UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FÍSICAS

TESIS DOCTORAL

Polinomios ortogonales, relaciones de recurrencia y ecuaciones no lineales

MEMORIA PARA OPTAR AL GRADO DE DOCTORA

PRESENTADA POR

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Índice general

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Summary

The purpose of this Thesis is to obtain non-linear relatiosnships for recurrence coefficients from several types of orthogonal polynomials using matrix analysis techniques in order to study the discrete orthogonality with one weight and to expand it to the multiple discrete orthogonality with weights that satisfy Pearson equations.

Firstly, for the purpose of studying the discrete orthogonality, Gauss-Borel factorization is applied to the moment matrix, which in this case is reduced to a Choslesky factorization. In this case, polynomials are expressed in terms of τ functions, which are generalized hypergeometric functions wronskians. Also, the Laguerre-Freud matrix is introduced, which is a banded matrix and its structure depends of the Pearson equation polynomials degrees. The Laguerre-Freud matrix codes the shifts for the independent variable of polynomials and it is used to obtain the Laguerre-Freud equations, which are fulfilled for the recurrence coefficients. Besides, several systems of equations and relevant equations are obtained: the recurrence coefficients form a Toda system, the τ functions fulfill a Toda equation and the squared norms of the polynomials satisfy Nijhoff-Capel Toda equations. In order to conclude the discrete orthogonality research, several orthogonal polynomial families are estudied: generalized Charlier, generalized Meixner, and those families in which the first moments are given by generalizaed hypergeometric functions ${}_1F_2$, ${}_2F_2$, ${}_3F_2$ y ${}_2F_1$ (which is also known as Hahn I case), and the Laguerre-Freud equations are obtained for them. For the generalized Charlier case second and third order ordinary differential equations, that are related to Painlevé equations, are obtained.

Subsequently, in order to expand that which has been studied for the discrete orthogonality to discrete multiple orthogonality, Gauss-Borel factorization is applied to the moment matrix. For the multiple orthogonality, polynomials are written in terms of τ functions which are multiple wronskians of hypergeometric equations. In addition, the Laguerre-Freud matrix, which is a banded matrix, is introduced and models spectral parameter shifts for type I and type II polynomials, and it is used to obtain the Laguerre-Freud equations. Analogously to the discrete orthogonality, multicomponent expansions of the Toda system are obtained for the recurrence coefficients, an expansion of the Toda equation for τ functions and an expansion of the Nijhoff-Capel Toda equations fulfilled by the squared norms of the polynomials. Finally, some families of multiple orthogonal polynomials are studied: Charlier, generalized Charlier, Meixner II and generalized Meixner II.

Conclusions obtained are:

- The Laguerre-Freud equations for the recurrence coefficients are related to Painlevé equations for the discrete orthogonality.
- The number of non-null diagonals that are contained in the Laguerre-

Freud matrix depends on Pearson equations polynomials degrees: for the discrete orthogonality the number of non-null sub-diagonals is given by the σ polynomial degree and the number of super-diagonals is given by the θ polynomial degree, and for the multiple orthogonality with q weights (where $q \geq 2$) the number of sub-diagonals is given by qM, where $M = \max\{\deg \sigma^{(1)}, \dots, \deg \sigma^{(q)}\}$ and the number of super-diagonals is given by the θ polynomial degree.

- The number of Toda equations and the number of involved variables increase as the number of weights increases.
- The τ function for the discrete orthogonality fulfills a second order difference-differential equation and for the discrete multiple orthogonality with two weights a third order difference-differential equation is fulfilled.
- The Toda type system which is fulfilled by the recurrence coefficients has $q + 1$ equations, where $q \geq 1$, and has a general form.

Resumen

El objetivo de esta tesis es obtener relaciones no lineales para los coeficientes de la recurrencia de varios tipos de polinomios ortogonales usando técnicas de análisis matricial en el estudio de la ortogonalidad discreta para el caso de un peso y su extensión a la ortogonalidad múltiple con pesos que satisfacen ecuaciones de Pearson.

En primer lugar, a la hora de estudiar la ortogonalidad discreta se aplica la factorización de Gauss-Borel, que en este caso se reduce a una factorización de Cholesky, a la matriz de momentos. En este caso, los polinomios se expresan en términos de funciones τ, que son wronskianos de funciones hipergeométricas generalizadas. Además, se introduce la matriz de Laguerre-Freud, que tiene una estructura a bandas y dependerá de los grados de los polinomios de la ecuación de Pearson, que modeliza los saltos en la variable independiente de los polinomios y se emplea para obtener las ecuaciones de Laguerre-Freud, que son relaciones que satisfacen los coeficientes de la recurrencia. Por otro lado, se obtienen varios sistemas de ecuaciones y ecuaciones relevantes: los coeficientes de la recurrencia forman un sistema de ecuaciones de Toda, las funciones τ satisfacen una ecuación de Toda y las normas al cuadrado de los polinomios satisfacen ecuaciones de Nijhoff-Capel Toda. Para culminar el estudio de la ortogonalidad discreta, se estudian las familias de polinomios de Charlier generalizado, Meixner generalizado, y aquellas en las que los primeros momentos son las funciones hipergeométricas generalizadas ${}_1F_2$, ${}_2F_2$, ${}_3F_2$ y ${}_2F_1$ (o caso de Hahn I), obteniéndose las ecuaciones de Laguerre-Freud. En el caso de Charlier generalizado se obtienen ecuaciones diferenciales ordinarias de segundo y tercer orden para los coeficientes de la recurrencia relacionadas con las ecuaciones de Painlevé.

Posteriormente, para extender lo dicho anteriormente a la ortogonalidad múltiple discreta, se aplica la factorización de Gauss-Borel a la matriz de momentos para polinomios ortogonales múltiples discretos. En el caso de la ortogonalidad múltiple, los polinomios se escriben en términos de funciones τ que son wronskianos múltiples de ecuaciones hipergeométricas. Asimismo, se introduce la matriz de Laguerre-Freud, que tiene una estructura a bandas, dicha matriz modeliza los saltos del parámetro espectral de los polinomios de tipo I y tipo II y se emplea en la obtención de las ecuaciones de Laguerre-Freud. De forma análoga al caso de la ortogonalidad discreta, se obtienen extensiones multicomponente del sistema de Toda que satisfacen los coeficientes de la recurrencia, de la ecuación de Toda que satisfacen las funciones τ y de las ecuaciones de Nijhoff-Capel Toda que satisfacen las normas al cuadrado de los polinomios obtenidas en el caso de la ortogonalidad discreta. Para finalizar, se estudian los casos de Charlier, Charlier generalizado, Meixner II y Meixner II generalizado.

Las conclusiones que se obtienen son:

- Las ecuaciones de Laguerre-Freud que satisfacen los coeficientes de la recu-

rrencia en la ortogonalidad discreta están relacionadas con las ecuaciones de Painlevé.

- El número de diagonales no nulas que tiene la matriz de Laguerre-Freud depende de los grados de los polinomios de las ecuaciones de Pearson: para la ortogonalidad discreta el número de subdiagonales no nulas viene dado por el grado del polinomio σ y el número de superdiagonales viene dado por el grado del polinomio θ, y en el caso de la ortogonalidad múltiple general respecto a q pesos, donde $q \geq 2$, el número de subdiagonales no nulas es qM, donde $M = \text{máx}\{\deg \sigma^{(1)}, \dots, \deg \sigma^{(q)}\}$ y el número de superdiagonales es el grado del polinomio θ.

- Las ecuaciones de Toda aumentan en número y el número de variables involucradas a medida que aumenta el número de pesos.

- La función τ para la ortogonalidad discreta satisface una ecuación diferencial en diferencias de segundo orden y en el caso de la ortogonalidad múltiple respecto a dos pesos satisface una ecuación diferencial en diferencias de tercer orden que es una expansión de la obtenida en el caso monopeso.

- El sistema de Toda que satisfacen los coeficientes de la recurrencia tiene $q+1$ ecuaciones, siendo $q \geq 1$, y tiene una forma general.

Lista de Publicaciones

- Fernández-Irisarri I., Mañas M. *Toda and Laguerre-Freud equations for multiple discrete orthogonal polinomials with an arbitrary number of weights*. arxiv. DOI 10.48550/arXiv.2407.00777.

- Fernández-Irisarri I., Mañas M. *Toda and Laguerre–Freud equations and tau functions for hypergeometric discrete multiple orthogonal polynomials*. Analysis and Mathematical Physics. 14, 30 (2024). DOI 10.1007/s13324-024-00876-4.

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Capítulo 1

Introducción

1.1. Objetivos de la tesis

En esta tesis se estudian las relaciones no lineales que satisfacen los coeficientes de la recurrencia y las funciones τ, para obtener dichas relaciones se define la matriz de Laguerre-Freud Ψ partiendo de la factorización LU de la matriz de momentos.

Inicialmente la factorización LU de la matriz de momentos se aplica al caso de la ortogonalidad discreta respecto a un peso, obteniedo sistemas y ecuaciones que satisfacen los coeficientes de la recurrencia y funciones τ y aplicándolo a casos prácticos. Posteriormente se estudia y generaliza lo estudiado para polinomios ortogonales múltiples discretos respecto de 2 pesos, y también se han obtenido resultados para la generalización al caso de la ortogonalidad múltiple discreta respecto a q pesos.

1.2. Polinomios Ortogonales

La Teoría de Polinomios Ortogonales nace a finales del siglo XVIII como disciplina enmarcada dentro del Análisis y la Matemática Aplicada. Tiene muchas aplicaciones en la resolución de problemas de diferentes ámbitos de la Física, como la Física Matemática, el Electromagnetismo, la Óptica o la Mecánica Cuántica, por ejemplo, en la resolución del problema del átomo de hidrógeno o el uso de los polinomios de Zernike para describir aberraciones ópticas en una superficie circular. También tiene aplicaciones dentro de las matemáticas [33], [37] como la Teoría Analítica de Números o la Teoría de la Aproximación, por ejemplo en la obtención de los armónicos esféricos o en la resolución de ecuaciones diferenciales.

Asociado a una **medida positiva** μ introducimos un funcional

$$\mathscr{L}[f(x)] = \int_a^b f(x)d\mu(x) = \langle \mu, f(x) \rangle.$$

Una medida positiva μ en un espacio medible $(X, \mathcal{A})$, donde $\mathcal{A}$ es una σ-álgebra [6], es una aplicación $\mu : \mathcal{A} \to \mathbb{R}_+$ tal que:

- $\mu(\emptyset) = 0$,
- dada una familia numerable $\{A_n : n \in \mathbb{N}\}$ de conjuntos de $\mathcal{A}$ disjuntos dos a dos, entonces $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

Los **momentos del funcional** se definen como

$$\mu_j = \langle \mu, x^j \rangle, \quad j \in \{0, 1, \dots\}.$$

Se suele normalizar tal que $\mu_0 = 1$. Dada una medida μ positiva (el funcional es definido positivo) y unos momentos finitos, existe una única secuencia de polinomios mónicos, es decir, polinomios donde el coeficiente del término de grado máximo es 1, $\{p_n(x)\}_0^{\infty}$ y una secuencia de números positivos $\{H_n\}_0^{\infty}$ con $H_0 = 1$ tales que

$$\langle \mu, p_m(x)p_n(x) \rangle = H_n \delta_{m,n}.$$

Para la aproximación matricial, se introduce la **matriz de momentos** que se define como

$$\mathscr{M} = \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \dots \\ \mu_1 & \mu_2 & \mu_3 & \dots \\ \mu_2 & \mu_3 & \mu_4 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

en términos de los momentos μ_j. La matriz de momentos es una **matriz de Hankel** [23], es decir, sus elementos cumplen

$$\mathscr{M}_{i+1,j} = \mathscr{M}_{i,j+1}$$

.

En el caso de la ortogonalidad en el círculo unidad el análogo es una matriz de Toeplitz [23], cuyos elementos son iguales para cada diagonal paralela a la diagonal principal, es decir

$$\mathscr{M}_{i,j} = \mathscr{M}_{i+1,j+1}$$

.

El funcional $\mathscr{L}$ es definido positivo si y sólo si los momentos son reales y los menores principales de la matriz de momentos son estrictamente positivos.

El funcional $\mathscr{L}$ será cuasi-definido si y solo si los menores principales de la matriz de momentos son no nulos. Lo dicho anteriormente es extensible a funcionales cuasi-definidos, ver [12].

La **factorización LU** de una matriz M se podrá realizar si los menores principales líderes de M son diferentes de cero, de forma que $M = LU$ donde L es una matriz triangular inferior cuyos elementos diagonales son 1 y U es una matriz triangular superior. Si la matriz M es simétrica la factorización LU se reduce a una **factorización de Cholesky** $M = LL^\top$, en el caso de que M tenga entradas reales la factorización de Cholesky [11] se podrá expresar como $M = LDL^\top$, donde D es una matriz diagonal que contiene los autovalores de M.

En esta tesis la matriz de momentos en el caso de la ortogonalidad simple tiene una factorización LU que se reduce a una factorización de Cholesky [30]

$$\mathscr{M} = S^{-1} H S^{-\top},$$

donde H es una matriz diagonal, S una matriz unitriangular inferior y el superíndice $-\top$ indica que es la matriz inversa de la matriz traspuesta. El vector de polinomios se puede expresar como

$$B := SX(x) = \begin{pmatrix} p_0(x) & p_1(x) & p_2(x) & \dots \end{pmatrix}^\top,$$

donde $X(x) := \begin{pmatrix} 1 & x & x^2 & x^3 & \dots \end{pmatrix}^\top$.

Los polinomios satisfacen la propiedad de ortogonalidad:

$$\int_a^b p_m(x) p_n(x) d\mu(x) = H_n \delta_{n,m}, \qquad n, m \in \{0, 1, 2, \dots\}$$

o, alternativamente

$$\left\langle \mu, BB^\top \right\rangle = H.$$

Cualquier polinomio q_n de grado n se podra escribir como combinación lineal de polinomios ortogonales p_k con $k \in \{0, 1, \dots, n\}$:

$$\begin{aligned} q_n(x) &= \sum_{k=0}^{n} c_{k,n} p_k(x), \\ c_{k,n} &= \frac{1}{d_k^2} \int_a^b q_n(x) p_k(x) d\mu(x), \end{aligned}$$

donde $d_k^2 = \int_a^b p_k^2(x) d\mu(x) = H_n$. Se satisface la propiedad

$$\int_a^b p_n(x) x^m d\mu(x) = 0, \quad 0 \leq m < n,$$

que es equivalente a la propiedad de ortogonalidad, es decir, los polinomios ortogonales son una base ortogonal del espacio vectorial de polinomios. Se puede probar escribiendo el polinomios x^m como combinación lineal de polinomios ortogonales

$$x^m = \sum_{k=0}^{m} c_{k,m} p_k(x),$$

de forma que

$$\int_a^b p_n(x) x^m d\mu = \sum_{k=0}^{m} c_{k,m} \int_a^b p_n(x) p_k(x) d\mu = 0, \quad 0 \leq m < n.$$

Los polinomios ortogonales satisfacen relaciones de **recurrencia a 3 términos**:

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x),$$

dicha relación se puede probar como sigue. El polinomio $xp_n(x)$ tiene grado $n+1$ y, dado que el teorema de Favard establece que una sucesión de polinomios que satisface una relación de recurrencia a tres términos como la anterior es una sucesión de polinomios ortogonales, se puede expresar en la base de polinomios ortogonales mónicos como

$$xp_n(x) = p_{n+1} + \sum_{k=0}^{n} c_{k,n+1} p_k(x)$$

donde

$$c_{k,n+1} = \frac{1}{d_k^2} \int_a^b xp_n(x) p_k(x) d\mu(x),$$

es decir, multiplica p_n por un polinomio de grado $j+1$, y por tanto, que se anulan para $k+1 < n$.

Las relaciones de recurrencia se pueden expresar matricialmente gracias a la **matriz de Jacobi**, definiendo la matriz de Jacobi como

$$J = S \Lambda S^{-1},$$

donde

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

que satisface

$$\Lambda X = xX,$$

siendo $X = \begin{pmatrix} 1 & x & x^2 & x^3 & \dots \end{pmatrix}^\top$.

Así, las relaciones de recurrencia vienen dadas por $JB = xB$, esto se puede deducir partiendo de la propiedad Hankel que satisface la matriz de momentos

$$\Lambda\mathscr{M} = \mathscr{M}\Lambda^{\top},$$

y aplicando la factorización de Cholesky para la matriz de momentos se obtiene la siguiente ecuación para la matriz de Jacobi

$$JH = HJ^{\top},$$

de la que se deduce que J es una matriz tridiagonal.

Los coeficientes de la recurrencia satisfacen las relaciones denominadas ecuaciones de Laguerre-Freud y tienen la forma

$$\beta_n = F_n(\gamma_n, \gamma_{n-1}, \dots, \beta_{n-1,\dots}), \qquad \gamma_n = G_n(\beta_{n-1}, \dots, \beta_{n-1}, \beta_{n-2}, \dots).$$

Como se puede ver en [28] dichas ecuaciones se pueden obtener para los casos semiclásicos mediente las integrales

$$I_n = \int_a^b \theta(x)p_n'(x)p_{n-1}(x)w(x)dx, \qquad J_n = \int_a^b \theta(x)p_n'(x)p_n(x)w(x)dx$$

haciéndolas de dos formas e igualando: por un lado se sustituye $\theta(x)p_n'(x)$ empleando la relación de estructura de primera clase y, por otro lado se hace directamente.

Un caso concreto es la teoría de **polinomios ortogonales discretos**, que es un pilar de esta tesis. Específicamente, el caso discreto, aparece en Mecánica Cuántica, Teoría de Codificación, Teoría de Representaciones de Grupos, en métodos de resolución de ecuaciones en derivadas parciales o en Teoría de Probabilidad [33], por ejemplo, en métodos de resolución de ecuaciones diferenciales parciales de diferencias finitas, fórmulas de cuadratura de tipo gaussiano o en representaciones del grupo de rotaciones en el espacio de cuatro dimensiones. En el caso discreto la medida μ adopta la forma donde usamos las deltas de Dirac

$$\mu = \sum_{k=0}^{\infty} w(k)\delta_{x_k},\ w(k) > 0,\ x_k \in \mathbb{R},$$

de forma que la propiedad de ortogonalidad se expresa como

$$\sum_{k=0}^{\infty} p_n(k)p_m(k)w(k) = 0,$$

y los momentos [35] son

$$\mu_n = \sum_{k=0}^{\infty} k^n w(k).$$

La ecuación diferencial hipergeométrica de Gauss [36], que será importante en lo que sigue,

$$\theta(x)y^{''} + \tau(x)y^{'} + \lambda y = 0, \tag{1.1}$$

donde θ y τ son polinomios a lo sumo de grados 2 y 1, y λ es una constante, tiene como soluciones las series hipergeométricas. Cuando

$$\lambda = \lambda_n = -n\tau^{'} - \frac{1}{2}n(n-1)\theta^{''}$$

la ecuación tiene como solución particular un polinomio de grado n, que se denotan como p_n.

Partiendo de la ecuación diferencial hipergeométrica y multiplicando por una función ρ, escribiéndola en forma autoadjunta

$$\left(\theta\rho y'\right)' + \lambda\rho y = 0, \tag{1.2}$$

se puede llegar a la conclusión de que ρ satisface la **ecuación de Pearson**

$$(\theta\rho)^{'} = \tau\rho. \tag{1.3}$$

Efectivamente, la ecuación (1.1) se puede escribir como

$$y''\theta\rho + y'\tau\rho + y\lambda\rho = 0 \tag{1.4}$$

y derivando en la ecuación (1.2) concluimos que

$$y''(\theta\rho) + y'(\theta\rho)' + y\lambda\rho = 0. \tag{1.5}$$

Ahora, observando e igualando los términos que acompañan a y' en (1.4) y (1.5), vemos que se cumple la ecuación de Pearson para los polinomios clásicos, que son las familias que satisfacen $\deg\theta \leq 2$ y $\deg\tau \leq 1$. Lo anterior se puede extender a las familias de polinomios ortogonales semiclásicos, que son aquellos que satisfacen $\deg\theta > 2$ o $\deg\tau \neq 1$ [40].

Se puede probar que la derivada de una función hipergeométrica también es una función hipergeométrica: si y es una función hipergeométrica y llamamos $u_1 = y'$ a su derivada, derivando la ecuación hipergeométrica que satisface y se obtiene

$$\theta y''' + (\theta' + \tau)y'' + (\tau' + \lambda)y' = 0$$

es decir que

$$\theta u_1'' + \tau_1 u_1' + \lambda_1 u_1 = 0,$$

donde $\tau_1 = \theta' + \tau$ y $\lambda_1 = \lambda + \tau'$. Para la n-ésima derivada de y, $u_n = y^{(n)}$ tendremos ecuaciones hipergeométricas

$$\theta u_n'' + \tau_n u_n' + \lambda_n u_n = 0,$$

donde $\tau_n = \tau + n\theta'$ y $\lambda_n = \lambda + n\tau' + \frac{1}{2}n(n-1)\theta''$.

Estos polinomios ortogonales continuos satisfacen la **fórmula de Rodrigues** [33]

$$p_n(x) = \frac{B_n}{\rho(x)}\big(\theta^n(x)\rho(x)\big)^{(n)}, \quad n \in \{0, 1, \dots\}$$

donde B_n es una constante de normalización. Para probarla escribimos las ecuaciones de Pearson correspondientes a ρ y ρ_n:

$$\frac{(\theta\rho)'}{\rho} = \tau,$$
$$\frac{(\theta\rho_n)'}{\rho_n} = \tau_n = \tau + n\theta'.$$

Sustituyendo la primera en la segunda:

$$\frac{(\theta\rho_n)'}{\rho_n} = \frac{(\theta\rho)'}{\rho} + n\theta',$$

simplificando llegamos a

$$\frac{\rho_n'}{\rho_n} = \frac{\rho'}{\rho} + n\frac{\theta'}{\theta}.$$

Es decir, se cumple

$$\rho_n = \theta^n\rho.$$

Si partimos de la ecuación en forma autoadjunta para u_n:

$$\big(\theta\rho_n u_n'\big)' + \lambda_n\rho_n u_n = 0,$$

aplicando la relación entre diferentes ρ deducida previamente se obtiene:

$$\rho_n u_n = -\frac{1}{\lambda_n}(\rho_{n+1}u_{n+1})',$$

entonces,

$$\rho y = \rho_0 u_0 = -\frac{1}{\lambda_0}(\rho_1 u_1)' = \frac{(-1)^n}{\prod_{i=0}^{n-1}\lambda_i}(\rho_n u_n)^{(n)}.$$

Si suponemos que y_n es un polinomio de grado n, entonces $u_n = y_n^{(n)} = cte$, donde $y_n^{(n)}$ denota la derivada n-ésima del polinomio y_n:

$$y_n = \frac{1}{\rho A_n}(\rho_n u_n)^{(n)} = \frac{B_n}{\rho}\big(\theta^n\rho\big)^{(n)},$$

donde

$$A_n = (-1)^n\prod_{i=0}^{n-1}\lambda_i \qquad \text{y} \qquad B_n = \frac{u_n}{A_n}.$$

Los polinomios ortogonales están relacionados con las series hipergeométricas generalizadas: en el caso de los polinomios ortogonales discretos veremos que sus momentos son series hipergeométricas.

Las **series hipergeométricas generalizadas** son series para las cuales el cociente entre dos términos consecutivos es una función racional [26]

$$\frac{c_{k+1}}{c_k} = \frac{(k+a_1)\dots(k+a_p)}{(b_1+k)\dots(k+b_q)(k+1)}.$$

Extienden las series geométricas, que son series en las que el cociente entre términos consecutivos es constante $\frac{c_{k+1}}{c_k} = r$. Las series hipergeométricas generalizadas [4] son

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) = {}_pF_q(\mathbf{a};\mathbf{b};z) = \sum_{k=0}^{\infty} c_k z^k = \sum_{k=0}^{\infty} \frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{z^k}{k!}$$

donde $(c)_k = c(c+1)(c+2)\dots(c+k-1)$ para $k>0$ y $(c)_0 = 1$ es el símbolo de Pochhammer. La convergencia de las series hipergeométricas depende de p, q y de los parámetros hipergeométricos:

- **$\mathbf{p} \leq \mathbf{q}$**
  La serie converge para todo valor finito de z y define una función entera.

- **$\mathbf{p} = \mathbf{q+1}$**
  Converge para todo valor finito de z si algún a_j es entero no positivo, y es un polinomio.
  Converge para $|z| < 1$ si todos los a_j son enteros y estrictamente positivos, y es un polinomio.

- **$\mathbf{p} > \mathbf{q+1}$**
  Es un polinomio si algún a_j es entero no positivo, si todos los a_j son enteros estrictamente positivos no converge.

Las funciones hipergeométricas satisfacen ecuaciones de contigüidad [38]:

$$\begin{aligned}
&\frac{d^n}{dz^n}{}_pF_q(\mathbf{a};\mathbf{b};z) \\
&= \frac{(a_1)_n(a_2)_n\dots(a_p)_n}{(b_1)_n(b_2)_n\dots(b_q)_n}{}_pF_q(a_1+n,\dots,a_p+n;b_1+n,\dots,b_q+n;z), \\
&\left(z\frac{d}{dz}+a_i\right){}_pF_q(\mathbf{a};\mathbf{b};z) = a_i{}_pF_q(a_1,\dots,a_i+1,\dots,a_p;\mathbf{b};z), \\
&\left(z\frac{d}{dz}+b_j-1\right){}_pF_q(\mathbf{a};\mathbf{b};z) = (b_j-1){}_pF_q(\mathbf{a};b_1,\dots,b_j-1,\dots,b_q;z).
\end{aligned}$$

El **teorema de Bochner** [12] establece que las únicas soluciones que son secuencias de polinomios ortogonales $\{P_n\}_{n=0}^{\infty}$, donde $\deg P_n = n$, a la ecuación diferencial

$$a_2(x)y^{''} + a_1(x)y' + a_0(x)y + \lambda y = 0$$

donde $a_2(x)$, $a_1(x)$ y $a_0(x)$ son polinomios de grados máximos 2, 1 y 0, respectivamente, son:

- Los polinomios de Jacobi.
- Los polinomios de Bessel.
- Los polinomios de Laguerre.
- Los polinomios de Hermite.

También existen otras soluciones a este tipo de ecuaciones, los polinomios excepcionales [22], que corresponden a secuencias de polinomios ortogonales $\{P_n\}_{n=l}^{\infty}$, donde $l \geq 1$.

Las familias de polinomios anteriores se denominan **polinomios ortogonales clásicos** (en algunos casos se denominan *very classical orthogonal polynomials*) [7], [24]:

- **Polinomios de Jacobi**

  En este caso los polinomios de la ecuación de Pearson y el peso son:

  $$\theta = 1 - x^2, \quad \tau = -(\alpha + \beta + 2)x + \beta - \alpha, \quad w(x) = (1-x)^{\alpha}(1+x)^{\beta}.$$

  Los límites del intervalo son $a = -1$ y $b = 1$, y se satisfacen las restricciones $\alpha > -1$ y $\beta > -1$.

  Se relacionan con las series hipergeométricas de la siguiente forma

  $$P_n^{(a,b)}(x) = \frac{(a+1)_n}{n!}\,{}_2F_1\left(-n, n+a+b+1; a+1; \frac{1-x}{2}\right).$$

  Se escriben como $P_n^{(\alpha,\beta)}(x)$, y dentro de los polinomios de Jacobi hay casos concretos relevantes como los polinomios de Legendre $P_n^{(0,0)}$, los polinomios de Chebyshev tipo I y II:

  $$T_n = \frac{n!}{(1/2)n} P_n^{(-1/2,-1/2)}(x), \qquad U_n = \frac{(n+1)!}{(3/2)_n} P_n^{(1/2,1/2)}(x),$$

  y los polinomios de Gegenbauer o ultraesféricos

  $$C_n^{\lambda}(x) = \frac{(2\lambda)_n}{(\lambda+1/2)_n} P_n^{(\lambda-1/2,\lambda-1/2)(x)}.$$

- **Polinomios de Bessel**
  Corresonden a los polinomios de la ecuación de Pearson y peso

$$\theta(x) = x, \qquad \tau(x) = (\alpha+2)x + 2, \qquad w(x) = \sum_{k=0}^{\infty} \frac{1}{(\alpha+1)_k} \Big(\frac{-2}{x}\Big)^k.$$

  Son soluciones a la ecuación

$$x^2 \frac{d^2y}{dx^2} + \big((\alpha+2)x+2\big)\frac{dy}{dx} - n(n+\alpha+1)y = 0.$$

- **Polinomios de Laguerre**
  Corresponden a los polinomios de la ecuación de Pearson y peso

$$\theta(x) = x, \qquad \tau(x) = -x + \alpha + 1, \qquad w(x) = x^{\alpha}e^{-x}.$$

  Los límites del intervalo son $a = 0$ y $b = \infty$, y cumple la restricción $\alpha > -1$.

  Se relacionan con las series hipergeométricas de la siguiente forma

$$L_n^{(a)}(x) = \frac{(a+1)_n}{n!} n!\,{}_1F_1(-n; a+1; x).$$

- **Polinomios de Hermite**
  Corresponden a los polinomios de la ecuación de Pearson y peso

$$\theta = 1, \qquad \tau(x) = -2x, \qquad w(x) = e^{-x^2}.$$

  Los límites del intervalo son $a = -\infty$ y $b = -\infty$. Se relacionan con las series hipergeométricas de la siguiente forma

$$H_n(x) = (2x)^n\, {}_2F_0\left(-\frac{n}{2}, -\frac{n}{2} + \frac{1}{2};\ ; -\frac{1}{x^2}\right).$$

Existen extensiones de las familias clásicas de polinomios ortogonales, se denominan polinomios ortogonales semiclásicos y satisfacen $\deg \theta > 2$ o $\deg \tau \neq 1$. Estas familias satisfacen una ecuación de Pearson, pero no son soluciones a la ecuación diferencial de segundo orden descrita en el teorema de Bochner.

Para establecer una jerarquía de las familias de polinomios ortogonales hipergeométricos se representa el esquema de Askey, como se puede consultar en la referencia [25].

Las ecuaciones de Laguerre-Freud que satisfacen los coeficientes de la recurrencia de los polinomios ortogonales están relacionadas con las **ecuaciones discretas de Painlevé**, que son ecuaciones en diferencias cuyo límite continuo es una de las ecuaciones de Painlevé, como se puede ver en [40]. La propiedad de Painlevé consiste en que la solución general de la ecuación diferencial $f(z)$ no tiene singularidades esenciales móviles p, es decir

$$\nexists \lim_{z\to p} f(z) \qquad \text{y} \qquad \nexists \lim_{z\to p} \frac{1}{f(z)}$$

y p no depende de los parámetros, las ecuaciones de Painlevé satisfacen dicha propiedad.

Existe una ecuación en diferencias [33] análoga a la ecuación diferencial hipergeométrica:

$$\theta(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda y(x) = 0$$

donde $\Delta f(x) = f(x+1) - f(x)$ y $\nabla f(x) = f(x) - f(x-1)$, análogamente al caso continuo, si aplicamos el operador Δ a una solución de la ecuación de arriba obtendremos otra solución.

Los pesos satisfacen la ecuación

$$\Delta\big(\theta(x)w(x)\big) = \tau(x)w(x)$$

donde σ y θ tienen la forma

$$\theta(k) = k(k + c_1 - 1)\ldots(k + c_N - 1), \qquad \sigma(k) = \eta(k + b_1)\ldots(b_M + k),$$

dicha ecuación se puede escribir como

$$\frac{w(x+1)}{w(x)} = \frac{\theta(x) + \tau(x)}{\theta(x+1)} = \frac{\sigma(x)}{\theta(x+1)},$$

que se conoce como **ecuación de Pearson discreta**. Análogamente al caso continuo existen polinomios de variable discreta en este caso que satisfacen la ecuación en diferencias hipergeométrica. Tienen propiedades análogas a las del caso continuo, en este caso la ecuación análoga a la **fórmula de Rodrigues** es:

$$p_n(x) = \frac{B_n}{w(x)}\Delta^n\left(w(x)\prod_{k=0}^{n-1}\theta(x-k)\right),$$

donde

$$B_n = \frac{\Delta^n y_n}{A_{nn}} \qquad \text{y} \qquad A_{nn} = n!\prod_{j=0}^{n-1}\big(\tau' + \frac{n+j-1}{2}\theta''\big),$$

se obtiene de forma análoga a la ecuación de Rodrigues del caso continuo.

Estos polinomios también satisfacen relaciones de ortogonalidad, en este caso se escriben como:

$$\sum_{x_i=a}^{b-1} p_m(x_i)p_n(x_i)w(x_i) = \delta_{mn}d_n^2.$$

También satisfacen una relación de recurrencia a 3 términos [5].

Un funcional semiclásico, introducido por P. Maroni en [1], L es todo funcional regular tal que

$$D(\theta L) + \tau L = 0,$$

donde θ y τ son polinomios que cumplen $0 \leq \deg \theta$ y $1 \leq \deg \tau$.
Para hacer una clasificación de polinomios ortogonales discretos se define la clase del funcional como [13], [14]

$$s = \max\{\deg(\theta) - 2, \deg(\theta - \sigma) - 1\},$$

para $s = 0$ tenemos los casos clásicos, donde $\deg(\tau) = 1$ y $1 \leq \deg(\theta) \leq 2$, existen 3 casos canónicos, es decir:

$\deg(\sigma)$	$\deg(\theta)$	$\deg(\tau)$
0	1	1
1	1	1
2	2	1

Los polinomios clásicos discretos son soluciones a ecuaciones en diferencias de segundo orden de la forma

$$\theta(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda = 0,$$

y dado que $\deg(\tau) = 1$ y $1 \leq \deg(\theta) \leq 2$, estos casos son análogos discretos a las soluciones que establece el teorema de Bochner en el caso continuo.

- **Polinomios de Charlier**
  Corresponde a la elección $\deg(\sigma) = 0$ y $\deg(\theta) = 1$, la solución de la ecuación de Pearson es

  $$\sigma(x) = \eta, \qquad \theta(x) = x, \qquad w(x) = \frac{\eta^x}{x!}.$$

- **Polinomios de Meixner**
  Corresponden a la elección $\deg(\sigma) = 1$ y $\deg(\theta) = 1$, la solución a la ecuación de Pearson es

  $$\sigma = \eta(x + \alpha), \qquad \theta = x, \qquad w = (\alpha)_x \frac{\eta^x}{x!}, \qquad \rho_0 = {}_1F_0(\alpha;\ ;\eta),$$

  donde se cumple $x \in \mathbb{N}_0$, $\alpha > 0$ y $0 < \eta < 1$.

- **Polinomios de Kravchouk**
  Al igual que en el caso de Meixner corresponde a $\deg(\sigma) = 1$ y $\deg(\theta) = 1$, la diferencia es que se considera $\eta < 0$, de forma que $\alpha = -N$ con $N \in \mathbb{N}$ y $x \in [0, N]$, donde el peso es

  $$w(x) = (-N)_x \frac{\eta^x}{x!}.$$

- **Polinomios de Hahn**
  Corresponde a la elección $\deg(\sigma) = 2$ y $\deg(\theta) = 2$, los polinomios de la ecuación de Pearson, el peso y el 0-momento son

$$\sigma = (x+\alpha_1)(x+\alpha_2), \qquad \theta = x(x+\beta),$$
$$w = \frac{(\alpha_1)_x(\alpha_2)_x}{(\beta+1)_x}\frac{1}{x!}, \qquad \rho_0 = {}_2F_1(\alpha_1, \alpha_2; \beta+1; 1)$$

  donde $x \in \mathbb{N}_0$; $\mathrm{Re}(\beta + 1 - \alpha_1 - \alpha_2) > 0$; $\alpha_1, \alpha_2 > 0$ y $\beta + 1 > \alpha_1 + \alpha_2$.

Para $s = 1$ obtenemos los casos semiclásicos de clase 1, en esta ocasión hay 5 casos canónicos, donde $\deg(\tau) = 2$ y $1 \leq \deg(\theta) \leq 3$, es decir:

$\deg(\sigma)$	$\deg(\theta)$	$\deg(\tau)$
0	2	2
1	2	2
2	1	2
2	2	2
3	3	2

- **Polinomios de Charlier generalizados**
  En este caso $\deg(\sigma) = 0$ y $\deg(\theta) = 2$, de forma que los polinomios, el peso y el 0-momento son

$$\sigma(x) = \eta, \quad \theta(x) = x(x+\beta), \quad w(x) = \frac{1}{(\beta+1)_x}\frac{\eta^x}{x!}, \quad \rho_0 = {}_0F_1(\ ; \beta+1; \eta)$$

  con $x \in \mathbb{N}_0$, $\beta > -1$ y $\eta > 0$.

- **Polinomios de Meixner generalizados**
  Corresponde a $\deg(\sigma) = 1$ y $\deg(\theta) = 2$. Los polinomios de la ecuación de Pearson, el peso y el 0-momento son

$$\sigma(x) = \eta(x+\alpha), \qquad \theta(x) = x(x+\beta),$$
$$w(x) = \frac{(\alpha)_x}{(\beta+1)_x}\frac{\eta^x}{x!}, \qquad \rho_0 = {}_1F_1(\alpha; \beta+1; \eta)$$

  donde $x \in \mathbb{N}_0$, $\alpha(\beta+1) > 0$ y $\eta > 0$.

- **Polinomios de Kravchouk generalizados**
  Corresponde a $\deg(\sigma) = 2$ y $\deg(\theta) = 1$. En este caso los polinomios, el peso y el 0-momento serán

$$\sigma = \eta(x+\alpha)(x-N), \quad \theta = x, \quad w = (\alpha)_x(-N)_x\frac{\eta^x}{x!}, \quad \rho_0 = {}_2F_0(\alpha, -N; ; \eta),$$

  donde $x \in [0, N]$, $\eta < 0$ y $\alpha > 0$

- **Polinomios de Hahn tipo I generalizados**
Corresponde a $\deg(\sigma) = 2$ y $\deg(\theta) = 2$. Los polinomios de la ecuación de Pearson, el peso y el 0-momento serán

$$\sigma = \eta(x+\alpha_1)(x+\alpha_2), \qquad \theta = x(x+\beta),$$
$$w = \frac{(\alpha_1)_x(\alpha_2)_x}{(\beta+1)_x}\frac{\eta^x}{x!}, \qquad \rho_0 = {}_2F_1(\alpha_1,\alpha_2;\beta+1;\eta),$$

donde $x \in \mathbb{N}_0$, $0 < \eta < 1$ y $\alpha_1\alpha_2(\beta+1) > 0$.

- **Polinomios de Hahn tipo II generalizado**
Corresponde a $\deg(\sigma) = 3$ y $\deg(\theta) = 3$, y en este caso, para que sea de clase 1 se debe cumplir $\deg(\tau) = 2$. Los polinomios de la ecuación de Pearson, el peso y el 0-momento serán

$$\sigma = (x+\alpha_1)(x+\alpha_2)(x+\alpha_3), \qquad \theta = x(x+\beta_1)(x+\beta_2),$$
$$w = \frac{(\alpha_1)_x(\alpha_2)_x(\alpha_3)_x}{(\beta_1+1)_x(\beta_2+1)_x}\frac{1}{x!}, \qquad \rho_0 = {}_3F_2(\alpha_1,\alpha_2,\alpha_3;\beta_1+1,\beta_2+1;1)$$

donde $x \in \mathbb{N}_0$, $\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3 > 0$ y $\alpha_1\alpha_2\alpha_3(\beta_1+1)(\beta_2+1) > 0$.

En el caso $s = 2$ se obtienen los casos semiclásicos de clase 2. Hay 7 casos canónicos que corresponden a los siguientes grados de los polinomios de la ecuación de Pearson:

$\deg(\sigma)$	$\deg(\theta)$	$\deg(\tau)$
0	3	3
1	3	3
2	3	3
3	1	3
3	2	3
3	3	3
4	4	4

Los polinomios ortogonales discretos semiclásicos de clase 1 y de clase 2 son extensiones de los casos clásicos. Satisfacen ecuaciones de Pearson pero no son soluciones a la ecuación discreta análoga a la descrita en el teorema de Bochner.

Los **polinomios ortogonales múltiples** [32] son familias de polinomios que son ortogonales respecto a varias varias medidas simultáneamente (vamos a suponer r medidas). Alguna de sus aplicaciones es en aproximaciones racionales de π o aproximaciones racionales de la función zeta de Riemann, $\zeta(x)$. Supongamos el multiíndice [23], [34] $\overrightarrow{n} = (n_1,\ldots,n_r)$, con $n_j \geq 0$, $\overrightarrow{n} \in \mathbb{N}^r$, donde $|\overrightarrow{n}| = \sum_{i=1}^{r} n_i$. En primer lugar, están los polinomios ortogonales de tipo I

$$(A_{\overrightarrow{n},1},\ldots,A_{\overrightarrow{n},r})$$

donde $\deg(A_{\overrightarrow{n},j}) \leq n_j - 1$. Dichos polinomios satisfacen las relaciones

$$\sum_{j=1}^{r} \int_{\mathbb{R}} x^k A_{\overrightarrow{n},j} dw_j(x) = 0, \quad k \in \{0, 1, 2, \ldots, |\overrightarrow{n}| - 2\}$$

$$\sum_{j=1}^{r} \int_{\mathbb{R}} x^{|\overrightarrow{n}|-1} A_{\overrightarrow{n},j} dw_j(x) = 1,$$

partiendo de ellas se obtiene un sistema de $|\overrightarrow{n}|$ ecuaciones y $|\overrightarrow{n}|$ coeficientes desconocidos de los $A_{\overrightarrow{n},j}$ $(j = 1, \ldots, r)$, donde la matriz de dicho sistema es:

$$M_{\overrightarrow{n}} = \begin{pmatrix} M_{n_1}^{(1)} & M_{n_2}^{(2)} & \ldots & M_{n_r}^{(r)} \end{pmatrix},$$

donde $M_{n_k}^{(k)} \in \mathbb{C}^{|\overrightarrow{n}| x n_k}$ (los momentos de w_k), son

$$M_{n_k}^{(k)} = \begin{pmatrix} m_0^k & m_1^k \cdots\cdots\cdots m_{n_{k-1}}^k \\ m_1^k \cdots\cdots\cdots\cdots\cdots\cdots & m_{n_k}^k \\ \vdots & \vdots \\ m_{|\overrightarrow{n}|-1}^k \cdots\cdots\cdots\cdots & m_{|\overrightarrow{n}|+n_k-2}^k \end{pmatrix},$$

si $\det(M_{\overrightarrow{n}}) \neq 0$, $\overrightarrow{n}$ es índice normal de tipo I, si todos los multiíndices son normales el sistema es perfecto. Originalmente, la condición de normalidad consiste en maximizar los grados de los polinomios de tipo I, es decir

$$\deg(A_{\overrightarrow{n},j}) = n_j - 1.$$

Por otro lado tenemos los polinomios de tipo II [3], un polinomio será de tipo II multiortogonal si $\deg(P_{\overrightarrow{n}}) = |\overrightarrow{n}|$ y

$$\int_{\mathbb{R}} P_{\overrightarrow{n}}(x) x^k dw_1(x) = 0, \quad k \in \{0, \ldots, n_1 - 1\},$$

$$\vdots$$

$$\int_{\mathbb{R}} P_{\overrightarrow{n}}(x) x^k dw_r(x) = 0, \quad k \in \{0, \ldots, n_r - 1\}.$$

Las ecuaciones anteriores constituyen un sistema de $|\overrightarrow{n}|$ ecuaciones y $|\overrightarrow{n}|$ coeficientes desconocidos del polinomios $P_{\overrightarrow{n}}$, si existe una única solución al sistema $\overrightarrow{n}$ es índice normal de tipo II. La matriz del sistema es

$$M'_{\overrightarrow{n}} = \begin{pmatrix} [M_{n_1}^1]' \\ \vdots \\ [M_{n_r}^r]' \end{pmatrix},$$

esta matriz es la traspuesta de la matriz de los polinomios tipo I.

Los polinomios ortogonales múltiples satisfacen relaciones de recurrencia pero el número de términos varía, ver referencias [18] y [19].

Para las familias de polinomios ortogonales múltiples discretos también se satisface la ecuación de Rodrigues como se puede observar en [3] y [23].

1.3. Resultados

En esta sección se muestran los resultados obtenidos en el estudio realizado de la ortogonalidad discreta en [31] donde se halla la matriz de Laguerre-Freud y su estructura, se obtienen las ecuaciones y sistemas de tipo Toda para los coeficientes de la recurrencia, ecuaciones diferenciales para las funciones τ y las ecuaciones de Nijhoff-Capel Toda. Se encuentran las ecuaciones de Laguerre-Freud para los coeficientes de la recurrencia en los casos donde el *0*-momento vienen dado por las familias hipergeométricas ${}_1F_2$, ${}_2F_2$ y ${}_3F_2$ en [16]; para la familia hipergeométrica ${}_2F_1$, también conocida como función hipergeometrica de Gauss, en [15]; para las familias hipergeométricas ${}_0F_1$ o Charlier generalizado y ${}_1F_1$ o Meixner generalizado, además de su relación con la ecuación de Painlevé deg-P_v en [17].

Posteriormente, se muestran los resultados obtenidos para la ortogonalidad múltiple discreta respecto de dos pesos estudiada en [18]: la matriz de Laguerre-Freud y su estructura, extensiones multicomponentes de las ecuaciones de Toda, extensiones de la ecuación de Nijhoff-Capel Toda, y se obtienen las ecuaciones de Laguerre-Freud para los casos de Charlier, Charlier generalizado, Meixner tipo II y Meixner tipo II generalizado.

Finalmente, se muestran los resultados obtenidos para la ortogonalidad múltiple discreta respecto a un número general de pesos obtenidos en [19], que generalizan lo obtenido en [18].

1.3.1. Polinomios ortogonales discretos

En el caso de polinomios ortogonales discretos la matriz de momentos $\mathscr{M}$ satisface la propiedad Hankel $\Lambda\mathscr{M} = \mathscr{M}\Lambda^\top$ y su factorización se puede expresar a partir de una matriz unitriangular inferior S y una matriz triangular H como $\mathscr{M} = S^{-1}HS^{-\top}$ [31]. El vector de polinomios se expresa como $B = SX(x)$ donde $X(x)$ es el vector de monomios de x.

En este caso los polinomios satisfacen una relación de recurrencia a 3 términos, que se puede expresar matricialmente gracias a la matriz

$$T = \Lambda^\top\alpha^{(1)} + \alpha^{(0)} + \Lambda$$

como $TB = xB$.

Las funciones τ [2] vienen dadas por wronskianos de funciones hipergeométricas y las funciones τ asociadas son las derivadas logarítmicas de las τ respecto de η.

Los conceptos mencionados anteriormente se desarrollan con más detalle en [31], [16], [15] y [17].

Las relaciones hipergeométricas de contigüidad para las funciones hipergeométricas se trasladan a la matriz de momentos de forma que se cumplen ecuaciones de compatibilidad para la matriz de momentos, ver referencia [31]. A partir de dichas ecuaciones de compatibilidad se definen las matrices de conexión [29] que relacionan el vector de polinomios B con dicho vector tras transformar los parámetros hipergeométricos. Dichas matrices de conexión tienen 2 diagonales no nulas.

Las relaciones de Laguerre-Freud [28] para los coeficientes de la recurrencia en este caso son de la forma

$$\begin{aligned}\alpha_n^{(0)} &= F_n\Big(\alpha_n^{(1)}, \alpha_{n-1}^{(1)}, \alpha_{n-1}^{(0)}, \dots\Big),\\ \alpha_n^{(1)} &= G_n\Big(\alpha_{n-1}^{(1)}, \alpha_{n-1}^{(0)}, \alpha_{n-2}^{(1)}, \dots\Big),\end{aligned}$$

dichas ecuaciones se obtienen empleando la matriz de Laguerre-Freud, que viene dada por

$$\begin{aligned}\Psi =&\Pi^{-1}H\theta\big(T^\top\big) = \sigma(T)H\Pi^\top = \Pi^{-1}\theta(T)H = H\sigma\big(T^\top\big)\Pi^\top\\ =&\theta(T+I)\Pi^{-1}H = H\Pi^\top\sigma(T^\top - I),\end{aligned}$$

que es una matriz banda con M subdiagonales y N superdiagonales

$$\Psi = \big(\Lambda^\top\big)^M\psi^{(-M)} + \cdots + \psi^{(N)}\Lambda^N,$$

donde N y M son los grados de θ y σ, respectivamente.

La matriz de Laguerre-Freud satisface la siguientes ecuaciones de compatibilidad

$$\Psi H^{-1} = [\Psi H^{-1}, T],$$

dicha expresión se emplea para calcular las ecuaciones de compatibilidad de los coeficientes de la recurrencia y, a partir de ellas, las relaciones de Laguerre-Freud en casos concretos de polinomios.

En este caso se obtienen la siguiente ecuación de Toda

$$\vartheta^2 q_n = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}},$$

donde $q_n = \log H_n$, siendo H_n los elementos de la matriz H. Para las funciones τ se obtiene una ecuación de segundo orden

$$\vartheta^2 \log \tau_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}.$$

Para los coeficientes de la recurrencia se satisface el siguiente sistema de tipo Toda de dos ecuaciones:

$$\begin{aligned}\vartheta\alpha^{(0)} &= \alpha^{(1)} - \mathfrak{a}_+\alpha^{(1)},\\ \vartheta \log \alpha^{(1)} &= \mathfrak{a}_-\alpha^{(0)} - \alpha^{(0)}.\end{aligned}$$

Para el caso de la ortogonalidad discreta se satisface la siguiente ecuación de Nijhoff-Capel Toda:

$$\frac{\bar{u}}{\tilde{u}} - \frac{\hat{\bar{u}}}{\hat{\tilde{u}}} = \frac{\bar{u}}{\hat{u}} - \frac{\tilde{\bar{u}}}{\tilde{\hat{u}}},$$

donde

$$\begin{aligned}&u_n = H_{n-1}(b_1,\dots,b_M;c_1,\dots,c_N;\eta), && \bar{u}_n = u_{n+1},\\ &\hat{u}_n = \hat{a}u_n, && \tilde{u}_n = \tilde{a}u_n,\end{aligned}$$

siendo $\hat{a} \in \{b_1 - 1,\dots,b_M - 1\}$, $\tilde{a} \in \{c_1,\dots,c_N\}$, $\hat{}$ y $\tilde{}$ transforman $\{b_i\}_{i=1}^M$ y $\{c_j\}_{j=1}^N$, respectivamente. En este trabajo se han estudiado varios casos concretos de polinomios que se comentan en los siguientes puntos.

Un desarrollo detallado de los resultados presentados anteriormente referentes a la matriz de Laguerre-Freud, ecuaciones y sistemas de Toda se puede consultar en [31].

Se conocen como redes de Toda a aquellas redes cuyo potencial de interacción es de forma exponencial [39]

$$\phi(r) = \frac{a}{b}e^{-bt} + ar$$

donde $ab > 0$.

Si consideramos $a = b = 1$ y $m = 1$, se obtienen las siguientes ecuaciones del movimiento:

$$\begin{aligned}\frac{d^2y_n}{dt^2} &= e^{-(y_n - y_{n-1})} - e^{-(y_{n+1}-y_n)},\\ \frac{d^2r_n}{dt^2} &= 2e^{-r_n} - e^{-r_{n-1}} - e^{-r_{n+1}},\end{aligned}$$

y si definimos las variables de Flaschka como

$$\alpha_n = e^{-(y_{n+1}-y_n)/2}, \qquad \beta_n = \frac{dy_n}{dt},$$

se obtienen las siguientes ecuaciones de Toda que describen el sistema en función de las nuevas variables

$$\begin{aligned}\frac{d}{dt}\alpha_n &= \frac{1}{2}\alpha_n(\beta_n - \beta_{n+1}),\\ \frac{d}{dt}\beta_n &= \alpha_{n-1}^2 - \alpha_n^2.\end{aligned}$$

Caso Charlier Generalizado

En este caso [21] los polinomios de la ecuación de Pearson son

$$\sigma(z) = \eta, \qquad \theta(z) = z(z+b);$$

el peso y el 0-momento son

$$w(z) = \frac{1}{(b+1)_z}\frac{\eta^z}{z!}, \qquad \rho_0(\eta) = {}_0F_1(;b+1;\eta).$$

En este caso [17] la matriz de Laguerre-Freud está dada por

$$\Psi = \psi^{(0)} + \psi^{(1)}\Lambda + \psi^{(2)}\Lambda^2,$$

donde

$$\psi^{(0)} = \eta H, \qquad \psi^{(1)} = \eta H D, \qquad \psi^{(2)} = H.$$

Las ecuaciones de Laguerre-Freud son

$$\begin{aligned}\alpha_{n+1}^{(1)} &= \alpha_{n-1}^{(1)} + \eta n\big(\alpha_{n-1}^{(1)}\big)^{-1}\big(1 + \alpha_{n-1}^{(0)} - \alpha_n^{(0)}\big),\\ \alpha_n^{(0)} &= 1 + \alpha_{n-2}^{(0)} + \eta\Big(n\big(\alpha_n^{(1)}\big)^{-1} - (n-1)\big(\alpha_{n-1}^{(1)}\big)^{-1}\Big).\end{aligned}$$

El coeficiente de la recurrencia $\alpha^{(1)}$ satisface la siguiente ecuación de tercer orden:

$$\vartheta\left(\frac{\alpha_n^{(1)}}{\eta}\Big(\vartheta^2\log\alpha_n^{(1)} + 2\alpha_n^{(1)}\Big) + n^2\frac{\eta}{\alpha^{(1)}}\right) = 2\alpha_n^{(1)}$$

dicha ecuación satisface la propiedad de Painlevé [40].

Los coeficientes de la recurrencia satisfacen el siguiente sistema de EDOs:

$$\begin{aligned}\vartheta\alpha_n^{(0)} &= \eta\frac{\eta + (b-n)n + \big(2n - b - \alpha_n^{(0)}\big)\alpha_n^{(0)} - \alpha_n^{(1)}}{\eta - \alpha_n^{(1)}},\\ \vartheta\alpha_n^{(1)} &= \big(b - n + 1 + 2\alpha_n^{(0)}\big)\alpha_n^{(1)} - n\eta.\end{aligned}$$

Caso de Meixner generalizado

Los polinomios de la ecuación de Pearson son

$$\sigma(z) = \eta(z+a), \qquad \theta(z) = z(z+b);$$

el peso es [20]

$$w = \frac{(a)_z}{(b+1)_z}\frac{\eta^z}{z!}$$

y el 0-momento, que corresponde con la función de Kummer, es

$$\rho_0 = {}_1F_1(a; b+1; \eta) = M(a, b+1, \eta).$$

La matriz de Laguerre-Freud [17] tiene cuatro diagonales no nulas:

$$\Psi = \Lambda^\top \psi^{(-1)} + \psi^{(0)} + \psi^{(1)}\Lambda + \psi^{(2)}\Lambda^2,$$

donde

$$\begin{aligned} \psi^{(-1)} &= \eta \mathfrak{a}_- H, \\ \psi^{(0)} &= \eta H(\alpha^{(0)} + aI + \mathfrak{a}_+ D), \\ \psi^{(1)} &= \mathfrak{a}_- H(\alpha^{(0)} + \mathfrak{a}_- \alpha^{(0)} + bI - \mathfrak{a}_+ D), \\ \psi^{(2)} &= \mathfrak{a}_-^2 H. \end{aligned}$$

Encontramos las siguientes ecuaciones de Laguerre-Freud:

$$\begin{aligned} \alpha^{(0)}_{n+1} = \frac{1}{\alpha^{(1)}_{n+1}} \Bigg(\eta \Big(\eta^{(1)}_n + (\alpha^{(0)}_n + b - n)\alpha^{(0)}_n - n(b-a-n+1) \Big) + \eta(\eta+1)(\alpha^{(0)}_n + a + n) \Bigg) \\ - \alpha^{(0)}_n - b + n + 1 + 2\eta, \end{aligned}$$

$$\begin{aligned} \alpha^{(1)}_{n+1} = -\alpha^{(1)}_n - (\alpha^{(0)}_n + b - n)\alpha^{(0)}_n + n(b-a-n+1) + \eta(\alpha^{(0)}_n + a + n) \\ + \eta^{-1}(\alpha^{(0)}_{n-1} + \alpha^{(0)}_n + b - n + 1 - \eta)\alpha^{(1)}_n. \end{aligned}$$

Caso de Hahn I

En este caso los polinomios de la ecuación de Pearson son

$$\sigma(z) = \eta(z+a)(z+b), \qquad \theta = z(z+c);$$

el peso es

$$w(z) = \frac{(a)_z (b)_z}{(c+1)z}\frac{\eta^z}{z!}$$

y el 0-momento viene dado por

$$\rho_0 = {}_2F_1(a, b; c+1; \eta).$$

En este caso la matriz de Laguerre-Freud [15] tiene 5 diagonales no nulas: primera y segunda subdiagonales, primera y segunda superdiagonales y diagonal principal. Las relaciones de Laguerre-Freud obtenidas se pueden ver en [15].

Caso de la familia hipergeométrica ${}_1F_2$

Los polinomios de la ecuación de Pearson son

$$\sigma(z) = \eta(z + a_1), \qquad \theta(z) = z(z + b_1)(z + b_2);$$

el peso es

$$w(z) = \frac{(a_1)_z}{(b_1 + 1)_z (b_2 + 1)_z} \frac{\eta^z}{z!},$$

y el momento-0 es

$$\rho_0 = {}_1F_2(a_1; b_1 + 1, b_2 + 1; \eta).$$

En este caso la matriz de Laguerre-Freud [16] tiene 5 diagonales no nulas: la primera subdiagonal, la diagonal principal y las 3 primeras superdiagonales.
Las relaciones de Laguerre-Freud tienen la forma:

$$\begin{aligned}
\alpha_{n+2}^{(1)} &= F\Big(\alpha_{n+1}^{(1)}, \alpha_{n+1}^{(0)}, \alpha_n^{(1)}, \alpha_n^{(0)}, \alpha_{n-1}^{(1)}, \alpha_{n-1}^{(0)}\Big),\\
\alpha_{n+2}^{(0)} &= G\Big(\alpha_{n+2}^{(1)}, \alpha_{n+1}^{(1)}, \alpha_{n+1}^{(0)}, \alpha_n^{(1)}, \alpha_n^{(0)}\Big).
\end{aligned}$$

Caso de la familia hipergeométrica ${}_2F_2$

Los coeficientes de la ecuación de Pearson son

$$\sigma(z) = \eta(z + a_1)(z + a_2), \qquad \theta(z) = z(z + b_1)(z + b_2);$$

el peso viene dado por

$$w(z) = \frac{(a_1)_z (a_2)_z}{(b_1 + 1)_z (b_2 + 1)_z} \frac{\eta^z}{z!},$$

y el momento-0 es la función hipergeométrica

$$\rho_0 = {}_2F_2(a_1, a_2; b_1 + 1, b_2 + 1; \eta).$$

La matriz de Laguerre-Freud [16] tiene 6 diagonales no nulas: las 2 primeras subdiagonales, la diagonal principal y las tres primeras superdiagonales.

En este caso las ecuaciones de Laguerre-Freud son de la forma:

$$\begin{aligned}\alpha_{n+2}^{(0)} &= G\Big(\alpha_{n+2}^{(1)}, \alpha_{n+1}^{(1)}, \alpha_{n+1}^{(0)}, \alpha_{n}^{(1)}, \alpha_{n}^{(0)}, \alpha_{n-1}^{(1)}, \alpha_{n-1}^{(0)}\Big),\\ \alpha_{n+2}^{(1)} &= F\Big(\alpha_{n+1}^{(1)}, \alpha_{n+1}^{(0)}, \alpha_{n}^{(1)}, \alpha_{n}^{(0)}, \alpha_{n-1}^{(1)}, \alpha_{n-1}^{(0)}\Big).\end{aligned}$$

Caso de la familia hipergeométrica ${}_3F_2$

En este caso los coeficientes de la ecuación de Pearson son

$$\sigma(z) = \eta(z+a_1)(z+a_2)(z+a_3), \qquad \theta(z) = z(z+b_1)(z+b_2);$$

el peso es

$$w(z) = \frac{(a_1)_z(a_2)_z(a_3)_z}{(b_1+1)_z(b_2+1)_z}\frac{\eta^z}{z!},$$

y el momento-0 es

$$\rho_0 = {}_3F_2(a_1, a_2, a_3; b_1+1, b_2+1; \eta).$$

En este caso la matriz de Laguerre-Freud [16] tiene 5 diagonales no nulas: la primera subdiagonal, la diagonal principal y las 3 primeras superdiagonales.

Se puede demostrar que las ecuaciones de Laguerre-Freud se pueden obtener despejando adecuadamente y sustituyendo a partir de las siguientes ecuaciones que ligan los siguientes coeficientes de la recurrencia:

$$\begin{aligned}0 &= F_1\Big(\alpha_{n+2}^{(1)}, \alpha_{n+2}^{(0)}, \alpha_{n+1}^{(1)}, \alpha_{n+1}^{(0)}, \alpha_{n}^{(1)}, \alpha_{n}^{(0)}, \alpha_{n-1}^{(0)}\Big),\\ 0 &= F_2\Big(\alpha_{n+2}^{(1)}, \alpha_{n+2}^{(0)}, \alpha_{n+1}^{(1)}, \alpha_{n+1}^{(0)}, \alpha_{n}^{(1)}, \alpha_{n}^{(0)}, \alpha_{n-1}^{(1)}, \alpha_{n-1}^{(0)}, \alpha_{n-2}^{(0)}\Big).\end{aligned}$$

1.3.2. Polinomios ortogonales múltiples discretos respecto a dos pesos

La matriz de momentos viene dada por

$$\mathscr{M} = \sum_{k=0}^{\infty} X(k)(X^{(1)}(k)w^{(1)}(k) + X^{(2)}(k)w^{(2)}(k))^{\top}.$$

La matriz de momentos cumple la propiedad bi-Hankel $\Lambda\mathscr{M} = \mathscr{M}\big(\Lambda^{\top}\big)^2$, la estructura es de forma que en las columnas impares aparecen las funciones de la primera familia y en las pares las funciones hipergeométricas de la otra familia

$$
\mathscr{M} = \left(\begin{array}{cc|cc|c}
\rho_0^{(1)} & \rho_0^{(2)} & \rho_1^{(1)} & \rho_1^{(2)} & \dots \\ \hline
\rho_1^{(1)} & \rho_1^{(2)} & \rho_2^{(1)} & \rho_2^{(2)} & \dots \\ \hline
\vdots & \vdots & \vdots & \vdots &
\end{array}\right),
$$

donde $\rho_n^{(a)} = (\vartheta^{(a)})^n \rho_0^{(a)}$, $\rho_0^{(a)} = {}_{M_a}F_N\left(b_1^{(a)}, \dots, b_{M_a}^{(a)}; c_1, \dots, c_N; \eta^{(a)}\right)$ y $a \in \{1, 2\}$.
La factorización de la matriz de momentos se puede escribir a partir de dos matrices unitriangulares inferiores S y $\tilde{S}$ y una matriz diagonal H como $\mathscr{M} = S^{-1} H \tilde{S}^{-\top}$.

En la ortogonalidad múltiple existen el vector de polinomios tipo I, $A^a(x)$, donde $a \in 1, 2$ y el vector de tipo II, $B(x)$, que es análogo al vector $B(x)$ del caso monopeso [10], los vectores de tipo I se expresan a partir de los vectores de monomios parciales

$$
X^{(1)} = \begin{bmatrix} 1 & 0 & x & 0 & \dots \end{bmatrix}^\top, \qquad X^{(2)} = \begin{bmatrix} 0 & 1 & 0 & x & \dots \end{bmatrix}^\top,
$$

de forma que $A^{(a)} = H^{-1} \tilde{S} X^{(a)}$ y $B = SX$, y para sistemas perfectos se cumple

$$
\deg B_n = n, \qquad \deg A_n^{(a)} = \left\lceil \frac{n + 2 - a}{2} \right\rceil - 1, \qquad a \in \{1, 2\}.
$$

Se cumplen las siguientes relaciones de ortogonalidad

$$
\begin{aligned}
&\sum_{a=1}^{2} \sum_{k=0}^{\infty} A_n^{(a)}(k) w^{(a)}(k) k^m, && k \in \{0, \dots, \deg B_{n-1}\} \\
&\sum_{k=0}^{\infty} B_n(k) w^{(a)}(k) k^m = 0, && k \in \{0, \dots, \deg A_{n-1}^{(a)}\} \qquad a \in \{1, 2\}.
\end{aligned}
$$

Los polinomios satisfacen una relación de recurrencia a 4 términos [9] que se puede expresar matricialemte de dos formas, a partir de la matriz

$$
T = \left(\Lambda^\top\right)^2 \alpha^{(2)} + \Lambda^\top \alpha^{(1)} + \alpha^{(0)} + \Lambda,
$$

como

$$
TB = xB, \qquad T^\top A^{(a)} = x A^{(a)}.
$$

Las funciones τ vienen dadas por wronskianos dobles de funciones hipergeométricas y las funciones τ^1 son derivadas logarítmicas de las funciones τ respecto a η.

Los polinomios de tipo I y de tipo II se pueden expresar como

$$A_n^{(a)} = \frac{1}{\tau_{n+1}} \left|\begin{array}{ccc|c} & \mathscr{M}^{[n]} & & \begin{matrix}\mathscr{M}_{0,n}\\ \vdots \\ \mathscr{M}_{n-1,n}\end{matrix} \\ \hline X_0^{(a)} & \dots & X_{n-1}^{(a)} & X_n^{(a)} \end{array}\right|, \qquad n\in\mathbb{N}_0 \qquad a\in\{1,2\}$$

y

$$B_n = \frac{1}{\tau_n} \left|\begin{array}{ccc|c} & \mathscr{M}^{[n]} & & \begin{matrix}1\\ \vdots \\ x^{n-1}\end{matrix} \\ \hline \mathscr{M}_{n,0} & \dots & \mathscr{M}_{n,n-1} & x^n \end{array}\right|, \qquad n\in\mathbb{N}_0.$$

Los conceptos mencionados anteriormente se presentan con mayor detalle en la referencia [18].

Ambos pesos satisfacen ecuaciones de Pearson de la forma

$$\theta(k+1)w^{(a)}(k+1) = \sigma^{(a)}(k)w^{(a)}(k),$$

de esta forma al estudiar cómo actúan las relaciones de compatibilidad de ambas familias de funciones hipergeométricas obtendremos matrices de conexión que relacionan los vectores B y $A^{(a)}$ con dichos vectores tras haber transformado los parámetros hipergeométricos. Las matrices de conexión tienen 3 diagonales no nulas.

Los coeficientes de la recurrencia satisfacen ecuaciones de Laguerre-Freud de la forma:

$$\begin{aligned}
\alpha_n^{(2)} &= F_n\Big(\alpha_{n-1}^{(2)}, \alpha_{n-1}^{(1)}, \alpha_{n-1}^{(0)}, \dots\Big),\\
\alpha_n^{(1)} &= G_n\Big(\alpha_n^{(2)}, \alpha_{n-1}^{(2)}, \alpha_{n-1}^{(1)}, \alpha_{n-1}^{(0)}, \dots\Big),\\
\alpha_n^{(0)} &= K_n\Big(\alpha_n^{(2)}, \alpha_n^{(1)}, \alpha_{n-1}^{(2)}, \alpha_{n-1}^{(1)}, \alpha_{n-1}^{(0)}, \dots\Big).
\end{aligned}$$

La matriz de Laguerre-Freud se puede obtener mediante las expresiones:

$$\Psi = \Pi^{-1}\theta(T) = \sum_{a=1}^{2} \sigma^{(a)}\big(T^{(a)}\big)^\top \big(\pi^{(a)}\big)^\top,$$

que es una matriz banda con 2M subdiagonales y N superdiagonales

$$\Psi = \big(\Lambda^\top\big)^{2M}\psi^{(-2M)} + \cdots + \psi^{(N)}\Lambda^N,$$

donde $N = \deg\theta$ y $M = \text{máx}\big(\deg\sigma^{(1)}, \deg\sigma^{(2)}\big)$.

La matriz de Laguerre-Freud satisface la ecuación de compatibilidad $\Psi = [\Psi, T]$, mediante la cual se podrán obtener las ecuaciones de compatibilidad y, despejando adecuadamente, las ecuaciones de Laguerre-Freud para casos concretos de polinomios. Como se puede observar es análoga a la de 1 peso.

