

# Commutators and commutator subgroups of the Riordan group

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## Abstract

We calculate the derived series of the Riordan group. To do that, we study a nested sequence of its subgroups, herein denoted by  $\mathcal{G}_k$ . By means of this sequence, we first obtain the  $n$ -th commutator subgroup of the Associated subgroup. This fact allows us to get some related results about certain groups of formal power series and to complete the proof of our main goal, Theorem 1 in this paper.

## 1 Introduction

If  $\mathcal{G}$  is a group and  $g, h \in \mathcal{G}$ ,  $[g, h] = g^{-1}h^{-1}gh$  is the commutator of  $g$  and  $h$ . Let  $\mathcal{C} = \{[g, h], | g, h \in \mathcal{G}\}$  be the subset of all commutators of  $\mathcal{G}$ . Denote, as usual, by  $[\mathcal{G}, \mathcal{G}]$  to the commutator subgroup of  $\mathcal{G}$ , that is,  $[\mathcal{G}, \mathcal{G}]$  is the subgroup generated by the set  $\mathcal{C}$ . We can define iteratively the  $n$ -th commutator subgroup, denoted by  $\mathcal{G}^{(n)}$ , as  $\mathcal{G}^{(1)} = [\mathcal{G}, \mathcal{G}]$  and  $\mathcal{G}^{(n)} = [\mathcal{G}^{(n-1)}, \mathcal{G}^{(n-1)}]$ . The sequence of groups  $\mathcal{G}^{(n)}$  is called *the derived series of  $\mathcal{G}$* . Recall that  $[\mathcal{G}, \mathcal{G}]$  is the smallest normal subgroup  $\mathcal{N}$  such that the quotient group  $\mathcal{G}/\mathcal{N}$  is abelian.

Following Guralnick [12], the problem of determining when  $[\mathcal{G}, \mathcal{G}] = \mathcal{C}$  is of particular interest. There are some examples in the literature where the commutator subgroup is not exclusively formed by commutators, see [5, 12, 15] and the references therein. Ore in [30] studied the commutator subgroup of certain groups. He also conjectured in [30] that any element in a finite non-abelian simple group is a commutator. This conjecture has been established in 2010 in [20]. In particular, for these kind of groups, all elements in the commutator subgroup are commutators. Thompson dealt with the corresponding problem for the special and general linear groups of finite matrices in [38]. Recently, it has been studied for certain groups of infinite matrices [11, 36].

Related to all above, the main result in this paper is:

**Theorem 1.** *Let  $\mathbb{K}$  be a field of characteristic 0. For  $n \geq 1$ :*

$$\mathcal{R}^{(n)} = \{(d, h) \in \mathcal{R} : d \in (1 + x^{2^n - n} \mathbb{K}[[x]]), h \in (x + x^{2^n} \mathbb{K}[[x]])\} \quad (1)$$

*and all of its elements are commutators of elements in  $\mathcal{R}^{(n-1)}$ .*

$\mathcal{R}$  above and in the rest of this paper denotes the Riordan group with entries in a field  $\mathbb{K}$  of characteristic 0 and  $\mathcal{R}^{(n)}$ , for  $n \geq 1$ , is the  $n$ -th commutator subgroup of  $\mathcal{R}$ . So, we describe the whole derived series of the Riordan group. A problem related to Theorem 1 was posed by Shapiro in [34]. In particular, our Theorem 1 for  $n = 2$  gives the whole answer to Shapiro's problem.

As a consequence of the above statement, the matrices in  $\mathcal{R}^{(n)}$  have all the elements in the main diagonal equal to 1 and the following  $2^n - n - 1$  diagonals have all their entries equal to 0. That is, they are of the form

The diagram shows a square matrix with a main diagonal of 1s. Below the main diagonal, there are several dotted lines representing other diagonals. Some of these dotted lines are labeled with 0. In the lower-left region of the matrix, there are several 'x' marks arranged in a pattern that suggests a specific structure of non-zero entries.

The Riordan group  $\mathcal{R}$  is a group whose elements are a special type of infinite lower triangular matrices with entries in  $\mathbb{K}$ , called Riordan matrices. The Riordan group was introduced, under this name and in a more restrictive context, by L. Shapiro and collaborators in [35]. Since then, many authors have developed this topic [14, 22, 27, 37, 40, 42]. Some previous related results can be found in [2, 16, 17, 32, 33, 39]. The concept was initially motivated by its applications to Combinatorics [37]. During the last few years, the Riordan group and some related topics have been studied in [3, 6–8, 13, 18, 25, 26, 31, 41]. For the purpose of this paper, it is of special interest the work [23], where the Riordan group is described as an inverse limit of an inverse sequence of groups of finite matrices.

The derived series is also an important aspect to understand the algebraic structure of the Riordan group. Moreover, analogous description of commutators subgroups for the Riordan groups of finite matrices introduced in [23] is also provided as a consequence of Theorem 1. These groups of finite matrices turn out to be solvable and, consequently, the Riordan group is a pro-solvable group.

In this work, two subgroups of the Riordan group, see Section 2, play an important role: the Toeplitz or Appell subgroup  $\mathcal{T}$  and the associated or Lagrange subgroup  $\mathcal{A}$ . They are also of interest because of their relationships with certain groups of formal power series.

Denote by  $\mathbb{K}[[x]]$  to the set of formal power series  $d = d_0 + d_1x + d_2x^2 + \dots$  with coefficients in  $\mathbb{K}$ . As it is detailed in the following section, each Riordan matrix can be identified with a pair of formal power series  $(d, h)$ . As a consequence, the Riordan group  $\mathcal{R}$  and its subgroups are also related to some groups of elements in  $\mathbb{K}[[x]]$ . Consider  $\mathcal{F}_0 \subset \mathbb{K}[[x]]$  to be the multiplicative group of units of formal power series. That is,  $\mathcal{F}_0$  is the multiplicative group of formal power series of order 0:  $f_0 + f_1x + f_2x^2 + \dots$  with  $f_0 \neq 0$ . Consider also  $\mathcal{F}_1 \subset \mathbb{K}[[x]]$  to be the group of invertible formal power series with respect to the composition. In this case,  $\mathcal{F}_1$  is the group of all series of order 1:  $g_1x + g_2x^2 + \dots$  ( $g_1 \neq 0$  and  $g_0 = 0$ ) with the composition. In this work, we denote by  $g^{-1}$  to the compositional inverse of  $g \in \mathcal{F}_1$ . Finally, denote by  $\mathcal{J}$  to the subgroup of  $\mathcal{F}_1$  of formal power series of the form:  $x + g_2x^2 + \dots$  which is known in the bibliography as the *substitution group* of formal power series (see [19]). We refer to [1] for an interesting survey concerning to the algebraic structure of  $\mathcal{F}_1$  and  $\mathcal{J}$ . In this paper, using the matricial approach, we obtain and improve some

of the results in [1] related to the derived subgroup of  $\mathcal{F}_1$ . The groups  $\mathcal{F}_0, \mathcal{F}_1, \mathcal{J}$  are isomorphic to certain subgroups of  $\mathcal{R}$  (as explained in Remark 2). Let us note that  $\mathcal{R}$  is isomorphic to a semidirect product  $\mathcal{F}_0 \rtimes \mathcal{F}_1$ .

