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ALGEBRAIC CORRESPONDENCES

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0. We have developed many years ago (see the References) a theory of the algebraic correspondences. The development of the commutative and local algebra in the last twenty years permits many simplifications of that theory and to obtain new results. We present in this paper a new elaboration of the main ideas of those papers.

1. Introduction. - Let k be an arbitrary field, X_i, Y_j, $i=0, \dots, n$; $j=0, \dots, m$ two sets of independent indeterminates. Let $A^* = k[X; Y] = k[X_0, \dots, X_n; Y_0, \dots, Y_m]$ and $I^* = A^*(F_1, \dots, F_r)$ an ideal of A^*, where F_i, $i=1, \dots, r$ are homogeneous polynomials relatively to the X and relatively to the Y. The ring $P^* = A^*/I^*$ is a bihomogeneous ring. Let be $\xi_i = X_i + I^*$, $\eta_j = Y_j + I^*$, $i=0, \dots, n$; $j=0, \dots, m$, then one can put $P^* = k[\xi_0, \dots, \xi_n; \eta_0, \dots, \eta_m]$.

The rings $P = k[\xi_0, \dots, \xi_n]$ and $P' = k[\eta_0, \dots, \eta_m]$ are homogeneous rings. The bihomogeneous ring P^* is called the definition's ring of the algebraic correspondence C. Let $A = k[X_0, \dots, X_n]$, $A' = k[Y_0, \dots, Y_m]$ and let us put $I = \{f(X) \mid f(\xi) = 0\}$, $I' = \{f(Y) \mid f(\eta) = 0\}$, one has $I = I^* \cap A$, $I' = I^* \cap A'$. One can give various definitions of algebraic correspondences, the most interesting are the following ones.

Definition I. - Let X be the set of all homogeneous ideals of the ring P, P excluded. Let $\mathfrak{R}(P)$ and $\mathfrak{R}(X)$ be the lattices of the subsets of P and X respectively. One can define two maps: $\mathfrak{R}(P) \overset{V}{\rightarrow} \mathfrak{R}(X)$ and $\mathfrak{R}(X) \overset{I}{\rightarrow} \mathfrak{R}(P)$, in the following way: $\forall E \in \mathfrak{R}(P)$, $V(E) = \{I \in X \mid E \subset I\}$ and $I(F) = \{f \in P \mid f \in I, \forall I \in F\} = \bigcap_{I \in F} \mathrm{I}$. The set $\{V(E)\}_{E \in \mathfrak{R}(P)} = C$, considered as the set of closed sets of X, defines on X a weak topology T. The set X with the weak topology T is called a total algebraic variety. The elements of X are called points of the t.a.v. $V = (X, T)$.

Let V, V', V^* be the t.a.v. corresponding to P, P', P^* (see over) respectively. V and V' are subvarieties of V^*, i.e. there are injections: $V \overset{i}{\rightarrow} V^*$, $V' \overset{j}{\rightarrow} V^*$ such that $im\, i$, with the w topology defined by $i(C)$ is a topological subspace of V^* and similarly for V'.

Let x be any point of V, we shall call complete transform of x by the total algebraic correspondence V^*, and we shall denote it by $C<x>$, the following point: $C<x> = P^*x \cap P'$, (1) where P^*x is the ideal of P^* generated by the set x. Hence, C is a map of V onto V', where $C<x>$ is a unic point of V'.

DEFINITION II. - Let X be the set of all homonogeneous prime ideals of the ring P. The set $\{V(E)\}_{E \in \mathfrak{R}(P)}$ defines in this case a ZARISKI's topology T on X. The set X with the ZARISKI topology is an algebraic variety.

The elements of X are called points of the a.v. $V = (X, T)$.

The ring P of definition of the a.v. V can have zero divisors and nilpotent elements, but since in X are only the prime ideals and since each prime ideal x of X is in one to one correspondence with the prime ideals $\mathfrak{P}$ of A that contain I, and since each prime ideal that divides I, divides also $\sqrt{I}$, it follows that the set X of all the prime ideals of P is the same set X' of all prime ideals of $A/\sqrt{I}$ and X and X' have the same ZARISKI topology. This permits, if one uses this second definition of algebraic correspondences, to substitute the ring P with nilpotent elements by the reduced ring $A/\sqrt{I} \simeq P/\mathfrak{N}$ where $\mathfrak{N}$ is the nilradical of P, without nilpotent elements.

