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SPACES OF APPROXIMATIVE MAPS, II

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Shape morphisms between compacta X and Y can be described as homotopy classes of approximative maps of X towards Y.

The notion of an approximative map was introduced by K. Borsuk in his foundational paper of shape theory [1] . This notion is strictly related to the concept of "a map towards a space" used by D.E. Christie [6] in his construction of the n-th weak homotopy groups. By an approximative map of a compactum X towards a compactum Y, which is a subset of the Hilbert cube Q, we understand a sequence of maps $f_k: X \longrightarrow Q$ satisfying the following condition:

For every neighborhood V of Y in Q the relation $f_k \simeq f_{k+1}$ in V holds for almost all k.

We shall denote this approximative map by $\underline{f}=\{f_k, X \to Y\}$.
In [9] the authors introduced two topologies on the set $A(X,Y)$ of all approximative maps of X towards Y. The first of them was induced by a metric $d: A(X,Y) \times A(X,Y) \to R$ given by

$$d(\underline{f},\underline{g})=\sup\{\text{dist.}(f_k,g_k) \mid k=1,2,\ldots\} \quad \text{for } \underline{f},\underline{g} \in A(X,Y)$$

The second one was induced by a pseudometric $d^*: A(X,Y) \times A(X,Y) \to R$ given by

$$d^*(\underline{f},\underline{g})=\inf\{\sup\{\text{dist.}(f_k,g_k) \mid k \geq k'\} \mid k'=1,2,\ldots\}.$$

The paper [9] was devoted to the study of some properties of the corresponding spaces, which were denoted by $A(X,Y)$ and $A^*(X,Y)$ respectively. In particular we associated with each map $h: Z \to A(X,Y)$, where Z is a compactum of trivial shape, an approximative map $\underline{H}=\{H_k, X \times Z \to Y\}$ in the following way. For each $z \in Z$ we have the approximative map $h(z)=\{h(z)_k, X \to Y\}$; then, we define $H_k: X \times Z \to Q$ by $H_k(x,z)=h(z)_k(x)$, for every $(x,z) \in X \times Z$. From the fact that $\underline{H}$ is an approximative map we deduced that two approximative maps lying in the same path-component of $A(X,Y)$ induce the same shape morphism.

In this paper we present an example which shows that it is not in general possible to associate, in the sense before explained, an approximative map $\underline{H}=\{H_k, X\times Z \longrightarrow Y\}$ to a continuous map $h\colon Z \longrightarrow A^*(X,Y)$ even when $Z=[0,1]$. However we still have that approximative maps lying in the same path-component of $A^*(X,Y)$ induce the same shape morphism (Proposition 1).

We also give an example showing that it is not in general true that homotopic approximative maps of X towards Y lie in the same path-component of $A^*(X,Y)$. As a consequence we have that it is not possible to represent the shape morphisms from X to Y as the path-components of $A(X,Y)$ or $A^*(X,Y)$. In a forthcoming paper the authors have defined a metric on the set of all approximative maps of X towards Y which makes this representation possible.

The reader is supposed to know the most basic facts of the theory of shape as can be found in [3], [7], [10]. All compacta are assumed to lie in the Hilbert cube, Q. We consider the usual metric on Q.

We begin with the announced example of a map $h\colon I=[0,1] \longrightarrow A^*(X,Y)$ such that the sequence of functions $H_k\colon X\times I \longrightarrow Q$ defined in the introduction is not an approximative map of $X\times I$ towards Y. This constitutes an important difference between the spaces $A(X,Y)$ and $A^*(X,Y)$.

Example 1. The Hilbert cube Q will be represented by the countable infinite product

$$Q=\prod_{i=1}^{\infty} I_i$$

where each I_i is the closed interval $[-1,1]$. Let $X=\{x_0\}$ (a point) and $Y=[-1,1]\times\{-1\}\times\{-1\}\times\{-1\}\times\ldots$. For each $k=1,2,\ldots$, consider a map $f_k:I=[0,1] \longrightarrow [-1,1]$ such that

$$f_k(t)=-1, \text{ whenever } t\notin [1/_{k+1},1/_k]$$

$$f_k(t)=1, \text{ for some } t\in [1/_{k+1},1/_k].$$

We define a function $h:I \longrightarrow A^*(X,Y)$ by

$$h(t)=\{h(t)_k,X \longrightarrow Y\}$$

where $h(t)_k(x_0)=(t,f_k(t),-1,-1,\ldots)$

for every $t\in I$ and $k=1,2,\ldots$

Then, it is easy to see that h is continuous and the associated sequence of maps $H_k:X\times I \longrightarrow Q$, $k=1,2,\ldots$, does not define an approximative map of $X\times I$ towards Y.

