dots, N. \end{cases}$$

□

We set

$$K_{\delta,\xi} := \mathrm{span}\{P\psi^j_{\delta,\xi} : j = 0,\dots,N\},$$
$$K^{\perp}_{\delta,\xi} := \{\phi \in H^1_0(\Omega) : \langle \phi, P\psi^j_{\delta,\xi}\rangle = 0,\ j = 0,\dots,N\},$$

and denote by

$$\Pi_{\delta,\xi} : H^1_0(\Omega) \to K_{\delta,\xi} \qquad \text{and} \qquad \Pi^{\perp}_{\delta,\xi} : H^1_0(\Omega) \to K^{\perp}_{\delta,\xi}$$

the orthogonal projections.

3 The finite dimensional reduction

To prove our main result we apply the well-known Ljapunov-Schmidt reduction procedure; see [19] and the references therein for a detailed discussion of this approach.

Let $i^* : L^{\frac{2^*}{2^*-1}}(\Omega) \to H^1_0(\Omega)$ be the adjoint operator of the embedding $i : H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$, *i.e.*, $i^*[v]$ is the unique solution to the problem

$$-\Delta u = v \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega.$$

It is well known that i^* is a continuous map and

$$\|i^*(v)\| \le c|v|_{\frac{2N}{N+2}} \text{ for any } v \in L^{\frac{2^*}{2^*-1}}(\Omega). \tag{3.1}$$

Then, problem (1.1) can be restated as

$$\begin{cases} u = i^*[f_\varepsilon(u)], \\ u \in H^1_0(\Omega). \end{cases} \tag{3.2}$$

Let $[0, \delta_N]$ be the largest interval in which the function $\delta \mapsto \delta^{N-2}|\ln\delta|$ is strictly increasing and, for $\delta \in (0, \delta_N)$ and $d \in (0,\infty)$ define

$$\varepsilon = \varepsilon(d,\delta) := d\,\delta^{N-2}|\ln\delta|. \tag{3.3}$$

For suitable $(d,\xi) \in (0,\infty)\times\Omega$ and ε small enough, we look for a solution to the problem (3.2) having the form

$$PU_{\delta,\xi} + \phi \qquad \text{with } \ \phi \in K^{\perp}_{\delta,\xi},$$

where δ and ε are related by (3.3).

Remark 3.1. Let $d_0 > 1$, $d \in (d_0^{-1}, d_0)$, $\varepsilon \in (0,1)$, and

$$\delta = \delta(d,\varepsilon) = \left(d\frac{\varepsilon}{|\ln\varepsilon|}\right)^{\frac{1}{N-2}}. \tag{3.4}$$

Then,

$$\delta^{N-2}|\ln\delta| = \frac{d}{N-2}\frac{\left|\ln\left(d\frac{\varepsilon}{|\ln\varepsilon|}\right)\right|}{|\ln\varepsilon|}\,\varepsilon = \kappa_{\varepsilon,d}\,\varepsilon, \tag{3.5}$$

where

$$\kappa_{\varepsilon,d} := \frac{d}{N-2}\frac{\left|\ln\left(d\frac{\varepsilon}{|\ln\varepsilon|}\right)\right|}{|\ln\varepsilon|} = \frac{d}{N-2}\frac{|\ln(d)+\ln(\varepsilon)-\ln|\ln\varepsilon||}{|\ln\varepsilon|}$$
$$= \frac{d}{N-2}\left|1-\frac{\ln(d)}{|\ln\varepsilon|}+\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|}\right| = \frac{d}{N-2}(1+o(1))$$

as $\varepsilon\to 0$. In particular, there is $\kappa_{\varepsilon,d}$ bounded away from zero and infinity such that the rate (3.4) satisfies (3.5).

Note that $PU_{\delta,\xi}+\phi$ satisfies (3.2) if and only if the following two identities hold true:

$$\Pi^{\perp}_{\delta,\xi}\big(PU_{\delta,\xi}+\phi-i^*[f_\varepsilon(PU_{\delta,\xi}+\phi)]\big)=0, \tag{3.6}$$
$$\Pi_{\delta,\xi}\big(PU_{\delta,\xi}+\phi-i^*[f_\varepsilon(PU_{\delta,\xi}+\phi)]\big)=0. \tag{3.7}$$

First, we show that, for any $(d,\xi)\in(0,\infty)\times\Omega$ and every ε small enough, there exists a unique $\phi\in K^{\perp}_{\delta,\xi}$ which satisfies (3.6). To this end, we consider the linear operator $L_{\delta,\xi}:K^{\perp}_{\delta,\xi}\to K^{\perp}_{\delta,\xi}$ given by

$$L_{\delta,\xi}(\phi):=\phi-\Pi^{\perp}_{\delta,\xi}i^*[f_0'(U_{\delta,\xi})\phi].$$

Proposition 3.2. *For any $\delta_0>0$ and for any compact subset D of Ω there exists $C>0$ such that, for every $\xi\in D$ and $\delta\in(0,\delta_0)$,*

$$\|L_{\delta,\xi}(\phi)\|\geq C\|\phi\| \qquad \text{for all } \phi\in K^{\perp}_{\delta,\xi}, \tag{3.8}$$

and the operator $L_{\delta,\xi}:K^{\perp}_{\delta,\xi}\to K^{\perp}_{\delta,\xi}$ is invertible.

Proof. For sake of completeness, we give a sketch of the proof which can also be found in [15, Lemma 1.7]. We argue by contradiction and suppose there exist sequences $\delta_n\to 0$, $\xi_n\to\xi\in\Omega$, and $\phi_n,z_n\in K^{\perp}_{\delta_n,\xi_n}$ such that $\|\phi_n\|=1$, $\|z_n\|\to 0$, and $z_n=L_{\delta_n,\xi_n}(\phi_n)$. In particular, there exists $w_n\in K_{\delta_n,\xi_n}$ such that

$$\int_\Omega \nabla\phi_n\nabla\varphi=\int_\Omega f_0'(U_{\delta_n,\xi_n})\phi_n\varphi+\int_\Omega\nabla(z_n+w_n)\nabla\varphi \text{ for any } \varphi\in H^1_0(\Omega). \tag{3.9}$$

First of all, we claim that $\|w_n\|\to 0$. Let, $w_n=\sum_{j=0}^N c_n^j P\psi^j_{\delta_n,\xi_n}$. By (2.11), $\|w_n\|=\sum_{j=0}^N|c_n^j|(1+o(1))$ and, by (3.9),

$$\begin{aligned}\|w_n\|^2&=\underbrace{\langle\phi_n-z_n,w_n\rangle}_{=0}-\int_\Omega f_0'(U_{\delta_n,\xi_n})\phi_n w_n\\&=-\sum_{j=0}^N c_n^j\underbrace{\int_\Omega f_0'(U_{\delta_n,\xi_n})\phi_n\psi^j_{\delta_n,\xi_n}}_{=\langle\phi_n,P\psi^j_{\delta_n,\xi_n}\rangle=0}-\sum_{j=0}^N c_n^j\int_\Omega f_0'(U_{\delta_n,\xi_n})\phi_n\left(P\psi^j_{\delta_n,\xi_n}-\psi^j_{\delta_n,\xi_n}\right)\\&\leq\sum_{j=0}^N|c_n^j|\,\big|f_0'(U_{\delta_n,\xi_n})\big|_{\frac{N}{2}}\underbrace{\left|P\psi^j_{\delta_n,\xi_n}-\psi^j_{\delta_n,\xi_n}\right|_{\frac{2N}{N-2}}}_{=o(1)}|\phi_n|_{\frac{2N}{N-2}}\end{aligned}$$

$$= o(\|w_n\|)$$

and the claim follows.

Now, we set $\tilde{h}(y) := \delta_n^{\frac{N-2}{2}} h(\delta_n y + \xi_n)$, for $y \in \Omega_n := \frac{\Omega - \xi_n}{\delta_n}$, so $|\nabla \tilde{h}|_2 = |\nabla h|_2$ and $|\tilde{h}|_{\frac{2N}{N-2}} = |h|_{\frac{2N}{N-2}}$. Then, by (3.9),

$$\int_{\Omega_n} \nabla \tilde{\phi}_n \nabla \tilde{\varphi} = \int_{\Omega_n} f_0'(U) \tilde{\phi}_n \tilde{\varphi} + \int_{\Omega_n} \nabla (\tilde{z}_n + \tilde{w}_n) \nabla \tilde{\varphi} \text{ for any } \tilde{\varphi} \in C_0^\infty(\mathbb{R}^N). \tag{3.10}$$

Now, up to a subsequence $\tilde{\phi}_n \to \tilde{\phi}$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, $\tilde{z}_n, \tilde{w}_n \to 0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, and from (3.10) we get that $\phi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ solves

$$-\Delta \tilde{\phi} = f_0'(U) \tilde{\phi} \text{ in } \mathbb{R}^N.$$

Moreover, since for any j

$$0 = \int_{\Omega} \nabla \tilde{\phi}_n \nabla P \psi^j_{\delta_n, \xi_n} = \int_{\Omega} f_0'(U_{\delta_n, \xi_n}) \tilde{\phi}_n \psi^j_{\delta_n, \xi_n} = \int_{\Omega_n} f'(U) \tilde{\phi}_n \psi^j \to \int_{\mathbb{R}^N} f'(U) \tilde{\phi} \psi^j,$$

we get $\tilde{\phi} = 0$.
On the other hand, testing (3.9) by ϕ_n and scaling, we have

$$1 = \int_{\Omega_n} f_0'(U) \tilde{\phi}_n^2 + \int_{\Omega_n} \nabla (\tilde{z}_n + \tilde{w}_n) \nabla \tilde{\phi}_n = o(1),$$

and a contradiction arises.

The invertibility follows from Fredholm's theory because $L_{\delta,\xi}$ is a compact perturbation of the identity. □

It is useful to recall the following well-known estimates.