Let V, V', V^* be the a.v. corresponding to P, P', P^* respectively. V and V' are subvarieties of V^*. Let x be any point of V and let

$$(2) \qquad P^*x = \mathfrak{Q}_1 \cap \cdots \cap \mathfrak{Q}_r \cap \mathfrak{Q}_{r+1} \cap \cdots \cap \mathfrak{Q}_s \cap \mathfrak{Q}_{s+1} \cap \cdots \cap \mathfrak{Q}_t \ ,$$

be the normal decomposition of P^*x in primary ideals. Let $\mathfrak{P}_i$

be the radical of $\mathfrak{Q}_i$ and let $\mathfrak{P}_i \cap P = x$, $i = 1, \ldots, r$; $\mathfrak{P}_j \cap P \supset x$, $\mathfrak{P}_j \cap P \neq x$, $\mathfrak{P}_j \cap P \neq x_0$, where $x_0 = P(\xi_0, \ldots, \xi_n)$ is the irrelevant prime ideal of P, $j = r+1, \ldots, s$ and let $\mathfrak{P}_l \cap P = x_0$, $l = s+1, \ldots, t$. Let $\mathfrak{P}_i = \sqrt{\mathfrak{Q}_i}$, $i = 1, \ldots, t$, and $x'_i = \mathfrak{P}_i \cap P'$, then one calls transform of the point x by the algebraic correspondence defined by V^*, and it is denoted by $C[x]$, the following set of points of V' (see (2))

$$C[x] = \{x_1', \ldots, x_r'\} \ , \tag{3}$$

each point of $C[x]$ is called a component of $C[x]$.

And one calls total transform of x, and it is denoted by $C\{x\}$ the set of points:

$$C\{x\} = \{x_1', \ldots, x_r', x'_{r+1}, \ldots, x_s'\} \ . \tag{4}$$

If P^* is an integrity domain, the correspondence C is called irreducible, if not, reducible. The study of the reducible correspondences one can essentially reduce to that of the irreducible ones. Hence, if one adopts the definition II, the essential problem is the study of the irreducible correspondences.

2. IRREDUCIBLE ALGEBRAIC CORRESPONDENCES. - Les us conserve the notations of the preceding section. Let Ω, Σ and Σ' be the fraction fields of P^*, P and P', respectively. Then $\Sigma \subset \Omega$ and $\Sigma' \subset \Omega$.

Let v^* be a valuation of Ω. Let $l(\xi) = l_0 \xi_0 + \ldots + l_n \xi_n$ be a k-linear form of minimal value in v^* in the set of all k-linear forms in the ξ. $l'(\eta) = l'_0 \eta_0 + \ldots + l'_m \eta_m$ has an analogous signification. Let $G_{v^*} = \{f_{\alpha,\beta}(\xi, \eta) \in P^* \mid v^*(f_{\alpha,\beta}/l^\alpha l'^\beta) > 0\}$, where $f_{\alpha,\beta}$ is a bihomogeneous form of degree α with respect to the ξ and with degree β with respect to the η. One calls centre of v^* in P^* the bihomogeneous ideal $z(v^*) = P^*(G_{v^*})$ generated by G_{v^*}. The ideal $z(v^*)$ is prime and distinct of the irrelevant prime ideal. Let K^* be the set of all bihomogeneous elements of Ω of degrees (0,0). K^* is a subfield of Ω and hence, v^* induces a valuation v_0^* in K^*. If v_0^* is a trivial valuation of K^* one says that v^* is a trivial valuation on Ω. If v^* is not a trivial valuation, the centre of v^* is not the zero ideal. Let v and v' be the valuations induced by v^* on Σ and Σ', respectively and let $z(v)$ and $z(v')$ be its centres.

Then it is verified that $z(v) = z(v^*) \cap P$, $z(v') = z(v^*) \cap P'$. The point $z(v')$ of V' is called an homologous point of $z(v)$ in the algebraic correspondence C and denoted by $C(z(v))$. If $C[x]$ is (3) the transform of x by C, and if $x' = C(x)$ is an homologous point of x, it is verified that $x' > x_i'$ for some i (where the relation $x' > x_i'$ corresponds to the inclusion relation betwen the sets x' and x_i').

