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Cardinal versus ordinal criteria in choice under risk with disconnected utility ranges☆

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ABSTRACT

This paper provides a formal justification for the existence of subjective random components intrinsic to the outcome evaluation process of decision makers and explicitly assumed in the stochastic choice literature. We introduce the concepts of *admissible error function* and *generalized certainty equivalent*, which allow us to analyze two different criteria, a cardinal and an ordinal one, when defining suitable approximations to expected utility values. Contrary to the standard literature requirements for irrational preferences, adjustment errors arise in a natural way within our setting, their existence following directly from the disconnectedness of the range of the utility functions. Conditions for the existence of minimal errors are also studied. Our results imply that neither the cardinal nor the ordinal criterion do necessarily provide the same evaluation for two or more different prospects with the same expected utility value. As a consequence, a *rational* decision maker may define two different generalized certainty equivalents when presented with the same prospect in two different occasions.

1. Motivation

There exists ample experimental evidence, obtained from individual decision making problems under risk, exhibiting violations of several expected utility theory basic axioms, see Starmer (2000) for an extensive survey. Two main theoretical branches have arisen that aim to explain these empirical phenomena. First, a great deal of effort has been employed to relax the independence axiom through subjective families of utility functions and probability decision weights that maintained the ordering and continuity axioms intact. The theories of stochastic choice under risk developed to explain mainly violations of the independence axiom that rely on random errors made by agents during their choice process. These errors may appear at either the preference selection stage (random preference model, Loomes and Sugden (1995)), the calculation stage (Fechner model, Hey and Orme (1994)), or the action stage (tremble or constant error model, Harless and Camerer (1994)). For example, Loomes and Sugden (1995, 1998) illustrate that no deterministic choice theory is totally successful in explaining the observed experimental behavior of agents, and introduce a stochastic choice function over acts to account for the observed anomalies in pairwise choice problems. Their random preference model derives from the inherent variability assumed or imprecision in the preferences of agents, which provide the foundation for the second branch of theoretical models. Both theoretical branches assume explicitly the existence of an error inherent to the choice process of agents but no formal theoretical explanation justifying the existence of such an error has been presented in the literature.

The subjectively defined inherent stochastic component of preferences is considered by the current experimental literature to be the main explanation for the errors observed in the choice process of agents. MacCrimmon and Smith (1986) noted that people have difficulties providing a single precise certainty equivalent and must approach their valuations using *equivalence intervals*, a tendency also observed in psychology by Krahnen et al. (1997). Imprecise preferences are to blame for the representation of the certainty equivalent as an interval, which grows wider as a bet becomes more dissimilar from certainty, see Loomes (2005). Loomes (1988) observed that agents tend to round valuations, which led him to conclude that individuals do not have easy access to a clear ordering of preferences, requiring a grid search to reach their valuations. He also proposed the incorporation of bounded rationality principles to the preference formation process of agents. Indeed, Loomes (1998) suggests that individuals may construct their preferences using rules of thumb together with

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basic principles specific to the particular structure of the decision task at hand. This idea was also expressed by Butler (2000), who stated that it is the incompleteness of a stable utility function what leads to attempts at preference construction and choice errors, which "result from the remaining gaps in [people's] preference ordering, after whatever cognitive effort to reduce the coarse-grading in [their] utility index has been attempted". Moreover, he stressed the important role played by presentational displays and perception effects in the process of preference construction, in accordance with the conclusions reached by similarity theory, see Leland (1998). In this regard, Rieskamp et al. (2006) present a literature review that highlights the dependence of choice on the environment and illustrates the explanatory role of complex psychological choice models. They follow a bounded rationality approach to explain the various violations of consistency displayed by the data, emphasizing the fact that people construct or discover their preferences when presented with different option sets, see also Vazquez and Watt (2003).

The importance of choice errors in economic experiments has also been remarked by Hey (2005), who distinguishes two different types: between subjects (people are different) and within subjects (people's behavior has some random component), generating a heterogeneity in choice already recognized by Ballinger and Wilcox (1997). Finally, Loomes et al. (2002) conclude that the model fitting the experimental data best combines the random preference model with a tremble mechanism. Besides, they show that agents do learn to eliminate the tremble through a dynamic choice process, though the random preference variation does not decay with experience. Similarly, Loomes and Sugden (1998) state that the patterns of behavior evolved over the course of their experiment in the direction of expected utility theory.

The experimental literature described above agrees on the fact that people have imprecise preferences, violating completeness or transitivity, and that no stochastic choice theory alone clearly dominates the others when it comes to explaining *all* the empirical regularities observed. However, this literature does not account for how the inherent generation of the choice errors observed takes place. Lacking a theoretical environment that reflects these errors does not only constrain the understanding of the basic principles underlying the formation of preferences, but also forces economists to define as erratic a behavior that does converge to the standard choice axioms if learning is allowed through a dynamic setting. That is, economic agents are required to learn the basic axioms that are assumed to define their choice behavior in the first place. For this paradox to work, either endogenously generated errors or incomplete (hence, irrational) preferences must be imposed on the choice process of agents.

This paper provides a formal justification for the existence of subjective random components intrinsic to the outcome evaluation process of decision makers that are reflected in the so-called erratic choices identified by the experimental literature.

We start by extending the notion of certainty equivalent to a very general setting, where not even a basic topological structure is assumed on the domain of the utility functions. Moreover, the domain set will not be subject to any cardinality or other structural constraints. We introduce the concept of "admissible error function", which, in turn, leads to that of "generalized certainty equivalent". The latter concept, which coincides with that of certainty equivalent under the standard expected utility theory assumptions, allows for rational prospect valuations (based on complete and transitive preference relations), even in the case when the preference relations of decision makers are not (or cannot be) represented by continuous utility functions. A *prospect* is to be understood as a list of consequences with associated probabilities, see Starmer (2000).

Adjustment errors arise in a natural way within the proposed setting, and their existence follows directly from the disconnectedness of the range of the utility functions. Conditions for the existence of a minimal error are also studied.

The newly introduced concepts of admissible error function and generalized certainty equivalent allow us to analyze two different criteria, a cardinal and an ordinal one, when defining suitable approximations to expected utility values. We show how these criteria lead to sets of approximations which are, in general, different from each other despite being induced by the same prospect.