Lemma 3.3.

$$\int_\Omega U^q_{\delta,\xi}(x) \, \mathrm{d}x = \begin{cases} \mathcal{O}\left(\delta^{\frac{N-2}{2} q}\right) & \text{if } 0 < q < \frac{N}{N-2}, \\ \mathcal{O}\left(\delta^{\frac{N}{2}} |\ln \delta|\right) & \text{if } q = \frac{N}{N-2}, \\ \mathcal{O}\left(\delta^{N - \frac{N-2}{2} q}\right) & \text{if } \frac{N}{N-2} < q \le 2^*, \end{cases} \tag{3.11}$$

$$\int_\Omega |\psi^0_{\delta,\xi}(x)|^q \, \mathrm{d}x = \begin{cases} \mathcal{O}\left(\delta^{\frac{N-2}{2} q}\right) & \text{if } 0 < q < \frac{N}{N-2}, \\ \mathcal{O}\left(\delta^{\frac{N}{2}} |\ln \delta|\right) & \text{if } q = \frac{N}{N-2}, \\ \mathcal{O}\left(\delta^{N - \frac{N-2}{2} q}\right) & \text{if } \frac{N}{N-2} < q \le 2^*, \end{cases} \tag{3.12}$$

and

$$\int_\Omega |\psi^j_{\delta,\xi}(x)|^q \, \mathrm{d}x = \begin{cases} \mathcal{O}\left(\delta^{\frac{N}{2} q}\right) & \text{if } 0 < q < \frac{N}{N-1}, \\ \mathcal{O}\left(\delta^{\frac{N^2}{2(N-1)}} |\ln \delta|\right) & \text{if } q = \frac{N}{N-1}, \\ \mathcal{O}\left(\delta^{N - \frac{N-2}{2} q}\right) & \text{if } \frac{N}{N-1} < q \le 2^*, \end{cases} \tag{3.13}$$

for $j = 1, \cdots, N$.

Proof. We prove the estimate (3.11). The other two are obtained in a similar way. In the following $C > 0$ denotes a constant independent of δ and ξ, not necessarily the same one. We perform the change of variable $x - \xi = \delta y$ and set $\Omega_\delta := \frac{1}{\delta}(\Omega - \xi)$. By (2.1), for $\delta \in (0, \delta_0)$ with δ_0 small enough, we obtain

$$\int_\Omega U_{\delta,\xi}^q(x)\,\mathrm{d}x = \delta^{N - \frac{N-2}{2}q} \int_{\Omega_\delta} U^q(y)\,\mathrm{d}y.$$

Assume now $0 < q < \frac{N}{N-2}$, since $1 + r^2 \geq \max\{1, r^2\}$, then

$$\begin{aligned}\int_{\Omega_\delta} U^q(y)\,\mathrm{d}y &\leq C \int_0^{c/\delta} \frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}q}}\,dr \leq C \left(\int_0^1 r^{N-1}\,dr + \int_1^{c/\delta} r^{N-1-(N-2)q}\,dr \right) \\ &\leq C\delta^{-N+(N-2)q}.\end{aligned}$$

On the other hand, if $q = \frac{N}{N-2}$, then

$$\int_{\Omega_\delta} U^q(y)\,\mathrm{d}y \leq C \int_0^{c/\delta} \frac{r^{N-1}}{(1+r^2)^{\frac{N}{2}}}\,dr \leq C \left(\int_0^1 r^{N-1}\,dr + \int_1^{c/\delta} r^{-1}\,dr \right) \leq C|\ln\delta|.$$

Finally, if $\frac{N}{N-2} < q \leq 2^*$, then

$$\int_{\Omega_\delta} U^q(y)\,\mathrm{d}y \leq C \int_0^{\infty} \frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}q}}\,dr = C.$$

This ends the proof. ∎

Lemma 3.4.

$$|f_0(PU_{\delta,\xi}) - f_0(U_{\delta,\xi})|_{\frac{2N}{N+2}} = \begin{cases} \mathcal{O}\left(\delta^{N-2}\right) & \text{if } 3 \leq N \leq 5, \\ \mathcal{O}\left(\delta^4 |\ln\delta|^{2/3}\right) & \text{if } N = 6, \\ \mathcal{O}\left(\delta^{\frac{N+2}{2}}\right) & \text{if } N \geq 7, \end{cases} \tag{3.14}$$

$$\left|f_0'(PU_{\delta,\xi}) - f_0'(U_{\delta,\xi})\right|_{\frac{N}{2}} = \begin{cases} \mathcal{O}(\delta) & \text{if } N = 3, \\ \mathcal{O}\left(\delta^2 |\ln\delta|^{1/2}\right) & \text{if } N = 4, \\ \mathcal{O}\left(\delta^2\right) & \text{if } N \geq 5, \end{cases} \tag{3.15}$$

and

(3.16)

$$\left|f_0(PU_{\delta,\xi}) - f_0(U_{\delta,\xi}) - f_0'(U_{\delta,\xi})(PU_{\delta,\xi}(x) - U_{\delta,\xi}(x))\right|_{\frac{N}{2}} = \begin{cases} \mathcal{O}\left(\delta^{\frac{N+2}{2}}\right) & \text{if } N = 3, \\ \mathcal{O}\left(\delta^{\frac{N+2}{2}} |\ln\delta|^{1/2}\right) & \text{if } N = 4, \\ \mathcal{O}\left(\delta^{\frac{N+2}{2}}\right) & \text{if } N \geq 5, \end{cases}$$

Proof. The following inequalities are well known. For any $a>0$ and $b\in\mathbb{R}$,

$$||a+b|^q-a^q|\le\begin{cases}c(q)\min\{|b|^q,a^{q-1}|b|\} & \text{if } 0<q<1,\\ c(q)\left(|b|^q+a^{q-1}|b|\right) & \text{if } q\ge 1,\end{cases}\tag{3.17}$$

and

$$\left||a+b|^q(a+b)-a^{q+1}-(1+q)a^qb\right|\le\begin{cases}c(q)\min\{|b|^{q+1},a^{q-1}b^2\} & \text{if } 0<q<1,\\ c(q)\left(|b|^{q+1}+a^{q-1}b^2\right) & \text{if } q\ge 1.\end{cases}\tag{3.18}$$

Estimates (3.14), (3.15), and (3.16) follow from these inequalities and Lemma 3.3. ☐

Lemma 3.5.

$$|f_\varepsilon\left(PU_{\delta,\xi}\right)-f_0\left(PU_{\delta,\xi}\right)|_{\frac{2N}{N+2}}=\mathcal{O}\left(\varepsilon\ln|\ln\delta|\right)\tag{3.19}$$

and

$$\left|f_\varepsilon'\left(PU_{\delta,\xi}\right)-f_0'\left(PU_{\delta,\xi}\right)\right|_{\frac{N}{2}}=\mathcal{O}\left(\varepsilon\ln|\ln\delta|\right).\tag{3.20}$$

Proof. In the following $C>0$ denotes a positive constant, independent of δ, ε, and $\xi\in(0,1)$, not necessarily the same one. We show first (3.19). By Lemma B.1 and the maximum principle,

$$\begin{aligned}|f_\varepsilon(PU_{\delta,\xi})-f_0(PU_{\delta,\xi})|&\le\varepsilon(PU_{\delta,\xi})^{2^*-1}\ln\ln(\mathrm{e}+PU_{\delta,\xi})\\&\le\varepsilon\,U_{\delta,\xi}^{2^*-1}\ln\ln(\mathrm{e}+U_{\delta,\xi}).\end{aligned}$$

Next, we scale $x-\xi=\delta y$, $y\in\Omega_\delta:=\frac{1}{\delta}(\Omega-\xi)$ and we get, for $\delta\in(0,1)$,

$$\begin{aligned}&\left(\int_\Omega\left|U_{\delta,\xi}^{2^*-1}(x)\ln\ln(\mathrm{e}+U_{\delta,\xi}(x))\right|^{\frac{2^*}{2^*-1}}\,\mathrm{d}x\right)^{\frac{2^*-1}{2^*}}\\&\qquad\le\left(\int_{\Omega_\delta}U^{2^*}(y)\left|\ln\ln\left(\mathrm{e}+\delta^{-\frac{N-2}{2}}U(y)\right)\right|^{\frac{2^*}{2^*-1}}\,\mathrm{d}y\right)^{\frac{2^*-1}{2^*}}\\&\qquad\le C\left|\ln\ln\left(\mathrm{e}+\delta^{-\frac{N-2}{2}}\alpha_N\right)\right|\le C\ln|\ln\delta|\end{aligned}$$

and (3.19) follows.

Now we show (3.20). By Lemma B.1, for ε small enough we have that

$$|f_\varepsilon'(u)-f_0'(u)|\le C\varepsilon|u|^{2^*-2}\big(\ln\ln(\mathrm{e}+|u|)+1\big).$$

Next, we scale $x-\xi=\delta y$, $y\in\Omega_\delta:=\frac{1}{\delta}(\Omega-\xi)$ and then, for $\delta\in(0,\frac{1}{2})$,

$$\begin{aligned}&\left(\int_\Omega|U_{\delta,\xi}^{2^*-2}\left(\ln\ln(\mathrm{e}+U_{\delta,\xi})+1\right)|^{\frac{2^*}{2^*-2}}\right)^{\frac{2^*-2}{2^*}}\\&\qquad=\left(\int_{\Omega_\delta}U^{2^*}(y)\left(\ln\ln(\mathrm{e}+\delta^{-\frac{N-2}{2}}U(y))+1\right)^{\frac{2^*}{2^*-2}}\,\mathrm{d}y\right)^{\frac{2^*-2}{2^*}}\\&\qquad\le C\left(\ln\ln(\mathrm{e}+\delta^{-\frac{N-2}{2}}\alpha_N)+1\right)\le C\ln|\ln\delta|\end{aligned}$$

and (3.20) follows.