Para las funciones $q_n = \log H_n$ y $f_n = S_n^{[1]}$ se satisface un sistema de ecuaciones de tipo Toda que es una extensión del resultado obtenido para el caso monopeso:

$$\begin{aligned}\vartheta q_n &= f_{n-1} - f_n,\\ \vartheta^2 f_n - (2f_n - f_{n+1} - f_{n-1})\vartheta f_n &= e^{q_{n+1}-q_{n-1}} - e^{q_{n-2}-q_n}.\end{aligned}$$

La función τ satisface una ecuación diferencial en diferencias de tercer orden que constituye una extensión del resultado obtenido para el caso monopeso

$$\begin{multline}\vartheta^3\tau_{n+1} - \left(\frac{\vartheta\tau_{n+2}}{\tau_{n+2}} + \frac{\vartheta\tau_{n+1}}{\tau_{n+1}} + \frac{\vartheta\tau_n}{\tau_n}\right)\vartheta^2\tau_{n+1} + \frac{(\vartheta\tau_{n+1})^2}{\tau_{n+1}}\left(\frac{\vartheta\tau_{n+2}}{\tau_{n+2}} - \frac{\vartheta\tau_n}{\tau_n}\right)\\ = \frac{\tau_{n+3}\tau_n}{\tau_{n+2}} - \frac{\tau_{n+2}\tau_{n-1}}{\tau_n}.\end{multline}$$

Los coeficientes de la recurrencia satisfacen el siguiente sistema de tipo Toda:

$$\begin{aligned}\vartheta\alpha^{(0)} &= \alpha^{(1)} - \mathfrak{a}_+\alpha^{(1)},\\ \vartheta\alpha^{(1)} &= \alpha^{(2)} - \mathfrak{a}_+\alpha^{(2)} + \alpha^{(1)}\big(\mathfrak{a}_-\alpha^{(0)} - \alpha^{(0)}\big),\\ \vartheta\alpha^{(2)} &= \alpha^{(2)}\big(\mathfrak{a}_-^2\alpha^{(0)} - \alpha^{(0)}\big),\end{aligned}$$

y se puede ver que es una extensión de lo obtenido en el caso de un peso [18].

Estudiando la compatibilidad entre transformaciones de las familias de parámetros hipergeométricos obtendremos los sistemas de Nijhoff-Capel Toda, se obtienen 3 sistemas de cuatro ecuaciones para cuatro funciones. Anulando ciertas funciones en unos de estos 3 sistemas podremos recuperar el resultado del caso monopeso.

La obtención de los resultados presentados en este apartado se desarrolla con detalle en [18].

Caso múltiple de Charlier

En el caso múltiple de Charlier los pesos son

$$w^{(a)}(k) = \frac{\big(\eta^{(a)}\big)^k}{k!}$$

donde $a \in \{1, 2\}$.

Los polinomios de la ecuación de Pearson son

$$\theta(k) = k, \qquad \sigma^{(a)}(k) = \eta^{(a)}$$

donde $a \in \{1, 2\}$.

La matriz de Laguerre-Freud tiene 2 diagonales no nulas, la diagonal principal y la primera superdiagonal, que son:

$$\psi^{(0)} = I^{(1)}\eta^{(1)} + I^{(2)}\eta^{(2)}, \qquad \psi^{(1)} = I.$$

En este caso se pueden obtener expresiones para los coeficientes de la recurrencia:

$$\begin{aligned} \alpha_{2n}^{(0)} &= 2n + \alpha_0^{(0)}, & \alpha_{2n+1}^{(0)} &= 2n + 1 + \eta^{(2)} - \eta^{(1)} + \alpha_0^{(0)}, \\ \alpha_{2n}^{(1)} &= n\big(\eta^{(1)} + \eta^{(2)}\big), & \alpha_{2n+1}^{(1)} &= n\big(\eta^{(1)} + \eta^{(2)}\big) + \eta^{(1)}, \\ \alpha_{2n+1}^{(2)} &= n\eta^{(2)}\big(\eta^{(2)} - \eta^{(1)}\big), & \alpha_{2n+2}^{(2)} &= (n+1)\eta^{(1)}\big(\eta^{(1)} - \eta^{(2)}\big). \end{aligned}$$

Este caso ha sido estudiado por otros autores, ver [3].

Caso múltiple de Meixner de tipo II

Los pesos son

$$w^{(a)}(k) = (b_a)_k \frac{\eta^k}{k!}.$$

Los polinomios de la ecuación de Pearson son

$$\theta(k) = k, \qquad \sigma^{(a)}(k) = \eta(k + b_a)$$

donde $a \in \{1, 2\}$.

La matriz de Laguerre-Freud tiene 4 diagonales no nulas: las 2 primeras subdiagonales, la diagonal principal y la primera superdiagonal. Dichas diagonales son:

$$\begin{aligned} \psi^{(1)} &= I, \\ \psi^{(0)} &= \alpha^{(0)} - \mathfrak{a}_+ D = \eta(\alpha^{(0)} + \mathfrak{a}_+^2 D_1^{[1]} I^{(1)} + \mathfrak{a}_+^2 D_2^{[1]} I^{(2)} + b_1 I^{(1)} + b_2 I^{(2)}), \\ \psi^{(-1)} &= \eta\alpha^{(1)}, \\ \psi^{(-2)} &= \eta\alpha^{(2)}. \end{aligned}$$

Al igual que en el caso de Charlier múltiple [8] se pueden hallar los coeficientes de la recurrencia, se obtiene el siguiente resultado:

$$
\begin{aligned}
\alpha_{2n}^{(0)} &= \frac{2n}{1-\eta} + \frac{\eta(n+b_1)}{1-\eta},\\
\alpha_{2n+1}^{(0)} &= \frac{(2n+1)}{1-\eta} + \frac{\eta(n+b_2)}{1-\eta},\\
\alpha_{2n}^{(1)} &= \frac{\eta}{(1-\eta)^2}\Big(3n^2 + n(b_1+b_2-2)\Big),\\
\alpha_{2n+1}^{(1)} &= \frac{\eta}{(1-\eta)^2}\Big(3n^2 + n(b_1+b_2+1) + b_1\Big),\\
\alpha_{2n}^{(2)} &= \frac{\eta^2}{(1-\eta)^3} n(n+b_1-1)(n+b_1-b_2),\\
\alpha_{2n+1}^{(2)} &= \frac{\eta^2}{(1-\eta)^3} n(n+b_2-1)(n+b_2-b_1).
\end{aligned}
$$

Este caso ha sido estudiado por otros autores, ver [3].

Caso de Charlier generalizado múltiple

Los pesos son

$$w^{(a)}(k) = \frac{1}{(c+1)_k}\frac{\left(\eta^{(a)}\right)^k}{k!}.$$

Los polinomios de la ecuación de Pearson son

$$\theta(k) = k(k+c), \qquad \sigma^{(a)}(k) = \eta^{(a)}$$

donde $a \in \{1,2\}$.

La matriz de Laguerre-Freud tiene 3 diagonales no nulas: diagonal principal y las dos primeras superdiagonales. Las diagonales son:

$$\psi^{(0)} = \eta^{(1)}I^{(1)} + \eta^{(2)}I^{(2)}, \quad \psi^{(1)} = \mathfrak{a}_-\alpha^{(0)} + \alpha^{(0)} + c - \mathfrak{a}_+D, \quad \psi^{(2)} = I.$$

A diferencia de los anteriores casos múltiples no podemos despejar los coeficientes de la recurrencia, pero sí podemos obtener ecuaciones de Laguerre-Freud:

$$
\begin{aligned}
\alpha_{n+2}^{(0)} &= n+1-c-\alpha_{n+1}^{(0)} + (\alpha_{n+2}^{(2)})^{-1}\Big((-1)^n\alpha_{n+1}^{(1)}\big(\eta^{(1)}-\eta^{(2)}\big) + \alpha_{n+1}^{(2)}(\alpha_n^{(0)} + \\
&\quad \alpha_{n-1}^{(0)} + c - n + 1)\Big),\\
\alpha_{n+2}^{(1)} &= \alpha_n^{(1)} + (\alpha_n^{(0)} + 1 + \alpha_{n+1}^{(0)})(\alpha_{n+1}^{(0)} + \alpha_n^{(0)} + c - n) + (-1)^{n+1}\big(\eta^{(1)} - \eta^{(2)}\big),\\
\alpha_{n+2}^{(2)} &= \alpha_n^{(2)} + \alpha_n^{(1)}(\alpha_n^{(0)} + \alpha_{n-1}^{(0)} + c - n + 1) - \alpha_{n+1}^{(1)}(\alpha_{n+1}^{(0)} + \alpha_n^{(0)} + c - n) + \\
&\quad \frac{\eta^{(1)}+\eta^{(2)}}{2} + (-1)^n\frac{\eta^{(1)}-\eta^{(2)}}{2}.
\end{aligned}
$$

Caso de Meixner tipo II generalizado múltiple

Los pesos son

$$w^{(a)}(k) = \frac{(b_1)_k}{(c+1)_k}\frac{\eta^k}{k!}.$$

Los polinomios de la ecuación de Pearson son

$$\theta(k) = k(k+c), \qquad \sigma^{(a)}(k) = \eta(k+b_a)$$

con $a \in \{1,2\}$.

La matriz de Laguerre-Freud tiene 5 diagonales no nulas: las 2 primeras subdiagonales, la diagonal principal y las dos primeras superdiagonales. Las diagonales son:

$$\begin{aligned}
\psi^{(2)} &= I,\\
\psi^{(1)} &= \alpha^{(0)} - \mathfrak{a}_-\alpha^{(0)} + c - \mathfrak{a}_+ D,\\
\psi^{(0)} &= \eta\Big(\alpha^{(0)} + I^{(1)}(\mathfrak{a}_+^2 D_1^{[1]} + b_1) + I^{(2)}(\mathfrak{a}_+^2 D_2^{[1]} + b_2)\Big),\\
\psi^{(-1)} &= \eta\alpha^{(1)},\\
\psi^{(-2)} &= \eta\alpha^{(2)}.
\end{aligned}$$

Encontramos ecuaciones de Laguerre-Freud para los coeficientes de la recurrencia de la forma:

$$\begin{aligned}
\alpha_{n+2}^{(0)} &= K\Big(\alpha_{n+2}^{(2)}, \alpha_{n+1}^{(2)}, \alpha_{n+1}^{(1)}, \alpha_{n+1}^{(0)}\alpha_n^{(0)}\Big),\\
\alpha_{n+2}^{(1)} &= G\Big(\alpha_{n+1}^{(0)}, \alpha_n^{(1)}, \alpha_n^{(0)}\Big),\\
\alpha_{n+2}^{(2)} &= F\Big(\alpha_{n+1}^{(1)}, \alpha_{n+1}^{(0)}, \alpha_n^{(2)}, \alpha_n^{(1)}, \alpha_n^{(0)}, \alpha_{n-1}^{(0)}\Big).
\end{aligned}$$

1.3.3. Extensión a la ortogonalidad múltiple respecto a q pesos

En esta tesis se ha generalizado la ortogonalidad múltiple respecto a 2 pesos al caso respecto a q pesos.
La matriz de momentos viene dada por

$$\mathscr{M} = \sum_{k=0}^{\infty} X(k)(X^{(1)}(k)w^{(1)}(k) + \cdots + X^{(q)}(k)w^{(q)}(k))^\top.$$

En este caso la matriz cumple la propiedad q-Hankel $\Lambda\mathscr{M} = \mathscr{M}\left(\Lambda^\top\right)^q$ [19], la estructura es de forma

$$\mathscr{M} = \begin{bmatrix} \rho_0^{(1)} & \rho_0^{(2)}\cdots\cdots\rho_0^{(q)} & \rho_1^{(1)}\cdots\cdots\rho_1^{(q)} & \cdots\cdots \\ \rho_1^{(1)} & \rho_1^{(2)}\cdots\cdots\rho_1^{(q)} & \rho_2^{(1)}\cdots\cdots\rho_2^{(q)} & \cdots\cdots \\ \vdots & \vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots & \end{bmatrix},$$

donde aparecen las q familias de funciones hipergeométricas distintas y se repiten periódicamente , y la factorización de la matriz de momentos se puede escribir a partir de dos matrices unitriangulares inferiores S y $\tilde{S}$ y una matriz diagonal H como $\mathscr{M} = S^{-1}H\tilde{S}^{-\top}$, igual que ocurría para la ortogonalidad múltiple respecto a 2 pesos.

En la ortogonalidad múltiple existen el vector de polinomios tipo I $A^a(x)$ donde $a \in 1, 2, \ldots, q$ y el vector de tipo II $B(x)$ que es análogo al vector $B(x)$ del caso monopeso, los vectores de tipo I se expresan a partir de los vectores de q monomios parciales $X^{(a)} = \begin{bmatrix} e_a & xe_a & x^2e_a \ldots \end{bmatrix}^\top$ donde $\{e_a\}_{a=1}^q$ es la base canónica en $\mathbb{R}^q$, de forma que $A^{(a)} = H^{-1}\tilde{S}X^{(a)}$ y $B = SX$. Para sistemas perfectos se cumple

$$\deg B_n = n, \qquad \deg A_n^{(a)} = \left\lceil \frac{n+2-a}{q} \right\rceil - 1, \qquad a \in \{1, \ldots, q\}.$$

Se cumplen las siguientes relaciones de ortogonalidad

$$\begin{aligned} &\sum_{a=1}^{q} \sum_{k=0}^{\infty} A_n^{(a)}(k) w^{(a)}(k) k^m, \qquad k \in \{0, \ldots, \deg B_{n-1}\} \\ &\sum_{k=0}^{\infty} B_n(k) w^{(a)}(k) k^m = 0, \qquad k \in \{0, \ldots, \deg A_{n-1}^{(a)}\} \qquad a \in \{1, \ldots, q\}. \end{aligned}$$

Los polinomios satisfacen una relación de recurrencia a $q+2$ términos que se puede expresar matricialemte de dos formas, a partir de la matriz

$$T = \left(\Lambda^\top\right)^q \alpha^{(q)} + \cdots + \alpha^{(0)} + \Lambda,$$

como

$$TB = xB, \qquad\qquad T^\top A^{(a)} = xA^{(a)}.$$

En este caso las funciones τ vienen dadas por q-wronskianos de funciones hipergeométricas y las funciones τ^1 son derivadas logarítmicas de las funciones τ respecto a η.

Los polinomios de tipo I y de tipo II se pueden expresar como

$$A_n^{(a)} = \frac{1}{\tau_{n+1}} \left| \begin{array}{ccc|c} & & & \mathscr{M}_{0,n} \\ & \mathscr{M}^{[n]} & & \vdots \\ & & & \mathscr{M}_{n-1,n} \\ \hline X_0^{(a)} & \dots & X_{n-1}^{(a)} & X_n^{(a)} \end{array} \right|, \quad n \in \mathbb{N}_0 \quad a \in \{1, \dots, q\}$$

y

$$B_n = \frac{1}{\tau_n} \left| \begin{array}{ccc|c} & & & 1 \\ & \mathscr{M}^{[n]} & & \vdots \\ & & & x^{n-1} \\ \hline \mathscr{M}_{n,0} & \dots & \mathscr{M}_{n,n-1} & x^n \end{array} \right|, \qquad n \in \mathbb{N}_0.$$

Los conceptos presentados anteriormente se desarrollan con más detalle en la referencia [19].

Ambos pesos satisfacen ecuaciones de Pearson de la forma

$$\theta(k+1)w^{(a)}(k+1) = \sigma^{(a)}(k)w^{(a)}(k)$$

donde $a \in \{1, \dots, q\}$.

De esta forma al estudiar cómo actúan las relaciones de compatibilidad de ambas familias de funciones hipergeométricas obtendremos matrices de conexión que relacionan los vectores B y $A^{(a)}$ con dichos vectores tras haber transformado los parámetros hipergeométricos, ver [19].

Los coeficientes de la recurrencia satisfacen ecuaciones de Laguerre-Freud de la forma:

$$\begin{aligned} \alpha_n^{(0)} &= F_n^{(0)}\Big(\alpha_n^{(q)}, \alpha_n^{(q-1)}, \dots, \alpha_n^{(1)}, \alpha_{n-1}^{(q)}, \dots\Big), \\ \alpha_n^{(m)} &= F_n^{(m)}\Big(\alpha_n^{(q)}, \dots, \alpha_n^{(m+1)}, \alpha_{n-1}^{(q)}, \dots, \alpha_{n-1}^{(0)}, \dots\Big), \\ \alpha_n^{(q)} &= F_n^{(q)}\Big(\alpha_{n-1}^{(q)}, \dots, \alpha_{n-1}^{(0)}, \alpha_{n-2}^{(q)}, \dots\Big), \end{aligned}$$

con $m \in \{1, \dots, q-1\}$.

La matriz de Laguerre-Freud se puede obtener mediante las expresiones:

$$\Psi = \Pi^{-1}\theta(T) = \sum_{a=1}^{q} \sigma^{(a)}\big(T^{(a)}\big)^\top \big(\pi^{(a)}\big)^\top,$$

es una matriz a bandas que viene dada por

$$\Psi = \big(\Lambda^\top\big)^{qM}\psi^{(-qM)} + \dots + \psi^{(N)}\Lambda^N,$$

donde $N = \deg \theta$ y $M = \text{máx}\big(\deg \sigma^{(1)}, \dots, \deg \sigma^{(q)}\big)$.
En este caso la matriz de Laguerre-Freud satisface la ecuación de compatibilidad

$$\Psi = [\Psi, T],$$

mediante la cual se podrán obtener las ecuaciones de compatibilidad y, despejando adecuadamente, las ecuaciones de Laguerre-Freud para casos concretos de polinomios. Como se puede observar es análoga a la de 1 peso e igual a la expresión para el caso bipeso.

Las ecuaciones de Toda en el caso de 3 pesos son 3 para 3 funciones, en el caso bipeso eran 2 ecuaciones para 2 funciones y en el caso monopeso era una ecuación pra una función, se puede observar que a medida que vayamos incrementando el número de pesos va aumentando el número de ecuaciones y funciones.

Se puede demostrar que la función τ satisface una ecuación diferencial en diferencias de cuarto orden para el caso de tres pesos, se observa que al incrementar el número de pesos se va incrementando el orden de la ecuación que satisfacen las funciones τ.

Los coeficientes de la recurrencia satisfacen el siguiente sistema de tipo Toda:

$$\begin{aligned}
\vartheta\alpha^{(0)} &= \alpha^{(1)} - \mathfrak{a}_+\alpha^{(1)},\\
\vartheta\alpha^{(n)} &= \alpha^{(n+1)} - \mathfrak{a}_+\alpha^{(n+1)} + \alpha^{(n)}\big(\mathfrak{a}_-^n\alpha^{(0)} - \alpha^{(0)}\big),\\
\vartheta\alpha^{(q)} &= \alpha^{(q)}\big(\mathfrak{a}_-^q\alpha^{(0)} - \alpha^{(0)}\big),
\end{aligned}$$

donde $n \in \{0, \dots, q-1\}$, que se puede observar que es una extensión de lo obtenido en los casos monopeso y bipeso.

La obtención de los resultados mostrados en este apartado se puede consultar en [19].

Caso de Charlier múltiple

Los pesos están dados por

$$w^{(a)}(k) = \frac{\big(\eta^{(a)}\big)^k}{k!}$$

donde $a \in \{1, \dots, q\}$. Los polinomios de la ecuación de Pearson son

$$\theta = k, \qquad\qquad \sigma^{(a)}(k) = \eta^{(a)},$$

donde $a \in \{1, \dots, q\}$.

De acuerdo con los grados de los polinomios de la ecuación de Pearson la estructura de la matriz de Laguerre-Freud es:

$$\Psi = \psi^{(0)} + \psi^{(1)}\Lambda.$$

donde las diagonales son:

$$\psi^{(0)} = \alpha^{(0)} - \mathfrak{a}_+ D = \sum_{a=1}^{p} \eta^{(a)} I^{(a)}, \quad \psi^{(1)} = I;$$

En este caso podemos despejar los coeficientes de la recurrencia

$$\begin{aligned}
\alpha^{(0)}_{qm+a} &= qm + a + \eta^{(a+1)}, && a \in \{0, \dots, q-1\},\\
\alpha^{(1)}_{qm+a} &= \sum_{b=1}^{a} \eta^{(b)} + m \sum_{b=1}^{p} \eta^{(b)}, && a \in \{1, 2, \dots, q-1\},\\
\alpha^{(l+1)}_{qm+a} &= \alpha^{(l+1)}_{qm+a-1} + \alpha^{(l)}_{qm+a-1}\big(\eta^{(a-l)} - \eta^{(a)}\big), && a \in \{l+1, \dots, q-1\},
\end{aligned}$$

donde $m \in \mathbb{N}_0$.

Caso de Charlier generalizado múltiple

Los pesos vienen dados por

$$w^{(a)}(k) = \frac{1}{(c+1)_k} \frac{\big(\eta^{(a)}\big)^k}{k!}.$$

Los polinomios de la ecuación de Pearson son

$$\theta(k) = k(k+c), \qquad \sigma^{(a)}(k) = \eta^{(a)}.$$

Dados los grados de los polinomios de la ecuación de Pearson, la matriz de Laguerre-Freud tiene la forma:

$$\Psi = \psi^{(0)} + \psi^{(1)}\Lambda + \psi^{(2)}\Lambda^2$$

donde

$$\psi^{(1)} = \alpha^{(0)} + \mathfrak{a}_- \alpha^{(0)} + c - \mathfrak{a}_+ D, \qquad \psi^{(2)} = I, \qquad \psi^{(0)} = \sum_{a=1}^{q} \eta^{(a)} I^{(a)}.$$

Se obtienen las siguientes ecuaciones de Laguerre-Freud:

$$
\begin{aligned}
\alpha^{(0)}_{m+q} &= m+q-1-c-\alpha^{(0)}_{m+q-1}+\frac{\alpha^{(q)}_{m+q-1}}{\alpha^{(q)}_{m+q}}(\alpha^{(0)}_{m-1}+\alpha^{(0)}_{m}+c-m)\\
&+\frac{\alpha^{(q-1)}_{m+q-1}}{\alpha^{(q)}_{m+q}}\big(r(m+1)-r(m)\big),\\
\alpha^{(1)}_{m+2} &= \alpha^{(1)}_{m}+\alpha^{(0)}_{m+1}+\alpha^{(0)}_{m}+c-m+(\alpha^{(0)}_{m}-\alpha^{(0)}_{m+1})(\alpha^{(0)}_{m}+\alpha^{(0)}_{m+1}+c-m)\\
&+r(m+2)-r(m+1),\\
\alpha^{(2)}_{m+2} &= \alpha^{(2)}_{m}+\alpha^{(1)}_{m}(\alpha^{(0)}_{m}+\alpha^{(0)}_{m-1}+c-m+1)-\alpha^{(1)}_{m+1}(\alpha^{(0)}_{m}+\alpha^{(0)}_{m+1}+c-m)\\
&+r(m+1),\\
\alpha^{(n+2)}_{n+2+m} &= \alpha^{(n+2)}_{n+m}-\alpha^{(n+1)}_{m+n+1}\big(\alpha^{(0)}_{a+m}+\alpha^{(0)}_{a+m+1}+c-(a+m)\big)+\alpha^{(a+1)}_{a+m}\big(\alpha^{(0)}_{m-1}+\alpha^{(0)}_{m}\\
&+c-(m-1)\big)+\alpha^{(n)}_{m+a}\big(r(m+1)-r(a+m+1)\big),
\end{aligned}
$$

donde $a\in\{1,\dots,q-2\}$, y $r(n)=n-q\lfloor\frac{n}{q}\rfloor$, $r(n)\in\{1,\dots,q\}$. $\lfloor x\rfloor$, es el mayor entero igual o menor que x.

Caso de Meixner II múltiple

En este caso los pesos son

$$w^{(a)}(k)=(b_a)_k\frac{\eta^k}{k!}.$$

Los polinomios de las ecuaciones de Pearson son

$$\theta(k)=k, \qquad \sigma^{(a)}(k)=\eta(k+b_a),$$

donde $a\in\{1,\dots,q\}$.

De acuerdo con los grados de los polinomios de las ecuaciones de Pearson sabemos que la matriz de Laguerre-Freud tiene la siguiente estructura:

$$\Psi=(\Lambda^\top)^q\psi^{(-q)}+\cdots+\psi^{(1)}\Lambda.$$

$$\psi^{(1)}=I, \qquad \psi^{(0)}=\alpha^{(0)}-\mathfrak{a}_+D, \qquad \psi^{(-m)}=\eta\alpha^{(m)}$$

con $m\in\{1,\dots,q\}$.

Se obtienen las siguientes ecuaciones para los coeficientes de la recurrencia, de forma que se pueden ir despejando iterativamente a partir del coeficiente anterior de forma general

$$
\begin{aligned}
\alpha^{(0)}_{qa+m} &= \frac{1}{1-\eta}\big(qa+m+\eta(b_{m+1}+a)\big),\\
\alpha^{(1)}_{qa+m} &= \frac{1}{1-\eta}\left(\frac{1}{1-\eta}\left((q+\eta)\frac{a(a+1)}{2}+\frac{m(m+1)}{2}+\eta\sum_{j=0}^{m}b_{j+1}\right)-\right.\\
&\qquad \left.\frac{(qa+m-1)(qa+m)}{2}\right),\\
\alpha^{(l+1)}_{qa+m+l+1} &= \alpha^{(l+1)}_{qa+m+l}+\frac{\eta}{1-\eta}\alpha^{(l)}_{qa+m+l}(1+b_{m+1}+a).
\end{aligned}
$$

Caso de Meixner II generalizado múltiple

Los pesos vienen dados por

$$
w^{(a)}(k)=\frac{(b_a)_k}{(c+1)_k}\frac{\eta^k}{k!}.
$$

Los polinomios de la ecuación de Pearson son:

$$
\theta(k)=k(k+c) \quad \text{y} \quad \sigma^{(a)}(k)=\eta(k+b_a),
$$

donde $a\in\{1,\dots,q\}$.

Debido a los grados de los polinomios de la ecuación de Pearson la estructura de la matriz de Laguerre-Freud es

$$
\Psi=(\Lambda^\top)^q\psi^{(-q)}+\cdots+\psi^{(2)}\Lambda^2.
$$

Las diagonales vienen dadas por

$$
\begin{aligned}
\psi^{(2)} &= I, & \psi^{(1)} &= \mathfrak{a}_-\alpha^{(0)}+\alpha^{(0)}+c-\mathfrak{a}_+D,\\
\psi^{(0)} &= \eta\left(\alpha^{(0)}+\sum_{a=1}^{p}I^{(a)}\big(b_a+\mathfrak{a}_+^pD^{(a)}\big)\right), & \psi^{(-m)} &= \eta\alpha^{(m)}
\end{aligned}
$$

donde $m\in\{1,\dots,p\}$.

Se tienen las siguientes ecuaciones de Laguerre-Freud para los coeficientes de la recurrencia:

$$
\begin{aligned}
\alpha^{(0)}_{(a+1)p+m} &= -\alpha^{(0)}_{(a+1)p+m-1} + a + p - 1 - c + (\alpha^{(p)}_{(a+1)p+m})^{-1}\Big(\eta\big(\alpha^{(p-1)}_{(a+1)p+m-1}(b_{m+1} \\
&\quad + a + 1) + \alpha^{(p)}_{(a+1)p+m} - \alpha^{(p)}_{(a+1)p+m-1}\big) + \alpha^{(p)}_{(a+1)p+m-1}\big(\alpha^{(0)}_{pa+m} + \alpha^{(0)}_{pa+m-1} \\
&\quad + c - (a-1)\big)\Big), \\
\alpha^{(1)}_{pa+m+2} &= \alpha^{(1)}_{pa+m} + \eta(\alpha^{(0)}_{pa+m+1} - \alpha^{(0)}_{pa+m} - b_m - a) + (\alpha^{(0)}_{pa+m} - \alpha^{(0)}_{pa+m+1} \\
&\quad + 1)\big(\alpha^{(0)}_{pa+m+1} + \alpha^{(0)}_{pa+m} + c - (m+pa)\big), \\
\alpha^{(l+2)}_{pa+m+l+2} &= \alpha^{(l+2)}_{pa+m+l} + \eta(\alpha^{(l+1)}_{pa+m+l+1} - \alpha^{(l+1)}_{pa+m+l}) - \alpha^{(l+1)}_{pa+m+l+1}\big(\alpha^{(0)}_{pa+m+l+1} \\
&\quad + \alpha^{(0)}_{pa+m+l} + c - (a+l)\big) + \alpha^{(l+1)}_{pa+m+l}\big(\alpha^{(0)}_{pa+m} + \alpha^{(0)}_{pa+m-1} + c - (a-1)\big), \\
\alpha^{(l+2)}_{pa+m+l+2} &= \alpha^{(l+2)}_{pa+m+l} + \eta\big(\alpha^{(l)}_{pa+m+l}(b_{m+1} + a + 1) + \alpha^{(l+1)}_{pa+m+l+1} - \alpha^{(l+1)}_{pa+m+l}\big) \\
&\quad + \alpha^{(l+1)}_{pa+m+l+1}\big(\alpha^{(0)}_{pa+m+l+1} + c - (a+l) + \alpha^{(0)}_{pa+m+l}\big) + \alpha^{(l+1)}_{pa+m+l}\big(\alpha^{(0)}_{pa+m} \\
&\quad + \alpha^{(0)}_{pa+m-1} + c - (a-1)\big).
\end{aligned}
$$

1.4. Conclusiones y problemas abiertos

Las conclusiones que se extraen de esta tesis son:

- Las ecuaciones de Laguerre-Freud que satisfacen los coeficientes de la recurrencia en las familias de polinomios ortogonales discretos están relacionadas con las ecuaciones de Painlevé.
- Las ecuaciones de Toda van aumentando en número y aumentando el número de funciones involucradas a medida que aumentan el número de pesos.
- La ecuación de Toda para las funciones τ aumenta el orden a medida que aumentamos el número de pesos.
- El sistema de tipo Toda que satisfacen los coeficientes de la recurrencia tiene una forma general y el número de ecuaciones depende del número de pesos, q, de forma que el sistema tiene $q+1$ ecuaciones.

A continuación se mencionan posibles lineas de investigación y problemas abiertos que ameritan un estudio más profundo:

- La relación que podrían tener las ecuaciones de Laguerre-Freud para los coeficientes de la recurrencia con las ecuaciones de Painlevé en las familias de polinomios ortogonales hipergeométricos múltiples.
- El estudio de las ecuaciones análogas a la ecuación de Nijhoff-Capel Toda para la ortogonalidad múltiple respecto a un número arbitrario de pesos.

- Estudio de ecuaciones de Laguerre-Freud para familias de polinomios hipergeométricas multiortogonales más complejas.
- Estudio de las ecuciones de Pearson para el caso múltiple en el círculo unidad.
- Extensión del caso múltiple a redes cuadriláteras.

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ORIGINAL ARTICLE

Pearson equations for discrete orthogonal polynomials: I. Generalized hypergeometric functions and Toda equations

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Abstract

The Cholesky factorization of the moment matrix is applied to discrete orthogonal polynomials on the homogeneous lattice. In particular, semiclassical discrete orthogonal polynomials, which are built in terms of a discrete Pearson equation, are studied. The Laguerre–Freud structure semiinfinite matrix that models the shifts by ± 1 in the independent variable of the set of orthogonal polynomials is introduced. In the semiclassical case it is proven that this Laguerre–Freud matrix is banded. From the well-known fact that moments of the semiclassical weights are logarithmic derivatives of generalized hypergeometric functions, it is shown how the contiguous relations for these hypergeometric functions translate as symmetries for the corresponding moment matrix. It is found that the 3D Nijhoff–Capel discrete Toda lattice describes the corresponding contiguous shifts for the squared norms of the orthogonal polynomials. The continuous 1D Toda equation for these semiclassical discrete orthogonal polynomials is discussed and the compatibility equations are derived. It is also shown that the Kadomtesev–Petviashvilii equation is connected to an adequate deformed semiclassical discrete weight, but in this case, the deformation does not satisfy a Pearson equation.

KEYWORDS
3D Nijhoff–Capel discrete Toda equation, Cholesky factorization, contigous relations, discrete orthogonal polynomials, generalized hypergeometric functions, Pearson equations, Toda hierarchy

1 | INTRODUCTION

In this paper, the Gauss–Borel factorization of the moment matrix is applied to study discrete orthogonal polynomials subject to a Pearson equation.

Discrete orthogonal polynomials is nowadays a well-established subject, the classical case has been treated extensively in Ref. 1 and the the Riemann–Hilbert problem has been used to study asymptotics and applications, see Ref. 2. Semiclassical reductions, in where a discrete Pearson equation is fulfilled by the weight has also been treated in the literature. See, for example, Refs. 3, 4 and 5, 6 and references therein for an comprehensive account. For some specific type of weights of generalized Charlier and Meixner types, the corresponding Freud–Laguerre-type equations for the coefficients of the three term recurrence have been studied, see, for example, Refs. 7–11. For a general account, see Refs. 12–14, and for discrete orthogonal polynomials and Painlevé equations, see Ref. 15.

With this introduction, we give a fast briefing of introductory character, of orthogonal polynomials, its discrete version, and the Cholesky factorization of the moment matrix. We also discuss for the first time in this context, to the best of our knowledge, the Pascal matrices and its dressed versions.

Then, in Section 2, we discuss the discrete Pearson equation and give Theorem 1, the first main result of the paper, which describes a new symmetry of the moment matrix, direct consequence of the Pearson equation, in terms of the Pascal matrix. We then deduce the corresponding symmetry for the Jacobi matrix, see Proposition 6. Then, in Theorem 2, the second main result of the paper, we describe a banded semi-infinite matrix, that we called Laguerre–Freud structure matrix, which models the shift in the spectral variable. The name is rooted on the fact that this banded matrix encodes Laguerre–Freud equations for the recurrence coefficients of the orthogonal polynomial sequence, see, for example, Ref. 16. The number of nontrivial superdiagonals and subdiagonals of this matrix is determined by the Pearson equation. Several properties involving the Jacobi and Laguerre–Freud matrices are derived. Of particular interest are those of compatibility type.

As the Pearson equation leads to generalized hypergeometric moments, we study the contiguous relations of these functions and its description as further symmetries of the moment matrix, see Theorem 4. In Theorem 5, the squared norms of these discrete orthogonal polynomials are shown to be solutions of a well-known discrete integrable multidimensional lattice, known as the Nijhoff–Capel lattice, see Refs. 17, 18. We also discuss the Toda hierarchy deformations, with only the first flow preserving the Pearson reduction. The compatibility with this first Toda flow and the Pearson equation is discussed in Proposition 21. The connection of deformations of these weights with solutions of the Kadomtsev–Petviashvilli (KP) equation is discussed as well.

1.1 | Linear functionals and orthogonal polynomials

Given a linear functional $\rho_z \in \mathbb{C}^*[z]$, here $\mathbb{C}^*[z]$ denotes the dual of the linear space of complex polynomials $\mathbb{C}[z]$, the corresponding moment matrix is

$$G = (G_{n,m}), \quad G_{n,m} = \rho_{n+m}, \quad \rho_n = \langle \rho_z, z^n \rangle, \quad n, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}, \tag{1}$$

with ρ_n the nth moment of the linear functional ρ_z. If the moment matrix is such that all its truncations, which are Hankel matrices, $G_{i+1,j} = G_{i,j+1}$,

$$G^{[k]} = \begin{pmatrix} G_{0,0} & \cdots & G_{0,k-1} \\ \vdots & & \vdots \\ G_{k-1,0} & \cdots & G_{k-1,k-1} \end{pmatrix} = \begin{pmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & \rho_2 & & \iddots & \rho_k \\ \rho_2 & & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & & \vdots \\ \rho_{k-1} & \rho_k & \cdots & \cdots & \rho_{2k-2} \end{pmatrix} \tag{2}$$

are nonsingular; that is, the Hankel determinants $\Delta_k := \det G^{[k]}$ do not cancel, $\Delta_k \neq 0$, $k \in \mathbb{N}_0$, then there exists monic polynomials

$$P_n(z) = z^n + p_n^1 z^{n-1} + \cdots + p_n^n, \quad n \in \mathbb{N}_0, \tag{3}$$

with $p_0^1 = 0$, such that the following orthogonality conditions are fulfilled:

$$\left\langle \rho, P_n(z) z^k \right\rangle = 0, \quad k \in \{0, \ldots, n-1\}, \quad \langle \rho, P_n(z) z^n \rangle = H_n \neq 0. \tag{4}$$

Moreover, the set $\{P_n(z)\}_{n\in\mathbb{N}_0}$ is an orthogonal set of polynomials $\langle \rho, P_n(z) P_m(z) \rangle = \delta_{n,m} H_n, n, m \in \mathbb{N}_0$. In this case, we have a symmetric bilinear form $\langle F, G \rangle_\rho := \langle \rho, FG \rangle$, such that the moment matrix is the Gram matrix of this bilinear form and $\langle P_n, P_m \rangle_\rho := \delta_{n,m} H_n$. In this paper, we will use both denominations, moment matrix, or Gram matrix, indistinctly to refer to the matrix G.

1.2 | The shift matrix Λ

In terms of the semi-infinite vector of monomials,

$$\chi(z) := \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \end{pmatrix} \tag{5}$$

the moment matrix is written as $G = \langle \rho, \chi\chi^\top \rangle$, and it becomes evident that this matrix is symmetric, that is, $G = G^\top$. The vector of monomials χ is an eigenvector of the *shift matrix*

$$\Lambda := \begin{pmatrix} 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ & & & \end{pmatrix} \tag{6}$$

that is, $\Lambda\chi = x\chi$. From here, it immediately follows that $\Lambda G = G\Lambda^\top$, that is, the Gram matrix is a Hankel matrix, as we previously said. The transposed matrix

$$\Lambda^\top = \begin{pmatrix} 0 & \cdots & \\ 1 & \ddots & \\ 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix} \tag{7}$$

satisfies $\Lambda\Lambda^\top = I$ and $\Lambda^\top\Lambda = I - E_{0,0}$, with $(E_{i,j})_{k,l} = \delta_{i,k}\delta_{j,l}, i, j \in \mathbb{N}_0$.

1.3 | The Cholesky factorization of the Gram matrix

Being the Gram matrix symmetric its Gauss–Borel factorization reduces to a Cholesky factorization

$$G = S^{-1}HS^{-\top}, \tag{8}$$

where S is a lower unitriangular matrix that can be written as

$$S = \begin{pmatrix} 1 & 0 & \cdots \\ S_{1,0} & 1 & \ddots \\ S_{2,0} & S_{2,1} & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix}, \tag{9}$$

and $H = \mathrm{diag}(H_0, H_1, \ldots)$ is a diagonal matrix, with $H_k \neq 0$, for $k \in \mathbb{N}_0$. The Cholesky factorization holds whenever the principal minors of the moment matrix; that is, the Hankel determinants Δ_k, do not cancel.

The components $P_n(z)$ of the semi-infinite vector of polynomials

$$P(z) := S\chi(z), \tag{10}$$

are the monic orthogonal polynomials of the functional ρ. Indeed, from the Cholesky factorization, we know that

$$\langle \rho, \chi\chi^\top \rangle = G = S^{-1} H S^{-\top} \tag{11}$$

so that $S\langle \rho, \chi\chi^\top \rangle S^\top = H$, and consequently, $\langle \rho, S\chi\chi^\top S^\top \rangle = H$ and we get $\langle \rho, PP^\top \rangle = H$, which recollects the orthogonality of the polynomials $\{P_n(z)\}_{n=0}^\infty$.

1.4 | The Jacobi matrix

Given this construction is natural to introduce the lower Hessenberg semi-infinite matrix

$$J = S\Lambda S^{-1} \tag{12}$$

that has the vector $P(z)$ as eigenvector with eigenvalue z, that is $JP(z) = zP(z)$. The Hankel condition of the Gram matrix $\Lambda G = G\Lambda^\top$ together with the Cholesky factorization leads to $\Lambda S^{-1} H S^{-\top} = S^{-1} H S^\top \Lambda^\top$, or, equivalently, $S\Lambda S^{-1} H = H S^{-\top} \Lambda^\top S^{-\top}$, that is,

$$JH = (JH)^\top = HJ^\top. \tag{13}$$

That is, the Hessenberg matrix JH is symmetric, thus being Hessenberg and symmetric we deduce that is tridiagonal. Hence, the Jacobi matrix J given in (12) reads

$$J = \begin{pmatrix} \beta_0 & 1 & 0 & \cdots & \\ \gamma_1 & \beta_1 & 1 & \ddots & \\ 0 & \gamma_2 & \beta_2 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{14}$$

and the eigenvalue equation $JP = zP$ is a three term recursion relation

$$zP_n(z) = P_{n+1}(z) + \beta_n P_n(z) + \gamma_n P_{n-1}(z), \tag{15}$$

that with the initial conditions $P_{-1} = 0$ and $P_0 = 1$ completely determines the sequence of orthogonal polynomials $\{P_n(z)\}_{n\in\mathbb{N}_0}$ in terms of the recursion coefficients β_n, γ_n. In terms of the Hankel and modified Hankel determinants

$$\Delta_k := \det \begin{pmatrix} \rho_0 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \vdots & \iddots & \iddots & \vdots \\ \rho_{k-2} & \iddots & & \rho_{2k-3} \\ \rho_{k-1} & \cdots & \rho_{2k-3} & \rho_{2k-2} \end{pmatrix}, \qquad \tilde{\Delta}_k := \det \begin{pmatrix} \rho_0 & \cdots & \rho_{k-2} & \rho_k \\ \vdots & \iddots & \rho_{k-1} & \vdots \\ \rho_{k-2} & \iddots & & \rho_{2k-2} \\ \rho_{k-1} & \cdots & \rho_{2k-3} & \rho_{2k-1} \end{pmatrix} \tag{16}$$

we find the following.

Proposition 1. *The recursion coefficients are given by*

$$\beta_n = p_n^1 - p_{n+1}^1 = -\frac{\tilde{\Delta}_n}{\Delta_n} + \frac{\tilde{\Delta}_{n+1}}{\Delta_{n+1}}, \quad \gamma_{n+1} = \frac{H_{n+1}}{H_n} = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}, \quad n \in \mathbb{N}_0, \tag{17}$$

Proof. For $n \in \mathbb{N} := \{1, 2, \ldots\}$, from $J^\top = H^{-1}JH$, we get that $\gamma_n = \frac{H_n}{H_{n-1}}$. On the other hand, as $J = S\Lambda S^{-1}$ and recalling that for the coefficients $S_{n,n-1}$ of the first subdiagonal of S, we have $S_{n,n-1} = p_n^1$, the first nontrivial leading coefficient of the monic polynomial P_n, we get $\beta_n = p_n^1 - p_{n+1}^1$. ■

For future use, we introduce the following diagonal matrices:

$$\gamma := \operatorname{diag}(\gamma_1, \gamma_2, \ldots), \quad \beta := \operatorname{diag}(\beta_0, \beta_1, \ldots) \tag{18}$$

and

$$J_- := \Lambda^\top \gamma, \quad J_+ := \beta + \Lambda, \tag{19}$$

so that we have the splitting

$$J = \Lambda^\top \gamma + \beta + \Lambda = J_- + J_+. \tag{20}$$

In general, given any semi-infinite matrix A, we will write $A = A_- + A_+$, where A_- is a strictly lower triangular matrix and A_+ an upper triangular matrix. Moreover, A_0 will denote the diagonal part of A.

1.5 | The Pascal matrices

The lower Pascal matrix, built up of binomial numbers, is defined by

$$B = (B_{n,m}), \quad B_{n,m} := \begin{cases} \binom{n}{m}, & n \geq m, \\ 0, & n < m, \end{cases} \tag{21}$$

so that

$$\chi(z+1) = B\chi(z). \tag{22}$$

Moreover,

$$B^{-1} = (\tilde{B}_{n,m}), \quad \tilde{B}_{n,m} := \begin{cases} (-1)^{n+m}\binom{n}{m}, & n \geq m, \\ 0, & n < m, \end{cases} \tag{23}$$

and

$$\chi(z-1) = B^{-1}\chi(z). \tag{24}$$

The lower Pascal matrix and its inverse are explicitly given by

$$B=\begin{pmatrix} 1 & 0 & \cdots & & & & & \\ 1 & 1 & 0 & \cdots & & & & \\ 1 & 2 & 1 & 0 & \cdots & & & \\ 1 & 3 & 3 & 1 & 0 & \cdots & & \\ 1 & 4 & 6 & 4 & 1 & 0 & \cdots & \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & \cdots \\ \vdots & & & & & & \ddots & \ddots \end{pmatrix}, \quad B^{-1}=\begin{pmatrix} 1 & 0 & \cdots & & & & & \\ -1 & 1 & 0 & \cdots & & & & \\ 1 & -2 & 1 & 0 & \cdots & & & \\ -1 & 3 & -3 & 1 & 0 & \cdots & & \\ 1 & -4 & 6 & -4 & 1 & 0 & \cdots & \\ -1 & 5 & -10 & 10 & -5 & 1 & 0 & \cdots \\ \vdots & & & & & & \ddots & \ddots \end{pmatrix}. \tag{25}$$

Let us introduce the lower unitriangular semi-infinite matrices, and let us refer to them as *dressed Pascal matrices,*

$$\Pi := SBS^{-1}, \quad \Pi^{-1} := SB^{-1}S^{-1}, \tag{26}$$

which are connection matrices; that is,

$$P(z+1) = \Pi P(z), \quad P(z-1) = \Pi^{-1}P(z). \tag{27}$$

The lower Pascal matrix can be expressed in terms of its subdiagonal structure as follows:

$$B^{\pm 1} = I \pm \Lambda^\top D + (\Lambda^\top)^2 D^{[2]} \pm (\Lambda^\top)^3 D^{[3]} + \cdots, \tag{28}$$

where the diagonal matrices $D, D^{[k]}$, with $k \in \mathbb{N}$, $(D = D^{[1]})$, are given by

$$D = \operatorname{diag}(1,2,3,\dots), \quad D^{[k]} = \frac{1}{k}\operatorname{diag}(k^{(k)}, (k+1)^{(k)}, (k+2)^{(k)} \cdots), \tag{29}$$

in terms of the falling factorials $x^{(k)} = x(x-1)(x-2)\cdots(x-k+1)$. That is,

$$D^{[k]}_n = \frac{(n+k)\cdots(n+1)}{k}, \quad k \in \mathbb{N}, \quad n \in \mathbb{N}_0. \tag{30}$$

The lower unitriangular factor can also be written in terms of its subdiagonals

$$S = I + \Lambda^\top S^{[1]} + (\Lambda^\top)^2 S^{[2]} + \cdots \tag{31}$$

with $S^{[k]} = \operatorname{diag}(S^{[k]}_0, S^{[k]}_1, \dots)$ diagonal matrices. From (10), it is clear the following connection between these subdiagonals coefficients and the coefficients of the orthogonal polynomials, given in (3):

$$S^{[k]}_k = p^k_{n+k}. \tag{32}$$

We will use the *shift operators* $T_\pm$ with action on the diagonal matrices given by

$$T_- \operatorname{diag}(a_0, a_1, \dots) := \operatorname{diag}(a_1, a_2, \dots), \quad T_+ \operatorname{diag}(a_0, a_1, \dots) := \operatorname{diag}(0, a_0, a_1, \dots), \tag{33}$$

where T_- is the lowering shift operator and T_+ the raising shift operator over the diagonal matrices. These shift operators have the following important properties, for any diagonal matrix $A = \operatorname{diag}(A_0, A_1, \dots)$:

$$\Lambda A = (T_- A)\Lambda, \quad A\Lambda = \Lambda(T_+ A), \quad A\Lambda^\top = \Lambda^\top(T_- A), \quad \Lambda^\top A = (T_+ A)\Lambda^\top. \tag{34}$$

Remark 1. Notice that the standard notation, see Ref. 1, for the differences of a sequence $\{f_n\}_{n\in\mathbb{N}_0}$,

$$\begin{aligned} \Delta f_n &:= f_{n+1} - f_n, \quad n \in \mathbb{N}_0, \\ \nabla f_n &= f_n - f_{n-1}, \quad n \in \mathbb{N}, \end{aligned} \tag{35}$$

and $\nabla f_0 = f_0$, connects with the shift operators by means of

$$T_- = I + \Delta, \quad T_+ = I - \nabla. \tag{36}$$

In terms of these shift operators, we find

$$2D^{[2]} = (T_- D)D, \quad 3D^{[3]} = (T_-^2 D)(T_- D)D = 2(T_- D^{[2]})D = 2D^{[2]}(T_-^2 D). \tag{37}$$

With these tools, we are ready to study the expansion in subdiagonals of S^{-1} in terms of the corresponding expansion of S.

Proposition 2. *The inverse matrix S^{-1} of the matrix S expands in terms of subdiagonals as follows:*

$$S^{-1} = I + \Lambda^\top S^{[-1]} + (\Lambda^\top)^2 S^{[-2]} + \cdots. \tag{38}$$

The subdiagonals $S^{[-k]}$ are explicitly given in terms of the subdiagonals of S, the first few are

$$\begin{aligned} S^{[-1]} &= -S^{[1]}, \\ S^{[-2]} &= -S^{[2]} + (T_- S^{[1]})S^{[1]}, \\ S^{[-3]} &= -S^{[3]} + (T_- S^{[2]})S^{[1]} + (T_-^2 S^{[1]})S^{[2]} - (T_-^2 S^{[1]})(T_- S^{[1]})S^{[1]}, \\ S^{[-4]} &= -S^{[4]} + (T_- S^{[3]})S^{[1]} + (T_-^2 S^{[2]})S^{[2]} - (T_-^2 S^{[2]})(T_- S^{[1])})S^{[1]} + (T_-^3 S^{[1]})S^{[3]} \\ &\quad -(T_-^3 S^{[1]})(T_- S^{[2])})S^{[1]} - (T_-^3 S^{[1]})(T_-^2 S^{[1])})S^{[2]} + (T_-^3 S^{[1]})(T_-^2 S^{[1]})(T_- S^{[1])})S^{[1]}. \end{aligned} \tag{39}$$

With all these at hand, we can also get expressions for the nontrivial coefficients corresponding to the highest powers of the monic orthogonal polynomials.

Proposition 3. *The following* nonlocal *expressions for the polynomial coefficients in terms of the recursion coefficients hold true:*

$$p^1_{n+1} = -\sum_{k=0}^{n} \beta_k, \quad p^2_{n+1} = -\sum_{k=1}^{n} \gamma_k + \sum_{0\le l<k<n} \beta_k \beta_l. \tag{40}$$

Moreover, it is also true that

$$p^3_{n+2} - p^3_{n+3} = \gamma_{n+2} p^1_{n+1} + (\beta_{n+2} + \beta_{n+1} + \beta_n) p^2_{n+2} - (\beta_{n+1} + \beta_n) p^1_{n+2} p^1_{n+1}. \tag{41}$$

Proof. We start by noticing that

$$\begin{aligned} J &= S\Lambda S^{-1} = (I + \Lambda^\top S^{[1]} + (\Lambda^\top)^2 S^{[2]} + \cdots)\Lambda(I + \Lambda^\top S^{[-1]} + (\Lambda^\top)^2 S^{[-2]} + \cdots) \\ &= \Lambda + T_+ S^{[1]} + S^{[-1]} + \Lambda^\top (T_+ S^{[2]} + S^{[-2]} + S^{[1]} S^{[-1]}) \\ &\quad + (\Lambda^\top)^2 (T_+ S^{[3]} + S^{[-3]} + S^{[2]} S^{[-1]} + (T_+ S^{[1]}) S^{[-2]}) + \cdots . \end{aligned} \tag{42}$$

Thus, we obtain

$$\beta = T_+ S^{[1]} - S^{[1]}, \tag{43}$$

$$\gamma = T_+ S^{[2]} - S^{[2]} + (T_- S^{[1]} - S^{[1]}) S^{[1]}, \tag{44}$$

and an infinite set of relations among subdiagonals of S, being the first

$$T_+ S^{[3]} + S^{[-3]} + S^{[2]} S^{[-1]} + (T_+ S^{[1]}) S^{[-2]} = 0. \tag{45}$$

The relation (43) componentwise is (17). Equation (44) reads componentwise as follows:

$$\gamma_{n+1} = S^{[2]}_{n-1} - S^{[2]}_n + \left(S^{[1]}_{n+1} - S^{[1]}_n \right) S^{[1]}_n = p^2_{n+1} - p^2_{n+2} - \beta_{n+1} p^1_{n+1}. \tag{46}$$

Hence, using telescoping series again, we find (40). Finally, Equation (45) reads

$$\begin{aligned} &T_+ S^{[3]} - S^{[3]} + (T_- S^{[2]}) S^{[1]} + (T_-^2 S^{[1]}) S^{[2]} - (T_-^2 S^{[1]})(T_- S^{[1]}) S^{[1]} - S^{[2]} S^{[1]} \\ &\quad + (T_+ S^{[1]}) \big(-S^{[2]} + (T_- S^{[1]}) S^{[1]} \big) = 0, \end{aligned} \tag{47}$$

that we can write

$$\begin{aligned} &T_+ S^{[3]} - S^{[3]} + T_- \big(S^{[2]} - T_+ S^{[2]} - (T_- S^{[1]} - S^{[1]}) S^{[1]} \big) S^{[1]} + (T_-^2 S^{[1]} - T_+ S^{[1]}) S^{[2]} \\ &\quad + (T_+ S^{[1]} - T_- S^{[1]})(T_- S^{[1]}) S^{[1]} = 0, \end{aligned} \tag{48}$$

so that

$$T_+ S^{[3]} - S^{[3]} = (T_- \gamma) S^{[1]} + (T_-^2 \beta + T_- \beta + \beta) S^{[2]} - (T_- \beta + \beta)(T_- S^{[1]}) S^{[1]} = 0, \tag{49}$$

and componentwise, we have (41). ■

Remark 2. In (41), we can sum up on the LHS, observe that is a telescoping series, to get a nonlocal nonlinear expression, in terms of the recursion coefficients, for p^3_n. A similar statement holds for every p^k_n with $k = 4, 5, \ldots$.

Now, we turn our attention to the dressed Pascal matrix. We expand the dressed Pascal matrices into subdiagonals

$$\Pi^{\pm 1} = I + \Lambda^{\top}\pi^{[\pm 1]} + (\Lambda^{\top})^2\pi^{[\pm 2]} + \cdots \tag{50}$$

with $\pi^{[\pm n]} = \operatorname{diag}(\pi_0^{[\pm n]}, \pi_1^{[\pm n]}, \ldots)$. Then, for these subdiagonals we find the following.

Proposition 4 (The dressed Pascal matrix coefficients). *The subdiagonal coefficients of the Pascal matrices satisfy*

$$\begin{aligned}\pi_n^{[\pm 1]} = \pm\,(n+1), \quad \pi_n^{[\pm 2]} &= \frac{(n+2)(n+1)}{2} \pm p^1_{n+2}(n+1) \mp (n+2)p^1_{n+1}\\ &= \frac{(n+2)(n+1)}{2} \mp (n+1)\beta_{n+1} \mp p^1_{n+1},\end{aligned} \tag{51}$$

$$\begin{aligned}\pi_n^{[\pm 3]} = &\pm \frac{(n+3)(n+2)(n+1)}{3} + \frac{(n+2)(n+1)}{2}p^1_{n+3} - \frac{(n+3)(n+2)}{2}p^1_{n+1}\\ &\pm (n+1)p^2_{n+3} \mp (n+3)p^2_{n+2} \pm (n+3)p^1_{n+2}p^1_{n+1} \mp (n+2)p^1_{n+3}p^1_{n+1}.\end{aligned} \tag{52}$$

Moreover, the following relations fulfill:

$$\begin{aligned}&\pi^{[1]} + \pi^{[-1]} = 0, \quad \pi^{[2]} + \pi^{[-2]} = 2D^{[2]}, \quad \pi^{[3]} + \pi^{[-3]} = 2((T_-^2 S^{[1]})D^{[2]} - (T_- D^{[2]})S^{[1]}),\\ &\pi^{[1]} - \pi^{[-1]} = 2D, \quad \pi^{[2]} - \pi^{[-2]} = 2((T_- S^{[1]})D - (T_- D)S^{[1]}),\\ &\pi^{[3]} - \pi^{[-3]} = 2\big(D^{[3]} + (T_- S^{[2]})D - (T_-^2 D)S^{[2]} + (T_-^2 D)(T_- S^{[1]})S^{[1]} - (T_-^2 S^{[1]})(T_- D)S^{[1]}\big).\end{aligned} \tag{53}$$

Proof. From (34), we get

$$\begin{aligned}\Pi^{\pm 1} &= (I + \Lambda^{\top}S^{[1]} + (\Lambda^{\top})^2 S^{[2]} + \cdots)(I \pm \Lambda^{\top}D + (\Lambda^{\top})^2 D^{[2]} \pm (\Lambda^{\top})^3 D^{[3]} + \cdots)(I + \Lambda^{\top}S^{[-1]} + (\Lambda^{\top})^2 S^{[-2]} + \cdots)\\ &= I + (I + \Lambda^{\top}S^{[1]} + (\Lambda^{\top})^2 S^{[2]} + \cdots)\big(\pm\Lambda^{\top}D + (\Lambda^{\top})^2 D^{[2]} \pm (\Lambda^{\top})^3 D^{[3]} + \cdots\big)(I + \Lambda^{\top}S^{[-1]} + (\Lambda^{\top})^2 S^{[-2]} + \cdots)\end{aligned} \tag{54}$$

so that

$$\begin{aligned}\Pi^{\pm 1} = I &\pm \Lambda^{\top}D + (\Lambda^{\top})^2\big(D^{[2]} \pm (T_- S^{[1]})D \pm (T_- D)S^{[-1]}\big) + (\Lambda^{\top})^3(\pm D^{[3]} + (T_-^2 S^{[1]})D^{[2]} + (T_- D^{[2]})S^{[-1]}\\ &\pm (T_- S^{[2]})D \pm (T_-^2 D)S^{[-2]} \pm (T_-^2 S^{[1]})(T_- D)S^{[-1]}) + \cdots.\end{aligned} \tag{55}$$

From (55) and Proposition 2, we obtain

$$\pi^{[\pm 1]} = \pm D, \tag{56}$$

$$\pi^{[\pm 2]} = D^{[2]} \pm (T_- S^{[1]})D \mp (T_- D)S^{[1]}, \tag{57}$$

$$\begin{aligned}\pi^{[\pm 3]} = &\pm D^{[3]} + (T_-^2 S^{[1]})D^{[2]} - (T_- D^{[2]})S^{[1]} \pm (T_- S^{[2]})D \mp (T_-^2 D)S^{[2]} \\ &\pm (T_-^2 D)(T_- S^{[1]})S^{[1]} \mp (T_-^2 S^{[1]})(T_- D)S^{[1]}.\end{aligned} \tag{58}$$

That component wise gives the desired result once we use the expressions for the β's in (17). ■

Proposition 5. *For any polynomial $R(z)$, we have*

$$\begin{aligned} R(\Lambda)B^{\pm 1} &= B^{\pm 1}R(\Lambda \pm I), \quad B^{\pm 1}R(\Lambda) = R(\Lambda \mp I)B^{\pm 1}, \\ R(J)\Pi^{\pm 1} &= \Pi^{\pm 1}R(J \pm I), \quad \Pi^{\pm 1}R(J) = R(J \mp I)\Pi^{\pm 1}. \end{aligned} \tag{59}$$

Proof. The compatibility of the couple of equations

$$\begin{cases} B^{\pm 1}\chi(z) = \chi(z \pm 1), \\ R(\Lambda)\chi(z) = R(z)\chi(z), \end{cases} \tag{60}$$

leads to

$$R(\Lambda)B^{\pm 1}\chi(z) = R(\Lambda)\chi(z \pm 1) = R(z \pm 1)\chi(z \pm 1) = R(z \pm 1)B^{\pm 1}\chi(z) = B^{\pm 1}R(\Lambda \pm I)\chi(z), \tag{61}$$

so that $R(\Lambda)B^{\pm 1} = B^{\pm 1}R(\Lambda \pm I)$, and consequently, $B^{\pm 1}R(\Lambda) = R(\Lambda \mp I)B^{\pm 1}$. Finally, a similarity transformation $\Lambda = S^{-1}JS$ gives the result. ■

2 | DISCRETE ORTHOGONAL POLYNOMIALS AND DISCRETE PEARSON EQUATIONS

2.1 | Discrete Pearson equation

We now assume that the functional is a sum of Dirac delta functions supported $\mathbb{N}_0$,

$$\rho = \sum_{k=0}^{\infty} \delta(z - k)w(k). \tag{62}$$

for some weight function $w(z)$ with finite values $w(k)$ for $k \in \mathbb{N}_0$. Hence, the bilinear form is $\langle F, G \rangle = \sum_{k=0}^{\infty} F(k)G(k)w(k)$. The moments are

$$\rho_n = \sum_{k=0}^{\infty} k^n w(k), \tag{63}$$

and, in particular, the 0th moment reads as follows:

$$\rho_0 = \sum_{k=0}^{\infty} w(k). \tag{64}$$

We will study families of weights that satisfy the following *discrete Pearson equation*:

$$\nabla(\sigma w) = \tau w, \tag{65}$$

that is, $\sigma(k)w(k) - \sigma(k-1)w(k-1) = \tau(k)w(k)$, for $k \in \{1, 2, \dots\}$, with $\sigma(z), \tau(z) \in \mathbb{R}[z]$ polynomials. If we write $\theta := \sigma - \tau$, the previous Pearson equation reads

$$\theta(k+1)w(k+1) = \sigma(k)w(k), \quad k \in \mathbb{N}_0. \tag{66}$$

Theorem 1 (Hypergeometric symmetry of the moment matrix)**.** *Let the weight w be subject to the discrete Pearson equation (66), where θ, σ are polynomials with $\theta(0) = 0$. Then, the corresponding moment matrix fulfills*

$$\theta(\Lambda)G = B\sigma(\Lambda)GB^\top. \tag{67}$$

Proof. The moment matrix is

$$G = \sum_{k=0}^{\infty} \chi(k)\chi(k)^\top w(k). \tag{68}$$

Thus,

$$\begin{aligned}
\theta(\Lambda)G &= \sum_{k=0}^{\infty} \theta(\Lambda)\chi(k)\chi(k)^\top w(k) \quad \text{use (68)} \\
&= \sum_{k=1}^{\infty} \chi(k)\chi(k)^\top \theta(k)w(k) \quad \text{use } \Lambda\chi = z\chi \quad and \quad \theta(0) = 0 \\
&= \sum_{k=0}^{\infty} \chi(k+1)\chi(k+1)^\top \theta(k+1)w(k+1) \quad \text{shift the summation variable} \\
&= \sum_{k=0}^{\infty} \chi(k+1)\chi(k+1)^\top \sigma(k)w(k) \quad \text{use Pearson equation (66)} \\
&= \sum_{k=0}^{\infty} B\chi(k)\chi(k)^\top B^\top \sigma(k)w(k) \quad \text{use (22)} \\
&= \sum_{k=0}^{\infty} B\sigma(\Lambda)\chi(k)\chi(k)^\top w(k)B^\top \quad \text{use } \Lambda\chi = z\chi \text{ again} \\
&= B\sigma(\Lambda)GB^\top \quad \text{use (68).}
\end{aligned} \tag{69}$$

■

Remark 3. This result extends to the case when θ and σ are entire functions, not necessarily polynomials, and we can ensure some meaning for $\theta(\Lambda)$ and $\sigma(\Lambda)$. Later on, see Section 2.5, we will show that this symmetry of the Gram matrix is a direct consequence of the generalized hypergeometric ODE satisfied by the the 0th moment.

2.2 | Consequences for the Jacobi matrix

We can use the Cholesky factorization (8) and the Jacobi matrix (12) to get the symmetrization matrix for the Jacobi matrix and an important constraint fulfilled by the dressed Pascal and Jacobi matrices.

Proposition 6. *Let the weight w be subject to the discrete Pearson equation (66), where the functions θ, σ are entire functions, not necessarily polynomials, with $\theta(0) = 0$. Then,*

(i) The matrices $H\theta(J^\top)$ and $\sigma(J)H$ are symmetric.
(ii) The following relation

$$\Pi^{-1}H\theta(J^\top) = \sigma(J)H\Pi^\top \tag{70}$$

holds.

Proof. Given the Hankel property, $\Lambda G = G\Lambda^\top$, we write (67) as $G\theta(\Lambda^\top) = B\sigma(\Lambda)GB^\top$, and using the Cholesky factorization (8), we get $S^{-1}HS^{-\top}\theta(\Lambda^\top) = B\sigma(\Lambda)S^{-1}HS^{-\top}B^\top$, so that $HS^{-\top}\theta(\Lambda^\top)S^\top = SBS^{-1}S\sigma(\Lambda)S^{-1}HS^{-\top}B^\top S^\top$, and we get Equation (70).

Let us prove that the matrices $H\theta(J^\top)$ and $\sigma(J)H$ are symmetric. This fact follows from (13), $JH = HJ^\top$. Indeed,

$$\begin{aligned}(H\theta(J^\top))^\top &= \theta(J)H = \theta(HJ^\top H^{-1})H = H\theta(J^\top)H^{-1}H = H\theta(J^\top),\\ (\sigma(J)H)^\top &= H\sigma(J^\top) = H\sigma(H^{-1}JH) = HH^{-1}\sigma(J)H = \sigma(J)H.\end{aligned} \tag{71}$$

■

2.3 | Generalized hypergeometric functions and the Pearson equation

Let us assume that θ, σ are polynomials, and denote their respective degrees by $N + 1 := \deg\theta(z)$ and $M := \deg\sigma(z)$. The roots of these polynomials are denoted by $\{-b_i + 1\}_{i=1}^N$ and $\{-a_i\}_{i=1}^M$, respectively. Following Ref. 3, we choose

$$\theta(z) = z(z + b_1 - 1)\cdots(z + b_N - 1), \quad \sigma(z) = \eta(z + a_1)\cdots(z + a_M). \tag{72}$$

Notice that we have normalized θ to be a monic polynomial, whereas σ is not monic, being the coefficient of the leading power denoted by η. This parameter η will be instrumental in what follows. Therefore, see Ref. 3, we have that the weight satisfying the Pearson equation (66) is proportional to

$$w(z)=\frac{(a_1)_z\cdots(a_M)_z}{\Gamma(z+1)(b_1)_z\cdots(b_N)_z}\eta^z, \tag{73}$$

here we use the Pochhammer symbol $(\alpha)_z=\frac{\Gamma(\alpha+z)}{\Gamma(\alpha)}$, that for $z\in\mathbb{N}$ reduces to the standard expression

$$(\alpha)_n=\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1),\quad (\alpha)_0=1. \tag{74}$$

The moments of this weight are finite whenever, see Ref. 3 and references therein,

(i) $M\leq N$ and $\eta\in\mathbb{C}$.
(ii) $M>N$, $\eta\in\mathbb{C}$ and one or more of the parameters a_i is a nonpositive integer.
(iii) $M=N+1$ and $|\eta|<1$.
(iv) $M=N+1$, $|\eta|=1$ and $\operatorname{Re}(b_1+\cdots+b_{N-1}-a_1-\cdots-a_M)>0$.

According to (64), the 0th moment

$$\begin{aligned}\rho_0&=\sum_{k=0}^{\infty}w(k)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_M)_k}{(b_1+1)_k\cdots(b_N+1)_k}\frac{\eta^k}{k!}\\&={}_MF_N(a_1,\dots,a_M;b_1,\dots,b_N;\eta)={}_MF_N\left[\begin{matrix}a_1&\cdots&a_M\\b_1&\cdots&b_N\end{matrix};\eta\right]\end{aligned} \tag{75}$$

is the generalized hypergeometric function, where we are using the two standard notations, see Ref. 19. Then, according to (63), for $n\in\mathbb{N}$, the corresponding higher moments, $\rho_n=\sum_{k=0}^{\infty}k^n w(k)$, are

$$\rho_n=\vartheta_\eta^n\rho_0=\vartheta_\eta^n\left({}_MF_N\left[\begin{matrix}a_1&\cdots&a_M\\b_1&\cdots&b_N\end{matrix};\eta\right]\right),\quad \vartheta_\eta:=\eta\frac{\partial}{\partial\eta}. \tag{76}$$

Given a function $f(\eta)$, for $n\in\mathbb{N}_0$, we consider the Wronskian

$$\mathscr{W}_{n+1}(f)=\det\begin{pmatrix}f&\vartheta_\eta f&\vartheta_\eta^2 f&\cdots&\vartheta_\eta^n f\\\vartheta_\eta f&\vartheta_\eta^2 f&&&\vartheta_\eta^{n+1}f\\\vartheta_\eta^2 f&&&&\\\vdots&&&&\\\vartheta_\eta^n f&\vartheta_\eta^{n+1}f&&&\vartheta_\eta^{2n}f\end{pmatrix}. \tag{77}$$

Then, for $n\in\mathbb{N}$, we have that the Hankel determinants $\Delta_n=\det G^{[n]}$, determined by the truncations of the corresponding moment matrix, are Wronskians of generalized hypergeometric functions

$$\Delta_n=\tau_n,\quad \tau_n:=\mathscr{W}_n\left({}_MF_N\left[\begin{matrix}a_1&\cdots&a_M\\b_1&\cdots&b_N\end{matrix};\eta\right]\right), \tag{78}$$

$$\tilde{\Delta}_n = \vartheta_\eta \tau_n. \tag{79}$$

Moreover, we also have for $n \in \mathbb{N}_0$ and taking $\tau_0 = 1$ that

$$H_n = \frac{\tau_{n+1}}{\tau_n}, \quad p^1_n = -\vartheta_\eta \log \tau_n, \tag{80}$$

as well as

$$\tau_{n+1} = H_n H_{n-1} \cdots H_0. \tag{81}$$

Remark 4. The functions τ_k are knwon in the theory of integrable systems as τ-functions.

2.4 | The Laguerre–Freud structure matrix and the Cholesky factorization

We are ready to prove one of the main results in the paper. We will get a banded semi-infinite matrix describing the action of multiplication by the polynomials $\theta(x)$ and $\sigma(x)$ the sequence of orthogonal polynomials $\{P_n(x)\}_{n=0}^\infty$.