Our results imply that neither the cardinal nor the ordinal criterion do necessarily provide the same evaluation for two or more different prospects with the same expected utility value. As a consequence, a *rational* decision maker may define two different generalized certainty equivalents when presented with the same prospect in two different occasions.

In conclusion, contrary to the literature requirement for imprecise preference orders, we show that adjustment errors derive from gaps in the range of rational decision makers' utility functions and how these errors induce different prospect evaluation criteria. One can easily build on the results obtained to provide a justification that, while preserving the rationality of preferences explains the violations of the expected utility axioms observed in the experimental literature of choice under risk. Besides, our results can be reinterpreted to account for Allais/Ellsberg/Kahneman–Tversky type issues arising in uncertain settings.

The paper proceeds as follows. Section 2 introduces basic definitions and fixes the notations. Section 3 defines the concepts of admissible error function and generalized certainty equivalent. Section 4 states the conditions for the existence of a minimal admissible error function. Section 5 describes both the cardinal and the ordinal approaches, compares them and highlights their differences in terms of generalized certainty equivalents sets. Section 6 presents the main conclusions.

2. Preliminaries and notations

Let X be a nonempty set. A *preference relation* on X is a binary relation $\succsim \subseteq X \times X$ satisfying reflexivity $(\forall x \in X,\ \langle x, x\rangle \in \succsim)$, completeness $(\forall x, y \in X,\ (\langle x, y\rangle \in \succsim \vee \langle y, x\rangle \in \succsim))$ and transitivity $(\forall x, y, z \in X,\ (\langle x, y\rangle \in \succsim \wedge \langle y, z\rangle \in \succsim \Rightarrow \langle x, z\rangle \in \succsim))$. We usually write $x \succsim y$ in place of $\langle x, y\rangle \in \succsim$ and read: *x is preferred or indifferent to y*.

Note that the angle brackets will continue to be used in what follows to denote ordered tuples of Cartesian products.

A *preference order* is a pair $(X, \succsim)$, where X is a nonempty set and $\succsim$ is a preference relation on X.

The strict preference and the indifference relations associated to a preference relation $\succsim$ are defined as follows:

$$x \succ y \Leftrightarrow x \succsim y \wedge y \not\succsim x, \quad \text{and} \quad x \sim y \Leftrightarrow x \succsim y \wedge y \succsim x.$$

We read $x \succ y$ as *x is preferred to y*, while $x \sim y$ is read *x is indifferent to y*.

From the definition, it is clear that preference relations are complete preorders. Also, preference relations which are complete and transitive are usually called *rational*. Hence, all the preference relations in this paper are rational.

A *utility function representing* $\succsim$ is a function $u : X \to \mathbb{R}$ such that:

$$\forall x, y \in X, \quad x \succsim y \Leftrightarrow u(x) \geq u(y).$$

The symbols $\geq$ and $>$ will denote the standard partial and linear order on the reals, respectively. It is known that a (rational) preference relation $\succsim$ on X is representable by a utility function (not necessarily continuous) if and only if it is *perfectly separable*, that is, if there exists a countable subset V of X such that for all $x \succ y$, there exists $z \in V$ with $x \succsim z \succsim y$ (Wakker, 1988). Henceforth, we will assume each preference relation $\succsim$ on X to be perfectly separable.

The continuity of utility functions is clearly subordinated to the existence of a topology on the set X.

A pair (X, τ), where X is a nonempty set and τ is a topology on X, is called a *topological space*. A topological space (X, τ) is *connected* if there are no two disjoint nonempty open subsets H and K such that $X = H \cup K$; and *disconnected* if it is not connected. Every preference relation on a nonempty set X induces a topology on X, called the *order topology induced by* $\succsim$ and denoted by $\tau_{\succsim}$. This topology has, as a subbase, all subsets of the form $\{y \in X : y \succ x\}$ and $\{y \in X : x \succ y\}$, where $x \in X$.

The triple $(X, \succsim, \tau)$ will be used to denote a nonempty set X endowed with both a preference relation $\succsim$ and a topology τ. Note that $\succsim$ and τ "are not necessarily compatible". This means that *the order topology* $\tau_{\succsim}$ *induced by* $\succsim$ *does not necessarily coincide with* τ. Consider, for example, the triple $(\mathbb{R}^2, >_{Lex}, \tau_e)$, where $>_{Lex}$ denotes the lexicographic order on $\mathbb{R}^2$ and τ_e the standard Euclidean topology. There exist, in fact, subsets which are open with respect to the order topology, but not with respect to τ_e.

Given $a, b \in \mathbb{R}$, with $a < b$, (a, b), $(a, b]$, $[a, b)$, $[a, b]$ will denote the open, half-open and closed real intervals of end-points a and b. We will also consider degenerate intervals, that is, intervals of the form $[a, a] = \{a\}$.

Given a preference order $(X, \succsim)$, we will denote by $\mathcal{U}(X, \succsim)$ the set of *all* bounded (above and below) utility functions representing $\succsim$ on X. It is well-known that the boundedness assumption is important in order to avoid the pathological and paradoxical situations where the expected utility takes infinite value, as the St. Petersburg paradox illustrates (see Mas-Colell et al. (1995, 27, Section 6.C)). Note that $\mathcal{U}(X, \succsim)$ is never empty since $\succsim$ is assumed to be perfectly separable.

For every $u \in \mathcal{U}(X, \succsim)$, *Range*$(u)$ will denote the range of the function u, that is *Range*$(u) = \{u(x) : x \in X\}$. By the boundedness assumption, inf *Range*(u) and sup *Range*(u) must be finite.

Abusing notation, X can also be considered as a random variable. In this case, $\mathcal{M}(X)$ will denote the set of all probability density functions μ of X such that μ is either a non-atomic probability density function if X is absolutely continuous, or a non-degenerate probability function if X is discrete. We do not consider degenerate discrete probability densities, since they do not induce risk on the choice process of a decision maker. The non-atomic requirement is imposed to allow for the expected utilities to be correctly defined.

The Cartesian product $\mathcal{M}(X) \times \mathcal{U}(X, \succsim)$ is the set of all prospects.

For every $\langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim)$, let $E(\mu, u)$ denote the expected utility induced by $\langle \mu, u\rangle$. $E(\mu, u) = \int_X u(x)\mu(x)\mathrm{d}x$, if X is absolutely continuous, and $E(\mu, u) = \sum_{x \in support(\mu)} u(x)\mu(x)$, if X is discrete.