Let  $\mathbb{F}_p$  be the finite field consisting of  $p$  elements for  $p$  prime. If we consider the set of formal power series  $\mathbb{F}_p[[x]]$  instead of  $\mathbb{K}[[x]]$ , the object analogue to the substitution group has also been extensively studied in the literature and it is known as the *Nottingham group* (see [4] for an overview of this topic).

The structure of this article is the following. We review the relevant basic facts about the Riordan group in Section 2. In Section 3, we introduce a family of nested subgroups of  $\mathcal{R}$ , denoted by  $\mathcal{G}_k$ , that helps us to understand the proof and statement of the main theorem. We also think that this sequence of subgroups is interesting itself. It is described in Theorem 4. We also prove therein a result, Proposition 8, concerning a functional equation in the context of formal power series (Weighted Schröder Equation). This result, which is itself interesting, is both necessary for Theorem 1 and a good example to understand the technique used in the rest of the proofs of this work: *induction in groups of finite matrices*. In Section 4, we prove that  $\mathcal{A}^{(n)} = \mathcal{G}_{2^n}$  (Theorem 9) and, as consequence, we give a description of the  $n$ -th commutator subgroup of  $\mathcal{F}_1$  (Theorem 11). In Section 5, we combine Proposition 8 and Theorem 11 to prove Theorem 1. Finally, we discuss the consequences of theorems 1 and 11 for the groups of finite Riordan matrices and we include some comments concerning future work.

## 2 Basic facts about formal power series and Riordan matrices

In this paper  $\mathbb{N}$  represents the set  $\{0, 1, 2, 3, \dots\} \subset \mathbb{K}$ ,  $[x^k]$  denotes the  $k$ -th coefficient in the series expansion.

The concept of a *Riordan matrix* and the related definition of *Riordan group* appeared in the foundational paper [35] due to Shapiro, Getu, Woan and Woodson. The original definition of a Riordan matrix given in [35] is more restrictive than that used currently in the literature, which is precisely that we are going to use herein. A Riordan matrix is a matrix  $D = (d_{i,j})_{i,j \in \mathbb{N}}$  whose columns are the coefficients of successive terms of a geometric progression, in  $\mathbb{K}[[x]]$ , where the initial term  $d$  is a formal power series of order 0 and with common ratio  $h$ , where  $h$  is a formal power series of order 1. From now on, we denote such a matrix  $D$  by  $(d, h)$ . An element  $(d, h) = (d_{i,j})_{i,j \in \mathbb{N}}$  in the Riordan group  $\mathcal{R}$  is an infinite matrix whose entries are  $d_{i,j} = [x^i]d(x)h^j(x)$ . Note that, by definition, these matrices are invertible infinite lower triangular. The set of all matrices  $(d, h)$  with the usual product of matrices forms a group called the *Riordan group* which is denoted by  $\mathcal{R}$ . In terms of the involved formal power series, the operations in the group are:

$$(d, h)(l, m) = (dl(h), m(h)), \quad (d, h)^{-1} = \left( \frac{1}{d(h^{-1})}, h^{-1} \right)$$

For any Riordan matrix  $(d, h)$  there is a formal power series  $A = \sum_{i=0}^{\infty} a_i x^i$  of order 0, called the  $A$ -sequence of  $(d, h)$ , with the property

$$d_{i,j} = \sum_{k=0}^{i-j} a_k d_{i-1, j-1+k} \quad i, j \geq 1 \quad (2)$$

It is known that  $h^{-1}(x) = \frac{x}{A(x)}$  or, equivalently,  $h(x) = x \cdot A(h(x))$ . See [21, 32]. See also page 401 in [24] for a proof of the existence and features of the A-sequence depending only on general group theoretic properties. The action induced by  $(d, h)$  in  $\mathbb{K}[[x]]$  is given by

$$(d, h)\alpha = d\alpha(h) \quad \text{for } \alpha \in \mathbb{K}[[x]].$$

Many authors call the above equality the Fundamental Theorem of Riordan Matrices. The expression  $(d, h)\alpha$  written matrixially corresponds to:

$$\begin{bmatrix} d_{00} & & & \\ d_{10} & d_{11} & & \\ d_{20} & d_{21} & d_{22} & \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \end{bmatrix} \quad \text{where } \alpha = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots$$

Note that  $(d, h)$  is a weighted composition operator in  $\mathbb{K}[[x]]$ .

For every  $n \in \mathbb{N}$  consider the general linear group  $GL(n+1, \mathbb{K})$  formed by all  $(n+1) \times (n+1)$  invertible matrices with coefficients in  $\mathbb{K}$ . Since every Riordan matrix is lower triangular we have a natural homomorphism  $\Pi_n : \mathcal{R} \rightarrow GL(n+1, \mathbb{K})$  given by  $\Pi_n((d_{i,j})_{i,j \in \mathbb{N}}) = (d_{i,j})_{i,j=0,1,\dots,n}$ , as considered in [23]. We denote by  $(d, h)_n = \Pi_n((d, h))$  and by  $\mathcal{R}_n = \Pi_n(\mathcal{R})$ . In the sequel we refer to these groups as Riordan groups of finite matrices.

We can recover the group  $\mathcal{R}$  as the inverse limit of the inverse sequence of groups  $\{(\mathcal{R}_n)_{n \in \mathbb{N}}, (P_n)_{n \in \mathbb{N}}\}$  where  $P_n : \mathcal{R}_{n+1} \rightarrow \mathcal{R}_n$  is such that if  $D \in \mathcal{R}_{n+1}$ ,  $P_n(D)$  is obtained from  $D$  by deleting its last row and its last column, i.e.  $P_n((d_{i,j})_{i,j=0,1,\dots,n+1}) = (d_{i,j})_{i,j=0,1,\dots,n}$ . See again [23]. Obviously if  $n = 0$  then  $\mathcal{R}_0 = \mathbb{K}^*$  with the usual product in  $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$ .