Theorem 2 (Laguerre–Freud structure matrix)**.** *Let us assume that the weight w is subject to the discrete Pearson equation (66) with θ, σ polynomials such that $\theta(0) = 0$, $\deg\theta(z) = N+1$, $\deg\sigma(z) = M$. Then, the Laguerre–Freud structure matrix*

$$\Psi := \Pi^{-1} H\theta(J^\top) = \sigma(J)H\Pi^\top = \Pi^{-1}\theta(J)H = H\sigma(J^\top)\Pi^\top \tag{82}$$

$$= \theta(J+I)\Pi^{-1}H = H\Pi^\top\sigma(J^\top - I), \tag{83}$$

has only $N+M+2$ possibly nonzero diagonals ($N+1$ superdiagonals, the main diagonal and M subdiagonals)

$$\Psi = (\Lambda^\top)^M\psi^{(-M)} + \cdots + \Lambda^\top\psi^{(-1)} + \psi^{(0)} + \psi^{(1)}\Lambda + \cdots + \psi^{(N+1)}\Lambda^{N+1}, \tag{84}$$

for some diagonal matrices $\psi^{(k)}$. In particular, the lowest subdiagonal and highest superdiagonal are given by

$$\begin{cases} (\Lambda^\top)^M\psi^{(-M)} = \eta(J_-)^M H, \quad \psi^{(-M)} = \eta H \prod_{k=0}^{M-1} T_-^k\gamma = \eta\operatorname{diag}\left(H_0\prod_{k=1}^{M}\gamma_k, H_1\prod_{k=2}^{M+1}\gamma_k, \dots\right), \\ \psi^{(N+1)}\Lambda^{N+1} = H(J_-^\top)^{N+1}, \quad \psi^{(N+1)} = H\prod_{k=0}^{N} T_-^k\gamma = \operatorname{diag}\left(H_0\prod_{k=1}^{N+1}\gamma_k, H_1\prod_{k=2}^{N+2}\gamma_k, \dots\right). \end{cases} \tag{85}$$

The vector $P(z)$ of orthogonal polynomials fulfills the following structure equations:

$$\theta(z)P(z-1) = \Psi H^{-1}P(z), \quad \sigma(z)P(z+1) = \Psi^\top H^{-1}P(z). \tag{86}$$

Proof. To get (83) just use (59). Now, notice that the matrices $H\theta(J^\top)$ and $\sigma(J)H$ are banded matrices with $2N+3$ diagonals ($N+1$ superdiagonals and subdiagonals) and $2M+1$ (M superdiagonals and subdiagonals), respectively. Thus, $\Pi^{-1}H\theta(J^\top)$ has at most $N+1$ superdiagonals, whereas $\sigma(J)H\Pi^\top$ has at most M subdiagonals. Consequently, Ψ given by (82) is a banded semi-infinite matrix as described. Then, (85) follow from (82) and the mentioned banded structure. To get the lowest subdiagonal, we use $\Psi = \sigma(J)H\Pi^\top$, so that the lowest subdiagonal will come from the lowest subdiagonal of $\sigma(J)$, namely, $\eta\,(J_-)^M$ right multiplied by the diagonal matrix H and then right multiplied the main diagonal of $\Pi^\top$, which happens to be the identity. To get the highest superdiagonal, we proceed similarly and use $\Psi = \Pi^{-1}H\theta(J^\top)$, so that the main superdiagonal will come from the product of the three factors I, H, and $\theta_N(J_-^\top)^{N+1}$, in this order. The componentwise expressions are direct computations.

Finally, we have

$$\begin{aligned}
\Psi H^{-1}P(z) &= \Pi^{-1}\theta(J)HH^{-1}P(z) & \Psi^\top H^{-1}P(z) &= \Pi\sigma(J)HH^{-1}P(z)\\
&= \Pi^{-1}\theta(J)P(z) & &= \Pi\sigma(J)P(z)\\
&= \Pi^{-1}\theta(z)P(z) & &= \Pi\sigma(z)P(z)\\
&= \theta(z)P(z-1), & &= \sigma(z)P(z+1).
\end{aligned} \tag{87}$$

■

Remark 5 (Laguerre–Freud equations). Equation(82) is instrumental in obtaining nonlinear recurrences for the recursion coefficients, see Ref. 16

$$\gamma_{n+1} = F_1(n, \gamma_n, \gamma_{n-1}, \dots, \beta_n, \beta_{n-1} \dots), \quad \beta_{n+1} = F_2(n, \gamma_{n+1}, \gamma_n, \dots, \beta_n, \beta_{n-1}, \dots), \tag{88}$$

for some functions F_1, F_2. These recurrences were named by Alphonse Magnus, attending to Refs. 20, 21, as Laguerre–Freud, see Refs. 22–25. This is the reason for the given name to Ψ.

The previous result extends to the Jacobi matrix, giving nontrivial nonlinear relations that link the Jacobi and Laguerre–Freud matrices.

Proposition 7. *The Laguerre–Freud and Jacobi matrices fulfill*

$$\sigma(J)\theta(J+I) = \Psi H^{-1}\Psi^\top H^{-1}, \quad \theta(J)\sigma(J-I) = \Psi^\top H^{-1}\Psi H^{-1}. \tag{89}$$

Proof. We have

$$\begin{aligned}
\sigma(J)\theta(J+I)P(z) &= \sigma(z)\theta(z+1)P(z) = \sigma(z)\Psi H^{-1}P(z+1) = \Psi H^{-1}\sigma(z)P(z+1)\\
&= \Psi H^{-1}\Psi^\top H^{-1}P(z),\\
\theta(J)\sigma(J-I)P(z) &= \theta(z)\sigma(z-1)P(z) = \theta(z)\Psi^\top H^{-1}P(z-1) = \Psi^\top H^{-1}\theta(z)P(z-1)\\
&= \Psi^\top H^{-1}\Psi H^{-1}P(z),
\end{aligned} \tag{90}$$

which hold for all $z \in \mathbb{C}$ and, consequently, imply the desired result. ■

Theorem 3 (Cholesky factorization)**.** *Let us assume that $H\theta(J^\top)$ and $\sigma(J)H$ have the following Cholesky factorizations:*

$$H\theta(J^\top) = \Theta^{-1}H_\theta\Theta^{-\top}, \quad \sigma(J)H = \Sigma^{-1}H_\sigma\Sigma^{-\top}, \tag{91}$$

with Θ and Σ lower unitriangular matrices and H_θ and H_σ diagonals matrices. Then,

(i) *Θ^{-1} and Σ^{-1} have only first $N+1$ and M subdiagonals possibly nonzero, respectively.*
(ii) *We have*

$$\Pi = \Theta^{-1}\Sigma, \quad H_\theta = H_\sigma =: h. \tag{92}$$

(iii) *The Laguerre–Freud matrix has the following Gauss–Borel factorization:*

$$\Psi = \Sigma^{-1}h\Theta^{-\top}. \tag{93}$$

Proof. Recall that according to Proposition 6, the matrices $H\theta(J^\top)$ and $\sigma(J)H$ are symmetric, and consequently, the corresponding Gauss–Borel factorizations, when they exist, will be Cholesky factorizations.

(i) If follows from the fact that J is tridiagonal and θ and σ polynomials of degrees $N+1$ and M, respectively.
(ii) The symmetry (70) yields

$$\Pi^{-1}\Theta^{-1}H_\theta\Theta^{-\top} = \Sigma^{-1}H_\sigma\Sigma^{-\top}\Pi^\top \tag{94}$$

and given the uniqueness of the Gauss–Borel factorization, the result follows.
(iii) From (82), we have different alternatives to show the result, let us see two of them

$$\begin{aligned}\Psi &= \Pi^{-1}H\theta(J^\top) = \Sigma^{-1}\Theta\Theta^{-1}H_\theta\Theta^{-\top} = \Sigma^{-1}h\Theta^{-\top},\\ \Psi &= \sigma(J)H\Pi^\top = \Sigma^{-1}H_\sigma\Sigma^{-\top}\Sigma^\top\Theta^{-1} = \Sigma^{-1}h\Theta^{-\top}.\end{aligned} \tag{95}$$

■

The compatibility with the recursion relation, that is, eigenfunctions of the Jacobi matrix, and the recursion matrix leads to some interesting equations.

Proposition 8. *The following compatibility conditions for the Laguerre–Freud and Jacobi matrices hold:*

$$[\Psi H^{-1}, J] = \Psi H^{-1}, \tag{96a}$$

$$[J, \Psi^\top H^{-1}] = \Psi^\top H^{-1}. \tag{96b}$$

Proof. To prove (96a), we evaluate the eigenvalue equation $JP(z) = zP(z)$ in $z-1$ to get $JP(z-1) = (z-1)P(z-1)$. Now multiply it by $\theta(z)$ to obtain $J\theta(z)P(z-1) = (z-1)\theta(z)P(z-1)$. Therefore, recalling (86), $J\Psi H^{-1}P(z) = (z-1)\Psi H^{-1}P(z) = \Psi H^{-1}(J-I)P(z)$ from where the relation follows.

Alternatively, observe that

$$\Psi H^{-1} = SB^{-1}\theta(\Lambda)S^{-1} \tag{97}$$

from where

$$[\Psi H^{-1}, J] = S[B^{-1}\theta(\Lambda), \Lambda]S^{-1} = S[B^{-1}, \Lambda]\theta(\Lambda)S^{-1} = SB^{-1}\theta(\Lambda)S^{-1} = \Psi H^{-1}. \tag{98}$$

To show (96b), we take the transpose of the proved one to get $[J^\top, H^{-1}\Psi^\top] = H^{-1}\Psi^\top$ so that

$$[HJ^\top H^{-1}, HH^{-1}\Psi^\top H^{-1}] = HH^{-1}\Psi^\top H^{-1} \tag{99}$$

and recalling that $HJ^\top H^{-1} = J$, see (13), we get the desired result. ■

2.5 | Contiguous hypergeometric relations

As we have seen, the polynomial discrete Pearson equation leads to (76), and the Hankel determinants are Wronskians of a generalized hypergeometric function, see (78) and (79). Hence, we expect that some properties of generalized hypergeometric functions may translate to the Gram matrix. To write this dictionary, we require of Proposition 10, discussed latter, which ensures

$$\vartheta_\eta G = \Lambda G = G\Lambda^\top. \tag{100}$$

We now study three important relations fulfilled by the generalized hypergeometric function, namely:

$$(\vartheta_\eta + a_i)\,{}_MF_N\left[\begin{matrix} a_1 \cdots a_i \cdots a_M \\ b_1 \cdots b_N \end{matrix};\eta\right] = a_i\,{}_MF_N\left[\begin{matrix} a_1 \cdots a_i + 1 \cdots a_M \\ b_1 \cdots b_N \end{matrix};\eta\right], \tag{101}$$

$$(\vartheta_\eta + b_j - 1)\,{}_MF_N\left[\begin{matrix} a_1 \cdots a_M \\ b_1 \cdots b_j \cdots b_N \end{matrix};\eta\right] = (b_j - 1)\,{}_MF_N\left[\begin{matrix} a_1 \cdots a_M \\ b_1 \cdots b_j - 1 \cdots b_N \end{matrix};\eta\right], \tag{102}$$

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\,{}_MF_N\left[\begin{matrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{matrix};\eta\right] = \kappa\,{}_MF_N\left[\begin{matrix} a_1+1 & \cdots & a_M+1 \\ b_1+1 & \cdots & b_N+1 \end{matrix};\eta\right], \quad \kappa := \frac{\prod_{i=1}^M a_i}{\prod_{j=1}^N b_j}. \tag{103}$$

From these three equations, we also derive

$$\eta\prod_{n=1}^{M}\left(\eta\frac{\mathrm{d}}{\mathrm{d}\eta} + a_n\right)u = \eta\frac{\mathrm{d}}{\mathrm{d}\eta}\prod_{n=1}^{N}\left(\eta\frac{\mathrm{d}}{\mathrm{d}\eta} + b_n - 1\right)u, \quad u := {}_MF_N\left[\begin{matrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{matrix};\eta\right]. \tag{104}$$

In (101) and (102), we have basic relations between contiguous generalized hypergeometric functions and its derivatives.

For the analysis of these equations, let us introduce the shift operators in the parameters $\{a_i\}_{i=1}^M$ and $\{b_j\}_{j=1}^N$. Thus, given a function $f[\begin{smallmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{smallmatrix}]$ of these parameters, we introduce the shifts ${}_iT$ and T_j as follows:

$$
\begin{aligned}
&{}_iTf\begin{bmatrix} a_1\cdots a_i\cdots a_M \\ b_1\cdots b_N \end{bmatrix} = f\begin{bmatrix} a_1\cdots a_i+1\cdots a_M \\ b_1\cdots b_N \end{bmatrix}, \quad T_jf\begin{bmatrix} a_1\cdots a_M \\ b_1\cdots b_j\cdots b_N \end{bmatrix} \\
&= f\begin{bmatrix} a_1\cdots a_M \\ b_1\cdots b_j-1\cdots b_N \end{bmatrix},
\end{aligned} \tag{105}
$$

and a total shift $T = {}_1T\cdots {}_MT\,T_1^{-1}\cdots T_N^{-1}$; that is,

$$
Tf\begin{bmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{bmatrix} := f\begin{bmatrix} a_1+1 & \cdots & a_M+1 \\ b_1+1 & \cdots & b_N+1 \end{bmatrix}. \tag{106}
$$

Then, we find the following.

Theorem 4 (Hypergeometric relations)**.** *The moment matrix $G = (\rho_{n+m})_{n,n\in\mathbb{N}_0}$ of a weight satisfying (66) fulfills the following hypergeometric relations:*

$$
(\Lambda + a_iI)G = a_i\ {}_iTG, \tag{107a}
$$

$$
(\Lambda + (b_j-1)I)G = (b_j-1)T_jG, \tag{107b}
$$

$$
\Lambda G = \kappa B(TG)B^\top \tag{107c}
$$

with B given in (22).

Proof. We first prove (107a). For that aim, we use that $G_{n,m} = \vartheta_\eta^{n+m}({}_MF_N[\begin{smallmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{smallmatrix};\eta])$ so that

$$
\begin{aligned}
(\vartheta_\eta + a_i)G_{n,m} &= (\vartheta_\eta + a_i)\vartheta_\eta^{n+m}\left({}_MF_N\begin{bmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{bmatrix};\eta\right) \\
&= \vartheta_\eta^{n+m}(\vartheta_\eta + a_i)\left({}_MF_N\begin{bmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{bmatrix};\eta\right) \\
&= a_i\vartheta_\eta^{n+m}\left({}_MF_N\begin{bmatrix} a_1\cdots a_i+1\cdots a_M \\ b_1\cdots b_N \end{bmatrix};\eta\right) \\
&= a_i\ {}_iTG_{n,m}.
\end{aligned} \tag{108}
$$

Thus, $(\vartheta_\eta + a_i)G = a_i\,{}_iTG$, and recalling $\vartheta_\eta G = \Lambda G$, we get the result. Relation (107b) is proved similarly. Finally, (107c) follows from (103), the novelty is that in (103), we have $\frac{\mathrm{d}}{\mathrm{d}\,\eta} = \eta^{-1}\vartheta_\eta$, which do not commute with ϑ_η. To overcome this difficulty, observe that the relation

$$\frac{\mathrm{d}}{\mathrm{d}\,\eta}\eta - \eta\frac{\mathrm{d}}{\mathrm{d}\,\eta} = 1 \tag{109}$$

can be written $\eta^{-1}\vartheta_\eta\eta = \vartheta_\eta + 1$. Hence, $\eta^{-1}\vartheta_\eta^n\eta = (\eta^{-1}\vartheta_\eta\eta)^n = (\vartheta_\eta + 1)^n$, that, in turn, implies

$$\frac{\mathrm{d}}{\mathrm{d}\,\eta}\vartheta_\eta^n = (\vartheta_\eta + 1)^n\frac{\mathrm{d}}{\mathrm{d}\,\eta}. \tag{110}$$

Thus, we get the equations

$$\begin{aligned}
\frac{\mathrm{d}\,G_{n,m}}{\mathrm{d}\,\eta} &= \frac{\mathrm{d}}{\mathrm{d}\,\eta}\vartheta_\eta^{n+m}\left({}_MF_N\begin{bmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{bmatrix};\eta\right]\right) = (\vartheta_\eta + 1)^{n+m}\frac{\mathrm{d}}{\mathrm{d}\,\eta}\left({}_MF_N\begin{bmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{bmatrix};\eta\right]\right) \\
&= (\vartheta_\eta + 1)^{n+m}\kappa\;{}_MF_N\left[\begin{matrix} a_1+1 & \cdots & a_M+1 \\ b_1+1 & \cdots & b_N+1 \end{matrix};\eta\right] \\
&= \kappa\sum_{k=0}^{n+m}\binom{n+m}{k}\vartheta_\eta^k\,{}_MF_N\left[\begin{matrix} a_1+1 & \cdots & a_M+1 \\ b_1+1 & \cdots & b_N+1 \end{matrix};\eta\right].
\end{aligned} \tag{111}$$

Using the Chu–Vandermonde identity,

$$\binom{n+m}{k} = \sum_{l=0}^{k}\binom{n}{l}\binom{m}{k-l}, \tag{112}$$

we find

$$\begin{aligned}
\frac{\mathrm{d}\,G_{n,m}}{\mathrm{d}\,\eta} &= \kappa\sum_{k=0}^{n+m}\sum_{l=0}^{k}\binom{n}{l}\binom{m}{k-l}\vartheta_\eta^{l+k-l}\,{}_MF_N\left[\begin{matrix} a_1+1 & \cdots & a_M+1 \\ b_1+1 & \cdots & b_N+1 \end{matrix};\eta\right] \\
&= \kappa\sum_{k=0}^{n+m}\sum_{l=0}^{k}\binom{n}{l}\binom{m}{k-l}TG_{l,k-l} \\
&= \kappa\sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}TG_{l,k}\binom{m}{k}
\end{aligned} \tag{113}$$

from where the result follows. ■

Remark 6. From (104), we derive, in an alternative manner, the relation (67). Indeed, using $\vartheta_\eta - 1 = \eta\vartheta_\eta\eta^{-1}$, we get $(\vartheta_\eta - 1)^n\eta = \eta(\vartheta_\eta)^n$ and we workout (104) as follows:

$$\begin{aligned}
\eta\prod_{n=1}^{M}(\vartheta_\eta+a_n)G_{n,m}&=\eta\prod_{n=1}^{M}(\vartheta_\eta+a_n)\vartheta_\eta^{n+m}\left({}_MF_N\begin{bmatrix}a_1&\cdots&a_M\\b_1&\cdots&b_N\end{bmatrix};\eta\right)\\
&=(\vartheta_\eta-1)^{n+m}\eta\prod_{n=1}^{M}(\vartheta_\eta+a_n)\left({}_MF_N\begin{bmatrix}a_1&\cdots&a_M\\b_1&\cdots&b_N\end{bmatrix};\eta\right)\\
&=\vartheta_\eta\prod_{n=1}^{N}(\vartheta_\eta+b_n-1)(\vartheta_\eta-1)^{n+m}\left({}_MF_N\begin{bmatrix}a_1&\cdots&a_M\\b_1&\cdots&b_N\end{bmatrix};\eta\right)\\
&=\vartheta_\eta\prod_{n=1}^{N}(\vartheta_\eta+b_n-1)\sum_{k=0}^{n+m}\binom{n+m}{k}(-1)^k\vartheta_\eta^k\left({}_MF_N\begin{bmatrix}a_1&\cdots&a_M\\b_1&\cdots&b_N\end{bmatrix};\eta\right)\\
&=\vartheta_\eta\prod_{n=1}^{N}(\vartheta_\eta+b_n-1)\sum_{l=0}^{n}\sum_{k=0}^{m}(-1)^l\binom{n}{l}G_{l,k}(-1)^k\binom{m}{k},
\end{aligned}\tag{114}$$

and consequently, we finally deduce (67).

2.6 | Discrete lattice Toda equation

The shifts in the hypergeometric parameters induce corresponding transformations on the discrete orthogonal polynomials. To describe them, we introduce the following semi-infinite matrices:

$${}_i\Omega := S({}_iTS)^{-1},\quad i\in\{1,\dots,M\},\quad \Omega_k := S\,(T_kS)^{-1},\quad k\in\{1,\dots,N\},\tag{115}$$

so that the following connection formulas are fulfilled:

$${}_i\Omega\,{}_iTP(z)=P(z),\quad i\in\{1,\dots,M\},\tag{116}$$

$$\Omega_j\,T_jP(z)=P(z),\quad j\in\{1,\dots,N\}.\tag{117}$$

In Ref. 26, we proved that

$${}_i\Omega=\begin{pmatrix}1&0&\cdots&\cdots\\ \frac{1}{a_i}\frac{H_1}{{}_iTH_0}&1&\ddots&\\ 0&\frac{1}{a_i}\frac{H_2}{{}_iTH_1}&\ddots&\ddots\\ \vdots&\ddots&\ddots&\ddots\end{pmatrix},\qquad \Omega_j=\begin{pmatrix}1&0&\cdots&\cdots\\ \frac{1}{b_j-1}\frac{H_1}{T_jH_0}&1&\ddots&\\ 0&\frac{1}{b_j-1}\frac{H_2}{T_jH_1}&\ddots&\ddots\\ \vdots&\ddots&\ddots&\ddots\end{pmatrix}.\tag{118}$$

The connection formulas (116) and (117) for contiguous hypergeometric parameters imply a nonlinear compatibility equations that lead to a generalized lattice Toda equation found in Ref. 18. We use the following notation:

$$u_n = H_{n-1}\begin{bmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{bmatrix}; \eta\Bigg]. \tag{119}$$

We now consider three variables (n, r, s), where r, s are any couples of variables taken from the set of hypergeometric parameters $\{a_1, \dots, a_M, b_1, \dots, b_N\}$ and denote the corresponding "shifts" in n, r, s as follows:

$$\bar{u}_n(r,s) = u_{n+1}(r,s), \quad \hat{u}_n(r,s) = \hat{a}u_n(r \pm 1, s), \quad \tilde{u}_n(r,s) = \tilde{a}u(r, s \pm 1), \tag{120}$$

where the $\pm$ signs is a $+$ if the corresponding variable belongs to $\{a_i\}_{i=1}^M$ or a $-$ if it belongs to $\{b_j\}_{j=1}^M$, and the constants $\hat{a}, \tilde{a}$ are taken from $\{a_i\}_{i=1}^N$ and $\{b_j - 1\}_{j=1}^N$ according to the choice selected for the variables r and s. We denote by $\Omega^{(r)}$ and $\Omega^{(s)}$, the corresponding matrices Ω's taken from $\{_i\Omega\}_{i=1}^M$ and $\{\Omega_j\}_{j=1}^N$, depending on the variables picked r and s.

Theorem 5 (Nijhoff–Capel discrete lattice Toda equations). *The squared norms satisfy the following nonlinear equations linking contiguous hypergeometric parameters:*

$$\frac{\hat{u} - \tilde{\hat{u}}}{\bar{u}} = \tilde{\hat{u}}\left(\frac{1}{\tilde{u}} - \frac{1}{\hat{u}}\right). \tag{121}$$

Proof. The connection formulas are

$$\Omega^{(r)}\hat{P} = P, \quad \Omega^{(s)}\tilde{P} = P. \tag{122}$$

Then, we have

$$\tilde{\Omega}^{(r)}\tilde{\hat{P}} = \tilde{P}, \tag{123}$$

so that

$$\Omega^{(s)}\tilde{\Omega}^{(r)}\tilde{\hat{P}} = \Omega^{(s)}\tilde{P} = P, \tag{124}$$

and the compatibility $\tilde{\hat{P}} = \hat{\tilde{P}}$ leads to the nonlinear condition

$$\Omega^{(s)}\tilde{\Omega}^{(r)} = \Omega^{(r)}\hat{\Omega}^{(s)}. \tag{125}$$

Then, as we can write

$$\Omega^{(r)} = \begin{pmatrix} 1 & 0 & \cdots & \\ \frac{\bar{u}_1}{\hat{u}_1} & 1 & \ddots & \\ 0 & \frac{\bar{u}_2}{\hat{u}_2} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \Omega^{(s)} = \begin{pmatrix} 1 & 0 & \cdots & \\ \frac{\bar{u}_1}{\tilde{u}_1} & 1 & \ddots & \\ 0 & \frac{\bar{u}_2}{\tilde{u}_2} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{126}$$

Equation (125) is

$$\begin{pmatrix} 1 & 0 & \cdots \\ \frac{\bar{u}_1}{\tilde{u}_1} & 1 & \ddots \\ 0 & \frac{\bar{u}_2}{\tilde{u}_2} & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots \\ \frac{\tilde{\bar{u}}_1}{\tilde{\hat{u}}_1} & 1 & \ddots \\ 0 & \frac{\tilde{\bar{u}}_2}{\tilde{\hat{u}}_2} & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots \\ \frac{\bar{u}_1}{\hat{u}_1} & 1 & \ddots \\ 0 & \frac{\bar{u}_2}{\hat{u}_2} & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots \\ \frac{\hat{\bar{u}}_1}{\hat{\tilde{u}}_1} & 1 & \ddots \\ 0 & \frac{\hat{\bar{u}}_2}{\hat{\tilde{u}}_2} & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix}, \tag{127}$$

and consequently, we deduce

$$\frac{\bar{u}}{\tilde{u}} + \frac{\tilde{\bar{u}}}{\tilde{\hat{u}}} = \frac{\bar{u}}{\hat{u}} + \frac{\hat{\bar{u}}}{\hat{\tilde{u}}}, \tag{128}$$

and the result follows immediately. ■

Remark 7. Equation (121) is the generalized discrete lattice Toda equation described for the first time by Nijhoff and Capel,[18] that taking adequate continuous limits in the r and s variables (hypergeometric parameters) recovers the 2D Toda equation. This is a canonical equation among the difference equations in three variables involving up to second-order differences of octahedral type, consistent on the 4D lattice,[27] it appears as the type V of the octahedron-type integrable discrete equations in Section 3.9 of the book.[17]

More compatibility conditions appear in terms of the matrices, see Ref. 26

$${}_i\omega := ({}_jTS)(\Lambda + a_iI)S^{-1}, \quad i \in \{1, \dots, M\}, \quad \omega_j := (T_kS)(\Lambda + (b_j - 1)I)S^{-1}, \quad j \in \{1, \dots, N\}, \tag{129}$$

related to the previous ones by

$${}_i\omega H = a_i({}_iTH)({}_i\Omega)^\top, \quad \omega_j H = (b_j - 1)(T_jH)(\Omega_j)^\top, \tag{130}$$

as a consequence of the following connection formulas:

$${}_i\omega P(z) = (z + a_i)_i TP(z), \quad i \in \{1, \dots, M\}, \quad \omega_j P(z) = (z + b_j - 1)T_jP(z), \quad j \in \{1, \dots, N\}. \tag{131}$$

Now we study compatibility of the recursion relation and the connection formula for contiguous parameters; that is, the compatibility of

$$\begin{aligned} zP &= JP, \\ \hat{P} &= \hat{a}\frac{\omega^{(r)}}{z + \hat{a}}P. \end{aligned} \tag{132}$$

We use the notation

$$u_n = H_{n-1}, \quad v_n = \beta_{n-1} \tag{133}$$

so that $\gamma_n = \frac{\bar{u}_n}{u_n}$ and we have the following expressions:

$$J = \begin{pmatrix} v_1 & 1 & 0 & \cdots & \\ \frac{\bar{u}_1}{u_1} & v_2 & 1 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \omega^{(r)} = \begin{pmatrix} \frac{\hat{u}_1}{u_1} & 1 & 0 & \cdots \\ 0 & \frac{\hat{u}_2}{u_2} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{134}$$

Let $\hat{J}$ stand for the Jacobi matrix J shifted in the corresponding hypergeometric parameter r, that is,

$$\hat{J} = \begin{pmatrix} \frac{\hat{v}_1}{\hat{a}} & 1 & 0 & \cdots & \\ \frac{\hat{\bar{u}}_1}{\bar{u}_1} & \frac{\hat{v}_1}{\hat{a}} & 1 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{135}$$

Then, the eigenvalue property $JP = zP$ when shifted reads $\hat{J}\hat{P}(z) = z\hat{P}(z)$. Hence,

$$\hat{J}\hat{a}\frac{\omega^{(r)}}{z+\hat{a}}P(z) = z\hat{a}\frac{\omega^{(r)}}{z+\hat{a}}P(z) = \hat{a}\frac{\omega^{(r)}}{z+\hat{a}}JP(z), \tag{136}$$

and therefore, we find the following compatibility equation:

$$\hat{J}\omega^{(r)} = \omega^{(r)}J. \tag{137}$$

the compatibility reads

$$\begin{pmatrix} \frac{\hat{v}_1}{\hat{a}} & 1 & 0 & \cdots & \\ \frac{\hat{\bar{u}}_1}{\hat{u}_1} & \frac{\hat{v}_2}{\hat{a}} & 1 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \frac{\hat{u}_1}{u_1} & 1 & 0 & \cdots \\ 0 & \frac{\hat{u}_2}{u_2} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} \frac{\hat{u}_1}{u_1} & 1 & 0 & \cdots \\ 0 & \frac{\hat{u}_2}{u_2} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} v_1 & 1 & 0 & \cdots & \\ \frac{\bar{u}_1}{u_1} & v_2 & 1 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{138}$$

and we find $\frac{\hat{v}_1}{\hat{a}}\frac{\hat{u}_1}{u_1} = v_1\frac{\hat{u}_1}{u_1} + \frac{\bar{u}_1}{u_1}$ and

$$\begin{aligned} \frac{\hat{\bar{u}}_n}{\hat{u}_n} + \frac{\hat{v}_n}{\hat{a}}\frac{\hat{\bar{u}}_n}{\bar{u}_n} &= \frac{\bar{\bar{u}}_n}{\bar{u}_n} + \bar{v}_n\frac{\hat{\bar{u}}_n}{\bar{u}_n}, \\ \frac{\hat{v}_n}{\hat{a}} + \frac{\hat{\bar{u}}_n}{\bar{u}_n} &= \bar{v}_n + \frac{\hat{u}_n}{u_n} \end{aligned} \tag{139}$$

for $n \in \mathbb{N}$. Thus, we find the following.

Proposition 9. *The squared norms $u_n = H_{n-1}$ and the recursion coefficients, $v_n = \beta_{n-1} = p^1_{n-1} - p^1_n$, satisfy the following system of nonlinear difference equations:*

$$\frac{\hat{v}}{\hat{a}} - \bar{v} = \frac{\bar{\bar{u}}}{\hat{\bar{u}}} - \frac{\bar{u}}{\hat{u}}, \tag{140a}$$

$$\frac{\hat{v}}{\hat{a}} - \bar{v} = \frac{\hat{u}}{u} - \frac{\hat{\bar{u}}}{\bar{u}}. \tag{140b}$$

2.7 | The Toda flows

Given a semi-infinite vector $\boldsymbol{\eta} := \{\eta_l\}_{l=1}^{\infty} \in \mathbb{C}^{\infty}$, for each $z \in \mathbb{C}$, we define $\mathscr{E}(z; \boldsymbol{\eta}) := \prod_{l=1}^{\infty} \eta_l^{z^l}$, and we assume a weight of the form $w(z) = v(z)\mathscr{E}(z; \boldsymbol{\eta})$, with v η-independent, so that the corresponding moment matrix is

$$G = \sum_{k=0}^{\infty} \chi(k)\chi(k)^{\top} v(k)\mathscr{E}(k; \boldsymbol{\eta}). \tag{141}$$

Observe that only the first flow preserves the Pearson reduction. Notice also that $\mathscr{E} = \mathrm{e}^{\sum_{l=1}^{\infty} t_l z^l}$, with $t_l := \log \eta_l$ the standard time flows in an integrable hierarchy. We will write $\eta_1 = \eta$; normally, t_1 is denoted by x, $t_1 = x$. Also, the following notation

$$\vartheta_l := \eta_l \frac{\partial}{\partial \eta_l} = \frac{\partial}{\partial t_l} \tag{142}$$

will be used. We notice that, in particular, $\vartheta_1 = \vartheta_\eta$. To ensure the converge of the corresponding series for the moments, we require $|\eta_k| \leq 1$ for $k \in \{2, 3, \ldots\}$.

If we take $\eta_k = 1$, $k \in \mathbb{N}$, then the corresponding moment matrix is

$$G_0 = \sum_{k=0}^{\infty} \chi(k)\chi(k)^{\top} v(k). \tag{143}$$

Moreover, we can write for the deformed Gram matrix

$$G = \mathscr{E}(\Lambda; \boldsymbol{\eta})G_0 = G_0\mathscr{E}(\Lambda^{\top}; \boldsymbol{\eta}), \quad \mathscr{E}(\Lambda; \boldsymbol{\eta}) := \prod_{l=1}^{\infty} \eta_l^{\Lambda^l} = \exp\left(\sum_{l=1}^{\infty} t_l \Lambda^l\right). \tag{144}$$

Proposition 10. *For a linear functional of the form $\rho = \sum_{k=0}^{\infty} \delta(z-k)v(z)\mathscr{E}(z; \boldsymbol{\eta})$, given $l \in \mathbb{N}$, the moment matrix satisfies*

$$\vartheta_l G = (\Lambda)^l G = G(\Lambda^{\top})^l. \tag{145}$$

Proof. It follows form $\vartheta_l \mathscr{E}(z;\boldsymbol{\eta}) = z^l \mathscr{E}(z;\boldsymbol{\eta})$; indeed,

$$\vartheta_l G = \sum_{k=0}^{\infty} \chi(k)\chi(k)^\top k^l v(k)\mathscr{E}(k;\boldsymbol{\eta}) = \Lambda^l G = G(\Lambda^\top)^l. \tag{146}$$

■

For $k \in \mathbb{N}$, let us define the strictly lower triangular matrix (strictly because it has zeros on the main diagonal)

$$\Phi_k := (\vartheta_k S)S^{-1}. \tag{147}$$

In particular, we denote $\Phi := \Phi_1 = (\vartheta_\eta S)S^{-1}$.

Proposition 11. *The semi-infinite vector P fulfills*

$$\vartheta_k P = \Phi_k P. \tag{148}$$

Proof. As $P = S\chi$, then $\vartheta_k P = (\vartheta_k S)\chi = (\vartheta_k S)S^{-1}S\chi = \Phi P$. ■

Proposition 12 (Sato–Wilson equations)**.** *The following equations holds:*

$$-\Phi_k H + \vartheta_k H - H\Phi_k^\top = J^k H, \quad k \in \mathbb{N}. \tag{149}$$

Consequently, for $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we have

$$\Phi_k = -(J^k)_-, \quad \vartheta_k \log H_n = (J^k)_{n,n}. \tag{150}$$

Proof. Using the Cholesky factorization (8) for the moment matrix, we get

$$\vartheta_k G = \vartheta_k(S^{-1}HS^{-\top}) = -S^{-1}(\vartheta_k S)S^{-1}HS^{-\top} + S^{-1}(\vartheta_k H)S^{-\top} - S^{-1}HS^{-\top}(\vartheta_k S)^\top S^{-\top}. \tag{151}$$

Hence, from the symmetry relation (145), we deduce

$$-S^{-1}(\vartheta_\eta S)S^{-1}HS^{-\top} + S^{-1}(\vartheta_\eta H)S^{-\top} - S^{-1}HS^{-\top}(\vartheta_\eta S)^\top S^{-\top} = \Lambda^k S^{-1}HS^{-\top}, \tag{152}$$

from where

$$-(\vartheta_k S)S^{-1}H + \vartheta_k H - H\big((\vartheta_k S)S^{-1}\big)^\top = J^k H, \tag{153}$$

for $J = S\Lambda S^{-1}$, and the result follows. ■

For $k = 1$; that is for the first flow $\eta_1 = \eta$, we have formulas (78), (79), and (80) in terms of the τ-function. Moreover,

Proposition 13 (Toda)**.** *The following equations hold:*

$$\Phi = (\vartheta_\eta S)S^{-1} = -\Lambda^\top \gamma, \tag{154a}$$

$$(\vartheta_\eta H)H^{-1} = \beta. \tag{154b}$$

In particular, for $n, k-1 \in \mathbb{N}$, we have

$$\vartheta_\eta p^1_n = -\gamma_n, \quad \vartheta_\eta p^k_{n+k} = -\gamma_{n+k} p^{k-1}_{n+k-1}, \tag{155a}$$

$$\vartheta_\eta \log H_n = \beta_n. \tag{155b}$$

The functions $q_n := \log H_n$, $n \in \mathbb{N}$, satisfy the Toda equations

$$\vartheta^2_\eta q_n = \mathrm{e}^{q_{n+1}-q_n} - \mathrm{e}^{q_n - q_{n-1}}. \tag{156}$$

For $n \in \mathbb{N}$, we also have

$$\vartheta_\eta P_n(z) = -\gamma_n P_{n-1}(z). \tag{157}$$

Proof. Equation (154a) is a direct consequence of (149) for $k = 1$. To obtain (155), we we write $\vartheta_\eta S = -\Lambda^\top \gamma S$ in terms of its subdiagonals, that is,

$$\vartheta_\eta (I + \Lambda^\top S^{[1]} + (\Lambda^\top)^2 S^{[2]} + \cdots) = -\Lambda^\top \gamma (I + \Lambda^\top S^{[1]} + (\Lambda^\top)^2 S^{[2]} + \cdots) \tag{158}$$

which, for $k \in \mathbb{N}$, gives

$$\vartheta_\eta S^{[k]} = -(T^{k-1}_{-}\gamma) S^{[k-1]} \tag{159}$$

with $S^{[0]} = I$. Now, recalling (32) we get (155). Equation (155b) follows component wise from (154b).

As $\beta_n = p^1_n - p^1_{n+1}$ and $\gamma_n = \frac{H_n}{H_{n-1}}$, we deduce that

$$\vartheta^2_\eta \log H_n = \vartheta_\eta p^1_n - \vartheta_\eta p^1_{n+1} = -\frac{H_n}{H_{n-1}} + \frac{H_{n+1}}{H_n}, \tag{160}$$

that is, the Toda equation (156) for $H_n = \mathrm{e}^{q_n}$. The last equation follows form $\vartheta_\eta P = \Phi P = -\Lambda^\top \gamma P$. ∎

Now, let us study the compatibility with the connection relations. From $\vartheta_\eta P = \Phi P$ and $\Omega \hat{P} = P$, we get

$$\vartheta_\eta \Omega = \Phi \Omega - \Omega \hat{\Phi}, \tag{161}$$

so that

Proposition 14. *The squared norms* $u_n = H_{n-1}$ *fulfill*

$$\vartheta_\eta\left(\frac{\bar{u}}{\hat{u}}\right) = \frac{\hat{u}}{\hat{u}} - \frac{\bar{u}}{u}. \tag{162}$$

Given the τ-function (78), we find the following.

Proposition 15 (τ-function expressions)**.** *In terms of the* τ*-function*

$$\tau_n := \mathscr{W}_n\left({}_MF_N\begin{bmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{bmatrix};\eta\right), \tag{163}$$

we find

$$H_n = \frac{\tau_{n+1}}{\tau_n}, \quad p_n^1 = -\vartheta_\eta \log\tau_n, \quad \gamma_n = \vartheta_\eta^2 \log\tau_n, \quad \beta_n = \vartheta_\eta \log\frac{\tau_{n+1}}{\tau_n}. \tag{164}$$

Proof. Collect together (80) and (155). ∎

We now give the differential relations involving the operator $\vartheta_\eta\eta$ and the Jacobi matrix as a Lax equation. Then, the 1D Toda equation emerges naturally.

Proposition 16. *The following Lax equation holds:*

$$\vartheta_\eta J = [J_+, J]. \tag{165}$$

The recursion coefficients satisfy the following Toda system:

$$\vartheta_\eta\beta_n = \gamma_{n+1} - \gamma_n, \tag{166a}$$

$$\vartheta_\eta \log\gamma_n = \beta_n - \beta_{n-1}, \tag{166b}$$

for $n \in \mathbb{N}_0$ *and* $\beta_{-1} = 0$*. Consequently, we get*

$$\vartheta_\eta^2 \log\gamma_n + 2\gamma_n = \gamma_{n+1} + \gamma_{n-1}. \tag{167}$$

Moreover,

$$\vartheta_\eta^2 \log\tau_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}. \tag{168}$$

Proof. As $J = S\Lambda S^{-1}$ clears that

$$\vartheta_\eta J = [\Phi, J] = [-J_-, J] = [J_+ - J, J] = [J_+, J]. \tag{169}$$

From this Lax equation, one could derive, component-wise, the Toda system (166). Alternatively, the Toda system (166) could be derived as follows. From (17) and (155), we obtain

$$\begin{aligned}\vartheta_\eta \beta_n &= \vartheta_\eta p_n^1 - \vartheta_\eta p_{n+1}^1 = -\gamma_n + \gamma_{n+1},\\ \vartheta_\eta \log \gamma_n &= \vartheta_\eta \log H_n - \vartheta_\eta \log H_{n-1} = \beta_n - \beta_{n-1}.\end{aligned} \tag{170}$$

Finally,

$$\vartheta_\eta^2 \log \tau_n = \gamma_n = \frac{H_n}{H_{n-1}} = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n}. \tag{171}$$

■

For higher Toda flows, which generically are not consistent with the hypergeometric reduction, we have the following.

Proposition 17 (Lax equations)**.** *The following Lax equation holds:*

$$\vartheta_k J = [(J^k)_+, J]. \tag{172}$$

Proof. As $J = S\Lambda S^{-1}$, we have

$$\vartheta_k J = [\Phi_k, J] = [-(J)_-^k, J] = [(J^k)_+ - J^k, J] = [(J^k)_+, J]. \tag{173}$$

■

A very important description of these integrable flows, and alternatively to the Lax equations previously discussed, are the zero-curvature conditions first discussed by Zakharov and Shabat in the context of the nonlinear Schrödinger equation.

Proposition 18 (Wave functions and Zakharov–Shabat equations)**.** *The wave function satisfies the linear system*

$$\vartheta_k \Psi = (J^k)_+ \Psi. \tag{174}$$

Moreover, the following Zakharov–Shabat equations hold:

$$\vartheta_k (J^l)_+ - \vartheta_l (J^k)_+ + [(J^l)_+, (J^k)_+] = 0. \tag{175}$$

Proof. The orthogonal polynomials we have

$$\vartheta_k P = \Phi_k P = ((J^k)_+ - J^k)P = ((J^k)_+ - z^k)P. \tag{176}$$

Thus, the wave function fulfills the following linear system:

$$\vartheta_k \Psi = \vartheta_k(P)\mathscr{E} + P\vartheta_k(\mathscr{E}) = (J^k)_+ \Psi. \tag{177}$$

Finally, the Zakharov–Shabat equations follow as the compatibility conditions for the previous linear system. ■

Finally, we connect the Toda equation with another very important soliton equation, the Kadomtsev–Petviashvili equation, or 2D KdV equation. For that aim, we require of the following technical result.

Proposition 19. *Let us assume that G_0, the moment matrix of the linear functional $\rho = \sum_{k=0}^{\infty} \delta(z-k)v(z)$, admits a Cholesky factorization, and that two semi-infinite matrices Z_1 and Z_2 are given, such that*

(i) $Z_1 \mathscr{E}(\Lambda, \boldsymbol{\eta})^{-1}$ *is strictly lower triangular,*
(ii) Z_2 *is upper triangular, and*
(iii) $Z_1(\boldsymbol{\eta}) G_0 = Z_2(\boldsymbol{\eta})$.

Then, we can ensure that $Z_1 = Z_2 = 0$.

Proof. We have

$$Z_1 G_0 = Z_1 \mathscr{E}(\Lambda; \boldsymbol{\eta})^{-1} G = Z_1 \mathscr{E}(\Lambda; \boldsymbol{\eta})^{-1} S^{-1} H S^{-\top}, \tag{178}$$

and we get

$$Z_1 \mathscr{E}(\Lambda; \boldsymbol{\eta})^{-1} S^{-1} H S^{-\top} = Z_2. \tag{179}$$

Consequently,

$$Z_1 \mathscr{E}(\Lambda; \boldsymbol{\eta})^{-1} S^{-1} = Z_2 S H^{-1}. \tag{180}$$

But $Z_1 \mathscr{E}(\Lambda; \boldsymbol{\eta})^{-1} S^{-1}$ is strictly lower triangular, whereas $Z_2 S H^{-1}$ is upper triangular. As both terms are equal, the only possibility is that both are the zero matrix

$$Z_1 \mathscr{E}(\Lambda; \boldsymbol{\eta})^{-1} S^{-1} = Z_2 S H^{-1} = 0, \tag{181}$$

and we get the result. ■

Using this results one proves that

Proposition 20 (KP equation)**.** *The wave function Ψ_n satisfies the linear equations*

$$\vartheta_m \Psi_n = \mathscr{P}_m(\vartheta_\eta) \Psi_n \tag{182}$$

with

$$\mathscr{P}_m(\vartheta_\eta) = \vartheta_\eta^m - m p_n^1 \vartheta_\eta^{m-2} + U_{n,m-3} \vartheta_\eta^{m-3} + \cdots + U_{n,0}, \tag{183}$$

where $\{U_{n,j}\}_{j=0}^{m-3}$ *are polynomials in* $\vartheta_\eta^l p_n^1, \vartheta_2 p_n^1, \dots, \vartheta_{m-1} p_n^1$. *In particular, the following linear equations hold:*

$$
\begin{aligned}
\vartheta_2 \Psi_n &= \vartheta_\eta^2 \Psi_n - 2\vartheta_\eta\big(p_n^1\big)\Psi_n,\\
\vartheta_3 \Psi_n &= \vartheta_\eta^3 \Psi_n - 3\vartheta_\eta\big(p_n^1\big)\vartheta_\eta \Psi_n - \frac{3}{2}\big(\vartheta_\eta^2\big(p_k^1\big) + \vartheta_2\big(p_k^1\big)\big)\Psi_n,
\end{aligned}
\tag{184}
$$

and consequently, its compatibility condition, which is the KP equation,

$$
\vartheta_\eta\Big(4\vartheta_3\big(p_n^1\big) + 6\big(\vartheta_\eta\big(p_n^1\big)\big)^2 - \vartheta_\eta^3\big(p_n^1\big)\Big) = \vartheta_2^2\big(p_n^1\big),
\tag{185}
$$

is fulfilled.

Remark 8. The weight w for which this KP equation appears is

$$
w(z) = \frac{(a_1)_z \cdots (a_M)_z}{\Gamma(z+1)(b_1)_z \cdots (b_N)_z} \eta^z \eta_2^{z^2} \eta_3^{z^3}, \quad |\eta_2|, |\eta_3| < 1,
\tag{186}
$$

and the corresponding 0th moment is the series

$$
\rho_0 = \sum_{k=0}^{\infty} w(k) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_M)_k}{(b_1+1)_k \cdots (b_{N-1}+1)_k} \frac{\eta^k \eta_2^{k^2} \eta_3^{k^3}}{k!}.
\tag{187}
$$

We see that only for $\eta_2, \eta_3 \to 1$, we recover a hypergeometric function.

2.8 | Toda–Pearson compatibility

For the compatibility of (27) and (148), that is, for the compatibility of the systems,

$$
\begin{cases}
P(z+1) = \Pi P(z),\\
\vartheta_\eta(P(z)) = \Phi P(z),
\end{cases}
\tag{188}
$$

we require

$$
\begin{aligned}
&\vartheta_\eta(P(z+1)) = \vartheta_\eta(\Pi P(z)) = \vartheta_\eta(\Pi)P(z) + \Pi\vartheta_\eta(P(z)) = (\vartheta_\eta(\Pi) + \Pi J \Phi)P(z),\\
&(\vartheta_\eta P)(z+1) = \Phi P(z+1) = \Phi \Pi P(z)
\end{aligned}
\tag{189}
$$

to be equal, and consequently, we obtain

$$
\vartheta_\eta(\Pi) = [\Phi, \Pi].
\tag{190}
$$

In the general case, the dressed Pascal matrix Π is a lower unitriangular semi-infinite matrix, which possibly has an infinite number of subdiagonals. However, for the case when the weight

$w(z) = v(z)\eta^z$ satisfies the Pearson equation (66), with v independent of η, that is,

$$\theta(k+1)v(k+1)\eta^{k+1} = \sigma(k)v(k)\eta^k \tag{191}$$

for $k \in \mathbb{N}_0$; that is,

$$\theta(k+1)v(k+1)\eta = \sigma(k)v(k), \tag{192}$$

the situation improves as we have the banded semi-infinite matrix Ψ that models the shift in the z variable as in (86). From the previous discrete Pearson equation, we see that $\sigma(z) = \eta\kappa(z)$ with κ, θ η-independent polynomials in z

$$\theta(k+1)v(k+1) = \eta\kappa(k)v(k). \tag{193}$$

Proposition 21. *Let us assume a weight w satisfying the Pearson equation (66), and consequently, of the form (73). Then, the Laguerre–Freud structure matrix Ψ given in (82) satisfies*

$$\vartheta_\eta(\eta^{-1}\Psi^\top H^{-1}) = [\Phi, \eta^{-1}\Psi^\top H^{-1}], \tag{194a}$$

$$\vartheta_\eta(\Psi H^{-1}) = [\Phi, \Psi H^{-1}]. \tag{194b}$$

Alternatively, the above equations can be written as follows:

$$\vartheta_\eta(\Psi^\top H^{-1}) = [J_+, \Psi^\top H^{-1}], \tag{195a}$$

$$\vartheta_\eta(\eta^{-1}\Psi H^{-1}) = [J_+, \eta^{-1}\Psi H^{-1}]. \tag{195b}$$

Relations (194a) and (194b) are gauge *equivalent.*

Proof. We look at the compatibility of (86) and (148); that is, for (194a), we consider

$$\begin{cases} \sigma(z)P(z+1) = \Psi^\top H^{-1}P(z), \\ \vartheta_\eta(P(z)) = \Phi P(z), \end{cases} \tag{196}$$

while for (194b), we consider

$$\begin{cases} \theta(z)P(z-1) = \Psi H^{-1}P(z), \\ \vartheta_\eta(P(z)) = \Phi P(z). \end{cases} \tag{197}$$

From (196), observing that $\vartheta_\eta(\sigma) = \sigma$, we deduce

$$\vartheta_\eta(\sigma(z)P(z+1)) = \vartheta_\eta(\Psi^\top H^{-1}P(z)) = \vartheta_\eta(\Psi^\top H^{-1})P(z) + \Psi^\top H^{-1}\vartheta_\eta(P(z))$$

$$= (\vartheta_\eta(\Psi^\top H^{-1}) + \Psi^\top H^{-1}\Phi)P(z),$$

$$\vartheta_\eta(\sigma(z)P(z+1)) = \sigma(z)P(z+1) + \sigma(z)\Phi P(z+1) = (\Phi + I)\Psi^\top H^{-1}P(z), \tag{198}$$

so that

$$\vartheta_\eta(\Psi^\top H^{-1}) - \Psi^\top H^{-1} = [\Phi, \Psi^\top H^{-1}], \tag{199}$$

and recalling $\vartheta_\eta(\eta^{-1}) = -\eta^{-1}$, we obtain (194a).

From (197), we find ($\vartheta_\eta \theta = 0$)

$$\vartheta_\eta(\theta(z)P(z-1)) = \vartheta_\eta(\Psi H^{-1}P(z)) = \vartheta_\eta(\Psi H^{-1})P(z) + \Psi H^{-1}\vartheta_\eta(P(z))$$
$$= (\vartheta_\eta(\Psi H^{-1}) + \Psi H^{-1}\Phi)P(z),$$
$$\vartheta_\eta(\theta(z)P(z-1)) = \theta(z)\Phi P(z-1) = \Phi\Psi H^{-1}P(z), \tag{200}$$

and (194b) follows immediately.

Notice that from (97), this equation (194b) follows immediately:

$$\vartheta_\eta(\Psi H^{-1}) = [(\vartheta_\eta S)S^{-1}, SB^{-1}\theta(\Lambda)S^{-1}] = [\Phi, \Psi H^{-1}]. \tag{201}$$

This comment applies similarly to (194a).

Using $\Phi = -J_- = J_+ - J$ in (194a) and (96a), we easily get (195a), and similarly from (194b) and (96b), we deduce (195b).

Finally, we study the relation between (194a) and (194b). Assume that (194b), $\vartheta_\eta(\Psi H^{-1}) = [\Phi, \Psi H^{-1}]$, holds and take its transpose to obtain $\vartheta_\eta(H^{-1}\Psi^\top) = -[\Phi^\top, H^{-1}\Psi^\top]$, which is not (194a). Observe that

$$\Psi^\top H^{-1} = H(H^{-1}\Psi^\top)H^{-1} \tag{202}$$

so that the following *gauge*-type transformation equation arises:

$$\vartheta_\eta(\Psi^\top H^{-1}) = \vartheta_\eta(H)(H^{-1}\Psi^\top)H^{-1} + H\vartheta_\eta(H^{-1}\Psi^\top)H^{-1} - H(H^{-1}\Psi^\top)H^{-1}\vartheta_\eta(H)H^{-1}$$
$$= \left[\vartheta_\eta(H)H^{-1}, H(H^{-1}\Psi^\top)H^{-1}\right] - H[\Phi^\top, H^{-1}\Psi^\top]H^{-1}$$
$$= \left[\vartheta_\eta(H)H^{-1} - H\Phi^\top H^{-1}, \Psi^\top H^{-1}\right]. \tag{203}$$

Now, attending to (149), we have $\vartheta_\eta(H)H^{-1} - H\Phi^\top H^{-1} = J + \Phi$ so that

$$\vartheta_\eta(\Psi^\top H^{-1}) = [J + \Phi, \Psi^\top H^{-1}] = [\Phi, \Psi^\top H^{-1}] + \Psi^\top H^{-1}, \tag{204}$$

and (194a) follows immediately. ∎

2.9 | An example: The Meixner polynomials

For the Meixner polynomials, we have

$$w(z)=\frac{(a)_z}{\Gamma(z+1)}\eta^z,\quad \theta=z,\quad \sigma=\eta(z+a), \tag{205}$$

with zero-order moment given by

$$\rho_0={}_1F_0\left[\begin{matrix}a\\ -\end{matrix};\eta\right]=\frac{1}{(1-\eta)^a}, \tag{206}$$

and the successive moments by, see Ref. 28,

$$\rho_n=\frac{1}{(1-\eta)^a}\sum_{m=0}^{n}\left\{\begin{matrix}n\\ m\end{matrix}\right\}(a)_m\frac{\eta^m}{(1-\eta)^m}, \tag{207}$$

with $\left\{\begin{smallmatrix}n\\ m\end{smallmatrix}\right\}$ the Stirling numbers of the second kind, $\left\{\begin{smallmatrix}n\\ m\end{smallmatrix}\right\}$ is the number of partitions of $\{1,\dots,n\}$ into exactly k parts, $1\leq k\leq n$. The moment matrix in this Meixner case satisfies

$$\Lambda G=B(\Lambda G+aG)B^{-\top}, \tag{208}$$

which implies for the moments ρ_n in (207) the following relation:

$$\rho_{n+m+1}=\sum_{k,l=0}^{n}(\rho_{k+l+1}+a\rho_{k+l})(-1)^{k+l}\binom{n}{k}\binom{m}{l}. \tag{209}$$

Alternatively, for $n,m\in\mathbb{N}_0$, we can write

$$\sum_{k=0}^{m}\binom{m}{k}\rho_{n+k+1}=\sum_{k=0}^{n}\binom{n}{k}(\rho_{m+k+1}+a\rho_{m+k}). \tag{210}$$

Hence, using the explicit expressions in terms of Stirling numbers (207), we get the following identities for binomial and Stirling numbers that hold for any a,η:

$$\begin{aligned}&\sum_{k=0}^{m}\sum_{l=0}^{n+k+1}\binom{m}{k}\left\{\begin{matrix}n+k+1\\ l\end{matrix}\right\}(a)_l\frac{\eta^l}{(1-\eta)^l}-\sum_{k=0}^{n}\sum_{l=0}^{m+k+1}\binom{n}{k}\left\{\begin{matrix}m+k+1\\ l\end{matrix}\right\}(a)_l\frac{\eta^l}{(1-\eta)^l}\\ &=a\sum_{k=0}^{n}\sum_{l=0}^{n+k}\binom{n}{k}\left\{\begin{matrix}m\\ l\end{matrix}\right\}(a)_l\frac{\eta^l}{(1-\eta)^l}.\end{aligned} \tag{211}$$

According to Ref. 5, we have the recursion coefficients

$$\beta_n=\frac{n+(n+a)\eta}{1-\eta},\quad \gamma_n=\frac{n(n+a-1)\eta}{(1-\eta)^2}. \tag{212}$$

It is easily checked, we used SageMath 9.4, that these functions are, for every value of a, solutions of the Toda system (166). The monic orthogonal polynomials are expressed in terms of the Gaussian hypergeometric function as follows:

$$P_n(x) = (a)_n \frac{\eta^n}{(\eta-1)^n} M_n(x;a,\eta), \quad M_n(x;a,\eta) := {}_2F_1\left[\begin{matrix}-n, -x \\ a\end{matrix}; \frac{\eta-1}{\eta}\right]. \tag{213}$$

Moreover, as we know that (Ref. 13, Theorem 6.1.1)

$$\sum_{k=0}^{\infty} M_n^2(k;a,\eta)w(k) = \frac{n!(1-\eta)^{-a}}{\eta^n (a)_n}, \tag{214}$$

we find

$$H_n = \sum_{k=0}^{\infty} P_n^2(k)w(k) = (a)_n^2 \frac{\eta^{2n}}{(\eta-1)^{2n}} \frac{n!(1-\eta)^{-a}}{\eta^n (a)_n} = n!(a)_n \frac{\eta^n}{(1-\eta)^{2n+a}}. \tag{215}$$

Observe that, as $\gamma_n = \frac{H_n}{H_{n-1}}$ and $\beta_n = \vartheta_\eta \log H_n$, we recover the previous expressions (212). Consequently, the function

$$q_n = \log n! + \log(a)_n + n \log \eta - (2n+a)\log(1-\eta) \tag{216}$$

satisfies the Toda equation (156). Now, for the Hankel τ-functions, we have

$$\tau_n = \frac{\eta^{\frac{n(n-1)}{2}}}{(1-\eta)^{n(n+a-1)}} \prod_{k=1}^{n-1} k!(a+k)^{n-k-1}, \tag{217}$$

see Ref. 7.

The Nijhoff–Capel equation requires of two or more hypergeometric parameters. Therefore, the classical discrete polynomials, as the Charlier or Meixner orthogonal polynomials, having 0 and 1 parameters, do not give solutions to that integrable discrete equation. Cases as the generalized Meixner or generalized Hahn[3] do have such number of parameters and do describe solutions of the Nijhoff–Capel equation.

The compatibility conditions (140) and (162) are fulfilled as can be checked using SageMath 9.4. However, we will compute them explicitly to show that they hold. For that aim, we consider

$$u_n = H_{n-1} = (n-1)!(a)_{n-1} \frac{\eta^{n-1}}{(1-\eta)^{2n+a-2}}, \quad v_n = \beta_{n-1} = \frac{n-1+(n+a-1)\eta}{1-\eta}, \tag{218}$$

Equation (140a) is fullfilled as a direct computation shows

$$\frac{\hat{\bar{v}}}{a} - \bar{v} = \frac{n+(n+a+1)\eta}{1-\eta} - \frac{n+(n+a)\eta}{1-\eta} = \frac{\eta}{1-\eta},$$

$$
\begin{aligned}
\frac{\bar{\bar{u}}}{\hat{\bar{u}}}-\frac{\bar{u}}{\hat{u}} &= \frac{(n+1)!(a)_{n+1}\frac{\eta^{n+1}}{(1-\eta)^{2n+a+2}}}{an!(a+1)_n\frac{\eta^{n}}{(1-\eta)^{2n+a+1}}}-\frac{n!(a)_n\frac{\eta^{n}}{(1-\eta)^{2n+a}}}{a(n-1)!(a+1)_{n-1}\frac{\eta^{n-1}}{(1-\eta)^{2n+a-1}}}\\
&=(n+1)\frac{\eta}{1-\eta}-n\frac{\eta}{1-\eta}=\frac{\eta}{1-\eta}.
\end{aligned}\tag{219}
$$

Also (140b) follows from a direct computation

$$
\begin{aligned}
\frac{\hat{v}}{a}-\bar{v} &= \frac{n-1+(n+a)\eta}{1-\eta}-\frac{n+(n+a)\eta}{1-\eta}=\frac{1}{\eta-1},\\
\frac{\hat{u}}{u}-\frac{\hat{\bar{u}}}{\bar{u}} &= \frac{a(n-1)!(a+1)_{n-1}\frac{\eta^{n-1}}{(1-\eta)^{2n+a-1}}}{(n-1)!(a)_{n-1}\frac{\eta^{n-1}}{(1-\eta)^{2n+a-2}}}-\frac{a\,n!(a+1)_n\frac{\eta^{n}}{(1-\eta)^{2n+a+1}}}{n!(a)_n\frac{\eta^{n}}{(1-\eta)^{2n+a+1}}}\\
&=\frac{a+n-1}{1-\eta}-\frac{a+n}{1-\eta}=\frac{1}{\eta-1}.
\end{aligned}\tag{220}
$$

Now, we check (162)

$$
\begin{aligned}
\vartheta_\eta\left(\frac{\bar{u}}{\hat{u}}\right) &= \vartheta_\eta\left(n\frac{\eta}{1-\eta}\right)=\frac{n\eta}{(1-\eta)^2}\\
\frac{\hat{\bar{u}}}{\hat{u}}-\frac{\bar{u}}{u} &= \frac{n!(a+1)_n\frac{\eta^{n}}{(1-\eta)^{2n+a+1}}}{(n-1)!(a+1)_{n-1}\frac{\eta^{n-1}}{(1-\eta)^{2n+a-1}}}-\frac{n!(a)_n\frac{\eta^{n}}{(1-\eta)^{2n+a}}}{(n-1)!(a)_{n-1}\frac{\eta^{n-1}}{(1-\eta)^{2n+a-2}}}\\
&=\frac{n(a+n)\eta}{(1-\eta)^2}-\frac{n(a+n-1)\eta}{(1-\eta)^2}=\frac{n\eta}{(1-\eta)^2}.
\end{aligned}\tag{221}
$$

The Laguerre–Freud structure matrix is a band matrix, see (85), $\Psi=\Lambda^\top\psi^{(-1)}+\psi^{(0)}+\psi^{(1)}\Lambda$. As $\Psi=\eta(J+aI)H\Pi^\top$, we find

$$
\begin{aligned}
\Psi &= \eta(\Lambda^\top\gamma+\beta+aI+\Lambda)H(I+D\Lambda+\pi^{[2]}\Lambda^2+\pi^{[3]}\Lambda^3+\cdots)\\
&= \underbrace{\eta\Lambda^\top\gamma H}_{\text{first subdiagonal}}+\underbrace{\eta(\Lambda^\top\gamma HD\Lambda+(\beta+aI)H)}_{\text{main diagonal}}+\underbrace{\eta(\Lambda^\top\gamma H\pi^{[2]}\Lambda^2+(\beta+aI)HD\Lambda+\Lambda H)}_{\text{first superdiagonal}}+\cdots
\end{aligned}\tag{222}
$$

and we get $\psi^{(-1)}=\eta\gamma H$. From the alternative expression $\Psi=\Pi^{-1}HJ^\top$, we find

$$
\begin{aligned}
\Psi &= (I-\Lambda^\top D+(\Lambda^\top)^2\pi^{[-2]}-(\Lambda^\top)^3\pi^{[-3]}+\cdots)H(\Lambda^\top+\beta+\gamma\Lambda)\\
&= \underbrace{H\gamma\Lambda}_{\text{first superdiagonal}}\ \underbrace{-\Lambda^\top DH\gamma\Lambda+H\beta}_{\text{main diagonal}}+\underbrace{H\Lambda^\top-\Lambda^\top DH\beta+(\Lambda^\top)^2\pi^{[-2]}H\gamma\Lambda}_{\text{first subdiagonal}}+\cdots.
\end{aligned}\tag{223}
$$

Hence, $\psi^{(1)} = H\gamma$, and there two alternative expressions for the main diagonal

$$\psi^{(0)} = \eta H T_+ D + \eta(\beta + aI)H = -HT_+D + H\beta, \tag{224}$$

which hold given the form of the β's. We finally get the following Laguerre–Freud structure matrix:

$$\Psi = \begin{pmatrix} \beta_0 H_0 & H_1 & 0 & \cdots & \\ \eta H_1 & (\beta_1 - 1)H_1 & H_2 & \ddots & \\ 0 & \eta H_2 & (\beta_2 - 2)H_2 & H_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{225}$$

The connection formulas (86) are, in this case,

$$zP(z-1) = \Psi H^{-1}P(z), \quad (z+a)P(z+1) = \Psi^\top H^{-1}P(z). \tag{226}$$

3 | CONCLUSIONS AND OUTLOOK

The use of the Gauss–Borel factorization of the moment matrix has been throughly used by Adler and van Moerbeke in their studies of integrable systems and orthogonal polynomials, see Refs. 29–31. We have extended these ideas and applied them in different contexts, CMV orthogonal polynomials, matrix orthogonal polynomials, multiple orthogonal polynomials, and multivariate orthogonal.[32–40] For an general overview, see Ref. 41.

In this paper, we extended those ideas to the discrete world. In particular, we applied this approach to the study of the consequences of the Pearson equation on the moment matrix and Jacobi matrices. For that description, a new banded matrix is required, the Laguerre–Freud structure matrix that encodes the Laguerre–Freud relations for the recurrence coefficients. We have also found that the contiguous relations fulfilled generalized hypergeometric functions determining the moments of the weight described for the squared norms of the orthogonal polynomials a discrete Toda hierarchy known as Nijhoff–Capel equation, see Ref. 18.

Further work in this direction is the study of the role of Christoffel and Geronimus transformations for the description of the mentioned contiguous relations, and the use of the Geronimus–Christoffel transformations to characterize the shifts in the spectral independent variable of the orthogonal polynomials.[26] In Ref. 16, these ideas are applied to generalized Charlier, Meixner, and type I Hahn discrete orthogonal polynomials extending the results of Refs. 5, 8–11.

For the future, we will study the type II generalized Hahn polynomials, and extend these techniques to multiple discrete orthogonal polynomials[42] and its relations with the transformations presented in Ref. 43 and quadrilateral lattices.[44,45]

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ORIGINAL ARTICLE

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Laguerre–Freud equations for three families of hypergeometric discrete orthogonal polynomials

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Abstract
The Cholesky factorization of the moment matrix is considered for discrete orthogonal polynomials of hypergeometric type. We derive the Laguerre–Freud equations when the first moments of the weights are given by the ${}_1F_2$, ${}_2F_2$, and ${}_3F_2$ generalized hypergeometric series.

KEYWORDS
banded matrices, discrete orthogonal polynomials, generalized hypergeometric series, Jacobi matrix, Laguerre–Freud equations, Pearson equations, semiclassical discrete orthogonal polynomials, recursion relations

JEL CLASSIFICATION
42C05, 33C45, 33C47

1 | INTRODUCTION

Discrete orthogonal polynomial is an important and active area of the theory of orthogonal polynomials see Refs. [18, 48] as well as Refs. [19, 36, 37, 51] When a discrete Pearson equation is fulfilled by the weight, we are dealing with semiclassical discrete orthogonal polynomials, see Refs. [25, 27–29]. For some specific type of weights of generalized Charlier and Meixner types, the corresponding Freud–Laguerre-type equations for the coefficients of the three-term recurrence have been studied, see, for Refs. [22, 31–33, 50].

This paper is the final one in our series of works on the Gauss–Borel factorization of the corresponding moment matrix, see Ref. [43] in standard discrete orthogonality. In Ref. [46], the Cholesky factorization of the moment matrix was used to study discrete orthogonal polynomials on the homogeneous lattice. For weights subject to a discrete Pearson equation, so that the moments are logarithmic derivatives of generalized hypergeometric series, a banded semi-infinite matrix Ψ, the so-called Laguerre–Freud structure matrix, which models the shifts by ± 1 in the independent variable of the sequence of orthogonal polynomials, was given. Contiguous relations for the generalized hypergeometric series translate into symmetries for the corresponding moment matrix, and the 3D Nijhoff–Capel discrete Toda lattice Refs. [35, 47] describes the corresponding contiguous shifts for the squared norms of the orthogonal polynomials. In Ref. [44], we gave an interpretation for the contiguous transformations for the generalized hypergeometric functions in terms of simple Christoffel and Geronimus transformations. Using the Geronimus–Uvarov perturbations, we got determinantal expressions for the shifted orthogonal polynomials. Then, in Ref. [30], we discussed the generalized Charlier, Meixner, and type I Hahn discrete orthogonal polynomials (according to the nomenclature in Ref. [28]), analyzed the Laguerre–Freud structure matrix Ψ, and derive from its banded structure and its compatibility with the Toda equation and the Jacobi matrix, a number of nonlinear equations for the coefficients $\{\beta_n, \gamma_n\}$ of the three-term recurrence relations $zP_n(z) = P_{n+1}(z) + \beta_n P_n(z) + \gamma_n P_{n-1}(z)$ satisfied by the orthogonal polynomial sequence. These Laguerre–Freud equations are of the form $\gamma_{n+1} = \mathscr{G}(n, \gamma_n, \gamma_{n-1}, \dots, \beta_n, \beta_{n-1} \dots)$ and $\beta_{n+1} = \mathscr{B}(n, \gamma_{n+1}, \gamma_n, \dots, \beta_n, \beta_{n-1}, \dots)$, for some functions $\mathscr{B}, \mathscr{G}$. Magnus[39–42] named these types of relations, attending to Refs. [34, 38] as Laguerre–Freud relations. There are several papers discussing Laguerre–Freud relations for the generalized Charlier, generalized Meixner, and type I generalized Hahn cases, see Refs. [22, 25, 31–33, 50].