3. Error functions and generalized certainty equivalents

Following the standard economic theory of choice under risk (refer to Chapter 6 in Mas-Colell et al. (1995)), we define a *certainty equivalent induced by* $\langle \mu, u\rangle$ for any pair $\langle \mu, u\rangle$ as follows.

Definition 3.1. Let $(X, \succsim)$ be a preference order and $\langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim)$. A *certainty equivalent value* for $\langle \mu, u\rangle$ is an element of X whose utility equals the expected utility $E(\mu, u)$, that is, any element of the set $u^{-1}(E(\mu, u))$.

In order to guarantee both the continuity of u and the existence of a certainty equivalent, the literature tacitly assumes the set X to be a convex subset (and hence a connected subspace) of a finite dimensional real space endowed with the standard Euclidean topology. However, as the authors show, see Di Caprio and Santos-Arteaga (2008), even though the continuity of u is an important requirement for the existence of a certainty equivalent, it is not essential in order to define it. Indeed, Proposition 3.2, Corollary 3.3, and Examples 3.4 and 3.5 in Di Caprio and Santos-Arteaga (2008) highlight the connectedness of the range of the utility function u as the main condition sufficient (but not necessary) for the existence of a certainty equivalent.

It clearly follows from the definition above that there exists *at least one* certainty equivalent for every pair $\langle \mu, u\rangle$ if and only if $u^{-1}(E(\mu, u)) \neq \emptyset$ (or equivalently, if and only if $E(\mu, u) \in$ *Range*(u)). In what follows, the sentence "$ce(\mu, u)$ exists" will be used to indicate the fact that $u^{-1}(E(\mu, u)) \neq \emptyset$.

If $u^{-1}(E(\mu, u)) = \emptyset$, the decision maker is forced to define subjectively a certainty equivalent substitute. This process leads to approximation errors inherent to the decision maker's preference relation, see Di Caprio and Santos-Arteaga (2009). The following definition constitutes a generalization of Definition 4.1 in Di Caprio and Santos-Arteaga (2009) that will prove useful in extending the formal results obtained by the authors to a more economical setting.

Definition 3.2. Let $(X, \succsim)$ be a preference order. An *upper error function for* $(X, \succsim)$ is a function $\epsilon^+ : \mathcal{M}(X) \times \mathcal{U}(X, \succsim) \to [0, +\infty)$ such that

$$E(\mu, u) + \epsilon^+(\mu, u) \le \sup\{u(x) : x \in support(\mu)\}.$$

A *lower error function for* $(X, \succsim)$ is a function $\epsilon^- : \mathcal{M}(X) \times \mathcal{U}(X, \succsim) \to [0, +\infty)$ such that

$$E(\mu, u) - \epsilon^-(\mu, u) \ge \inf\{u(x) : x \in support(\mu)\}.$$

A function $\epsilon : \mathcal{M}(X) \times \mathcal{U}(X, \succsim) \to [0, +\infty) \times [0, +\infty)$ will be called an *error function for* $(X, \succsim)$ if it is the product of an upper error function and a lower error function for $(X, \succsim)$; that is, if for every $\langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim)$

$$\epsilon(\mu, u) = \langle \epsilon^+(\mu, u), \epsilon^-(\mu, u)\rangle.$$

An error function ϵ is *admissible* if $\forall \langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim)$:

(i) $E(\mu, u) \in Range(u) \Longrightarrow \epsilon(\mu, u) = \langle 0, 0\rangle$;
(ii) $u^{-1}([E(\mu, u), E(\mu, u) + \epsilon^+(\mu, u)]) \neq \emptyset$;
(iii) $u^{-1}([E(\mu, u) - \epsilon^-(\mu, u), E(\mu, u)]) \neq \emptyset$.

$\mathcal{AE}(X, \succsim)$ will denote the set of all admissible error functions for $(X, \succsim)$.

Definition 3.2 allows us to generalize the notion of induced certainty equivalent that was first introduced by Di Caprio and Santos-Arteaga (2008, Definition 5.1).

Definition 3.3. Let $(X, \succsim)$ be a preference order, $\langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim)$ and ϵ be an error function. A *generalized certainty equivalent of* $\langle \mu, u\rangle$ *determined by* ϵ is an element of X, denoted by $gce(\mu, u, \epsilon)$, whose utility belongs to the bounded closed real interval $[E(\mu, u) - \epsilon^-(\mu, u), E(\mu, u) + \epsilon^+(\mu, u)]$, that is:

$$gce(\mu, u, \epsilon) \in u^{-1}([E(\mu, u) - \epsilon^-(\mu, u), E(\mu, u) + \epsilon^+(\mu, u)]).$$

To be consistent with the notations, we will write "$gce(\mu, u, \epsilon)$ exists" to indicate the fact that $u^{-1}([E(\mu, u) - \epsilon^-(\mu, u), E(\mu, u) + \epsilon^+(\mu, u)]) \neq \emptyset$.

It is immediate to check that Definitions 3.2 and 3.3 yield the following.

Proposition 3.4. *Let* $(X, \succsim)$ *be a preference order. Then,* $gce(\mu, u, \epsilon)$ *exists for every* $\langle \mu, u, \epsilon\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim) \times \mathcal{AE}(X, \succsim)$.

Proposition 3.4, states that generalized certainty equivalents determined by admissible errors always exist. Moreover, given any admissible error, the notion of generalized certainty equivalent induced by a pair $\langle\mu,u\rangle$ coincides with the one of certainty equivalent if X is a connected topological space and u is continuous with respect to the topology on X.

Proposition 3.5. *Let $(X,\succsim,\tau)$ be given and $\langle\mu,u\rangle\in\mathcal{M}(X)\times\mathcal{U}(X,\succsim)$. If:*
(a) *(X,τ) is connected,*
(b) *u is continuous with respect to τ,*

then, for every $\epsilon\in\mathcal{AE}(X,\succsim)$, $gce(\mu,u,\epsilon)$ exists if and only if $ce(\mu,u)$ exists.