If the fields Σ and Σ' are isomorphic, the correspondence C defined by V^* is called a birational correspondence.

Other assumptions: Let trans . degree $(\Sigma : k) = r+1$, trans . degree $(\Sigma' : k) = s+1$, trans . degree $(\Omega : \Sigma) = a+1$, trans . degree $(\Omega : \Sigma') = b+1$.

One denotes by $\mathfrak{A}$ the ring of polynomials over $\Sigma : \mathfrak{A} = \Sigma[\eta_0, \ldots, \eta_m]$.

1. *There exists* $a+1$ *elements*:

$$\theta_i = \sum_{j=0}^{m} \varphi_{ij}(\xi)\, \eta_j \,, \qquad i = 0, \ldots, a, \tag{5}$$

where the φ_{ij} *are homogeneous polynomials of the same degree such that the ideal* $\mathfrak{A}(\theta_0, \ldots, \theta_a)$ *is an irrelevant ideal of* $\mathfrak{A}$. [1].

From 1. and from a lemma of I. Cohen it follows that:

2. $\mathfrak{A}$ *depends integrally of* $\Sigma\,[\theta_0, \ldots, \theta_a]$.

This proposition implies:

3. *There is an homogeneous polynomial* $H(\xi) \in P$ *such that, if*

$$\zeta_i = \frac{\theta_i}{H(\xi)} \,, \tag{6}$$

it is verified that

$$\eta_i^{\varrho_i} + c_1^{(i)}(\zeta)\, \eta_i^{\varrho_i - 1} + \ldots + c_{\varrho_i}^{(i)}(\zeta) = 0 \,, \qquad i = 0, \ldots, m, \tag{7}$$

where the $c_j^{(i)}$ *are bihomogeneous polynomials relatively to the* ξ, ζ, *of degree* i *with respect to the* ζ.

Analogously, *there are* $b+1$ *elements*:

$$\zeta_i' = \frac{\theta_i'}{H'(\eta)} \,, \qquad \theta_i' = \sum_{j=0}^{n} \varphi_{ij}'(\eta)\, \xi_j \,, \qquad i = 0, \ldots, b, \tag{8}$$

such that:

$$\xi_i^{\tau_i} + d_1^{(i)}(\zeta')\,\xi_i^{\tau_i - 1} + \dots + d_{\tau_i}^{(i)}(\zeta') = 0 \;, \qquad i = 0, \dots, n, \tag{9}$$

where the $d_j^{(i)}$ are bihomogeneous polynomials in the η, ζ' of degree i with respect to the ζ'.

NOTATIONS. - Let $P_1 = P[\zeta_0, \dots, \zeta_a]$, $P_1^* = P^*[\zeta_0, \dots, \zeta_a]$, $P_2 = P'[\zeta'_0, \dots, \zeta'_b]$, $P_2^* = P^*[\zeta'_0, \dots, \zeta'_b]$. Let $S(H, H')$ be the multiplicatively closed system generated by H and H' and let $P^*_{S(H,H')}$ be the fractions ring of P^* relatively to $S(H, H')$.

It is verified that

$$P^*_{S(H,H')} = P^*_{1S(H,H')} = P^*_{2S(H,H')} \;, \tag{10}$$

hence, *one verifies the following inclusion's relations:*

$$\begin{array}{ccccccccccccc} P & \longrightarrow & P_1 & \longrightarrow & P_1^* & \longrightarrow & P^*_{S(H,H')} & \longleftarrow & P_2^* & \longleftarrow & P_2 & \longleftarrow & P' \\ & \searrow & & & & & \uparrow & & & & & \swarrow & \\ & & & & & & P^* & & & & & & \end{array} \tag{11}$$

where $\rightarrow = \subset$. *One verifies that: a) P_1 and P_2 are pure transcendental extensions of P and P' respectively. b) P_1^* and P_2^* are integrally dependent over P_1 and P_2, respectively. c) $P^*_{S(H,H')}$ is the fractions ring of P^*, P_1^* and P_2^* relatively to the multiplicatively closed system $S(H, H')$.*

Let $x \in V$ be any point of V, let x^* be a minimal prime divisor of the ideal P^*x that lies over x. Let K and K^* be the fractions fields of P/x and P^*/x^*, respectively.

4. *Trans. degree* $(K^* : K) \geqslant a + 1$. [1].