Related to the aims of this contribution, in Ref. [26], the author finds a power series expansion for the Hankel determinants whose entries are the moments of a linear functional related to discrete semiclassical orthogonal polynomials. Explicit formulas to compute the coefficients of arbitrary order as well as some examples for the first few terms in the series are given. On the other hand, polynomial sequences corresponding to hypergeometric weights have been analyzed in Ref. [24].

As mentioned above, in Ref. [30], we successfully studied the generalized Charlier, Meixner, and type Hahn orthogonal polynomials corresponding to the generalized hypergeometric series ${}_0F_1$, ${}_1F_1$, and ${}_2F_1$, respectively. It is not clear if the Laguerre–Freud relations can be found in a general setting in which the first moment is a generalized hypergeometric series. To get a better understanding, we decided to explore some examples and see if the construction can be applied. Consequently, we continue in this paper with our studies on these topics, and extend these methods to the next three families of discrete orthogonal polynomials with first moments given by the generalized hypergeometric series ${}_1F_2$, ${}_2F_2$, and ${}_3F_2$. For the three families, we explicitly compute the Laguerre-Freud structure matrix that happens to be a pentadiagonal-, hexadiagonal-, or heptadiagonal-banded matrix, respectively. We also compute the Laguerre–Freud equations and give explicit expressions for the leading nontrivial coefficients of the orthogonal polynomials in terms of the recursion coefficients. Therefore, our impression is that the method can be extended to a generalized hypergeometric series.

The layout of the paper is as follows. We first complete this introduction by discussing, without proof but referring to the appropriate sources, some preliminary material. Then, we have three sections devoted each of them to the corresponding generalized hypergeometric series, ${}_1F_2$, ${}_2F_2$, and ${}_3F_2$, finding in each case the Laguerre–Freud structure matrix and corresponding Laguerre–Freud equations.

1.1 Pearson equations and discrete orthogonal polynomials

Let us recall some relevant facts of the theory of orthogonal polynomials. Given a linear functional $\rho_z \in (\mathbb{C}[z])^*$, the corresponding moment matrix is

$$G = (G_{n,m}), \qquad G_{n,m} = \rho_{n+m}, \qquad \rho_n = \left\langle \rho_z, z^n \right\rangle, \qquad n,m \in \mathbb{N}_0 := \{0,1,2,\dots\},$$

with ρ_n being the nth moment of the linear functional ρ_z. Let us assume that the moment matrix is such that all its leading principal submatrices, which are Hankel matrices, $G_{i+1,j} = G_{i,j+1}$,

$$G^{[k]} = \begin{bmatrix} G_{0,0} & \cdots & G_{0,k-1} \\ \vdots & & \vdots \\ G_{k-1,0} & \cdots & G_{k-1,k-1} \end{bmatrix} = \begin{bmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & \rho_2 & & \iddots & \rho_k \\ \rho_2 & & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & & \vdots \\ \rho_{k-1} & \rho_k & \cdots & \cdots & \rho_{2k-2} \end{bmatrix}$$

are nonsingular; that is, we have for the Hankel determinants that $\Delta_k := \det G^{[k]} \neq 0$, $k \in \mathbb{N}_0$. A particularly relevant quasi-definite case is the positive-definite case in which all Hankel determinants are positive. The corresponding linear functional ρ is said to be quasi-definite or positive-definite, respectively, cf. Ref. [21]. From hereon, we will assume that ρ is quasi-definite. In the positive-definite scenario, the linear functional happens to be a positive measure and the corresponding moment matrix G has $\Delta_n > 0$, $n \in \mathbb{N}$.

Then, for the quasi-definite situation, there are monic polynomials

$$P_n(z) = z^n + p_n^1 z^{n-1} + \cdots + p_n^n, \qquad n \in \mathbb{N}_0, \tag{1}$$

satisfying the following orthogonality relations $\langle \rho, P_n(z) z^k \rangle = 0$, for $k \in \{0,\dots,n-1\}$ and $\langle \rho, P_n(z) z^n \rangle = H_n \neq 0$. Thus, $\{P_n(z)\}_{n\in\mathbb{N}_0}$ is a sequence of orthogonal polynomials, that is, $\langle \rho, P_n(z) P_m(z) \rangle = \delta_{n,m} H_n$ for $n,m \in \mathbb{N}_0$. The symmetric bilinear form $\langle F, G \rangle_\rho := \langle \rho, FG \rangle$, is such that the moment matrix is the Gram matrix of this bilinear form and $\langle P_n, P_m \rangle_\rho := \delta_{n,m} H_n$.

Introducing the monomial sequence $\chi(z) := \begin{bmatrix} 1 & z & z^2 & \cdots \end{bmatrix}^\top$, the moment matrix is $G = \langle \rho, \chi\chi^\top \rangle$, and χ is an eigenvector of the *shift matrix*, $\Lambda\chi = x\chi$, where

$$\Lambda := \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Hence $\Lambda G = G\Lambda^\top$, this equation is equivalent to the fact that the moment matrix is a Hankel matrix. As the moment matrix is symmetric its Borel–Gauss factorization is a Cholesky factorization, that is,

$$G = S^{-1} H S^{-\top}, \tag{2}$$

where S is a lower unitriangular matrix that can be written as

$$S = \begin{bmatrix} 1 & 0 & \cdots \\ S_{1,0} & 1 & \ddots \\ S_{2,0} & S_{2,1} & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix},$$

and $H = \operatorname{diag}(H_0, H_1, \dots)$ is a diagonal matrix, with $H_k \neq 0$, for $k \in \mathbb{N}_0$. The Cholesky factorization of these moment matrices exists if and only if ρ is quasi-definite.

The entries $P_n(z)$ of the polynomial sequence

$$P(z) := S\chi(z), \tag{3}$$

are the monic orthogonal polynomials of the functional ρ.

We have the determinantal expressions $H_k = \frac{\Delta_{k+1}}{\Delta_k}$ and $p^1_k = -\frac{\tilde{\Delta}_k}{\Delta_k}$, with

$$\Delta_k := \begin{vmatrix} \rho_0 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \vdots & & & \vdots \\ \rho_{k-2} & & & \rho_{2k-3} \\ \rho_{k-1} & \cdots & \rho_{2k-3} & \rho_{2k-2} \end{vmatrix}, \qquad \tilde{\Delta}_k := \begin{vmatrix} \rho_0 & \cdots & \rho_{k-2} & \rho_k \\ \vdots & & \rho_{k-1} & \vdots \\ \rho_{k-2} & & & \rho_{2k-2} \\ \rho_{k-1} & \cdots & \rho_{2k-3} & \rho_{2k-1} \end{vmatrix}.$$

The lower Hessenberg semi-infinite matrix

$$J = S\Lambda S^{-1} \tag{4}$$

has the polynomial sequence $P(z)$ as eigenvector with eigenvalue z. The Hankel condition $\Lambda G = G\Lambda^\top$ and the Cholesky factorization give

$$JH = (JH)^\top = HJ^\top. \tag{5}$$

For $T_+\gamma := \operatorname{diag}(0, \gamma_1, \gamma_2, \dots)$, we find $H\Lambda^\top = (T_+\gamma)\Lambda^\top H$ and its transpose, that is, $H_{n+1} = \gamma_{n+1} H_n$. As the Hessenberg matrix JH is symmetric, the Jacobi matrix J is tridiagonal, that is,

$$J = \begin{bmatrix} \beta_0 & 1 & 0 & \cdots \\ \gamma_1 & \beta_1 & 1 & \ddots \\ 0 & \gamma_2 & \beta_2 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

and we find the three-term recurrence relation $zP_n(z) = P_{n+1}(z) + \beta_n P_n(z) + \gamma_n P_{n-1}(z)$, which with the initial conditions $P_{-1} = 0$ and $P_0 = 1$ completely determines the sequence of orthogonal polynomials $\{P_n(z)\}_{n\in\mathbb{N}_0}$ in terms of the recursion coefficients β_n, γ_n. The recursion coefficients are given by

$$\beta_n = p_n^1 - p_{n+1}^1 = -\frac{\tilde{\Delta}_n}{\Delta_n} + \frac{\tilde{\Delta}_{n+1}}{\Delta_{n+1}}, \qquad \gamma_{n+1} = \frac{H_{n+1}}{H_n} = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}, \qquad n \in \mathbb{N}_0. \tag{6}$$

We introduce the following diagonal matrices $\gamma := \operatorname{diag}(\gamma_1, \gamma_2, \dots)$ and $\beta := \operatorname{diag}(\beta_0, \beta_1, \dots)$. In general, given any semi-infinite matrix A, we will write $A = A_- + A_+$, where A_- is a strictly lower triangular matrix and A_+ an upper triangular matrix. Moreover, A_0 will denote the diagonal part of A.

The lower Pascal matrix is

$$B = (B_{n,m}), \qquad B_{n,m} := \begin{cases} \binom{n}{m}, & n \geq m, \\ 0, & n < m, \end{cases} \tag{7}$$

so that

$$\chi(z+1) = B\chi(z). \tag{8}$$

The inverse of this Pascal matrix is

$$B^{-1} = [\tilde{B}_{n,m}], \qquad \tilde{B}_{n,m} := \begin{cases} (-1)^{n+m}\binom{n}{m}, & n \geq m, \\ 0, & n < m, \end{cases}$$

and $\chi(z-1) = B^{-1}\chi(z)$. The *dressed Pascal matrices* are $\Pi := SBS^{-1}$ and $\Pi^{-1} := SB^{-1}S^{-1}$ and give the connection between shifted and nonshifted polynomials, that is,

$$P(z+1) = \Pi P(z), \qquad P(z-1) = \Pi^{-1}P(z). \tag{9}$$

The lower Pascal matrix can be expressed as follows:

$$B^{\pm 1} = I \pm \Lambda^\top D + \left(\Lambda^\top\right)^2 D^{[2]} \pm \left(\Lambda^\top\right)^3 D^{[3]} + \cdots,$$

where $D = \operatorname{diag}(1,2,3,\dots)$ and $D^{[k]} := \frac{1}{k}\operatorname{diag}(k^{(k)}, (k+1)^{(k)}, (k+2)^{(k)} \cdots)$, in terms of the falling factorials $x^{(k)} := x(x-1)(x-2)\cdots(x-k+1)$. That is, in terms of the Pochhammer symbols $(z)_n := z(z+1)\cdots(z+n-1)$, for $z \in \mathbb{C}$ and $n \in \mathbb{N}$, and $(z)_0 = 1$, we have

$$D^{[k]}_n = \frac{(n+1)_k}{k}, \qquad k \in \mathbb{N}, \qquad n \in \mathbb{N}_0.$$

The lower unitriangular factor can be written as

$$S = I + \Lambda^\top S^{[1]} + \left(\Lambda^\top\right)^2 S^{[2]} + \cdots, \qquad S^{[k]} = \operatorname{diag}\left(S^{[k]}_0, S^{[k]}_1, \dots\right).$$

For any diagonal matrix $A = \operatorname{diag}(A_0, A_1, \dots)$, we will use the *shift operators* $T_\pm$ acting as follows:

$$T_- \operatorname{diag}(A_0, A_1, \dots) := \operatorname{diag}(A_1, A_2, \dots), \qquad T_+ \operatorname{diag}(A_0, A_1, \dots) := \operatorname{diag}(0, A_0, A_1, \dots),$$

so that

$$\Lambda A = (T_- A)\Lambda, \qquad A\Lambda = \Lambda(T_+ A), \qquad A\Lambda^\top = \Lambda^\top(T_- A), \qquad \Lambda^\top A = (T_+ A)\Lambda^\top. \tag{10}$$

In terms of these shift operators, we find

$$2D^{[2]} = (T_- D)D, \qquad 3D^{[3]} = (T_-^2 D)(T_- D)D = 2(T_- D^{[2]})D = 2D^{[2]}(T_-^2 D).$$

For the inverse matrix

$$S^{-1} = I + \Lambda^\top S^{[-1]} + \left(\Lambda^\top\right)^2 S^{[-2]} + \cdots,$$

we have

$$\begin{aligned}
S^{[-1]} &= -S^{[1]},\\
S^{[-2]} &= -S^{[2]} + (T_- S^{[1]})S^{[1]},\\
S^{[-3]} &= -S^{[3]} + (T_- S^{[2]})S^{[1]} + (T_-^2 S^{[1]})S^{[2]} - (T_-^2 S^{[1]})(T_- S^{[1]})S^{[1]}.
\end{aligned}$$

Corresponding expansions for the dressed Pascal matrices are

$$\Pi^{\pm 1} = I + \Lambda^\top \pi^{[\pm 1]} + (\Lambda^\top)^2 \pi^{[\pm 2]} + \cdots$$

with $\pi^{[\pm n]} = \operatorname{diag}(\pi^{[\pm n]}_0, \pi^{[\pm n]}_1, \dots)$. We have

$$\begin{aligned}
\pi^{[\pm 1]}_n = \pm(n+1), \pi^{[\pm 2]}_n &= \frac{(n+2)(n+1)}{2} \pm p^1_{n+2}(n+1) \mp (n+2)p^1_{n+1}\\
&= \frac{(n+2)(n+1)}{2} \mp (n+1)\beta_{n+1} \mp p^1_{n+1},
\end{aligned} \tag{11}$$

$$\pi_n^{[\pm 3]} = \pm\frac{(n+3)(n+2)(n+1)}{3} + \frac{(n+2)(n+1)}{2}p^1_{n+3} - \frac{(n+3)(n+2)}{2}p^1_{n+1}$$
$$\pm (n+1)p^2_{n+3} \mp (n+3)p^2_{n+2} \pm (n+3)p^1_{n+2}p^1_{n+1} \mp (n+2)p^1_{n+3}p^1_{n+1}. \tag{12}$$

Moreover, the following relations are fulfilled

$$\pi^{[1]} + \pi^{[-1]} = 0, \quad \pi^{[2]} + \pi^{[-2]} = 2D^{[2]}, \quad \pi^{[3]} + \pi^{[-3]} = 2((T_-^2 S^{[1]})D^{[2]} - (T_- D^{[2]})S^{[1]}). \tag{13}$$

We now recall some facts regarding discrete orthogonality. Hence, we focus on measures with support on the uniform lattice $\mathbb{N}_0$, that is, $\rho = \sum_{k=0}^{\infty}\delta(z-k)w(k)$, with moments given by

$$\rho_n = \sum_{k=0}^{\infty} k^n w(k), \tag{14}$$

and, in particular, with 0th moment given by

$$\rho_0 = \sum_{k=0}^{\infty} w(k). \tag{15}$$

The weights we consider in this paper satisfy the following *discrete Pearson equation*:

$$\nabla(\sigma w) = \tau w, \tag{16}$$

that is, $\sigma(k)w(k) - \sigma(k-1)w(k-1) = \tau(k)w(k)$, for $k \in \mathbb{N}$, with $\sigma(z), \tau(z) \in \mathbb{R}[z]$. If we write $\theta := \tau - \sigma$, the previous Pearson equation reads

$$\theta(k+1)w(k+1) = \sigma(k)w(k), \qquad\qquad k \in \mathbb{N}_0. \tag{17}$$

We assume that $\theta(0) = 0$. If $N+1 := \deg\theta(z)$ and $M := \deg\sigma(z)$, and the zeros of these polynomials are $\{-b_i+1\}_{i=1}^N$ and $\{-a_i\}_{i=1}^M$, we write $\theta(z) = z(z+b_1)\cdots(z+b_N)$ and $\sigma(z) = \eta(z+a_1)\cdots(z+a_M)$.

The weight can be expressed in terms of Pochhammer symbols as follows:

$$w(k) = \frac{(a_1)_k\cdots(a_M)_k}{(b_1+1)_k\cdots(b_N+1)_k}\frac{\eta^k}{k!},$$

which are positive whenever

$$a_1, \dots, a_M, b_1+1, \dots b_N+1, \eta \geq 0.$$

To ensure positivity of the corresponding moment matrix is enough to request $a_1, \dots, a_M, b_1+1, \dots b_N+1, \eta > 0$, and in that case, the orthogonal polynomials do exist. According to (15), the

0th moment

$$\rho_0 = \sum_{k=0}^{\infty} w(k) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_M)_k}{(b_1+1)_k \cdots (b_N+1)_k} \frac{\eta^k}{k!}$$
$$= {}_MF_N(a_1, \dots, a_M; b_1+1, \dots, b_N+1; \eta) = {}_MF_N\left[\begin{matrix} a_1 & \cdots & a_M \\ b_1+1 & \cdots & b_N+1 \end{matrix}; \eta\right],$$

is the generalized hypergeometric series, where we are using the two standard notations, see Refs. [17, 49]. Then, according to (14), for $n \in \mathbb{N}$, the corresponding higher moments $\rho_n = \sum_{k=0}^{\infty} k^n w(k)$ are

$$\rho_n = \vartheta_\eta^n \rho_0 = \vartheta_\eta^n \left({}_MF_N\left[\begin{matrix} a_1 & \cdots & a_M \\ b_1+1 & \cdots & b_N+1 \end{matrix}; \eta\right]\right), \qquad \vartheta_\eta := \eta \frac{\partial}{\partial \eta}.$$

Given a function $f = f(\eta)$ depending on the independent variable η, we consider its kth Wronskian

$$\mathscr{W}_k(f) := \begin{vmatrix} f & \vartheta_\eta f & \vartheta_\eta^2 f & \cdots & \vartheta_\eta^k f \\ \vartheta_\eta f & \vartheta_\eta^2 f & & \iddots & \vartheta_\eta^{k+1} f \\ \vartheta_\eta^2 f & & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & & \vdots \\ \vartheta_z^k f & \vartheta_\eta^{k+1} f & \cdots & \cdots & \vartheta_\eta^{2k} f \end{vmatrix}.$$

Let us introduce the following tau-functions

$$\tau_k := \mathscr{W}_k \left({}_MF_N\left[\begin{matrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{matrix}; \eta\right]\right),$$

that is, Wronskians of generalized hypergeometric series. Then, we have that

$$\Delta_k = \tau_k, \qquad \tilde{\Delta}_k = \vartheta_\eta \tau_k, \qquad H_k = \frac{\tau_{k+1}}{\tau_k}, \qquad p_k^1 = -\vartheta_\eta \log \tau_k.$$

In this discrete context, the moment matrix reads

$$G = \sum_{n=0}^{\infty} \chi(k) \chi^\top(k) w(k).$$

In Ref. [46] it was shown that:

Theorem 1 (Hypergeometric symmetries). *Let the weight w be subject to a discrete Pearson equation of the type (17), where the functions θ, σ are polynomials, with $\theta(0) = 0$. Then,* *i) The moment matrix fulfills*

$$\theta(\Lambda)G = B\sigma(\Lambda)GB^{\top}. \tag{18}$$

ii) The Jacobi matrix satisfies

$$\Pi^{-1}H\theta(J^{\top}) = \sigma(J)H\Pi^{\top}, \tag{19}$$

and the matrices $H\theta(J^{\top})$ and $\sigma(J)H$ are symmetric.

Theorem 2 (Laguerre–Freud structure matrix). *Let us assume that the weight w solves the discrete Pearson equation (17) with θ, σ polynomials such that $\theta(0) = 0$, $\deg\theta(z) = N+1$, $\deg\sigma(z) = M$. Then,*

i) The Laguerre–Freud structure matrix

$$\Psi := \Pi^{-1}H\theta(J^{\top}) = \sigma(J)H\Pi^{\top} = \Pi^{-1}\theta(J)H = H\sigma(J^{\top})\Pi^{\top} \tag{20}$$

$$= \theta(J+I)\Pi^{-1}H = H\Pi^{\top}\sigma(J^{\top}-I), \tag{21}$$

has only $N+M+2$ possibly nonzero diagonals ($N+1$ superdiagonals and M subdiagonals)

$$\Psi = (\Lambda^{\top})^M\psi^{(-M)} + \cdots + \Lambda^{\top}\psi^{(-1)} + \psi^{(0)} + \psi^{(1)}\Lambda + \cdots + \psi^{(N+1)}\Lambda^{N+1},$$

for some diagonal matrices $\psi^{(k)}$. In particular, the lowest subdiagonal and highest superdiagonal are given by

$$\begin{cases} (\Lambda^{\top})^M\psi^{(-M)} = \eta(J_-)^M H, \quad \psi^{(-M)} = \eta H\prod_{k=0}^{M-1} T_-^k\gamma = \eta\operatorname{diag}\left(H_0\prod_{k=1}^{M}\gamma_k, H_1\prod_{k=2}^{M+1}\gamma_k, \dots\right), \\ \psi^{(N+1)}\Lambda^{N+1} = H(J_-^{\top})^{N+1}, \quad \psi^{(N+1)} = H\prod_{k=0}^{N} T_-^k\gamma = \operatorname{diag}\left(H_0\prod_{k=1}^{N+1}\gamma_k, H_1\prod_{k=2}^{N+2}\gamma_k, \dots\right). \end{cases} \tag{22}$$

ii) The vector $P(z)$ of orthogonal polynomials fulfills the following structure equations:

$$\theta(z)P(z-1) = \Psi H^{-1}P(z), \qquad \sigma(z)P(z+1) = \Psi^{\top}H^{-1}P(z). \tag{23}$$

iii) The following compatibility conditions for the Laguerre–Freud and Jacobi matrices hold:

$$[\Psi H^{-1}, J] = \Psi H^{-1}, \tag{24a}$$

$$[J, \Psi^{\top}H^{-1}] = \Psi^{\top}H^{-1}. \tag{24b}$$

We now recall some facts regarding the Toda flows appearing in this discrete orthogonal polynomial setting. Let us define the strictly lower triangular matrix $\Phi := (\vartheta_\eta S)S^{-1}$. Then, see Ref. [46], we have that the orthogonal polynomial sequence P fulfills $\vartheta_\eta P = \Phi P$, and the Sato–Wilson equations $-\Phi H + \vartheta_\eta H - H\Phi^\top = JH$ are satisfied. Consequently, $\Phi = -J_-$ and, for $n \in \mathbb{N}_0$, we have $\vartheta_\eta \log H_n = J_{n,n}$. Moreover, $\Phi = (\vartheta_\eta S)S^{-1} = -\Lambda^\top\gamma$ and $(\vartheta_\eta H)H^{-1} = \beta$ are satisfied. The functions $q_n := \log H_n$, $n \in \mathbb{N}$, satisfy the Toda equations

$$\vartheta_\eta^2 q_n = \mathrm{e}^{q_{n+1}-q_n} - \mathrm{e}^{q_n - q_{n-1}}. \tag{25}$$

For $n \in \mathbb{N}$, we also have $\vartheta_\eta P_n(z) = -\gamma_n P_{n-1}(z)$. Additionally, the Lax equation $\vartheta_\eta J = [J_+, J]$ is fulfilled, and recursion coefficients satisfy the following Toda system, $\vartheta_\eta \beta_n = \gamma_{n+1} - \gamma_n$, and $\vartheta_\eta \log \gamma_n = \beta_n - \beta_{n-1}$, for $n \in \mathbb{N}_0$ and $\beta_{-1} = 0$. Thus, we get $\vartheta_\eta^2 \log \gamma_n + 2\gamma_n = \gamma_{n+1} + \gamma_{n-1}$.

From the compatibility of $P(z+1) = \Pi P(z)$ and $\vartheta_\eta(P(z)) = \Phi P(z)$, we get $\vartheta_\eta \Pi = [\Phi, \Pi]$. In the general case, the dressed Pascal matrix Π is a lower unitriangular semi-infinite matrix, which possibly has an infinite number of subdiagonals. However, when the weight $w(z) = v(z)\eta^z$ satisfies the Pearson equation (17), with v independent of η, that is, $\theta(k+1)v(k+1)\eta = \sigma(k)v(k)$, the situation improves as we have the banded semi-infinite matrix Ψ that models the shift in the z variable as in (23). From the previous discrete Pearson equation, we see that $\sigma(z) = \eta\kappa(z)$ with κ, θ η-independent polynomials in z

$$\theta(k+1)v(k+1) = \eta\kappa(k)v(k). \tag{26}$$

Proposition 1 (Ref. [46]). *Let us assume a weight w satisfying the Pearson equation (17). Then, the Laguerre–Freud structure matrix Ψ given in (20) satisfies*

$$\vartheta_\eta(\eta^{-1}\Psi^\top H^{-1}) = [\Phi, \eta^{-1}\Psi^\top H^{-1}], \tag{27a}$$

$$\vartheta_\eta(\Psi H^{-1}) = [\Phi, \Psi H^{-1}]. \tag{27b}$$

Relations (27a) and (27b) are gauge *equivalent.*

2 | THE ${}_1F_2$ HYPERGEOMETRIC FAMILY

We choose $\sigma(z) = \eta(z + a_1)$ and $\theta(z) = z(z + b_1)(z + b_2)$ with corresponding weight given by

$$w(z) = \frac{(a_1)_z}{(b_1+1)_z(b_2+1)_z}\frac{\eta^z}{z!},$$

with first moment given by the generalized hypergeometric series: $\rho_0 = {}_1F_2\left[\begin{smallmatrix} a_1 \\ b_1+1, b_2+1 \end{smallmatrix}; \eta\right]$, and subsequent moments $\rho_n = \vartheta_\eta^n {}_1F_2\left[\begin{smallmatrix} a_1 \\ b_1+1, b_2+1 \end{smallmatrix}; \eta\right]$. The corresponding moments exist for any $\eta \in \mathbb{C}$.

Theorem 3 (The generalized Laguerre–Freud structure matrix). *For $\sigma(z) = \eta(z + a_1)$ and $\theta(z) = z(z + b_1)(z + b_2)$, we find for the subleading coefficients the following expression:*

$$p_n^1 = \frac{n(n+1)}{2} - n\beta_n - \frac{1}{\eta + \gamma_{n+1}}(\gamma_{n+1}(\gamma_n + \gamma_{n+1} + \gamma_{n+2} + (\beta_{n+1} + b_1)(\beta_{n+1} + b_2)$$

$$+ \beta_n(\beta_{n+1} + \beta_n + b_1 + b_2) - n(2\beta_n + \beta_{n+1} + b_1 + b_2 - n) - \eta) - \eta(n+1)(\beta_n + a_1))$$

as well as

$$\pi_{n-1}^{[2]} = \frac{1}{\eta + \gamma_{n+1}}(\gamma_{n+1}(\gamma_n + \gamma_{n+1} + \gamma_{n+2} + (\beta_{n+1} + b_1)(\beta_{n+1} + b_2) + \beta_n(\beta_{n+1} + \beta_n + b_1 + b_2)$$

$$-n(2\beta_n + \beta_{n+1} + b_1 + b_2 - 2n) - \eta) - \eta(n+1)(a_1 + a_2)).$$

The Laguerre–Freud structure matrix is the following pentadiagonal matrix:

$$\Psi = \begin{bmatrix} \psi_0^{(0)} & \psi_0^{(1)} & \psi_0^{(2)} & H_3 & 0 & \cdots \\ \eta H_1 & \psi_1^{(0)} & \psi_1^{(1)} & \psi_1^{(2)} & H_4 & \ddots \\ 0 & \eta H_2 & \psi_2^{(0)} & \psi_2^{(1)} & \psi_2^{(2)} & H_5 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

with

$$\psi_n^{(0)} = \eta H_n(n + \beta_n + a_1),$$

$$\psi_n^{(1)} = \eta\Big(H_{n+1} + H_n\Big(\pi_{n-1}^{[2]} + (n+1)(\beta_n + a_1)\Big)\Big),$$

$$\psi_n^{(2)} = H_{n+2}(\beta_n + \beta_{n+1} + \beta_{n+2} + b_1 + b_2 - n).$$

Proof. As $\deg \sigma = 1$ and $\deg \theta = 3$ for the Lagurre–Freud matrix, we have

$$\Psi = \Lambda^\top \psi^{(-1)} + \psi^{(0)} + \psi^{(1)}\Lambda + \psi^{(2)}\Lambda^2 + \psi^{(3)}\Lambda^3.$$

From $\Psi = \sigma(J)H\Pi^\top$, we get the first subdiagonal, the main diagonal, and an expression for the first superdiagonal depending upon $\pi^{[2]}$:

$$\psi^{(-1)} = \eta T_- H,$$

$$\psi^{(0)} = \eta H(T_+ D + \beta + a_1),$$

$$\psi^{(1)} = \eta\big(T_- H + H\big(T_+ \pi^{[2]} + D(\beta + a_1)\big)\big).$$

From $\Psi = \Pi^{-1}H\theta(J^\top)$, we get an expression for the first superdiagonal depending on de $T_+^2\pi^{[-2]}$ and the other two nonzero superdiagonals

$$\begin{aligned}\psi^{(1)} &= T_-H[T_+\gamma + \gamma + T_-\gamma + (T_-\beta + b_1)(T_-\beta + b_2) + \beta(T_-\beta + \beta + b_1 + b_2) - T_+D(T_+\beta + \beta \\ &\quad + b_1 + b_2 + T_-\beta) + T_+^2\pi^{[-2]}],\\ \psi^{(2)} &= T_-^2H[\beta + T_-\beta + T_-^2\beta + b_1 + b_2 - T_+D],\\ \psi^{(3)} &= T_-^3H.\end{aligned}$$

To obtain the diagonal matrix $\pi^{[\pm 2]}$, we combine $T_+^2\pi^{[-2]} - T_+\pi^{[-2]} = T_+D(T_+\beta - \beta - 1)$ with $\pi^{[-2]} = 2D^{[2]} - \pi^{[2]}$ to get

$$T_+^2\pi^{[-2]} = T_+D(T_+\beta - \beta - 1) + 2T_+D^{[2]} - T_+\pi^{[2]}.$$

Now, equating the two formulas for $\psi^{(1)}$ and using the previous results, we find

$$\begin{aligned}T_+\pi^{[2]}(\eta H + T_-H) &= T_-H[T_+\gamma + \gamma + T_-\gamma + (T_-\beta + b_1)(T_-\beta + b_2) + \beta(T_-\beta + \beta + b_1 + b_2)\\ &\quad - T_+D(\beta + T_-\beta + b_1 + b_2) - T_+D(\beta + 1) + 2T_+D^{[2]} - \eta I] - \eta HD(\beta + a_1).\end{aligned}$$

Consequently, we derive the given expression for p_n^1. □

We now explore the consequences of the compatibility relation $[\Psi H^{-1}, J] = \Psi H^{-1}$. We get trivial relations and $T_+\pi^{[2]} - T_+^2\pi^{[2]} = T_+D(T_+\beta - \beta + I)$. However, a new formula appears from the first superdiagonal:

$$\begin{aligned}\eta(\beta + T_-\beta - 1) &= T_-\gamma(T_+D - 2T_-\beta - T_-^2\beta + b_1 + b_2 + 1) + \gamma(\beta + 1 - T_-\beta)\\ &\quad + T_+\gamma(T_+\beta + 2\beta + b_1 + b_2 - T_+^2D + 1) + \beta^3 - (T_-\beta)^3 + (T_-\beta)^2 + \beta^2 + \beta T_-\beta\\ &\quad + \$b_1 + b_2)(\beta^2 - (T_-\beta)^2 + T_-\beta + \beta) - T_+D(2\beta^2 - \beta T_-\beta - (T_-\beta)^2 + 3\beta)\\ &\quad + (\beta + 1 - T_-\beta)[2T_+D^{[2]} - T_+D(b_1 + b_2) + b_1b_2 - T_+\pi^{[2]}].\end{aligned}$$

No new relations are obtained from the second compatibility (27a). Using this compatibility, we seek for Laguerre–Freud equations

$$\beta_{n+1} = \mathscr{B}_n(\gamma_{n+1}, \gamma_n, \dots, \beta_n, \beta_{n-1}, \dots), \qquad \gamma_{n+2} = \mathscr{G}_n(\gamma_{n+1}, \gamma_n, \dots, \beta_{n+1}, \beta_n, \dots). \tag{28}$$

Theorem 4. *We find the following Laguerre–Freud equations:*

$$\begin{aligned}\gamma_{n+2} &= \frac{\eta + \gamma_{n+1}}{\gamma_{n+1}} n(\beta_{n-1} - \beta_n + 1) - (\gamma_n + \gamma_{n+1} + (\beta_{n+1} + b_1)(\beta_{n+1} + b_2)\\ &\quad + \beta_n(\beta_{n+1} + \beta_n + b_1 + b_2) - n(2\beta_n + \beta_{n+1} + b_1 + b_2 - n) - \eta) + \frac{\eta}{\gamma_{n+1}}(n+1)(\beta_n + a_1)\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\eta+\gamma_{n+1}}{(\eta+\gamma_n)\gamma_{n+1}}(\gamma_n(\gamma_{n-1}+\gamma_n+\gamma_{n+1}+(\beta_n+b_1)(\beta_n+b_2)+\beta_{n-1}(\beta_n+\beta_{n-1}+b_1+b_2)\\
&-(n-1)(2\beta_{n-1}+\beta_n+b_1+b_2-n+1)-\eta)-\eta n(\beta_{n-1}+a_1)),
\end{aligned}
$$

$$
\begin{aligned}
\beta_{n+2} = {}& n-2\beta_{n+1}+b_1+b_2+1+\frac{1}{\gamma_{n+2}}(\eta(1-\beta_n-\beta_{n+1})+\gamma_{n+1}(\beta_n+1-\beta_{n+1})\\
&+\gamma_n(\beta_{n-1}+2\beta_n+b_1+b_2-n+2)+\beta_n^3-\beta_{n+1}^3+\beta_{n+1}^2+\beta_n^2+\beta_n\beta_{n+1}\\
&+(b_1+b_2)(\beta_n^2-\beta_{n+1}^2+\beta_{n+1}+\beta_n)-n(2\beta_n^2-\beta_n\beta_{n+1}-\beta_{n+1}^2+3\beta_n)\\
&+(\beta_n-\beta_{n+1}+1)(n(n+1)-n(b_1+b_2)+b_1b_2\\
&-\frac{1}{\eta+\gamma_{n+1}}(\gamma_{n+1}(\gamma_n+\gamma_{n+1}+\gamma_{n+2}+(\beta_{n+1}+b_1)(\beta_{n+1}+b_2)\\
&+\beta_n(\beta_{n+1}+\beta_n+b_1+b_2)-n(2\beta_n+\beta_{n+1}+b_1+b_2-n)-\eta)-\eta(n+1)(\beta_n+a_1)))).
\end{aligned}
$$

Proof. To get $\gamma_{n+2}=F_1(n,\gamma_{n+1},\gamma_n,\dots,\beta_{n+1},\beta_n,\dots)$, we consider the compatibility $T_+\pi^{[2]}-T_+^2\pi^{[2]}=T_+D(T_+\beta+1-\beta)$ and solve γ_{n+2}. Also, we obtain $\beta_{n+2}=F_2(n,\gamma_{n+2},\gamma_{n+1},\dots,\beta_{n+1},\beta_n,\dots)$ by solving for β_{n+2} in the first subdiagonal of the first type of compatibility. □

3 | THE $_2F_2$ HYPERGEOMETRIC FAMILY

Let us take $\sigma(z)=\eta(z+a_1)(z+a_2)$ and $\theta(z)=z(z+b_1)(z+b_2)$ and consider the corresponding Pearson equation

$$(k+1)(k+1+b_1)(k+1+b_2)w(k+1)=\eta(k+a_1)(k+a_2)w(k),$$

whose solutions are proportional to $w(z)=\frac{(a_1)_z(a_2)_z}{(b_1+1)_z(b_2+1)_z}\frac{\eta^z}{z!}$, so that the discrete orthogonality measure is

$$\rho_z=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k}{(b_1+1)_k(b_2+1)_k}\frac{\eta^k}{k!},$$

with first moment given by the generalized hypergeometric series: $\rho_0={}_2F_2\left[{a_1,a_2 \atop b_1+1,b_2+1};\eta\right]$, and subsequent moments $\rho_n=\vartheta_\eta^n{}_2F_2\left[{a_1,a_2 \atop b_1+1,b_2+1};\eta\right]$. The corresponding moments exist for any $\eta\in\mathbb{C}$.

Definition 1. Let us introduce

$$
\begin{aligned}
A_n :={}& -\eta^2\left(\frac{n(n+1)}{2}+\beta_{n-1}+\beta_n+\gamma_{n+2}+\gamma_{n+1}+(n+1)(\beta_{n+1}+a_1+a_2)+(\beta_{n+1}+a_1)(\beta_{n+1}+a_2)\right)\\
&+\eta(4\gamma_n+\gamma_{n+1}(a_1+a_2-b_1-b_2+2(n+1)+\beta_{n-1}-\beta_{n+1})+\gamma_{n+2}(n-1-2\beta_{n+1}-\beta_{n+2}-b_1-b_2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{n(n+1)}{2}(a_1+a_2-b_1-b_2-\beta_{n+1})+(\beta_{n+1}+b_1+b_2+a_1+a_2-n^2)(\beta_n+\beta_{n-1})+2(\beta_n^2+\beta_{n-1}^2)\\
&-(\beta_{n+1}+b_1)(\beta_{n+1}+b_2)(\beta_{n+1}-n-1))\\
&-\gamma_{n+1}\left(\frac{n(n-1)}{2}+\gamma_{n+2}+\gamma_{n+1}+\gamma_n+\beta_{n-1}+(\beta_{n+1}+b_1)(\beta_{n+1}+b_2)+(\beta_n-n)(\beta_{n+1}+\beta_n+b_1+b_2)\right),
\end{aligned}
$$

and

$$
B_n := \eta^2+\eta(2\beta_{n-1}+2\beta_n+\beta_{n+1}+a_1+a_2+b_1+b_2+2n)+\gamma_{n+1}. \tag{29}
$$

Theorem 5 (The generalized Laguerre–Freud structure matrix). *For $\sigma(z)=\eta(z+a_1)(z+a_2)$ and $\theta(z)=z(z+c_1)(z+c_2)$, we find for the subleading coefficients the following expression:*

$$
p^1_{n-1}=\frac{A_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\beta_{n+2},\gamma_n,\gamma_{n+1},\gamma_{n+2})}{B_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_{n+1})}. \tag{30}
$$

The Laguerre–Freud structure matrix is the following hexadiagonal matrix:

$$
\Psi=\begin{bmatrix}
\psi_0^{(0)} & \psi_0^{(1)} & \psi_0^{(2)} & H_3 & 0 & \cdots & \\
\psi_0^{(-1)} & \psi_1^{(0)} & \psi_1^{(1)} & \psi_1^{(2)} & H_4 & \ddots & \\
\eta H_2 & \psi_1^{(-1)} & \psi_2^{(0)} & \psi_2^{(1)} & \psi_2^{(2)} & H_5 & \ddots \\
0 & \eta H_3 & \psi_2^{(-1)} & \psi_3^{(0)} & \psi_3^{(1)} & \psi_3^{(2)} & H_6 \; \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix},
$$

with

$$
\begin{aligned}
\psi_n^{(2)}&=(\beta_n+\beta_{n+1}+\beta_{n+2}+b_1+b_2-n)H_{n+2},\\
\psi_n^{(0)}&=\eta\left(\frac{n(n-1)}{2}+\beta_{n-1}+n(\beta_n+a_1+a_2)+\gamma_n+\gamma_{n+1}+(\beta_n+a_1)(\beta_n+a_2)-p^1_{n-1}\right)H_n,\\
\psi_n^{(1)}&=\left(\frac{n(n-1)}{2}+\gamma_n+\gamma_{n+1}+\gamma_{n+2}+(\beta_{n+1}+b_1)(\beta_{n+1}+b_2)\right.\\
&\qquad\left.+(\beta_n-n)(\beta_n+\beta_{n+1}+b_1+b_2)-\beta_{n-1}+p^1_{n-1}\right)H_{n+1},\\
\psi_n^{(-1)}&=\eta(\beta_n+\beta_{n+1}+a_1+a_2+n)H_{n+1}.
\end{aligned}
$$

Proof. We write (20) and (21) as

$$
\Psi=\sigma(J)H\Pi^\top, \tag{31}
$$

$$\Psi = \Pi^{-1} H \theta(J^\top). \tag{32}$$

The Laguerre–Freud matrix Ψ is a hexadiagonal matrix

$$\Psi = (\Lambda^\top)^2 \psi^{(-2)} + \Lambda^\top \psi^{(-1)} + \psi^{(0)} + \psi^{(1)}\Lambda + \psi^{(2)}\Lambda^2 + \psi^{(3)}\Lambda^3,$$

with $\psi^{(n)}$ diagonal matrices. From Equation (31), we get

$$\psi^{(-2)} = \eta T_-^2 H, \qquad \psi^{(-1)} = \eta T_- H(T_+ D + \beta + T_-\beta + a_1 + a_2).$$

From Equation (32), we obtain

$$\psi^{(3)} = T_-^3 H, \qquad \psi^{(2)} = T_-^2 H(T_-^2\beta + \beta + T_-\beta - T_+ D + b_1 + b_2).$$

We also have expressions for the diagonal matrices $\psi^{(0)}, \psi^{(1)}$ in terms of $\pi^{[\pm 2]}, \pi^{[\pm 3]}$. From Equation (31), we get

$$\psi^{(0)} = \eta H[T_+^2 \pi^{[2]} + T_+ D(T_+\beta + \beta + a_1 + a_2) + \gamma + T_+\gamma + (\beta + a_1)(\beta + a_2)], \tag{33}$$

$$\begin{aligned}\psi^{(1)} = \eta H[&\gamma(\beta + T_-\beta + a_1 + a_2) + D(\gamma + T_+\gamma + (\beta + a_1)(\beta + a_2)) + T_+^2\pi^{[3]} + T_+\pi^{[2]}(T_+\beta \\ &+ \beta + a_1 + a_2)],\end{aligned} \tag{34}$$

and Equation (32) leads to

$$\begin{aligned}\psi^{(0)} = H[&(\beta + b_1)(\beta + b_2)(\beta - T_+ D) + \gamma(2\beta + T_-\beta + b_1 + b_2 - T_+ D) \\ &+ T_+\gamma(2\beta + T_+\beta + b_1 + b_2 - T_+ D) - T_+ D[T_+^2\gamma + T_+\beta(\beta + T_+\beta + b_1 + b_2)] \\ &+ T_+^2\pi^{[-2]}(\beta + T_+\beta + T_+^2\beta + b_1 + b_2) - T_+^3\pi^{[-3]}],\end{aligned} \tag{35}$$

$$\begin{aligned}\psi^{(1)} = T_- H[&T_+^2\pi^{[-2]} - T_+ D(T_-\beta + \beta + T_+\beta + b_1 + b_2) \\ &+ T_-\gamma + \gamma + T_+\gamma + (T_-\beta + b_1)(T_-\beta + b_2) + \beta(T_-\beta + \beta + b_1 + b_2)].\end{aligned} \tag{36}$$

We get a system of two equations by equating the RHS of (33) and (35) and the RHS of (34) and (36). Using the shift $n \to n+1$ in the first relation obtained from the two possible expressions for $\psi^{(0)}$, we now show that the π matrices involved in the system can be expressed uniquely in terms of p_n^2, p_{n-1}^1. From the relations

$$p_n^1 = p_{n-1}^1 - \beta_{n-1}, \tag{37}$$

$$p_{n+1}^1 = p_{n-1}^1 - \beta_{n-1} - \beta_n, \tag{38}$$

$$p_{n+1}^2 = p_n^2 - \gamma_n - \beta_n p_{n-1}^1 + \beta_n\beta_{n-1}, \tag{39}$$

we get

$$
\begin{aligned}
\pi^{[\pm 2]}_{n-1} &= \frac{n(n+1)}{2} \mp (n\beta_n + p^1_{n-1} - \beta_{n-1}),\\
\pi^{[\pm 2]}_{n-2} &= \frac{n(n-1)}{2} \mp (n-1)\beta_{n-1} \mp p^1_{n-1},\\
\pi^{[\pm 3]}_{n-2} &= \pm\frac{n(n+1)(n-1)}{3} - \frac{n(n-1)}{2}(\beta_{n-1}+\beta_n) \pm (n-1)(\beta_n\beta_{n-1} - \gamma_n)\\
&\quad + p^1_{n-1}(\pm\beta_n \mp \beta_{n-1} - n) + \pm(p^1_{n-1})^2 \mp 2p^2_n.
\end{aligned}
$$

Then, we solve the equation for p^2_n in the equation obtained for the two expressions (33) and (35) for $\psi^{(0)}$ and substitute in the Equation gotten from the two expressions (34) and (36) for $\psi^{(1)}$ to deduce

$$
\begin{aligned}
&p^1_{n-1}\big(\eta^2 + \eta(2\beta_{n-1} + 2\beta_n + \beta_{n+1} + a_1 + a_2 + b_1 + b_2 + 2n) + \gamma_{n+1}\big)\\
&\quad = -\eta^2\bigg(\frac{n(n+1)}{2} + \beta_{n-1} + \beta_n + \gamma_{n+2} + \gamma_{n+1} + (n+1)(\beta_{n+1} + a_1 + a_2)\\
&\qquad + (\beta_{n+1} + a_1)(\beta_{n+1} + a_2))\\
&\qquad + \eta(\gamma_{n+1}(a_1 + a_2 - b_1 - b_2 + 2(n+1) + \beta_{n-1} - \beta_{n+1}) + \gamma_{n+2}(n - 1 - 2\beta_{n+1} - \beta_{n+2} - b_1 - b_2)\\
&\qquad + n(\beta_{n-1} + \beta_n) + (a_1 + a_2)\beta_{n-1} + 4\gamma_n + 2(\beta_n^2 + \beta_{n-1}^2) + \beta_n(\beta_{n+1} + a_1 + a_2 + b_1 + b_2)\\
&\qquad + (n+1)a_1a_2 + \beta_{n-1}(\beta_{n+1} + b_1 + b_2) - (\beta_{n+1} + b_1)(\beta_{n+1} + b_2)(\beta_{n+1} - n - 1)\\
&\qquad + \frac{n(n+1)}{2}(a_1 + a_2 - b_1 - b_2 - \beta_{n+1}) - n^2(\beta_n + \beta_{n-1})\bigg)\\
&\qquad - \gamma_{n+1}\bigg(\frac{n(n-1)}{2} + \gamma_{n+2} + \gamma_{n+1} + \gamma_n + \beta_{n-1} - n(\beta_{n+1} + \beta_n + b_1 + b_2) + (\beta_{n+1} + b_1)(\beta_{n+1} + b_2)\\
&\qquad + \beta_n(\beta_{n+1} + \beta_n + b_1 + b_2)\bigg).
\end{aligned} \tag{40}
$$

That is, we have proven Equation (30) for p^1_n. It is not necessary at this point to find p^2_n in order to get the Freud–Laguerre matrix, as we can get $\psi^{(0)}$ from (33) and $\psi^{(1)}$ from (36). The entries in the diagonal matrices $\psi^{(0)}, \psi^{(1)}$ can be written in terms of p^1_{n-1} as:

$$
\begin{aligned}
\psi^{(0)}_n &= \eta H_n\bigg(\frac{n(n-1)}{2} + \beta_{n-1} + n(\beta_n + a_1 + a_2) + \gamma_n + \gamma_{n+1} + (\beta_n + a_1)(\beta_n + a_2) - p^1_{n-1}\bigg),\\
\psi^{(1)}_n &= \gamma_{n+1}H_n\bigg(\frac{n(n-1)}{2} + \gamma_{n+1} + \gamma_{n+2} + \gamma_n + (\beta_{n+1} + b_1)(\beta_{n+1} + b_2)\\
&\qquad + (\beta_n - n)(\beta_n + \beta_{n+1} + b_1 + b_2) - \beta_{n-1} + p^1_{n-1}\bigg).
\end{aligned}
$$

□

Definition 2. Let us consider

$$
\begin{aligned}
C_n :=\ & \eta((n+\beta_n+\beta_{n+1}+a_1+a_2)(\beta_n-\beta_{n+1}-1)-\beta_n-\beta_{n+1}+\gamma_n)\\
&-n(\beta_{n+1}-\beta_n-1)(\beta_{n+1}+\beta_n+\beta_{n-1})-\beta_n(\beta_{n+1}+\beta_n+b_1+b_2)\\
&-\gamma_n(\beta_{n-1}+2\beta_n+b_1+b_2-n+2)+\gamma_{n+2}(2\beta_{n+1}+b_1+b_2-n-1),
\end{aligned} \tag{41}
$$

$$
D_n := \frac{n(n-1)}{2}+\beta_{n+1}^2+(\beta_{n+1}-n)(b_1+b_2)+b_1b_2+(n-1)\beta_{n-1}+\gamma_{n+1}, \tag{42}
$$

and

$$
\begin{aligned}
E_n :=\ & -\eta\bigg(\frac{n(n-1)}{2}+\beta_{n-1}+n(\beta_n+a_1+a_2)+(\beta_n+a_1)(\beta_n+a_2)\\
&-\gamma_n(n-2+\beta_{n-1}+\beta_n+a_1+a_2)+\gamma_{n+1}(n+1+\beta_n+\beta_{n+1}+a_1+a_2)\bigg)\\
&+\gamma_{n+1}\bigg(\frac{n(n-1)}{2}+\gamma_{n+1}-\beta_{n-1}+(\beta_n-n)(\beta_{n+1}+\beta_n+b_1+b_2)+(\beta_{n+1}+b_1)(\beta_{n+1}+b_2)\bigg)\\
&-\gamma_n\bigg(\frac{(n-1)(n-2)}{2}+\gamma_n+\gamma_{n-1}+(\beta_n+b_1)(\beta_n+b_2)+(\beta_{n-1}-n+1)(\beta_n+\beta_{n-1}+b_1+b_2)\bigg).
\end{aligned} \tag{43}
$$

Theorem 6 (Consequences of compatibility conditions). *The compatibility conditions lead to two alternative relations expressions for the subleading coefficient p^1_{n-1} of the orthogonal polynomials:*

$$
p^1_{n-1} = \frac{C_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_n)+\gamma_{n+2}(\beta_{n+2}-\eta)}{1+\beta_n-\beta_{n+1}} - D_n(\beta_{n-1},\beta_{n+1},\gamma_{n+1}), \tag{44}
$$

$$
p^1_{n-1} = \frac{E_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_{n-1},\gamma_n,\gamma_{n+1})+\gamma_{n+1}\gamma_{n+2}}{\gamma_n-\gamma_{n+1}-\eta}, \tag{45}
$$

where p^1_n is given in Equation (30).

Proof. We compute explicitly the compatibility equation $[\Psi H^{-1}, J] = \Psi H^{-1}$. On the one hand, ΨH^{-1} can be expressed as

$$
\Psi H^{-1} = (\Lambda^\top)^2 M^{(-2)} + \Lambda^\top M^{(-1)} + M^{(0)} + M^{(1)}\Lambda + M^{(2)}\Lambda^2 + M^{(3)}\Lambda^3
$$

with $M^{(n)}$ being the following diagonal matrices

$$
\begin{aligned}
M^{(3)} &= I,\\
M^{(2)} &= T_-^2\beta + T_-\beta + \beta - T_+D + b_1 + b_2,\\
M^{(1)} &= T_+^2\pi^{[-2]} - T_+D(T_-\beta+\beta+T_+\beta+b_1+b_2) + T_-\gamma + \gamma + T_+\gamma + (T_-\beta+b_1)(T_-\beta+b_2)
\end{aligned}
$$

$$
\begin{aligned}
&+\ \beta(T_-\beta+\beta+b_1+b_2),\\
M^{(0)} &= \eta\big(T_+^2\pi^{[2]}+T_+D(T_+\beta+\beta+a_1+a_2)+\gamma+T_+\gamma+(\beta+a_1)(\beta+a_2)\big),
\end{aligned}
$$

or, alternatively,

$$
\begin{aligned}
M^{(0)} &= \beta^3+(b_1+b_2)\beta^2+b_1b_2\beta-T_+D[\beta^2+(b_1+b_2)\beta+b_1b_2+T_+\beta(\beta+T_+\beta+b_1+b_2)]\\
&\quad+\ \gamma(2\beta+T_-\beta+b_1+b_2-T_+D)+T_+\gamma(2\beta+T_+\beta+b_1+b_2-T_+D)\\
&\quad-\ T_+^2\gamma T_+D+T_+^2\pi^{[-2]}(\beta+T_+\beta+T_+^2\beta+b_1+b_2)-T_+^3\pi^{[-3]},\\
M^{(-1)} &= \eta\gamma(T_+D+\beta+T_-\beta+a_1+a_2),\\
M^{(-2)} &= \eta\gamma(T_-\gamma).
\end{aligned}
$$

On the other hand, the commutator can be written as follows:

$$
[\Psi H^{-1},J]=(\Lambda^\top)^2\tilde{M}^{(-2)}+\Lambda^\top\tilde{M}^{(-1)}+\tilde{M}^{(0)}+\tilde{M}^{(1)}\Lambda+\tilde{M}^{(2)}\Lambda^2+\tilde{M}^{(3)}\Lambda^3,
$$

where the diagonal matrices $\tilde{M}^{(n)}$ are

$$
\begin{aligned}
\tilde{M}^{(3)} &= I,\\
\tilde{M}^{(2)} &= T_+^2\pi^{[-2]}-T_+\pi^{[-2]}+T_+D(\beta-T_+\beta)+T_-^2\beta+T_-\beta+\beta+b_1+b_2,\\
\tilde{M}^{(1)} &= \eta\big(T_+^2\pi^{[2]}-T_+\pi^{[2]}-(\beta+a_1+a_2)+T_+(D\beta)-DT_-\beta\big)\\
&\quad+\ (T_-\beta-\beta)\big(T_+^2\pi^{[-2]}-T_+D(T_+\beta+b_1+b_2)+\gamma+(T_-\beta+b_1)(T_-\beta+b_2)-\eta(a_1+a_2)\big)\\
&\quad+\ (\beta^2-(T_-\beta)^2)(T_+D+\eta)+T_-\gamma(T_-^2\beta+2T_-\beta-T_+D+b_1+b_2-\eta)\\
&\quad-\ T_+\gamma(2\beta+T_+\beta-T_+^2D+b_1+b_2-\eta),\\
\tilde{M}^{(0)} &= \eta T_+\gamma\big(T_+^2D+T_+\beta+\beta+a_1+a_2\big)\\
&\quad+\ \gamma\big(T_+^2\pi^{[-2]}-T_+D(T_-\beta+\beta+T_+\beta+b_1+b_2)+T_-\gamma+\gamma+(T_-\beta+b_1)(T_-\beta+b_2)\\
&\quad+\beta(T_-\beta+\beta+b_1+b_2)\big)-T_+\gamma(T_+^3\pi^{[-2]}-T_+^2D(\beta+T_+\beta+T_+^2\beta+b_1+b_2)\\
&\quad+\ T_+\gamma+T_+^2\gamma+(\beta+b_1)(\beta+b_2)\mathbb{T}_+\beta(\beta+T_+\beta+b_1+b_2)-\eta\gamma(T_+D+\beta+T_-\beta+a_1+a_2),\\
\tilde{M}^{(-1)} &= \eta\gamma(T_+\pi^{[2]}-T_+^2\pi^{[2]}+T_+D(-T_-\beta-T_+\beta)+D(\beta+T_-\beta)+(a_1+a_2)),\\
\tilde{M}^{(-2)} &= \eta\gamma(T_-\gamma).
\end{aligned}
$$

From the equations derived from second superdiagonal and first subdiagonal, one obtains, respectively, the following equations:

$$
T_+^2\pi^{[-2]}-T_+\pi^{[-2]}=T_+D(T_+\beta-\beta-1),\qquad T_+\pi^{[2]}-T_+^2\pi^{[2]}=T_+D(1-\beta+T_+\beta).
$$

Hence , they can be gotten directly from the expressions for the diagonal matrices $\pi^{[\pm 2]}$.

From the first superdiagonal, we get (44), and using the alternative expression of the main diagonal, it can be seen that it corresponds to the identity:

$$\pi^{[-3]}_{n-2}-\pi^{[-3]}_{n-3}=\pi^{[-2]}_{n-2}(1+\beta_n-\beta_{n-2})-(n-1)\gamma_n+n\gamma_{n-1}.$$

Compatibility in the main diagonal is (45). □

We discuss now about Laguerre–Freud equations (28).

Definition 3. Let us introduce

$$\begin{aligned}\hat{A}_n:=&-\eta^2\left(\frac{n(n+1)}{2}+\beta_{n-1}+\beta_n+\gamma_{n+1}+(n+1)(\beta_{n+1}+a_1+a_2)+(\beta_{n+1}+a_1)(\beta_{n+1}+a_2)\right)\\&+\eta(4\gamma_n+\gamma_{n+1}(a_1+a_2-b_1-b_2+2(n+1)+\beta_{n-1}-\beta_{n+1})\\&+\frac{n(n+1)}{2}(a_1+a_2-b_1-b_2-\beta_{n+1})+(\beta_{n+1}+b_1+b_2+a_1+a_2-n^2)(\beta_n+\beta_{n-1})+2(\beta_n^2+\beta_{n-1}^2)\\&-(\beta_{n+1}+b_1)(\beta_{n+1}+b_2)(\beta_{n+1}-n-1))\\&-\gamma_{n+1}\left(\frac{n(n-1)}{2}+\gamma_{n+1}+\gamma_n+\beta_{n-1}+(\beta_{n+1}+b_1)(\beta_{n+1}+b_2)+(\beta_n-n)(\beta_{n+1}+\beta_n+b_1+b_2)\right),\end{aligned}\tag{46}$$

$$F_n:=-\frac{E_n}{\gamma_n-\gamma_{n+1}-\eta}+\frac{C_n}{1+\beta_n-\beta_{n+1}}-D_n+\frac{1}{\eta(1+\beta_n-\beta_{n+1})}\left(\hat{A}_n-\frac{E_n}{\gamma_n-\gamma_{n+1}-\eta}B_n\right),\tag{47}$$

$$G_n:=\frac{\gamma_{n+1}}{\gamma_n-\gamma_{n+1}-\eta}+\frac{\eta}{1+\beta_n-\beta_{n+1}}+\frac{\eta^2-\eta(n-1-2\beta_{n+1}-b_1-b_2)+\gamma_{n+1}+\frac{\gamma_{n+1}B_n}{\gamma_n-\gamma_{n+1}-\eta}}{\eta(1+\beta_n-\beta_{n+1})}.\tag{48}$$

Then, we have the following result:

Theorem 7 (Laguerre–Freud equations). *Given the functions $\hat{A}_n, B_n, C_n, D_n, E_n, F_n$, and G_n, as in (46), (29), (41), (42), (43), (47), and (48), respectively, the following nonlinear equations of Laguerre–Freud type are satisfied:*

$$\begin{aligned}\beta_{n+2}=&\frac{\hat{A}_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_n,\gamma_{n+1})-(\eta^2-\eta(n-1-2\beta_{n+1}-b_1-b_2)+\gamma_{n+1})\gamma_{n+2}}{\eta\gamma_{n+2}}\\&-\frac{E_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_{n-1},\gamma_n,\gamma_{n+1})+\gamma_{n+1}\gamma_{n+2}}{\gamma_n-\gamma_{n+1}-\eta}\frac{B_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_{n+1})}{\eta\gamma_{n+2}},\end{aligned}\tag{49}$$

$$\gamma_{n+2}=\frac{F_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_{n-1},\gamma_n,\gamma_{n+1})}{G_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_n,\gamma_{n+1})}.\tag{50}$$

Proof. From (30), we get

$$\beta_{n+2}=\frac{\tilde{A}_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_n,\gamma_{n+1},\gamma_{n+2})-p^1_{n-1}B_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_{n+1})}{\eta\gamma_{n+2}}, \tag{51}$$

with

$$\tilde{A}_n:=\hat{A}_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_n,\gamma_{n+1})-(\eta^2-\eta(n-1-2\beta_{n+1}-b_1-b_2)+\gamma_{n+1})\gamma_{n+2}.$$

Therefore, recalling (44), we get the Laguerre–Freud equation (49).