Proof. Conditions (a) and (b) imply that $E(\mu,u)\in Range(u)$. Hence, by Definition 3.2, for every $\epsilon\in\mathcal{AE}(X,\succsim)$, $\epsilon(\mu,u)=\langle 0,0\rangle$. Consequently, for every $\epsilon\in\mathcal{AE}(X,\succsim)$, $u^{-1}([E(\mu,u)-\epsilon^-(\mu,u),E(\mu,u)+\epsilon^+(\mu,u)])=u^{-1}(E(\mu,u))$. □

4. Minimal error functions

For every $\langle\mu,u\rangle\in\mathcal{M}(X)\times\mathcal{U}(X,\succsim)$, let:
$\mathbb{E}^{\mu,u}=\{\epsilon(\mu,u):\epsilon\in\mathcal{AE}(X,\succsim)\}$.

Definition 4.1. Let $(X,\succsim)$ be given. Fix $\langle\mu,u\rangle\in\mathcal{M}(X)\times\mathcal{U}(X,\succsim)$. For every $\epsilon_1,\epsilon_2\in\mathcal{AE}(X,\succsim)$, we say that ϵ_1 *is smaller than* ϵ_2 *with respect to* $\langle\mu,u\rangle$, and write $\epsilon_1(\mu,u)\lhd_{\mu,u}\epsilon_2(\mu,u)$, if

$$u^{-1}([E(\mu,u)-\epsilon_1^-(\mu,u),E(\mu,u)+\epsilon_1^+(\mu,u)]) \subseteq u^{-1}([E(\mu,u)-\epsilon_2^-(\mu,u),E(\mu,u)+\epsilon_2^+(\mu,u)]).$$

It is easy to check that the relation $\lhd_{\mu,u}$ defines a preorder on the set $\mathbb{E}^{\mu,u}$ and that allowing μ and u to vary produces a preorder on $\mathcal{AE}(X,\succsim)$.

Definition 4.2. Let $(X,\succsim)$ be given and let $\epsilon_1,\epsilon_2\in\mathcal{AE}(X,\succsim)$. We say that ϵ_1 *is smaller than* ϵ_2, and write $\epsilon_1\lhd\epsilon_2$, if

$$\forall\langle\mu,u\rangle\in\mathcal{M}(X)\times\mathcal{U}(X,\succsim),\quad \epsilon_1(\mu,u)\lhd_{\mu,u}\epsilon_2(\mu,u).$$

Definitions 4.1 and 4.2 naturally pose the problem of the existence of a minimal admissible error function from a local (i.e. for a fixed pair $\langle\mu,u\rangle$) and a global viewpoint, respectively. Clearly, providing conditions for a $\lhd_{\mu,u}$-minimal element to exist for $\mathbb{E}^{\mu,u}$ becomes an essential issue.

Let $(X,\succsim)$ be a preference order. For every $\langle\mu,u\rangle\in\mathcal{M}(X)\times\mathcal{U}(X,\succsim)$, let:

$$\lambda^+(E(\mu,u)) = \inf\{Range(u)\setminus(-\infty,E(\mu,u))\} = \inf\{u(x):u(x)\geq E(\mu,u)\}$$

and

$$\lambda^-(E(\mu,u)) = \sup\{Range(u)\setminus(E(\mu,u),+\infty)\} = \sup\{u(x):u(x)\leq E(\mu,u)\}.$$

Proposition 4.3. *Let $(X,\succsim)$ be a preference order. For every $\langle\mu,u\rangle\in\mathcal{M}(X)\times\mathcal{U}(X,\succsim)$,*
(a) *$\lambda^-(E(\mu,u))\leq E(\mu,u)\leq\lambda^+(E(\mu,u))$;*
(b) *if $E(\mu,u)\in Range(u)$, then $\lambda^-(E(\mu,u))=E(\mu,u)=\lambda^+(E(\mu,u))$.*

If $E(\mu,u)\notin Range(u)$, $E(\mu,u)$ belongs to a jump of u whose supremum and infimum are $\lambda^+(E(\mu,u))$ and $\lambda^-(E(\mu,u))$, respectively. Note that the converse of Proposition 4.3(b) is not always true, as the following example illustrates.

Example 4.4. Let $K=\{\frac{1}{n}:n\in\mathbb{N}\wedge n\geq 2\}$ and let $X=[-1,1]\setminus K$. Endow X with the strict linear order of the reals. The function $u:X\to\mathbb{R}$ defined by $u(x)=x+1$ is a utility function representing the preference relation given on X and $Range(u)=[0,2]\setminus\{1+\frac{1}{n}:n\in\mathbb{N}\wedge n\geq 2\}$.

Define on X the following probability density function:

$$\mu(x)=\begin{cases}\frac{2}{3}, & \text{if } x=1,\\ \frac{1}{3}, & \text{if } x=-1,\\ 0, & \text{if } x\in X\setminus\{-1,1\}.\end{cases}$$

Then, $E(\mu,u)=\frac{4}{3}=1+\frac{1}{3}$ and $\lambda^+(E(\mu,u))=\lambda^-(E(\mu,u))=E(\mu,u)\notin Range(u)$. □

Given a preference order $(X,\succsim)$, let $\epsilon_0^+,\epsilon_0^-:\mathcal{M}(X)\times\mathcal{U}(X,\succsim)\to[0,+\infty)$ be defined by:

$$\epsilon_0^+(\mu,u)=\lambda^+(E(\mu,u))-E(\mu,u)$$

and

$$\epsilon_0^-(\mu,u)=E(\mu,u)-\lambda^-(E(\mu,u))$$

respectively, and $\epsilon_0:\mathcal{M}(X)\times\mathcal{U}(X,\succsim)\to[0,+\infty)\times[0,+\infty)$ be defined by:

$$\epsilon_0(\mu,u)=\langle\epsilon_0^+(\mu,u),\epsilon_0^-(\mu,u)\rangle.$$

Also, let:

$$\Lambda(X)=\{\langle\mu,u\rangle\in\mathcal{M}(X)\times\mathcal{U}(X,\succsim):\text{both }\min\{u(x):u(x)\geq E(\mu,u)\}\text{ and }\max\{u(x):u(x)\leq E(\mu,u)\}\text{ exist}\}.$$

Equivalently, $\Lambda(X)$ is the set of all pairs $\langle\mu,u\rangle$ such that both $\lambda^+(E(\mu,u))\in Range(u)$ and $\lambda^-(E(\mu,u))\in Range(u)$.

It is clear that $\Lambda(X)\neq\emptyset$ whatever is the preorder $(X,\succsim)$. Note also that $\Lambda(X)$ contains all pairs $\langle\mu,u\rangle$ where u is a continuous utility function representing $\succsim$, provided that X is endowed with a connected topology.