☐

Proposition 3.6. *For any compact subset X of $(0,\infty)\times\Omega$ there is $\delta_0(X)=\delta_0>0$ such that, for every $(d,\xi)\in X$ and $\delta\in(0,\delta_0)$, there exists a unique $\phi=\phi_{\delta,\xi}\in K^{\perp}_{\delta,\xi}$ which solves equation (3.6) with $\varepsilon=d\,\delta^{N-2}|\ln\delta|$ and satisfies*

$$\|\phi_{\delta,\xi}\| = \begin{cases} \mathcal{O}\left(\delta^{N-2}\,|\ln\delta|\,(\ln|\ln\delta|)\right) & \text{if } 3\le N\le 6,\\ \mathcal{O}\left(\delta^{\frac{N+2}{2}}\right) & \text{if } N\ge 7.\end{cases} \tag{3.21}$$

Proof. Let $(d,\xi)\in X$, $\delta\in(0,1)$. Note that $\phi\in K^{\perp}_{\delta,\xi}$ solves (3.6) if and only if ϕ is a fixed point of the operator $T_{\delta,\xi}:K^{\perp}_{\delta,\xi}\to K^{\perp}_{\delta,\xi}$ defined by

$$\begin{aligned} T_{\delta,\xi}(\phi) := L^{-1}_{\delta,\xi}\,\Pi^{\perp}_{\delta,\xi}\, i^*\Big\{ &\left[f_\varepsilon(PU_{\delta,\xi}+\phi)-f_\varepsilon(PU_{\delta,\xi})-f'_\varepsilon(PU_{\delta,\xi})\phi\right]\\ &+\left[f'_\varepsilon(PU_{\delta,\xi})-f'_0(PU_{\delta,\xi})\right]\phi+\left[f'_0(PU_{\delta,\xi})-f'_0(U_{\delta,\xi})\right]\phi\\ &+\left[f_\varepsilon(PU_{\delta,\xi})-f_0(PU_{\delta,\xi})\right]+\left[f_0(PU_{\delta,\xi})-f_0(U_{\delta,\xi})\right]\ \Big\}.\end{aligned}$$

We shall prove that $T_{\delta,\xi}$ is a contraction in a suitable ball. Hereafter $C>0$ denotes a positive constant, independent of $(d,\xi)\in X$ and $\delta\in(0,1)$, not necessarily the same one. Proposition 3.2 and Sobolev's inequality yield

$$\begin{aligned}\|T_{\delta,\xi}(\phi)\| &\le C\left|f_\varepsilon(PU_{\delta,\xi}+\phi)-f_\varepsilon(PU_{\delta,\xi})-f'_\varepsilon(PU_{\delta,\xi})\phi\right|_{\frac{2^*}{2^*-1}}\\ &\quad+C\left|\left(f'_\varepsilon(PU_{\delta,\xi})-f'_0(PU_{\delta,\xi})\right)\phi\right|_{\frac{2^*}{2^*-1}}+C\left|\left(f'_0(PU_{\delta,\xi})-f'_0(U_{\delta,\xi})\right)\phi\right|_{\frac{2^*}{2^*-1}}\\ &\quad+C|f_\varepsilon(PU_{\delta,\xi})-f_0(PU_{\delta,\xi})|_{\frac{2^*}{2^*-1}}+C|f_0(PU_{\delta,\xi})-f_0(U_{\delta,\xi})|_{\frac{2^*}{2^*-1}}\\ &=: A_1+A_2+A_3+A_4+A_5.\end{aligned}$$

By (3.14) and (3.19),

$$|A_4|_{\frac{2N}{N+2}}+|A_5|_{\frac{2N}{N+2}}\le\mathcal{O}(R_\delta),$$

where

$$R_\delta=\begin{cases}\delta^{N-2}\,|\ln\delta|\ \ln|\ln\delta| & \text{if } 3\le N\le 6,\\ \delta^{\frac{N+2}{2}} & \text{if } N\ge 7.\end{cases}$$

Next, we estimate the other terms.

A_1) By the mean value theorem, there exists $\theta=\theta(x)\in(0,1)$ such that

$$\begin{aligned}A_1&=\left|f_\varepsilon(PU_{\delta,\xi}+\phi)-f_\varepsilon(PU_{\delta,\xi})-f'_\varepsilon(PU_{\delta,\xi})\phi\right|_{\frac{2^*}{2^*-1}}\\ &=\left|\left(f'_\varepsilon(PU_{\delta,\xi}+\theta\phi)-f'_\varepsilon(PU_{\delta,\xi})\right)\phi\right|_{\frac{2^*}{2^*-1}}.\end{aligned}\tag{3.22}$$

If $N<6$, from Lemma B.1 and Hölder's inequality,

$$\begin{aligned}A_1&\le C\left(\left||\phi|^{2^*-1}\right|_{\frac{2^*}{2^*-1}}+\left|U^{2^*-3}_{\delta,\xi}\phi^2\right|_{\frac{2^*}{2^*-1}}\right)=C\left[|\phi|^{2^*-1}_{2^*}+\left(\int_\Omega\left(U^{2^*-3}_{\delta,\xi}\phi^2\right)^{\frac{2^*}{2^*-1}}\right)^{\frac{2^*-1}{2^*}}\right]\\ &\le C\left(|\phi|^{2^*-1}_{2^*}+|U_{\delta,\xi}|^{2^*-3}_{2^*}|\phi|^2_{2^*}\right),\end{aligned}$$

while if $N=6$,

$$A_1 \le C\left(\left||\phi|^{2^*-1}\right|_{\frac{2^*}{2^*-1}} + \left|\phi^2\right|_{\frac{2^*}{2^*-1}}\right) = C\left[|\phi|_{2^*}^{2^*-1} + \left(\int_\Omega |\phi|^{2^*}\right)^{\frac{2^*-1}{2^*}}\right] = 2C\,|\phi|_{2^*}^{2^*-1}.$$

On the other hand, if $N>6$, we obtain

$$\begin{aligned} A_1 &\le C\left(\left||\phi|^{2^*-1}\right|_{\frac{2^*}{2^*-1}} + \varepsilon\left|U_{\delta,\xi}^{2^*-2}\phi\right|_{\frac{2^*}{2^*-1}}\right) = C\left[|\phi|_{2^*}^{2^*-1} + \varepsilon\left(\int_\Omega \left(U_{\delta,\xi}^{2^*-2}|\phi|\right)^{\frac{2^*}{2^*-1}}\right)^{\frac{2^*-1}{2^*}}\right] \\ &\le C\left(|\phi|_{2^*}^{2^*-1} + \varepsilon|U_{\delta,\xi}|_{2^*}^{2^*-2}|\phi|_{2^*}\right). \end{aligned}$$

Now, Sobolev's inequality gives

$$A_1 \le \begin{cases} C\left(1+\|\phi\|^{2^*-3}\right)\|\phi\|^2 & \text{if } 3\le N\le 5,\\ C\,\|\phi\|^2 & \text{if } N=6,\\ C\left(\varepsilon+\|\phi\|^{2^*-2}\right)\|\phi\| & \text{if } N\ge 7. \end{cases} \tag{3.23}$$

A_2) By Holder's inequality and (3.20),

$$\left|\left(f_\varepsilon'(PU_{\delta,\xi}) - f_0'(PU_{\delta,\xi})\right)\phi\right|_{\frac{2^*}{2^*-1}} \le \left|f_\varepsilon'(PU_{\delta,\xi}) - f_0'(PU_{\delta,\xi})\right|_{\frac{N}{2}}|\phi|_{2^*} \le C\varepsilon\ln|\ln\delta|\ \|\phi\|.$$

A_3) By Holder's inequality and (3.15),

$$\left|\left(f_0'(PU_{\delta,\xi}) - f_0'(U_{\delta,\xi})\right)\phi\right|_{\frac{2^*}{2^*-1}} \le \left|f_0'(PU_{\delta,\xi}) - f_0'(U_{\delta,\xi})\right|_{\frac{N}{2}}|\phi|_{2^*} \le \begin{cases} C\delta\,\|\phi\| & \text{if } N=3,\\ C\delta^2|\ln\delta|^{1/2}\|\phi\| & \text{if } N=4,\\ C\delta^2\|\phi\| & \text{if } N\ge 5. \end{cases}$$

Collecting all the previous estimates, we deduce that there exist $R^*>0$ and $\delta_0>0$ such that for any $\delta\in(0,\delta_0)$

$$\|T_{\delta,\xi}(\phi)\| \le R^*R_\delta \text{ for any } \phi\in B_\delta := \{\phi\in K_{\delta,\xi}^\perp : \|\phi\|\le R^*R_\delta\}.$$

Next, we show that $T_{\delta,\xi}$ is a contraction. To this end, let $\phi_1,\phi_2\in B_\delta$. We have

$$\begin{aligned} \|T_{\delta,\xi}(\phi_1) - T_{\delta,\xi}(\phi_2)\| \le{}& C\left|f_\varepsilon(PU_{\delta,\xi}+\phi_1) - f_\varepsilon(PU_{\delta,\xi}+\phi_2) - f_\varepsilon'(PU_{\delta,\xi})(\phi_1-\phi_2)\right|_{\frac{2^*}{2^*-1}} \\ &+ C\left|[f_\varepsilon'(PU_{\delta,\xi}) - f_0'(PU_{\delta,\xi})](\phi_1-\phi_2)\right|_{\frac{2^*}{2^*-1}} \\ &+ C\left|[f_0'(PU_{\delta,\xi}) - f_0'(U_{\delta,\xi})](\phi_1-\phi_2)\right|_{\frac{2^*}{2^*-1}} =: a_1+a_2+a_3. \end{aligned}$$

To estimate a_1, a_2, and a_3 we argue as we did above for A_1, A_2, and A_3.

a_1) By the mean value theorem, there exists $\theta=\theta(x)\in(0,1)$ such that

$$a_1 = \left|f_\varepsilon(PU_{\delta,\xi}+\phi_1) - f_\varepsilon(PU_{\delta,\xi}+\phi_2) - f_\varepsilon'(PU_{\delta,\xi})(\phi_1-\phi_2)\right|_{\frac{2^*}{2^*-1}}$$

$$= \left|\big(f_\varepsilon'(PU_{\delta,\xi}+\phi_\theta)-f_\varepsilon'(PU_{\delta,\xi})\big)(\phi_2-\phi_1)\right|_{\frac{2^*}{2^*-1}},$$

where $\phi_\theta := (1-\theta)\phi_1+\theta\phi_2$.