To prove the second Laguerre–Freud equation (50), we first notice that (49) can be written as

$$\begin{aligned}2\eta\beta_{n+2}\gamma_{n+2}&=\tilde{A}_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_n,\gamma_{n+1},\gamma_{n+2})\\&\quad-\frac{E_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_{n-1},\gamma_n,\gamma_{n+1})+\gamma_{n+1}\gamma_{n+2}}{\gamma_n-\gamma_{n+1}-\eta}B_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_{n+1}).\end{aligned}$$

Consequently, recalling (44), we find

$$\begin{aligned}p^1_{n-1}&=\frac{C_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_n)-\eta\gamma_{n+2}}{1+\beta_n-\beta_{n+1}}-D_n(\beta_{n-1},\beta_{n+1},\gamma_{n+1})+\frac{\gamma_{n+2}\beta_{n+2}}{1+\beta_n-\beta_{n+1}}\\&=\frac{C_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_n)-\eta\gamma_{n+2}}{1+\beta_n-\beta_{n+1}}-D_n(\beta_{n-1},\beta_{n+1},\gamma_{n+1})\\&\quad+\frac{1}{\eta(1+\beta_n-\beta_{n+1})}\bigg(\tilde{A}_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_n,\gamma_{n+1},\gamma_{n+2})\\&\quad-\frac{E_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_{n-1},\gamma_n,\gamma_{n+1})+\gamma_{n+1}\gamma_{n+2}}{\gamma_n-\gamma_{n+1}-\eta}B_n(\eta,\beta_{n-1},\beta_n,\beta_{n+1},\gamma_{n+1})\bigg).\end{aligned}$$

Thus, we see that the following relation is fulfilled:

$$\begin{aligned}p^1_{n-1}&=\frac{C_n-\eta\gamma_{n+2}}{1+\beta_n-\beta_{n+1}}-D_n+\frac{1}{\eta(1+\beta_n-\beta_{n+1})}\\&\quad\times\left(\hat{A}_n-(\eta^2-\eta(n-1-2\beta_{n+1}-b_1-b_2)+\gamma_{n+1})\gamma_{n+2}-\frac{E_n+\gamma_{n+1}\gamma_{n+2}}{\gamma_n-\gamma_{n+1}-\eta}B_n\right).\end{aligned}$$

Now, from Equation (44), we get

$$\begin{aligned}\frac{E_n+\gamma_{n+1}\gamma_{n+2}}{\gamma_n-\gamma_{n+1}-\eta}&=\frac{C_n-\eta\gamma_{n+2}}{1+\beta_n-\beta_{n+1}}-D_n+\frac{1}{\eta(1+\beta_n-\beta_{n+1})}\\&\quad\times\left(\hat{A}_n-(\eta^2-\eta(n-1-2\beta_{n+1}-b_1-b_2)+\gamma_{n+1})\gamma_{n+2}-\frac{E_n+\gamma_{n+1}\gamma_{n+2}}{\gamma_n-\gamma_{n+1}-\eta}B_n\right).\end{aligned}$$

Hence, we find the second Laguerre–Freud equation (50). □

4 | THE ${}_3F_2$ HYPERGEOMETRIC CASE

Let us take $\sigma(z) = \eta(z + a_1)(z + a_2)(z + a_3)$ and $\theta(z) = z(z + b_1)(z + b_2)$ and consider the corresponding Pearson equation

$$(k + 1)(k + 1 + b_1)(k + 1 + b_2)w(k + 1) = \eta(k + a_1)(k + a_2)(k + a_3)w(k)$$

whose solutions are proportional to $w(z) = \frac{(a_1)_z(a_2)_z(a_3)_z}{(b_1+1)_z(b_2+1)_z}\frac{\eta^z}{z!}$, so that the discrete orthogonality measure is

$$\rho_z = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k(a_3)_k}{(b_1 + 1)_k(b_2 + 1)_k}\frac{\eta^k}{k!},$$

with first moment given by the generalized hypergeometric series: $\rho_0 = {}_3F_2\left[\begin{smallmatrix} a_1, a_2, a_3 \\ b_1+1, b_2+1 \end{smallmatrix}; \eta\right]$, and subsequent moments $\rho_n = \vartheta_\eta^n {}_3F_2\left[\begin{smallmatrix} a_1, a_2, a_3 \\ b_1+1, b_2+1 \end{smallmatrix}; \eta\right]$. The corresponding moments exist for any $\eta \in \mathbb{C}$, whenever one of the a's is a nonpositive integer, when $|\eta| < 1$ or when $|\eta| = 1$ and

$$\operatorname{Re}(b_1 + b_2 - a_1 - a_2 - a_3) > 0.$$

Definition 4. We consider

$$\begin{aligned}
A_n &:= n(n + 1) + n(\beta_{n-1} + \beta_n + \beta_{n+1}) + (a_1 + a_2 + a_3)(n + \beta_n + \beta_{n+1}) + a_1a_2 + a_2a_3 + a_3a_1 \\
&\quad + \gamma_n + \gamma_{n+1} + \gamma_{n+2} + \beta_n^2 + \beta_{n+1}^2 + \beta_n\beta_{n+1}, \\
B_n &= -n(\beta_n^2 - \beta_{n+1}^2 + \beta_{n+1} + \beta_n - \gamma_{n+2}) + (\beta_{n+1} - \beta_n - 1)(n(b_1 + b_2 - \beta_{n-1}) - b_1b_2) \\
&\quad - (b_1 + b_2)(\beta_{n+1}^2 - \beta_n^2 - \beta_n - \beta_{n+1}) - \gamma_{n+2}(\beta_{n+2} + 2\beta_{n+1} + b_1 \\
&\quad + b_2 - 1) - \gamma_{n+1}(\beta_{n+1} - \beta_n - 1) + \gamma_n(\beta_{n-1} + 2\beta_n + b_1 + b_2 - (n - 2)) \\
&\quad - \beta_{n+1}^3 + \beta_n^3 + \beta_n^2 + \beta_{n+1}^2 + \beta_n\beta_{n+1}, \\
C_n &= \eta + \beta_{n+1} - \beta_{n-2}, \\
D_n &= \beta_n^3 + (b_1 + b_2)\left(\beta_n^2 - \beta_{n-1} - \beta_{n-2} + \frac{n(n-1)}{2}\right) + b_1b_2\beta_n \\
&\quad - n[\beta_n^2 - \beta_{n-1}\beta_{n-2} + (b_1 + b_2)\beta_n + \gamma_{n-1}] - \beta_n(\beta_{n-1} + \beta_{n-2}) - 2\beta_{n-1}\beta_{n-2} - \beta_{n-1}^2 - \beta_{n-2}^2 \\
&\quad + \gamma_n[b_1b_2 + (b_1 + b_2)(\beta_n + \beta_{n-1} - n + 1) - n + \beta_{n-1} + 2\beta_n + b_1 + b_2] \\
&\quad - \gamma_{n+1}[(b_1 + b_2)(\beta_n + \beta_{n+1} - n) + b_1b_2 + n - (\beta_{n+1} + 2\beta_n + b_1 + b_2)], \\
E_n &= n\gamma_{n+1}(\beta_n + \beta_{n+1}) + \gamma_{n+1}(\beta_{n-1} + \beta_{n-2}) - \gamma_n\beta_{n-2} - (n - 1)\gamma_n(\beta_{n-1} + \beta_n), \\
F_n &= \gamma_{n+1}\left(\frac{n(n-1)}{2} + \gamma_{n+1} + \gamma_{n+2} + \beta_n^2 + \beta_{n+1}^2 + \beta_n\beta_{n+1}\right)
\end{aligned}$$

$$- \gamma_n\left(\frac{(n-1)(n-2)}{2} + \gamma_{n-1} + \gamma_n + \beta_n^2 + \beta_{n-1}^2 + \beta_n\beta_{n-1}\right),$$

$$G_n = (\gamma_{n+1} - \gamma_n)(a_1a_2 + a_2a_3 + a_3a_1 + (a_1 + a_2 + a_3)(\beta_n + n + \gamma_{n+1}\beta_{n+1} - \gamma_n(\beta_{n-1} - 1)).$$

Theorem 8. *(The Laguerre–Freud structure matrix) The Laguerre–Freud structure matrix is the following heptadiagonal matrix:*

$$\Psi = \begin{bmatrix} \psi_0^{(0)} & \psi_0^{(1)} & \psi_0^{(2)} & H_3 & 0 & \cdots & \cdots \\ \psi_0^{(-1)} & \psi_1^{(0)} & \psi_1^{(1)} & \psi_1^{(2)} & H_4 & \ddots & \\ \psi_0^{(-2)} & \psi_1^{(-1)} & \psi_2^{(0)} & \psi_2^{(1)} & \psi_2^{(2)} & H_5 & \ddots \\ \eta H_3 & \psi_1^{(-2)} & \psi_2^{(-1)} & \psi_3^{(0)} & \psi_3^{(1)} & \psi_3^{(2)} & H_6 & \\ 0 & \eta H_4 & \psi_2^{(-2)} & \psi_3^{(-1)} & \psi_4^{(0)} & \psi_4^{(1)} & \psi_4^{(2)} & H_7 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

where

$$\psi_n^{(-2)} = \eta H_{n+2}(\beta_+ + \beta_{n+1} + \beta_{n+2} + a_1 + a_2 + a_3 + n),$$

$$\begin{aligned}\psi_n^{(-1)} &= \eta H_{n+1}(\pi_{n-2}^{[2]} + n(\beta_{n-1} + \beta_n + \beta_{n+1} + a_1 + a_2 + a_3) + \gamma_n + \gamma_{n+1} + \gamma_{n+2} + \beta_{n+1}^2 + \beta_n^2 \\ &\quad + \beta_{n+1}\beta_n + (\beta_n + \beta_{n+1})(a_1 + a_2 + a_3) + a_1a_2 + a_2a_3 + a_3a_1),\end{aligned}$$

$$\begin{aligned}\psi^{(0)} &= H_n(\beta_n(\beta_n + b_1)(\beta_n + b_2) + \gamma_n(\beta_{n-1} + 2\beta_n + b_1 + b_2) + \gamma_{n+1}(\beta_{n+1} + 2\beta_n + b_1 + b_2) \\ &\quad - n(\beta_n^2 + \beta_{n-1}^2 + \beta_n\beta_{n-1} + (\beta_n + \beta_{n-1})(b_1 + b_2)b_1b_2 + \gamma_{n-1} + \gamma_n + \gamma_{n+1}) \\ &\quad + \pi_{n-2}^{[-2]}(\beta_n + \beta_{n-1} + \beta_{n-2} + b_1 + b_2) - \pi_{n-3}^{[-3]}),\end{aligned}$$

or, alternatively,

$$\begin{aligned}\psi_n^{(0)} &= \eta H_n(\pi_{n-3}^{[3]} + \pi_{n-2}^{[2]}(\beta_{n-2} + \beta_{n-1} + \beta_n + a_1 + a_2 + a_3) \\ &\quad + n(\beta_{n-1}^2 + \beta_n^2 + \beta_n\beta_{n-1} + a_1a_2 + a_2a_3 + a_3a_1 \\ &\quad + (\beta_n + \beta_{n-1})(a_1 + a_2 + a_3) + \gamma_{n-1} + \gamma_n + \gamma_{n+1}) + \gamma_n(\beta_{n-1} + 2\beta_n + a_1 + a_2 + a_3) \\ &\quad + \gamma_{n+1}(\beta_{n+1} + 2\beta_n + a_1 + a_2 + a_3) + (\beta_n + a_1)(\beta_n + a_2)(\beta_n + a_3)),\end{aligned}$$

$$\begin{aligned}\psi_n^{(1)} &= H_{n+1}(\beta_{n+1}^2 + \beta_n^2 + \beta_n\beta_{n+1} + (\beta_n + \beta_{n+1})(b_1 + b_2) + b_1b_2 + \gamma_{n+2} + \gamma_{n+1} + \gamma_n \\ &\quad - n(\beta_{n+1} + \beta_N + \beta_{n-1} + b_1 + b_2) + \pi_{n-2}^{[-2]}),\end{aligned}$$

$$\psi_n^{(2)} = H_{n+2}(\beta_n + \beta_{n+1} + \beta_{n+2} + b_1 + b_2 - n).$$

For the diagonal matrix $\pi^{[-2]}$, the nontrivial entries are

$$\pi^{[-2]}_{n-2} = \frac{\eta A(\beta_{n-1},\beta_n,\beta_{n+1},,\beta_{n+2},\gamma_n,\gamma_{n+1},\gamma_{n+2}) + B(\beta_{n-1},\beta_n,\beta_{n+1},,\beta_{n+2},\gamma_n,\gamma_{n+1},\gamma_{n+2})}{C(\beta_n,\beta_{n+1},\eta)} \tag{52}$$

and the subleading coefficient of the corresponding hypergeometric discrete orthogonal polynomials is

$$\begin{aligned} p^1_{n-2} = {} & \frac{\eta A(\beta_{n-1},\beta_n,\beta_{n+1},,\beta_{n+2},\gamma_n,\gamma_{n+1},\gamma_{n+2}) + B(\beta_{n-1},\beta_n,\beta_{n+1},,\beta_{n+2},\gamma_n,\gamma_{n+1},\gamma_{n+2})}{C(\beta_n,\beta_{n+1},\eta)} \\ & + \beta_{n-2} - (n-1)\beta_{n-1} - \frac{(n-3)(n-4)}{2}, \end{aligned}$$

and the $\pi^{[-3]}$ matrix entries are

$$\begin{aligned} \pi^{[-3]}_{n-3} = {} & p^1_{n-2}[(\eta+1)(\gamma_n-\gamma_{n+1}) + \beta_n + \beta_{n-1} + \beta_{n-2} + b_1 + b_2] \\ & + D_n(\beta_{n-2},\beta_{n-1},\beta_n,\beta_{n+1},\gamma_{n-1},\gamma_n,\gamma_{n+1}) + (\eta+1)E_n(\beta_{n-2},\beta_{n-1},\beta_n,\beta_{n+1},\gamma_n,\gamma_{n+1}) \\ & + (\eta-1)F_n(\beta_{n-1},\beta_n,\beta_{n+1},\gamma_{n-1},\gamma_n,\gamma_{n+1},\gamma_{n+2}) + \eta G_n(\beta_{n-1},\beta_n,\beta_{n+1},\gamma_n,\gamma_{n+1}). \end{aligned} \tag{53}$$

Proof. Matrix elements of $\psi^{(-3)},\psi^{(-2)},\psi^{(-1)},\psi^{(0)}$ are obtained using $\Psi = \sigma(J)H\Pi^\top$. On the other hand, the entries of $\psi^{(3)},\psi^{(2)},\psi^{(1)}$ and the alternative expression of $\psi^{(0)}$ are gotten from $\Psi = \Pi^{-1}H\theta(J^\top)$. The matrix $\pi^{[-2]}$ entries follow from the equation from the first subdiagonal obtained from $[\Psi H^{-1},\Lambda^\top\gamma] = \vartheta_\eta(\Psi H^{-1})$. Applying ϑ_η to $\psi^{(0)}H^{-1}$, where $\psi^{(0)}$ is the first expression for the main diagonal, and equating to $\psi^{(-1)}H^{-1}$, after cleaning, it is obtained $\pi^{[-2]}$. From $\pi^{[-2]}$ using $p^1_{n-2} = \pi^{[-2]}_{n-2} - \frac{n(n-1)}{2} - (n-1)\beta_{n-1} + \beta_{n-2}$, it is obtained p^1_{n-2}. Finally, the matrix $\pi^{[-3]}$ is obtained from the main diagonal of $[\Psi H^{-1}, J] = \Psi H^{-1}$. □

Theorem 9. *(Compatibility conditions) The following relations are satisfied:*

$$\pi^{[2]}_{n-1} - \pi^{[2]}_{n-2} = n(1-\beta_n+\beta_{n-1}), \tag{54}$$

$$\pi^{[3]}_{n-2} - \pi^{[3]}_{n-3} = \pi^{[2]}_{n-2}(\beta_{n-2}-\beta_n+1) + n\gamma_{n-1} - (n-1)\gamma_n, \tag{55}$$

$$\vartheta_\eta \pi^{[-2]}_{n-2} = (n-1)\gamma_n - n\gamma_{n-1}, \tag{56}$$

$$\vartheta_\eta \pi^{[-3]}_{n-3} = (\gamma_n-\gamma_{n-2})\pi^{[-2]}_{n-2} + (n-1)(\beta_{n-2}-\beta_{n-1}-1)\gamma_n. \tag{57}$$

Proof. Equation (54) is obtained from the second subdiagonal and the second superdiagonal of the resulting matrix of $[\Psi H^{-1}, J] = \Psi H^{-1}$, and it is also obtained from the second subdiagonal of $[\Psi H^{-1},\Lambda^\top\gamma] = \vartheta_\eta(\Psi H^{-1})$. Equation (55) is obtained from the first superdiagonal and from the first subdiagonal of $[\Psi H^{-1}, J] = \Psi H^{-1}$, and it is also obtained from the first subdiagonal of $[\Psi H^{-1},\Lambda^\top\gamma] = \vartheta_\eta(\Psi H^{-1})$ using the alternative expression of $\psi^{(0)}$. Then, Equation (56) is a derived from the first superdiagonal of $[\Psi H^{-1},\Lambda^\top\gamma] = \vartheta_\eta(\Psi H^{-1})$ and Equation (57) is obtained from the main diagonal of $[\Psi H^{-1},\Lambda^\top\gamma] = \vartheta_\eta(\Psi H^{-1})$. □

Remark 1.

i) The first three equations are equivalent to

$$\pi^{[-2]}_{n-2} - \pi^{[-2]}_{n-1} = n(\beta_{n-1} - \beta_n - 1), \tag{58}$$

$$\pi^{[-3]}_{n-2} - \pi^{[-3]}_{n-3} = \pi^{[-2]}_{n-2}(1 + \beta_n - \beta_{n-2}) - (n-1)\gamma_n + n\gamma_{n-1}, \tag{59}$$

$$\vartheta_\eta p^1_{n-1} = -\gamma_{n-1}.$$

ii) Notice that compatibility conditions do not depend of hypergeometric parameters a's or b's.
iii) Equations (52) and (58) combine and give a constraint among $(\beta_{n-1}, \beta_n, \beta_{n+1}, \beta_{n+2}, \gamma_n, \gamma_{n+1}, \gamma_{n+2})$, while Equations (53) and (59) lead to a ligature between $(\beta_{n-2}, \beta_{n-1}, \beta_n, \beta_{n+1}, \beta_{n+2}, \gamma_{n-1}, \gamma_n, \gamma_{n+1}, \gamma_{n+2})$. With these two equations at hand, two Laguerre–Freud equations of the type (28) can be derived.

5 | CONCLUSIONS AND OUTLOOK

Adler and van Moerbeke have thoroughly used the Gauss–Borel factorization of the moment matrix, see Ref. [2, 4] in their studies of integrable systems and orthogonal polynomials. We have applied these ideas in different contexts, cantero–moral–velázquez (CMV) orthogonal polynomials, matrix orthogonal polynomials, multiple orthogonal polynomials, and multivariate orthogonal, see Refs [5–15] For a general overview, see Ref. [43].

Recently, in Ref. [46] we applied that approach to study the consequences of the Pearson equation on the moment matrix and Jacobi matrices. For that description, a new banded matrix is required, the Laguerre–Freud structure matrix, which encodes the Laguerre–Freud relations for the recurrence coefficients. We have also found that the contiguous relations fulfilled generalized hypergeometric series determining the moments of the weight described for the squared norms of the orthogonal polynomials a discrete Toda hierarchy known as Nijhoff–Capel equation, see Ref. [47]. In Ref. [44], we studied the role of Christoffel and Geronimus transformations for the description of the mentioned contiguous relations, and the use of the Geronimus–Christoffel transformations to characterize the shifts in the spectral independent variable of the orthogonal polynomials. In Ref. [30], we deepened in that program and searched further for the discrete semiclassical cases, finding Laguerre–Freud relations for the recursion coefficients for three types of discrete orthogonal polynomials of generalized Charlier, generalized Meixner, and generalized Hahn of type I cases. In this paper, we concluded this program by getting the Laguerre–Freud structure matrices and equations for three families of hypergeometric discrete orthogonal polynomials.

As an outlook, we point the following three lines of research:

i) A question of general character that remains open is the possibility of explicit determination (using the methods of Ref. [30] and this paper) of Laguerre–Freud equations for arbitrary hypergeometric families ${}_pF_q$ of discrete orthogonal polynomials. In Ref. [30] and this paper, we have shown that it is possible for the cases $(p, q) = (0, 1), (1, 1), (2, 1), (1, 2), (2, 2), (3, 2)$. For the future, we will also extend these techniques to multiple discrete orthogonal polynomials Ref. [16] and study its relations with the transformations presented in Ref. [20] and quadrilateral lattices.[23,45]

ii) An important issue is the connection with discrete and continuous Painlevé equations for the coefficients β_n and γ_n, taking into account the study of such a question when you deal with discrete semiclassical linear functionals given in Ref. [51] for generalized Charlier and generalized Meixner as well as in Ref. [33] The results that are presented in this paper are a step forward in this direction because this identification is not a trivial problem and using these given relations could be reached in future works.

iii) The uniform lattice with distance ω and difference operator $(D_\omega p)(x) = p(x+\omega) - p(x)$ is of interest for future research. Notice that D_ω-classical linear functionals have been studied in Ref. [1] and as limit case, when $\omega \to 0$, you get the classical ones (Hermite, Laguerre, Jacobi, and Bessel). An open question is to get by a change of variable and a limit what are the sequences of *continuous* orthogonal polynomials associated with the three generalized hypergeometric examples studied.

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CONFLICT OF INTEREST STATEMENT

The authors declare no conflict of interest.

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This paper has no associated data.

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mathematics

Article

Laguerre–Freud Equations for the Gauss Hypergeometric Discrete Orthogonal Polynomials †

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† Dedicated to the memory of Diego Dominici.

Abstract: The Cholesky factorization of the moment matrix is considered for the Gauss hypergeometric discrete orthogonal polynomials. This family of discrete orthogonal polynomials has a weight with first moment given by $\rho_0 = {}_2F_1\left[\begin{matrix} a,b \\ c+1 \end{matrix};\eta\right]$. For the Gauss hypergeometric discrete orthogonal polynomials, also known as generalized Hahn of type I, Laguerre–Freud equations are found, and the differences with the equations found by Dominici and by Filipuk and Van Assche are provided.

Keywords: discrete orthogonal polynomials; Pearson equations; Cholesky factorization; Laguerre–Freud equations; hypergeometric functions; tau functions; Gauss hypergeometric orthogonal polynomials

MSC: 42C05; 33C45; 33C47

1. Introduction

Discrete orthogonal polynomials constitute a well-established and distinguished branch of orthogonal polynomial theory. Various classical monographs have been dedicated to this category of orthogonal polynomials. For instance, the classical case is thoroughly examined in [1], and the Riemann–Hilbert problem has been employed to investigate asymptotics and applications, as discussed in [2]. Authoritative discussions of the subject can be found in [3–6].

Discrete orthogonal polynomials play a role in designing digital filters for signal processing applications. They help in representing signals in a more compact and efficient manner, facilitating analysis and manipulation. Discrete orthogonal polynomials are employed in approximating functions. They provide a basis for representing functions in terms of a series expansion, making it easier to approximate complex functions. Discrete orthogonal polynomials are used in coding theory for designing error-correcting codes. They help in constructing efficient encoding and decoding algorithms, improving the reliability of data transmission in communication systems. Discrete orthogonal polynomials are often involved in combinatorics, providing tools for solving combinatorial problems and deriving combinatorial identities.

Semiclassical discrete orthogonal polynomials, characterized by the satisfaction of a discrete Pearson equation by their weight functions, have received extensive attention in the literature. A comprehensive overview of this topic, along with extensive references, can be found in [7–10]. Furthermore, for certain specific weight types, such as the generalized Charlier and Meixner weights, the corresponding Freud–Laguerre-type equations governing the coefficients of the three-term recurrence have been investigated. Notable works in this area include [11–15].

In our work presented in [16], we harnessed the Cholesky factorization of the moment matrix to delve into the realm of discrete orthogonal polynomials denoted as $\{P_n(x)\}_{n=0}^{\infty}$ on a homogeneous lattice. We directed our focus on semiclassical discrete orthogonal

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polynomials, defined by weight functions constrained by a discrete Pearson equation. This constraint led to the moments being expressible in terms of generalized hypergeometric functions.

We introduced a banded semi-infinite matrix named the Laguerre–Freud structure matrix Ψ, designed to model shifts by ± 1 in the independent variable of the sequence of orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$. Our study also unveiled that the contiguous relations governing the generalized hypergeometric functions have corresponding symmetries within the moment matrix. Furthermore, we established that the 3D Nijhoff–Capel discrete Toda lattice [17,18] offers a description of the contiguous shifts pertaining to the squared norms of the orthogonal polynomials.

In [19], we provided an interpretation for the contiguous transformations of generalized hypergeometric functions by invoking simple Christoffel and Geronimus transformations. Leveraging Geronimus–Uvarov perturbations, we derived determinantal expressions for the shifted orthogonal polynomials. Additionally, our exploration extended to three hypergeometric families and their associated Laguerre–Freud equations in [20].

In the research presented here, we extend and complete the investigations in [16,19,20], adding a new weight not studied so far with the similar techniques adapted to this situation. We delve into the realm of Gauss hypergeometric discrete orthogonal polynomials, which are also known as generalized Hahn polynomials of type I, as discussed in [9,10]. Our primary focus is an in-depth analysis of the Laguerre–Freud structure matrix Ψ. By examining its unique banded structure and its inherent compatibility with both the Toda equation and the Jacobi matrix, we uncover a set of nonlinear equations that govern the coefficients β_n, γ_n in the three-term recursion relations for the orthogonal polynomial sequence.

These nonlinear recurrences for the recursion coefficients take the following form:

$$\begin{aligned}
\gamma_{n+1} &= F_1(n, \gamma_n, \gamma_{n-1}, \dots, \beta_n, \beta_{n-1}, \dots), \\
\beta_{n+1} &= F_2(n, \gamma_{n+1}, \gamma_n, \dots, \beta_n, \beta_{n-1}, \dots),
\end{aligned}$$

with F_1 and F_2 being specific functions. Notably, these relations, referred to as Laguerre–Freud relations by Magnus [21] in reference to [22,23], have been explored in numerous papers for cases like generalized Charlier, generalized Meixner, and Gauss hypergeometric, as documented in [7,14].

A crucial insight that emerges is the role of the τ-function, defined as the Wronskian of the Gauss hypergeometric functions. This τ-function proves to be a valuable solution for each of these systems of nonlinear Laguerre–Freud equations governing the recursion coefficients.

This paper's structure is outlined as follows: In Section 2, we delve into the world of Gauss hypergeometric discrete orthogonal polynomials, as initially introduced in [7,9]. We present the pentadiagonal Laguerre–Freud structure matrix in Theorem 3. Additionally, we delve into the Laguerre–Freud relations in Theorem 4, providing a comparative analysis with Dominici's findings in [7] and those of Filipuk & Van Assche in [14], as well as the work presented in [24].

To round off this introduction, let us begin by summarizing the foundational concepts of discrete orthogonal polynomials. Afterward, we will provide a brief overview of the significant findings from our previous work in [16].

1.1. Basics on Orthogonal Polynomials

Given a linear functional $\rho_z \in \mathbb{C}^*[z]$, the corresponding moment matrix is

$$\begin{aligned}
G &= (G_{n,m}), \\
G_{n,m} &= \rho_{n+m}, \\
\rho_n &= \langle \rho_z, z^n \rangle, & n, m \in \mathbb{N}_0 := \{0, 1, 2, \dots\},
\end{aligned}$$

with ρ_n the n-th moment of the linear functional ρ_z. If the moment matrix is such that all its truncations, which are Hankel matrices, $G_{i+1,j} = G_{i,j+1}$,

$$G^{[k]} = \begin{bmatrix} G_{0,0} & \cdots & G_{0,k-1} \\ \vdots & & \vdots \\ G_{k-1,0} & \cdots & G_{k-1,k-1} \end{bmatrix} = \begin{bmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & \rho_2 & & \iddots & \rho_k \\ \rho_2 & & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & & \vdots \\ \rho_{k-1} & \rho_k & \cdots & \cdots & \rho_{2k-2} \end{bmatrix}$$

are nonsingular; i.e., the Hankel determinants $\Delta_k := \det G^{[k]}$ do not cancel, $\Delta_k \neq 0$, $k \in \mathbb{N}_0$. If this is the case, we have monic polynomials.

$$P_n(z) = z^n + p_n^1 z^{n-1} + \cdots + p_n^n, \quad n \in \mathbb{N}_0, \tag{1}$$

with $p_0^1 = 0$, fulfilling the orthogonality relations

$$\langle \rho, P_n(z) z^k \rangle = 0, \quad k \in \{0, \ldots, n-1\}, \quad \langle \rho, P_n(z) z^n \rangle = H_n \neq 0,$$

and $\{P_n(z)\}_{n \in \mathbb{N}_0}$ is a sequence of orthogonal polynomials, i.e., $\langle \rho, P_n(z) P_m(z) \rangle = \delta_{n,m} H_n$ for $n, m \in \mathbb{N}_0$. The symmetric bilinear form $\langle F, G \rangle_\rho := \langle \rho, FG \rangle$, is such that the moment matrix is the Gram matrix of this bilinear form and $\langle P_n, P_m \rangle_\rho := \delta_{n,m} H_n$.

Introducing $\chi(z) := \begin{pmatrix} 1 & z & z^2 \cdots \end{pmatrix}^\top$ the moment matrix is $G = \langle \rho, \chi \chi^\top \rangle$, and χ is an eigenvector of the *shift matrix*, $\Lambda \chi = x \chi$, where

$$\Lambda := \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ & & & \end{bmatrix}.$$

Hence, $\Lambda G = G \Lambda^\top$, and the moment matrix is a Hankel matrix.

As the moment matrix symmetric its Borel–Gauss factorization is a Cholesky factorization

$$G = S^{-1} H S^{-\top},$$

where S is a lower unitriangular matrix that can be written as

$$S = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ S_{1,0} & 1 & 0 & \ddots \\ S_{2,0} & S_{2,1} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

and $H = \operatorname{diag}(H_0, H_1, \ldots)$ is a diagonal matrix, with $H_k \neq 0$, for $k \in \mathbb{N}_0$. The Cholesky factorization does hold whenever the principal minors of the moment matrix; i.e., the Hankel determinants Δ_k, do not cancel.

The components $P_n(z)$ of

$$P(z) := S \chi(z), \tag{2}$$

are the monic orthogonal polynomials of the functional ρ.

Proposition 1. *We have the determinantal expressions*

$$H_n = \frac{\Delta_{n+1}}{\Delta_n}, \quad p^1_n = -\frac{\tilde{\Delta}_n}{\Delta_n},$$

with the Hankel determinants given by

$$\Delta_n := \begin{vmatrix} \rho_0 & \cdots & \rho_{n-2} & \rho_{n-1} \\ \vdots & \iddots & \iddots & \vdots \\ \rho_{n-2} & \rho_{n-1} & \cdots & \rho_{2n-3} \\ \rho_{n-1} & \cdots & \rho_{2n-3} & \rho_{2n-2} \end{vmatrix}, \quad \tilde{\Delta}_n := \begin{vmatrix} \rho_0 & \cdots & \rho_{n-2} & \rho_{n-1} \\ \vdots & \iddots & \iddots & \vdots \\ \rho_{n-2} & \rho_{n-1} & \cdots & \rho_{2n-3} \\ \rho_{n} & \cdots & \rho_{2n-2} & \rho_{2n-1} \end{vmatrix}.$$

We introduce the lower Hessenberg semi-infinite matrix

$$J = S\Lambda S^{-1} \tag{3}$$

that has the vector $P(z)$ as eigenvector with eigenvalue z, $JP(z) = zP(z)$. The Hankel condition $\Lambda G = G\Lambda^\top$ and the Cholesky factorization gives

$$JH = (JH)^\top = HJ^\top.$$

As the Hessenberg matrix JH is symmetric, the Jacobi matrix J is tridiagonal. The Jacobi matrix J given in (3) reads

$$J = \begin{bmatrix} \beta_0 & 1 & 0 & \cdots \\ \gamma_1 & \beta_1 & 1 & \ddots \\ 0 & \gamma_2 & \beta_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

The eigenvalue equation $JP = zP$ is a three term recursion relation $zP_n(z) = P_{n+1}(z) + \beta_n P_n(z) + \gamma_n P_{n-1}(z)$, that with the initial conditions $P_{-1} = 0$ and $P_0 = 1$ completely determines the sequence of orthogonal polynomials $\{P_n(z)\}_{n=0}^\infty$ in terms of the recursion coefficients β_n, γ_n. The recursion coefficients, in terms of the Hankel determinants, are given by

$$\beta_n = p^1_n - p^1_{n+1} = -\frac{\tilde{\Delta}_n}{\Delta_n} + \frac{\tilde{\Delta}_{n+1}}{\Delta_{n+1}}, \quad \gamma_{n+1} = \frac{H_{n+1}}{H_n} = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}, \quad n \in \mathbb{N}_0. \tag{4}$$

For future use, we introduce the following diagonal matrices $\gamma{:=}\operatorname{diag}(\gamma_1, \gamma_2, \dots)$ and $\beta{:=}\operatorname{diag}(\beta_0, \beta_1, \dots)$ and $J_-{:=}\Lambda^\top\gamma$ and $J_+{:=}\beta + \Lambda$, so that we have the splitting $J = \Lambda^\top\gamma + \beta + \Lambda = J_- + J_+$. In general, given any semi-infinite matrix A, we will write $A = A_- + A_+$, where A_- is a strictly lower triangular matrix and A_+ an upper triangular matrix. Moreover, A_0 will denote the diagonal part of A.

The lower Pascal matrix is defined by

$$B = (B_{n,m}), \quad B_{n,m} := \begin{cases} \binom{n}{m}, & n \geq m, \\ 0, & n < m, \end{cases}$$

so that

$$\chi(z+1) = B\chi(z).$$

Moreover,

$$B^{-1} = (\tilde{B}_{n,m}), \quad \tilde{B}_{n,m} := \begin{cases} (-1)^{n+m}\binom{n}{m}, & n \geq m, \\ 0, & n < m, \end{cases}$$

and $\chi(z-1) = B^{-1}\chi(z)$. The lower Pascal matrix and its inverse are explicitly given by

$$B = \begin{bmatrix} 1 & 0 & \cdots & & & & & \\ 1 & 1 & 0 & \cdots & & & & \\ 1 & 2 & 1 & 0 & \cdots & & & \\ 1 & 3 & 3 & 1 & 0 & \cdots & & \\ 1 & 4 & 6 & 4 & 1 & 0 & \cdots & \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix},$$

$$B^{-1} = \begin{bmatrix} 1 & 0 & \cdots & & & & & \\ -1 & 1 & 0 & \cdots & & & & \\ 1 & -2 & 1 & 0 & \cdots & & & \\ -1 & 3 & -3 & 1 & 0 & \cdots & & \\ 1 & -4 & 6 & -4 & 1 & 0 & \cdots & \\ -1 & 5 & -10 & 10 & -5 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix},$$

in terms of which we introduce the dressed Pascal matrices, $\Pi := SBS^{-1}$ and $\Pi^{-1} := SB^{-1}S^{-1}$, which are connection matrices; i.e.,

$$P(z+1) = \Pi P(z), \quad P(z-1) = \Pi^{-1} P(z). \tag{5}$$

The lower Pascal matrix can be expressed in terms of its subdiagonal structure as follows

$$B^{\pm 1} = I \pm \Lambda^{\top} D + \left(\Lambda^{\top}\right)^2 D^{[2]} \pm \left(\Lambda^{\top}\right)^3 D^{[3]} + \cdots,$$

where $D = \operatorname{diag}(1, 2, 3, \dots)$ and $D^{[k]} = \frac{1}{k}\operatorname{diag}\left(k^{(k)}, (k+1)^{(k)}, (k+2)^{(k)} \cdots\right)$, in terms of the falling factorials $x^{(k)} = x(x-1)(x-2)\cdots(x-k+1)$. That is,

$$D^{[k]}_n = \frac{(n+k)\cdots(n+1)}{k}, \quad k \in \mathbb{N}, \quad n \in \mathbb{N}_0.$$

The lower unitriangular factor can be also written in terms of its subdiagonals $S = I + \Lambda^{\top} S^{[1]} + \left(\Lambda^{\top}\right)^2 S^{[2]} + \cdots$ with $S^{[k]} = \operatorname{diag}\left(S^{[k]}_0, S^{[k]}_1, \dots\right)$. From (2) is clear the following connection between these subdiagonals entries and the coefficients of the orthogonal polynomials given in (1)

$$S^{[k]}_n = p^k_{n+k}.$$

We will use the *shift operators* $T_{\pm}$ acting over the diagonal matrices as follows

$$T_- \operatorname{diag}(a_0, a_1, \dots) := \operatorname{diag}(a_1, a_2, \dots), \qquad T_+ \operatorname{diag}(a_0, a_1, \dots) := \operatorname{diag}(0, a_0, a_1, \dots).$$

These shift operators have the following important properties, for any diagonal matrix $A = \operatorname{diag}(A_0, A_1, \dots)$

$$\Lambda A = (T_- A)\Lambda, \qquad A\Lambda = \Lambda(T_+ A), \qquad A\Lambda^{\top} = \Lambda^{\top}(T_- A), \qquad \Lambda^{\top} A = (T_+ A)\Lambda^{\top}.$$

In terms of these shift operators, we find

$$2D^{[2]} = (T_- D)D, \quad 3D^{[3]} = (T_-^2 D)(T_- D)D = 2(T_- D^{[2]})D = 2D^{[2]}(T_-^2 D).$$

Proposition 2. *The inverse matrix S^{-1} of the matrix S expands as follows*

$$S^{-1} = I + \Lambda^\top S^{[-1]} + \left(\Lambda^\top\right)^2 S^{[-2]} + \cdots .$$

The subdiagonals $S^{[-k]}$ are given in terms of the subdiagonals of S. In particular,

$$\begin{aligned}
S^{[-1]} &= -S^{[1]}, \\
S^{[-2]} &= -S^{[2]} + (T_- S^{[1]})S^{[1]}, \\
S^{[-3]} &= -S^{[3]} + (T_- S^{[2]})S^{[1]} + (T_-^2 S^{[1]})S^{[2]} - (T_-^2 S^{[1]})(T_- S^{[1]})S^{[1]}.
\end{aligned}$$

Remark 1. *Corresponding expansions for the dressed Pascal matrices are*

$$\Pi^{\pm 1} = I + \Lambda^\top \pi^{[\pm 1]} + (\Lambda^\top)^2 \pi^{[\pm 2]} + \cdots$$

with $\pi^{[\pm n]} = \operatorname{diag}(\pi_0^{[\pm n]}, \pi_1^{[\pm n]}, \dots)$.

Proposition 3 (The dressed Pascal matrix coefficients)**.** *We have*

$$\begin{aligned}
\pi_n^{[\pm 1]} = \pm(n+1), \quad \pi_n^{[\pm 2]} &= \frac{(n+2)(n+1)}{2} \pm p^1_{n+2}(n+1) \mp (n+2)p^1_{n+1} \\
&= \frac{(n+2)(n+1)}{2} \mp (n+1)\beta_{n+1} \mp p^1_{n+1},
\end{aligned} \tag{6}$$

$$\begin{aligned}
\pi_n^{[\pm 3]} = &\pm \frac{(n+3)(n+2)(n+1)}{3} + \frac{(n+2)(n+1)}{2} p^1_{n+3} - \frac{(n+3)(n+2)}{2} p^1_{n+1} \\
&\pm (n+1)p^2_{n+3} \mp (n+3)p^2_{n+2} \pm (n+3)p^1_{n+2}p^1_{n+1} \mp (n+2)p^1_{n+3}p^1_{n+1}.
\end{aligned}$$

Moreover, the following relations are fulfill

$$\pi^{[1]} + \pi^{[-1]} = 0, \quad \pi^{[2]} + \pi^{[-2]} = 2D^{[2]}, \quad \pi^{[3]} + \pi^{[-3]} = 2((T_-^2 S^{[1]})D^{[2]} - (T_- D^{[2]})S^{[1]}).$$

1.2. Discrete Orthogonal Polynomials and Pearson Equation

We are interested in measures with support on the homogeneous lattice $\mathbb{N}_0$ as follows $\rho = \sum_{k=0}^{\infty} \delta(z-k)w(k)$, with moments given by

$$\rho_n = \sum_{k=0}^{\infty} k^n w(k), \tag{7}$$

and, in particular, with 0-th moment given by

$$\rho_0 = \sum_{k=0}^{\infty} w(k). \tag{8}$$

The weights we consider in this paper satisfy the following *discrete Pearson equation*

$$\theta(k+1)w(k+1) = \sigma(k)w(k), \qquad k \in \mathbb{N}_0. \tag{9}$$

Theorem 1 (Hypergeometric symmetries)**.** *Let the weight w be subject to a discrete Pearson equation of the type (9), where the functions θ, σ are polynomials, with $\theta(0) = 0$. Then,* *(i)* *The moment matrix fulfills*

$$\theta(\Lambda)G = B\sigma(\Lambda)GB^\top.$$

(ii) *The Jacobi matrix satisfies*

$$\Pi^{-1}H\theta(J^\top) = \sigma(J)H\Pi^\top,$$

and the matrices $H\theta(J^\top)$ and $\sigma(J)H$ are symmetric.

If $N+1{:=}\deg\theta(z)$ and $M{:=}\deg\sigma(z)$, and zeros of these polynomials are $\{-b_i+1\}_{i=1}^N$ and $\{-a_i\}_{i=1}^M$ we write $\theta(z) = z(z+b_1-1)\cdots(z+b_N-1)$ and $\sigma(z) = \eta(z+a_1)\cdots(z+a_M)$. According to (8) the 0-th moment

$$\rho_0 = \sum_{k=0}^{\infty} w(k) = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_M)_k}{(b_1+1)_k\cdots(b_N+1)_k}\frac{\eta^k}{k!} = {}_MF_N\left[\begin{matrix}a_1 \cdots\cdots a_M\\ b_1\cdots\cdots b_N\end{matrix};\eta\right].$$

is the generalized hypergeometric function, where we are using the standard notationamc+amc+, see [25]. Then, according to (7), for $n\in\mathbb{N}$, the corresponding higher moments $\rho_n = \sum_{k=0}^{\infty} k^n w(k)$, are

$$\rho_n = \vartheta_\eta^n\rho_0 = \vartheta_\eta^n\left({}_MF_N\left[\begin{matrix}a_1 \cdots\cdots a_M\\ b_1\cdots\cdots b_N\end{matrix};\eta\right]\right),\quad \vartheta_\eta := \eta\frac{\partial}{\partial\eta}.$$

Given a function $f(\eta)$, we consider the Wronskian of the covector

$$\delta(f) := \begin{bmatrix} f & \vartheta f \cdots\cdots \vartheta^n f\end{bmatrix}$$

given by

$$\mathscr{W}_n(f) := \begin{vmatrix} f & \vartheta_\eta f & \vartheta_\eta^2 f & \cdots & \vartheta_\eta^n f\\ \vartheta_\eta f & \vartheta_\eta^2 f & & \iddots & \vartheta_\eta^{n+1} f\\ \vartheta_\eta^2 f & & \iddots & \iddots & \vdots\\ \vdots & \iddots & \iddots & & \vdots\\ \vartheta_\eta^n f & \vartheta_\eta^{n+1} f & \cdots & \cdots & \vartheta_\eta^{2n} f\end{vmatrix}.$$

We refer to this Wronskian as the δ-Wronskian of f. Then, we have that the Hankel determinants $\Delta_n = \det G^{[n]}$ determined by the truncations of the corresponding moment matrix are δ-Wronskians of generalized hypergeometric functions,

$$\Delta_n = \tau_n,\quad \tilde{\Delta}_n = \vartheta_\eta\tau_n, \tag{10}$$

where

$$\tau_n := \mathscr{W}_n\left({}_MF_N\left[\begin{matrix}a_1 \cdots\cdots a_M\\ b_1\cdots\cdots b_N\end{matrix};\eta\right]\right).$$

Moreover, using Proposition 1 we get

$$H_n = \frac{\tau_{n+1}}{\tau_n},\quad p_n^1 = -\vartheta_\eta\log\tau_n,\quad n\in\mathbb{N}_0. \tag{11}$$

The functions τ_k are the well known tau functions [17]. In terms of these τ-functions we have

$$\beta_n = \vartheta_\eta\log\frac{\tau_{n+1}}{\tau_n},\quad \gamma_{n+1} = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2},\quad n\in\mathbb{N}_0, \tag{12}$$

were we have used (4), (10) and (11).

Theorem 2 (Laguerre–Freud structure matrix)**.** *Let us assume that the weight w solves the discrete Pearson Equation (9) with θ,σ polynomials such that $\theta(0)=0$, $\deg\theta(z)=N+1$, $\deg\sigma(z)=M$. Then, the Laguerre–Freud structure matrix*

$$\begin{aligned}\Psi &:= \Pi^{-1}H\theta(J^\top)=\sigma(J)H\Pi^\top=\Pi^{-1}\theta(J)H=H\sigma(J^\top)\Pi^\top \\ &=\theta(J+I)\Pi^{-1}H=H\Pi^\top\sigma(J^\top-I),\end{aligned} \tag{13}$$

has only $N+M+2$ possibly nonzero diagonals ($N+1$ superdiagonals and M subdiagonals)

$$\Psi=(\Lambda^\top)^M\psi^{(-M)}+\cdots+\Lambda^\top\psi^{(-1)}+\psi^{(0)}+\psi^{(1)}\Lambda+\cdots+\psi^{(N+1)}\Lambda^{N+1},$$

for some diagonal matrices $\psi^{(k)}$. In particular, the lowest subdiagonal and highest superdiagonal are given by

$$\begin{cases}(\Lambda^\top)^M\psi^{(-M)}=\eta(J_-)^MH, & \psi^{(-M)}=\eta H\displaystyle\prod_{k=0}^{M-1}T_-^k\gamma=\eta\operatorname{diag}\Big(H_0\prod_{k=1}^{M}\gamma_k,H_1\prod_{k=2}^{M+1}\gamma_k,\dots\Big),\\ \psi^{(N+1)}\Lambda^{N+1}=H(J_-^\top)^{N+1}, & \psi^{(N+1)}=H\displaystyle\prod_{k=0}^{N}T_-^k\gamma=\operatorname{diag}\Big(H_0\prod_{k=1}^{N+1}\gamma_k,H_1\prod_{k=2}^{N+2}\gamma_k,\dots\Big).\end{cases}$$

The vector $P(z)$ of orthogonal polynomials fulfill the following structure equations

$$\theta(z)P(z-1)=\Psi H^{-1}P(z), \qquad \sigma(z)P(z+1)=\Psi^\top H^{-1}P(z). \tag{14}$$

The compatibility of the recursion relation, i.e., Eigenfunctions of the Jacobi matrix, and the recursion matrix leads to some interesting equations:

Proposition 4. *The following compatibility conditions for the Laguerre–Freud and Jacobi matrices hold*

$$[\Psi H^{-1},J]=\Psi H^{-1}, \quad [J,\Psi^\top H^{-1}] \quad =\Psi^\top H^{-1}.$$

1.3. The Toda Flows

Let us define the strictly lower triangular matrix $\Phi:=(\vartheta_\eta S)S^{-1}$.

Proposition 5.

(i) *The semi-infinite vector P fulfills*

$$\vartheta_\eta P=\Phi P. \tag{15}$$

(ii) *The Sato–Wilson equations holds*

$$-\Phi H+\vartheta_\eta H-H\Phi^\top=JH.$$

Consequently, $\Phi=-J_-$ and $n\in\mathbb{N}_0$ we have $\vartheta_\eta\log H_n=J_{n,n}$.

Moreover,

Proposition 6 (Toda)**.** *The following equations hold*

$$\Phi=(\vartheta_\eta S)S^{-1}=-\Lambda^\top\gamma, \quad (\vartheta_\eta H)H^{-1}=\beta.$$

In particular, for $n,k-1\in\mathbb{N}$, we have

$$\vartheta_\eta p^1_n = -\gamma_n, \quad \vartheta_\eta p^k_{n+k} = -\gamma_{n+k} p^{k-1}_{n+k-1}, \quad \vartheta_\eta \log H_n = \beta_n.$$

The functions $q_n := \log H_n$, $n \in \mathbb{N}$, *satisfy the Toda equations*

$$\vartheta^2_\eta q_n = \mathrm{e}^{q_{n+1}-q_n} - \mathrm{e}^{q_n - q_{n-1}}.$$

For $n \in \mathbb{N}$, *we also have* $\vartheta_\eta P_n(z) = -\gamma_n P_{n-1}(z)$.

Proposition 7. *The following Lax equation holds* $\vartheta_\eta J = [J_+, J]$. *The recursion coefficients satisfy the following Toda system*

$$\vartheta_\eta \beta_n = \gamma_{n+1} - \gamma_n, \tag{16a}$$
$$\vartheta_\eta \log \gamma_n = \beta_n - \beta_{n-1}, \tag{16b}$$

for $n \in \mathbb{N}_0$ *and* $\beta_{-1} = 0$. *Consequently, we get*

$$\vartheta^2_\eta \log \gamma_n = \gamma_{n+1} - 2\gamma_n + \gamma_{n-1}.$$

For the compatibility of (5) and (15) we obtain $\vartheta_\eta(\Pi) = [\Phi, \Pi]$. In the general case the dressed Pascal matrix Π is a lower unitriangular semi-infinite matrix that possibly has an infinite number of subdiagonals. However, for the case when the weight $w(z) = v(z)\eta^z$ satisfies the Pearson Equation (9), with v independent of η, that is $\theta(k+1)v(k+1)\eta = \sigma(k)v(k)$, the situation improves as we have the banded semi-infinite matrix Ψ that models the shift in the z variable as in (14). From the previous discrete Pearson equation, we see that $\sigma(z) = \eta\kappa(z)$ with κ, θ η-independent polynomials in z

$$\theta(k+1)v(k+1) = \eta\kappa(k)v(k).$$

Proposition 8. *Let us assume a weight w satisfying the Pearson equation* (9)*. Then, the Laguerre–Freud structure matrix Ψ given in* (13) *satisfies*

$$\vartheta_\eta(\eta^{-1}\Psi^\top H^{-1}) = [\Phi, \eta^{-1}\Psi^\top H^{-1}], \tag{17a}$$
$$\vartheta_\eta(\Psi H^{-1}) = [\Phi, \Psi H^{-1}]. \tag{17b}$$

Relations (17a) *and* (17b) *are* gauge *equivalent.*

2. Gauss Hypergeometric Weights

The choice of $\sigma(z) = \eta(z+a)(z+b)$ and $\theta(z) = z(z+c)$ results in the following Pearson equation:

$$(k+1)(k+1+c)w(k+1) = \eta(k+a)(k+b)w(k)$$

with solutions proportional to: $w(z) = \frac{(a)_z(b)_z}{(c+1)_z}\frac{\eta^z}{z!}$. As documented in [7,9], this weight function falls under the category of generalized Hahn weight of type I. The first moment is given by $\rho_0 = {}_2F_1\left[\begin{matrix} a, b \\ c+1 \end{matrix}; \eta\right]$, which is expressed as the Gauss hypergeometric function. For the specific case when $\eta = 1$, Hahn introduced these discrete orthogonal polynomials in [26]. Notably, the standard "Hahn polynomials" commonly found in the literature have parameters set as $a = \alpha + 1$, $b = -N$, and $c = -N - 1 - \beta$, where N is a positive integer.

The Gauss hypergeometric series is convergent for $|\eta| < 1$ and divergent for $|\eta| > 1$. For $|\eta| = 1$, $\eta \neq 1$, the series is absolutely convergent whenever $\operatorname{Re}(c + 1 - a - b) > 0$ and convergent and not absolutely convergent for $-1 < \operatorname{Re}(c + 1 - a - b) \leq 0$. Finally, for $\operatorname{Re}(c + 1 - a - b) = -1$, the series is convergent for $\operatorname{Re}(a + b) > \operatorname{Re} ab$ and divergent for $\operatorname{Re}(a + b) \leq \operatorname{Re} ab$. See [27].

Remark 2. *By referring to previous comments and utilizing (12), we establish that in terms of the following δ-Wronskian of the Gauss hypergeometric function*

$$\tau_n := \mathscr{W}_n\left({}_2F_1\left[\begin{matrix} a, b \\ c+1 \end{matrix}\, ; \eta\right]\right),$$

we can derive explicit expressions for the recursion coefficients:

$$\beta_n = \vartheta_\eta \log \frac{\tau_{n+1}}{\tau_n}, \quad \gamma_{n+1} = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2},$$

where n belongs to the set of non-negative integers, denoted as $\mathbb{N}_0$.

Theorem 3 (The Gauss hypergeometric Laguerre–Freud structure matrix)**.** *For a Gauss hypergeometric weight; i.e., $\sigma = \eta(z+a)(z+b)$ and $\theta = z(z+c)$, we find*

$$\begin{aligned} p^1_{n+1} &= (n+2)\beta_{n+2} + \beta_{n+1} + \frac{\eta-1}{\eta+1}\Big(\gamma_{n+3} + \gamma_{n+2} + \beta^2_{n+2} + \frac{(n+2)(n+1)}{2}\Big) \\ &\qquad + \frac{1}{\eta+1}\big(\eta ab + (\eta(a+b)-c)\beta_{n+2} + (\eta(a+b)+c)(n+2)\big), \\ \pi^{[2]}_n &= \frac{1}{1+\eta}\big((n+2)(n+1) - \eta ab - (\eta(a+b)-c)\beta_{n+2} - (\eta(a+b)+c)(n+2)\big) \\ &\qquad + \frac{1-\eta}{1+\eta}(\gamma_{n+3} + \gamma_{n+2} + \beta^2_{n+2}) - (n+2)(\beta_{n+2} + \beta_{n+1}). \end{aligned}$$

The Laguerre–Freud structure matrix is

$$\Psi = \begin{bmatrix} \eta(\gamma_1+(\beta_0+a)(\beta_0+b))H_0 & (\beta_0+\beta_1+c)H_1 & H_2 & 0 & \cdots \\ \eta(\beta_0+\beta_1+a+b)H_1 & \begin{matrix}\eta(\gamma_1+\gamma_2+(\beta_1+a)(\beta_1+b)\\+\beta_0+\beta_1+a+b)H_1\end{matrix} & (\beta_1+\beta_2+c-1)H_2 & H_3 & \ddots \\ \eta H_2 & \eta(\beta_1+\beta_2+a+b+1)H_2 & \begin{matrix}\eta(\gamma_2+\gamma_3+(\beta_2+a)(\beta_2+b)\\+2(\beta_1+\beta_2+a+b)+\pi^{[2]}_0)H_2\end{matrix} & (\beta_2+\beta_3+c-2)H_3 & \ddots \\ 0 & \eta H_3 & \eta(\beta_2+\beta_3+a+b+2)H_3 & \begin{matrix}\eta(\gamma_3+\gamma_4+(\beta_3+a)(\beta_3+b)\\+3(\beta_2+\beta_3+a+b)+\pi^{[2]}_1)H_1\end{matrix} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Proof. In this case, due to the properties of the polynomials σ and θ, the Freud–Laguerre matrix exhibits a specific diagonal structure:

$$\Psi = (\Lambda^\top)^2\psi^{(-2)} + \Lambda^\top\psi^{(-1)} + \psi^{(0)} + \psi^{(1)}\Lambda + \psi^{(2)}\Lambda^2.$$

This structure consists of two subdiagonals, two superdiagonals, and the main diagonal.

On one hand, starting with the equation $\Psi = \sigma(J)H\Pi^\top$, we can derive the Laguerre–Freud structure matrix as follows:

$$\begin{aligned} \Psi = {} & \underbrace{\eta\Lambda^\top\gamma\Lambda^\top\gamma H}_{\text{second subdiagonal}} + \underbrace{\eta(\Lambda^\top\gamma\Lambda^\top\gamma HD\Lambda + \Lambda^\top\gamma(\beta+b)H + (\beta+a)\Lambda^\top\gamma H)}_{\text{first subdiagonal}} \\ & + \underbrace{\eta\big(\Lambda^\top\gamma\Lambda H + \Lambda\Lambda^\top\gamma H + (\beta+a)(\beta+b)H + (\Lambda^\top\gamma(\beta+b) + (\beta+a)\Lambda^\top\gamma)HD\Lambda}_{\text{main diagonal}} \\ & + \underbrace{\Lambda^\top\gamma\Lambda^\top\gamma H\pi^{[2]}\Lambda^2\big)}_{\text{main diagonal}} \end{aligned}$$

$$+\eta\Big(\underbrace{\big((\beta+a)(\beta+b)+\Lambda^\top\gamma\Lambda+\Lambda\Lambda^\top\gamma\big)HD\Lambda+(\beta+a)\Lambda H+\Lambda(\beta+b)H}_{\text{first superdiagonal}}$$
$$\underbrace{+\Lambda^\top(\beta+b)H\pi^{[2]}\Lambda^2+(\beta+a)\Lambda^\top H\pi^{[2]}\Lambda^2+\Lambda^\top\gamma\Lambda^\top\gamma H\pi^{[3]}\Lambda^3\Big)}_{\text{first superdiagonal}}$$
$$+\eta\Big(\underbrace{\Lambda^\top\gamma\Lambda^\top\gamma H\pi^{[4]}\Lambda^4+\big(\Lambda^\top\gamma(\beta+b)+(\beta+a)\Lambda^\top\gamma\big)H\pi^{[3]}\Lambda^3}_{\text{second superdiagonal}} \tag{18}$$
$$\underbrace{+\big(\Lambda^\top\gamma\Lambda+\Lambda\Lambda^\top\gamma+(\beta+a)(\beta+b)\big)H\pi^{[2]}\Lambda^2+\big(\Lambda(\beta+b)+(\beta+a)\Lambda\big)HD\Lambda+\Lambda^2 H\Big)}_{\text{second superdiagonal}}.$$

On the other hand, by using the equation $\Psi=\Pi^{-1}H\theta(J^\top)$, we can deduce:

$$\Psi=\underbrace{H(\Lambda^\top)^2+\Lambda^\top DH\Lambda^\top(\beta+T_-\beta+c)+(\Lambda^\top)^2\pi^{[-2]}H(\gamma+T_+\gamma+\beta^2+c\beta)}_{\text{second subdiagonal}}$$
$$\underbrace{-(\Lambda^\top)^3\pi^{[-3]}H(\beta+T_-\beta+c)\gamma\Lambda+(\Lambda^\top)^4\pi^{[-4]}H(T_-\gamma)\gamma\Lambda^2}_{\text{second subdiagonal}}$$
$$+\underbrace{H\Lambda^\top(\beta+T_-\beta+c)-\Lambda^\top DH(\gamma+T_+\gamma+\beta^2+c\beta)}_{\text{first subdiagonal}}$$
$$+\underbrace{(\Lambda^\top)^2\pi^{[-2]}H(\beta+T_-\beta+c)\gamma\Lambda-(\Lambda^\top)^3\pi^{[-3]}H(T_-\gamma)\gamma\Lambda^2}_{\text{first subdiagonal}}$$
$$+\underbrace{H(\gamma+T_+\gamma+\beta^2+c\beta)-\Lambda^\top DH(\beta+T_-\beta+c)\gamma\Lambda+(\Lambda^\top)^2\pi^{[-2]}H(T_-\gamma)\gamma\Lambda^2}_{\text{main diagonal}}$$
$$+\underbrace{H(\beta+T_-\beta+c)\gamma\Lambda-\Lambda^\top DH(T_-\gamma)\gamma\Lambda^2}_{\text{first superdiagonal}}+\underbrace{H(T_-\gamma)\gamma\Lambda^2}_{\text{second superdiagonal}} \tag{19}$$

From (18), we can extract the first two subdiagonals of Ψ, which are as follows:

$$\psi^{(-2)}=\eta H\gamma T_-\gamma,\qquad \psi^{(-1)}=\eta\gamma T_+\gamma T_+H+\gamma(\beta+b)H+(T_-\beta+a)\gamma H,$$

From (19), we can determine the first two superdiagonals of Ψ, which are as follows:

$$\psi^{(1)}=(\beta+T_-\beta+c)\gamma-(T_+D)(T_+H)\gamma T_+\gamma,\qquad \psi^{(2)}=H(T_-\gamma)\gamma.$$

We can derive an expression for the main diagonal that is independent of $\pi^{[+2]}$ or $\pi^{[-2]}$ by equating the terms associated with the main diagonal in the two previous expressions, namely (18) and (19). This yields:

$$(\eta-1)(T_+\gamma+\gamma+\beta^2)+\beta[\eta(a+b)-c]+\eta ab$$
$$+(\eta+1)(T_+D)[T_+\beta+\beta]+(T_+D)[\eta(a+b)+c]=T_+^2(\pi^{[-2]}-\eta\pi^{[2]}).$$

Referring back to (6), which states that $\pi_n^{[\pm 2]}=\frac{(n+2)(n+1)}{2}\mp(n+1)\beta_{n+1}\mp p_{n+1}^1$, and substituting this into the previous expression, we can eliminate p_{n+1}^1. We can then replace this result in the expression for $\pi^{[2]}$, ensuring that the main diagonal is no longer dependent on it. Examining each term in the first equation and isolating p_{n+1}^1, we obtain

$$p^1_{n+1} = (n+2)\beta_{n+2} + \beta_{n+1} + \frac{\eta-1}{\eta+1}\Big(\gamma_{n+3} + \gamma_{n+2} + \beta^2_{n+2} + \frac{(n+2)(n+1)}{2}\Big) \\ + \frac{1}{\eta+1}\big(\eta ab + (\eta(a+b)-c)\beta_{n+2} + (\eta(a+b)+c)(n+2)\big),$$

and the diagonal matrix entries of $\pi^{[2]}$ are

$$\pi^{[2]}_n = \frac{1}{1+\eta}\big((n+2)(n+1) - \eta ab - (\eta(a+b)-c)\beta_{n+2} - (\eta(a+b)+c)(n+2)\big) \\ + \frac{1-\eta}{1+\eta}(\gamma_{n+3} + \gamma_{n+2} + \beta^2_{n+2}) - (n+2)(\beta_{n+2} + \beta_{n+1}).$$

Simplifying further, we obtain the Laguerre–Freud matrix as

$$\Psi = \eta(\Lambda^\top)^2 T_-^2 H + \eta\Lambda^\top\big(a+b+T_+D+\beta+T_-\beta\big)T_-H \\ + \eta\big(T_+\gamma + \gamma + (\beta+a)(\beta+b) + T_+(D)(a+b+T_+\beta+\beta) + T_+^2\pi^{[2]})\big)H \\ + (c - T_+D + \beta + T_-\beta)T_-H\Lambda + T_-^2 H\Lambda^2.$$

□

Now, let's investigate the compatibility condition $[\Psi H^{-1}, J] = \Psi H^{-1}$.

Theorem 4 (Laguerre–Freud equations for Gauss hypergeometric)**.** *The Gauss hypergeometric recursion coefficients satisfy the following Laguerre–Freud relations*

$$(\eta^2-1)\big((\beta_{n+1}+\beta_n)\gamma_{n+1} - (\beta_{n-1}+\beta_n)\gamma_n\big) \\ + \eta\big(\beta_n(2\beta_n + a + b + c) + 2(\gamma_{n+1}+\gamma_n) + n(a+b-c+n-1) + ab\big) \\ + (\eta+1)\big(\big(\eta(a+b) - c - (\eta+1)n\big)(\gamma_{n+1}-\gamma_n) + (\eta+1)\gamma_n\big) = 0, \tag{20a}$$

$$(\eta+1)((n-1)\beta_n + (n+1)\beta_{n+1}) + (\eta-1)(\gamma_{n+2} - \gamma_n + \beta^2_{n+1} - \beta^2_n + n) \\ + (\eta(a+b)-c)(\beta_{n+1}-\beta_n) + \eta(a+b) + c = 0. \tag{20b}$$

Proof. We analyze the compatibility $[\Psi H^{-1}, J] = \Psi H^{-1}$ by diagonals. In both sides of the equation, we find matrices whose only non-zero diagonals are the main diagonal, the first and second subdiagonals, and the first and second superdiagonals. Equating the non-zero diagonals of both matrices, two identities for the second superdiagonal and subdiagonal are obtained. From the remaining diagonals, we obtain the two Laguerre–Freud equations (we obtain the same equality from the first subdiagonal and from the first superdiagonal). Firstly, by simplifying, we obtain that

$$\Psi H^{-1} = \eta(\Lambda^\top)^2(T_-\gamma)\gamma + \eta\Lambda^\top\gamma(T_+(D) + (\beta+b) + T_-(\beta+a)) \\ + \frac{\eta}{\eta+1}\big(2(T_+\gamma + \gamma + \beta^2) + (a+b+c)\beta + ab + T_+D(a+b-c) + T_+DT_+^2D\big) \\ + (\beta + T_-\beta + c - T_+(D))\Lambda + \Lambda^2.$$

From the main diagonal, clearing, we obtain

$$(1-\eta)\beta_{n-1}\gamma_n + (\eta-1)\beta_{n+1}\gamma_{n+1} + \beta_n(\frac{\eta}{\eta+1}(2\beta_n + a + b + c) + (\eta-1)(\gamma_{n+1} - \gamma_n)) \\ + (\frac{\eta}{\eta+1}(2(\gamma_{n+1}+\gamma_n) - nc + n(a+b+n-1) + ab) - (\eta+1)(n(\gamma_n - \gamma_{n+1}) - \gamma_n) \\ + (\eta(a+b)-c)(\gamma_{n+1}-\gamma_n)) = 0$$

and we get Equation (20a).

We obtain the following expressions from the first superdiagonal and the first subdiagonal:

$$\begin{aligned}((\eta+1)(\beta_n+\beta_{n+1}+n(\beta_{n+1}-\beta_n))+(\eta-1)(\gamma_{n+2}-\gamma_n+\beta_{n+1}^2-\beta_n^2+n)\\+c(1+\beta_n-\beta_{n+1})+\eta(a+b)(1+\beta_{n+1}-\beta_n)=0\end{aligned}$$

which results in Equation (20b). □

We now proceed with the compatibility condition $[\Psi H^{-1}, J_-] = \vartheta_\eta(\Psi H^{-1})$. Recall that $J_-:=\Lambda^\top\gamma$ and $\vartheta_\eta = \eta\frac{\mathrm{d}}{\mathrm{d}\eta}$. As we will see, we do not obtain any further equations beyond those already obtained in Theorem 4.

Proposition 9. *The recursion coefficients for the Gauss hypergeometric discrete orthogonal polynomials satisfy the following Laguerre–Freud relations:*

$$\vartheta_\eta(\beta_n+\beta_{n+1}+c-n)=\gamma_{n+2}-\gamma_n, \tag{21a}$$

$$\begin{aligned}\vartheta_\eta\Big(\frac{\eta}{\eta+1}(2(\gamma_{n+1}+\gamma_n+\beta_n^2)+c(\beta_n-n)+n(n-1)+(a+b)(\beta_n+n)+ab)\Big)=\\ \gamma_{n+1}(\beta_n+\beta_{n+1}+c-n)-\gamma_n(\beta_{n-1}+\beta_n+c-(n-1)),\end{aligned} \tag{21b}$$

$$\begin{aligned}&\vartheta_\eta(\eta\gamma_{n+1}(n+a+b+\beta_n+\beta_{n+1}))=\\ &\frac{\eta}{\eta+1}\gamma_{n+1}(2(\gamma_{n+2}-\gamma_n+\beta_{n+1}^2-\beta_n^2)+(a+b+c)(\beta_{n+1}-\beta_n)+2n+(a+b-c))\end{aligned} \tag{21c}$$

$$\vartheta_\eta(\eta\gamma_{n+1}\gamma_{n+2})=\eta\gamma_{n+1}\gamma_{n+2}(\beta_{n+2}-\beta_n+1). \tag{21d}$$

Proof. From the diagonals of $[\Psi H^{-1}, J_-] = \vartheta_\eta(\Psi H^{-1})$ we get

(i) From the first superdiagonal we obtain (21a)
(ii) From the main diagonal cleaning up we get (21b).
(iii) From the first subdiagonal we get, symplifying (21c).
(iv) Finally, from the second subdiagonal we get (21d).

□

Remark 3. *We observe that* (21a) *follow from the Toda Equations* (16a)*, and* (21d) *follow from Toda Equation* (16b)*. Moreover, the two remaining equations (from the main diagonal and the first subdiagonal) coincide with those presented in Theorem* 4*.*

Remark 4. *Dominici in* [7] *(Theorem 4) found the following Laguerre–Freud equations*

$$\begin{aligned}(1-\eta)\nabla(\gamma_{n+1}+\gamma_n)&=\eta v_n\nabla(\beta_n+n)-u_n\nabla(\beta_n-n),\\ \Delta\nabla(u_n-\eta v_n)\gamma_n&=u_n\nabla(\beta_n-n)+\nabla(\gamma_{n+1}+\gamma_n),\end{aligned}$$

with $u_n:=\beta_n+\beta_{n+1}-n+c+1$ *and* $v_n:=\beta_n+\beta_{n-1}+n-1+a+b$*. Therefore, the first one is of type* $\gamma_{n+1}=F_1(n,\gamma_n,\gamma_{n-1},\beta_n,\beta_{n-1})$*, of length two, and the second of the form* $\beta_{n+1}=F_2(n,\gamma_{n+1},\gamma_n,\gamma_{n-1},\beta_n,\beta_{n-1},\beta_{n-2})$*, is of length three. .*

Remark 5. *Filipuk and Van Assche in* [14] *(Equations (3.6) and (3.9)) introduce new non local variables* (x_n, y_n)*,*

$$\begin{aligned}\beta_n&=x_n+\frac{n+(n+a+b)\eta-c-1}{1-\eta},\\ \frac{1-\eta}{\eta}\gamma_n&=y_n+\sum_{k=0}^{n-1}x_k+\frac{n(n+a+b-c-2)}{1-\eta}.\end{aligned}$$

Then, in [14] *(Theorem 3.1) Equations (3.13) and (3.14) for* (x_n, y_n) *are found, of length* 0 *and* 1 *respectively, in the new variables. Recall that these new variables are non-local and involve all* *the previous recursion coefficients. In this respect, the meaning of length is not so clear. The nice feature in this case is that [14] (Equations (3.13) and (3.14)) are discrete Painlevé equations that, combined with the Toda equations, lead to a differential system for the new variables x_n and y_n that, after suitable transformation, can be reduced to Painlevé VI σ-equation. Recently, it has been shown in [24] that this system is equivalent to $dP(D_4^{(1)}/D_4^{(1)})$, known as the difference Painlevé V.*

3. Conclusions and Outlook

In their studies of integrable systems and orthogonal polynomials, Adler and van Moerbeke have thoroughly used the Gauss–Borel factorization of the moment matrix (see [28–30]). This strategy has been extended and applied by us in different contexts, such as CMV orthogonal polynomials, matrix orthogonal polynomials, multiple orthogonal polynomials, and multivariate orthogonal polynomials (see [31–33]). For a general overview, see [34].

Recently, we extended those ideas to the discrete world (see [16]). In particular, we applied that approach to the study of the consequences of the Pearson equation on the moment matrix and Jacobi matrices. For that description, a new banded matrix is required, the Laguerre–Freud structure matrix, which encodes the Laguerre–Freud relations for the recurrence coefficients. We have also found that the contiguous relations fulfilled by generalized hypergeometric functions, which determine the moments of the weight, describe a discrete Toda hierarchy known as the Nijhoff–Capel equation (see [18]). In [19], we study the role of Christoffel and Geronimus transformations in the description of the mentioned contiguous relations, as well as the use of Geronimus–Christoffel transformations to characterize the shifts in the spectral independent variable of the orthogonal polynomials.

In this paper, we delve deeper into that program and further explore the discrete semiclassical cases. We find Laguerre–Freud relations for the recursion coefficients of Gauss hypergeometric discrete orthogonal polynomials. We observe that a solution of the Laguerre–Freud-type nonlinear equations for the recursion coefficients is provided by the τ-function, which is defined as a Wronskian of the Gauss hypergeometric functions, respectively. Notice that in [20], we presented a study similar to the one in this paper for the hypergeometric cases ${}_1F_2, {}_2F_2$ and ${}_3F_2$.

For the future, we will extend these techniques to multiple discrete orthogonal polynomials [35] and their relations with the transformations presented in [36], as well as quadrilateral lattices [37,38].

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Matrix factorizations for the generalized Charlier and Meixner orthogonal polynomials

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A B S T R A C T

The Cholesky factorization of the moment matrix is considered for the generalized Charlier and generalized Meixner discrete orthogonal polynomials. For the generalized Charlier, we present an alternative derivation of the Laguerre–Freud relations found by Smet and Van Assche. Third-order and second-order nonlinear ordinary differential equations are found for the recursion coefficient γ_n, that happen to be forms of the Painlevé deg-$\mathrm{P_V}$ in disguise. New Laguerre–Freud relations are also found for the generalized Meixner case, which are compared with those of Smet and Van Assche.

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Generalized Meixner orthogonal polynomials

1. Introduction

Discrete orthogonal polynomials constitute a distinguished and well-established field within the theory of orthogonal polynomials. Classical monographs have been dedicated to this particular class of orthogonal polynomials. For instance, the classical case is meticulously explored in [47], while the Riemann–Hilbert problem has been considered for investigating asymptotics and applications, as discussed in [14]. For a comprehensive and authoritative discussion on this subject, please refer to [33,34,15,51].

In addition to classical discrete orthogonal polynomials, there is a significant body of literature addressing semiclassical discrete orthogonal polynomials. These involve scenarios where a discrete Pearson equation is satisfied by the weight function. A thorough treatment of this topic can be found in the works of [25,26,23,24], along with relevant references therein that provide a comprehensive overview.

For certain specific types of weight functions of generalized Charlier and Meixner types, extensive research has been carried out to study the corresponding Freud–Laguerre type equations governing the coefficients of the three-term recurrence. To delve into this area, you can explore works such as [20,28–30,50].

To gain a comprehensive understanding of the subject matter discussed in this paper, please refer to [16] for insights into Laguerre–Freud equations pertaining to D-semiclassical linear functionals, and consult [17,44] for a detailed exploration of discrete semiclassical linear functionals.

In our work presented in [43], we employed matrix factorizations to deal with discrete orthogonal polynomials. To be more precise, we used the Cholesky factorization of the moment matrix to investigate discrete orthogonal polynomials denoted as $\{P_n(x)\}_{n=0}^{\infty}$ on a uniform lattice. Specifically, we focused on semiclassical discrete orthogonal polynomials, where the weight functions are constrained by a discrete Pearson equation, resulting in moments expressed in terms of generalized hypergeometric functions.

In this study, we introduced a banded semi-infinite matrix called the Laguerre–Freud structure matrix, which serves as a model for the shifts by ± 1 in the independent variable of the sequence of orthogonal polynomials, $\{P_n(x)\}_{n=0}^{\infty}$. Furthermore, our research in that paper demonstrated that the contiguous relationships associated with the generalized hypergeometric functions correspond to symmetries within the corresponding moment matrix. Moreover, we established that the 3D Nijhoff–Capel discrete Toda lattice, as discussed in [46] and [32], describes the contiguous shifts for the squared norms of the orthogonal polynomials.

In our subsequent work presented in [41], we provided an interpretation of the contiguous transformations for the generalized hypergeometric functions in terms of simple Christoffel and Geronimus transformations. By utilizing Geronimus–Uvarov perturbations, we derived determinantal expressions for the shifted orthogonal polynomials.

Additionally, in our investigation detailed in [27], we explored three distinct hypergeometric families along with their associated Laguerre–Freud equations.

In this paper, using matrix factorization techniques, we delve into the realm of discrete orthogonal polynomials, specifically focusing on the generalized Charlier and generalized Meixner. Our investigation centers on the analysis of the Laguerre–Freud structure matrix denoted as Ψ. We aim to deduce a set of nonlinear equations for the coefficients β_n, γ_n in the context of the three-term recursion relations, given by the equation

$$zP_n(z) = P_{n+1}(z) + \beta_n P_n(z) + \gamma_n P_{n-1}(z), \qquad n \in \mathbb{N}_0, \quad P_{-1}(z) = 0, \quad P_0(z) = 1,$$

governing the monic orthogonal polynomial sequence. These nonlinear recurrence relations for the recursion coefficients can be expressed as follows:

$$\begin{aligned}\gamma_{n+1} &= F_1(n, \gamma_n, \gamma_{n-1}, \dots, \beta_n, \beta_{n-1} \dots),\\ \beta_{n+1} &= F_2(n, \gamma_{n+1}, \gamma_n, \dots, \beta_n, \beta_{n-1}, \dots),\end{aligned}$$

where F_1 and F_2 are certain functions. This type of relationship has been coined as Laguerre–Freud relations, as introduced by Magnus in reference to [35] and [31], and further explored in his works [36–39].

The literature contains several papers that discuss Laguerre–Freud relations in the context of the generalized Charlier, generalized Meixner, and Gauss hypergeometric cases. These can be found in references such as [50,20,28–30,23].

In both instances, it is noteworthy that the τ-function, defined as the Wronskian of the modified Bessel and Kummer functions, serves as a solution for each set of nonlinear Laguerre–Freud equations governing the recursion coefficients.

The paper's layout is organized as follows: Following this introduction, Section 2 is dedicated to the study of generalized Charlier orthogonal polynomials. Specifically, Theorem 2.5 presents the tridiagonal Laguerre–Freud structure matrix. In Proposition 2.11, we identify a set of Laguerre–Freud equations, which we demonstrate to be equivalent to those outlined in [50]. Within this section, we also derive third-order and second-order ordinary differential equations for the coefficient γ_n, as detailed in Theorems 2.7 and 2.17, respectively. It is worth noting that these equations, as pointed out by Peter Clarkson, can be considered as a disguised Painlevé equation deg-$\mathrm{P_V}$ [21].

Moving on to Section 3, we shift our focus to the examination of generalized Meixner orthogonal polynomials. In Theorem 3.2, we introduce the tetradiagonal Laguerre–Freud structure matrix. After a series of simplifications, we present the corresponding Laguerre–Freud relations in Theorem 3.7. Furthermore, this section includes a comparative discussion with the Laguerre–Freud equations put forth by Smet & Van Assche in [50].