We now describe the two subgroups announced in the introduction. First, we have the *Toeplitz subgroup*  $\mathcal{T}$ , made up by the elements of the type  $(d, x)$ . This subgroup is normal and abelian. Second, consider also the *associated subgroup*  $\mathcal{A}$ , formed by the elements of the type  $(1, h)$ . Recall that  $\mathcal{R}$  is isomorphic to a semidirect product  $\mathcal{T} \rtimes \mathcal{A}$ . Another important fact relating the Riordan group with some groups of formal power series is:

**Remark 2.** *The groups  $\mathcal{F}_0$  and  $\mathcal{F}_1$  described in the introduction (and so any of their subgroups) are isomorphic to some subgroups of the Riordan group. In fact, there is a natural isomorphism between  $\mathcal{F}_0$  and the Toeplitz subgroup  $\mathcal{T}$  given by*

$$f \mapsto (f, x)$$

and another one between  $\mathcal{F}_1$  and the associated subgroup  $\mathcal{A}$

$$g \mapsto (1, g^{-1}).$$

### 3 The groups $\mathcal{G}_k$

This section is devoted to understand some aspects about a family of nested subgroups of  $\mathcal{A}$ , with a band of null diagonals under the main one, that are present in the proofs of the main theorems of this article. These groups are also related to the balls with respect to the ultrametric introduced in [22]. Moreover, the group  $\mathcal{G}_{k+1}$  is isomorphic, via the isomorphism described in Remark 2, to the groups  $\mathcal{G}_k$  introduced by Jennings in [19].

**Definition 3.** For  $k \geq 2$ , we define  $\mathcal{G}_k = \{(1, h) : h \in (x + x^k \mathbb{K}[[x]])\}$ .

The following proposition clarifies the shape of the matrices in each  $\mathcal{G}_k$ :

**Theorem 4.**

- (1)  $\{\mathcal{G}_k\}_{k \geq 2}$  is a nested family of normal subgroups of  $\mathcal{G}_2$ .  
(2) Let  $(1, h) = (d_{ij})_{0 \leq i, j < \infty} \in \mathcal{G}_k$ , such that  $h = x + h_k x^k + h_{k+1} x^{k+1} + \dots$ . Then

$$d_{j+m, j} = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } 1 \leq m \leq k - 2 \\ j h_{j+1} & \text{if } k - 1 \leq m \leq 2k - 2 \end{cases}$$

- (3) Matrices in  $\mathcal{G}_k$  have an A-sequence of the type  $(1, 0, \dots, 0, \alpha_{k-1}, \alpha_k, \dots)$  where  $\alpha_{l-1} = h_l$  for  $1 \leq l \leq 2k - 2$ .  
(4) Let  $(d, h) \in \mathcal{R}$ , such that  $(1, h) \in \mathcal{G}_k$  and  $d = d_0 + d_1 x + \dots$ . Then:

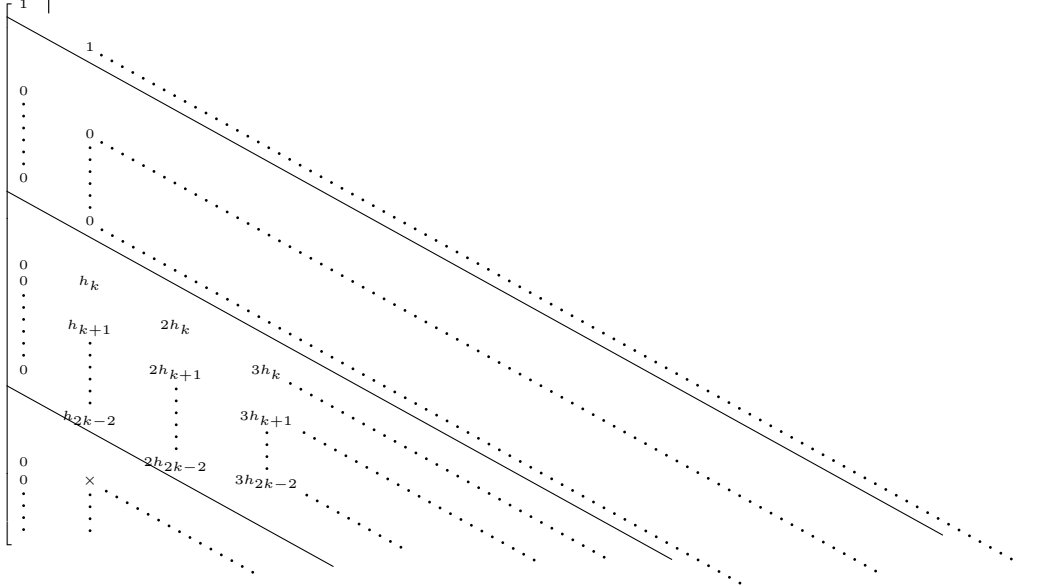
$$d_{j+m, j} = \begin{cases} d_0 & \text{if } m = 0 \\ d_m & \text{if } 1 \leq m \leq k - 2 \\ d_m + j h_{j+1} & \text{if } k - 1 \leq m \leq 2k - 2 \end{cases}$$

*Proof:* Consider the homomorphisms  $\Pi_l|_{\mathcal{G}_2} : \mathcal{G}_2 \rightarrow \mathcal{R}_l$  defined in Section 2. Then  $\mathcal{G}_k = \ker(\Pi_{k-1}|_{\mathcal{G}_2})$ , and so all of them are normal subgroups of  $\mathcal{G}_2$ . The subgroups are nested because the diagram

$$\begin{array}{ccc} \mathcal{G}_2 & & \\ \downarrow \Pi_{k-1} & \searrow \Pi_k & \\ \mathcal{R}_{k-1} & \xleftarrow{P_{k-1}} & \mathcal{R}_k \end{array}$$

is commutative.

Note that conditions in (2) give rise to the following picture of the matrix  $(1, h)$ :



The main diagonal has all its elements equal to 1. After this, the matrix has a band of  $k - 2$  null diagonals, followed by a band of  $k - 1$  diagonals which entries form an arithmetic progression. This is a consequence of being  $(1, h)$  a Riordan matrix.