Remark 4.5. It is easy to show that ϵ_0 is an admissible error function for $(X,\succsim)$ if and only if $\mathcal{M}(X)\times\mathcal{U}(X,\succsim)=\Lambda(X)$. Furthermore,

$$\forall\langle\mu,u\rangle\in\Lambda(X),\ \forall\epsilon\in\mathcal{AE}(X,\succsim),\quad \epsilon_0(\mu,u)\lhd_{\mu,u}\epsilon(\mu,u).$$

We can now state our main result on the minimality of admissible error functions and error values. This result identifies the equality $\mathcal{M}(X)\times\mathcal{U}(X,\succsim)=\Lambda(X)$ as the natural condition characterizing the existence of minimal errors both locally (Theorem 4.6(a)) and globally (Theorem 4.6(b)). The proof follows from Remark 4.5 and it is left to the reader.

Theorem 4.6. *Let $(X,\succsim)$ be a preference order. The following are equivalent:*
(a) *$\forall\langle\mu,u\rangle\in\mathcal{M}(X)\times\mathcal{U}(X,\succsim)$, $\epsilon_0(\mu,u)$ is the minimal element of $\mathbb{E}^{\mu,u}$ with respect to $\lhd_{\mu,u}$;*
(b) *ϵ_0 is the minimal element of $\mathcal{AE}(X,\succsim)$ with respect to $\lhd$;*
(c) *$\mathcal{M}(X)\times\mathcal{U}(X,\succsim)=\Lambda(X)$.*

Note that condition (c) of Theorem 4.6 is satisfied by all discrete real subsets, which are the type of sets used by the experimental literature described in the introductory section to define their respective choice sets.

5. Maximal approximation sets

The concept of admissible error function, see Definition 3.2, and that of generalized certainty equivalent, see Definition 3.3, yield two different criteria when defining suitable approximations to $E(\mu,u)$, where $\langle\mu,u\rangle\in\mathcal{M}(X)\times\mathcal{U}(X,\succsim)$.

The first criterion consists of a pure cardinal approach in expected utility terms, and reduces to the standard expected utility theory setting when $E(\mu,u)\in Range(u)$. The second criterion is based on an ordinal interpretation of the set of possible generalized certainty equivalents associated with a pair $\langle \mu, u\rangle$. Both criteria lead to the same set of generalized certainty equivalents in the standard economic literature case, i.e. when $E(\mu, u) \in Range(u)$. This is not necessarily true in the more general setting considered in this paper.

In this section, we describe both approaches, compare them and highlight their differences in terms of generalized certainty equivalents sets.

5.1. Cardinal criterion

The cardinal approach is based on assigning an ordering on the set of all possible subjective approximations to a certain $E(\mu, u)$, that is, all generalized certainty equivalents associated to a given pair $\langle \mu, u\rangle$.

Definition 5.1. Let $(X, \succsim)$ be a preference order. Let $\langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim)$ and $\epsilon_1, \epsilon_2 \in \mathcal{AE}(X, \succsim)$. We say that $gce(\mu, u, \epsilon_1)$ *induces a better cardinal approximation to* $E(\mu, u)$ than $gce(\mu, u, \epsilon_2)$, and we write $gce(\mu, u, \epsilon_1) \ggg_{\mu,u} gce(\mu, u, \epsilon_2)$, if

(a) $\epsilon_1(\mu, u) \lhd_{\mu,u} \epsilon_2(\mu, u)$;
(b) $|u(gce(\mu, u, \epsilon_1)) - E(\mu, u)| < |u(gce(\mu, u, \epsilon_2)) - E(\mu, u)|$.

For every $\langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim)$, let

$$\mathbb{G}^{\mu,u} = \{gce(\mu, u, \epsilon) : \epsilon \in \mathcal{AE}(X, \succsim)\}.$$

The binary relation $\ggg_{\mu,u}$ defines a strict preorder on each $\mathbb{G}^{\mu,u}$. Note that condition (b) above is essential to guarantee the transitivity of $\ggg_{\mu,u}$. Furthermore, $\ggg_{\mu,u}$ does not necessarily preserve the initial preference order $\succsim$, as the following example illustrates.

Example 5.2. Let $X = [0, 4]$ and the preference relation on X be the strict linear order of the reals. Let $\langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, >)$ be defined by:

$$u(x) = \begin{cases} x & \text{if } x \in [0, 1], \\ x+2 & \text{if } x \in (1, 4], \end{cases}$$

and:

$$\mu(x) = \begin{cases} \dfrac{6}{7}, & \text{if } x = 1, \\ \dfrac{1}{7}, & \text{if } x = 4, \\ 0, & \text{if } x \in X \setminus \{1, 4\}. \end{cases}$$

Clearly, $Range(u) = [0, 1] \cup (3, 6]$, $E(\mu, u) = \frac{12}{7} \in (1, 3]$, $\lambda^+(E(\mu, u)) = 3$ and $\lambda^-(E(\mu, u)) = 1$.

Let $\{a_n\}_{n\in\mathbb{N}}$ be a real sequence converging to 3. Fix two elements of this sequence, $a_1 < a_2$. Let $\epsilon_1(\mu, u) = \langle a_1 - \frac{12}{7}, \frac{12}{7} - 1\rangle$ and $\epsilon_2(\mu, u) = \langle a_2 - \frac{12}{7}, \frac{12}{7} - 1\rangle$. Two possible generalized certainty equivalents are $gce(\mu, u, \epsilon_1) = 1$ and $gce(\mu, u, \epsilon_2) = u^{-1}(a_2)$. It is easy to check that $\epsilon_1(\mu, u) \lhd_{\mu,u} \epsilon_2(\mu, u)$. Also, $|u(gce(\mu, u, \epsilon_1)) - E(\mu, u)| = |1 - \frac{12}{7}| < |u(gce(\mu, u, \epsilon_2)) - E(\mu, u)| = |a_2 - \frac{12}{7}|$. Thus, $gce(\mu, u, \epsilon_1) \ggg_{\mu,u} gce(\mu, u, \epsilon_2)$, but $gce(\mu, u, \epsilon_2) \succ gce(\mu, u, \epsilon_1)$. □