To conclude this introduction, we provide a concise summary of the fundamental aspects concerning discrete orthogonal polynomials, followed by a brief overview of the pertinent details found in [43].

1.1. Basics on orthogonal polynomials

For a linear functional $\rho_z \in \mathbb{C}^*[z]$, the corresponding moment matrix is defined as:

$$G = (G_{n,m}), \qquad G_{n,m} = \rho_{n+m}, \qquad \rho_n = \left\langle \rho_z, z^n \right\rangle, \qquad n, m \in \mathbb{N}_0 := \{0, 1, 2, \dots\}.$$

Here, ρ_n represents the n-th moment of the linear functional ρ_z. If the moment matrix satisfies the condition where all its truncations, which are Hankel matrices, have the property $G_{i+1,j} = G_{i,j+1}$,

$$G^{[k]} = \begin{bmatrix} G_{0,0} & \cdots\cdots & G_{0,k-1} \\ \vdots & & \vdots \\ G_{k-1,0} & \cdots\cdots & G_{k-1,k-1} \end{bmatrix} = \begin{bmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & \rho_2 & & \iddots & \rho_k \\ \rho_2 & & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & & \vdots \\ \rho_{k-1} & \rho_k & \cdots & \cdots & \rho_{2k-2} \end{bmatrix}.$$

If the Hankel determinants $\varDelta_k := \det G^{[k]}$ are all non-zero, i.e., $\varDelta_k \neq 0$ for all $k \in \mathbb{N}_0$, then we obtain monic polynomials described by:

$$P_n(z) = z^n + p_n^1 z^{n-1} + \cdots + p_n^n, \quad n \in \mathbb{N}_0, \tag{1}$$

where $p_0^1 = 0$. These polynomials satisfy the orthogonality relations:

$$\left\langle \rho, P_n(z) z^k \right\rangle = 0, \quad k \in \{0, \dots, n-1\}, \qquad \left\langle \rho, P_n(z) z^n \right\rangle = H_n \neq 0,$$

and the sequence $\{P_n(z)\}_{n=0}^{\infty}$ is a sequence of monic orthogonal polynomials, i.e.:

$$\left\langle \rho, P_n(z) P_m(z) \right\rangle = \delta_{n,m} H_n$$

for $n, m \in \mathbb{N}_0$. The symmetric bilinear form $\langle F, G \rangle_\rho := \langle \rho, FG \rangle$ is structured in a way that the moment matrix serves as the Gram matrix of this bilinear form, and $\langle P_n, P_m \rangle_\rho := \delta_{n,m} H_n$.

We introduce the function $\chi(z) := \begin{bmatrix} 1 & z & z^2 & \cdots \end{bmatrix}^\top$. The moment matrix can be expressed as $G = \left\langle \rho, \chi \chi^\top \right\rangle$. Notably, χ serves as an eigenvector of the "shift matrix," denoted as $\Lambda \chi(z) = z \chi(z)$, where Λ is defined as:

$$\Lambda := \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

As a result, it follows that $\Lambda G = G\Lambda^\top$, and therefore, the moment matrix takes the form of a Hankel matrix.

Given the moment matrix's symmetry, it can be factorized using the Borel–Gauss factorization, which takes the form of a Cholesky factorization

$$G = S^{-1} H S^{-\top}.$$

In this factorization, S is a lower unitriangular matrix represented by:

$$S = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ S_{1,0} & 1 & 0 & \ddots \\ S_{2,0} & S_{2,1} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Furthermore, the matrix $H = \operatorname{diag}(H_0, H_1, \dots)$ is a diagonal matrix, with $H_k \neq 0$, for $k \in \mathbb{N}_0$. The Cholesky factorization remains valid as long as the principal minors of the moment matrix, i.e., the Hankel determinants Δ_k, do not equal zero.

The entries $P_n(z)$ in the expression:

$$P(z) := S\chi(z), \tag{2}$$

constitute the monic orthogonal polynomials sequence associated with the functional ρ.

Proposition 1.1. *We can express the following determinantal relationships:*

$$H_n = \frac{\Delta_{n+1}}{\Delta_n}, \quad p_n^1 = -\frac{\tilde{\Delta}_n}{\Delta_n},$$

where

$$\Delta_n := \begin{vmatrix} \rho_0 & \cdots & \rho_{n-2} & \rho_{n-1} \\ \vdots & \iddots & \iddots & \vdots \\ \rho_{n-2} & \rho_{n-1} & \cdots & \rho_{2n-3} \\ \rho_{n-1} & \cdots & \rho_{2n-3} & \rho_{2n-2} \end{vmatrix}, \quad \tilde{\Delta}_n := \begin{vmatrix} \rho_0 & \cdots & \rho_{n-2} & \rho_{n-1} \\ \vdots & \iddots & \iddots & \vdots \\ \rho_{n-2} & \rho_{n-1} & \cdots & \rho_{2n-3} \\ \rho_n & \cdots & \rho_{2n-2} & \rho_{2n-1} \end{vmatrix}.$$

We introduce the lower Hessenberg semi-infinite matrix

$$J = S\Lambda S^{-1} \tag{3}$$

that has the vector $P(z)$ as eigenvector with eigenvalue z, $JP(z) = zP(z)$. The Hankel condition $\Lambda G = G\Lambda^\top$ and the Cholesky factorization give $JH = (JH)^\top = HJ^\top$. Since the Hessenberg matrix JH is symmetric, the Jacobi matrix J becomes tridiagonal. The Jacobi matrix J described in Equation (3) is formulated as follows:

$$J = \begin{bmatrix} \beta_0 & 1 & 0 & \cdots \\ \gamma_1 & \beta_1 & 1 & \ddots \\ 0 & \gamma_2 & \beta_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

The eigenvalue equation $JP = zP$ manifests as a three-term recursion relation: $zP_n(z) = P_{n+1}(z) + \beta_n P_n(z) + \gamma_n P_{n-1}(z)$, and when equipped with the initial conditions $P_{-1} = 0$ and $P_0 = 1$, it fully specifies the sequence of orthogonal polynomials $\{P_n(z)\}_{n=0}^{\infty}$ in terms of the recursion coefficients β_n and γ_n. For $n \in \mathbb{N}_0$, the expressions for the recursion coefficients, derived from the Hankel determinants, are as follows:

$$\begin{aligned} \beta_n &= p_n^1 - p_{n+1}^1 = -\frac{\tilde{\varDelta}_n}{\varDelta_n} + \frac{\tilde{\varDelta}_{n+1}}{\varDelta_{n+1}}, \\ \gamma_{n+1} &= \frac{H_{n+1}}{H_n} = \frac{\varDelta_{n+1}\varDelta_{n-1}}{\varDelta_n^2}. \end{aligned} \tag{4}$$

To facilitate future use, we introduce the following diagonal matrices: $\gamma := \operatorname{diag}(\gamma_1, \gamma_2, \dots)$ and $\beta := \operatorname{diag}(\beta_0, \beta_1, \dots)$. Additionally, we define: $J_- := \Lambda^\top \gamma$ and $J_+ := \beta + \Lambda$. This establishes a convenient split for J as: $J = \Lambda^\top \gamma + \beta + \Lambda = J_- + J_+$. In a more general context, when dealing with any semi-infinite matrix A, we'll denote it as $A = A_- + A_+$, where A_- represents a strictly lower triangular matrix, and A_+ is an upper triangular matrix. Furthermore, we'll use A_0 to refer to the diagonal part of matrix A.

The lower Pascal matrix is defined as:

$$B = (B_{n,m}), \quad B_{n,m} := \begin{cases} \dbinom{n}{m}, & n \geq m, \\ 0, & n < m. \end{cases}$$

This matrix is instrumental in the equation: $\chi(z+1) = B\chi(z)$. Furthermore, the inverse of the lower Pascal matrix is expressed as:

$$B^{-1} = (\tilde{B}_{n,m}), \quad \tilde{B}_{n,m} := \begin{cases} (-1)^{n+m}\dbinom{n}{m}, & n \geq m, \\ 0, & n < m. \end{cases}$$

This allows us to represent: $\chi(z-1) = B^{-1}\chi(z)$.

The explicit forms of the lower Pascal matrix and its inverse are:

$$B=\begin{bmatrix}1 & 0 & \cdots & & & & \\ 1 & 1 & 0 & \cdots & & & \\ 1 & 2 & 1 & 0 & \cdots & & \\ 1 & 3 & 3 & 1 & 0 & \cdots & \\ 1 & 4 & 6 & 4 & 1 & 0 & \cdots \\ 1 & 5 & 10 & 10 & 5 & 1 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix}, \qquad B^{-1}=\begin{bmatrix}1 & 0 & \cdots & & & & \\ -1 & 1 & 0 & \cdots & & & \\ 1 & -2 & 1 & 0 & \cdots & & \\ -1 & 3 & -3 & 1 & 0 & \cdots & \\ 1 & -4 & 6 & -4 & 1 & 0 & \cdots \\ -1 & 5 & -10 & 10 & -5 & 1 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix}.$$

In this context, we introduce the concept of dressed Pascal matrices: $\Pi := SBS^{-1}$ and $\Pi^{-1} := SB^{-1}S^{-1}$, which are connection matrices, i.e.,

$$P(z+1)=\Pi P(z), \quad P(z-1)=\Pi^{-1}P(z). \tag{5}$$

The lower Pascal matrix can be represented in terms of its subdiagonal structure as follows:

$$B^{\pm 1}=I\pm\Lambda^\top D+\left(\Lambda^\top\right)^2 D^{[2]}\pm\left(\Lambda^\top\right)^3 D^{[3]}+\cdots.$$

Here, we define: $D=\operatorname{diag}(1,2,3,\dots)$ and $D^{[k]}=\frac{1}{k}\operatorname{diag}\left(k^{(k)},(k+1)^{(k)},(k+2)^{(k)},\cdots\right)$, in terms of the falling factorials: $x^{(k)}=x(x-1)(x-2)\cdots(x-k+1)$. This means:

$$D^{[k]}_n=\frac{(n+k)\cdots(n+1)}{k}, \quad k\in\mathbb{N}, \quad n\in\mathbb{N}_0.$$

The lower unitriangular factor can also be expressed in terms of its subdiagonals as: $S=I+\Lambda^\top S^{[1]}+\left(\Lambda^\top\right)^2 S^{[2]}+\cdots$ Here, we define $S^{[k]}:=\operatorname{diag}\left(S^{[k]}_0,S^{[k]}_1,\dots\right)$.

From Equation (2), it is evident that a connection exists between these subdiagonal entries and the coefficients of the orthogonal polynomials described in Equation (1). This connection can be expressed as: $S^{[k]}_n=p^k_{n+k}$. We will employ the "shift operators," denoted as $T_\pm$, which act on diagonal matrices as follows:

$$T_-\operatorname{diag}(a_0,a_1,\dots):=\operatorname{diag}(a_1,a_2,\dots), \qquad T_+\operatorname{diag}(a_0,a_1,\dots):=\operatorname{diag}(0,a_0,a_1,\dots).$$

These shift operators possess crucial properties, particularly when applied to any diagonal matrix $A=\operatorname{diag}(A_0,A_1,\dots)$, as expressed in the following equations:

$$\Lambda A=(T_-A)\Lambda, \qquad A\Lambda=\Lambda(T_+A), \qquad A\Lambda^\top=\Lambda^\top(T_-A), \qquad \Lambda^\top A=(T_+A)\Lambda^\top. \tag{6}$$

Remark 1.2. It's worth noting that the standard notation, as seen in [47], for the differences within a sequence $\{f_n\}_{n\in\mathbb{N}_0}$ is given by:

$$\Delta f_n:=f_{n+1}-f_n, \quad n\in\mathbb{N}_0, \quad \nabla f_n:=f_n-f_{n-1}, \quad n\in\mathbb{N}.$$

Additionally, by convection $\nabla f_0 = f_0$, and these notations can be related to the shift operators using the following expressions:

$$T_- = I + \Delta, \quad T_+ = I - \nabla.$$

Using these shift operators, we can derive the following expressions:

$$2D^{[2]} = (T_- D)D, \quad 3D^{[3]} = (T_-^2 D)(T_- D)D = 2(T_- D^{[2]})D = 2D^{[2]}(T_-^2 D). \tag{7}$$

Proposition 1.3. *The inverse matrix S^{-1} of matrix S can be expanded as follows:*

$$S^{-1} = I + \Lambda^\top S^{[-1]} + \left(\Lambda^\top\right)^2 S^{[-2]} + \cdots .$$

The subdiagonals $S^{[-k]}$ are expressed in terms of the subdiagonals of S. Specifically:

$$\begin{aligned}
S^{[-1]} &= -S^{[1]}, \\
S^{[-2]} &= -S^{[2]} + (T_- S^{[1]})S^{[1]}, \\
S^{[-3]} &= -S^{[3]} + (T_- S^{[2]})S^{[1]} + (T_-^2 S^{[1]})S^{[2]} - (T_-^2 S^{[1]})(T_- S^{[1]})S^{[1]}.
\end{aligned}$$

Remark 1.4. The expansions for the dressed Pascal matrices are as follows:

$$\Pi^{\pm 1} = I + \Lambda^\top \pi^{[\pm 1]} + (\Lambda^\top)^2 \pi^{[\pm 2]} + \cdots ,$$

where $\pi^{[\pm n]} = \operatorname{diag}(\pi_0^{[\pm n]}, \pi_1^{[\pm n]}, \dots)$.

Proposition 1.5 *(The dressed Pascal matrix coefficients). The specific values for $\pi^{[\pm n]}$ are:*

$$\begin{aligned}
\pi_n^{[\pm 1]} &= \pm(n+1), \\
\pi_n^{[\pm 2]} &= \frac{(n+2)(n+1)}{2} \pm p_{n+2}^1 (n+1) \mp (n+2) p_{n+1}^1 \\
&= \frac{(n+2)(n+1)}{2} \mp (n+1)\beta_{n+1} \mp p_{n+1}^1, \\
\pi_n^{[\pm 3]} &= \pm \frac{(n+3)(n+2)(n+1)}{3} + \frac{(n+2)(n+1)}{2} p_{n+3}^1 - \frac{(n+3)(n+2)}{2} p_{n+1}^1 \\
&\quad \pm (n+1) p_{n+3}^2 \mp (n+3) p_{n+2}^2 \pm (n+3) p_{n+2}^1 p_{n+1}^1 \mp (n+2) p_{n+3}^1 p_{n+1}^1.
\end{aligned} \tag{8}$$

Furthermore, the following relations hold:

$$\begin{aligned}
\pi^{[1]} + \pi^{[-1]} &= 0, \\
\pi^{[2]} + \pi^{[-2]} &= 2D^{[2]}, \\
\pi^{[3]} + \pi^{[-3]} &= 2\left((T_-^2 S^{[1]})D^{[2]} - (T_- D^{[2]})S^{[1]}\right).
\end{aligned} \tag{9}$$

1.2. Discrete orthogonal polynomials and Pearson equation

We are interested in measures with support on the uniform lattice $\mathbb{N}_0$, which are defined as follows: $\rho = \sum_{k=0}^{\infty} \delta(z-k)w(k)$. The moments of such measures are:

$$\rho_n = \sum_{k=0}^{\infty} k^n w(k). \tag{10}$$

In particular, the 0-th moment is:

$$\rho_0 = \sum_{k=0}^{\infty} w(k). \tag{11}$$

The weights considered in this paper satisfy the following discrete Pearson equation:

$$\nabla(\sigma w) = \tilde{\tau} w.$$

This equation can be expressed as: $\sigma(k)w(k) - \sigma(k-1)w(k-1) = \tilde{\tau}(k)w(k)$, where $\sigma(z)$ and $\tilde{\tau}(z)$ are polynomials with real numbers as coefficients. If we define $\theta := \tilde{\tau} - \sigma$, the previous Pearson equation can be written as:

$$\theta(k+1)w(k+1) = \sigma(k)w(k), \quad k \in \mathbb{N}_0. \tag{12}$$

Theorem 1.6 *(Hypergeometric symmetries). Suppose the weight function w satisfies the discrete Pearson equation of the type (12), where the functions θ and σ are polynomials, with $\theta(0) = 0$. In this context, the following statements hold:*

i) *The moment matrix satisfies:*

$$\theta(\Lambda)G = B\sigma(\Lambda)GB^{\top}.$$

ii) *The Jacobi matrix complies with:*

$$\Pi^{-1}H\theta(J^{\top}) = \sigma(J)H\Pi^{\top}.$$

Additionally, both matrices $H\theta(J^{\top})$ and $\sigma(J)H$ are symmetric.

If we denote the degrees of the polynomials $\theta(z)$ and $\sigma(z)$ as $N+1$ and M respectively, and if their zeros are $\{-b_i+1\}_{i=1}^{N}$ and $\{-a_i\}_{i=1}^{M}$, we can express them as: $\theta(z) = z(z+b_1-1)\cdots(z+b_N-1)$ and $\sigma(z) = \eta(z+a_1)\cdots(z+a_M)$.

According to (11), the 0-th moment, represented as ρ_0, can be calculated as:

$$\rho_0 = \sum_{k=0}^{\infty} w(k) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_M)_k}{(b_1+1)_k \cdots (b_N+1)_k} \frac{\eta^k}{k!} = {}_MF_N \left[\begin{matrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{matrix} ; \eta \right],$$

where we are using the generalized hypergeometric function with the two standard notations, as described in [13].

Then, based on (10), for any natural number n, the corresponding higher moments, denoted as ρ_n, can be expressed as:

$$\rho_n = \vartheta_\eta^n \rho_0 = \vartheta_\eta^n \left({}_M F_N \begin{bmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{bmatrix} ; \eta \right),$$

where $\vartheta_\eta := \eta \frac{\partial}{\partial \eta}$.

Given a function $f(\eta)$, we define the Wronskian of the covector $\delta(f)$ as follows:

$$\mathscr{W}_n(f) := \begin{vmatrix} f & \vartheta_\eta f & \vartheta_\eta^2 f & \cdots & \vartheta_\eta^n f \\ \vartheta_\eta f & \vartheta_\eta^2 f & & \iddots & \vartheta_\eta^{n+1} f \\ \vartheta_\eta^2 f & & \iddots & & \vdots \\ \vdots & \iddots & & & \vdots \\ \vartheta_\eta^n f & \vartheta_\eta^{n+1} f & \cdots & \cdots & \vartheta_\eta^{2n} f \end{vmatrix}.$$

We refer to this Wronskian as the δ-Wronskian of f.

The Hankel determinants $\Delta_n = \det G^{[n]}$ are δ-Wronskians of generalized hypergeometric functions, given by:

$$\Delta_n = \tau_n, \quad \tilde{\Delta}_n = \vartheta_\eta \tau_n, \tag{13}$$

where

$$\tau_n := \mathscr{W}_n \left({}_M F_N \begin{bmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{bmatrix} ; \eta \right).$$

Using Proposition 1.1 we find:

$$H_n = \frac{\tau_{n+1}}{\tau_n}, \quad p_n^1 = -\vartheta_\eta \log \tau_n, \quad n \in \mathbb{N}_0. \tag{14}$$

The functions τ_k are the well-known tau functions [32]. In terms of these τ-functions, we can represent the recursion coefficients as:

$$\beta_n = \vartheta_\eta \log \frac{\tau_{n+1}}{\tau_n}, \quad \gamma_{n+1} = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \quad n \in \mathbb{N}_0, \tag{15}$$

where we have utilized equations (13), (4), and (14).

Remark 1.7. Let's observe that this Wronskian structure is not limited to weights that only solve Pearson's equations. In fact, the key requirement is that the weight depends on a parameter η in such a way that $\vartheta_\eta w(k) = k w(k)$ for $k \in \mathbb{N}_0$. In these cases, the tau function can be expressed as $\tau_n = \mathscr{W}_n(\rho_0)$, with $\rho_0 = \sum_{k=0}^{\infty} w(k)$. This allows us to apply Equations (13), (14), and (15) without any restrictions.

Theorem 1.8 *(Laguerre–Freud structure matrix). Assuming that the weight w satisfies the discrete Pearson equation (12) with polynomial functions θ and σ such that $\theta(0) = 0$, $\deg\theta(z) = N + 1$, and $\deg\sigma(z) = M$, the Laguerre–Freud structure matrix is given by:*

$$\begin{aligned}\Psi &:= \Pi^{-1}H\theta(J^\top) = \sigma(J)H\Pi^\top = \Pi^{-1}\theta(J)H = H\sigma(J^\top)\Pi^\top \\ &= \theta(J+I)\Pi^{-1}H = H\Pi^\top\sigma(J^\top - I).\end{aligned} \tag{16}$$

This matrix has only $N + M + 2$ possibly nonzero diagonals, consisting of $N + 1$ superdiagonals and M subdiagonals:

$$\Psi = (\Lambda^\top)^M\psi^{(-M)} + \cdots + \Lambda^\top\psi^{(-1)} + \psi^{(0)} + \psi^{(1)}\Lambda + \cdots + \psi^{(N+1)}\Lambda^{N+1}.$$

Here, $\psi^{(k)}$ are diagonal matrices. The lowest subdiagonal and highest superdiagonal are defined as follows:

$$\begin{cases}(\Lambda^\top)^M\psi^{(-M)} = \eta(J_-)^M H, \\ \psi^{(-M)} = \eta H\displaystyle\prod_{k=0}^{M-1}T_-^k\gamma = \eta\operatorname{diag}\Big(H_0\prod_{k=1}^{M}\gamma_k, H_1\prod_{k=2}^{M+1}\gamma_k, \dots\Big), \\ \psi^{(N+1)}\Lambda^{N+1} = H(J_-^\top)^{N+1}, \\ \psi^{(N+1)} = H\displaystyle\prod_{k=0}^{N}T_-^k\gamma = \operatorname{diag}\Big(H_0\prod_{k=1}^{N+1}\gamma_k, H_1\prod_{k=2}^{N+2}\gamma_k, \dots\Big).\end{cases} \tag{17}$$

The vector of orthogonal polynomials, denoted as $P(z)$, satisfies the following structure equations:

$$\theta(z)P(z-1) = \Psi H^{-1}P(z), \qquad \sigma(z)P(z+1) = \Psi^\top H^{-1}P(z). \tag{18}$$

The compatibility of the recursion relation, which represents the eigenfunctions of the Jacobi matrix, and the recursion matrix leads to some intriguing equations:

Proposition 1.9. *The following compatibility conditions between the Laguerre–Freud and Jacobi matrices hold:*

$$[\Psi H^{-1}, J] = \Psi H^{-1}, \tag{19a}$$

$$[J, \Psi^\top H^{-1}] = \Psi^\top H^{-1}. \tag{19b}$$

1.3. The Toda flows

Let us define the strictly lower triangular matrix Φ as:

$$\Phi := (\vartheta_\eta S)\, S^{-1}.$$

Proposition 1.10.

i) *The semi-infinite vector P satisfies:*

$$\vartheta_\eta P = \Phi P. \tag{20}$$

ii) *The Sato–Wilson equations hold as:*

$$-\Phi H + \vartheta_\eta H - H\Phi^\top = JH.$$

Consequently, $\Phi = -J_-$ and for $n \in \mathbb{N}_0$ we have $\vartheta_\eta \log H_n = J_{n,n}$.

As a result, we conclude that $\Phi = -J_-$, and for any $n \in \mathbb{N}_0$, we can establish that $\vartheta_\eta \log H_n = J_{n,n}$.

Furthermore,

Proposition 1.11 *(Toda). The following equations hold:*

$$\Phi = (\vartheta_\eta S)S^{-1} = -\Lambda^\top \gamma, \quad (\vartheta_\eta H)H^{-1} = \beta.$$

In particular, for n and $k-1$ in $\mathbb{N}$, we have:

$$\vartheta_\eta p^1_n = -\gamma_n, \quad \vartheta_\eta p^k_{n+k} = -\gamma_{n+k} p^{k-1}_{n+k-1}.$$

The functions $q_n = \log H_n$ for n in $\mathbb{N}$ satisfy the Toda equations:

$$\vartheta^2_\eta q_n = \mathrm{e}^{q_{n+1}-q_n} - \mathrm{e}^{q_n - q_{n-1}}.$$

Additionally, for n in $\mathbb{N}$, we have:

$$\vartheta_\eta P_n(z) = -\gamma_n P_{n-1}(z).$$

Proposition 1.12. *The following Lax equation holds: $\vartheta_\eta J = [J_+, J]$. The recursion coefficients satisfy the following Toda system:*

$$\vartheta_\eta \beta_n = \gamma_{n+1} - \gamma_n, \tag{21a}$$

$$\vartheta_\eta \log \gamma_n = \beta_n - \beta_{n-1}, \tag{21b}$$

for n in $\mathbb{N}_0$, with $\beta_{-1} = 0$. Consequently, we get:

$$\vartheta^2_\eta \log \gamma_n = \gamma_{n+1} - 2\gamma_n + \gamma_{n-1}. \tag{22}$$

For the compatibility of (5) and (20), which means the compatibility of the system:

$$\begin{cases} P(z+1) = \Pi P(z), \\ \vartheta_\eta(P(z)) = \Phi P(z), \end{cases}$$

we obtain:

$$\vartheta_\eta(\Pi) = [\Phi, \Pi].$$

In the general case, the dressed Pascal matrix Π is a lower unitriangular semi-infinite matrix, possibly having an infinite number of subdiagonals. However, in the case when the weight $w(z) = v(z)\eta^z$ satisfies the Pearson equation (12) with v independent of η, such that: $\theta(k+1)v(k+1)\eta = \sigma(k)v(k)$, the situation improves. We have the banded semi-infinite matrix Ψ that models the shift in the z variable as in (18). From the previous discrete Pearson equation, we see that $\sigma(z) = \eta\kappa(z)$ with κ and θ being η-independent polynomials in z:

$$\theta(k+1)v(k+1) = \eta\kappa(k)v(k).$$

Proposition 1.13. *Let us assume a weight w satisfying the Pearson equation* (12)*. Then, the Laguerre–Freud structure matrix Ψ given in* (16) *satisfies the following relations:*

$$\vartheta_\eta(\eta^{-1}\Psi^\top H^{-1}) = [\Phi, \eta^{-1}\Psi^\top H^{-1}], \tag{23a}$$

$$\vartheta_\eta(\Psi H^{-1}) = [\Phi, \Psi H^{-1}]. \tag{23b}$$

Relations (23a) *and* (23b) *are* gauge *equivalent.*

2. Generalized Charlier weights

For the generalized Charlier weight, we have $\sigma(z) = \eta$ and $\theta(z) = z(z+b)$. Weight functions that satisfy the Pearson equation

$$(k+1)(k+1+b)w(k+1) = \eta w(k)$$

are proportional to the generalized Charlier weight, given by

$$w(z) = \frac{\Gamma(b+1)\eta^z}{\Gamma(z+b+1)\Gamma(z+1)} = \frac{1}{(b+1)_z}\frac{\eta^z}{z!}.$$

The finiteness of the moments requires $b > -1$ and $\eta > 0$, see [25]. The generalized Charlier 0-th moment, as we have discussed, is the confluent hypergeometric limit function $\rho_0(\eta) = {}_0F_1(;b+1;\eta)$.

It's worth noting that the finiteness of the moments requires that $b > -1$ and $\eta > 0$, as discussed in [25].

Remark 2.1. It is well known that this function is related to Bessel functions, namely, J_ν and I_ν. The relation is given by:

$$\frac{(\frac{\eta}{2})^b}{\Gamma(b+1)}\rho_0\Big(-\frac{\eta^2}{4}\Big)=J_b(\eta),\quad \frac{(\frac{\eta}{2})^b}{\Gamma(b+1)}\rho_0\Big(\frac{\eta^2}{4}\Big)=I_b(\eta).$$

Here, J_ν represents the ν-th Bessel function, and I_ν is the ν-th modified Bessel function. It's important to note that these two types of Bessel functions are connected: $\mathrm{e}^{\pm\,\mathrm{i}\,\frac{\nu\pi}{2}}\,I_\nu(x)=J_\nu(\mathrm{e}^{\pm\,\mathrm{i}\,\frac{\nu\pi}{2}}\,x)$. Therefore, in terms of these modified Bessel functions, you can express it as, [25]:

$$\rho_0(\eta)=\frac{\Gamma(b+1)}{\sqrt{\eta}^b}I_b(2\sqrt{\eta}). \tag{24}$$

This expression establishes a direct connection between the 0-th moment of the generalized Charlier weight and modified Bessel functions.

Remark 2.2. From (24) and (15), we obtain explicit expressions for the recursion coefficients in terms of the following δ-Wronskian of a modified Bessel function:

$$\tau_n:=\mathscr{W}_n\left(\frac{\Gamma(b+1)}{\sqrt{\eta}^b}I_b(2\sqrt{\eta})\right).$$

These expressions for the recursion coefficients are:

$$\beta_n=\vartheta_\eta\log\frac{\tau_{n+1}}{\tau_n},\quad \gamma_{n+1}=\frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2},\qquad n\in\mathbb{N}_0.$$

Remark 2.3. The Charlier polynomials or Poisson–Charlier polynomials were introduced by Charlier in [19].

Proposition 2.4. *In the case of an extended Charlier-type weight, i.e., when $\theta(k+1)w(k+1)=\eta w(k)$ with $\theta(0)=0$, the semi-infinite matrix $H\theta(J^\top)$ allows for a Cholesky factorization as follows:*

$$\theta(J)H=H\theta(J^\top)=\Theta^{-1}H\Theta^{-\top}=\Pi H\Pi^\top.$$

Here, the dressed Pascal semi-infinite matrix $\Pi=\Theta^{-1}$ is a lower unitriangular semi-infinite matrix with only its first $\deg\theta$ *subdiagonals different from zero.*

Proof. From (16), we have $\Pi^{-1}H\theta(J^\top)=H\Pi^\top$, which means that $H\theta(J^\top)=\Pi H\Pi^\top$. Given the uniqueness of the Cholesky factorization and its band structure, we can deduce the result. $\square$

Theorem 2.5 *(The generalized Charlier Laguerre–Freud structure matrix). For a generalized Charlier weight, which is characterized by $\sigma = \eta$ and $\theta(z) = z(z+b)$, the Laguerre–Freud structure matrix is as follows:*

$$\Psi = \begin{bmatrix} \eta H_0 & \eta H_0 & H_2 & 0 & 0 & \cdots \\ 0 & \eta H_1 & 2\eta H_1 & H_3 & 0 & \ddots \\ 0 & 0 & \eta H_2 & 3\eta H_2 & H_4 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Proof. For the structure matrix Ψ, as shown in (17), we can express it as follows: $\Psi = \psi^{(0)} + \psi^{(1)}\Lambda + \psi^{(2)}\Lambda^2$. Let us find these diagonal matrix coefficients. In the one hand, as

$$\Psi = \eta H \Pi^\top = \underbrace{\eta H}_{\text{main diagonal}} + \underbrace{\eta H D\Lambda}_{\text{first superdiagonal}} + \underbrace{\eta H \pi^{[2]}\Lambda^2}_{\text{second superdiagonal}} + \cdots, \tag{25}$$

we get from the main diagonal that $\psi^{(0)} = \eta H$, and from the first superdiagonal that $\psi^{(1)} = \eta H D$. In the other hand, we observe that

$$\begin{aligned}\Psi &= \Pi^{-1} H\Big((J^\top)^2 + bJ^\top\Big)\\ &= \big(I - \Lambda^\top D + (\Lambda^\top)^2\pi^{[-2]} + \cdots\big)H\\ &\times \big((\Lambda^\top)^2 + \Lambda^\top(\beta + T_-\beta + bI) + \gamma + T_+\gamma + \beta^2 + b\beta + (\beta + T_-\beta + bI)\gamma\Lambda + (T_-\gamma)\gamma\Lambda^2\big)\end{aligned}$$

so that

$$\begin{aligned}\Psi = \cdots + &\underbrace{H(\gamma + T_+\gamma + \beta^2 + b\beta) - \Lambda^\top DH(\beta + T_-\beta + bI)\gamma\Lambda + (\Lambda^\top)^2\pi^{[-2]}H(T_-\gamma)\gamma\Lambda^2}_{\text{main diagonal}}\\ &+ \underbrace{H(\beta + T_-\beta + bI)\gamma\Lambda - \Lambda^\top DH(T_-\gamma)\gamma\Lambda^2}_{\text{first superdiagonal}} + \underbrace{H(T_-\gamma)\gamma\Lambda^2,}_{\text{second superdiagonal}}\end{aligned} \tag{26}$$

and the result is proven. □

Proposition 2.6 *(Compatibility). For a generalized Charlier weight, the recursion coefficients satisfy the following equation:*

$$n\eta(\beta_n - \beta_{n-1} - 1) = (-\gamma_{n+1} + \gamma_{n-1})\gamma_n. \tag{27}$$

This equation holds for $n \in \mathbb{N}$ with the initial condition $\gamma_0 = 0$.

Alternatively, we can express this equation in different forms:

$$n\vartheta_\eta\Big(\frac{\eta}{\gamma_n}\Big) = \gamma_{n+1} - \gamma_{n-1}, \tag{28a}$$

$$\vartheta_\eta\Big(\frac{\gamma_n\gamma_{n+1}}{\eta}\Big) = (n+1)\gamma_n - n\gamma_{n+1}. \tag{28b}$$

These alternative forms are valid for $n \in \mathbb{N}$ with the same initial condition $\gamma_0 = 0$.

Additionally, we have the relationship:

$$\beta_n - \beta_{n-2} - 1 = \eta\big(n\gamma_n^{-1} - (n-1)\gamma_{n-1}^{-1}\big), \quad n \geq 2. \tag{29}$$

Proof. Recalling that $\gamma = (T_- H)H^{-1}$, we can write:

$$\begin{aligned}
\Psi H^{-1} &= \eta + \eta H D\Lambda H^{-1} + H(T_-\gamma)\gamma\Lambda^2 H^{-1} \\
&= \eta + \eta H(T_- H)^{-1} D\Lambda + H(T_-{}^2 H)^{-1}(T_-\gamma)\gamma\Lambda^2 \\
&= \eta + \eta\gamma^{-1} D\Lambda + \Lambda^2 \\
&= \begin{bmatrix} \eta & \frac{\eta}{\gamma_1} & 1 & 0 & \cdots \\ 0 & \eta & \frac{2\eta}{\gamma_2} & 1 & \ddots \\ 0 & 0 & \eta & \frac{3\eta}{\gamma_3} & \ddots \\ 0 & 0 & 0 & \eta & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.
\end{aligned}$$

The compatibility equation (19b), i.e. $[\Psi H^{-1}, J] = \Psi H^{-1}$, can be expressed as follows:

$$\begin{aligned}
\begin{bmatrix} \eta & \frac{\eta}{\gamma_1} & 1 & 0 & \cdots \\ 0 & \eta & \frac{2\eta}{\gamma_2} & 1 & \ddots \\ 0 & 0 & \eta & \frac{3\eta}{\gamma_3} & \ddots \\ 0 & 0 & 0 & \eta & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} = &- \begin{bmatrix} \beta_0 & 1 & 0 & \cdots \\ \gamma_1 & \beta_1 & 1 & \ddots \\ 0 & \gamma_2 & \beta_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \eta & \frac{\eta}{\gamma_1} & 1 & 0 & \cdots \\ 0 & \eta & \frac{2\eta}{\gamma_2} & 1 & \ddots \\ 0 & 0 & \eta & \frac{3\eta}{\gamma_3} & \ddots \\ 0 & 0 & 0 & \eta & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \\
&+ \begin{bmatrix} \eta & \frac{\eta}{\gamma_1} & 1 & 0 & \cdots \\ 0 & \eta & \frac{2\eta}{\gamma_2} & 1 & \ddots \\ 0 & 0 & \eta & \frac{3\eta}{\gamma_3} & \ddots \\ 0 & 0 & 0 & \eta & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \beta_0 & 1 & 0 & \cdots \\ \gamma_1 & \beta_1 & 1 & \ddots \\ 0 & \gamma_2 & \beta_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}
\end{aligned}$$

As a result, we obtain (27) from the first superdiagonal, and from the second superdiagonal, we derive (29).

Based on the Toda system (21b), it becomes evident that (28a) is equivalent to (27). From (21b) we get

$$n\vartheta_\eta\Big(\frac{\eta}{\gamma_n}\Big)=\frac{n\eta}{\gamma_n}-\frac{n\eta\vartheta_\eta\gamma_n}{\gamma_n^2}=-\frac{n\eta(\beta_n-\beta_{n-1}-1)}{\gamma_n},$$

and the statement follows. □

Theorem 2.7 *(Third order ODE for generalized Charlier). The recursion coefficient γ_n of the generalized Charlier polynomials is governed by the following third-order nonlinear ordinary differential equation:*

$$\vartheta_\eta\Big(\frac{\gamma_n}{\eta}(\vartheta_\eta^2\log\gamma_n+2\gamma_n)+n^2\frac{\eta}{\gamma_n}\Big)=2\gamma_n. \tag{30}$$

Proof. On one hand, Equation (28b) can be expressed as:

$$\pm\vartheta_\eta\left(\frac{\gamma_n\gamma_{n\pm1}}{\eta}\right)=(n\pm1)\gamma_n-n\gamma_{n\pm1}.$$

Hence, we can derive:

$$\begin{aligned}\vartheta_\eta\left(\frac{\gamma_n\gamma_{n+1}+\gamma_{n-1}\gamma_n}{\eta}\right)&=(n+1)\gamma_n-n\gamma_{n+1}-(n-1)\gamma_n+n\gamma_{n-1}\\&=2\gamma_n-n(\gamma_{n+1}-\gamma_{n-1})=2\gamma_n-n^2\vartheta_\eta\left(\frac{\eta}{\gamma_n}\right),\end{aligned}$$

where we've utilized Equation (28a).

On the other hand:

$$\vartheta_\eta\left(\frac{\gamma_n\gamma_{n+1}+\gamma_{n-1}\gamma_n}{\eta}\right)=\vartheta_\eta\left(\frac{\gamma_n}{\eta}(\gamma_{n+1}+\gamma_{n-1})\right)=\vartheta_\eta\left(\frac{\gamma_n}{\eta}\left(\vartheta_\eta^2\log\gamma_n+2\gamma_n\right)\right).$$

Here, we've employed the Toda equation (22) for the γ's. By comparing these two expressions, we arrive at Equation (30). □

Remark 2.8. From:

$$\vartheta_\eta^2\log\gamma_n=\vartheta_\eta\left(\frac{\eta}{\gamma_n}\right)\frac{\mathrm{d}\gamma_n}{\mathrm{d}\eta}+\frac{\eta^2}{\gamma_n}\frac{\mathrm{d}^2\gamma_n}{\mathrm{d}\eta^2},\quad \vartheta_\eta\left(\frac{\eta}{\gamma_n}\right)=\frac{\eta}{\gamma_n}-\frac{\eta^2}{\gamma_n^2}\frac{\mathrm{d}\gamma_n}{\mathrm{d}\eta},$$

we can express (30) as:

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\mathcal{P}'_{\mathrm{III},n}(\gamma_n)\right)=2\frac{\gamma_n}{\eta},\quad \mathcal{P}'_{\mathrm{III},n}(\gamma_n):=\frac{\mathrm{d}^2\gamma_n}{\mathrm{d}\eta^2}-\frac{1}{\gamma_n}\left(\frac{\mathrm{d}\gamma_n}{\mathrm{d}\eta}\right)^2+\frac{1}{\eta}\frac{\mathrm{d}\gamma_n}{\mathrm{d}\eta}+\frac{2\gamma_n^2}{\eta^2}+\frac{n^2}{\gamma_n}.$$

The notation used here is inspired by Okamoto's alternative form $\mathrm{P}'_{\mathrm{III}}$ of the Painlevé III equation ($\mathcal{P}'_{\mathrm{III},n}(u)=0$), as found in the Digital Library of Mathematical Functions (DLMF) at NIST, under entry 32.2.9. In this notation, parameters are specified as $\alpha=8$,

$\gamma = \beta = 0$, and $\delta = 4n^2$. This corresponds to the Painlevé equation $\mathrm{P}'_{\mathrm{III}}(D7)$ equation (9) in Okamoto's work [48], where $\beta = 0$. See Theorem 1 (iv) in [48], along with the subsequent comment.

For a connection to $\mathrm{P}_{\mathrm{III}}$, refer to [50], [51], and [20]. In particular, Remark 4.5 in [20] provides functions p and q for the Hamiltonian system $\mathcal{H}'_{\mathrm{III}}$, as defined by Okamoto [49], in terms of $S_n = -p^1_n = \vartheta_\eta \tau_n$:

$$q = \frac{\eta \frac{\mathrm{d}^2 S_n}{\mathrm{d}\,\eta^2} - 2(n+b)\frac{\mathrm{d}\,S_n}{\mathrm{d}\,\eta} + 2n}{4\frac{\mathrm{d}\,S_n}{\mathrm{d}\,\eta}\left(1 - \frac{\mathrm{d}\,S_n}{\mathrm{d}\,\eta}\right)}, \quad p = \frac{\mathrm{d}\,S_n}{\mathrm{d}\,\eta},$$

with parameters $\theta_0 = n + b$ and $\theta_\infty = n - b$. In terms of $\gamma_n = \eta \frac{\mathrm{d}\,S_n}{\mathrm{d}\,\eta}$:

$$q = \frac{\eta^2 \frac{\mathrm{d}\,\gamma_n}{\mathrm{d}\,\eta} - (2n+2b+1)\eta\gamma_n + 2n\eta^2}{4\gamma_n\left(\eta^2 - \gamma_n\right)}, \quad p = \frac{\gamma_n}{\eta}.$$

As established in [20], q satisfies $\mathrm{P}'_{\mathrm{III}}$ with $\alpha = -4(n-b)$, $\beta = 4(n+b+1)$, $\gamma = 4$, and $\delta = -4$, as found in entry 32.2.9 in DLMF. This corresponds to Equation (2.4) in [20] with $A = -2\theta_\infty = -2(n-b)$ and $B = 2(\theta_0 + 1) = 2(n+b+1)$:

$$\frac{\mathrm{d}^2\,q}{\mathrm{d}\,\eta^2} = \frac{1}{q}\left(\frac{\mathrm{d}\,q}{\mathrm{d}\,\eta}\right)^2 - \frac{1}{\eta}\frac{\mathrm{d}\,q}{\mathrm{d}\,\eta} - (n-b)\frac{q^2}{\eta^2} + \frac{n+b+1}{\eta} + \frac{q^3}{\eta} - \frac{1}{q}.$$

Additionally, it's worth noting that p^1_n is a solution to the $\mathrm{P}'_{\mathrm{III}}$ σ-equation.

Remark 2.9. In terms of $p = \frac{\gamma_n}{\eta} = -\frac{\mathrm{d}\,p^1_n}{\mathrm{d}\,\eta}$, Equation (30) can be expressed as follows

$$\vartheta_\eta\Big(\vartheta^2_\eta p - \frac{(\vartheta_\eta p)^2}{p} + 2\eta p^2 + \frac{n^2}{p}\Big) = 2\eta p.$$

This equation can be further expanded to:

$$p^2\vartheta^3_\eta p - 2p(\vartheta_\eta p)(\vartheta^2_\eta p) + (\vartheta_\eta p)^3 + (2\eta p^3 - n^2)\vartheta_\eta p + 2p^3(p - \eta) = 0.$$

It's worth noting that in [20], it is demonstrated that the function $\frac{p-1}{p}$ satisfies an instance of Painlevé V, another member of the Painlevé equations, which are known for their special properties and integrability.

Remark 2.10 *(The appearance of deg-P_V).* Based on these observations, one is naturally led to conjecture that Equation (30) possesses the Painlevé property and could potentially be solvable in terms of the $\mathrm{P}_{\mathrm{III}}$ transcendents.

This conjecture indeed holds true, as it was recently demonstrated by Peter Clarkson in [21]. An integration of this equation leads to an expression equivalent to deg-P_{V}, specifically [21, Equation (83)], which is itself equivalent to $\mathrm{P_V}$, as indicated in [21, Equation (38)].

Proposition 2.11 *(Laguerre–Freud relations for the generalized Charlier case). For $n \in \mathbb{N}_0$, we can derive a set of Laguerre–Freud relations that govern the recursion coefficients, as follows:*

$$\beta_{n+1} = \frac{\eta(n+1)}{\gamma_{n+1}} - \beta_n + n - b, \tag{31a}$$

$$\gamma_{n+1} = \eta - \gamma_n - \beta_n^2 - b\beta_n + \frac{\gamma_{n-1}\gamma_n}{\eta} + \frac{\eta n^2}{\gamma_n}, \qquad \gamma_{-1} = \gamma_0 = 0. \tag{31b}$$

Moreover, for $n \in \mathbb{N}$, we have the following expression for the coefficient of the subleading term:

$$p_n^1 = \frac{n(n+1)}{2} - n\beta_n - \frac{\gamma_n\gamma_{n+1}}{\eta}. \tag{32}$$

Proof. From equations (25) and (26), and by applying (6), we obtain two different expressions for the first superdiagonal, which only involve the recursion coefficients. These expressions must be equated, leading to the following relationship:

$$\eta H D = H(\beta + T_-\beta + bI)\gamma - T_+\big(DH(T_-\gamma)\gamma\big).$$

This results in:

$$\eta D\gamma^{-1} = \beta + T_-\beta + bI - (T_+D)\big(T_+T_-{}^2H\big)(T_-H)^{-1} = \beta + T_-\beta + bI - T_+D.$$

Here, we have used the expression $\gamma = (T_-H)H^{-1}$, as defined componentwise in (31a).

Alternatively, it is worth noting that equation (29), after summation and simplification via a telescopic series on the right-hand side, leads to the expression in (31a).

From the main diagonal and the second superdiagonal, again using (6), we obtain the following two expressions. However, one depends on $\pi^{[-2]}$:

$$\begin{aligned} \eta H &= H(\gamma + T_+\gamma + \beta^2 + b\beta) - T_+(DH(\beta + T_-\beta + bI)\gamma) + T_+^2\big(\pi^{[-2]}H(T_-\gamma)\gamma\big), \\ \eta H\pi^{[2]} &= H(T_-\gamma)\gamma. \end{aligned} \tag{33}$$

From (33) and (8), we obtain (32). Once again, using $\gamma = (T_-H)H^{-1}$, we have:

$$\begin{aligned} \eta &= \gamma + T_+\gamma + \beta^2 + b\beta - (T_+D)(T_+H)H^{-1}T_+(\beta + T_-\beta + bI)(T_+T_-H)T_+(H^{-1}) \\ &\quad + T_+^2\big(\pi^{[-2]}T_-{}^2H\big)H^{-1} \\ &= \gamma + T_+\gamma + \beta^2 + b\beta - (T_+D)(T_+^2D) - \eta(T_+D)^2T_+(\gamma^{-1}) + T_+^2\pi^{[-2]}. \end{aligned}$$

Noting that $2D^{[2]} = DT_{-}D$, we can see that (9) implies $\pi^{[-2]} = D(T_{-}D) - \pi^{[2]}$. Furthermore, (33) gives $\pi^{[-2]} = D(T_{-}D) - \eta^{-1}(T_{-}{}^{2}H)H^{-1}$. Therefore, $T_{+}^{2}\pi^{[-2]} = (T_{+}^{2}D)(T_{+}D) - \eta^{-1}HT_{+}(H^{-1})$. From this, we arrive at:

$$\eta = \gamma + T_{+}\gamma + \beta^2 + b\beta - \eta(T_{+}D)^2 T_{+}(\gamma^{-1}) - \eta^{-1}HT_{+}^{2}(H^{-1}).$$

We finally find:

$$\gamma + T_{+}\gamma + \beta^2 + b\beta - \eta = \eta(T_{+}D)^2 T_{+}(\gamma^{-1}) + \eta^{-1}HT_{+}^{2}(H^{-1}),$$

which componentwise yields (31b). □

Remark 2.12. Smet and Van Assche found, as stated in [50, Theorem 2.1], that for the generalized Charlier weight, the corresponding recursion coefficients satisfy (31a) and, instead of (31b), they derived the following relation:

$$(\gamma_{n+1} - \eta)(\gamma_n - \eta) = \eta(\beta_n - n)(\beta_n - n + b). \tag{34}$$

It's worth noting that (34) is an equation of the form $\gamma_{n+1} = g(\gamma_n, \beta_n)$, which only involves a step backward in terms of the index n. This can be considered an advantage compared to (31b), which is of the form $\gamma_{n+1} = f(\gamma_n, \gamma_{n-1}, \beta_n)$. However, (27) gives $\gamma_{n-1} = F(\gamma_{n+1}, \gamma_n, \beta_n, \beta_{n-1})$. Combining this with (31a), which gives $\beta_{n-1} = G(\beta_n, \gamma_n)$, we can derive $\gamma_{n-1} = F(\gamma_{n+1}, \gamma_n, \beta_n, G(\beta_n, \gamma_n))$. This allows us to find a relation that involves only γ_{n+1}, γ_n, and β_n.

Proposition 2.13. *If (27) holds, then equations (31a) and (31b) are equivalent to (31a) and (34).*

Proof. For $n \in \mathbb{N}$, we rewrite (27) and (31a) as indicated:

$$\gamma_{n-1} = \gamma_{n+1} + \frac{n\eta(\beta_n - \beta_{n-1} - 1)}{\gamma_n}, \quad \beta_{n-1} + 1 = -\beta_n + \frac{\eta n}{\gamma_n} + n - b.$$

Hence, for $n \in \mathbb{N}$, and taking $\gamma_{-1} = \gamma_0 = 0$, we can write (31b) as follows:

$$\begin{aligned}
\gamma_{n+1} &= \eta - \gamma_n - \beta_n^2 - b\beta_n + \frac{\gamma_n}{\eta}\left(\gamma_{n+1} + \frac{n\eta(\beta_n - \beta_{n-1} - 1)}{\gamma_n}\right) + \frac{\eta n^2}{\gamma_n} \\
&= \eta + \frac{\gamma_n\gamma_{n+1}}{\eta} - \gamma_n + \frac{\eta n^2}{\gamma_n} - \beta_n^2 - b\beta_n + n(\beta_n - \beta_{n-1} - 1) \\
&= \eta + \frac{\gamma_n\gamma_{n+1}}{\eta} - \gamma_n + \frac{\eta n^2}{\gamma_n} - \beta_n^2 - b\beta_n + n\left(2\beta_n - \frac{\eta n}{\gamma_n} - n + b\right) \\
&= \eta + \frac{\gamma_n\gamma_{n+1}}{\eta} - \gamma_n - \beta_n^2 - b\beta_n + 2n\beta_n - n^2 + bn.
\end{aligned} \tag{35}$$

This is equivalent to (34). The inverse statement is easily proven to hold. If we assume (34), (31a), and (27) to be true, we can go backward in the chain of equalities leading to (35) and obtain the stated result. □

Corollary 2.14. *The β's are subject to the following nonlinear recursion:*

$$\begin{aligned}\frac{\eta(n+1)}{\beta_{n+1}+\beta_n-n+b}\\ &= \eta + \left(\frac{n-1}{\beta_{n-1}+\beta_{n-2}-n+2+b}-1\right)\frac{\eta n}{\beta_n+\beta_{n-1}-n+1+b}-\beta_n^2-b\beta_n\\ &\quad + n(\beta_n+\beta_{n-1}-n+1+b). \qquad (36)\end{aligned}$$

Proof. Notice that we can write (31a) as follows

$$\gamma_{n+1}=\frac{\eta(n+1)}{\beta_{n+1}+\beta_n-n+b},$$

that we can introduce in (31b) to get (36). □

We now seek for a second order ODE for γ_n. From the above results it easily follows that (this was previously found in [28]),

Lemma 2.15. *The recursion coefficients β_n and γ_n satisfy the following system of first-order nonlinear ODEs:*

$$\vartheta_\eta\beta_n=\eta\frac{\eta+(b-n)n+(2n-b-\beta_n)\beta_n-\gamma_n}{\eta-\gamma_n}-\gamma_n, \tag{37a}$$

$$\vartheta_\eta\gamma_n=(b-n+1+2\beta_n)\gamma_n-n\eta. \tag{37b}$$

Proof. Equation (35), after some simplification, can be expressed as:

$$\gamma_{n+1}=\eta\frac{\eta+(b-n)n+(2n-b-\beta_n)\beta_n-\gamma_n}{\eta-\gamma_n}, \tag{38}$$

which matches [28, Equation (16)].

On one hand, using (38) and the Toda equation (21a), we can deduce (37a), as seen in Equation (18) of [28]. On the other hand, from (31a) and the Toda equation (21b), we can obtain (37b). □

Remark 2.16. The differential system (37) was utilized in [28] to derive a second-order nonlinear ODE for β_n. Additionally, in [28, Theorem 2.1], it was demonstrated that an auxiliary function y (as shown in [28, Equation (20)]) satisfies an instance of the $\mathrm{P_V}$ equation, which is related to $\mathrm{P_{III}}$. It's worth noting that y is a solution to a Riccati equation in which β_n appears in the coefficients.

Theorem 2.17 *(Second order ODE for generalized Charlier). The recursion coefficient γ_n satisfies the second-order nonlinear ODE:*

$$\begin{aligned}\Big(1-\frac{\gamma_n}{\eta}\Big)\Big(\vartheta_\eta\Big(\frac{\vartheta_\eta\gamma_n}{\gamma_n}+\frac{n\eta}{\gamma_n}\Big)+2\gamma_n\Big)+2(\gamma_n-\eta+(n-b)n)\\ =-\frac{1}{2}\Big(\frac{\vartheta_\eta\gamma_n}{\gamma_n}+\frac{n\eta}{\gamma_n}\Big)^2+(n+1)\Big(\frac{\vartheta_\eta\gamma_n}{\gamma_n}+\frac{n\eta}{\gamma_n}\Big)+(-b+n-1)(-b+3n+1).\end{aligned}\tag{39}$$

Proof. Observe that Equation (37b) leads to

$$\beta_n=\frac{1}{2}\Big(\vartheta_\eta\log\gamma_n+\frac{n\eta}{\gamma_n}-b+n-1\Big),\tag{40}$$

so that

$$\vartheta_\eta\beta_n=\frac{1}{2}\vartheta_\eta\Big(\vartheta_\eta\log\gamma_n+\frac{n\eta}{\gamma_n}\Big).\tag{41}$$

From (37a) and (41) we get

$$\Big(1-\frac{\gamma_n}{\eta}\Big)\Big(\frac{1}{2}\vartheta_\eta\Big(\vartheta_\eta\log\gamma_n+\frac{n\eta}{\gamma_n}\Big)+\gamma_n\Big)+\gamma_n-\eta+(n-b)n=(2n-b)\beta_n-\beta_n^2.$$

Replacing β_n with the expression provided in (40), we obtain (39). To verify this, let's elaborate on the RHS of the equation.

$$\begin{aligned}(2n-b)\beta_n-\beta_n^2&=\frac{2n-b}{2}\Big(\vartheta_\eta\log\gamma_n+\frac{n\eta}{\gamma_n}-b+n-1\Big)\\&\quad-\frac{1}{4}\Big(\vartheta_\eta\log\gamma_n+\frac{n\eta}{\gamma_n}-b+n-1\Big)^2\\&=\frac{2n-b}{2}\Big(\vartheta_\eta\log\gamma_n+\frac{n\eta}{\gamma_n}-b+n-1\Big)-\frac{1}{4}\Big(\vartheta_\eta\log\gamma_n+\frac{n\eta}{\gamma_n}\Big)^2\\&\quad-\frac{(b-n+1)^2}{4}+\frac{b-n+1}{2}\Big(\vartheta_\eta\log\gamma_n+\frac{n\eta}{\gamma_n}\Big)\\&=-\frac{1}{4}\Big(\vartheta_\eta\log\gamma_n+\frac{n\eta}{\gamma_n}\Big)^2+\frac{n+1}{2}\Big(\vartheta_\eta\log\gamma_n+\frac{n\eta}{\gamma_n}\Big)\\&\quad+\frac{(-b+n-1)(-b+3n+1)}{2},\end{aligned}$$

and the statement is proven. □

3. Generalized Meixner weights

Another interesting case arises when we consider $\sigma=z+a$, which we refer to as the extended Meixner type. The generalized Meixner weight serves as the primary example of these extended Meixner weights, corresponding to the choices $\sigma(z) = \eta(z + a)$ and $\theta(z) = z(z + b)$. A weight satisfying the Pearson equation

$$(k + 1)(k + 1 + b)w(k + 1) = \eta(k + a)w(k)$$

is proportional to the generalized Meixner weight

$$w(z) = \frac{\Gamma(b + 1)\Gamma(z + a)\eta^z}{\Gamma(a)\Gamma(z + b + 1)\Gamma(z + 1)} = \frac{(a)_z}{(b + 1)_z}\frac{\eta^z}{z!}.$$

Note that the moments are finite if $a(b+1) > 0$ and $\eta > 0$ [25]. In this case, the moments can be expressed in terms of

$$\rho_0 = {}_1F_1(a; b + 1; \eta) = M(a, b + 1, \eta),$$

also known as the confluent hypergeometric function or Kummer function.

Remark 3.1. The Meixner polynomials, associated with the Pascal distribution, i.e. $w = (a)_z \frac{\eta^z}{z!}$ where introduced in [45].

Theorem 3.2 *(The generalized Meixner Laguerre–Freud structure matrix). For a generalized Meixner weight, where $\sigma = \eta(z+a)$ and $\theta = z(z+b)$, the structure matrix is given by:*

$$\Psi = \begin{bmatrix} \eta(\beta_0 + a)H_0 & (\beta_0 + \beta_1 + b)H_1 & H_2 & 0 & \cdots & & \\ \eta H_1 & \eta(\beta_1 + a + 1)H_1 & (\beta_1 + \beta_2 + b - 1)H_2 & H_3 & \ddots & & \\ 0 & \eta H_2 & \eta(\beta_2 + a + 2)H_2 & (\beta_2 + \beta_3 + b - 2)H_3 & H_4 & \ddots & \\ \vdots & \ddots & \eta H_3 & \eta(\beta_3 + a + 3)H_3 & (\beta_3 + \beta_4 + b - 3)H_4 & \ddots & \\ & & \ddots & \ddots & \ddots & & \end{bmatrix} \tag{42}$$

Proof. The structure matrix can be diagonally expanded (17), $\Psi = \Lambda^\top \psi^{(-1)} + \psi^{(0)} + \psi^{(1)}\Lambda + \psi^{(2)}\Lambda^2$. Since $\Psi = \eta(J + aI)H\Pi^\top$ we obtain

$$\begin{aligned} \Psi &= \eta(\Lambda^\top \gamma + \beta + aI + \Lambda)H(I + D\Lambda + \pi^{[2]}\Lambda^2 + \pi^{[3]}\Lambda^3 + \cdots) \\ &= \underbrace{\eta\Lambda^\top \gamma H}_{\text{first subdiagonal}} + \underbrace{\eta(\Lambda^\top \gamma H D\Lambda + (\beta + aI)H)}_{\text{main diagonal}} \\ &\quad + \underbrace{\eta(\Lambda^\top \gamma H\pi^{[2]}\Lambda^2 + (\beta + aI)HD\Lambda + \Lambda H)}_{\text{first superdiagonal}} \\ &\quad + \underbrace{\eta(\Lambda^\top \gamma H\pi^{[3]}\Lambda^3 + (\beta + aI)H\pi^{[2]}\Lambda^2 + \Lambda HD\Lambda)}_{\text{second superdiagonal}} + \cdots \end{aligned} \tag{43}$$

and conclude that $\psi^{(0)} = \eta T_+(\gamma HD) + \eta(\beta + aI)H$. Additionally, recalling that

$$
\begin{aligned}
J^2 + bJ = {} & (\Lambda^\top)^2(T_-\gamma)\gamma + \Lambda^\top(\beta + T_-\beta + bI)\gamma + \gamma + T_+\gamma + \beta^2 + b\beta \\
& + (\beta + T_-\beta + bI)\Lambda + \Lambda^2,
\end{aligned}
$$

from the alternative expression $\Psi = \Pi^{-1} H\big((J^\top)^2 + bJ^\top\big)$, we derive

$$
\begin{aligned}
\Psi = (I - \Lambda^\top D + (\Lambda^\top)^2\pi^{[-2]} - (\Lambda^\top)^3\pi^{[-3]} + \cdots)H\big((\Lambda^\top)^2 + \Lambda^\top(\beta + T_-\beta + bI) \\
+\gamma + T_+\gamma + \beta^2 + b\beta + (\beta + T_-\beta + bI)\gamma\Lambda + (T_-\gamma)\gamma\Lambda^2\big)
\end{aligned}
$$

that splits into diagonals

$$
\begin{aligned}
\Psi = {} & \underbrace{H(T_-\gamma)\gamma\Lambda^2}_{\text{second superdiagonal}} \\
& \underbrace{-\Lambda^\top DH(T_-\gamma)\gamma\Lambda^2 + H(\beta + T_-\beta + bI)\gamma\Lambda}_{\text{first superdiagonal}} \\
& \underbrace{+H(\gamma + T_+\gamma + \beta^2 + b\beta) - \Lambda^\top DH(\beta + T_-\beta + bI)\gamma\Lambda + (\Lambda^\top)^2\pi^{[-2]}H(T_-\gamma)\gamma\Lambda^2}_{\text{main diagonal}} \\
& \underbrace{+H\Lambda^\top(\beta + T_-\beta + bI) - \Lambda^\top DH(\gamma + T_+\gamma + \beta^2 + b\beta) + (\Lambda^\top)^2\pi^{[-2]}H(\beta + T_-\beta + bI)\gamma\Lambda - (\Lambda^\top)^3\pi^{[-3]}H(T_-\gamma)\gamma\Lambda^2.}_{\text{first subdiagonal}}
\end{aligned}
\tag{44}
$$

Thus, we have $\psi^{(1)} = -T_+(DH(T_-\gamma)\gamma) + H(\beta + T_-\beta + bI)\gamma$. Finally, the Laguerre–Freud matrix has the form:

$$
\Psi = \eta\Lambda^\top T_- H + \eta(\beta + aI + T_+D)H + \big(\beta + T_-\beta + bI - T_+D\big)T_-H\Lambda + T_-{}^2 H\Lambda^2.
$$

When expressed componentwise, we obtain the form presented in (42). □

Proposition 3.3 *(Compatibility). For the generalized Meixner case, the recursion coefficients satisfy the following equations:*

$$
(\beta_n + \beta_{n+1} + b - n - \eta)\gamma_{n+1} - (\beta_{n-1} + \beta_n + b - n + 1 - \eta)\gamma_n = \eta(\beta_n + a + n), \tag{45a}
$$

$$
\gamma_{n+2} - \gamma_n - 2\eta + (\beta_n + \beta_{n+1} + b - n - \eta)(\beta_{n+1} - \beta_n - 1) = 0. \tag{45b}
$$

Proof. The matrix

$$
\Psi H^{-1} = \begin{bmatrix}
\eta(\beta_0 + a) & \beta_0 + \beta_1 + b & 1 & 0 & \cdots \\
\eta\gamma_1 & \eta(\beta_1 + a + 1) & \beta_1 + \beta_2 + b - 1 & 1 & \ddots \\
0 & \eta\gamma_2 & \eta(\beta_2 + a + 2) & \beta_2 + \beta_3 + b - 2 & 1 & \ddots \\
\vdots & \ddots & \eta\gamma_3 & \eta(\beta_3 + a + 3) & \beta_3 + \beta_4 + b - 3 & \ddots \\
 & & \ddots & \ddots & \ddots &
\end{bmatrix}
$$

satisfies the compatibility condition (19a). Since

$$
\begin{aligned}
&\left[\Psi H^{-1}, J\right]\\
&=\begin{bmatrix}
(\beta_0+\beta_1+b-\eta)\gamma_1 & \begin{gathered}\gamma_2-\eta\\+(\beta_0+\beta_1+b-\eta)(\beta_1-\beta_0)\end{gathered} & 1 & 0 & \cdots \\
\eta\gamma_1 & \begin{gathered}(\beta_1+\beta_2+b-1-\eta)\gamma_2\\-(\beta_0+\beta_1+b-\eta)\gamma_1\end{gathered} & \begin{gathered}\gamma_3-\gamma_1-\eta\\+(\beta_1+\beta_2+b-1-\eta)(\beta_2-\beta_1)\end{gathered} & 1 & \ddots \\
0 & \eta\gamma_2 & \begin{gathered}(\beta_2+\beta_3+b-2-\eta)\gamma_3\\-(\beta_1+\beta_2+b-1-\eta)\gamma_2\end{gathered} & \begin{gathered}\gamma_4-\gamma_2-\eta\\+(\beta_2+\beta_3+b-2-\eta)(\beta_3-\beta_2)\end{gathered} & \ddots \\
\vdots & \ddots & \ddots & \ddots &
\end{bmatrix}
\end{aligned}
$$

from the first row we obtain that

$$
\gamma_1=\frac{\eta(\beta_0+a)}{\beta_0+\beta_1+b-\eta}, \qquad \gamma_2=\beta_0+\beta_1+b-(\beta_0+\beta_1+b-\eta)(\beta_1-\beta_0)+\eta,
$$

and from the next rows we deduce that the following equations are satisfied

$$
\begin{aligned}
(\beta_n+\beta_{n+1}+b-n-\eta)\gamma_{n+1}-(\beta_{n-1}+\beta_n+b-n+1-\eta)\gamma_n&=\eta(\beta_n+a+n),\\
\gamma_{n+2}-\gamma_n-\eta+(\beta_n+\beta_{n+1}+b-n-\eta)(\beta_{n+1}-\beta_n)&=\beta_n+\beta_{n+1}+b-n.
\end{aligned}
$$

These are represented by equations (45a) and (45b), the latter of which requires some elaboration. □

Proposition 3.4 *(Laguerre–Freud relations). For $n\in\mathbb{N}$, we find the following Laguerre–Freud equations*

$$
\begin{aligned}
\gamma_{n+1}={}&-\gamma_n-(\beta_n+b-n)\beta_n+n(b-a-n+1)+\eta(\beta_n+a+n)\\
&+\eta^{-1}(\beta_{n-1}+\beta_n+b-n+1-\eta)\gamma_n, \qquad (46)\\
0={}&(n+3)(n+2)(n+1)(\beta_n-\beta_{n+2}+1)-\big(\eta^{-1}\gamma_{n+3}-n-3\big)\gamma_{n+2} \qquad (47)\\
&+(n+3)(n+2)\Big(\eta^{-1}(\beta_{n+1}+\beta_{n+2}+b-n-1-\eta)\gamma_{n+2}-(n+2)(\beta_{n+1}+a)\Big)\\
&-(n+2)(n+1)\Big(\eta^{-1}(\beta_{n+3}+\beta_{n+4}+b-n-3-\eta)\gamma_{n+4}-(n+4)(\beta_{n+3}+a)\Big)\\
&+(-\beta_{n+2}+a-b)\big(\eta^{-1}(\beta_{n+2}+\beta_{n+3}+b-n-2-\eta)\gamma_{n+3}-(n+3)(\beta_{n+2}+a)\big)\\
&+(\beta_{n+2}+\beta_{n+3}+b-n-3-\eta)\gamma_{n+3}-(n+3)(\gamma_{n+2}+\beta_{n+2}^2+b\beta_{n+2})\\
&+(n+3)(n+2)(\beta_{n+1}+\beta_{n+2}+b).
\end{aligned}
$$

We also find

$$
p^1_{n+1}=-\eta^{-1}(\beta_{n+1}+\beta_{n+2}+b-n-1-\eta)\gamma_{n+2}+\beta_{n+1}+a(n+2)+\frac{(n+2)(n+1)}{2}. \qquad (48)
$$

Proof. By comparing (43) with (44) and referring to (6), we can derive the following result:

$$\begin{aligned}\eta T_- H &= (\beta + T_-\beta + bI)T_- H - DH(\gamma + T_+\gamma + \beta^2 + b\beta) \\ &\quad + T_+\big(\pi^{[-2]}(\beta + T_-\beta + bI)H\gamma\big) + T_+^2\big(\pi^{[-3]}H(T_-\gamma)\gamma\big), \qquad (49)\end{aligned}$$

$$\begin{aligned}\eta(T_+(\gamma HD) + (\beta + aI)H) &= H(\gamma + T_+\gamma + \beta^2 + b\beta) \\ &\quad - T_+(DH(\beta + T_-\beta + bI)\gamma) + T_+^2(\pi^{[-2]}H(T_-\gamma)\gamma), \qquad (50)\end{aligned}$$

$$\eta\big(T_+(\gamma H\pi^{[2]}) + (\beta + aI)HD + T_- H\big) = -T_+\big(DH(T_-\gamma)\gamma\big) + H(\beta + T_-\beta + bI)\gamma, \qquad (51)$$

$$\eta\big(T_+(\gamma H\pi^{[3]}) + (\beta + aI)H\pi^{[2]} + T_-(HD)\big) = H(T_-\gamma)\gamma. \qquad (52)$$

Let's examine relations (50) and (51). On the one hand, if we opt to solve (51) for $\pi^{[2]}$, we obtain (recalling that $T_- T_+ = I$):

$$\begin{aligned}\pi^{[2]} &= -H^{-1}\gamma^{-1}T_-((\beta + aI)HD + T_- H) - \eta^{-1}DT_-\gamma \\ &\quad + \eta^{-1}H^{-1}\gamma^{-1}T_-(H\gamma(\beta + T_-\beta + bI)) \\ &= \eta^{-1}(T_-\beta + T_-^2\beta + bI - D - \eta)T_-\gamma - (T_- D)(T_-\beta + aI), \qquad (53)\end{aligned}$$

so that

$$\pi_n^{[2]} = \eta^{-1}(\beta_{n+1} + \beta_{n+2} + b - n - 1 - \eta)\gamma_{n+2} - (n+2)(\beta_{n+1} + a).$$

Moreover, from $\beta = T_+ S^{[1]} - S^{[1]}$ and $\pi^{[2]} = D^{[2]} + (T_- S^{[1]} - S^{[1]})D - S^{[1]}$ we obtain

$$S^{[1]} = -\pi^{[2]} + D^{[2]} - DT_-\beta. \qquad (54)$$

This equation, when expressed componentwise, corresponds to (48). Now, taking into account (9), we can rewrite (50) as follows:

$$\eta(\beta + aI + T_+ D) = \gamma + T_+\gamma + \beta^2 + b\beta - T_+(D(\beta + T_-\beta + bI)) + T_+^2(-\pi^{[2]} + 2D^{[2]}).$$

From (7), this leads to

$$\begin{aligned}\eta(T_-^2\beta + aI + T_- D) &= T_-^2\gamma + T_-\gamma + T_-^2\beta^2 + bT_-^2\beta - (T_- D)(T_-\beta + T_-^2\beta + bI - D) - \pi^{[2]} \\ &= T_-^2\gamma + T_-\gamma + T_-^2\beta^2 + bT_-^2\beta - (T_- D)(T_-\beta + T_-^2\beta + bI - D) \\ &\quad - \eta^{-1}(T_-\beta + T_-^2\beta + bI - D - \eta)T_-\gamma + (T_- D)(T_-\beta + aI) \\ &= T_-^2\gamma + T_-\gamma + T_-^2\beta^2 + bT_-^2\beta - (T_- D)(T_-^2\beta + (b-a)I - D) \\ &\quad - \eta^{-1}(T_-\beta + T_-^2\beta + bI - D - \eta)T_-\gamma,\end{aligned}$$

that componentwise reads

$$\begin{aligned}\eta(\beta_{n+2} + a + n + 2) &= \gamma_{n+3} + \gamma_{n+2} + \beta_{n+2}^2 + b\beta_{n+2} - (n+2)(\beta_{n+2} + b - a - n - 1) \\ &\quad - \eta^{-1}(\beta_{n+1} + \beta_{n+2} + b - n - 1 - \eta)\gamma_{n+2},\end{aligned}$$

which is (46).