Concerning the third part of the result, since the generating function of the A-sequence is  $\frac{x}{h-x}$ , and  $h \in (x + x^k \mathbb{K}[[x]])$  the statement is clear.

Note that the A-sequence of  $(d, h)$  is the same as that of  $(1, h)$ . Then, by the construction pattern of Riordan matrices in terms of the A-sequence, we can finally prove (4) as a consequence of (2). □

For  $i \geq 1$ , let us denote by  $H_i$  the map  $H_i : \mathcal{R} \rightarrow \mathbb{K}$ , given by  $H_i((d, h)) = [x^i]h$ . First we have:

**Lemma 5.** *For  $k \geq 2$  and  $2 \leq i \leq 2k - 2$ , the restriction  $H_i|_{\mathcal{G}_k}$  is a homomorphism between  $\mathcal{G}_k$  and the abelian group  $(\mathbb{K}, +)$ .*

*Proof:* Let  $(1, v), (1, w) \in \mathcal{G}_2$ , where  $v(x) = x + v_2x^2 + v_3x^3 + \dots$  and  $w(x) = x + w_2x^2 + w_3x^3 + \dots$ . The product  $(1, v) \cdot (1, w)$  is equal to:

$$\begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & v_2 & 1 & & \\ 0 & v_3 & 2v_2 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & w_2 & 1 & & \\ 0 & w_3 & 2w_2 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 0 & & 1 & & \\ 0 & & v_2 + w_2 & & 1 \\ 0 & & v_3 + w_3 + 2v_2w_2 & & 2(v_2 + w_2) & 1 \\ \vdots & & \vdots & & \vdots & \vdots & \ddots \end{bmatrix} \quad (3)$$

From this expression we can easily check that  $H_2((1, v) \cdot (1, w)) = H_2((1, v)) + H_2((1, w))$ .

Let us now consider the general case  $k \geq 3$ . In order to see that for  $2 \leq i \leq 2k - 2$ , the restriction  $H_i|_{\mathcal{G}_k}$  is an homomorphism, we distinguish two cases.



Now using Theorem 4 and that  $a_{2k-1,1} = v_{2k-1}$ , this yields:

$$c_{2k-1,1} = v_{2k-1} + w_{2k-1} + kv_k w_k + \sum_{j=k+1}^{2k-2} (2k-j-1)v_{2k-j}w_j$$

Finally, since  $(1, v) \in \mathcal{G}_k$ ,  $c_{2k-1,1} = v_{2k-1} + w_{2k-1} + kv_k w_k$ . □

Theorem 6 implies that  $[\mathcal{G}_k, \mathcal{G}_k] < \mathcal{G}_{2k}$ . As a corollary we finally obtain:

**Corollary 7.** *For all  $n \geq 1$ ,  $\mathcal{A}^{(n)} \subseteq \mathcal{G}_{2^n}$ .*

*Proof:* We proceed by induction. The result is true for  $n = 1$ : just see that if  $A, B$  are lower triangular finite or infinite matrices, the elements in the main diagonal of the product  $AB$ , are the product of the corresponding elements in the main diagonals of  $A, B$ . Also the elements in the main diagonal of  $A^{-1}$  are the multiplicative inverses of the elements in the main diagonal of  $A$ . So  $[A, B] \in \mathcal{G}_2$ .

Now let us go for the case  $n \geq 2$ , assuming the result is true for  $n - 1$ . If  $A, B \in \mathcal{A}^{(n-1)}$ , we have that  $A, B \in \mathcal{G}_{2^{n-1}}$ . According to the previous lemma  $\Pi_{2^{n-1}}(A), \Pi_{2^{n-1}}(B)$  commute, so  $\Pi_{2^{n-1}}(A^{-1}B^{-1}AB)$  is the identity matrix, and  $[A, B] \in \mathcal{G}_{2^n}$  by definition. □

We stop now to put the above results in context with some previous works. Babenko in [1] deeply studied the *substitution group*  $\mathcal{J}$  of formal power series. He also defined, for  $k \geq 2$ , the groups  $\mathcal{J}^k = (x + x^k \mathbb{K}[[x]])$ . In that article it is already proved, by a direct computation, that  $[\mathcal{J}^k, \mathcal{J}^k] \subset \mathcal{J}^{2k}$ , and then that the  $(n - 1)$ -th commutator  $\mathcal{J}^{(n-1)} = [\mathcal{J}^{(n-2)}, \mathcal{J}^{(n-2)}]$  is contained in  $\mathcal{J}^{2^n}$  (Lemma 2.3, and Section 2.3 in [1]).

Via the isomorphism described in Remark 2 between  $\mathcal{A}$  and  $\mathcal{F}_1$ , we can see that the groups  $\mathcal{J}^k$  and  $\mathcal{G}_k$  are isomorphic. Those related results appearing in [1] and our Corollary 7 are, in some sense, equivalent. However, they have been proved with different methods. But using Riordan matrices in the associated subgroup (Theorem 11) we strength the result contained in Babenko [1] about the substitution group, proving actually that  $\mathcal{J}^{(n-1)} = \mathcal{J}^{2^n}$ .

To conclude this section we prove one more result concerning the so called *weighted Schröder equation*. It is important for the rest of the paper and it is closely related to the groups  $\mathcal{G}_k$ . The name of this equation comes from the analysis of the *weighted composition operators*. It is the functional equation  $du(h) = \lambda u$ . In this case, it is considered in the formal power series context in the indeterminate  $u = u_0 + u_1 t + u_2 t^2 + \dots \in \mathbb{K}[[t]]$  and for some given  $d \in \mathcal{F}_0$ ,  $h \in \mathcal{F}_1$ ,  $\lambda \in \mathbb{K}$ . Weighted Schröder equations are suitable to be adressed in terms of Riordan matrices, because they are eigenvector problems:

$$du(h) = \lambda u \iff (d, h) \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \end{bmatrix} = \lambda \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \end{bmatrix}$$

If the previous weighted Schröder equation has a solution, then  $\lambda = d(0)$ . So just dividing both sides of the equation by  $\lambda$ , we can reduce our study, changing suitably  $d$ , to equations of the type:

$$du(h) = u, \quad \text{with } d(0) = 1 \tag{5}$$

The following Proposition is also a detailed example of how the rest of the results in this article are proved using *induction in the size of the matrices in Riordan groups of finite matrices*.