The approximative map $\underline{H}:X\times I \longrightarrow Y$ associated with a map $h:I \longrightarrow A(X,Y)$ was used in [9] to prove that approximative maps lying in the same path-component of $A(X,Y)$ induce the same shape morphism. We now prove in a direct way that this is also true for the space $A^*(X,Y)$.

Proposition 1. If the approximative maps $\underline{f}$ and $\underline{g}$ lie in the same path-component of $A^*(X,Y)$ then $\underline{f}$ and $\underline{g}$ induce the same shape morphism.

Proof. Let V be a compact neighborhood of Y in Q. We can assume that $V \in ANR$. Hence there is an $\varepsilon>0$ such that any two ε-close maps from an arbitrary space into V are homotopic. Take an index $k_0 \geq 1$ such that

$f_k \simeq f_{k+1}$ in V for every $k \geq k_0$
and
$g_k \simeq g_{k+1}$ in V for every $k \geq k_0$.

By hipothesis, there exists a map $h: I \longrightarrow A^*(X,Y)$ such that $h(0)=\underline{f}$ and $h(1)=\underline{g}$. So, for each $t \in I$ we have an approximative map $h(t)=\{h(t)_k, X \longrightarrow Y\}$. Then, there exists a $\delta>0$ such that

$$|t-t'|<\delta \text{ in I implies } d^*(h(t),h(t'))<\varepsilon.$$

Consider a finite ordered family of numbers $0=t_1<t_2<\ldots<t_{n-1}<t_n=1$ such that $t_{i+1}-t_i<\delta$, for $i=1,2,\ldots,n-1$. Since $h(t_i)$ and $h(t_{i+1})$ are ε-close we have that for every $i=1,2,\ldots,n-1$, there is a $k_i \geq k_0$ such that

$$\text{dist.}(h(t_i)_k, h(t_{i+1})_k)<\varepsilon \quad \text{for } k \geq k_i;$$

we can also suppose that

$$h(t_i)_k \simeq h(t_i)_{k+1} \quad \text{in } V \text{ for } k \geq k_i$$

and, as a consequence

$$h(t_i)_k \simeq h(t_{i+1})_k \quad \text{in } V \text{ for } k \geq k_i.$$

Take $k' = \max\{k_i \mid i=1,2,\ldots,n-1\}$. Then

$$f_k = h(t_1)_k \simeq h(t_2)_k \simeq \ldots \simeq h(t_n)_k = g_k \quad \text{in } V \text{ for } k \geq k'.$$

This proves that $\underline{f}$ is homotopic to $\underline{g}$ and as a consequence they define the same shape morphism.

We now show that there are compacta X and Y and homotopic approximative maps of X towards Y which do not lie in the same path-component of $A^*(X,Y)$. The example is the same as that of the analogous statement for $A(X,Y)$, [9], Remark 2. However in that case the proof was immediate.

Example 2. Let $X=\{x_0\}$ (a point) and consider the following compacta in the Hilbert cube, Q.

$$Y_0 = \{(0,t,-1,-1,\ldots) \mid -1 \leq t \leq 1\}$$
$$Y = Y_0 \cup \{(x, \sin \pi/x, -1, -1, \ldots) \mid 0 < x \leq 1\}.$$

Define approximative maps $\underline{f}=\{f_k, X \to Y\}$, $\underline{g}=\{g_k, X \to Y\}$ by

$$f_k(x_0)=(0,1,-1,-1,\ldots), g_k(x_0)=(1,0,-1,-1,\ldots)$$
for $k=1,2,\ldots\ldots$

Obviously $\underline{f}$ and $\underline{g}$ induce the same shape morphism and we claim that they do not belong to the same path-component of $A^*(X,Y)$. Suppose on the contrary that there is a map $h: I \to A^*(X,Y)$ such that $h(0)=\underline{f}$ and $h(1)=\underline{g}$. Since for each $t \in I$ $h(t)$ is an approximative map, it can be expressed as $\{h(t)_k, X \to Y\}$ and we can define a non-empty set

$$K_t=\{y \in Q \mid \text{there exists a subsequence of } h(t)_k(x_0) \text{ whose limit is } y\}.$$