For every $\langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim)$ and for every error function ϵ, define the set:

$$\mathcal{X}^{\mu,u}_{\epsilon} = \begin{cases} u^{-1}([E(\mu, u), E(\mu, u) + \epsilon^+(\mu, u)]) \\ \quad \text{if } \epsilon^+(\mu, u) < \epsilon^-(\mu, u), \\ u^{-1}([E(\mu, u) - \epsilon^-(\mu, u), E(\mu, u)]) \\ \quad \text{if } \epsilon^+(\mu, u) > \epsilon^-(\mu, u), \\ u^{-1}([E(\mu, u) - \epsilon^-(\mu, u), E(\mu, u) + \epsilon^+(\mu, u)]) \\ \quad \text{if } \epsilon^+(\mu, u) = \epsilon^-(\mu, u). \end{cases}$$

In particular, the set determined by the error function ϵ_0 is:

$$\mathcal{X}^{\mu,u}_{0} = \begin{cases} u^{-1}(\lambda^+(E(\mu, u))) \\ \quad \text{if } \epsilon^+_0(\mu, u) < \epsilon^-_0(\mu, u), \\ u^{-1}(\lambda^-(E(\mu, u))) \\ \quad \text{if } \epsilon^+_0(\mu, u) > \epsilon^-_0(\mu, u), \\ u^{-1}(\lambda^+(E(\mu, u))) \cup u^{-1}(\lambda^-(E(\mu, u))) \\ \quad \text{if } \epsilon^+_0(\mu, u) = \epsilon^-_0(\mu, u). \end{cases}$$

Remark 5.3. By (ii) and (iii) of Definition 3.2, each set of the form $\mathcal{X}^{\mu,u}_{\epsilon}$ is not empty if and only if the error function ϵ is admissible.

Note that $\mathcal{X}^{\mu,u}_{\epsilon}$ defines a set of cardinal approximations to $E(\mu, u)$. In particular, if not empty, $\mathcal{X}^{\mu,u}_{0}$ can be interpreted as the set of maximal cardinal approximations to $E(\mu, u)$ in the sense of Definition 5.1. The following result formalizes this fact.

Proposition 5.4. *Let $(X, \succsim)$ be a preference order. For every $\langle \mu, u\rangle \in \Lambda(X)$, $\mathcal{X}^{\mu,u}_{0}$ is the set of maximal elements of $\mathbb{G}^{\mu,u}$ with respect to $\ggg_{\mu,u}$.*

In order to complete our discussion from a global point of view (i.e. when $\langle \mu, u\rangle$ varies in $\mathcal{M}(X) \times \mathcal{U}(X, \succsim)$), we need to define the following Cartesian products:

$$\mathbb{G} = \prod_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)} \mathbb{G}^{\mu,u}, \qquad \mathbb{G} \restriction \Lambda(X) = \prod_{\langle \mu,u\rangle \in \Lambda(X)} \mathbb{G}^{\mu,u}$$

$$\mathcal{X}_0 = \prod_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)} \mathcal{X}^{\mu,u}_0, \qquad \mathcal{X}_0 \restriction \Lambda(X) = \prod_{\langle \mu,u\rangle \in \Lambda(X)} \mathcal{X}^{\mu,u}_0.$$

A preorder can be defined on $\mathbb{G}$, and consequently on $\mathbb{G} \restriction \Lambda(X)$, as follows.

Definition 5.5. Let $(X, \succsim)$ be a preference order. For every

$$\langle a_{\langle \mu,u\rangle}\rangle_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)}, \langle b_{\langle \mu,u\rangle}\rangle_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)} \in \mathbb{G},$$

we say that $\langle a_{\langle \mu,u\rangle}\rangle_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)}$ *is a better global approximation than* $\langle b_{\langle \mu,u\rangle}\rangle_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)}$, and we write

$$\langle a_{\langle \mu,u\rangle}\rangle_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)} \ggg \langle b_{\langle \mu,u\rangle}\rangle_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)},$$

if:

$$\forall \langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim), \quad a_{\langle \mu,u\rangle} \ggg_{\mu,u} b_{\langle \mu,u\rangle}.$$

The following proposition provides a $\Lambda(X)$-dimensional equivalent of Proposition 5.4.

Proposition 5.6. *Let $(X, \succsim)$ be a preference order. Then, $\mathcal{X}_0 \restriction \Lambda(X)$ is the set of maximal elements of $\mathbb{G} \restriction \Lambda(X)$ with respect to $\ggg$.*

A global extension of Propositions 5.4 and 5.6 is possible under the assumption that $\mathcal{M}(X) \times \mathcal{U}(X, \succsim) = \Lambda(X)$, as Theorem 5.11 will show.

5.2. Ordinal criterion

In the ordinal approach, we propose that the decision maker does not discriminate between different possible generalized certainty equivalents as long as they are determined by the same pair $\langle \mu, u\rangle$ and the same error function ϵ. Definition 5.7 expresses this idea formally by introducing a quotient set associated to a given triple $\langle \mu, u, \epsilon\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim) \times \mathcal{AE}(X, \succsim)$.

Definition 5.7. Let $(X, \succsim)$ be a preference order. Let $\langle \mu, u, \epsilon\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim) \times \mathcal{AE}(X, \succsim)$. We denote by $X/\bowtie_{\epsilon(\mu,u)}$ the quotient set of X determined by the following equivalence relation:

$$\forall x, y \in X, \quad x \bowtie_{\epsilon(\mu,u)} y \iff x, y \in u^{-1}([E(\mu, u) - \epsilon^-(\mu, u), E(\mu, u) + \epsilon^+(\mu, u)]).$$

Note that our ordinal approach corresponds to the basic concept introduced in standard undergraduate microeconomic textbooks. Indeed, when the range of the utility function u is connected, the set of approximations defined following our cardinal approach and the equivalence class defined through our ordinal criterion are the same set, namely $u^{-1}(E(\mu, u))$, see Proposition 3.5. Clearly, this is also the case in the standard microeconomic literature. However, in situations where the range of u is disconnected, such as those included in this paper, these approaches lead to sets of approximations and equivalence classes which are, in general, different from each other (the former being subsets of the latter).

Every quotient set of the form $X/\bowtie_{\epsilon(\mu,u)}$ can be regarded as the *set of ordinal approximations* relative to the triple $\langle \mu, u, \epsilon\rangle$. By construction, the most natural way of comparing these sets is using the preorder of Definition 4.1.