**Proposition 8.**

- (1) Suppose  $h = rx + h_2x^2 + \dots$  where  $r \in \mathbb{K}$  and it is not a root of unity. Then there is a unique solution  $u \in (1 + x\mathbb{K}[[x]])$  of (5), for any  $d$ .
- (2) Suppose  $k \geq 2$  and  $h = x + h_kx^k + \dots$ , with  $h_k \neq 0$ . Then, there exists a solution  $u \in (1 + x\mathbb{K}[[x]])$  of (5) if and only if  $d \in (1 + x^k\mathbb{K}[[x]])$ . When this  $u$  exists, it is unique.
- (3) Moreover, in the previous case, for  $i \geq 1$ ,  $u \in (1 + x^i\mathbb{K}[[x]])$  if and only if  $d \in (1 + x^{i+k-1}\mathbb{K}[[x]])$ .

*Proof:* Let  $(d, h) = (d_{ij})_{0 \leq i, j < \infty}$ . We have that:

- (1) Equation (5) in matricial form is:

$$(d, h) \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ d_{10} & r & & & \\ \vdots & \vdots & \ddots & & \\ d_{n0} & d_{n1} & \dots & r^n & \\ \vdots & \vdots & & \vdots & \ddots \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \\ \vdots \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \\ \vdots \end{bmatrix}$$

Now we solve the equations corresponding to each row by the method of forward substitution:

$$\begin{cases} u_0 = u_0 \\ d_{10}u_0 + ru_1 = u_1 \\ \vdots \\ d_{n0}u_0 + d_{n1}u_1 + \dots + d_{n-1,n}u_{n-1} + r^nu_n = u_n \\ \vdots \end{cases}$$

From the fact that  $r^n \neq 1$  for all  $n$ , we have that the infinite system above has solution. In fact, for each  $u_0$  given there is a unique solution of the system. Choose  $u_0 = 1$ .

- (2) If there is a solution, then necessarily  $d \in (1 + x^k\mathbb{K}[[x]])$  because if  $0 < i < k$ , then:

$$\begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ \vdots & \ddots & \ddots & & \\ 0 & \dots & 0 & 1 & \\ d_{i0} & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ u_1 \\ \vdots \\ u_{i-1} \\ u_i \end{bmatrix} = \begin{bmatrix} 1 \\ u_1 \\ \vdots \\ u_{i-1} \\ u_i \end{bmatrix} \Rightarrow d_{i0} = 0$$

To prove the converse, we show by induction over  $n$ , using the projections  $\Pi_n(d, h) = (d, h)_n$ , that there exist a unique sequence  $(1, u_1, \dots, u_{n-k+1})$  such that:

$$(d, h)_n \begin{bmatrix} 1 \\ u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} 1 \\ u_1 \\ \vdots \\ u_n \end{bmatrix} \tag{6}$$

independently on the values of the terms in the sequence  $(u_{n-k+2}, \dots, u_n)$ . According to Theorem 4, if we write (6) as a system of linear equations in the indeterminates  $u_1, \dots, u_n$ , then the values  $u_{n-k+2}, \dots, u_n$  do not appear in them.

The first case,  $n = k - 1$ , is obviously true since  $(d, h)_n = I_{n+1}$  (identity matrix). Now consider the case  $n > k - 1$ . By induction hypothesis there exists a unique sequence  $(1, u_1, \dots, u_{n-k})$  such that:

$$(d, h)_{n-1} \begin{bmatrix} 1 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} 1 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix} \quad (7)$$

independently on the sequence  $(u_{n-k+1}, \dots, u_{n-1})$ . The system (6) has only one more linear equation than (7). So, we only need to check that there exists a unique  $u_{n-k+1}$  such that the last equation in (6) holds:

$$d_{n0} + \left( \sum_{m=1}^{n-k} d_{nm} u_m \right) + d_{n, n-k+1} u_{n-k+1} + u_n = u_n$$

The elements  $d_{ij}$  with  $i - j = k - 1$  form an arithmetic progression, according to Theorem 4. Recall that the first term is 0 and its common difference is  $d_{k1} = h_k \neq 0$ . So, we can see that this equation is equivalent to:

$$(n - k + 1)h_k u_{n-k+1} = -d_{n0} - \left( \sum_{m=1}^{n-k} d_{nm} u_m \right) \quad (8)$$

Then, there exists a unique value  $u_{n-k+1}$  such that this equation holds. See that the right hand side in the equation above is completely determined by the previous step.

- (3) The case  $i = 1$  have already been discussed in (2). Now to see the general case  $i > 1$  see that in (8), taking  $i = n - k + 1$ ,

$$ih_k u_i = -d_{i+k-1,0} - \left( \sum_{m=1}^{i-1} d_{i+k-1,m} u_m \right)$$

By hypothesis,  $u \in (1 + x^{i-1}\mathbb{K}[[x]])$  and  $d \in (1 + x^{i+k-2}\mathbb{K}[[x]])$ , so  $u_i \neq 0$  if and only if  $d_{i+k-1,0} \neq 0$ .

□

## 4 The Derived Series of the associated subgroup of the Riordan group

The main result in this section is the following:

**Theorem 9.** *Let  $\mathcal{A}$  be the associated subgroup of the Riordan group  $\mathcal{R}$ . Then for  $n \geq 1$ :*

$$\mathcal{A}^{(n)} = \mathcal{G}_{2^n} \tag{9}$$

and all of its elements are commutators of elements in  $\mathcal{A}^{(n-1)}$ .

Via the isomorphism between  $\mathcal{A}$  and  $\mathcal{F}_1$  explained in Remark 2, this result implies Theorem 11 below.

The case  $n = 1$  in the theorem states that  $\mathcal{A}' = \mathcal{G}_2$ , that is,  $\mathcal{A}'$  is the subgroup of matrices of  $\mathcal{A}$  with all the entries in the main diagonal equal to one. We will start the proof of Theorem 9 showing this fact. The statement of the analogous result for formal power series is that  $\mathcal{F}'_1 = \mathcal{J}$ .

**Lemma 10.**  *$\mathcal{A}' = \mathcal{G}_2$ . Moreover, any element in  $\mathcal{G}_2$  is a commutator of elements in  $\mathcal{A}$ .*

*Proof:* In Corollary 7 we already proved that  $\mathcal{A}' \subset \mathcal{G}_2$ . So, the only thing we need to see is that any element  $(1, g) \in \mathcal{G}_2$  (that is, with  $g = x + g_2x^2 + g_3x^3 + \dots$  or equivalently an element in  $\mathcal{A}$  with all the entries in the main diagonal equal to 1) is a commutator of elements in  $\mathcal{A}$ .