If $N<6$, from Lemma B.1 and Hölder's inequality we get

$$\begin{aligned} a_1 &\le C\left(\left||\phi_\theta|^{2^*-2}(\phi_2-\phi_1)\right|_{\frac{2^*}{2^*-1}} + \left|U_{\delta,\xi}^{2^*-3}\phi_\theta(\phi_2-\phi_1)\right|_{\frac{2^*}{2^*-1}}\right) \\ &\le C\left[|\phi_\theta|_{2^*}^{2^*-2}|\phi_2-\phi_1|_{2^*} + \left(\int_\Omega \left(U_{\delta,\xi}^{2^*-3}\phi_\theta(\phi_2-\phi_1)\right)^{\frac{2^*}{2^*-1}}\right)^{\frac{2^*-1}{2^*}}\right] \\ &\le C\left(|\phi_\theta|_{2^*}^{2^*-2}+|U_{\delta,\xi}|_{2^*}^{2^*-3}|\phi_\theta|_{2^*}\right)|\phi_2-\phi_1|_{2^*}, \end{aligned}$$

while, if $N=6$, we obtain

$$a_1 \le C\,|\phi_\theta|_{2^*}\,|\phi_2-\phi_1|_{2^*}.$$

On the other hand, if $N>6$,

$$\begin{aligned} a_1 &\le C\left(\left||\phi_\theta|^{2^*-2}(\phi_2-\phi_1)\right|_{\frac{2^*}{2^*-1}} + \varepsilon\left|U_{\delta,\xi}^{2^*-2}(\phi_2-\phi_1)\right|_{\frac{2^*}{2^*-1}}\right) \\ &\le C\left[|\phi_\theta|_{2^*}^{2^*-2}\,|\phi_2-\phi_1|_{2^*} + \varepsilon\left(\int_\Omega\left(U_{\delta,\xi}^{2^*-2}|\phi_2-\phi_1|\right)^{\frac{2^*}{2^*-1}}\right)^{\frac{2^*-1}{2^*}}\right] \\ &\le C\left(|\phi_\theta|_{2^*}^{2^*-2}+\varepsilon\right)|\phi_2-\phi_1|_{2^*}. \end{aligned}$$

Now, Sobolev's inequality gives

$$a_1 \le C\left(\|\phi_\theta\|^{2^*-2}+\max\{\|\phi_\theta\|,\varepsilon\}\right)\|\phi_2-\phi_1\|.$$

a_2) By Holder's inequality and (3.20),

$$\begin{aligned} a_2 &= \left|\big(f_\varepsilon'(PU_{\delta,\xi})-f_0'(PU_{\delta,\xi})\big)(\phi_2-\phi_1)\right|_{\frac{2^*}{2^*-1}} \\ &\le \left|f_\varepsilon'(PU_{\delta,\xi})-f_0'(PU_{\delta,\xi})\right|_{\frac{N}{2}}|\phi_2-\phi_1|_{2^*} \le C\varepsilon\ln|\ln\delta|\|\phi_2-\phi_1\|. \end{aligned}$$

a_3) By Holder's inequality and (3.15),

$$\begin{aligned} a_3 &= \left|\big(f_0'(PU_{\delta,\xi})-f_0'(U_{\delta,\xi})\big)(\phi_2-\phi_1)\right|_{\frac{2^*}{2^*-1}} \\ &\le \left|f_0'(PU_{\delta,\xi})-f_0'(U_{\delta,\xi})\right|_{\frac{N}{2}}|\phi_2-\phi_1|_{2^*} \le \begin{cases} C\delta\,\|\phi_2-\phi_1\| & \text{if } N=3, \\ C\delta^2|\ln\delta|^{1/2}\|\phi_2-\phi_1\| & \text{if } N=4, \\ C\delta^2\|\phi_2-\phi_1\| & \text{if } N\ge 5. \end{cases} \end{aligned}$$

From the above estimates we conclude that

$$\|T_{\delta,\xi}(\phi_2)-T_{\delta,\xi}(\phi_1)\| \le C\Big(R^*R_\delta+(R^*R_\delta)^{2^*-2}+\varepsilon\ln|\ln\delta|+\delta\Big)\|\phi_2-\phi_1\|.$$

Hence, there exists $\delta_0\in(0,1)$ such that $T_{\delta,\xi}: B_\delta\to B_\delta$ is a contraction for all $\delta\in(0,\delta_0)$. By Banach's fixed point theorem, $T_{\delta,\xi}: B_\delta\to B_\delta$ has a unique fixed point, as claimed. ☐

4 The finite dimensional problem

In the previous section we proved that, if $\varepsilon = d\,\delta^{N-2}|\ln\delta|$ is small enough, then, for each $d>0$ and $\xi\in\Omega$, there exists a unique $\phi=\phi_{\delta,\xi}\in K^{\perp}_{\delta,\xi}$ which solves equation (3.6), *i.e.*,

$$PU_{\delta,\xi}+\phi-i^*[f_\varepsilon(PU_{\delta,\xi}+\phi)]\in K_{\delta,\xi}.$$

Hence, there exist $c^0_{\delta,\xi},c^1_{\delta,\xi},\dots,c^N_{\delta,\xi}\in\mathbb{R}$ such that

$$PU_{\delta,\xi}+\phi-i^*[f_\varepsilon(PU_{\delta,\xi}+\phi)]=\sum_{i=0}^{N}c^i_{\delta,\xi}P\psi^i_{\delta,\xi}.$$

In order to show that $PU_{\delta,\xi}+\phi$ solves (3.2), we need to prove that it solves (3.7). That is, we need to show that there exists $d_\varepsilon>0$ and $\xi_\varepsilon\in\Omega$ such that the $c^i_{\delta_\varepsilon,\xi_\varepsilon}$'s are zero for ε small enough.

Proposition 4.1. *Let $\xi_0\in\Omega$ be a non-degenerate critical point of the Robin function ϱ_Ω. Then there exist $\xi_\varepsilon\to\xi_0$ and $\delta_\varepsilon\to0$ given by $\varepsilon=d_\varepsilon\delta_\varepsilon^{N-2}|\ln\delta_\varepsilon|$ with $d_\varepsilon\to d_0>0$ and such that*

$$PU_{\delta_\varepsilon,\xi_\varepsilon}+\phi-i^*[f_\varepsilon(PU_{\delta_\varepsilon,\xi_\varepsilon}+\phi)]=0,$$

where $\phi=\phi_{\delta_\varepsilon,\xi_\varepsilon}$ is given by Proposition 3.6.

Proof. We split the proof in two steps.

Step 1. We take the inner product of (4.1) with $P\psi^j_{\delta,\xi}$ and compute each side of the identity

$$\langle PU_{\delta,\xi}+\phi-i^*[f_\varepsilon(PU_{\delta,\xi}+\phi)],P\psi^j_{\delta,\xi}\rangle=\sum_{i=0}^{N}c^i_{\delta,\xi}\langle P\psi^i_{\delta,\xi},P\psi^j_{\delta,\xi}\rangle. \tag{4.1}$$

The left-hand side is

$$\begin{aligned}
&LHS:=\langle PU_{\delta,\xi}+\phi-i^*[f_\varepsilon(PU_{\delta,\xi}+\phi)],P\psi^j_{\delta,\xi}\rangle\\
&=\langle PU_{\delta,\xi},P\psi^j_{\delta,\xi}\rangle-\int_\Omega f_\varepsilon(PU_{\delta,\xi}+\phi)\,P\psi^j_{\delta,\xi}=\int_\Omega f_0(U_{\delta,\xi})\,P\psi^j_{\delta,\xi}-\int_\Omega f_\varepsilon(PU_{\delta,\xi}+\phi)\,P\psi^j_{\delta,\xi}\\
&=\int_\Omega\left(f_0(U_{\delta,\xi})-f_0(PU_{\delta,\xi})\right)\psi^j_{\delta,\xi}+\int_\Omega\left(f_0(U_{\delta,\xi})-f_0(PU_{\delta,\xi})\right)\left(P\psi^j_{\delta,\xi}-\psi^j_{\delta,\xi}\right)\\
&+\int_\Omega\left(f_0(PU_{\delta,\xi})-f_\varepsilon(PU_{\delta,\xi})\right)\psi^j_{\delta,\xi}+\int_\Omega\left(f_0(PU_{\delta,\xi})-f_\varepsilon(PU_{\delta,\xi})\right)\left(P\psi^j_{\delta,\xi}-\psi^j_{\delta,\xi}\right)\\
&-\int_\Omega\big(f_\varepsilon(PU_{\delta,\xi}+\phi)-f_\varepsilon(PU_{\delta,\xi})-f_\varepsilon'(PU_{\delta,\xi})\phi\big)\,P\psi^j_{\delta,\xi}-\int_\Omega\big(f_\varepsilon'(PU_{\delta,\xi})-f_0'(PU_{\delta,\xi})\big)\,\phi\,P\psi^j_{\delta,\xi}\\
&-\int_\Omega\big(f_0'(PU_{\delta,\xi})-f_0'(U_{\delta,\xi})\big)\,\phi\,P\psi^j_{\delta,\xi}-\int_\Omega f_0'(U_{\delta,\xi})\phi\,(P\psi^j_{\delta,\xi}-\psi^j_{\delta,\xi})-\underbrace{\int_\Omega f_0'(U_{\delta,\xi})\phi\,\psi^j_{\delta,\xi}}_{=0}\\
&=:I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8.
\end{aligned}$$

Next, we estimate each summand. The leading terms are I_1 and I_3.

(I_1) We have

$$\int_\Omega (f_0(U_{\delta,\xi}) - f_0(PU_{\delta,\xi}))\,\psi^j_{\delta,\xi}$$
$$= -\int_\Omega f_0'(U_{\delta,\xi})(PU_{\delta,\xi}(x) - U_{\delta,\xi}(x))\psi^j_{\delta,\xi}$$
$$- \int_\Omega \big(f_0(PU_{\delta,\xi}) - f_0(U_{\delta,\xi}) - f_0'(U_{\delta,\xi})(PU_{\delta,\xi}(x) - U_{\delta,\xi}(x))\big)\,\psi^j_{\delta,\xi}.$$

By (3.12), (3.13), and (3.16),

$$\int_\Omega \Big(f_0(PU_{\delta,\xi}) - f_0(U_{\delta,\xi}) - f_0'(U_{\delta,\xi})(PU_{\delta,\xi}(x) - U_{\delta,\xi}(x))\Big)\psi^j_{\delta,\xi}$$
$$\leq \left|\Big(f_0(PU_{\delta,\xi}) - f_0(U_{\delta,\xi}) - f_0'(U_{\delta,\xi})(PU_{\delta,\xi}(x) - U_{\delta,\xi}(x))\Big)\right|_{\frac{N}{2}} \left|\psi^j_{\delta,\xi}\right|_{\frac{N}{N-2}} = o(\delta^{N-1}).$$

Moreover,

$$-\int_\Omega f_0'(U_{\delta,\xi})(PU_{\delta,\xi}(x) - U_{\delta,\xi}(x))\psi^j_{\delta,\xi}$$
$$= p\int_\Omega U^{p-1}_{\delta,\xi}(x)\delta^{\frac{N-2}{2}}\big(\alpha_N H(x,\xi) + \mathcal{O}(\delta)\big)\psi^j_{\delta,\xi}(x)\,dx.$$

If $j = 0$, we scale $x = \xi + \delta y$ to obtain

$$p\,\alpha_N\,\delta^{\frac{N-2}{2}}\int_\Omega U^{p-1}_{\delta,\xi}(x)H(x,\xi)\psi^0_{\delta,\xi}(x)dx$$
$$= p\,\alpha_N\,\delta^{\frac{N-2}{2}}\,\delta^{N-\frac{N-2}{2}\frac{N+2}{N-2}}\int_{\frac{\Omega-\xi}{\delta}} U^{p-1}(y)H(\xi+\delta y,\xi)\psi^0(y)dy$$
$$= \alpha_N\,A\delta^{N-2}\big(H(\xi,\xi) + \mathcal{O}(\delta)\big),$$

where $A := p\int_{\mathbb{R}^N} U^{p-1}(y)\psi^0(y)dy$.