DEFINITIONS. - One says that the point x of V is *not fundamental relatively to the component* $x^* \cap P' = x'$ of its transform $C[x]$ by the a.c. C, when trans. degree $(K^* : K) = a + 1$, in other case x *is fundamental relatively to* x'.

5. x is not fundamental relatively to $x' \Longleftrightarrow \exists H(\xi) \mid H(\xi) \notin x$. [1].

6. *If* $\{x_1^*, ..., x_r^*\}$ *are all the m.p.d. of* P^*x *that lie over* x *and if* x *is not fundamental relatively to* x_j^*, $1 \leq j \leq r$, *then* x *is not fundamental relatively to any one* x_i^*, $i=1, ..., r$. [2].

Hence one can say simply that x is fundamental or that x is not fundamental.

The a.v. V^* define also an algebraic correspondence $C^{-1}: V' \to V$ called the *inverse correspondence of* C. If x' is any point of V' and if $x_i'^*$, $i=1, ..., r'$, are all the m.p.d. of P^*x' that lie over x', one has $C^{-1}[x'] = \{x_1'^* \cap P, ..., x_{r'}'^* \cap P\}$ and $C^{-1}\{x'\} = \{x_1'^* \cap P, ..., x_{r'}'^* \cap P, ..., x_{s'}'^* \cap P\}$, where $x_{r'+1}'^* \cap P', ..., x_{s'}'^* \cap P'$ are all distinct of the irrelevant prime ideal of P'.

Definition. - If x' is a component of the transform $C[x]$ of the point x of V and if x is not fundamental by C and x' is not fundamental by C^{-1}, one say that x *is regular relatively to* x'. If x is not fundamental by C and x' is fundamental by C^{-1} one says that x *is irregular relatively to* x'.

7. *If* x *is regular relatively to* x', *there exist* $H(\xi)$ *and* $H'(\eta)$ *such that* $H(\xi)$, $H'(\eta) \notin x^*$, *where* x^* *is the m.p.d. of* P^*x *that lies over* x *and* x' *and reciprocally.* [1].

8. *If* x *is regular relatively to* x', *and* x^* *is the m.p.d. of* P^*x *that lies over* x *and* x', *then there exists a m.p.d.* x^{**} *of* $P^*_{S(H,H')}x$ *such that* x^{**} *lies over* x *and* x' *being* $H, H' \notin x^*$. *If* x^{**} *is a m.p.d. of* $P^*_{S(H,H')}x$ *that lies over* x, *then* $x' = x^{**} \cap P' \in C[x]$ *and* x *is regular relatively to* x'.

Proof. Let $P^*x = Q^* \cap ...$ be a normal decomposition of P^*x and let $x^* = \sqrt{Q^*}$. Since $S(H, H') \cap x^* = \emptyset$, it follows that $P^*_{S(H,H')}x = P^*_{S(H,H')}(P^*x) = P^*_{S(H,H')}Q^* \cap ...$ is a normal decomposition of $P^*_{S(H,H')}x$, $\sqrt{P^*_{S(H,H')}Q^*} = P^*_{S(H,H')}x^* = x^{**}$ and $x^{**} \cap P^* = x^*$.

Since $x^{**} \cap P = x$, it follows that $x^{**} \neq P^*_{S(H,H')}$ and hence $H, H' \in x^{**} \cap P^*$.

9. As a consequence of 8. and 6., the components of the transform of a regular point of V can be obtained in the way that

shows the following diagram:

$$(12)\quad \begin{array}{l} P \to P_1 = P[\zeta] \to P_1^* = P_1[\eta] \to P^*_{S(H,H')} \qquad \to P_2^* = \\ \qquad = P_2[\xi] \quad \to P_2 = P'[\zeta'] \to P. \\ x \to x_1 = P_1 x \quad \to x_1^* \qquad \to x^{**} = P^*_{S(H,H')}\, x_1^* \to x_2^* = \\ \qquad = x^{**} \cap P_2^* \to x_2 = x_2^* \cap P_2 \to x' = x_2 \cap P' \end{array}$$

where $H \notin x$, $H' \notin x'$ and x_1^* is a m.p.d. of $P_1^* x$ that lies over x.

DEFINITION. - A *normal* irreducible algebraic correspondence is an i.a.c. defined by a bihomogeneous, integrally closed, integral domain P^*.