We will prove a stronger fact. Fix this  $(1, g)$ . For any  $r \in \mathbb{K}$  not a root of unity, there exists a unique  $(1, v) \in \mathcal{G}_2 \subset \mathcal{A}$  (and so with all the entries in its main diagonal equal to 1) such that:

$$(1, g) = (1, rx)^{-1}(1, v)^{-1}(1, rx)(1, v)$$

Note that  $(1, rx)$  is a diagonal Riordan matrix.

Using the inverse limit approach to the Riordan group (see again [23]), this is equivalent to showing that there exists a unique sequence of finite Riordan matrices  $(B_0, B_1, B_2, \dots, B_i, \dots)$  satisfying the following conditions.

For all  $i \geq 0$ ,  $B_i \in \Pi_i(\mathcal{A})$ ,  $B_i = p_i(B_{i+1})$ ,  $B_i$  has all the entries in the main diagonal equal to one and  $(1, g)_i = (1, rx)_i^{-1} B_i^{-1} (1, rx)_i B_i$ .

Recall that any element in  $\Pi_i(\mathcal{A})$  is determined by its second column. So, if we denote by  $B_i = (b_{lm})_{0 \leq l, m < i}$ , to solve the last equation in the previous paragraph is equivalent to solve:

$$B_i(1, rx)_i \begin{bmatrix} 0 \\ 1 \\ g_2 \\ \vdots \\ g_i \end{bmatrix} = (1, rx)_i \begin{bmatrix} 0 \\ 1 \\ b_{21} \\ \vdots \\ b_{i1} \end{bmatrix} \tag{10}$$

For  $i = 1$ , (10) is a trivial equality, independently on  $r$ . The first case with special meaning is  $i = 2$ . We have:

$$\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & b_{21} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 & r \\ 0 & 0 & r^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & r & \\ 0 & 0 & r^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ b_{21} \end{bmatrix}$$

This matricial equation is equivalent to a system of three linear equations. The first two of them correspond to the case  $i = 1$ . The last one is:

$$rb_{21} + r^2g_2 = r^2b_{21}$$

and, since  $r \neq 1$ , then the solution is:

$$b_{21} = \frac{r}{r-1}g_2.$$

In the general case  $i > 2$ , we need to show that there is a unique  $b_{i1}$  satisfying the last equation in the system (10). Note that the previous case have already fixed the unique possible choice of  $b_{21}, \dots, b_{i-1,1}$  satisfying the rest of the equations. This last equation is:

$$\begin{aligned} [0 \quad b_{i1} \quad \dots \quad b_{ii}] (1, rx)_i \begin{bmatrix} 0 \\ 1 \\ g_2 \\ \vdots \\ g_i \end{bmatrix} &= [0 \quad \dots \quad 0 \quad r^i] \begin{bmatrix} 0 \\ 1 \\ b_{21} \\ \vdots \\ b_{i1} \end{bmatrix} \implies \\ \implies rb_{i1} + \sum_{k=2}^i (b_{ik} r^k g_k) &= r^i b_{i1} \implies (1 - r^{i-1}) b_{i1} = - \sum_{k=2}^i (b_{ik} r^{k-1} g_k) \end{aligned}$$

We have to recall at this point that, for  $2 \leq m \leq i$ , all the entries  $b_{im}$  are determined by the elements  $b_{l1}$  with  $1 \leq l \leq i-1$ , because  $B_i$  is a finite Riordan matrix. Thus, they are determined by induction hypothesis. So, nothing in the right hand side of the last equation depends on  $b_{i1}$  and all the terms are known. Since the coefficient  $(1 - r^{i-1})$  is not 0 ( $r$  is not a root of unity) the equation has a unique solution, and the proof is complete.  $\square$

Finally, we are ready to prove the main result of this section:

*Proof of Theorem 9:* We have already proved that  $\mathcal{A}' = \mathcal{G}_2$ . Let us proceed by induction. Assume that  $\mathcal{A}^{(n-1)} = \mathcal{G}_{2^{n-1}}$ . Corollary 7 guarantees that  $\mathcal{A}^{(n)} \subseteq \mathcal{G}_{2^n}$ . So, we only need to prove that every element in  $\mathcal{G}_{2^n}$  is a commutator of elements in  $\mathcal{A}^{(n-1)}$ .

Again, we will prove a stronger fact. Let us fix an element  $(1, g) \in \mathcal{G}_{2^n}$  and suppose that  $A = (1, x + \lambda x^{2^{n-1}})$  with  $\lambda \neq 0$ . In particular,  $A$  in  $\mathcal{A}^{(n-1)}$  and not in  $\mathcal{G}_{2^{n-1}+1}$ . Then, we have to prove that there exists a unique  $B \in \mathcal{G}_{2^{n-1}+1}$  such that:

$$(1, g) = A^{-1} B^{-1} A B$$

As in the previous lemma, this is equivalent to prove that

**Claim (n):** Fixed  $n$ , there exists a unique sequence of finite Riordan matrices  $(B_0, B_1, B_2, \dots, B_i, \dots)$  satisfying the following conditions.  $B_{2^{n-1}}$  must be the identity matrix of the corresponding size and for all  $i \geq 0$ ,  $B_i \in \Pi_i(\mathcal{A})$ ,  $B_i = p_i(B_{i+1})$ . Finally, for each  $i$ ,  $B_i$  must satisfy:

$$B_i \Pi_i(A) (1, g)_i = \Pi_i(A) B_i \tag{11}$$

Denote  $A = (a_{lm})_{0 \leq l, m < \infty}$ ,  $B_i = (b_{lm})_{0 \leq l, m < i}$ . For  $i \geq 2^n$ , Equation (11) is equivalent to

$$\left[ \begin{array}{c|c} B_{i-1} & \\ \hline b_{i0} \dots b_{i,i-1} & 1 \end{array} \right] \left[ \begin{array}{c|c} \Pi_{i-1}(A) & \\ \hline a_{i0} \dots a_{i,i-1} & 1 \end{array} \right] \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ g_{2^n} \\ \vdots \\ g_i \end{bmatrix} = \left[ \begin{array}{c|c} \Pi_{i-1}(A) & \\ \hline a_{i0} \dots a_{i,i-1} & 1 \end{array} \right] \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ b_{2^{n-1},1} \\ \vdots \\ b_{i1} \end{bmatrix}$$