Now, (52) can be written as follows

$$\begin{aligned}\pi^{[3]} &= -(T_-\beta + aI)T_-\pi^{[2]} + \big(\eta^{-1}T_-^2\gamma - T_-^2 D\big)T_-\gamma \\ &= -(T_-\beta + aI)\big(\eta^{-1}(T_-^2\beta + T_-^3\beta + bI - T_-D - \eta)T_-^2\gamma - (T_-^2 D)(T_-^2\beta + aI)\big) \\ &\quad + \big(\eta^{-1}T_-^2\gamma - T_-^2 D\big)T_-\gamma,\end{aligned}$$

where (53) has been used. From (9), $\pi^{[3]} + \pi^{[-3]} = 2((T_-{}^2 S^{[1]})D^{[2]} - (T_-D^{[2]})S^{[1]})$, and (54) (noticing that $3D^{[3]} = 2D^{[2]}(T_-^2 D^{[2]} - T_-D^{[2]})$) we get

$$\begin{aligned}\pi^{[-3]} =&2D^{[2]}(-T_-^2\pi^{[2]} + T_-^2 D^{[2]} - (T_-^2 D)T_-^3\beta) - 2(T_-D^{[2]})(-\pi^{[2]} + D^{[2]} - DT_-\beta) - \pi^{[3]} \\ =&3D^{[3]} + 2(T_-D^{[2]})(\pi^{[2]} + DT_-\beta) - 2D^{[2]}(T_-^2\pi^{[2]} + (T_-^2 D)T_-^3\beta) - \pi^{[3]} \\ =&3D^{[3]}T_-(\beta - T_-{}^2\beta + I) + 2(T_-D^{[2]})\pi^{[2]} - 2D^{[2]}T_-^2\pi^{[2]} - \pi^{[3]} \\ =&3D^{[3]}T_-(\beta - T_-^2\beta + I) - \big(\eta^{-1}T_-^2\gamma - T_-^2 D\big)T_-\gamma \\ &+ 2(T_-D^{[2]})\Big(\eta^{-1}(T_-\beta + T_-^2\beta + bI - D - \eta)T_-\gamma - (T_-D)(T_-\beta + aI)\Big) \\ &- 2D^{[2]}\Big(\eta^{-1}(T_-^3\beta + T_-^4\beta + bI - T_-^2 D - \eta)T_-^3\gamma - (T_-^3 D)(T_-^3\beta + aI)\Big) \\ &+ (T_-\beta + aI)\big(\eta^{-1}(T_-^2\beta + T_-^3\beta + bI - T_-D - \eta)T_-^2\gamma - (T_-^2 D)(T_-^2\beta + aI)\big).\end{aligned}$$

We now write (49) as follows

$$\begin{aligned}\pi^{[-3]} =& -(T_-^2\beta + T_-^3\beta + bI - T_-^2 D - \eta)T_-^2\gamma + (T_-^2 D)(T_-\gamma + T_-^2\beta^2 + bT_-^2\beta) \\ &- (T_-\beta + T_-^2\beta + bI)T_-(2D^{[2]} - \pi^{[2]}) \\ =& -(T_-^2\beta + T_-^3\beta + bI - T_-^2 D - \eta)T_-^2\gamma + (T_-^2 D)(T_-\gamma + T_-^2\beta^2 + bT_-^2\beta) \\ &- (T_-\beta + T_-^2\beta + bI)T_- \\ &\times \big(2D^{[2]} - \eta^{-1}(T_-\beta + T_-^2\beta + bI - D - \eta)T_-\gamma + (T_-D)(T_-\beta + aI)\big).\end{aligned}$$

Thus, we get the following equation

$$\begin{aligned}0 =&3D^{[3]}(T_-\beta - T_-^3\beta + I) - \big(\eta^{-1}T_-^2\gamma - T_-^2 D\big)T_-\gamma \\ &+ 2(T_-D^{[2]})\Big(\eta^{-1}(T_-\beta + T_-^2\beta + bI - D - \eta)T_-\gamma - (T_-D)(T_-\beta + aI)\Big) \\ &- 2D^{[2]}\Big(\eta^{-1}(T_-^3\beta + T_-^4\beta + bI - T_-^2 D - \eta)T_-^3\gamma - (T_-^3 D)(T_-^3\beta + aI)\Big) \\ &+ (T_-\beta + aI)\big(\eta^{-1}(T_-^2\beta + T_-^3\beta + bI - T_-D - \eta)T_-^2\gamma - (T_-^2 D)(T_-^2\beta + aI)\big) \\ &+ (T_-^2\beta + T_-^3\beta + bI - T_-^2 D - \eta)T_-^2\gamma - (T_-^2 D)(T_-\gamma + T_-^2\beta^2 + bT_-^2\beta) \\ &+ (T_-\beta + T_-^2\beta + bI)T_- \\ &\times \big(2D^{[2]} - \eta^{-1}(T_-\beta + T_-^2\beta + bI - D - \eta)T_-\gamma + (T_-D)(T_-\beta + aI)\big).\end{aligned}$$

Hence, componentwise we obtain (47). □

Remark 3.5. Notice that the long and cumbersome expression in (47) is a ligature for the recursion coefficients, given by:

$$(\beta_{n+4}, \beta_{n+2}, \beta_{n+1}, \beta_n, \gamma_{n+4}, \gamma_{n+3}, \gamma_{n+2}).$$

Our intention was to demonstrate that the method leads to these equations. Indeed, (46) represents a more compact Laguerre–Freud equation for the β's. However, it's important to note that while (46) provides an expression for β_{n+1} in terms of $(\beta_{n-1}, \beta_n, \gamma_{n+1}, \gamma_n)$, (45a) offers an expression for β_n in terms of $(\beta_n, \beta_{n-1}, \gamma_n, \gamma_{n+1}))$. In both cases, we need to look two steps down, and we refer to these as "length two relations." However, we can combine both Laguerre–Freud equations to derive an "improved" expression for β_{n+1} which only requires one step down, namely $(\beta_n, \gamma_n, \gamma_{n+1})$, creating a "length one relation." The same reasoning applies to (45b).

Proposition 3.6. *The generalized Meixner recursion coefficients are governed by the following length-one Laguerre–Freud equation.*

$$\begin{aligned}\beta_{n+1} = {} & \frac{1}{\gamma_{n+1}}\Big(\eta\big(\gamma_n + (\beta_n + b - n)\beta_n - n(b - a - n + 1)\big) + \eta(\eta+1)(\beta_n + a + n)\Big) \\ & - \beta_n - b + n + 1 + 2\eta.\end{aligned} \tag{55}$$

Proof. We can express (46) as follows:

$$\begin{aligned}(\beta_{n-1} + \beta_n + b - n + 1 - \eta)\gamma_n = {} & \eta\left(\gamma_{n+1} + \gamma_n + (\beta_n + b - n)\beta_n - n(b - a - n + 1)\right) \\ & + \eta^2(\beta_n + a + n).\end{aligned}$$

This expression, when substituted into (45a), yields the following result.

$$\begin{aligned}(\beta_n + \beta_{n+1} + b - n - \eta)\gamma_{n+1} = {} & (\beta_{n-1} + \beta_n + b - n + 1 - \eta)\gamma_n + \eta(\beta_n + a + n) \\ = {} & \eta\big(\gamma_{n+1} + \gamma_n + (\beta_n + b - n)\beta_n - n(b - a - n + 1)\big) \\ & + \eta(\eta+1)(\beta_n + a + n),\end{aligned}$$

and we obtain Laguerre–Freud equation (55).

Let us shift the relation (46) by $n \to n + 1$:

$$\begin{aligned}\gamma_{n+2} = {} & -\gamma_{n+1} - (\beta_{n+1} + b - n - 1)\beta_{n+1} + n(b - a - n) \\ & + \eta(\beta_{n+1} + a + n + 1) + \eta^{-1}(\beta_n + \beta_{n+1} + b - n - \eta)\gamma_{n+1}.\end{aligned}$$

Now, let's introduce the expression for γ_{n+2} into (45b). This leads to:

$$\gamma_n+2\eta-(\beta_n+\beta_{n+1}+b-n-\eta)(\beta_{n+1}-\beta_n-1)=-(\beta_{n+1}+b-n-1)\beta_{n+1}+n(b-a-n)\\ +\eta(\beta_{n+1}+a+n+1)+\eta^{-1}(\beta_n+\beta_{n+1}+b-n-2\eta)\gamma_{n+1}.$$

Now, (55) can be written as:

$$\gamma_{n+1}(\beta_{n+1}+\beta_n+b-n-2\eta)\\ =\eta\big(\gamma_n+(\beta_n+b-n)\beta_n-n(b-a-n+1)\big)+\eta(\eta+1)(\beta_n+a+n)+\gamma_{n+1},$$

so that

$$\gamma_n+2\eta-(\beta_n+\beta_{n+1}+b-n-\eta)(\beta_{n+1}-\beta_n-1)\\ =-(\beta_{n+1}+b-n-1)\beta_{n+1}+n(b-a-n)+\eta(\beta_{n+1}+a+n+1)\\ +\big(\gamma_n+(\beta_n+b-n)\beta_n-n(b-a-n+1)\big)+(\eta+1)(\beta_n+a+n)+\eta^{-1}\gamma_{n+1}. \quad \square$$

Our most significant Laguerre–Freud equations are (55) and (46), with lengths one and two, respectively. For the reader's convenience, we restate them as a theorem.

Theorem 3.7 *(Laguerre–Freud equations for generalized Meixner). The generalized Meixner recursion coefficients satisfy the following Laguerre–Freud relations*

$$\beta_{n+1}=\frac{1}{\gamma_{n+1}}\Big(\eta\big(\gamma_n+(\beta_n+b-n)\beta_n-n(b-a-n+1)\big)+\eta(\eta+1)(\beta_n+a+n)\Big)\\ -\beta_n-b+n+1+2\eta,\\ \gamma_{n+1}=-\gamma_n-(\beta_n+b-n)\beta_n+n(b-a-n+1)+\eta(\beta_n+a+n)\\ +\eta^{-1}(\beta_{n-1}+\beta_n+b-n+1-\eta)\gamma_n.$$

Remark 3.8. In [50], a system of Laguerre–Freud equations was presented. There are two functions, denoted as u_n and v_n, which are connected to β_n and γ_n as follows:

$$\gamma_n=n\eta-(a-1)u_n, \qquad \beta_n=n+a-b-1+\eta-\frac{(a-1)v_n}{\eta}.$$

The nonlinear system, as described in equations (3.2) and (3.3) in [50], is defined by:

$$(u_n+v_n)(u_{n+1}+v_n)=\frac{a-1}{\eta^2}v_n(v_n-\eta)\Big(v_n-\eta\frac{a-b-1}{a-1}\Big),\\ (u_n+v_n)(u_n+v_{n-1})=\frac{u_n}{u_n-\frac{\eta n}{a-1}}(u_n+\eta)\Big(u_n+\eta\frac{a-b-1}{a-1}\Big).$$

It is worth noting that the first relation (3.2) in [50] expresses γ_{n+1} as a rational function of (γ_n,β_n), constituting a length one relation. This rational function is a cubic polynomial divided by a linear function in terms of the recursion coefficients.

In contrast, our equation (46) represents a length two relation. It involves quadratic polynomials in the recursion coefficients, not rational functions, and includes cubic polynomials as (3.1) does.

The second relation (3.3) in [50] expresses β_{n+1} in terms of a rational function of (γ_{n+1}, β_n), which is a length one relation as well. The rational function in this case is a cubic polynomial divided by a quadratic one.

On the other hand, our relation (55) is also a length one relation, but the rational function it involves is a quadratic polynomial divided by a linear one.

Remark 3.9. A notable feature of the Smet–Van Assche system is that, for a specific value of $a = 1$, it provides an explicit expression for β_n and γ_n, which becomes $\beta_n = n - b + \eta$ and $\gamma_n = n\eta$. This is confirmed through verification using SageMath 9.0, where equations (45a), (45b), (46), and (47) are satisfied. In the case of $a = 1$, the weight function is defined as: $w(z) = \frac{\eta^z}{(b+1)_z}$. The 0-th moment of this weight function is given by the Kummer function: $\rho_0 = M(1, b+1, \eta) = \frac{b}{\eta^b} e^{\eta}(\Gamma(b) - \Gamma(b, \eta))$, where $\Gamma(b, \eta) = \int_\eta^\infty t^{b-1} e^{-t} dt$ represents the incomplete Gamma function.

According to [29], this particular case corresponds to the Charlier case on the shifted lattice $\mathbb{N} - b$.

4. Conclusions and outlook

In their studies of integrable systems and orthogonal polynomials, Adler and van Moerbeke have extensively employed the Gauss-Borel factorization of the moment matrix, as seen in [1–3]. This strategy has been further developed and applied in various contexts by us, including CMV orthogonal polynomials, matrix orthogonal polynomials, multiple orthogonal polynomials, and multivariate orthogonal polynomials, as detailed in [5,4,6–11]. For a comprehensive overview, please refer to [40].

Recently, we extended these ideas into the discrete realm, as described in [43]. Specifically, we applied this approach to examine the implications of the Pearson equation on the moment matrix and Jacobi matrices. This description is based on a new banded matrix, the Laguerre-Freud structure matrix, which encodes the Laguerre-Freud relations for the recurrence coefficients. Furthermore, we discovered that the contiguous relations satisfied by generalized hypergeometric functions, which dictate the moments of the weight, correspond to a discrete Toda hierarchy known as the Nijhoff–Capel equation (see [46]). In our paper [41], we delve into the role of Christoffel and Geronimus transformations in explaining these contiguous relations and use Geronimus-Christoffel transformations to characterize shifts in the spectral independent variable of orthogonal polynomials.

In this paper, we delve deeper into this program and further explore discrete semiclassical cases. We establish Laguerre-Freud relations for the recursion coefficients of two types of discrete orthogonal polynomials: generalized Charlier and generalized Meixner. We observe that in both cases, a solution to the system of nonlinear Laguerre-Freud equations for the recursion coefficients can be provided by the τ-function. This τ-function is defined as the Wronskian of modified Bessel and Kummer functions, respectively. It is worth noting that a similar study to the one presented in this paper for the hypergeometric cases ${}_1F_2$, ${}_2F_2$, and ${}_3F_2$ was previously presented in [27].

Looking ahead, we plan to extend these techniques to multiple discrete orthogonal polynomials [12] and explore their relationships with the transformations presented in [18]. Additionally, we aim to investigate quadrilateral lattices [22,42].

Declarations

Ethical approval

Not applicable.

Contributions

All the authors have contributed equally.

Declaration of competing interest

The authors declare no conflict of interest.

Data availability

No data was used for the research described in the article.

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used ChatGPT in order to improve English grammar, syntax, spelling and wording. After using this tool/service, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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Toda and Laguerre–Freud equations and tau functions for hypergeometric discrete multiple orthogonal polynomials

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Abstract

In this paper, the authors investigate the case of discrete multiple orthogonal polynomials with two weights on the step line, which satisfy Pearson equations. The discrete multiple orthogonal polynomials in question are expressed in terms of τ-functions, which are *double* Wronskians of generalized hypergeometric series. The shifts in the spectral parameter for type II and type I multiple orthogonal polynomials are described using banded matrices. It is demonstrated that these polynomials offer solutions to multicomponent integrable extensions of the nonlinear Toda equations. Additionally, the paper characterizes extensions of the Nijhoff–Capel totally discrete Toda equations. The hypergeometric τ-functions are shown to provide solutions to these integrable nonlinear equations. Furthermore, the authors explore Laguerre–Freud equations, nonlinear equations for the recursion coefficients, with a particular focus on the multiple Charlier, generalized multiple Charlier, multiple Meixner II, and generalized multiple Meixner II cases.

Keywords Pearson equations · Discrete multiple orthogonal polynomials · Toda type integrable equations · Tau functions · Generalized hypergeometric series · Laguerre–Freud equations

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In memory of the late Diego Dominici.

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1 Introduction

Discrete multiple orthogonal polynomials, τ-functions, generalized hypergeometric series, Pearson equations, Laguerre–Freud equations, and Toda type equations are intriguing mathematical objects that have attracted considerable attention due to their deep connections with various areas of mathematics and physics. These intertwined concepts play a fundamental role in understanding the underlying structures and dynamics of discrete systems, integrable models, and special functions. In this paper, we delve into the rich mathematical theory of discrete multiple orthogonal polynomials, explore their relationship with τ-functions and generalized hypergeometric series, and investigate their applications in Laguerre–Freud equations and Toda type equations.

Orthogonal polynomials [18, 26, 46] are a fascinating and powerful tool in Applied Mathematics and Theoretical Physics, with a wide range of applications in various fields. They possess remarkable properties and have been extensively studied for their applications in approximation theory, numerical analysis, and many other areas. Among the vast family of orthogonal polynomials, two special classes that have attracted significant attention are multiple orthogonal polynomials and discrete orthogonal polynomials.

Back in 1895, Pearson [62], in the context of fitting curves to real data, introduced his famous family of frequency curves by means of the differential equation $\frac{f'(x)}{f(x)}=\frac{p_1(x)}{p_2(x)}$, where f is the probability density and p_i is a polynomial in x of degree at most i, $i=1,2$. Since then, a vast bibliography has been developed regarding the properties of Pearson distributions. The connection between Pearson equations and orthogonal polynomials lies in the fact that the solutions to Pearson equations often coincide with or are closely related to certain families of orthogonal polynomials. They are characterized by a weight function, which determines the orthogonality properties of the associated orthogonal polynomials. These equations have been extensively studied due to their close relationship with special functions, such as the classical orthogonal polynomials (e.g., Legendre, Hermite, Laguerre) and their generalizations. In this work, we will use the Gauss–Borel approach to orthogonal polynomials, for a review paper see [53]. In previous papers, we have explored the application of this technique to the study of discrete hypergeometric orthogonal polynomials, Toda equations, τ-functions, and Laguerre–Freud equations, see [35–37, 55].

Multiple orthogonal polynomials [5, 46, 57, 61] arise when multiple weight functions are involved in the orthogonality conditions. Unlike classical orthogonal polynomials, where a single weight function is used, multiple orthogonal polynomials exhibit a more intricate structure due to the interplay of multiple weight functions. These polynomials have been studied extensively and find applications in areas such as simultaneous approximation, random matrix theory, and statistical mechanics. For the appearance and use of the Pearson equation in multiple orthogonality, see [21, 29], see also [22]. On the other hand, discrete orthogonal polynomials [46, 60] emerge when the orthogonality conditions are defined on a discrete set of points. These polynomials find applications in various discrete systems, such as combinatorial optimization, signal processing, and coding theory. Discrete orthogonal polynomials provide a valuable tool for analyzing and solving problems in discrete settings, where the underlying structure is often characterized by a finite or countable set of points. Discrete Pearson equations play an important role in the classification of classical discrete orthogonal polynomials [60]. When a discrete Pearson equation is fulfilled by the weight, we are dealing with semiclassical discrete orthogonal polynomials, see [30, 32–34]. Multiple discrete orthogonal polynomials were discussed in [17].

Integrable discrete equations [45] have emerged as a fascinating and important field of study in Mathematical Physics, providing insights into the behavior of discrete dynamical systems with remarkable properties. These equations possess a rich algebraic and geometric structure, allowing for the existence of soliton solutions, conservation laws, and integrability properties. The concept of integrability in discrete equations goes beyond mere solvability and extends to the existence of an abundance of conserved quantities and symmetries [4]. This integrability property enables the development of powerful mathematical methods for studying and understanding their behavior. One of the key features of integrable discrete equations is their connection to the theory of orthogonal polynomials. These polynomials play a central role in the construction of discrete equations with special properties. The link between discrete equations and orthogonal polynomials provides a deep understanding of their solutions, recursion relations, and symmetry properties.

The τ-function [44] is a key concept in the theory of integrable systems and serves as a bridge between the spectral theory of linear operators and the dynamics of soliton equations. It has profound connections with algebraic geometry, representation theory, and special functions. Semiclassical orthogonal polynomials, along with their corresponding functionals supported on complex curves, have been extensively studied in relation to isomonodromic tau functions and matrix models by Bertola, Eynard, and Harnad [20]. In the context of discrete multiple orthogonal polynomials, the τ-function captures the underlying integrable structure and provides insights into the dynamics and symmetries of the corresponding discrete systems. Its study enables us to uncover deep connections between different mathematical objects and uncover hidden patterns.

Generalized hypergeometric series, another important topic in this research, constitute a class of special functions that arise in diverse mathematical contexts [6, 24]. They possess remarkable properties and have been extensively studied due to their connections with combinatorics, number theory, and mathematical physics. Generalized hypergeometric series play a key role in expressing solutions of differential equations, evaluating integrals, and understanding the behavior of various mathematical models. They have a pivotal role in the Askey scheme that gathers together different important families of orthogonal families within a chain of limits [47]. See [19] for an extension of the Askey scheme to multiple orthogonality. Recently, we have completed a part of the multiple Askey scheme by finding hypergeometric expressions for the Hahn type I multiple orthogonal polynomials and their descendants in the multiple Askey scheme [23] (but for the multiple Hermite case).

Laguerre–Freud equations are a family of differential equations that emerge in the study of orthogonal polynomials associated with exponential weights. For orthogonal polynomials, these Laguerre–Freud equations represent nonlinear relations for the recursion coefficients in the three-term recurrence relation. In other words, we are dealing with a set of nonlinear equations for the coefficients $\{\beta_n, \gamma_n\}$ of the three-term recursion relations $zP_n(z) = P_{n+1}(z) + \beta_n P_n(z) + \gamma_n P_{n-1}(z)$ satisfied by the orthogonal polynomial sequence. These Laguerre–Freud equations take the form $\gamma_{n+1} = \mathcal{G}(n, \gamma_n, \gamma_{n-1}, \dots, \beta_n, \beta_{n-1}, \dots)$ and $\beta_{n+1} = \mathcal{B}(n, \gamma_{n+1}, \gamma_n, \dots, \beta_n, \beta_{n-1}, \dots)$. Magnus [49–52], referring to [42, 48], named these types of relations Laguerre–Freud relations. Several papers discuss Laguerre–Freud relations for the generalized Charlier, generalized Meixner, and type I generalized Hahn cases [27, 30, 38–40, 64].

Laguerre–Freud equations are also connected to Painlevé equations. The Painlevé equations are a set of nonlinear ordinary differential equations that were first introduced by the French mathematician Paul Painlevé in the early 20th century. These equations have attracted significant attention due to their integrability properties and the rich mathematical structures associated with their solutions. The connection between orthogonal polynomials and Painlevé equations lies in the fact that certain families of orthogonal polynomials can be related to solutions of specific Painlevé equations. This relationship provides deep insights into the solvability and analytic properties of the Painlevé equations and allows for the construction of explicit solutions using the theory of orthogonal polynomials. For an account of the connection between these fields of study, see [27, 28, 65].

Moreover, building upon the mentioned studies [35–37, 55], this paper investigates the relationship between discrete multiple orthogonal polynomials, τ-functions, generalized hypergeometric series, and specific equations such as Laguerre–Freud equations and Toda type equations.

Toda type equations, on the other hand, are nonlinear partial differential-difference equations with wide-ranging applications in mathematical physics and integrable systems [44, 45]. Exploring the connections between these equations and the aforementioned mathematical objects sheds light on the interplay between discrete systems, special functions, and integrable models.

In this paper, we aim to provide a comprehensive overview of the theory of discrete multiple orthogonal polynomials, τ-functions, generalized hypergeometric series, Laguerre–Freud equations, and Toda type equations. We will delve into the mathematical properties, explicit representations, and recurrence relations of discrete multiple orthogonal polynomials. Furthermore, we will explore the relationship between these polynomials, τ-functions, and generalized hypergeometric series. Finally, we will investigate the applications of these concepts to Laguerre–Freud equations and Toda type equations, including both continuous and totally discrete types, showcasing their relevance in the field of mathematical physics and integrable systems.

The layout of the paper is as follows. We continue this section with a basic introduction to discrete multiple orthogonal polynomials and their associated tau functions. In the second section, we delve into Pearson equations for multiple orthogonal polynomials, generalized hypergeometric series, and the corresponding hypergeometric discrete multiple orthogonal polynomials. We discuss their properties, including the Laguerre–Freud matrix that models the shift in the spectral variable, as well as contiguity relations and their consequences.

In the third section, we present two of the main findings discussed in this paper. In Theorem 3.9, we present nonlinear partial differential equations of the Toda type, specifically of the third order, for which the hypergeometric tau functions provide solutions. Additionally, in Theorem 3.20, we provide a completely discrete system, extending the completely discrete Nikhoff–Capel Toda equation, and its solutions in terms of the hypergeometric tau functions.

Finally, in the fourth section, we present explicit Laguerre–Freud equations for the first time for the multiple versions of the generalized Charlier and generalized Meixner II orthogonal polynomials.

1.1 Discrete multiple orthogonality on the step line with two weights

Let us introduce the following vectors of monomials in the independent variable x

$$X=\begin{bmatrix}1 & x & x^2 \cdots\end{bmatrix}^\top, \qquad X^{(1)}=\begin{bmatrix}1 & 0 & x & 0\cdots\end{bmatrix}^\top, \qquad X^{(2)}=\begin{bmatrix}0 & 1 & 0 & x\cdots\end{bmatrix}^\top.$$

Let us also consider a couple of weights $w^{(1)}, w^{(2)} : \mathbb{N}_0 \to \mathbb{C}$ defined on the homogeneous lattice $\mathbb{N}_0$. The corresponding moment matrix is given by

$$\mathscr{M}:=\sum_{k=0}^{\infty} X(k)\big(X^{(1)}(k)w^{(1)}(k)+X^{(2)}(k)w^{(2)}(k)\big)^\top.$$

We also consider the matrices

$$I^{(1)}:=\operatorname{diag}(1,0,1,0,\dots), \qquad I^{(2)}:=\operatorname{diag}(0,1,0,1,\dots).$$

and shift matrices

$$\Lambda := \begin{bmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

and $\Lambda^{(a)} := \Lambda^2 I^{(a)}$, $a \in \{1, 2\}$. The following equations are satisfied

$$\begin{aligned} &\Lambda X(x) = xX(x), \quad \Lambda^{(1)} X^{(1)}(x) = xX^{(1)}(x), \quad \Lambda^{(2)} X^{(2)}(x) = xX^{(2)}(x), \\ &\Lambda^{(1)} X^{(2)}(x) = 0, \qquad \Lambda^{(2)} X^{(1)}(x) = 0. \end{aligned}$$

The moment matrix satisfies

$$\Lambda \mathscr{M} = \mathscr{M} (\Lambda^\top)^2,$$

so that we say that a matrix is bi-Hankel matrix.

The Gauss-Borel or LU factorization of the moment matrix is

$$\mathscr{M} = S^{-1} H \tilde{S}^{-\top} \tag{1}$$

where S and $\tilde{S}$ are lower unitriangular matrices and H is a diagonal matrix.

We define the truncated matrix $\mathscr{M}^{[n]}$ as the n-th leading principal submatrix of the moment matrix $\mathscr{M}$, i.e. built up with the first n rows and columns. Then, we introduce the so called τ-function, as

$$\tau_n = \det \mathscr{M}^{[n]}. \tag{2}$$

Factorization (1) exists whenever $\tau_n \neq 0$ for $n \in \mathbb{N}_0$.

Matrices S and $\tilde{S}$ expand as power series in $\Lambda^\top$ with diagonal matrices coefficients

$$\begin{aligned} S &= I + \Lambda^\top S^{[1]} + (\Lambda^\top)^2 S^{[2]} + \cdots, & S^{-1} &= I + \Lambda^\top S^{[-1]} + (\Lambda^\top)^2 S^{[-2]} + \cdots, \\ \tilde{S} &= I + \Lambda^\top \tilde{S}^{[1]} + (\Lambda^\top)^2 \tilde{S}^{[2]} + \cdots, & \tilde{S}^{-1} &= I + \Lambda^\top \tilde{S}^{[-1]} + (\Lambda^\top)^2 \tilde{S}^{[-2]} + \cdots. \end{aligned}$$

In particular, the following equations hold

$$S^{[-1]} = -S^{[1]}, \qquad S^{[-2]} = -S^{[2]} + (\mathfrak{a}_- S^{[1]}) S^{[1]},$$

and matrices $\tilde{S}^{[-1]}$ and $\tilde{S}^{[-2]}$ fulfill analogous relations.

In what follows we will use lowering and rising shift matrices given by

$$\begin{aligned} \mathfrak{a}_- \operatorname{diag}(m_0, m_1, \dots) &= \operatorname{diag}(m_1, m_2, \dots), \\ \mathfrak{a}_+ \operatorname{diag}(m_0, m_1, \dots) &= \operatorname{diag}(0, m_0, \dots). \end{aligned}$$

Any diagonal matrix D fulfills the following algebraic properties:

$$\Lambda D=(\mathfrak{a}_-D)\Lambda, \qquad D\Lambda=\Lambda\mathfrak{a}_+D, \qquad D\Lambda^\top=\Lambda^\top\mathfrak{a}_-D, \qquad \Lambda^\top D=\mathfrak{a}_+D\Lambda^\top.$$

Multiple orthogonal polynomials of type II and I on the step-line are given as entries of certain vector of polynomials. Type II multiple orthogonal polynomial B_n are the entries of the vector B, while type I multiple orthogonal polynomials $A^{(a)}_n$, $a\in\{1,2\}$, are the entries of the vectors $A^{(a)}$ where

$$B:=SX, \qquad\qquad A^{(a)}:=H^{-1}\tilde{S}X^{(a)}, \qquad\qquad a\in\{1,2\}.$$

The degrees of these polynomials, for perfect sytems of weights, are easily seen to be

$$\deg B_n=n, \qquad\qquad \deg A^{(a)}_n=\left\lceil\frac{n+2-a}{2}\right\rceil-1, \qquad\qquad a\in\{1,2\},$$

here $\lceil x\rceil$ is the the smallest integer that is greater than or equal to x. The following multiple orthogonality relations are satisfied

$$\begin{aligned}\sum_{a=1}^{2}\sum_{k=0}^{\infty}A^{(a)}_n(k)w^{(a)}(k)k^m&=0, \qquad k\in\{0,\dots,\deg B_{n-1}\},\\ \sum_{k=0}^{\infty}B_n(k)w^{(a)}(k)k^m&=0, \qquad k\in\left\{0,\dots,\deg A^{(a)}_{n-1}\right\}, \qquad a\in\{1,2\}.\end{aligned}$$

According to [5, Proposition 7] the type II polynomials can be expressed as the following quotient of determinants

$$B_n=\frac{1}{\tau_n}\left|\begin{array}{c|c}\mathscr{M}^{[n]} & \begin{matrix}1\\ \vdots\\ x^{n-1}\end{matrix}\\ \hline \mathscr{M}_{n,0}\cdots\cdots\cdots\mathscr{M}_{n,n-1} & x^n\end{array}\right|, \qquad\qquad n\in\mathbb{N}_0. \tag{3}$$

Hence, for the coefficients p^j_n of the type II multiple orthogonal polynomials

$$B_n=x^n+p^1_nx^{n-1}+\cdots+p^{n-1}_nx+p^n_n$$

we get determinantal expressions. For that aim, let us consider the *associated τ-functions* τ^j_n, $j\in\{1,\dots,n-1\}$, as the determinants of the matrix $\mathscr{M}^{[n,j]}$, $j\in\{1,\dots,n-1\}$, obtained from $\mathscr{M}^{[n]}$ by removing the $(n-j)$-th row $\left[\mathscr{M}_{n-j,0}\cdots\cdots\mathscr{M}_{n-j,n-1}\right]$ and adding as last row the row $\left[\mathscr{M}_{n,0}\cdots\cdots\mathscr{M}_{n,n-1}\right]$. Then, by expanding the determinant in the RHS of (3) by the last column we get

$$p^j_n=(-1)^j\frac{\tau^j_n}{\tau_n}. \tag{4}$$

Similarly [5] we have

$$A_n^{(a)} = \frac{1}{\tau_{n+1}} \left| \begin{array}{c|c} \mathscr{M}^{[n]} & \begin{matrix} \mathscr{M}_{0,n} \\ \vdots \\ \mathscr{M}_{n-1,n} \end{matrix} \\ \hline X_0^{(a)} \cdots\cdots\cdots X_{n-1}^{(a)} & X_n^{(a)} \end{array} \right|, \qquad n \in \mathbb{N}_0. \tag{5}$$

The Hessenberg matrix $T{:=}S\Lambda S^{-1}$ and its transposed matrix is $T^{\top} = H^{-1}\tilde{S}\Lambda^2\tilde{S}^{-1}H$ give the 4-term recursion relations related to these multiple orthogonal polynomials. Indeed, the following recurrence relations hold

$$TB = zB, \qquad T^{\top}A^{(a)} = zA^{(a)}, \qquad a \in \{1, 2\}. \tag{6}$$

Notice that T is a tetradiagonal Hessenberg matrix, i.e.,

$$T = (\Lambda^{\top})^2\gamma + \Lambda^{\top}\beta + \alpha + \Lambda, \tag{7}$$

where α, β, and γ are the following diagonal matrices

$$\alpha = \operatorname{diag}(\alpha_0, \alpha_1, \dots), \qquad \beta = \operatorname{diag}(\beta_1, \beta_2, \dots), \qquad \gamma = \operatorname{diag}(\gamma_2, \gamma_3, \dots).$$

The diagonal matrices α, β, and γ are related with S, $\tilde{S}$ and H matrices by the following equations:

$$\begin{cases} \alpha = S^{[-1]} + \mathfrak{a}_+ S^{[1]} = \mathfrak{a}_+ S^{[1]} - S^{[1]} = \mathfrak{a}_+^2 \tilde{S}^{[2]} - \tilde{S}^{[2]} + \tilde{S}^{[1]}(\mathfrak{a}_- \tilde{S}^{[1]} - \mathfrak{a}_+ \tilde{S}^{[1]}), \\ \beta = H^{-1}\mathfrak{a}_- H(\mathfrak{a}_+ \tilde{S}^{[1]} + \mathfrak{a}_- \tilde{S}^{[-1]}) = H^{-1}\mathfrak{a}_- H(\mathfrak{a}_+ \tilde{S}^{[1]} - \mathfrak{a}_- \tilde{S}^{[1]}) = \mathfrak{a}_+ S^{[2]} - S^{[2]} - S^{[1]}\mathfrak{a}_- \alpha, \\ \gamma = H^{-1}\mathfrak{a}_-^2 H. \end{cases} \tag{8}$$

We introduce the following matrices $T^{(1)}$ and $T^{(2)}$

$$T^{(a)}{:=}H^{-1}\tilde{S}\Lambda^{(a)}\tilde{S}^{-1}H = T^{\top}M^{(a)}, \quad M^{(a)}{:=}H^{-1}\tilde{S}I^{(a)}\tilde{S}^{-1}H, \quad a \in \{1, 2\}.$$

These matrices satisfy

$$T^{(a)}A^{(b)} = \delta_{a,b}zA^{(b)}, \qquad a, b \in \{1, 2\}.$$

as well as $T^{\top} = T^{(1)} + T^{(2)}$. Hence, $T^{(a)}$ are matrices with all its superdiagonals but for the first and the second have zero entries.

The Pascal matrices $L = (L_{n,m})$ have entries given by

$$L_{n,m} = \begin{cases} \binom{n}{m}, & n \geq m, \\ 0, & n < m, \end{cases}$$

while its inverse $L^{-1}=(\tilde{L}_{n,m})$ have entries

$$\tilde{L}_{n,m}=\begin{cases}(-1)^{n+m}\binom{n}{m}, & n\geq m,\\ 0 & n<m.\end{cases}$$

These matrices act on the monomial vectors as follows

$$X(z+1)=LX(z), \qquad\qquad X(z-1)=L^{-1}X(z).$$

and can be expanded as

$$L^{\pm 1}=I\pm\Lambda^{\top}D+(\Lambda^{\top})^2D^{[2]}+\cdots$$

with $D{:=}\operatorname{diag}(1,2,3,\dots)$ and $D^{[n]}{:=}\frac{1}{n}\operatorname{diag}(n^{(n)},(n+1)^{(n)},\dots)$ where $x^{(n)}=x(x-1)\cdots(x-n+1)$. In terms of Pascal matrices we define the dressed Pascal matrices

$$\Pi{:=}SLS^{-1}, \qquad\qquad \Pi^{-1}{:=}SL^{-1}S^{-1}.$$

For type II multiple orthogonal polynomials we find the following properties:

$$B(z+1)=\Pi B(z), \qquad\qquad B(z-1)=\Pi^{-1}B(z).$$

We can proceed similarly for the type I multiple orthogonal polynomials. We introduce the matrices $L^{\pm(a)}$, $a\in\{1,2\}$, with nonzero entries

$$L^{\pm(a)}_{2n+a-1,2m+a-1}=(\pm 1)^{n+m}\binom{n}{m}, \qquad a\in\{1,2\}, \qquad n,m\in\mathbb{N}_0, \qquad n\geq m,$$

and zero for any other couple sub-indexes. We will also use $L^{(a)}=L^{+(a)}$. These matrices satisfy

$$L^{(a)}L^{-(a)}=I^{(a)}, \qquad\qquad a\in\{1,2\}.$$

The partial Pascal matrices can be expressed as band matrices, but in this case the even or odd diagonals are null, i.e.,

$$L^{\pm(a)}=I^{(a)}\pm(\Lambda^{\top})^2D^{(a),[1]}+(\Lambda^{\top})^4D^{(a),[2]}+\cdots,$$

where we have

$$D_1^{(1),[n]}=\frac{1}{n}\operatorname{diag}(n^{(n)},0,(n+1)^{(n)},0,\dots), \qquad D^{(2),[n]}=\frac{1}{n}\operatorname{diag}(0,n^{(n)},0(n+1)^{(n)},\dots).$$

Analogously, dressed Pascal matrices are defined by

$$\Pi^{\pm(a)}{:=}H^{-1}\tilde{S}L^{\pm(a)}\tilde{S}^{-1}H,$$

and the following relations are satisfied

$$A^{(a)}(z \pm 1) = \Pi^{\pm(a)} A^{(a)}(z).$$

2 Discrete Multiple Orthogonal Polynomials and Pearson equations

2.1 Pearson equation

Let us consider that weights $w^{(1)}, w^{(2)}$ subject to the following discrete Pearson equations

$$\theta(k+1)w^{(a)}(k+1) = \sigma^{(a)}(k)w^{(a)}(k), \qquad a \in \{1, 2\}. \tag{9}$$

where θ, $\sigma^{(1)}$ and $\sigma^{(2)}$ are polynomials, and $\theta(0) = 0$. An important result regarding these type of weights is:

Theorem 2.1 *Let us assume that the weights $w^{(1)}, w^{(2)}$ solve the Pearson equations (9). Then, the corresponding moment matrix $\mathscr{M}$ satisfy the following matrix equation*

$$\theta(\Lambda)\mathscr{M} = L\mathscr{M}\left(L^{(1)}\sigma^{(1)}(\Lambda^{(1)}) + L^{(2)}\sigma^{(2)}(\Lambda^{(2)})\right)^{\mathsf{T}}. \tag{10}$$

Proof For the moment matrix is $\mathscr{M}$ we find

$$\begin{aligned}
\theta(\Lambda)\mathscr{M} &= \sum_{k=1}^{\infty} \theta(k)X(k)\big(w^{(1)}X^{(1)}(k) + w^{(2)}X^{(2)}(k)\big)^{\mathsf{T}} \\
&= \sum_{k=0}^{\infty} \theta(k+1)X(k+1)\big(w^{(1)}(k+1)X^{(1)}(k+1) + w^{(2)}(k+1)X^{(2)}(k+1)\big)^{\mathsf{T}}) \\
&= \sum_{k=0}^{\infty} X(k+1)\big(\sigma^{(1)}(k)w^{(1)}(k)X^{(1)}(k+1) + \sigma^{(2)}(k)w^{(2)}(k)X^{(2)}(k+1)\big)^{\mathsf{T}} \\
&= \sum_{k=0}^{\infty} LX(k)\big(\sigma^{(1)}(k)w^{(1)}(k)X^{(1)}(k)^{\mathsf{T}}L^{(1)^{\mathsf{T}}} + \sigma^{(2)}(k)w^{(2)}(k)X^{(2)}(k)^{\mathsf{T}}L^{(2)^{\mathsf{T}}}\big) \\
&= \sum_{k=0}^{\infty} LX(k)\big(X^{(1)}(k)^{\mathsf{T}}w^{(1)}(k) + X^{(2)}(k)^{\mathsf{T}}w^{(2)}(k)\big)\big(\sigma^{(1)}(\Lambda^{(1)^{\mathsf{T}}})L^{(1)^{\mathsf{T}}} \\
&\quad + \sigma^{(2)}(\Lambda^{(2)^{\mathsf{T}}})L^{(2)^{\mathsf{T}}}\big) \\
&= L\mathscr{M}[L^{(1)}\sigma^{(1)}(\Lambda^{(1)}) + L^{(2)}\sigma^{(2)}(\Lambda^{(2)})]^{\mathsf{T}}.
\end{aligned}$$

□

This matrix property implies for the tetradiagonal Hessenberg recursion matrix T the following

Proposition 2.2 *Let us assume that the weights $w^{(1)}$, $w^{(2)}$ solve the Pearson equations (9), for the recursion matrix T in (7), the following relation holds*

$$\Pi^{-1}\theta(T) = \sigma^{(1)}\big(T^{(1)^{\top}}\big)\Pi^{(1)^{\top}} + \sigma^{(2)}\big(T^{(2)^{\top}}\big)\Pi^{(2)^{\top}}. \tag{11}$$

Proof Using Eq. (10) and the Gauss–Borel factorization we get

$$\theta(\Lambda)S^{-1}H\tilde{S}^{-\top} = LS^{-1}H\tilde{S}^{-\top}\big(\sigma^{(1)}(\Lambda^{(1)^{\top}})L^{(1)^{\top}} + \sigma^{(2)}(\Lambda^{(2)^{\top}})L^{(2)^{\top}}\big),$$

grouping S on one side and $\tilde{S}$ on the other side of the equation:

$$S\theta(\Lambda)S^{-1} = SLS^{-1}H\tilde{S}^{\top}(\sigma^{(1)}(\Lambda^{(1)^{\top}})L^{(1)^{\top}} + \sigma^{(2)}(\Lambda^{(2)^{\top}})L^{(2)^{\top}})\tilde{S}^{\top}H^{-1},$$

it can be written as

$$\Pi^{-1}\theta(T) = H\tilde{S}^{-\top}\Big(\sigma^{(1)}(T_1^{\top})(\tilde{S}^{\top}H^{-1})(H\tilde{S}^{-\top})L^{(1)^{\top}} + \sigma^{(2)}(T_2^{\top})(\tilde{S}^{\top}H^{-1})(H\tilde{S}^{-\top})L^{(2)^{\top}}\Big)\tilde{S}^{\top}H^{-1},$$

and finally we get Eq. (11). □

2.2 Hypergeometric discrete multiple orthogonal polynomials

Let us write the polynomials θ and $\sigma^{(a)}$, $a \in \{1, 2\}$, in Eq. (9) as follows

$$\begin{aligned}\theta(z) &= z(z + c_1 - 1)\cdots(z + c_N - 1), \qquad \sigma^{(a)}(z)\\ &= \eta^{(a)}\big(z + b_1^{(a)}\big)\cdots\big(z + b_{M^{(a)}}^{(a)}\big), \qquad M^{(a)}, N \in \mathbb{N}_0, \qquad a \in \{1, 2\}.\end{aligned} \tag{12}$$

In the subsequent discussion we require of generalized hypergeometric series, see [6, 63],

$${}_MF_N\left[\begin{matrix} b_1, \dots, b_M \\ c_1, \dots, c_N \end{matrix}; \eta\right] := \sum_{k=0}^{\infty} \frac{(b_1)_k \cdots (b_M)_k}{(c_1)_k \cdots (c_N)_k} \frac{\eta^k}{k!},$$

where we use the Pochhammer symbol $(b)_k := b(b+1)\cdots(b+k-1)$ and $(b)_0 = 1$,

Pearson equations (9) determine uniquely the weights $w^{(1)}$, $w^{(2)}$, up to multiplicative constant, to be

$$w^{(a)}(k) = \frac{\big(b_1^{(a)}\big)_k \cdots \Big(b_{M^{(a)}}^{(a)}\Big)_k}{(c_1)_k \cdots (c_N)_k} \frac{(\eta^{(a)})^k}{k!}, \qquad a \in \{1, 2\}. \tag{13}$$

In what follows we will use logarithmic derivatives

$$\vartheta^{(a)} := \eta^{(a)} \frac{\partial}{\partial \eta^{(a)}}, \qquad a \in \{1, 2\},$$

as well as

$$\vartheta := \vartheta^{(1)} + \vartheta^{(2)}, \qquad \tilde{\vartheta} := \vartheta^{(1)} - \vartheta^{(2)}.$$

We introduce $t^{(a)} := \log \eta^{(a)}$, $a \in \{1, 2\}$ and

$$t := \frac{t^{(1)} + t^{(2)}}{2}, \qquad \tilde{t} := \frac{t^{(1)} - t^{(2)}}{2},$$

so that $\eta^{(1)} = \mathrm{e}^{t+\tilde{t}}$ and $\eta^{(2)} = \mathrm{e}^{t-\tilde{t}}$

Lemma 2.3 *For the differential operators ϑ and $\tilde{\vartheta}$ we find*

$$\vartheta = \frac{\partial}{\partial t}, \qquad \tilde{\vartheta} = \frac{\partial}{\partial \tilde{t}}.$$

Proof It follows from

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t^{(1)}} + \frac{\partial}{\partial t^{(2)}} = \vartheta, \qquad \frac{\partial}{\partial \tilde{t}} = \frac{\partial}{\partial t^{(1)}} - \frac{\partial}{\partial t^{(2)}} = \tilde{\vartheta}.$$

□

An open question is whether the system of weights $\{w^{(1)}, w^{(2)}\}$ is perfect. That is the case if we are dealing with AT systems, see [46]. Following [17, Examples 2.1 and 2.2] we can give two families satisfying this condition:

Lemma 2.4 (Perfect & AT systems) *The system of weights $\{w^{(1)}, w^{(2)}\}$ with:*

(i)

$$w^{(a)}(k) = \frac{\big(b_1\big)_k \cdots \big(b_M\big)_k}{(c_1)_k \cdots (c_N)_k} \frac{(\eta^{(a)})^k}{k!}, \qquad a \in \{1, 2\},$$

where $\eta^{(a)}, b_i, c_j > 0$, and

(ii)

$$w^{(a)}(k) = \frac{\big(b_1\big)_k \cdots \big(b_{M-1}\big)_k (b^{(a)})_k}{(c_1)_k \cdots (c_N)_k} \frac{\eta^k}{k!}, \qquad a \in \{1, 2\},$$

with $\eta, b_i, c_j, b^{(a)} > 0$

are AT systems and consequently a perfect systems.

Proof (i) This fit in [17, Examples 2.1].
(ii) This fit in [17, Examples 2.2].

□

Proposition 2.5 *In terms of generalized hypergeometric functions, the moment matrix can be written as a block matrix*

$$\mathscr{M}=\left[\begin{array}{cc|cc|cc|c}
\rho_0^{(1)} & \rho_0^{(2)} & \rho_1^{(1)} & \rho_1^{(2)} & \rho_2^{(1)} & \rho_2^{(2)} & \cdots\cdots\cdots \\ \hline
\rho_1^{(1)} & \rho_1^{(2)} & \rho_2^{(1)} & \rho_2^{(2)} & \rho_3^{(1)} & \rho_3^{(2)} & \cdots\cdots\cdots \\ \hline
\rho_2^{(1)} & \rho_2^{(2)} & \rho_3^{(1)} & \rho_3^{(2)} & \rho_4^{(1)} & \rho_4^{(2)} & \cdots\cdots\cdots \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}\right], \tag{14}$$

where the moments are expressed in terms of generalized hypergeometric series as follows

$$\rho_n^{(a)}=(\vartheta^{(a)})^n\rho_0^{(a)},\quad \rho_0^{(a)}:={}_{M_a}F_N\left[\begin{matrix} b_1^{(a)},\dots,b_{M^{(a)}}^{(a)} \\ c_1,\dots,c_N\end{matrix};\eta^{(a)}\right],\quad a\in\{1,2\},\quad n\in\mathbb{N}_0. \tag{15}$$

Proof It follows from the definition of the moment matrix $\mathscr{M}$. Equation (15) determining the moments in terms of two families of generalized hypergeometric series follows from Eq. (14) and [55]. □

Given a function $f=f(\eta)$ we consider the covector $\delta_n(f):=\big[f\ \vartheta f\cdots\cdots\vartheta^{n-1}f\big]$. For two functions $f_1(\eta^{(1)})$ we consider the double Wronskian of the covectors $\delta_n(f_1)$ and $\delta_n(f_2)$ given by

$$\mathscr{W}_{2n}[f_1,f_2]:=\left|\begin{array}{cc|cc|cc|c|cc}
f_1 & f_2 & \vartheta f_1 & \vartheta f_2 & \vartheta^2 f_1 & \vartheta^2 f_2 & \cdots\cdots & \vartheta^{n-1}f_1 & \vartheta^{n-1}f_2 \\ \hline
\vartheta f_1 & \vartheta f_2 & \vartheta^2 f_1 & \vartheta^2 f_2 & \vartheta^3 f_1 & \vartheta^3 f_1 & \cdots\cdots & \vartheta^{n}f_1 & \vartheta^{n}f_2 \\ \hline
\vartheta^2 f_1 & \vartheta^2 f_2 & \vartheta^3 f_1 & \vartheta^3 f_2 & \vartheta^4 f_1 & \vartheta^4 f_2 & \cdots\cdots & \vartheta^{n}f_1 & \vartheta^{n}f_2 \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \hline
\vartheta^{2n-1}f_1 & \vartheta^{2n-1}f_2 & \vartheta^{2n}f_1 & \vartheta^{2n}f_2 & \vartheta^{2n+1}f_1 & \vartheta^{2n+1}f_2 & \cdots\cdots & \vartheta^{3n-1}f_1 & \vartheta^{3n-1}f_2
\end{array}\right|,$$

the double Wronskian of the covectors $\delta_{n+1}(f_1)$ and $\delta_n(f_2)$ given by

$$\mathscr{W}_{2n+1}[f_1,f_2]:=$$

$$\left|\begin{array}{cc|cc|cc|c|cc|c}
f_1 & f_2 & \vartheta f_1 & \vartheta f_2 & \vartheta^2 f_1 & \vartheta^2 f_2 & \cdots\cdots & \vartheta^{n-1}f_1 & \vartheta^{n-1}f_2 & \vartheta^{n}f_1 \\ \hline
\vartheta f_1 & \vartheta f_2 & \vartheta^2 f_1 & \vartheta^2 f_2 & \vartheta^3 f_1 & \vartheta^3 f_1 & \cdots\cdots & \vartheta^{n}f_1 & \vartheta^{n}f_2 & \vartheta^{n+1}f_1 \\ \hline
\vartheta^2 f_1 & \vartheta^2 f_2 & \vartheta^3 f_1 & \vartheta^3 f_2 & \vartheta^4 f_1 & \vartheta^4 f_2 & \cdots\cdots & \vartheta^{n+1}f_1 & \vartheta^{n+1}f_2 & \vartheta^{n+2}f_1 \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \hline
\vartheta^{2n-1}f_1 & \vartheta^{2n-1}f_2 & \vartheta^{2n}f_1 & \vartheta^{2n}f_2 & \vartheta^{2n+1}f_1 & \vartheta^{2n+1}f_2 & \cdots\cdots & \vartheta^{3n-1}f_1 & \vartheta^{3n-1}f_2 & \vartheta^{3n}f_1 \\ \hline
\vartheta^{2n}f_1 & \vartheta^{2n}f_2 & \vartheta^{2n+1}f_1 & \vartheta^{2n+1}f_2 & \vartheta^{2n+2}f_1 & \vartheta^{2n+2}f_2 & \cdots\cdots & \vartheta^{3n}f_1 & \vartheta^{3n}f_2 & \vartheta^{3n+1}f_1
\end{array}\right|.$$

We refer to these objects as double δ-Wronskians. Then, the τ-function (2) can be written as the double δ-Wronskian of two generalized hypergeometric series as follows

$$\tau_n = \mathscr{W}_n\left[{}_{M_1}F_N\left[\begin{matrix} b_1^{(1)},\dots,b_{M^{(1)}}^{(1)} \\ c_1,\dots,c_N \end{matrix};\eta^{(1)}\right], {}_{M_2}F_N\left[\begin{matrix} b_1^{(2)},\dots,b_{M^{(2)}}^{(2)} \\ c_1,\dots,c_N \end{matrix};\eta^{(2)}\right]\right].$$

Double or 2-component Wronskians were first considered, within the context of integrable systems, by Freeman, Gilson and Nimmo in [41], see also [43].

Lemma 2.6 *For the associated τ-function we find*

$$\tau_n^1 = \vartheta\tau_n.$$

Proof It follows from the multi-linearity of the determinant and Equation (14) in where the even columns depend only on $\eta^{(1)}$ and the odd columns on $\eta^{(2)}$. □

Proposition 2.7 *The following expressions in terms of the τ functions hold true*

$$H_n = \frac{\tau_{n+1}}{\tau_n}, \qquad p_n^1 = -\vartheta \log \tau_n. \tag{16}$$

Proof It can be proven [5]

$$H_n = \frac{\tau_{n+1}}{\tau_n}, \qquad p_n^1 = -\frac{\tau_n^1}{\tau_n}. \tag{17}$$

Then, using the previous Lemma 2.6 we get the result. □

Proposition 2.8 *The moment matrix satisfies*

$$\vartheta^{(a)}\mathscr{M} = \mathscr{M}\Lambda^{(a)^\top}, \qquad a \in \{1,2\}.$$

Proof It follows from Eq. (15). □

Remark 2.9 The previous results apply to more general situations than those provided by the Pearson equation and the generalized hypergeometric series. We only need to assume that $\vartheta w^{(a)}(k) = kw^{(a)}(k)$. Using the double δ-Wronskian, we can construct a tau function as follows:

$$\tau_n = \mathscr{W}_n[\rho_0^{(1)},\rho_0^{(2)}], \qquad \rho_0^{(a)} = \sum_{n=0}^{\infty} w^{(a)}(k), \qquad a \in \{1,2\}.$$

Then, Lemma 2.6 and Propositions 2.7 and 2.8 hold true.

2.3 Laguerre-Freud matrix

Theorem 2.10 (Laguerre–Freud matrix)*Let us assume that the weights fulfill discrte Pearson equation (9), where θ, $\sigma^{(1)}$ and $\sigma^{(2)}$ are polynomials with* $\deg\theta = N$ *and* $\max(\deg\sigma^{(1)}, \deg\sigma^{(2)}) = M$*.Then, the Laguerre-Freud matrix given by*

$$\Psi = \Pi^{-1}\theta(T) = \sigma^{(1)}(T^{(1)^\top})\Pi^{(1)^\top} + \sigma^{(2)}(T^{(2)^\top})\Pi^{(2)^\top} \tag{18}$$

is a banded matrix with lower bandwidth $2M$ and upper bandwidth N, as follows:

$$\Psi = (\Lambda^\top)^{2M}\psi^{(-2M)} + \cdots + \psi^{(N)}\Lambda^N.$$

Moreover, the following connection formulas are fulfilled:

$$\theta(z)B(z-1) = \Psi B(z), \tag{19}$$

$$(A^{(a)}(z+1))^\top\sigma^{(a)}(z) = (A^{(a)}(z))^\top\Psi, \qquad a \in \{1,2\}. \tag{20}$$

Proof If $\deg\theta = N$ then $\theta(T) = \Lambda^N + M^{(N-1)}\Lambda^{N-1} + \cdots$ so it will have N possible nonzero superdiagonals and using the same argument we conclude that it has $2M$ subdiagonals possibly nonzero.

For the type II polynomials $B(z)$ we have

$$\Psi B(z) = \Pi^{-1}\theta(T)B(z) = \Pi^{-1}\theta(z)B(z) = \theta(z)B(z-1)$$

and for the type I polynomials $A^{(a)}$ we find

$$\begin{aligned}\Psi^\top A^{(a)}(z) &= \left(\Pi^{(1)}\sigma^{(1)}(T^{(1)}) + \Pi^{(2)}\sigma^{(2)}(T^{(2)})\right)A^{(a)}(z)\\ &= \Pi^{(a)}\sigma^{(a)}(z)A^{(a)}(z) = \sigma^{(a)}(z)A^{(a)}(z+1).\end{aligned}$$

□

Lemma 2.11 *If the matrix M is such that $MX(z) = 0$ then $M = 0$.*

Proof From $MX(z) = 0$ we conclude that $M\frac{1}{n!}X^{(n)}(0) = 0$ but $\frac{1}{n!}X^{(n)}(0) = e_n$, where e_n is the vector with all its entries zero but for a unit at the n-th entry. Therefore, we conclude that the n-th column of M has all its entries equal to zero. Consequently, $M = 0$ □

Proposition 2.12 (Compatibility I) *For recursion and Freud–Laguerre matrices the following compatibility relation is fulfilled*

$$[\Psi, T] = \Psi. \tag{21}$$

Proof Using (19) and (6) we find

$$T\Psi B(z) = T\theta(z)B(z-1) = \theta(z)(z-1)\theta(z)B(z-1),$$

$$\Psi(T-I)B(z)=\Psi(z-1)B(z)=(z-1)\theta(z)B(z-1).$$

Therefore, $([\Psi,T]-\Psi)B(z)=0$. Consequently, we get $([\Psi,T]-\Psi)SX(z)=0$ and according to Lemma 2.11 we deduce that $([\Psi,T]-\Psi)S=0$, and as S is an unitriangular matrix, with inverses, we deduce that $[\Psi,T]-\Psi$ is the zero matrix. □

2.4 Contiguous hypergeometric relations

For generalized hypergeometric series we have the following contiguous relations

$$\begin{aligned}
&\left(\eta\frac{\mathrm{d}}{\mathrm{d}\,\eta}+b_i\right){}_MF_N\left[\begin{matrix}b_1,\dots,b_M\\c_1,\dots,c_N\end{matrix};\eta\right]\\
&\quad=b_i\,{}_MF_N\left[\begin{matrix}b_1,\dots,b_i+1,\dots,b_M\\c_1,\dots,c_N\end{matrix};\eta\right], && i\in\{1,\dots,M\}\\
&\left(\eta\frac{\mathrm{d}}{\mathrm{d}\,\eta}+c_j-1\right){}_MF_N\left[\begin{matrix}b_1,\dots,b_M\\c_1,\dots,c_N\end{matrix};\eta\right]\\
&\quad=(c_j-1)\,{}_MF_N\left[\begin{matrix}b_1,\dots,b_M\\c_1,\dots,c_j-1,\dots,c_N\end{matrix};\eta\right], && j\in\{1,\dots,N\}
\end{aligned}$$

Let us denote by ${}_i\Theta^{(a)}$ the shifting the $b_i^{(a)}\to b_i^{(a)}+1$ and by Θ_j the shifting $c_j\to c_j+1$. These contiguity relations translate into the moment matrix as follows:

Theorem 2.13 (Contiguous relations for the moment matrix) *The following equations for the moment matrix hold:*

$$\mathscr{M}\Lambda^{(a)^\top}+b_i^{(a)}\mathscr{M}=b_i^{(a)}{}_i\Theta^{(a)}\mathscr{M},\qquad i\in\{1,\dots,M^{(a)}\},\quad a\in\{1,2\},\tag{22}$$
$$\mathscr{M}\Lambda^{2^\top}+(c_j-1)\mathscr{M}=(c_j-1)\Theta_j\mathscr{M},\quad j\in\{1,\dots,N\}\tag{23}$$

Proof To prove Eq. (22) we notice that, according to (14), we can write the following equations for moment matrix entries

$$\begin{aligned}
\big(\vartheta^{(1)}+b_i^{(1)}\big)\mathscr{M}_{n,2m}&=b_i^{(1)}\Theta_i^{(1)}\mathscr{M}_{n,2m},\\
\big(\vartheta^{(1)}+b_i^{(1)}\big)\mathscr{M}_{n,2m+1}&=b_i^{(1)}\mathscr{M}_{n,2m+1}=\Theta_i^{(1)}\mathscr{M}_{n,2m+1},\\
\big(\vartheta^{(2)}+b_i^{(2)}\big)\mathscr{M}_{n,2m}&=b_i^{(2)}\mathscr{M}_{n,2m}=b_i^{(2)}\Theta_i^{(2)}\mathscr{M}_{n,2m+1},\\
\big(\vartheta^{(2)}+b_i^{(2)}\big)\mathscr{M}_{n,2m+1}&=b_i^{(2)}\Theta_i^{(2)}\mathscr{M}_{n,2m+1}.
\end{aligned}$$

with $n,m\in\mathbb{N}_0$. Let us prove Eq. (23) by noticing that the entries of the moment matrix satisfy

$$\begin{aligned}
(\vartheta+c_j-1)\mathscr{M}_{(n,2m)}&=(c_j-1)\Theta_i\mathscr{M}_{(n,2m)},\\
(\vartheta+c_j-1)\mathscr{M}_{(n,2m+1)}&=(c_j-1)\Theta_i\mathscr{M}_{(n,2m+1)}.
\end{aligned}$$

so that

$$\mathscr{M}(\Lambda^{(1)^\top}+\Lambda^{(2)^\top})+(c_j-1)\mathscr{M}=(c_j-1)\Theta_j\mathscr{M}.$$

□

2.5 Connection matrices

Definition 2.14 Connection matrices are defined as

$${}_i\omega^{(a)}:=({}_i\Theta^{(a)}H)^{-1}({}_i\Theta^{(a)}\tilde{S})(\Lambda^{(a)}+b_i^{(a)})\tilde{S}^{-1}H,\quad i\in\{1,\dots,M^{(a)}\},\quad a\in\{1,2\},\tag{24}$$

$${}_i\Omega^{(a)}:=S({}_i\Theta^{(a)}S)^{-1},\quad i\in\{1,\dots,M^{(a)}\},\quad a\in\{1,2\},\tag{25}$$

$$\omega_j:=(\Theta_jH)^{-1}(\Theta_j\tilde{S})(\Lambda^2+c_j-1)\tilde{S}^{-1}H,\quad j\in\{1,\dots,N\},\tag{26}$$

$$\Omega_j:=S(\Theta_jS)^{-1},\quad j\in\{1,\dots,N\}.\tag{27}$$

Lemma 2.15 *The following relations between connection matrices are fulfilled*

$${}_i\omega^{(a)^\top}=b_i^{(a)}{}_i\Omega^{(a)},\quad i\in\{1,\dots,M^{(a)}\},\quad a\in\{1,2\},\tag{28}$$

$${\omega_j}^\top=(c_j-1)\Omega_j,\quad j\in\{1,\dots,N\}.\tag{29}$$

Proof Let us prove Eq. (28). From the definition (22) and the Gauss–Borel factorization (1) of the moment matrix $\mathscr{M}$ we find

$$S^{-1}H\tilde{S}^{-\top}\Lambda^{(a)^\top}+b_i^{(a)}S^{-1}H\tilde{S}^{-\top}=b_i^{(a)}({}_i\Theta^{(a)}S)^{-1}({}_i\Theta^{(a)}H)({}_i\Theta^{(a)}\tilde{S})^{-\top},\\ i\in\{1,\dots,M^{(a)}\},a\in\{1,2\},$$

that after some cleaning leads to

$$H\tilde{S}^{-\top}(\Lambda^{(a)^\top}+b_i^{(a)})({}_i\Theta^{(a)}\tilde{S})^\top({}_i\Theta^{(a)}H)^{-1}=b_i^{(a)}S({}_i\Theta^{(a)}S)^{-1},i\in\{1,\dots,M^{(a)}\},\\ a\in\{1,2\}.$$

Analogously, Eq. (29) can be proved using (23) and (1). □

Proposition 2.16 *Connection matrices have a banded triangular structure with three non-null diagonals. In the one hand, for $i\in\{1,\dots,M^{(a)}\}$, $a\in\{1,2\}$, we have*

$${}_i\omega^{(a)}=b_i^{(a)}I+\big({}_i\omega^{(a)}\big)^{[1]}\Lambda+\big({}_i\omega^{(a)}\big)^{[2]}\Lambda^2,\tag{30}$$

$${}_i\Omega^{(a)}=\Lambda^{\top2}\frac{1}{b_i^{(a)}}\big({}_i\omega^{(a)}\big)^{[2]}+\Lambda^\top\frac{1}{b_i^{(a)}}\big({}_i\omega^{(a)}\big)^{[1]}+I,\tag{31}$$

with

$$\begin{aligned}&\big({}_i\omega^{(a)}\big)^{[1]}:=b_i^{(a)}(S^{[1]}-{}_i\Theta^{(a)}S^{[1]})=({}_i\Theta^{(a)}H)^{-1}(\mathfrak{a}_-H)\big(\mathfrak{a}_+({}_i\Theta^{(a)}\tilde S^{[1]})I^{(\pi(a))}-I^{(a)}(\mathfrak{a}_-\tilde S^{[1]})\big),\\&\big({}_i\omega^{(a)}\big)^{[2]}:=({}_i\Theta^{(a)}H)^{-1}I^{(a)}\mathfrak{a}_-^2H,\end{aligned}$$

where $\pi\in S_2$ is the permutation $\pi(1)=2$ and $\pi(2)=1$. In the other hand, for $j\in\{1,\dots,N\}$, we find

$$\omega_j=(c_j-1)I+\omega_j^{[1]}\Lambda+\omega_j^{[2]}\Lambda^2, \tag{32}$$

$$\Omega_j=I+\Lambda^\top\frac{1}{c_j-1}\omega_j^{[1]}+(\Lambda^\top)^2\frac{1}{c_j-1}\omega_j^{[2]}, \tag{33}$$

with

$$\begin{aligned}&\omega_j^{[1]}:=(c_j-1)(S^{[1]}-\Theta_jS^{[1]})=\mathfrak{a}_-H(\Theta_jH)^{-1}\big(\mathfrak{a}_+(\Theta_j\tilde S^{[1]})-\mathfrak{a}_-\tilde S^{[1]}\big),\\&\omega_j^{[2]}:=(\Theta_jH)^{-1}\mathfrak{a}_-^2H.\end{aligned}$$

Proof Above relations are found using Eq. (28), (24) and (25). Let us show only the cases ${}_i\omega^{(1)}$ and ${}_i\Omega^{(1)}$, as the others cases are proven similarly. Departing from Eqs. (24) and (25) we see that

$$\begin{aligned}{}_i\omega^{(a)}&=({}_i\Theta^{(a)}H)^{-1}{}_i\Theta^{(a)}(I+\Lambda^\top\tilde S^{[1]}+(\Lambda^\top)^2\tilde S^{[2]}+\cdots)(I^{(a)}\Lambda^2+b_i^{(a)})(I+\Lambda^\top\tilde S^{[-1]}\\&\quad+(\Lambda^\top)^2\tilde S^{[-2]}+\cdots)H,\\{}_i\Omega^{(a)}&=(I+\Lambda^\top S^{[1]}+\cdots){}_i\Theta^{(a)}(I+\Lambda^\top S^{[-1]}+\cdots),\end{aligned}$$

and using (28) we get the result. □

Theorem 2.17 *Vectors of polynomials $A^{(a)}$, $a\in\{1,2\}$, and B fulfill the following connection formulas*

$${}_i\omega^{(a)}A^{(a)}(z)=(z+b_i^{(a)})\big({}_i\Theta^{(a)}A^{(a)}(z)\big), \tag{34}$$

$${}_i\Omega^{(1)}{}_i\Theta^{(a)}B(z)=B(z), \tag{35}$$

$$\omega_jA^{(a)}(z)=(z+c_i-1)\big(\Theta_iA^{(a)}(z)\big), \tag{36}$$

$$\Omega_j\Theta_jB(z)=B(z), \tag{37}$$

with $i\in\{1,\dots,M^{(a)}\}$, $a\in\{1,2\}$, and $j\in\{1,\dots,N\}$.