If $j = 1,\dots,N$, taking into account that $\psi^j_{\delta,\xi}(x) = \delta\partial_{\xi_j}U_{\delta,\xi}(x)$ and setting $x = \xi + \delta y$, we get

$$p\,\alpha_N\,\delta^{\frac{N-2}{2}}\int_\Omega U^{p-1}_{\delta,\xi}(x)H(x,\xi)\psi^j_{\delta,\xi}(x)dx = \alpha_N\,\delta^{\frac{N}{2}}\int_\Omega \partial_{\xi_j}U^p_{\delta,\xi}(x)H(x,\xi)dx$$
$$= \alpha_N\,\delta^{\frac{N}{2}}\left(\partial_{\xi_j}\int_\Omega U^p_{\delta,\xi}(x)H(x,\xi)dx - \int_\Omega U^p_{\delta,\xi}(x)\partial_{\xi_j}H(x,\xi)dx\right)$$
$$= \alpha_N\,\delta^{\frac{N}{2}}\left(\delta^{\frac{N-2}{2}}\partial_{\xi_j}\int_{\frac{\Omega-\xi}{\delta}} U^p(y)H(\xi+\delta y,\xi)dy - \delta^{\frac{N-2}{2}}\int_{\frac{\Omega-\xi}{\delta}} U^p(y)\partial_{\xi_j}H(\xi+\delta y,\xi)dy\right)$$
$$= \alpha_N B\delta^{N-1}\Big(\underbrace{\partial_{\xi_j}\big(H(\xi,\xi)\big) - \partial_{\xi_j}H(\xi,\xi)}_{=\frac{1}{2}\partial_{\xi_j}\varrho(\xi)} + \mathcal{O}(\delta)\Big)$$

where $B := \int\limits_{\mathbb{R}^N} U^p(y)dy$. A straightforward computation shows that $A = \frac{N-2}{2}B$ (see also Remark B.2 in [15]). Hence,

$$-\int_\Omega f_0'(U_{\delta,\xi})(PU_{\delta,\xi}(x) - U_{\delta,\xi}(x))\psi^j_{\delta,\xi} = \begin{cases} \alpha_N\, A\delta^{N-2}H(\xi,\xi) + \mathcal{O}\left(\delta^{N-1}\right) & \text{if } j=0,\\ \frac{1}{2}\alpha_N\, B\delta^{N-1}\partial_{\xi_j}\varrho(\xi) + \mathcal{O}\left(\delta^{N}\right) & \text{if } j=1,\dots,N.\end{cases}$$

Consequently,

$$I_1 = \begin{cases} \alpha_N\, A\delta^{N-2}H(\xi,\xi) + o\left(\delta^{N-2}\right) & \text{if } j=0,\\ \frac{1}{2}\alpha_N\, B\delta^{N-1}\partial_{\xi_j}\varrho(\xi) + o\left(\delta^{N-1}\right) & \text{if } j=1,\dots,N.\end{cases} \tag{4.2}$$

(I_2) By (2.10) and (3.14) we deduce

$$\begin{aligned} I_2 &= \int_\Omega \left(f_0(U_{\delta,\xi}) - f_0(PU_{\delta,\xi})\right)\left(P\psi^j_{\delta,\xi} - \psi^j_{\delta,\xi}\right)\\ &= \mathcal{O}\left(\left|f_0\left(PU_{\delta,\xi}\right) - f_0\left(U_{\delta,\xi}\right)\right|_{\frac{2N}{N+2}}\left|P\psi^j_{\delta,\xi} - \psi^j_{\delta,\xi}\right|_{\frac{2N}{N-2}}\right) = \begin{cases} o(\delta^{N-2}) & \text{if } j=0,\\ o(\delta^{N-1}) & \text{if } j=1,\cdots,N.\end{cases}\end{aligned}$$

(I_3) The proof of this estimate is long, so we postpone the details to an appendix. If $j=0$, by Lemma A.1, we obtain that

$$\int_\Omega \left(f_0(PU_{\delta,\xi}) - f_\varepsilon(PU_{\delta,\xi})\right)\psi^0_{\delta,\xi} = -\frac{2d}{N-2}\mathfrak{B}\frac{\varepsilon}{|\ln\delta|} + o\left(\frac{\varepsilon}{|\ln\delta|}\right),$$

where $\mathfrak{B} > 0$, and, if $j = 1,\dots,N$, using Remark A.2, we get

$$\int_\Omega \left(f_0(PU_{\delta,\xi}) - f_\varepsilon(PU_{\delta,\xi})\right)\psi^j_{\delta,\xi} = \begin{cases} o(\delta^{N-2}) & \text{if } 3\le N\le 4,\\ o(\delta^{N-1}) & \text{if } N\ge 5.\end{cases}$$

(I_4) By (2.10) and (3.19),

$$\begin{aligned} &\int_\Omega \left(f_\varepsilon(PU_{\delta,\xi}) - f_0(PU_{\delta,\xi})\right)\left(P\psi^j_{\delta,\xi} - \psi^j_{\delta,\xi}\right)\\ &= \mathcal{O}\left(\left|f_\varepsilon\left(PU_{\delta,\xi}\right) - f_0\left(PU_{\delta,\xi}\right)\right|_{\frac{2N}{N+2}}\left|P\psi^j_{\delta,\xi} - \psi^j_{\delta,\xi}\right|_{\frac{2N}{N-2}}\right)\\ &= \begin{cases} \mathcal{O}\left(\varepsilon\delta^{\frac{N-2}{2}}\ln|\ln\delta|\right) & \text{if } j=0,\\ \mathcal{O}\left(\varepsilon\delta^{\frac{N}{2}}\ln|\ln\delta|\right) & \text{if } j=1,\dots,N,\end{cases}\\ &= \begin{cases} o\left(\delta^{N-2}\right) & \text{if } j=0 \text{ and } 3\le N\le 4,\\ o\left(\delta^{N-1}\right) & \text{if } j=0 \text{ and } N\ge 5,\\ o\left(\delta^{N-1}\right) & \text{if } j=1,\dots,N, \text{ and } 3\le N\le 4,\\ o\left(\delta^{N}\right) & \text{if } j=1,\dots,N, \text{ and } N\ge 5.\end{cases}\end{aligned}$$

(I_5) By (2.10), (3.12), (3.13), (3.22), and (3.23),

$$\int_\Omega \left(f_\varepsilon(PU_{\delta,\xi}+\phi) - f_\varepsilon(PU_{\delta,\xi}) - f_\varepsilon'(PU_{\delta,\xi})\phi\right)P\psi^j_{\delta,\xi}$$

$$
\begin{aligned}
&= \mathcal{O}\left(\left| f_\varepsilon(PU_{\delta,\xi}+\phi) - f_\varepsilon(PU_{\delta,\xi}) - f'_\varepsilon(PU_{\delta,\xi})\phi \right|_{\frac{2N}{N+2}} \left| P\psi^j_{\delta,\xi} \right|_{\frac{2N}{N-2}} \right) \\
&\le \left.\begin{cases} C\left(1+\|\phi\|^{2^*-3}\right)\|\phi\|^2 & \text{if } 3\le N\le 5, \\ C\,\|\phi\|^2 & \text{if } N=6, \\ C\left(\varepsilon+\|\phi\|^{2^*-2}\right)\|\phi\| & \text{if } N\ge 7, \end{cases}\right\} = \begin{cases} \mathcal{O}\left(\delta^2|\ln\delta|^2(\ln|\ln\delta|)^2\right) & \text{if } N=3, \\ o\left(\delta^{N-1}\right) & \text{if } N\ge 4, \end{cases} \\
&= \begin{cases} o\left(\delta^{N-2}\right) & \text{if } N=3, \\ o\left(\delta^{N-1}\right) & \text{if } N\ge 4. \end{cases}
\end{aligned}
$$

(I_6) By (3.20) and (3.21),

$$
\begin{aligned}
&\int_\Omega \left(f'_\varepsilon(PU_{\delta,\xi}) - f'_0(PU_{\delta,\xi})\right)\phi\, P\psi^j_{\delta,\xi} \\
&= \mathcal{O}\left(\left| f'_\varepsilon\left(PU_{\delta,\xi}\right) - f'_0\left(PU_{\delta,\xi}\right) \right|_{\frac{N}{2}} |\phi|_{\frac{2N}{N-2}} \left| P\psi^j_{\delta,\xi} \right|_{\frac{2N}{N-2}} \right) \\
&= \mathcal{O}\left(\varepsilon \ln|\ln\delta|\,\|\phi\|\right) = \begin{cases} \mathcal{O}\left(\delta^2|\ln\delta|^2(\ln|\ln\delta|)^2\right) & \text{if } N=3, \\ o\left(\delta^{N-1}\right) & \text{if } N\ge 4, \end{cases} \\
&= \begin{cases} o\left(\delta^{N-2}\right) & \text{if } N=3, \\ o\left(\delta^{N-1}\right) & \text{if } N\ge 4. \end{cases}
\end{aligned}
$$