Note that the last equation in the above linear system, in the indeterminates  $b_{2^{n-1},1}, \dots, b_{i1}$ , is:

$$\underbrace{[b_{i0} \dots b_{i,i-1}] \Pi_{i-1}(A)}_{(I)} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ g_{2^n} \\ \vdots \\ g_{i-1} \end{bmatrix} + [a_{i0} \dots a_{i,i-1}] \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ g_{2^n} \\ \vdots \\ g_{i-1} \end{bmatrix} + g_i = [a_{i0} \dots a_{i,i-1}] \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ b_{2^{n-1},1} \\ \vdots \\ b_{i-1,1} \end{bmatrix}}_{(II)} + b_{i1} \quad (12)$$

Appart from  $b_{i1}$ , only the terms (I) and (II) in the above equation contain indeterminates.

Now we will see, as happened in some of the previous proofs and because of the structure of the matrices, that in the linear system above only the unknowns  $b_{11}, \dots, b_{i-2^{n-1}+1,1}$  are involved.

First, note that  $A = (1, x + \lambda x^{2^{n-1}}) \in \mathcal{G}_{2^n-1}$ . So, according to Theorem 4, we obtain

$$a_{j+m,j} = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } 1 \leq m \leq 2^{n-1} - 2 \\ j\lambda & \text{if } m = 2^{n-1} - 1 \\ 0 & \text{if } 2^{n-1} \leq m \leq 2^n - 2 \end{cases}, \quad b_{j+m,j} = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } 1 \leq m \leq 2^{n-1} - 1 \\ jb_{j+1,1} & \text{if } 2^{n-1} \leq m \leq 2^n - 2 \end{cases} \quad (13)$$

We have that:

$$(I) = \left[ 0, b_{i1} + b_{i,2^{n-1}}\lambda, b_{i2} + \sum_{k=2^{n-1}+1}^{i-1} b_{ik}a_{k2}, \dots, b_{i,i-1} + \sum_{k=2^{n-1}+i-2}^{i-1} b_{ik}a_{k,i-1} \right] \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ g_{2^n} \\ \vdots \\ g_{i-1} \end{bmatrix}$$

Note that all the elements  $b_{ik}$  in any of the sums  $\sum_{k=2^{n-1}+1}^{i-1} b_{ik}a_{k2}, \dots, \sum_{k=2^{n-1}+i-2}^{i-1} b_{ik}a_{k,i-1}$  depend only on  $b_{11}, \dots, b_{i-2^{n-1},1}$ . So:

$$(I) = b_{i1} + \lambda b_{i,2^{n-1}} + [C_1] \quad (14)$$

where nothing in  $[C_1]$  depends on  $b_{i-2^{n-1}+1,1}, b_{i-2^{n-1}+2,1}, \dots$

Moreover,

$$(II) = [a_{i0} \dots a_{i,i-1}] \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ b_{2^{n-1},1} \\ \vdots \\ b_{i-1,1} \end{bmatrix} = \left[ \sum_{k=1}^{i-2^{n-1}} a_{ik} b_{k1} \right] + a_{i,i-2^{n-1}+1} b_{i-2^{n-1}+1,1} \quad (15)$$

where, again, nothing inside the term  $\left[ \sum_{k=1}^{i-2^{n-1}} a_{ik} b_{k1} \right]$  depends on  $b_{i-2^{n-1}+1,1}, b_{i-2^{n-1}+2,1}, \dots$ . Substituting (14) and (15) in (12), reorganizing and cancelling when needed, we obtain

$$a_{2^{n-1},1} b_{i,2^{n-1}} - a_{i,i-2^{n-1}+1} b_{i-2^{n-1}+1,1} = -[C_1] - [a_{i0} \dots a_{i,i-1}] \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ g_{2^n} \\ \vdots \\ g_{i-1} \end{bmatrix}}_{[C_2]} - g_i + \left[ \sum_{k=1}^{i-2^{n-1}} a_{ik} b_{k1} \right] \quad (16)$$

The right hand side in (16), i.e.  $[C_2]$ , does not depend on  $b_{i-2^{n-1}+1,1}, b_{i-2^{n-1}+2,1}, \dots$ .

Following (13) we get  $a_{2^{n-1},1} = \lambda$ ,  $a_{i,i-2^{n-1}+1} = (i - 2^{n-1} + 1)\lambda$ . On the other hand, by definition of finite Riordan matrix, there exist a matrix  $B = (1, h) \in \mathcal{A}$  such that  $\Pi_i(B) = B_i$ . The entries in the columns in  $B$  are the coefficients of a geometric sequence with first term the formal power series 1 with common ratio the formal power series  $h$ . Since  $b_{i-2^{n-1}+1,1} = [x^{i-2^{n-1}+1}]h$  and  $b_{i,2^{n-1}} = [x^i]h^{2^{n-1}}$ , then

$$b_{i,2^{n-1}} = b_{i-2^{n-1}+1,1} + [C_3]$$

where nothing in the term  $[C_3]$  depends on  $b_{i-2^{n-1}+1,1}, b_{i-2^{n-1}+2,1}, \dots$ .

So, finally, (16) is equivalent to:

$$\lambda(b_{i-2^{n-1}+1,1} + [C_3]) - (i - 2^{n-1} + 1)\lambda b_{i-2^{n-1}+1,1} = [C_2] \quad (17)$$

which has a unique solution in the indeterminate  $b_{i-2^{n-1}+1,1}$ , provided that  $b_{k1}$  is known for  $1 \leq k \leq i - 2^{n-1}$ .

We can now prove Claim (n) by induction over  $i$ . Let us begin imposing that  $B_{2^{n-1}}$  is the identity matrix. We trivially have (Theorem 6) that (11) holds for  $i \leq 2^n - 1$ . Assume that  $B_{i-1} \Pi_{i-1}(A)(1, g)_{i-1} = \Pi_{i-1}(A) B_{i-1}$  for some  $i \geq 2^n$  for a fixed value of the indeterminates  $b_{21}, \dots, b_{i-2^{n-1},1}$  indedntently on the value of the indeterminates  $b_{i-2^{n-1}+1,1}, b_{i-2^{n-1}+2,1}, \dots$ . Then, as explained before, Equation (11) holds if and only if its last equation holds. We have just showed

that there is a unique solution  $b_{i-2^{n-1}+1,1}$  of this equation and that this solution does not depend on  $b_{i-2^{n-1}+2,1}, b_{i-2^{n-1}+3,1}, \dots$

□

**Theorem 11.** For  $n \geq 1$ :

$$\mathcal{F}_1^{(n)} = \mathcal{J}^{(n-1)} = \{g \in \mathcal{J} : g \in (x + x^{2^n} \mathbb{K}[[x]])\} \quad (18)$$

and all of its elements are commutators in  $\mathcal{F}_1^{(n-1)}$ .