Definition 5.8. Let $(X, \succsim)$ be a preference order. Let $\langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim)$ and $\epsilon_1, \epsilon_2 \in \mathcal{AE}(X, \succsim)$. We say that $X/\bowtie_{\epsilon_1(\mu,u)}$ *ordinally* $\langle \mu, u\rangle$*-dominates* $X/\bowtie_{\epsilon_2(\mu,u)}$, and we write $X/\bowtie_{\epsilon_1(\mu,u)} \sqsupset_{\mu,u} X/\bowtie_{\epsilon_2(\mu,u)}$, if $\epsilon_1(\mu, u) \lhd_{\mu,u} \epsilon_2(\mu, u)$.

For every $\langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim)$, the relation $\sqsupset_{\mu,u}$ is a preorder on the set:

$$\Theta^{\mu,u} = \{X/\bowtie_{\epsilon(\mu,u)} : \epsilon \in \mathcal{AE}(X, \succsim)\}.$$

As in the cardinal criterion case, we can consider the approximation problem from a global point of view. The set providing ordinal approximations as the pair $\langle \mu, u\rangle$ varies is the following Cartesian product:

$$\Theta = \prod_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)} \Theta^{\mu,u}.$$

The preorders $\sqsupset_{\mu,u}$ can be combined together to define a preorder on Θ. This preorder is the corresponding ordinal version of the preorder $\ggg$ introduced in Definition 5.5.

Definition 5.9. Let $(X, \succsim)$ be a preference order. For every

$$\langle A_{\mu,u}\rangle_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)},\ \langle B_{\mu,u}\rangle_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)} \in \Theta,$$

we say that $\langle A_{\mu,u}\rangle_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)}$ *ordinally dominates* $\langle B_{\mu,u}\rangle_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)}$, and we write

$$\langle A_{\mu,u}\rangle_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)} \sqsupset \langle B_{\mu,u}\rangle_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)},$$

if:

$$\forall \langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim), \quad A_{\mu,u} \sqsupset_{\mu,u} B_{\mu,u}.$$

5.3. Existence of maximal approximations

Given $\langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim)$, let $\mathcal{S}^{\mu,u}$ be a subset of $\mathcal{AE}(X, \succsim)$ of any cardinality. Denote by $\mathcal{S}$ the family of all the sets $\mathcal{S}^{\mu,u}$. The following sets can be considered:

$$\mathbb{G}^{\mu,u} \upharpoonright \mathcal{S}^{\mu,u} = \{gce(\mu, u, \epsilon) : \epsilon \in \mathcal{S}^{\mu,u}\}$$

$$\mathbb{G} \upharpoonright \mathcal{S} = \prod_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)} \mathbb{G}^{\mu,u} \upharpoonright \mathcal{S}^{\mu,u}$$

$$\Theta^{\mu,u} \upharpoonright \mathcal{S}^{\mu,u} = \{X/\bowtie_{\epsilon(\mu,u)} : \epsilon \in \mathcal{S}^{\mu,u}\}$$

$$\Theta \upharpoonright \mathcal{S} = \prod_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)} \Theta^{\mu,u} \upharpoonright \mathcal{S}^{\mu,u}.$$

In general, there is no guarantee that a $\lhd$-minimal admissible error exists for a given $\mathcal{S}^{\mu,u}$. In fact, the error functions in $\mathcal{S}^{\mu,u}$ do not even need to be pairwise comparable with respect to $\lhd$.

As a consequence, the decision maker may be unable to define a set of $\ggg$-maximal elements for $\mathbb{G} \upharpoonright \mathcal{S}$ (i.e. maximal cardinal approximations) or a set of $\sqsupset$-maximal elements for $\Theta \upharpoonright \mathcal{S}$ (i.e. maximal ordinal approximations).

An intuitive and admissible economic assumption supporting the existence of sets of $\sqsupset$-maximal elements for $\Theta \upharpoonright \mathcal{S}$ would consist of imposing either time or information processing constraints on the decision maker. Thus, by relying on bounded rationality, we are able to restrict our attention to a family of finite sets $\mathcal{S}^{\mu,u}$, where $\langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim)$.

Bounded rationality is formally introduced by means of the following condition.

($\star$) For every $\langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim)$, let $BR^{\mu,u}$ be a fixed finite subset of $\mathcal{AE}(X, \succsim)$, that is,

$$BR^{\mu,u} = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{n_{\mu,u}}, [\eta^{\mu,u}]\},$$

where $[\eta^{\mu,u}]$ is the admissible error function defined by:

$$[\eta^{\mu,u}]^+ = \min_{i \le n_{\mu,u}} \epsilon_i^+ \quad \text{and} \quad [\eta^{\mu,u}]^- = \min_{i \le n_{\mu,u}} \epsilon_i^-.$$

The following result is easy to prove.

Theorem 5.10. *Let* $(X, \succsim)$ *be a preference order. Suppose that condition* ($\star$) *holds. Then:*

(a) *for every* $\langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim)$, $X/\bowtie_{[\eta^{\mu,u}](\mu,u)}$ *is the* $\sqsupset_{\mu,u}$*-maximal element of* $\Theta^{\mu,u} \upharpoonright BR^{\mu,u}$;
(b) *the tuple* $\langle X/\bowtie_{[\eta^{\mu,u}](\mu,u)}\rangle_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)}$ *is the* $\sqsupset$*-maximal element of* $\Theta \upharpoonright BR$.

The analogous of Theorem 5.10 cannot be stated in the cardinal case. Indeed, it can be shown that condition ($\star$) does not suffice to guarantee that the sets $\mathbb{G}^{\mu,u} \upharpoonright BR^{\mu,u}$ and $\mathbb{G} \upharpoonright BR$ have a maximal element.

An alternative to bounded rationality, condition ($\star$), guaranteeing the existence of maximal approximations in both the cardinal and ordinal approaches and both locally and globally is provided by condition (c) of Theorem 4.6. Indeed, Theorem 5.11 below completes Theorem 4.6 showing the necessity and sufficiency of this condition not only for identifying ϵ_0 as the minimal error function both locally and globally, but also for the corresponding sets of approximations to be maximal. More precisely, statements (a), (a′) and (a″) assert the existence of minimal/maximal elements locally, while (b), (b′) and (b″) provide their corresponding global versions.