(I_7) By Hölder's inequality,

$$
I_7 = \int_\Omega \left(f'_0(PU_{\delta,\xi}) - f'_0(U_{\delta,\xi})\right)\phi\, P\psi^j_{\delta,\xi} = \mathcal{O}\Big(\big| \left(f'_0\left(PU_{\delta,\xi}\right) - f'_0\left(U_{\delta,\xi}\right)\right) P\psi^j_{\delta,\xi} \big|_{\frac{2N}{N+2}} |\phi|_{\frac{2N}{N-2}} \Big).
$$

By (3.17),

$$
|(f'_0\left(PU_{\delta,\xi}\right) - f'_0\left(U_{\delta,\xi}\right))P\psi^j_{\delta,\xi}| \le C \begin{cases} \left(\delta^2 + U^{2^*-3}_{\delta,\xi}\delta^{\frac{N-2}{2}}\right)|P\psi^j_{\delta,\xi}| & \text{if } 3\le N\le 6, \\ \min\{\delta^2, U^{2^*-3}_{\delta,\xi}\delta^{\frac{N-2}{2}}\}|P\psi^j_{\delta,\xi}| & \text{if } N\ge 7. \end{cases}
$$

Note that, by definition $|\psi^j_{\delta,\xi}| \le (N-2)|U_{\delta,\xi}|$ for $j=0,1,\dots,N$, and by the maximum principle, there is $C\ge 1$ such that

$$
|P\psi^j_{\delta,\xi}| \le C|U_{\delta,\xi}| \quad \text{in } \Omega, \qquad \text{for } j=0,1,\dots,N. \tag{4.3}
$$

Now we estimate I_7 using (4.3) and Lemma 3.3. If $N\ge 7$ (since $(2^*-2)\frac{2N}{N+2} < \frac{N}{N-2}$),

$$
I_7 = \mathcal{O}(\|\phi\|\delta^{\frac{N-2}{2}}|U^{2^*-2}_{\delta,\xi}|_{\frac{2N}{N+2}}) = \mathcal{O}(\delta^{\frac{N+2}{2}}\delta^{\frac{N-2}{2}}\delta^{\frac{N-2}{2}(2^*-2)}) = \mathcal{O}(\delta^{N+2}) = o(\delta^{N-1}),
$$

whereas, if $3\le N\le 5$ (since $(2^*-2)\frac{2N}{N+2} > \frac{N}{N-2}$),

$$
\begin{aligned}
I_7 &= \mathcal{O}\Big(\|\phi\|\delta^2|U_{\delta,\xi}|_{\frac{2N}{N+2}} + \|\phi\|\delta^{\frac{N-2}{2}}|U^{2^*-2}_{\delta,\xi}|_{\frac{2N}{N+2}}\Big) \\
&= \mathcal{O}\left(\left(\delta^{N+\frac{N-2}{2}} + \delta^{N-2+\frac{N-2}{2}}\delta^{(N-\frac{N-2}{2}(2^*-2)\frac{2N}{N+2})\frac{N+2}{2N}}\right)|\ln\delta|(\ln|\ln\delta|)\right)
\end{aligned}
$$

$$= \mathcal{O}\Big((\delta^{\frac{3N-2}{2}} + \delta^{2(N-2)})|\ln\delta|(\ln|\ln\delta|)\Big) = o(\delta^{N-1}).$$

Similarly, if $N = 6$ (since $\frac{2N}{N+2} = \frac{N}{N-2} = (2^*-2)\frac{2N}{N+2} = \frac{3}{2}$),

$$\begin{aligned} I_7 &= \mathcal{O}(\|\phi\|\delta^2|U_{\delta,\xi}|_{\frac{2N}{N+2}} + \|\phi\|\delta^{\frac{N-2}{2}}|U_{\delta,\xi}^{2^*-2}|_{\frac{2N}{N+2}}) \\ &= \mathcal{O}\left(\left(\delta^N\delta^{\frac{N+2}{4}}|\ln\delta|^{\frac{N+2}{2N}} + \delta^{N-2+\frac{N-2}{2}}\delta^{\frac{N}{2}\frac{N+2}{2N}}|\ln\delta|^{\frac{N+2}{2N}}\right)|\ln\delta|(\ln|\ln\delta|)\right) \\ &= \mathcal{O}\Big((\delta^8+\delta^8)(\ln|\ln\delta|)|\ln\delta|^{\frac{5}{3}}\Big) = o(\delta^{N-1}). \end{aligned}$$

In any case, we conclude that $I_7 = o(\delta^{N-1})$.

(I_8) If $j = 0$, by (3.11), (3.21), and (2.10),

$$\begin{aligned} &\int_\Omega f_0'(U_{\delta,\xi})\phi\,(P\psi^0_{\delta,\xi} - \psi^0_{\delta,\xi}) \\ &= \mathcal{O}\left(\left|f_0'(U_{\delta,\xi})\right|_{\frac{N}{2}}|\phi|_{\frac{2N}{N-2}}\left|P\psi^j_{\delta,\xi} - \psi^j_{\delta,\xi}\right|_{\frac{2N}{N-2}}\right) = \mathcal{O}\left(\delta^{\frac{N-2}{2}}\|\phi\|\right) \\ &= \begin{cases} \mathcal{O}\left(\delta^{\frac{3}{2}N-3}|\ln\delta|\ \ln|\ln\delta|\right) & \text{if } 3\le N\le 4, \\ o\left(\delta^{N-1}\right) & \text{if } N\ge 5, \end{cases} = \begin{cases} o\left(\delta^{N-2}\right) & \text{if } 3\le N\le 4, \\ o\left(\delta^{N-1}\right) & \text{if } N\ge 5. \end{cases} \end{aligned}$$

On the other hand, if $j = 1,\dots,N$, by (3.11), (3.21), and (2.10),

$$\begin{aligned} &\int_\Omega f_0'(U_{\delta,\xi})\phi\,(P\psi^j_{\delta,\xi} - \psi^j_{\delta,\xi}) \\ &= \mathcal{O}\left(\left|f_0'(U_{\delta,\xi})\right|_{\frac{N}{2}}|\phi|_{\frac{2N}{N-2}}\left|P\psi^j_{\delta,\xi} - \psi^j_{\delta,\xi}\right|_{\frac{2N}{N-2}}\right) = \mathcal{O}\left(\delta^{\frac{N}{2}}\|\phi\|\right) \\ &= \begin{cases} \mathcal{O}\left(\delta^{\frac{3}{2}N-2}|\ln\delta|\ \ln|\ln\delta|\right) & \text{if } 3\le N\le 4, \\ o\left(\delta^{N}\right) & \text{if } N\ge 5, \end{cases} = \begin{cases} o\left(\delta^{N-1}\right) & \text{if } 3\le N\le 4, \\ o\left(\delta^{N}\right) & \text{if } N\ge 5. \end{cases} \end{aligned}$$

Step 2. If $\varepsilon = d\delta^{N-2}|\ln\delta|$, taking into account all the previous estimates, the left hand side of equation (4.1) can be rewritten as

$$L.H.S. = \begin{cases} \delta^{N-2}F^0_\varepsilon(d,\xi) & \text{if } j = 0, \\ \delta^{N-1}F^j_\varepsilon(d,\xi) & \text{if } j\ge 1, \end{cases} \tag{4.4}$$

where $F_\varepsilon : [0,+\infty]\times\Omega\to\mathbb{R}\times\mathbb{R}^N$ is defined by

$$F^0_\varepsilon(d,\xi) := \mathfrak{A}_1\ \varrho(\xi) - \mathfrak{A}_2 d + o(1) \quad \text{and} \quad F^j_\varepsilon(d,\xi) := \mathfrak{A}_3\partial_{\xi_j}\varrho(\xi) + o(1),\ j = 1,\dots,N,$$

and the $\mathfrak{A}_i$'s are positive constants.
We remark that $F_\varepsilon \to F$ uniformly on compact sets of $[0,+\infty]\times\Omega$ where the function $F : [0,+\infty]\times\Omega\to\mathbb{R}\times\mathbb{R}^N$ is defined by

$$F(\xi,d) = (\ \mathfrak{A}_1\ \varrho(\xi) - \mathfrak{A}_2 d\ ,\ \mathfrak{A}_3\nabla\varrho(\xi)\ ).$$

Now, let ξ_0 be a non-degenerate critical point of the Robin function ϱ and let $d_0 := \frac{\mathfrak{A}_1}{\mathfrak{A}_2}\varrho(\xi_0)$. It is easy to check that (ξ_0, d_0) is an isolated zero of F whose Brouwer degree is not zero. Then, if ε is small enough, there exist $\xi_\varepsilon \to \xi_0$ and $d_\varepsilon \to d_0$ such that $F_\varepsilon(\xi_\varepsilon, d_\varepsilon) = 0$. Therefore, also the right hand side of (4.1) is zero, *i.e.*,

$$\sum_{i=0}^{N} c^i_{\delta,\xi}\langle P\psi^i_{\delta,\xi}, P\psi^j_{\delta,\xi}\rangle = 0.$$

Finally, from (2.11) we immediately deduce that all the $c^i_{\delta,\xi}$'s are zero. That concludes the proof. □

Proof of Theorem 1.1. Proposition 4.1 implies that $u_\varepsilon = PU_{\delta,\xi} + \phi_{\delta,\xi}$ is a solution of (1.1) (see equation (3.2)). Moreover, by Lemma 2.1, Proposition 4.1, and Remark 3.1, statement (1.4) holds true, where the function Φ_ε in (1.4) is given by

$$\Phi_\varepsilon = PU_{\delta_\varepsilon,\xi_\varepsilon} - U_{\delta_\varepsilon,\xi_\varepsilon} + \phi_{\delta_\varepsilon,\xi_\varepsilon}.$$

□

A Appendix: The proof of estimate I_3

This section is devoted to the proof of the following estimate.