If C is a normal i.a.c. defined by P^* then P and P' are also integrally closed.

10. *If x is a regular point by the n.i.a.c. C defined by P^* and if x' is a component of $C[x]$, it is verified that*

$$(13)\qquad \dim. x' \geq \dim. x + a - b,$$

where $\dim.\ x = $ trans. degree $(K: k) - 1$, being K the fractions field of P/x and analogously by $\dim. x'$.

PROOF. By 9. one can obtain x' according to the diagram (12). Since P and P' are integrally closed rings, its pure transcendental extensions P_1 and P_2, respectively, are also integrally closed rings. P_1^* and P_2^* are integrally dependent over P_1 and P_2, respectively. Hence one can apply the KRULL-COHEN-SEIDENBERG theorems (see f.i. [3]) and one obtains: $\dim. x_1 = \dim. x + a + 1$, $\dim. x_1^* = \dim. x_1$, $\dim. x^{**} = \dim. x_1^*$, $\dim. x_2^* = \dim. x^{**}$ and $\dim. x_2 = \dim. x_2^*$. But $\dim. x' \geqslant \dim. x_2 - (b+1)$ and therefore (13).

11. *In the case for wich $b = 0$ it follows that:*

$$(14)\qquad \dim. x' = \dim. x + a.$$

12. *If $b \neq 0$*, the equality in (13) doesn't hold, as proves the following example. Let $A^* = k[x_0, x_1, x_2; y_0, y_1, y_2]$, $A = k[x_0, x_1, x_2]$

and $A'=k[y_0, y_1, y_2]$, being the $x's$ and the $y's$ independent indeterminates. Let $I^*=A^*(x_0y_0+x_1y_1+x_2y_2)$ be the prime ideal that defines the algebraic correspondence C. Let $P^*=A^*/I^*=$ $=k[\xi_0, \xi_1, \xi_2; \eta_0, \eta_1, \eta_2]$, being $\xi_0\eta_0+\xi_1\eta_1+\xi_2\eta_2=0$. C is a n.i.a.c. It is verified that $r=s=2$ and $a=b=1$. Let $x=P(a_1\xi_0-a_0\xi_1,$ $a_2\xi_0-a_0\xi_2)$, where $a_0\neq 0$, $a_1\neq 0$, and $a_2\neq 0$, are elements of k. x is a point of V and is a regular point by C. Actually, $C[x]=P'(a_0\eta_0+a_1\eta_1+a_2\eta_2)$, $P/x\simeq k[\xi_0]$ and hence $K\simeq k(\xi_0)$. Since $x^*=P^*(a_1\xi_0-a_0\xi_1, a_2\xi_0-a_0\xi_2, a_0\eta_0+a_1\eta_1+a_2\eta_2)$, it follows that $K^*=k(\xi_0, \eta_1, \eta_2)$. Hence trans. degree $(K^*:K)=2=a+1$, that proves that x is not fundamental. Since $x'=P'(a_0\eta_0+a_1\eta_1+$ $+a_2\eta_2)$ it follows that $x^{*\prime}=P^*x'=P^*(a_0\eta_0+a_1\eta_1+a_2\eta_2)$ and $x^{*\prime}\cap P=0$. Therefore $C^{-1}[x']=0$. Then one has: trans. degree (fract. field $(P^*/x^{*\prime}):k$) = trans. degree $(k(\xi_0, \xi_1, \xi_2, \eta_0):k)=4$, and trans. degree (fract. field $(P'/x'):k$) = trans. degree $(k(\eta_1,\eta_2):k)=$ $=2$ whose difference is $2=a+1$. But dim $x'=1$, dim. $x=0$ and hence dim. $x'\neq$ dim. $x+a-b$.

13. *The transform of a regular variety is a pure variety.* [4], [2].

14. *If x is a regular point by C, x^* is a simple point of V^* that lies over x and $x'=x^*\cap P'$, then x and x' are simple points of V and V'.* [4].

15. *If $x_1, \ldots, x_k$, are points of an irreducible algebraic variety V of dimension r, if dim. $x_i\leqslant r-2$, $i=1, \ldots, k$, then there exists a point x of V such that:* 1) *x is a principal ideal.* 2) *The ideals x_i, $i=1, \ldots, k$, contains x.*

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