Theorem 5.11. *Let* $(X, \succsim)$ *be a preference order. The following are equivalent:*

(a) $\forall \langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim)$, $\epsilon_0(\mu, u)$ *is the minimal element of* $\mathbb{E}^{\mu,u}$ *with respect to* $\lhd_{\mu,u}$;
(a′) $\forall \langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim)$, $\mathcal{X}_0^{\mu,u}$ *is the set of maximal elements of* $\mathbb{G}^{\mu,u}$ *with respect to* $\ggg_{\mu,u}$;
(a″) $\forall \langle \mu, u\rangle \in \mathcal{M}(X) \times \mathcal{U}(X, \succsim)$, $X/\bowtie_{\epsilon_0(\mu,u)}$ *is the maximal element of* $\Theta^{\mu,u}$ *with respect to* $\sqsupset_{\mu,u}$;
(b) ϵ_0 *is the minimal element of* $\mathcal{AE}(X, \succsim)$ *with respect to* $\lhd$;
(b′) $\mathcal{X}_0$ *is the set of maximal elements of* $\mathbb{G}$ *with respect to* $\ggg$;
(b″) $\langle X/\bowtie_{\epsilon_0(\mu,u)}\rangle_{\langle \mu,u\rangle \in \mathcal{M}(X)\times\mathcal{U}(X,\succsim)}$ *is the maximal element of* Θ *with respect to* $\sqsupset$;
(c) $\mathcal{M}(X) \times \mathcal{U}(X, \succsim) = \Lambda(X)$.

Even though Theorem 5.11 states the equivalence between the existence of a maximal cardinal approximation and that of a maximal ordinal one, whenever condition (c) holds, these criteria generally lead the decision maker to consider different generalized certainty equivalents. In fact, excluding the standard case where $E(\mu, u) \in Range(u)$, given a pair $\langle \mu, u\rangle$, the set of maximal approximations reduces under both criteria to the set $u^{-1}(\lambda^+(E(\mu, u))) \cup u^{-1}(\lambda^-(E(\mu, u)))$ if $\epsilon_0^+(\mu, u) = \epsilon_0^-(\mu, u)$. Alternatively, the inequality $\epsilon_0^+(\mu, u) \neq \epsilon_0^-(\mu, u)$ produces a subset of $u^{-1}(\lambda^+(E(\mu, u))) \cup u^{-1}(\lambda^-(E(\mu, u)))$ in the cardinal case, i.e. either $u^{-1}(\lambda^+(E(\mu, u)))$ or $u^{-1}(\lambda^-(E(\mu, u)))$, while leaving unchanged the set of ordinal approximations.

In any of the more general situations where ϵ_0 is not admissible, that is, when condition (c) of Theorem 5.11 does not hold (see Remark 4.5), a maximal cardinal and/or ordinal approximation may not exist.

Consider, for instance, a preference order $(X, >)$, where $X \subseteq \mathbb{R}$ and $>$ is the strict linear order of the reals. Let a pair $\langle \mu, u \rangle$ be given such that $\lambda^+(E(\mu, u)) \notin Range(u)$ and $\lambda^-(E(\mu, u)) \in Range(u)$. If $E(\mu, u) - \lambda^-(E(\mu, u)) < \lambda^+(E(\mu, u)) - E(\mu, u)$, then for every $\epsilon \in \mathcal{AE}(X, \succsim)$, the pair of error values $\langle \epsilon^+(\mu, u), \epsilon_0^-(\mu, u) \rangle$ generates a one-point set consisting of the only possible maximal cardinal approximation with respect to $\ggg_{\mu,u}$, namely the set $u^{-1}(\lambda^-(E(\mu, u)))$. However, there exist no maximal ordinal approximation with respect to $\sqsupset_{\mu,u}$. In this case, imposing bounded rationality, i.e. condition $(\star)$, would not modify the maximal cardinal approximation, but would allow for a maximal ordinal approximation to exist. Indeed, the maximal ordinal approximation with respect to $\sqsupset_{\mu,u}$ would be given by the set $u^{-1}(\lambda^-(E(\mu, u))) \cup u^{-1}([E(\mu, u), E(\mu, u) + [\eta^{\mu,u}]^+(\mu, u)])$.

6. Conclusion

In this paper, we have introduced the concept of generalized certainty equivalent in situations of choice under risk, and shown that it remains defined when we get rid of core expected utility theory classical assumptions such as the continuity of the utility functions and the connectedness of their domains.

The set of possible generalized certainty equivalents associated with a pair $\langle \mu, u \rangle$ has been obtained by means of a pair of admissible error values, $\epsilon(\mu, u) = \langle \epsilon(\mu, u)^+, \epsilon(\mu, u)^- \rangle$. The values in $\epsilon(\mu, u)$ can be easily interpreted as subjective errors inherent to the decision maker's preference relation and their size depends both on the graph of u and the support of μ. These errors, which arise quite naturally in our setting, are artificially imposed in the economic literature to justify the various preference puzzles observed in experimental choice models, see Harless and Camerer (1994), Hey and Orme (1994), and Loomes and Sugden (1995).

We have also illustrated how the concept of admissible error function and that of generalized certainty equivalent yield either a cardinal or an ordinal criterion when defining suitable approximations to the expected utility value of a given pair $\langle \mu, u \rangle$. These criteria lead to sets of approximations which are, in general, different from each other despite being induced by the same pair $\langle \mu, u \rangle$. As a consequence, the cardinal evaluation of a given prospect would generally differ from the ordinal one.

Our results immediately imply that neither the cardinal nor the ordinal criterion do necessarily provide the same evaluation for two or more different prospects with the same $E(\mu, \overline{u})$, where $\overline{u}$ is a fixed utility function. Assume, for instance, that $E(\mu, \overline{u}) \notin Range(\overline{u})$ and the minimal admissible error exists. Even in this case, the evaluation of two different prospects within either the cardinal or the ordinal criterion does not need to be unique.

Our results also imply that a *rational* decision maker may define two different generalized certainty equivalents when presented with the same prospect in two different occasions.

One can easily build on the results obtained to provide a justification that, while preserving the rationality of preferences, explains the violations of the expected utility axioms observed in the experimental literature of choice under risk.

Finally, the paper can be reinterpreted to account for an uncertain setting, where the utility function is given but the probability density function is unknown. In this sense, this paper offers a novel approach to "fuzziness" of preferences under conditions of uncertainty, where preferences are the "fuzziest" of all, relating directly to Allais/Ellsberg/Kahneman–Tversky type issues.

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