Lemma A.1. *As $\delta \to 0$,*

$$I_3 = \int_\Omega \left(f_0(PU_{\delta,\xi}) - f_\varepsilon(PU_{\delta,\xi})\right)\, \psi^j_{\delta,\xi} = \begin{cases} -\frac{2d}{N-2}\mathfrak{B}\,\frac{\varepsilon}{|\ln\delta|} + o\big(\frac{\varepsilon}{|\ln\delta|}\big), & \text{if } j=0,\\ \mathcal{O}\big(\varepsilon\delta^{\frac{N-2}{2}}\ln|\ln\delta|\big), & \text{if } j=1,\dots,N,\end{cases}$$

where

$$\mathfrak{B} := -\int_{\mathbb{R}^N} U^p\,[\ln U]\psi^0 = \frac{\Gamma(\frac{N}{2})\pi^{\frac{N}{2}}}{4\Gamma(N+1)}N^{\frac{N}{2}}(N-2)^{\frac{N+4}{2}} > 0. \tag{A.1}$$

Remark A.2. Observe that

$$\varepsilon\delta^{\frac{N-2}{2}}\ln|\ln\delta| = d\delta^{\frac{3}{2}N-3}\,|\ln\delta|\,\ln|\ln\delta| = \begin{cases} o(\delta^{N-2}) & \text{if } 3\le N\le 4,\\ o(\delta^{N-1}) & \text{if } N\ge 5.\end{cases}$$

Proof of Lemma A.1. Taylor's expansion with respect to ε yields,

$$\begin{aligned}&\int_\Omega \big(f_0(PU_{\delta,\xi}) - f_\varepsilon(PU_{\delta,\xi})\big)\psi^j_{\delta,\xi}\\ &= \varepsilon\int_\Omega (PU_{\delta,\xi})^p\ln\ln(\mathrm{e}+PU_{\delta,\xi})\psi^j_{\delta,\xi} - \varepsilon^2\int_\Omega \frac{(PU_{\delta,\xi})^p[\ln\ln(\mathrm{e}+PU_{\delta,\xi})]^2\psi^j_{\delta,\xi}}{1+\varepsilon\ln(\mathrm{e}+PU_{\delta,\xi})},\end{aligned}$$

because, by (2.1), and Lemma B.2, it holds that

$$\left|\int_\Omega \frac{(PU_{\delta,\xi})^p(x)[\ln\ln(\mathrm{e}+PU_{\delta,\xi}(x))]^2\psi^j_{\delta,\xi}(x)}{1+\varepsilon\ln(\mathrm{e}+PU_{\delta,\xi})}\,\mathrm{d}x\right|$$

$$\leq \int_\Omega U^p_{\delta,\xi}(x)[\ln\ln(\mathrm{e}+U_{\delta,\xi}(x))]^2|\psi^j_{\delta,\xi}(x)|\,\mathrm{d}x = \mathcal{O}\big((\ln|\ln\delta|)^2\big).$$

Next, we set $g(u) := u^p \ln\ln(\mathrm{e}+u)$. Then, the mean value theorem yields

$$0 \leq g(u) - g(v) \leq C\, u^{p-1}\ (\ln\ln(\mathrm{e}+u)+1)\,[u-v], \qquad \text{if } 0 \leq v \leq u.$$

It follows that

$$\int_\Omega PU^p_{\delta,\xi}[\ln\ln(\mathrm{e}+PU_{\delta,\xi})]\psi^j_{\delta,\xi} = \int_\Omega U^p_{\delta,\xi}[\ln\ln(\mathrm{e}+U_{\delta,\xi})]\psi^j_{\delta,\xi} + \mathcal{O}\big(\delta^{\frac{N-2}{2}}\ln|\ln\delta|\big),$$

because, by (2.1), (2.5), and Lemmas B.2 and 2.1, it holds, for $\delta\in(0,\delta_0)$ small, that

$$\begin{aligned}
&\int_\Omega U^{p-1}_{\delta,\xi}(x)\big(\ln\ln(\mathrm{e}+U_{\delta,\xi}(x))+1\big)[U_{\delta,\xi}(x)-PU_{\delta,\xi}(x)]|\psi^j_{\delta,\xi}(x)|\\
&\leq C\delta^{N+\frac{N-2}{2}-\frac{N-2}{2}-\frac{N-2}{2}(p-1)}(\ln|\ln\delta|+1)\int_{\mathbb{R}^N}U^{p-1}(y)\ \big(H(\xi+\delta y,\xi)+\mathcal{O}(\delta)\big)|\psi^j(y)|\\
&= C\delta^{\frac{N-2}{2}}(\ln|\ln\delta|+1)\int_{\mathbb{R}^N}U^{p-1}(y)\ \big(H(\xi+\delta y,\xi)+\mathcal{O}(\delta)\big)|\psi^j(y)|\\
&\leq C\delta^{\frac{N-2}{2}}\ln|\ln\delta|\left(\int_{\mathbb{R}^N}U^{p-1}(y)\,|\psi^j(y)|\ \mathrm{d}y\right)\ \big(H(\xi,\xi)+\mathcal{O}(\delta)\big).
\end{aligned}$$

Now, using (2.1), (2.5), Lemma B.2, and statement (2.6), we get

$$\begin{aligned}
&\int_\Omega U^p_{\delta,\xi}(x)[\ln\ln(\mathrm{e}+U_{\delta,\xi}(x))]\psi^j_{\delta,\xi}(x)\,\mathrm{d}x\\
&= \delta^{N-\frac{N+2}{2}-\frac{N-2}{2}}\int_{\Omega_\delta}U^p(y)[\ln\ln(\mathrm{e}+\delta^{-\frac{N-2}{2}}U(y))]\psi^j(y)\,\mathrm{d}y\\
&= \ln\ln(\delta^{-\frac{N-2}{2}})\int_{\Omega_\delta}U^p(y)\psi^j(y)dy\\
&\quad+\frac{1}{|\ln\delta|}\int_{\Omega_\delta}U^p(y)\left(|\ln\delta|\ \ln\left[1+\frac{\ln(\mathrm{e}^{1-\frac{N-2}{2}|\ln\delta|}+U(y))}{\frac{N-2}{2}|\ln\delta|}\right]\right)\psi^j(y)dy.
\end{aligned}\tag{A.2}$$

Note that, for $j=1,\dots,N$, the function

$$y\mapsto\varphi(y):=U^p(y)\left(|\ln\delta|\ \ln\left[1+\frac{\ln(\mathrm{e}^{1-\frac{N-2}{2}|\ln\delta|}+U(y))}{\frac{N-2}{2}|\ln\delta|}\right]\right)\psi^j(y)$$

is odd. Hence, its integral over $\mathbb{R}^N$ is equal to zero and, by (B.4),

$$\begin{aligned}
&\int_{\mathbb{R}^N}\varphi(y)-\int_{\Omega_\delta}\varphi(y)=\int_{\mathbb{R}^N\setminus\Omega_\delta}\varphi(y)\\
&\leq C\left(|\ln\delta|\ \ln\left[1+\frac{\ln(\mathrm{e}^{1-\frac{N-2}{2}|\ln\delta|}+\alpha_N)}{\frac{N-2}{2}|\ln\delta|}\right]\right)\int_{\mathbb{R}^N\setminus\Omega_\delta}U^p(y)|\psi^j(y)|\\
&\leq C\left(\frac{2}{N-2}\ln\alpha_N+o(1)\right)\delta^{N+1}=\mathcal{O}(\delta^{N+1}).
\end{aligned}$$

Consequently,

$$\int_\Omega U^p_{\delta,\xi}(x)[\ln\ln(\mathrm{e}+U_{\delta,\xi}(x))]\psi^j_{\delta,\xi}(x)\,\mathrm{d}x = \mathcal{O}\big(\delta^{N+1}\ln|\ln(\delta)|\big) = o(\delta^N),$$

and the claim concerning $j = 1,\dots,N$ follows.

On the other hand, for $j = 0$ we also have $\int_{\mathbb{R}^N} U^p\psi^0 = 0$ (see (2.6)), and it can also be proved that $\int_{\mathbb{R}^N\setminus\Omega_\delta} U^p\psi^0 = \mathcal{O}(\delta^N)$. Then, by (A.2) and (B.4),

$$\begin{aligned}&\int_\Omega U^p_{\delta,\xi}(x)[\ln\ln(\mathrm{e}+U_{\delta,\xi}(x))]\psi^0_{\delta,\xi}(x)\,\mathrm{d}x\\ &= \frac{1}{|\ln\delta|}\,\frac{2}{N-2}\int_{\mathbb{R}^N} U^p(y)\,[\ln U(y)]\,\psi^0(y)dy + o\left(\frac{1}{|\ln\delta|}\right),\end{aligned}$$

as claimed. To finish the proof, we show (A.1). Indeed, passing to polar coordinates, integrating by parts, changing variables ($s = r^2$, $dr = \frac{1}{2}s^{-\frac{1}{2}}\,ds$), and using that $|\partial B_1(0)| = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}$,

$$\begin{aligned}\mathfrak{B} &= -|\partial B_1(0)|\frac{\alpha_N^{2^*}(N-2)}{2}\int_0^\infty \frac{r^{N-1}}{(1+r^2)^{\frac{N+2}{2}}}\ln\left(\frac{1}{(1+r^2)^{\frac{N-2}{2}}}\right)\frac{r^2-1}{(1+r^2)^{\frac{N}{2}}}\,dr\\ &= \frac{\alpha_N^{2^*}(N-2)^2}{4}\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\int_0^\infty \frac{r^{N-1}(r^2-1)}{(1+r^2)^{N+1}}\ln(1+r^2)\,dr\\ &= \frac{\alpha_N^{2^*}(N-2)^2\pi^{\frac{N}{2}}}{2\Gamma(\frac{N}{2})}\int_0^\infty \frac{1}{N}\left(\frac{r}{1+r^2}\right)^N\frac{2r}{1+r^2}\,dr\\ &= \frac{\alpha_N^{2^*}(N-2)^2\pi^{\frac{N}{2}}}{2N\Gamma(\frac{N}{2})}\int_0^\infty \frac{s^{\frac{N}{2}}}{(1+s)^{N+1}}\,ds = \frac{\alpha_N^{2^*}(N-2)^2\pi^{\frac{N}{2}}}{2N\Gamma(\frac{N}{2})}B\left(\frac{N}{2}+1,\frac{N}{2}\right),\\ &= \frac{\alpha_N^{2^*}(N-2)^2\pi^{\frac{N}{2}}}{2N\Gamma(\frac{N}{2})}\frac{\Gamma(\frac{N}{2}+1)\Gamma(\frac{N}{2})}{\Gamma(N+1)} = \frac{\Gamma(\frac{N}{2})\pi^{\frac{N}{2}}}{4\Gamma(N+1)}N^{\frac{N}{2}}(N-2)^{\frac{N+4}{2}},\end{aligned}$$

where $B(\cdot,\cdot)$ denotes the usual Beta function. □

B Appendix: Further estimates

Lemma B.1.