**Remark 12.** In the case in which  $\mathbb{K}$  is a finite field, there is an analogue of Theorem 11, in the context of the so called Nottingham groups (see [4]). Consequently, we could also obtain an analogue of Theorem 9.

## 5 Proof of Theorem 1 and some consequences

Finally we can prove Theorem 1 as a consequence of Proposition 8 and Theorem 11.

*Proof of Theorem 1:* We are going to prove by induction over  $n$  that:

$$\mathcal{R}^{(n)} = \begin{cases} \mathcal{R} & \text{if } n = 0 \\ \{(d, h) \in \mathcal{R} : d \in (1 + x^{2^n - n} \mathbb{K}[[x]]), h \in (x + x^{2^n} \mathbb{K}[[x]])\} & \text{if } n > 0 \end{cases}$$

The case  $n = 0$  is just a notational fact. Let us prove the case  $n > 0$ , assuming that the case  $n - 1$  holds. The set:

$$\{(d, h) \in \mathcal{R} : d \in (1 + x^{2^n - n} \mathbb{K}[[x]]), h \in (x + x^{2^n} \mathbb{K}[[x]])\}$$

is a group. So we only need to prove that any commutator of elements in  $\mathcal{R}^{(n-1)}$  is of this type.

Using the multiplication and inversion formula in the Riordan group, we have that the matricial equation  $(d, h) = (u, v)^{-1}(f, g)^{-1}(u, v)(f, g)$  is equivalent to the following equation in formal power series

$$\begin{cases} h = g \circ v \circ g^{-1} \circ v^{-1} \\ \left[ \frac{u}{u(g)d(v(g))} \right] f(v) = f \end{cases} \quad (19)$$

If  $v, g \in (x + x^{2^{n-1}} \mathbb{K}[[x]])$  we have already proved that  $h \in (x + x^{2^n} \mathbb{K}[[x]])$  (consequence of Theorem 6), and that in this case, for any fixed  $h$  we can find such a pair  $v, g$  (Theorem 11).

On the other hand, according to Proposition 8, there exists  $f \in (1 + x^{2^{n-1} - n + 1} \mathbb{K}[[x]])$ , satisfying the second condition in (19) if and only if

$$\left[ \frac{u}{u(g)d(v(g))} \right] \in (x + x^{2^n - n} \mathbb{K}[[x]]) \quad (20)$$

To check (20), see that, as soon as  $u \in (1 + x^{2^{n-1} - n + 1} \mathbb{K}[[x]])$  we obtain that  $\frac{u}{u(g)} \in (1 + x^{2^n - n} \mathbb{K}[[x]])$ . Note that  $u$  and  $u(g)$  only differ from the  $(x^{2^n - n})$ -th term onwards. On the other hand,  $\frac{1}{d(v(g))} \in (x + x^{2^n - n} \mathbb{K}[[x]])$  if and only if  $d \in (x + x^{2^n - n} \mathbb{K}[[x]])$ . As a consequence, (20) holds if and only if  $d \in (x + x^{2^n - n} \mathbb{K}[[x]])$  and the proof is finished.

□

The case  $n = 1$  in Theorem 1 was previously proved in [25].

Recall that  $\Pi_k(\mathcal{R}) = \mathcal{R}_k$  and  $\Pi_k$  is a group homomorphism. So, using Lema 2.1 in [28] we get

**Corollary 13.** *For every  $n, k \geq 1$ ,  $\mathcal{R}_k^{(n)} = \Pi_k(\mathcal{R}^{(n)})$ .*

**Remark 14.** (a) *Note that, as soon as  $k < 2^n - n$ ,  $\mathcal{R}_k^{(n)}$  is formed only by the identity matrix in  $\mathcal{R}_k$ .*

(b) *If  $2^n - n \leq k < 2^n$ ,  $\mathcal{R}_k^{(n)}$  is formed only by Toeplitz matrices and then in this case  $\mathcal{R}_k^{(n)}$  is abelian.*

(c) *Note also that  $\mathcal{R}_k$  is solvable.*

**Remark 15. (Some prospects)** *For any integer  $k \geq 0$  we denote by  $\varphi(k)$  to the derived length or solvable length of  $\mathcal{R}_k$ , that is, the lowest integer  $n$  such that  $\mathcal{R}_k^{(n)}$  is the trivial subgroup. The first terms of the sequence  $\varphi$  are*

$$1, 2, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 5, \dots$$

*They are the first terms of the sequence A103586 in OEIS. See [9,10,29], and the references therein, for some featured properties of the solvable length in special cases.*

*A description of the derived series of the Riordan group has not been given yet for other choices of the field  $\mathbb{K}$  (cases in which  $\mathbb{K}$  is not of characteristic 0). See Remark 12.*

*For a better understanding of the algebraic structure of the Riordan group it would be desirable to determine the quotient groups  $\mathcal{R}^{(n)}/\mathcal{R}^{(n+1)}$ . It is easy to see, and it is proved in [25], that  $\mathcal{R}/\mathcal{R}'$  is isomorphic to  $\mathbb{K}^* \times \mathbb{K}^*$ . But the rest of the elements in this sequence of quotients do not seem to have such an easy description.*

*We would like to point out the following. Let  $\mathfrak{R}$  be a commutative ring with a unit, denoted by 1. Consider a coherent definition of Riordan matrix with entries in  $\mathfrak{R}$  and such that all the entries in the main diagonal equal to 1. The set of such matrices should be also a group (with a certain operation). For this group, we can study the corresponding derived subgroup. The case where  $\mathfrak{R} = \mathbb{Z}$  (the ring of integers) is of special interest in combinatorics. It would be interesting, due to the possible applications, to compute the derived series. The arguments followed herein do not apply for rings, since we usually need to take multiplicative inverses. See (17) and (8).*

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