It is easy to see that K_t is contained in Y for every $t \in I$. Consider now the set

$$J=\{t \in I \mid K_t \text{ is contained in } Y_0\}.$$

Obviously $0 \in J$ and it is not difficult to see that $t_0=\sup J$ also lies in J. Choose now an $\varepsilon>0$ such that for each $y \in Y_0$ the closed ball $B(y,\varepsilon)$ in Y with center y and radius ε consists of an infinity of (connected) components. Since h is continuous there exists a closed inter val $T=[t_0,t_1]$ with $t_0<t_1\leq 1$ such that

$$d^*(h(t),h(t_0))<\varepsilon \quad \text{for every } t\in T.$$

The fact that $t_1\notin J$ implies that there is a subsequence $h(t_1)_{k_i}(x_0)$ of $h(t_1)_k(x_0)$ such that

$$\lim_{i\to\infty} h(t_1)_{k_i}(x_0)=y_1\in Y-Y_0.$$

We can also suppose that

$$\lim_{i\to\infty} h(t_0)_{k_i}(x_0)=y_0\in Y_o.$$

Obviously $y_1\in B(y_0,\varepsilon)$. Denote by L the component of $B(y_0,\varepsilon)$ which contains y_1 and consider the sets

$T_1=\{t\in T \mid h(t)_{k_i}(x_0)$ has a subsequence that conver_ges to a point lying in L$\}$

and

$T_2=T-T_1$.

It is not difficult to see that T_1 and T_2 are open sets in T and since $t_1\in T_1$, $t_0\in T_2$, we deduce that T is non-connected. This contradiction shows that <u>f</u> and <u>g</u> do not lie in the same path-component of $A^*(X,Y)$.

We finish this note by describing some important families of approximative maps which are closed subsets of $A^*(X,Y)$ or $A(X,Y)$. Other closed families have been studied in [9]. Previously, we recall some definitions. Consider compacta X,Y lying in the Hilbert cube, Q. We denote by $S(\underline{f})$ the shape morphism induced by an approximative map $\underline{f}$.

An approximative map $\underline{f}=\{f_k,X \longrightarrow Y\}$ is said to be accessible [11] provided for every $\varepsilon>0$ and every neighborhood U of X in Q there exists approximative maps $\underline{f}'=\{f'_k,X \longrightarrow Y\}$ and $\underline{g}=\{g_k,Y \longrightarrow X\}$ such that a) $d^*(\underline{f},\underline{f}')<\varepsilon$ and b) $S(j)S(\underline{g})S(\underline{f}')=S(j)$, where $j:X \longrightarrow U$ is the inclusion map. We say that $\underline{f}$ is strongly accessible provided for every $\varepsilon>0$ and every neighborhood (U,V) of (X,Y) in (Q,Q) there are approximative maps $\underline{f}'$ and $\underline{g}$ satisfying a) b) and c) $S(j')S(\underline{f}')S(\underline{g})=S(j')$, where $j':Y \longrightarrow V$ is the inclusion.

An approximative map $\underline{f}=\{f_k,X \longrightarrow Y\}$ is called an approximative e-map [5] provided the family $\{f_k\}$ satisfies the following condition

For every $\varepsilon>0$ there is a $\delta>0$ such that
$\mathrm{dist}(x,x')<\delta$ in X implies
$\mathrm{dist}(f_k(x),f_k(x'))<\varepsilon$ in Q for almost all k.

Let n be a positive integer. We say that the shape category (see [2], [4], [12]) of the approximative map $\underline{f}=\{f_k,X \longrightarrow Y\}$ is less or equal than n if for every neighborhood V of Y in Q, X can be expressed as a union of subcompacta $X_1,\dots,X_n$ such that $f_{k|X_i}$ is homotopically trivial in V for every $i\leq n$ and almost all k.

Finally an approximative map $\underline{f}=\{f_k,X \longrightarrow Y\}$ is said to be internal [8] if $f_k(X)\subset Y$ for each $k=1,2,\dots$.

The proof of the next result is left to the reader.

Proposition 2. Let X,Y be compacta lying in the Hilbert cube, Q.

a) The following sets are closed in $A^*(X,Y)$ and hence in $A(X,Y)$.

   1) The accessible approximative maps of X towards Y.
   2) The strongly accessible approximative maps of X towards Y.
   3) The approximative e-maps of X towards Y.
   4) The approximative maps of X towards Y with shape category less or equal than a fixed positive integer n.

b) The set of all internal approximative maps of X towards Y is closed in A(X,Y) but it is not in general closed in A*(X,Y).

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