(i) $|f_\varepsilon(u) - f_0(u)| \le \varepsilon\,|u|^{2^*-1}\,\ln\ln(\mathrm{e}+|u|)$.

(ii) *For ε small enough, and any $u\in\mathbb{R}$,*

(B.1) $$|f'_\varepsilon(u)| \le C|u|^{2^*-2},$$

and

$$|f'_\varepsilon(u) - f'_0(u)| \le \varepsilon|u|^{2^*-2}\left((2^*-1)\ln\ln(\mathrm{e}+|u|) + \frac{1}{\ln(\mathrm{e}+|u|)}\right).$$

(iii) *There exists $C>0$ such that, for ε small enough and any $u,v\in\mathbb{R}$,*

(B.2) $$|f'_\varepsilon(u+v) - f'_\varepsilon(u)| \le \begin{cases} C(|u|^{2^*-3}+|v|^{2^*-3})|v| & \text{if } N\le 6,\\ C(|v|^{2^*-2}+\varepsilon|u|^{2^*-2}) & \text{if } N>6.\end{cases}$$

Proof. (i) : Since

$$\frac{\partial f_\varepsilon(u)}{\partial \varepsilon} = -\frac{|u|^{2^*-2}u}{[\ln(\mathrm{e}+|u|)]^\varepsilon}\, \ln\ln(\mathrm{e}+|u|),$$

we have that

$$\left|\frac{\partial f_\varepsilon(u)}{\partial \varepsilon}\right| \leq |u|^{2^*-1}\ln\ln(\mathrm{e}+|u|),$$

and the statement follows easily from the mean value theorem.

(ii) : As

$$f'_\varepsilon(u) = \frac{|u|^{2^*-2}}{[\ln(\mathrm{e}+|u|)]^\varepsilon}\left(2^*-1-\frac{\varepsilon|u|}{(\mathrm{e}+|u|)\ln(\mathrm{e}+|u|)}\right), \tag{B.3}$$

then (B.1) holds due to $0 \leq \frac{|u|}{\mathrm{e}+|u|} \leq 1$, and $0 \leq \frac{1}{\ln(\mathrm{e}+|u|)} \leq 1$.

Since (B.3), we have that

$$\begin{aligned}\frac{\partial f'_\varepsilon(u)}{\partial \varepsilon} = &-\frac{|u|^{2^*-2}\ln\ln(\mathrm{e}+|u|)}{[\ln(\mathrm{e}+|u|)]^\varepsilon}\left(2^*-1-\frac{\varepsilon|u|}{(\mathrm{e}+|u|)\ln(\mathrm{e}+|u|)}\right)\\ &-\frac{|u|^{2^*-1}}{(\mathrm{e}+|u|)[\ln(\mathrm{e}+|u|)]^{\varepsilon+1}}.\end{aligned}$$

Hence, for ε small enough,

$$\begin{aligned}\left|\frac{\partial f'_\varepsilon(u)}{\partial \varepsilon}\right| &\leq (2^*-1)\frac{|u|^{2^*-2}\ln\ln(\mathrm{e}+|u|)}{[\ln(\mathrm{e}+|u|)]^\varepsilon} + \frac{|u|^{2^*-2}}{[\ln(\mathrm{e}+|u|)]^{\varepsilon+1}}\\ &\leq |u|^{2^*-2}\left((2^*-1)\ln\ln(\mathrm{e}+|u|) + \frac{1}{\ln(\mathrm{e}+|u|)}\right).\end{aligned}$$

Now the claim follows from the mean value theorem.

(iii) : Setting $p := 2^*-1$, we see that

$$\begin{aligned}f''_\varepsilon(u) = &\frac{\varepsilon\,|u|^{2^*-3}u}{[\ln(\mathrm{e}+|u|)]^\varepsilon}\left(\frac{|u|-\mathrm{e}\ln(\mathrm{e}+|u|)}{(\mathrm{e}+|u|)^2(\ln(\mathrm{e}+|u|))^2}\right)\\ &+\frac{|u|^{2^*-4}u}{[\ln(\mathrm{e}+|u|)]^\varepsilon}\left(p-1-\frac{\varepsilon|u|}{(\mathrm{e}+|u|)\ln(\mathrm{e}+|u|)}\right)\left(p-\frac{\varepsilon|u|}{(\mathrm{e}+|u|)\ln(\mathrm{e}+|u|)}\right).\end{aligned}$$

So, for ε small enough,

$$|f''_\varepsilon(u)| \leq C|u|^{2^*-3}.$$

Since $2^*-3 \geq 0$ for $N \leq 6$, the mean value theorem yields

$$|f'_\varepsilon(u+v) - f'_\varepsilon(u)| = |f''_\varepsilon(u+tv)||v| \leq C|u+tv|^{2^*-3}|v| \leq C(|u|^{2^*-3}+|v|^{2^*-3})|v|$$

for some $t \in (0,1)$, as stated in (iii) for $N \leq 6$.

Next, assume $N > 6$. Then, $q := 2^*-2 \in (0,1)$. We write

$$|f'_\varepsilon(u+v) - f'_\varepsilon(u)| \leq p\left|\frac{|u+v|^q}{[\ln(\mathrm{e}+|u+v|)]^\varepsilon} - \frac{|u|^q}{[\ln(\mathrm{e}+|u|)]^\varepsilon}\right|$$

$$
\begin{aligned}
&+\varepsilon\left|\frac{|u+v|^p}{(\mathrm{e}+|u+v|)[\ln(\mathrm{e}+|u+v|)]^{\varepsilon+1}}-\frac{|u|^p}{(\mathrm{e}+|u|)[\ln(\mathrm{e}+|u|)]^{\varepsilon+1}}\right|\\
&=: F_1+F_2.
\end{aligned}
$$

Clearly,

$$F_2 \le C\varepsilon(|u|^q+|v|^q).$$

Now, for any fixed $v \neq 0$, setting $x := \frac{u}{v}$ we have that

$$
\begin{aligned}
&\frac{1}{|v|^q}\left|\frac{|u+v|^q}{[\ln(\mathrm{e}+|u+v|)]^{\varepsilon}}-\frac{|u|^q}{[\ln(\mathrm{e}+|u|)]^{\varepsilon}}\right|\\
&\qquad=\left|\frac{|x+1|^q}{[\ln(\mathrm{e}+|v||x+1|)]^{\varepsilon}}-\frac{|x|^q}{[\ln(\mathrm{e}+|v||x|)]^{\varepsilon}}\right| =: g(x).
\end{aligned}
$$

The function g is symmetric with respect to $-\frac{1}{2}$, *i.e.*, $g(x)=g(-x-1)$, it is increasing in $[-\frac{1}{2},0]$ and decreasing in $[0,\infty)$, and $g(0)\le 1$. Hence,

$$F_1 \le p|v|^q.$$

This proves (B.2), concluding the proof of statement (*iii*). $\square$

Lemma B.2. *Let $r>0$. Then, for any $u>0$ and $\delta\in(0,1)$,*

$$\ln\ln\left(\mathrm{e}+\delta^{-r}u\right)=\ln\ln\left(\delta^{-r}\right)+\ln\left(1+\frac{\ln(\mathrm{e}^{1-r|\ln\delta|}+u)}{r|\ln\delta|}\right),$$

and

$$\lim_{\delta\to 0}\left(|\ln\delta|\,\ln\left[1+\frac{\ln(\mathrm{e}^{1-r|\ln\delta|}+u)}{r|\ln\delta|}\right]\right)=\frac{1}{r}\ln u. \tag{B.4}$$

Proof. We have that

$$
\begin{aligned}
&\ln\ln\left(\mathrm{e}+\delta^{-r}u\right)=\ln\ln(\delta^{-r}(\delta^{r}\mathrm{e}+u))=\ln\left(\ln\delta^{-r}+\ln(\mathrm{e}^{1-r|\ln\delta|}+u)\right)\\
&=\ln\left[\ln\delta^{-r}\left(1+\frac{\ln(\mathrm{e}^{1-r|\ln\delta|}+u)}{\ln\delta^{-r}}\right)\right]\\
&=\ln\ln\left(\delta^{-r}\right)+\ln\left(1+\frac{\ln(\mathrm{e}^{1-r|\ln\delta|}+u)}{r|\ln\delta|}\right).
\end{aligned}
$$

Set

$$g(t):=\ln\left[1+\frac{t}{r}\ln(\mathrm{e}^{1-\frac{r}{t}}+u)\right],\qquad t>0.$$

Applying L'Hôpital's rule we obtain

$$
\begin{aligned}
\lim_{t\to 0}\frac{g(t)}{t}&=\lim_{t\to 0}g'(t)=\lim_{t\to 0}\frac{\frac{1}{r}\ln(\mathrm{e}^{1-\frac{r}{t}}+u)+\frac{1}{t}\mathrm{e}^{1-\frac{r}{t}}(\mathrm{e}^{1-\frac{r}{t}}+u)^{-1}}{1+\frac{t}{r}\ln(\mathrm{e}^{1-\frac{r}{t}}+u)}\\
&=\frac{1}{r}\ln u.
\end{aligned}
$$

Taking $t:=|\ln\delta|^{-1}$, we obtain the claim. $\square$

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Mónica Clapp and Alberto Saldaña
Instituto de Matemáticas,
Universidad Nacional Autónoma de México,
Circuito Exterior, Ciudad Universitaria,
04510 Coyoacán, Ciudad de México, Mexico.
`monica.clapp@im.unam.mx, alberto.saldana@im.unam.mx`

Rosa Pardo
Departamento de Análisis Matemático y Matemática Aplicada,
Facultad de Ciencias Químicas,
Universidad Complutense de Madrid,
28040 Madrid, Spain.
`rpardo@ucm.es`

Angela Pistoia
Dipartimento di Metodi e Modelli Matematici,
La Sapienza Università di Roma,
Via Antonio Scarpa 16,
00161 Roma, Italy.
`angela.pistoia@uniroma1.it`