Proof Equations (34), (35), (36) and (37) can be proved through the action of connection matrix on vectors $A^{(a)}$ and B. Let us check (34):

$$\begin{aligned}{}_i\omega^{(a)}A^{(a)}(z)&=({}_i\Theta^{(a)}H)^{-1}({}_i\Theta^{(a)}\tilde S)(\Lambda^{(a)}+b_i^{(a)})X^{(a)}(z)\\&=({}_i\Theta^{(a)}H)^{-1}({}_i\Theta^{(a)}\tilde S)\big(z+b_i^{(a)}\big)X^{(a)}(z),\end{aligned}$$

and immediately (34) is proved. □

3 Multiple Toda systems and hypergeometric τ-functions

3.1 Continuous case

Let us now explore how the hypergeometric discrete multiple orthogonal polynomials leads to solutions, in terms of τ-functions, which are double Wronskians of generalized hypergeometric series, to multiple Toda type systems.

Let us start by introducing the strictly lower triangular matrices

$$\phi^{(a)}:=(\vartheta^{(a)}S)S^{-1}, \qquad \tilde{\phi}^{(a)}:=(\vartheta^{(a)}\tilde{S})\tilde{S}^{-1}, \qquad a\in\{1,2\}.$$

For them we have the following two lemmas

Lemma 3.1 *The following relations are fulfilled*

$$\vartheta^{(a)}B=\phi^{(a)}B, \qquad a\in\{1,2\}. \tag{38}$$

Proof From $B=SX$ we get

$$\vartheta^{(a)}B=(\vartheta^{(a)}S)X=(\vartheta^{(a)}S)S^{-1}SX=(\vartheta^{(a)}S)S^{-1}B.$$

□

Lemma 3.2 *The following relations are fulfilled*

$$\vartheta^{(a)}(HA^{(a)})=\tilde{\phi}^{(a)}HA^{(a)}, \qquad a\in\{1,2\}. \tag{39}$$

Proof We have that

$$\vartheta^{(a)}(HA^{(a)})=\vartheta(\tilde{S})X^{(a)}=\tilde{\phi}HA^{(a)}.$$

□

Proposition 3.3 *The following equations are fulfilled*

$$\vartheta^{(a)}H-\phi^{(a)}H-H\tilde{\phi}^{(a)\top}=T^{(a)\top}H, \qquad a\in\{1,2\}. \tag{40}$$

Proof Recall that $\mathscr{M}=S^{-1}H\tilde{S}^{-\top}$ so that

$$\begin{aligned}\vartheta^{(a)}(S^{-1}H\tilde{S}^{-\top})&=-S^{-1}(\vartheta^{(a)}S)S^{-1}H\tilde{S}^{-\top}+S^{-1}(\vartheta^{(a)}H)\tilde{S}^{-\top}-S^{-1}H\tilde{S}^{-\top}(\vartheta^{(a)}\tilde{S})^{\top}\tilde{S}^{-\top}\\&=S^{-1}H\tilde{S}^{-\top}\Lambda^{(a)\top}\end{aligned}$$

so

$$\vartheta^{(a)}H-(\vartheta^{(a)}S)S^{-1}H-H\tilde{S}^{-\top}(\vartheta^{(a)}\tilde{S})^{\top}=H\tilde{S}^{-\top}\Lambda^{(a)\top}\tilde{S}^{\top}H^{-1}H=T^{(a)\top}H.$$

□

We will denote:

$$\phi := \phi^{(1)} + \phi^{(2)}, \qquad \tilde{\phi} := \tilde{\phi}^{(1)} + \tilde{\phi}^{(2)}.$$

Proposition 3.4 *The following equations hold*

$$(\vartheta H)H^{-1} = \alpha, \tag{41a}$$

$$-\phi = (\Lambda^\top)^2\gamma + \Lambda^\top\beta, \tag{41b}$$

$$-\tilde{\phi} = \Lambda^\top\mathfrak{a}_- H H^{-1} \tag{41c}$$

Proof From (40) we obtain

$$\vartheta H - \phi H - H\tilde{\phi}^\top = TH = (\Lambda^\top)^2\gamma H + \Lambda^\top\beta H + \alpha H + \mathfrak{a}_- H\Lambda$$

Diagonal by diagonal we get the equations. □

Proposition 3.5 *The following relations are fulfilled*

$$\vartheta S^{[n]} = -(S^{[n-2]}\mathfrak{a}_-^{n-2}\gamma + S^{[n-1]}\mathfrak{a}_-^{n-1}\beta), \tag{42a}$$

$$\vartheta S^{[1]} = -\beta, \tag{42b}$$

$$\vartheta \tilde{S}^{[n]} = -\mathfrak{a}_-^n H\mathfrak{a}_-^{n-1}H^{-1}\tilde{S}^{[n-1]}, \tag{42c}$$

$$\vartheta \tilde{S}^{[1]} = -H^{-1}\mathfrak{a}_- H. \tag{42d}$$

Proof Relations (42a) and (42b) can be obtained from (41b) if we consider the equation in the form $\vartheta S = -((\Lambda^\top)^2\gamma + \Lambda^\top)S$ equating by diagonals. Analogously, (42d) and (42c) con be obtained from (41c). □

Proposition 3.6 *For the step-line recursion coefficients the following expressions hold*

$$\begin{cases} \alpha = (\vartheta H)H^{-1} = \mathfrak{a}_+ S^{[1]} - S^{[1]}, \\ \beta = -\vartheta S^{[1]} = \mathfrak{a}_+ S^{[2]} - S^{[2]} - S^{[1]}\mathfrak{a}_-\big((\vartheta H)H^{-1}\big), \\ \gamma = -\vartheta S^{[2]} + (\vartheta\mathfrak{a}_- S^{[1]})S^{[1]} = H^{-1}\mathfrak{a}_-^2 H, \end{cases} \tag{43}$$

Proof It is proven using Eq. (8) and Proposition 3.3, just split (41b) by diagonals. □

Corollary 3.7 *In terms of tau functions the entries in recursion Hessenberg matrix can be expressed as follows*

$$\alpha_n = \vartheta \log\frac{\tau_{n+1}}{\tau_n}, \qquad \beta_n = \vartheta^2 \log\tau_n, \qquad \gamma_n = \frac{\tau_{n+3}\tau_n}{\tau_{n+2}\tau_{n+1}}.$$

Proof It follows from Eqs. (43) and (16). □

Proposition 3.8 (Multiple Toda equations) *The functions* $q_n := \log H_n$ *and* $f_n := S^{[1]}_n = p^1_{n+1}$ *solve the system*

$$\begin{cases} \vartheta q_n = f_{n-1} - f_n, \\ \vartheta^2 f_n - (2f_n - f_{n+1} - f_{n-1})\vartheta f_n = \mathrm{e}^{q_{n+1}-q_{n-1}} - \mathrm{e}^{q_{n+2}-q_n}. \end{cases} \tag{44}$$

Proof Using Eq. (43) we write

$$\begin{aligned} \vartheta S^{[1]} &= -\mathfrak{a}_+ S^{[2]} + S^{[2]} + S^{[1]}(\vartheta \mathfrak{a}_- H)(\mathfrak{a}_- H)^{-1}, \\ \vartheta S^{[2]} &= (\vartheta \mathfrak{a}_- S^{[1]})S^{[1]} - H^{-1}\mathfrak{a}_-^2 H. \end{aligned}$$

Acting with ϑ on the first equation an using then the second equation we get

$$\begin{aligned} &(\vartheta \mathfrak{a}_- S^{[1]})S^{[1]} - (\vartheta S^{[1]})\mathfrak{a}_+ S^{[1]} + \mathfrak{a}_+ H^{-1}\mathfrak{a}_- H - H^{-1}\mathfrak{a}_-^2 H \\ &\quad = \vartheta^2 S^{[1]} - \vartheta\left(S^{[1]}\big(S^{[1]} - \mathfrak{a}_- S^{[1]}\big)\right) \end{aligned}$$

Then, the equation reads

$$\begin{aligned} &f_{n-1}\vartheta f_n - f_n \vartheta f_{n+1} + \mathrm{e}^{q_{n+2}-q_n} - \mathrm{e}^{q_{n+1}-q_{n-1}} \\ &\quad = -\vartheta^2 f_n + (f_n - f_{n+1})\vartheta f_n + f_n(\vartheta f_n - \vartheta f_{n+1}) \end{aligned}$$

that after some clearing leads to Eq. (44). □

Theorem 3.9 *The* τ*-function satisfies the following nonlinear third order differential–difference equation*

$$\begin{aligned} &\vartheta^3 \tau_{n+1} - \left(\frac{\vartheta \tau_{n+2}}{\tau_{n+2}} + \frac{\vartheta \tau_{n+1}}{\tau_{n+1}} + \frac{\vartheta \tau_n}{\tau_n}\right)\vartheta^2 \tau_{n+1} + \frac{(\vartheta \tau_{n+1})^2}{\tau_{n+1}}\left(\frac{\vartheta \tau_{n+2}}{\tau_{n+2}} - \frac{\vartheta \tau_n}{\tau_n}\right) \\ &\quad = \frac{\tau_{n+3}\tau_n}{\tau_{n+2}} - \frac{\tau_{n+2}\tau_{n-1}}{\tau_n}. \end{aligned} \tag{45}$$

Proof Notice that the first equation $\vartheta q_n = f_{n-1} - f_n$, when the τ function is used, becomes an identity, indeed

$$\vartheta q_n = \vartheta \log \tau_{n+1} - \vartheta \log \tau_n, \quad f_{n-1} - f_n = p^1_n - p^1_{n+1} = -\vartheta \log \tau_n + \vartheta \log \tau_{n+1}.$$

Then, the second equation reads

$$-\vartheta^3 \log \tau_{n+1} - \vartheta(2\log \tau_{n+1} - \log \tau_{n+2} - \log \tau_n)\vartheta^2 \log \tau_{n+1} = \frac{\tau_{n+2}\tau_{n-1}}{\tau_{n+1}\tau_n} - \frac{\tau_{n+3}\tau_n}{\tau_{n+2}\tau_{n+1}},$$

that expanding the differentiations leads to

$$\frac{2(\vartheta \tau_{n+1})^3}{\tau^3_{n+1}} - 3\frac{(\vartheta \tau_{n+1})\vartheta^2 \tau_{n+1}}{\tau^2_{n+1}} + \frac{\vartheta^3 \tau_{n+1}}{\tau_{n+1}}$$

$$+\left(-\frac{(\vartheta\tau_{n+1})^2}{\tau_{n+1}^2}+\frac{\vartheta^2\tau_{n+1}}{\tau_{n+1}}\right)\left(2\frac{\vartheta\tau_{n+1}}{\tau_{n+1}}-\frac{\vartheta\tau_{n+2}}{\tau_{n+2}}-\frac{\vartheta\tau_n}{\tau_n}\right)=\frac{\tau_{n+3}\tau_n}{\tau_{n+2}\tau_{n+1}}-\frac{\tau_{n+2}\tau_{n-1}}{\tau_{n+1}\tau_n},$$

and after some clearing we get

$$\begin{aligned}-&\frac{(\vartheta\tau_{n+1})\vartheta^2\tau_{n+1}}{\tau_{n+1}^2}+\frac{\vartheta^3\tau_{n+1}}{\tau_{n+1}}+\left(-\frac{(\vartheta\tau_{n+1})^2}{\tau_{n+1}^2}+\frac{\vartheta^2\tau_{n+1}}{\tau_{n+1}}\right)\left(-\frac{\vartheta\tau_{n+2}}{\tau_{n+2}}-\frac{\vartheta\tau_n}{\tau_n}\right)\\&=\frac{\tau_{n+3}\tau_n}{\tau_{n+2}\tau_{n+1}}-\frac{\tau_{n+2}\tau_{n-1}}{\tau_{n+1}\tau_n},\end{aligned}$$

from where we immediately find Eq. (45). □

Remark 3.10 For the discrete hypergeometric orthogonal polynomials the standard Toda equation, see for example [55, Equation (168)], can be written in terms of the τ-function as the following nonlinear second order differential–difference equation

$$\vartheta^2\tau_{n+1}-\frac{(\vartheta\tau_{n+1})^2}{\tau_{n+1}}=\frac{\tau_{n+2}\tau_n}{\tau_{n+1}}.$$

We clearly see that (45) is an extension to the multiple orthogonal realm of the standard Toda equation.

Next we write this multiple Toda equations as the following Toda system for the recursion coefficients α, β and γ:

Proposition 3.11 *For multiple orthogonality with two weights we have the following Toda type system:*

$$\vartheta\alpha=\beta-\mathfrak{a}_+\beta, \tag{46}$$
$$\vartheta\beta=\gamma-\mathfrak{a}_+\gamma+\beta(\mathfrak{a}_-\alpha-\alpha), \tag{47}$$
$$\vartheta\gamma=\gamma(\mathfrak{a}_-^2\alpha-\alpha). \tag{48}$$

Proof Equation (46) is immediately obtained from (41a). From (41b), analyzing diagonal by diagonal we can obtain two different equations:

$$\begin{aligned}\vartheta S^{[1]}&=-\beta,\\ \gamma&=-\vartheta S^{[2]}+(\vartheta\mathfrak{a}_-S^{[1]})S^{[1]},\end{aligned}$$

using equations above and expressions for α, β and γ depending of $S^{[2]}$ and $S^{[1]}$ we obtain (46), (47) and (48). □

Proposition 3.12 (Lax pair) *The matrices T and ϕ are a Lax pair, i.e. the following Lax equation is satisfied*

$$\vartheta T=[\phi,T]=[T_+,T], \tag{49}$$

where

$$
\begin{aligned}
T_- &= (\Lambda^\top)^2\gamma + \Lambda^\top\beta,\\
T_+ &= T - T_- = \alpha + \Lambda.
\end{aligned}
$$

Proof We have $T = S\Lambda S^{-1}$ so that

$$
\vartheta^{(a)}T = (\vartheta^{(a)}S)\Lambda S^{-1} - S\Lambda S^{-1}(\vartheta^{(a)}S)S^{-1} = \underbrace{(\vartheta^{(a)}S)S^{-1}}_{\phi^{(a)}}\underbrace{S\Lambda S^{-1}}_{T} - \underbrace{S\Lambda S^{-1}}_{T}\underbrace{(\vartheta^{(a)}S)S^{-1}}_{\phi^{(a)}}
$$

and we get Eq. (49). The other form can be proved with (41b) and:

$$
[-T_-, T] = [T_+ - T, T] = [T_+, T].
$$

□

Remark 3.13 As mentioned in Remark 2.9, the previous results are applicable to more general situations beyond generalized hypergeometric series. The only requirement is that $\vartheta w^{(a)}(k) = kw^{(a)}(k)$. Once again, we utilize the tau function defined as $\tau_n = \mathscr{W}_n[\rho_0^{(1)}, \rho_0^{(2)}]$, where $\rho_0^{(a)} = \sum_{n=0}^{\infty} w^{(a)}(k)$.

Coming back to the hypergeometric situation, there is another compatibility equation for Laguerre-Freud matrix, when we make calculations for concrete cases of multiple orthogonal polynomials we will name it *Compatibility II*.

Proposition 3.14 *The following compatibility conditions hold for recursion and Laguerre-Freud matrices*

$$
\vartheta\Psi = [\phi, \Psi] = [-T_-, \Psi] = [\Psi, T_-], \tag{50}
$$
$$
\vartheta\Psi^\top = \Psi^\top + [\mu, \Psi^\top] = \Psi^\top + [-T_+^\top, \Psi^\top], \tag{51}
$$

with lower triangular matrices $\mu := \mu^{(1)} + \mu^{(2)}$ *and* $\mu^{(a)} := (\vartheta_{(a)}H^{-1})H + H^{-1}\tilde{\phi}^{(a)}H$.

Proof We have that

$$
\begin{aligned}
\theta(z)B(z-1) &= \Psi B(z),\\
\vartheta B(z) &= \phi B(z).
\end{aligned}
$$

Therefore,

$$
\vartheta(\theta(z)B(z-1)) = \vartheta(\Psi B(z)) = (\vartheta\Psi)B(z) + \Psi\vartheta B(z) = ((\vartheta\Psi) - \Psi\phi)B(z),
$$

then

$$
\vartheta(\theta(z)B(z-1)) = ((\vartheta\Psi) + \Psi\phi)B(z).
$$

We also have

$$\vartheta(\theta(z)B(z-1)) = \theta(z)\phi B(z-1) = \phi\Psi B(z).$$

Equating above relations and grouping terms and using Lemma 2.11 we find that

$$\vartheta\Psi = [\phi, \Psi].$$

To prove (51) we need to show first $\vartheta^{(1)}A^{(1)} = \mu^{(1)}A^{(1)}$, $\vartheta^{(2)}A^{(1)} = \mu^{(2)}A^{(1)}$, $\vartheta^{(1)}A^{(2)} = \mu^{(1)}A^{(2)}$ and $\vartheta^{(2)}A^{(2)} = \mu^{(2)}A^{(2)}$.We will prove equations for $A^{(1)}$ because for $A^{(2)}$ it is analogous:

$$\begin{aligned}\vartheta^{(2)}A^{(1)} &= \vartheta^{(2)}(H^{-1}\tilde{S})X^{(1)} = ((\vartheta^{(2)}H^{-1})\tilde{S} + H^{-1}(\vartheta^{(2)}\tilde{S}))X^{(1)}\\ &= (\vartheta^{(2)}H^{-1})HH^{-1}\tilde{S}X^{(1)}\\ &\quad +H^{-1}(\vartheta^{(2)}\tilde{S})(H^{-1}\tilde{S})^{-1}(H^{-1}\tilde{S})X^{(1)}\\ &= ((\vartheta^{(2)}H^{-1})H + H^{-1}(\vartheta^{(2)}\tilde{S})\tilde{S}^{-1}H)A^{(1)}.\end{aligned}$$

Hence, we get

$$\vartheta^{(2)}A^{(1)} = \underbrace{((\vartheta^{(2)}H^{-1})H + H^{-1}\tilde{\phi}^{(2)}H)}_{\mu^{(2)}}A^{(1)},$$

and

$$\begin{aligned}\vartheta^{(1)}A^{(1)} &= \vartheta^{(1)}(H^{-1}\tilde{S})X^{(1)} = ((\vartheta^{(1)}H^{-1})\tilde{S} + H^{-1}(\vartheta^{(1)}\tilde{S}))X^{(1)}\\ &= (\vartheta^{(1)}H^{-1})HH^{-1}\tilde{S}X^{(1)}\\ &\quad +H^{-1}(\vartheta^{(1)}\tilde{S})(H^{-1}\tilde{S})^{-1}(H^{-1}\tilde{S})X^{(1)}\\ &= ((\vartheta^{(1)}H^{-1})H + H^{-1}(\vartheta^{(1)}\tilde{S})\tilde{S}^{-1}H)A^{(1)}.\end{aligned}$$

Consequently,

$$\vartheta^{(1)}A^{(1)} = \underbrace{((\vartheta^{(1)}H^{-1})H + H^{-1}\tilde{\phi}^{(1)}H)}_{\mu^{(1)}}A^{(1)},$$

if we add both last equations we have $\vartheta A^{(1)} = \mu A^{(1)}$ (and for $A^{(2)}$ there is the same equation). The compatibility Toda–Pearson must hold:

$$\begin{aligned}\sigma^{(1)}(z)A^{(1)}(z+1) &= \Psi^{\mathsf{T}}A^{(1)}(z),\\ \vartheta A^{(1)}(z) &= \mu A^{(1)}(z),\end{aligned}$$

On the one hand

$$\begin{aligned}\vartheta(\sigma^{(1)}(z)A^{(1)}(z+1)) &= \vartheta(\Psi^{\mathsf{T}}A^{(1)}(z)) = \vartheta(\Psi^{\mathsf{T}})A^{(1)}(z) + \Psi^{\mathsf{T}}\vartheta A^{(1)}(z)\\ &= \vartheta(\Psi^{\mathsf{T}})A^{(1)}(z) + \Psi^{\mathsf{T}}\mu A^{(1)}(z).\end{aligned}$$

On the other hand

$$\vartheta(\sigma^{(1)}(z)A^{(1)}(z+1)) = \sigma^{(1)}(z)A^{(1)}(z+1) + \sigma^{(1)}(z)\mu A^{(1)}(z+1) = (I+\mu)\Psi^{\mathsf{T}}A^{(1)}(z).$$

Consequently, equating both:

$$\vartheta(\Psi^{\mathsf{T}})A^{(1)}(z) + \Psi^{\mathsf{T}}\mu A^{(1)}(z) = (I+\mu)\Psi^{\mathsf{T}}A^{(1)}(z),$$

and using Lemma 3.21 we finally get

$$\vartheta(\Psi^{\mathsf{T}}) = \Psi^{\mathsf{T}} + [\mu, \Psi^{\mathsf{T}}].$$

□

3.2 Multiple Nijhoff–Capel discrete Toda equations

We will now study the compatibility conditions between shifts in the three families of hypergeometric parameters:

$$\{b_i^{(1)}\}_{i=1}^{M^{(1)}}, \{b_i^{(2)}\}_{i=1}^{M^{(2)}}, \{c_j\}_{j=1}^{N},$$

for functions u_n. We will use the following notation:

(i) $\hat{u}$ denotes a shift in one and only one parameter in the first family of parameters, i.e., $b_r^{(1)} \to b_r^{(1)} + 1$ for only one given $r \in \{1, \dots, M^{(1)}\}$. The corresponding tridiagonal connection matrix is denoted by $\Omega^{(r)}$, and we define $\hat{d}{:=}b_r^{(1)}$.

(ii) $\check{u}$ denotes a shift in one and only one parameter in the second family of parameters, i.e., $b_s^{(2)} \to b_s^{(2)} + 1$ for only one given $q \in \{1, \dots, M^{(1)}\}$. The corresponding tridiagonal connection matrix is denoted by $\Omega^{(q)}$, and we define $\check{d}{:=}b_q^{(2)}$.

(iii) $\tilde{u}$ denotes a shift in one and only one parameter in the third family of parameters, i.e., $c_s \to c_s - 1$ for only one given $s \in \{1, \dots, N\}$. The corresponding tridiagonal connection matrix is denoted by $\Omega^{(s)}$, and we define $\tilde{d}{:=}c_s - 1$.

(iv) $\bar{u}$ denotes the shift $n \to n + 2$.

Lemma 3.15 *Connection matrices fulfill the following compatibility conditions:*

$$\Omega^{(s)}\tilde{\Omega}^{(r)} = \Omega^{(r)}\hat{\Omega}^{(s)}, \tag{52a}$$

$$\Omega^{(s)}\tilde{\Omega}^{(q)} = \Omega^{(q)}\check{\Omega}^{(s)}, \tag{52b}$$

$$\Omega^{(r)}\hat{\Omega}^{(q)} = \Omega^{(q)}\check{\Omega}^{(r)}. \tag{52c}$$

Proof In the one hand, connection formulas can be written as follows

$$\Omega^{(r)}\hat{B} = B, \qquad\qquad \Omega^{(s)}\tilde{B} = B,$$

so that $\tilde{\Omega}^{(r)}\hat{\tilde{B}} = \tilde{B}$ and, consequently, $\Omega^{(s)}\tilde{\Omega}^{(r)}\hat{\tilde{B}} = \Omega^{(s)}\tilde{B} = B$. On the other hand, we have $\hat{\Omega}^{(s)}\tilde{\hat{B}} = \hat{B}$ so that $\Omega^{(r)}\hat{\Omega}^{(s)}\tilde{\hat{B}} = \Omega^{(r)}\hat{B} = B$. Hence, the compatibility $\hat{\tilde{P}} = \tilde{\hat{P}}$ leads to (52a). Relations (52b) and (52c) are proven analogously. □

From this point we will use the following notation:

$$H_{2n}=:u, \quad H_{2n+1}=:v, \quad S^{[1]}_{2n}=:f, \quad S^{[1]}_{2n+1}=:g, \quad S^{[2]}_{2n}=:F, \quad S^{[2]}_{2n+1}=:G.$$

Theorem 3.16 (Multiple Nijhoff–Capel Toda systems) *The four functions u, v, f and g satisfy the following 3D system of four nonlinear difference equations:*

(i) For the shifts $\hat{}$, $\tilde{}$ and $\bar{}$:

$$g(\tilde{f} - \hat{f}) + \tilde{g}(\tilde{\hat{f}} - \tilde{f}) + \hat{g}(\hat{f} - \hat{\tilde{f}}) = \frac{1}{\tilde{d}}\left(\frac{\hat{\bar{u}}}{\hat{\tilde{u}}} - \frac{\bar{u}}{\tilde{u}}\right) + \frac{1}{\hat{d}}\left(\frac{\bar{u}}{\hat{u}} - \frac{\tilde{\bar{u}}}{\tilde{\hat{u}}}\right), \tag{53a}$$

$$\bar{f}(\hat{g} - \tilde{g}) + \tilde{\bar{f}}(\tilde{g} - \tilde{\hat{g}}) + \hat{\bar{f}}(\hat{\tilde{g}} - \hat{g}) = \frac{1}{\tilde{d}}\left(\frac{\bar{v}}{\tilde{v}} - \frac{\hat{\bar{v}}}{\hat{\tilde{v}}}\right), \tag{53b}$$

$$\frac{1}{\tilde{d}}\frac{\hat{\bar{u}}}{\hat{\tilde{u}}}(\hat{\bar{f}} - \bar{f}) + \frac{1}{\hat{d}}\frac{\tilde{\bar{u}}}{\tilde{\hat{u}}}(\bar{f} - \tilde{\bar{f}}) + \frac{1}{\tilde{d}}\frac{\bar{v}}{\tilde{v}}(\tilde{f} - \tilde{\hat{f}}) = 0, \tag{53c}$$

$$\frac{1}{\hat{d}}\frac{\bar{\bar{u}}}{\hat{\bar{u}}}(\hat{g} - \tilde{\hat{g}}) + \frac{1}{\tilde{d}}\frac{\bar{\bar{u}}}{\tilde{\bar{u}}}(\tilde{\hat{g}} - \tilde{g}) + \frac{1}{\tilde{d}}\frac{\hat{\bar{v}}}{\hat{\tilde{v}}}(\bar{g} - \hat{\bar{g}}) = 0. \tag{53d}$$

(ii) For the shifts $\check{}$, $\tilde{}$ and $\bar{}$:

$$g(\tilde{f} - \check{f}) + \tilde{g}(\check{\tilde{f}} - \tilde{f}) + \check{g}(\check{f} - \check{\tilde{f}}) = \frac{1}{\tilde{d}}\left(\frac{\check{\bar{u}}}{\check{\tilde{u}}} - \frac{\bar{u}}{\tilde{u}}\right) \tag{54a}$$

$$\bar{f}(\tilde{g} - \check{g}) + \tilde{\bar{f}}(\check{\tilde{g}} - \tilde{g}) + \check{\bar{f}}(\check{g} - \check{\tilde{g}}) = \frac{1}{\tilde{d}}\left(\frac{\check{\bar{v}}}{\check{\tilde{v}}} - \frac{\bar{v}}{\tilde{v}}\right) + \frac{1}{\check{d}}\left(\frac{\bar{v}}{\check{v}} - \frac{\tilde{\bar{v}}}{\check{\tilde{v}}}\right), \tag{54b}$$

$$\frac{1}{\tilde{d}}\frac{\check{\bar{u}}}{\check{\tilde{u}}}(\bar{f} - \check{\bar{f}}) + \frac{1}{\check{d}}\frac{\bar{v}}{\check{v}}(\check{f} - \check{\tilde{f}}) + \frac{1}{\tilde{d}}\frac{\bar{v}}{\tilde{v}}(\tilde{\check{f}} - \tilde{f}) = 0, \tag{54c}$$

$$\frac{1}{\tilde{d}}\frac{\bar{\bar{u}}}{\tilde{\bar{u}}}(\tilde{g} - \check{\tilde{g}}) + \frac{1}{\check{d}}\frac{\tilde{\bar{v}}}{\tilde{\check{v}}}(\bar{g} - \tilde{\bar{g}}) + \frac{1}{\tilde{d}}\frac{\check{\bar{v}}}{\check{\tilde{v}}}(\check{\tilde{g}} - \bar{g}) = 0. \tag{54d}$$

(iii) For the shifts $\hat{}$, $\check{}$ and $\bar{}$:

$$\frac{1}{\hat{d}}\frac{\check{\bar{u}}}{\check{\hat{u}}}(\bar{f} - \check{\bar{f}}) + \frac{1}{\check{d}}\frac{\bar{v}}{\check{v}}(\check{f} - \check{\hat{f}}) = 0, \tag{55a}$$

$$\frac{1}{\hat{d}}\frac{\bar{\bar{u}}}{\hat{\bar{u}}}(\hat{\check{g}}-\hat{g})+\frac{1}{\check{d}}\frac{\hat{\bar{v}}}{\check{\hat{v}}}(\hat{\bar{g}}-\bar{g})=0, \tag{55b}$$

$$g(\check{f}-\hat{f})+\check{g}(\check{\hat{f}}-\check{f})+\hat{g}(\hat{f}-\check{\hat{f}})=\frac{1}{\hat{d}}\left(\frac{\bar{u}}{\hat{u}}-\frac{\check{\bar{u}}}{\hat{\check{u}}}\right), \tag{55c}$$

$$\bar{f}(\check{g}-\hat{g})+\check{\bar{f}}(\check{\hat{g}}-\check{g})+\hat{\bar{f}}(\hat{g}-\check{\hat{g}})=\frac{1}{\check{d}}\left(\frac{\hat{\bar{v}}}{\check{\hat{v}}}-\frac{\bar{v}}{\check{v}}\right). \tag{55d}$$

Proof (i) From (52a) we can find the following two non-trivial equations for matrices $S^{[1]}$ and H from second and third subdiagonals:

$$\begin{aligned}&\frac{1}{\tilde{d}}(\mathfrak{a}_-^2\hat{H})\hat{\tilde{H}}^{-1}+(\mathfrak{a}_-S^{[1]}-\mathfrak{a}_-\hat{S}^{[1]})(\hat{S}^{[1]}-\hat{\tilde{S}}^{[1]})+\frac{1}{\hat{d}}I^{(1)}\hat{H}^{-1}\mathfrak{a}_-^2H\\&\quad=\frac{1}{\hat{d}}\tilde{\hat{H}}^{-1}(\mathfrak{a}_-^2\tilde{H})I^{(1)}+(\mathfrak{a}_-S^{[1]}-\mathfrak{a}_-\hat{S}^{[1]})(\tilde{S}^{[1]}-\tilde{\hat{S}}^{[1]})+\frac{1}{\tilde{d}}(\mathfrak{a}_-^2H)\tilde{H}^{-1}\end{aligned} \tag{56}$$

and

$$\begin{aligned}&\frac{1}{\tilde{d}}(\mathfrak{a}_-^2\hat{H})\hat{\tilde{H}}^{-1}(\mathfrak{a}_-^2S^{[1]}-\mathfrak{a}_-^2\hat{S}^{[1]})+\frac{1}{\hat{d}}I^{(2)}(\mathfrak{a}_-^3H)(\mathfrak{a}_-\hat{H}^{-1})(\hat{S}^{[1]}-\tilde{\hat{S}}^{[1]})\\&\quad=\frac{1}{\hat{d}}I^{(1)}(\mathfrak{a}_-^2\tilde{H})\tilde{\hat{H}}^{-1}+\frac{1}{\tilde{d}}(\mathfrak{a}_-^3H)(\mathfrak{a}_-\tilde{H}^{-1})(\tilde{S}^{[1]}-\tilde{\hat{S}}^{[1]}).\end{aligned} \tag{57}$$

From (56), equating by one side the terms that multiplied $I^{(1)}$ we obtain (53a) and from the terms multiplied by $I^{(2)}$ it is obtained (53b). From (57), equating terms in $I^{(1)}$ it is obtained (53c) and from terms in $I^{(2)}$ we get (53d).

(ii) From (52b), we find two non-trivial equations for matrices $S^{[1]}$ and H from second and third subdiagonals:

$$\begin{aligned}&\frac{1}{\tilde{d}}(\mathfrak{a}_-^2\check{H})\check{\tilde{H}}^{-1}+(\mathfrak{a}_-S^{[1]}-\mathfrak{a}_-\check{S}^{[1]})(\check{S}^{[1]}-\check{\tilde{S}}^{[1]})+\frac{1}{\check{d}}I^{(2)}\check{H}^{-1}\mathfrak{a}_-^2H\\&\quad=\frac{1}{\check{d}}I^{(2)}\tilde{\check{H}}^{-1}\mathfrak{a}_-^2\tilde{H}+(\mathfrak{a}_-S^{[1]}-\mathfrak{a}_-\tilde{S}^{[1]})(\tilde{S}^{[1]}-\tilde{\check{S}}^{[1]})+\frac{1}{\tilde{d}}(\mathfrak{a}_-^2H)\tilde{H}^{-1}\end{aligned} \tag{58}$$

and

$$\begin{aligned}&\frac{1}{\tilde{d}}(\mathfrak{a}_-^2\check{H})\check{\tilde{H}}^{-1}(\mathfrak{a}_-^2S^{[1]}-\mathfrak{a}_-^2\check{S}^{[1]})+\frac{1}{\check{d}}I^{(1)}(\mathfrak{a}_-\check{H}^{-1})(\mathfrak{a}_-^3H)(\check{S}^{[1]}-\check{\tilde{S}}^{[1]})\\&\quad=\frac{1}{\check{d}}I^{(2)}\tilde{\check{H}}^{-1}\mathfrak{a}_-^2\tilde{H}+\frac{1}{\tilde{d}}(\mathfrak{a}_-^3H)(\mathfrak{a}_-\tilde{H}^{-1})(\tilde{S}^{[1]}-\tilde{\check{S}}^{[1]}).\end{aligned} \tag{59}$$

From (58), equating by one side the terms that multiplied $I^{(1)}$ we obtain (54a) and from the terms multiplied by $I^{(2)}$ it is obtained (53b). From Eq. (59), equating terms in $I^{(1)}$ it is obtained (54c) and from terms in $I^{(2)}$ we obtain (54d).

(iii) From (52c), from third subdiagonal we can obtain

$$\begin{aligned}&\frac{1}{\tilde{d}}(\mathfrak{a}_-^2\tilde{H}^{-1})I^{(1)}(\mathfrak{a}_-^3 H)(\tilde{S}^{[1]}-\tilde{\hat{S}}^{[1]})+\frac{1}{\hat{d}}I^{(1)}\tilde{\hat{H}}^{-1}(\mathfrak{a}_-^2\tilde{H})(\mathfrak{a}_-^2 S^{[1]}-\mathfrak{a}_-^2\tilde{S}^{[1]})\\&\quad=\frac{1}{\hat{d}}I^{(2)}(\mathfrak{a}_-\hat{H}^{-1})(\mathfrak{a}_-^3 H)(\hat{S}^{[1]}-\hat{\tilde{S}}^{[1]})+\frac{1}{\tilde{d}}I^{(2)}\hat{\tilde{H}}^{-1}\mathfrak{a}_-^2\hat{H}\end{aligned}\tag{60}$$

and from the terms of $I^{(1)}$ we can obtain (55a), and from terms of $I^{(2)}$ we can obtain (55b).

From second subdiagonal we get the following equation:

$$\begin{aligned}&\frac{1}{\tilde{d}}I^{(2)}\tilde{H}^{-1}\mathfrak{a}_-^2 H+(\mathfrak{a}_- S^{[1]})\tilde{S}^{[1]}-(\mathfrak{a}_-\tilde{S}^{[1]})(\tilde{S}^{[1]}-\tilde{\hat{S}}^{[1]})+\frac{1}{\hat{d}}I^{(1)}\tilde{\hat{H}}^{-1}\mathfrak{a}_-^2\tilde{H}\\&\quad=\frac{1}{\hat{d}}I^{(1)}\hat{H}^{-1}\mathfrak{a}_-^2 H+(\mathfrak{a}_- S^{[1]})\hat{S}^{[1]}-(\mathfrak{a}_-\hat{S}^{[1]})(\hat{S}^{[1]}-\hat{\tilde{S}}^{[1]})+\frac{1}{\tilde{d}}I^{(2)}\hat{\tilde{H}}^{-1}\mathfrak{a}_-^2\hat{H},\end{aligned}\tag{61}$$

from terms of $I^{(1)}$ we obtain (55c) and from terms of $I^{(2)}$ we deduce (55d).

□

Remark 3.17 Each set of Eqs. (53), (54) and (55) is a nonlinear system of four equations of discrete partial difference equations in three variables for four functions. The first two systems (53) and (54) are equivalent under the interchange of $\hat{}\longleftrightarrow\check{}$, $u\longleftrightarrow v$ and $f\longleftrightarrow g$.

Remark 3.18 Notice that Eqs. (53) and (54) can be considered for the AT systems of weights described Lemma 2.4. Therefore, we are sure that the multiple orthogonal polynomials do exist. For other cases, one needs to proof perfectness or that the system is AT, which is still a open issue.

Remark 3.19 In [55], we discovered the Nijhoff–Capel Toda equation (Equation (128)) for standard non-multiple hypergeometric orthogonal polynomials [55]. It can be expressed as:

$$\frac{\bar{u}}{\bar{\tilde{u}}}-\frac{\hat{\bar{u}}}{\hat{\tilde{u}}}=\frac{\bar{u}}{\hat{u}}-\frac{\tilde{\bar{u}}}{\tilde{\hat{u}}}$$

This equation is obtained by considering Equation (53a) with $v=g=0$. Consequently, we refer to these systems as multiple Nijhoff–Capel Toda systems, namely (53), (54), and (55). The mentioned equation is the fifth equation among the canonical types for consistency equations corresponding to an octahedral sub-lattice, as described in [4] and [45, Equation (3.107c) and §3.9].

Theorem 3.20 *The multiple Nijhoff–Capel system* (54) *reads*

$$\frac{\tau^1_{2n+1}}{\tau_{2n+1}}\left(\frac{\tilde{\tau}^1_{2n}}{\tilde{\tau}_{2n}}-\frac{\hat{\tau}^1_{2n}}{\hat{\tau}_{2n}}\right)+\frac{\tilde{\tau}^1_{2n+1}}{\tilde{\tau}_{2n+1}}\left(\frac{\tilde{\tilde{\tau}}^1_{2n}}{\tilde{\tilde{\tau}}_{2n}}-\frac{\tilde{\tau}^1_{2n}}{\tilde{\tau}_{2n}}\right)+\frac{\hat{\tau}^1_{2n+1}}{\hat{\tau}_{2n+1}}\left(\frac{\hat{\tau}^1_{2n}}{\hat{\tau}_{2n}}-\frac{\hat{\tilde{\tau}}^1_{2n}}{\hat{\tilde{\tau}}_{2n}}\right)$$

$$
\begin{aligned}
&= \frac{1}{\tilde{d}}\left(\frac{\hat{\bar{\tau}}_{2n+1}}{\hat{\bar{\tau}}_{2n}}\frac{\hat{\tilde{\tau}}_{2n}}{\hat{\tilde{\tau}}_{2n+1}} - \frac{\bar{\tau}_{2n+1}}{\bar{\tau}_{2n}}\frac{\tilde{\tau}_{2n}}{\tilde{\tau}_{2n+1}}\right) + \frac{1}{\hat{d}}\left(\frac{\bar{\tau}_{2n+1}}{\bar{\tau}_{2n}}\frac{\hat{\tau}_{2n}}{\hat{\tau}_{2n+1}} - \frac{\tilde{\bar{\tau}}_{2n+1}}{\tilde{\bar{\tau}}_{2n}}\frac{\tilde{\hat{\tau}}_{2n}}{\tilde{\hat{\tau}}_{2n+1}}\right),\\
&\frac{\bar{\tau}^1_{2n}}{\bar{\tau}_{2n}}\left(\frac{\tilde{\tau}^1_{2n+1}}{\tilde{\tau}_{2n+1}} + \frac{\hat{\tau}^1_{2n+1}}{\hat{\tau}_{2n+1}}\right) + \frac{\tilde{\bar{\tau}}^1_{2n}}{\tilde{\bar{\tau}}_{2n}}\left(\frac{\tilde{\hat{\tau}}^1_{2n+1}}{\tilde{\hat{\tau}}_{2n+1}} - \frac{\tilde{\tau}^1_{2n+1}}{\tilde{\tau}_{2n+1}}\right) + \frac{\hat{\bar{\tau}}^1_{2n}}{\hat{\bar{\tau}}_{2n}}\left(\frac{\hat{\tau}^1_{2n+1}}{\hat{\tau}_{2n+1}} - \frac{\hat{\tilde{\tau}}^1_{2n+1}}{\hat{\tilde{\tau}}_{2n+1}}\right)\\
&= \frac{1}{\tilde{d}}\left(\frac{\hat{\bar{\bar{\tau}}}_{2n}}{\hat{\bar{\tau}}_{2n+1}}\frac{\hat{\tilde{\tau}}_{2n+1}}{\hat{\tilde{\bar{\tau}}}_{2n}} - \frac{\bar{\bar{\tau}}_{2n}}{\bar{\tau}_{2n+1}}\frac{\tilde{\tau}_{2n+1}}{\tilde{\bar{\tau}}_{2n}}\right),\\
&\frac{1}{\tilde{d}}\frac{\hat{\bar{\tau}}_{2n+1}}{\hat{\bar{\tau}}_{2n}}\frac{\hat{\tilde{\tau}}_{2n}}{\hat{\tilde{\tau}}_{2n+1}}\left(\frac{\hat{\bar{\tau}}^1_{2n}}{\hat{\bar{\tau}}_{2n}} - \frac{\bar{\tau}^1_{2n}}{\bar{\tau}_{2n}}\right) + \frac{1}{\hat{d}}\frac{\tilde{\bar{\tau}}_{2n+1}}{\tilde{\bar{\tau}}_{2n}}\frac{\tilde{\hat{\tau}}_{2n}}{\tilde{\hat{\tau}}_{2n+1}}\left(\frac{\bar{\tau}^1_{2n}}{\bar{\tau}_{2n}} - \frac{\tilde{\bar{\tau}}^1_{2n}}{\tilde{\bar{\tau}}_{2n}}\right)\\
&+ \frac{1}{\tilde{d}}\frac{\bar{\bar{\tau}}_{2n}}{\bar{\tau}_{2n+1}}\frac{\tilde{\tau}_{2n+1}}{\tilde{\bar{\tau}}_{2n}}\left(\frac{\tilde{\tau}^1_{2n}}{\tilde{\tau}_{2n}} - \frac{\tilde{\hat{\tau}}^1_{2n}}{\tilde{\hat{\tau}}_{2n}}\right) = 0,\\
&\frac{1}{\hat{d}}\frac{\bar{\bar{\tau}}_{2n+1}}{\bar{\bar{\tau}}_{2n}}\frac{\hat{\bar{\tau}}_{2n}}{\hat{\bar{\tau}}_{2n+1}}\left(\frac{\hat{\tau}^1_{2n+1}}{\hat{\tau}_{2n+1}} - \frac{\hat{\tilde{\tau}}^1_{2n+1}}{\hat{\tilde{\tau}}_{2n+1}}\right) + \frac{1}{\tilde{d}}\frac{\bar{\bar{\tau}}_{2n+1}}{\bar{\bar{\tau}}_{2n}}\frac{\tilde{\bar{\tau}}_{2n}}{\tilde{\bar{\tau}}_{2n+1}}\left(\frac{\tilde{\hat{\tau}}^1_{2n+1}}{\tilde{\hat{\tau}}_{2n+1}} - \frac{\tilde{\tau}^1_{2n+1}}{\tilde{\tau}_{2n+1}}\right)\\
&+ \frac{1}{\tilde{d}}\frac{\tilde{\bar{\tau}}_{2n+1}}{\tilde{\bar{\tau}}_{2n}}\frac{\tilde{\hat{\tau}}_{2n}}{\tilde{\hat{\tau}}_{2n+1}}\left(\frac{\bar{\tau}^1_{2n+1}}{\bar{\tau}_{2n+1}} - \frac{\hat{\bar{\tau}}^1_{2n+1}}{\hat{\bar{\tau}}_{2n+1}}\right) = 0.
\end{aligned}
$$

Notice that we have large families of solutions to this system of equations in terms of double Wronskians of generalized hypergeometric functions.

Proof Use the τ-function parametrization

$$
u = \frac{\tau_{2n+1}}{\tau_{2n}}, \qquad v = \frac{\tau_{2n+2}}{\tau_{2n+1}}, \qquad f = \frac{\tau^1_{2n}}{\tau_{2n}}, \qquad g = \frac{\tau^1_{2n+1}}{\tau_{2n+1}}.
$$

□

Lemma 3.21 *If a matrix M is such that $MX^{(1)}(z) = MX^{(2)}(z) = 0$ then $M = 0$.*

Proof We have that

$$
M\frac{1}{n!}\frac{\mathrm{d}^n X^{(1)}(z)}{\mathrm{d}\, z^n}\Big|_{z=0} = 0, \qquad M\frac{1}{n!}\frac{\mathrm{d}^n X^{(2)}(z)}{\mathrm{d}\, z^n}\Big|_{z=0} = 0.
$$

Observe that $\frac{1}{n!}\frac{\mathrm{d}^n X^{(1)}(z)}{\mathrm{d}\, z^n}\big|_{z=0}$ is the vector with all its entries equal to zero but for a 1 in the $2n$-th entry. Similarly, $\frac{1}{n!}\frac{\mathrm{d}^n X^{(2)}(z)}{\mathrm{d}\, z^n}\big|_{z=0}$ is the vector with all its entries equal to zero but for a 1 in the $(2n+1)$-th entry. Therefore, all the columns of M have 0 all its entries. □

Lemma 3.22 *The following equations are satisfied*

$$
\hat{T}^{\top}\omega^{(r)} = \omega^{(r)}T^{\top} \tag{62}
$$

$$
\check{T}^{\top}\omega^{(q)} = \omega^{(q)}T^{\top} \tag{63}
$$

$$\tilde{T}^{\top}\omega^{(s)} = \omega^{(s)}T^{\top} \tag{64}$$

holds.

Proof From eigenvalue equation for $T^{\top}$ and $A^{(a)}$, $A^{(a)\top}T = zA^{(a)\top}$, and (34) for this case, $\hat{A}^{(a)}(z) = \frac{\omega^{(r)}}{z+\hat{d}}A^{(a)}(z)$ it can be seen

$$\hat{A}^{(a)\top}\hat{T} = z\hat{A}^{(a)\top},$$

so that

$$\hat{T}^{\top}\frac{\omega^{(r)}}{z+\hat{d}}A^{(a)}(z) = z\frac{\omega^{(r)}}{z+\hat{d}}A^{(a)}(z) = \frac{\omega^{(r)}}{z+\hat{d}}T^{\top}A^{(a)}.$$

Hence, if we denote $M = (\hat{T}^{\top}\omega^{(r)} - \hat{T}^{\top}\omega^{(r)}T)H^{-1}\tilde{S}$ we find $MX^{(a)} = 0$ so that, according to Lemma 3.21, we get that $M = 0$ and as $H^{-1}\tilde{S}$ is a lower triangular invertible matrix we conclude the result. and finally we get

$$\hat{T}^{\top}\omega^{(r)} = \omega^{(r)}T^{\top}.$$

For Eqs. (63) and (64) it can be proven analogously. □

Theorem 3.23 *The six functions u, v, f, g, F and G fulfill the following system of six 2D nonlinear difference equations*

(i) For the shifts $\tilde{}$ and $\bar{}$:

$$\tilde{d}(G - \bar{F} + \tilde{\bar{F}} - \tilde{G} + \tilde{\bar{f}}(\bar{g} - \tilde{\bar{g}}) + \tilde{g}(\tilde{\bar{f}} - \bar{f})) = \frac{\bar{v}}{\tilde{v}} - \frac{\bar{\bar{u}}}{\bar{\tilde{\bar{u}}}}, \tag{65a}$$

$$\tilde{d}(F - G + \tilde{G} - \tilde{F} + \tilde{g}(\bar{f} - \tilde{\bar{f}}) + \tilde{f}(\tilde{g} - g)) = \frac{\bar{u}}{\tilde{u}} - \frac{\bar{v}}{\tilde{v}}, \tag{65b}$$

$$\begin{aligned}\frac{\bar{\bar{u}}}{\bar{\tilde{\bar{u}}}}(\tilde{g} - \tilde{\bar{f}}) + \tilde{d}(\bar{g} - \tilde{\bar{g}})(\tilde{G} - \tilde{\bar{F}} - \tilde{\bar{f}}(\tilde{\bar{f}} - \tilde{g})) + \tilde{d}\frac{\tilde{\bar{\bar{u}}}}{\bar{\tilde{\bar{u}}}} &= \tilde{d}\frac{\bar{\bar{u}}}{\bar{u}} + \frac{\bar{\bar{u}}}{\bar{\tilde{\bar{u}}}}(\bar{g} - \bar{\bar{f}}) \\ &+ \tilde{d}(\bar{f} - \tilde{\bar{f}})(\bar{F} - \bar{G} - \bar{g}(\bar{g} - \bar{\bar{f}})),\end{aligned} \tag{65c}$$

$$\begin{aligned}\frac{\bar{v}}{\tilde{v}}(\tilde{f} - \tilde{g}) + \tilde{d}(\bar{f} - \tilde{\bar{f}})(\tilde{F} - \tilde{G} - \tilde{g}(\tilde{g} - \tilde{\bar{f}})) + \tilde{d}\frac{\tilde{\bar{v}}}{\tilde{v}} &= \tilde{d}\frac{\bar{v}}{v} + \frac{\bar{v}}{\tilde{v}}(\bar{f} - \bar{g}) \\ &+ \tilde{d}(g - \tilde{g})(G - \bar{F} - \bar{f}(\bar{f} - \bar{g})),\end{aligned} \tag{65d}$$

$$\frac{\bar{\bar{v}}}{\bar{\tilde{\bar{v}}}}(\tilde{G} - \tilde{\bar{\bar{F}}} - \tilde{\bar{\bar{f}}}(\tilde{\bar{\bar{f}}} - \tilde{\bar{g}})) + \tilde{d}\frac{\tilde{\bar{\bar{u}}}}{\bar{\tilde{\bar{u}}}}(\bar{f} - \tilde{\bar{f}}) = \tilde{d}\frac{\bar{\bar{v}}}{\bar{v}}(\bar{f} - \tilde{\bar{f}}) + \frac{\bar{\bar{u}}}{\bar{\tilde{\bar{u}}}}(\bar{G} - \bar{\bar{F}} - \bar{\bar{f}}(\bar{\bar{f}} - \tilde{\bar{g}})), \tag{65e}$$

$$\frac{\bar{\bar{u}}}{\bar{\tilde{\bar{u}}}}(\tilde{F} - \tilde{G} - \tilde{g}(\tilde{g} - \tilde{\bar{f}})) + \tilde{d}\frac{\tilde{\bar{v}}}{\tilde{v}}(\bar{g} - \tilde{\bar{g}}) = \tilde{d}\frac{\bar{\bar{u}}}{\bar{u}}(g - \tilde{g}) + \frac{\bar{v}}{\tilde{v}}(\bar{F} - \bar{G} - \bar{g}(\bar{g} - \tilde{\bar{f}})). \tag{65f}$$

(ii) For the shifts $\check{}$ *and* $\bar{}$*:*

$$\frac{\bar{v}}{\check{v}}=\check{d}(G-\bar{F}-\check{G}+\check{\bar{F}}+\check{\bar{f}}(\bar{g}-\check{\bar{g}})+\check{g}(\check{\bar{f}}-\bar{f})), \tag{66a}$$

$$\check{d}(F-G-\check{F}+\check{G}+\check{g}(\bar{f}-\check{\bar{f}})+\check{f}(\check{g}-g))=-\frac{\bar{v}}{\check{v}}, \tag{66b}$$

$$\frac{\check{\bar{\bar{u}}}}{\check{\bar{u}}}+(\bar{g}-\check{\bar{g}})(\check{G}-\check{\bar{F}}-\check{\bar{f}}(\check{\bar{f}}-\check{\bar{g}}))=\frac{\bar{\bar{u}}}{\bar{u}}+(\bar{f}-\check{\bar{f}})(\bar{F}-\bar{G}-\bar{g}(\bar{g}-\bar{\bar{f}})), \tag{66c}$$

$$\begin{aligned}&\frac{\bar{v}}{\check{v}}(\check{f}-\check{g})+\check{d}(\bar{f}-\check{\bar{f}})(\check{F}-\check{G}-\check{g}(\check{g}-\check{\bar{f}}))+\check{d}\frac{\check{\bar{v}}}{\check{v}}\\&\quad=\check{d}\frac{\bar{v}}{v}+\check{d}(g-\check{g})(G-\bar{F}-\bar{f}(\bar{f}-\bar{g}))+\frac{\bar{v}}{\check{v}}(\bar{f}-\bar{g}),\end{aligned} \tag{66d}$$

$$\frac{\bar{\bar{v}}}{\check{\bar{v}}}(\check{G}-\check{\bar{F}}-\check{\bar{f}}(\check{\bar{f}}-\check{\bar{g}}))+\check{d}\frac{\check{\bar{\bar{u}}}}{\check{\bar{u}}}(\bar{\bar{f}}-\check{\bar{\bar{f}}})=\check{d}\frac{\bar{\bar{v}}}{\bar{v}}(\bar{f}-\check{\bar{f}}), \tag{66e}$$

$$\check{d}\frac{\check{\bar{v}}}{\check{v}}(\bar{g}-\check{\bar{g}})=\check{d}\frac{\bar{\bar{u}}}{\bar{u}}(g-\check{g})+\frac{\bar{v}}{\check{v}}(\bar{F}-\bar{G}-\bar{g}(\bar{g}-\bar{\bar{f}})). \tag{66f}$$

(iii) For the shifts $\hat{}$ *and* $\bar{}$*:*

$$\hat{d}(G-\bar{F}-\hat{G}+\hat{\bar{F}}+\hat{g}(\hat{\bar{f}}-\bar{f})+\hat{\bar{f}}(\bar{g}-\hat{\bar{g}}))=-\frac{\bar{\bar{u}}}{\hat{\bar{u}}}, \tag{67a}$$

$$\hat{d}(F-G-\hat{\bar{F}}+\hat{G}+\hat{f}(\hat{g}-g)+\hat{g}(\bar{f}-\hat{\bar{f}}))=\frac{\bar{u}}{\hat{u}}, \tag{67b}$$

$$\begin{aligned}&\hat{d}\left(\frac{\hat{\bar{\bar{u}}}}{\hat{\bar{u}}}+(\bar{g}-\hat{\bar{g}})(\hat{G}-\hat{\bar{F}}-\hat{\bar{f}}(\hat{\bar{f}}-\hat{\bar{g}}))\right)+\frac{\bar{\bar{u}}}{\hat{\bar{u}}}(\hat{g}-\hat{\bar{f}})\\&\quad=\frac{\bar{\bar{u}}}{\hat{\bar{u}}}(\bar{g}-\bar{\bar{f}})+\hat{d}\left(\frac{\bar{\bar{u}}}{\bar{u}}+(\bar{f}-\hat{\bar{f}})(\bar{F}-\bar{G}-\bar{g}(\bar{g}-\bar{\bar{f}}))\right),\end{aligned} \tag{67c}$$

$$\frac{\hat{\bar{v}}}{\hat{v}}+(\bar{f}-\hat{\bar{f}})(\hat{F}-\hat{G}-\hat{g}(\hat{g}-\hat{\bar{f}}))=\frac{\bar{v}}{v}+(g-\hat{g})(G-\bar{F}-\bar{f}(\bar{f}-\bar{g})), \tag{67d}$$

$$\hat{d}\frac{\hat{\bar{u}}}{\hat{u}}(\bar{f}-\hat{\bar{f}})=\hat{d}\frac{\bar{v}}{v}(f-\hat{f})+\frac{\bar{u}}{\hat{u}}(G-\bar{F}-\bar{f}(\bar{f}-\bar{g})), \tag{67e}$$

$$\frac{\bar{v}}{\hat{v}}(\hat{F}-\hat{G}-\hat{g}(\hat{g}-\hat{\bar{f}}))+\hat{d}\frac{\hat{\bar{v}}}{\hat{v}}(\bar{g}-\hat{\bar{g}})=\hat{d}\frac{\bar{\bar{u}}}{\bar{u}}(g-\hat{g}). \tag{67f}$$

Proof From (64) we obtained three non-null diagonals. From first diagonal

$$\begin{aligned}&\tilde{d}(\mathfrak{a}_+S^{[2]}-S^{[2]}+\tilde{S}^{[2]}-\mathfrak{a}_+\tilde{S}^{[2]}+\tilde{S}^{[1]}(\mathfrak{a}_-S^{[1]}-\mathfrak{a}_-\tilde{S}^{[1]})+\mathfrak{a}_+\tilde{S}^{[1]}(\tilde{S}^{[1]}-S^{[1]}))\\&=\frac{\mathfrak{a}_-H}{\mathfrak{a}_+\tilde{H}}-\frac{\mathfrak{a}_-^2H}{\tilde{H}},\end{aligned}$$

if we take $I^{(1)}$ part we obtain (65a) and from $I^{(2)}$ we obtain (65b); from second diagonal

$$\begin{aligned}
&(\mathfrak{a}_+\tilde{S}^{[1]}-\tilde{S}^{[1]})\frac{\mathfrak{a}_-^2 H}{\tilde{H}}+\tilde{d}(\mathfrak{a}_- S^{[1]}-\mathfrak{a}_-\tilde{S}^{[1]})(\mathfrak{a}_+\tilde{S}^{[2]}-\tilde{S}^{[2]}-\tilde{S}^{[1]}(\tilde{S}^{[1]}-\mathfrak{a}_-\tilde{S}^{[1]}))\\
&\quad+\tilde{d}\frac{\mathfrak{a}_-^2\tilde{H}}{\tilde{H}}=\tilde{d}\frac{\mathfrak{a}_-^2 H}{H}+\frac{\mathfrak{a}_-^2 H}{\tilde{H}}(\mathfrak{a}_- S^{[1]}-\mathfrak{a}_-^2 S^{[1]})\\
&\quad+\tilde{d}(S^{[1]}-\tilde{S}^{[1]})(S^{[2]}-\mathfrak{a}_- S^{[2]}-\mathfrak{a}_- S^{[1]}(\mathfrak{a}_- S^{[1]}-\mathfrak{a}_-^2 S^{[1]})),
\end{aligned}$$

from part coming from $I^{(1)}$ it can be obtained (65c) and from part of $I^{(2)}$ it can be obtained (65d); from third superdiagonal

$$\begin{aligned}
&\frac{\mathfrak{a}_-^3 H}{\mathfrak{a}_-\tilde{H}}(\mathfrak{a}_+\tilde{S}^{[2]}-\tilde{S}^{[2]}-\tilde{S}^{[1]}(\tilde{S}^{[1]}-\mathfrak{a}_-\tilde{S}^{[1]}))+\tilde{d}\frac{\mathfrak{a}_-^2\tilde{H}}{\tilde{H}}(\mathfrak{a}_-^2 S^{[1]}-\mathfrak{a}_-^2\tilde{S}^{[1]})\\
&\quad=\tilde{d}\frac{\mathfrak{a}_-^3 H}{\mathfrak{a}_- H}(S^{[1]}-\tilde{S}^{[1]})\\
&\quad+\frac{\mathfrak{a}_-^2 H}{\tilde{H}}(\mathfrak{a}_- S^{[2]}-\mathfrak{a}_-^2 S^{[2]}-\mathfrak{a}_-^2 S^{[1]}(\mathfrak{a}_-^2 S^{[1]}-\mathfrak{a}_-^3 S^{[1]})),
\end{aligned}$$

from part of $I^{(1)}$ it can be obtained (65e) and from part of $I^{(2)}$ it can be obtained (65f). From (63) we can obtain three non-null diagonals. From first superdiagonal

$$\begin{aligned}
I^{(1)}\frac{\mathfrak{a}_- H}{\mathfrak{a}_+\check{H}}-I^{(2)}\frac{\mathfrak{a}_-^2 H}{\check{H}}&=\check{d}(\mathfrak{a}_+ S^{[2]}-S^{[2]}-\mathfrak{a}_+\check{S}^{[2]}+\check{S}^{[2]}+\check{S}^{[1]}(\mathfrak{a}_- S^{[1]}-\mathfrak{a}_-\check{S}^{[1]})\\
&\quad+\mathfrak{a}_+\check{S}^{[1]}(\check{S}^{[1]}-S^{[1]}))
\end{aligned}$$

from part of $I^{(1)}$ it can be obtained (66a) and from part of $I^{(2)}$ it can be obtained (66b); from second superdiagonal

$$\begin{aligned}
&I^{(2)}\frac{\mathfrak{a}_-^2 H}{\check{H}}(\mathfrak{a}_+\check{S}^{[1]}-\check{S}^{[1]})+\check{d}(\mathfrak{a}_- S^{[1]}-\mathfrak{a}_-\check{S}^{[1]})(\mathfrak{a}_+\check{S}^{[2]}-\check{S}^{[2]}-\check{S}^{[1]}(\check{S}^{[1]}-\mathfrak{a}_-\check{S}^{[1]}))\\
&\quad+\check{d}\frac{\mathfrak{a}_-^2\check{H}}{\check{H}}\\
&=\check{d}\frac{\mathfrak{a}_-^2 H}{H}+\check{d}(S^{[1]}-\check{S}^{[1]})(S^{[2]}-\mathfrak{a}_- S^{[2]}-\mathfrak{a}_- S^{[1]}(\mathfrak{a}_- S^{[1]}-\mathfrak{a}_-^2 S^{[1]}))\\
&\quad+I^{(2)}\frac{\mathfrak{a}_-^2 H}{\check{H}}(\mathfrak{a}_- S^{[1]}-\mathfrak{a}_-^2 S^{[1]})
\end{aligned}$$

from part of $I^{(1)}$ it can be obtained (66c) and from part of $I^{(2)}$ it can be obtained (66d); and from third superdiagonal

$$I^{(1)}\frac{\mathfrak{a}_-^3 H}{\mathfrak{a}_-\check{H}}(\mathfrak{a}_+\check{S}^{[2]}-\check{S}^{[2]}-\check{S}^{[1]}(\check{S}^{[1]}-\mathfrak{a}_-\check{S}^{[1]}))+\check{d}\frac{\mathfrak{a}_-^2\check{H}}{\check{H}}(\mathfrak{a}_-^2 S^{[1]}-\mathfrak{a}_-^2\check{S}^{[1]})$$

$$= \check{d}(S^{[1]} - \check{S}^{[1]})\frac{\mathfrak{a}_-^3 H}{\mathfrak{a}_- H} + I^{(2)}\frac{\mathfrak{a}_-^2 H}{\check{H}}(\mathfrak{a}_- S^{[2]} - \mathfrak{a}_-^2 S^{[2]]} - \mathfrak{a}_-^2 S^{[1]}](\mathfrak{a}_-^2 S^{[1]} - \mathfrak{a}_-^3 S^{[1]}))$$

from part of $I^{(1)}$ we can obtain (66e) and from $I^{(2)}$ we can obtain (66f). □

Remark 3.24 Notice that

$$F = \frac{\tau_{2n}^2}{\tau_{2n}}, \qquad G = \frac{\tau_{2n+1}^2}{\tau_{2n+1}},$$

As we did in Proposition 3.20 for the multiple Nijhoff–Capel Toda system we can get τ-function representation of the system (65).

4 Laguerre–Freud equations for generalized multiple Charlier and Meixner II

We will begin by applying the previous developments to the multiple orthogonal polynomials discussed in [17], specifically the multiple Charlier and Meixner II sequences of multiple orthogonal polynomials. It has been proven in [17] that in all these cases, the system is AT, indicating that we are dealing with perfect systems. Subsequently, we will proceed with the study of generalizations of these two cases. The original polynomials were found in [25] and [58], respectively.

We will demonstrate that it is possible to discuss Laguerre–Freud equations in all these examples. In this multiple scenario, we extend the concept of Laguerre–Freud equations to encompass nonlinear equations satisfied by the coefficients of the recursion relation in the form:

$$\gamma_n = \mathscr{F}_n(\gamma_{n-1}, \beta_{n-1}, \alpha_{n-1}, \dots), \tag{68a}$$

$$\beta_n = \mathscr{G}_n(\gamma_n, \gamma_{n-1}, \beta_{n-1}, \alpha_{n-1}, \dots), \tag{68b}$$

$$\alpha_n = \mathscr{H}_n(\gamma_n, \beta_n, \gamma_{n-1}, \beta_{n-1}, \alpha_{n-1}, \dots). \tag{68c}$$

4.1 Charlier multiple orthogonal polynomials

In this case we set

$$w^{(1)}(k) = \frac{{\eta^{(1)}}^k}{k!}, \qquad w^{(2)}(k) = \frac{{\eta^{(2)}}^k}{k!},$$

if $\eta^{(1)}, \eta^{(2)} > 0$, it is an AT system, and polynomials $\theta, \sigma^{(1)}$ and $\sigma^{(2)}$ are:

$$\theta(k) = k, \qquad \sigma^{(1)}(k) = \eta^{(1)}, \qquad \sigma^{(2)}(k) = \eta^{(2)}.$$

Now, $\sigma^{(1)}$ and $\sigma^{(2)}$ are degree zero polynomials and θ is a degree one polynomial, consequently there are not non-zero subdiagonals and there is only one non-zero superdiagonal, this is:

$$\Psi=\psi^{(0)}+\psi^{(1)}.$$

To obtain Laguerre-Freud matrix firstly it is used $\Psi=\sigma^{(1)}(T_1^\intercal)\Pi_1^\intercal+\sigma^{(2)}(T_2^\intercal)\Pi_2^\intercal$, from this expression it is obtained:

$$\psi^{(0)}=I^{(1)}\eta^{(1)}+I^{(2)}\eta^{(2)}, \qquad \psi^{(1)}=\eta^{(1)}D_1+\eta^{(2)}D_2.$$

From the expression $\Psi=\Pi^{-1}\theta(T)$ we get

$$\psi^{(1)}=I, \qquad \psi^{(0)}=\alpha-\mathfrak{a}_+D.$$

Attending to the above results the Laguerre-Freud structure matrix is

$$\Psi=\eta^{(1)}I^{(1)}+\eta^{(2)}I^{(2)}+\Lambda.$$

Let us discuss $[\Psi,T]=\Psi$. First we obtain that $[\Psi,T]=\Lambda^\intercal\tilde{\psi}^{(-1)}+\tilde{\psi}^{(0)}+\tilde{\psi}^{(1)}\Lambda$, i.e.,

$$\begin{aligned}\tilde{\psi}^{(-1)}&=\gamma-\mathfrak{a}_+\gamma-\beta(\eta^{(1)}-\eta^{(2)})(I^{(1)}-I^{(2)}),\\ \tilde{\psi}^{(0)}&=\beta-\mathfrak{a}_+\beta,\\ \tilde{\psi}^{(1)}&=\mathfrak{a}_-\alpha-\alpha+(\eta^{(1)}-\eta^{(2)})(I^{(1)}-I^{(2)}),\end{aligned}$$

equating these to Laguerre–Freud matrix Ψ we obtain three non-trivial equations:

$$\begin{aligned}\mathfrak{a}_-\alpha-\alpha&=I+(\eta^{(1)}-\eta^{(2)})(I^{(2)}-I^{(1)}),\\ \beta-\mathfrak{a}_+\beta&=\eta^{(1)}I^{(1)}+\eta^{(2)}I^{(2)},\\ \gamma-\mathfrak{a}_+\gamma&=\beta(\eta^{(1)}-\eta^{(2)})(I^{(1)}-I^{(2)}).\end{aligned}$$

From compatibility equations we get

$$\begin{aligned}\alpha_{2n}&=2n+\alpha_0, & \alpha_{2n+1}&=2n+1+\eta^{(2)}-\eta^{(1)}+\alpha_0,\\ \beta_{2n}&=n(\eta^{(1)}+\eta^{(2)}), & \beta_{2n+1}&=n((\eta^{(1)}+\eta^{(2)})+\eta^{(1)},\\ \gamma_{2n+1}&=n\eta^{(2)}(\eta^{(2)}-\eta^{(1)}), & \gamma_{2n+2}&=(n+1)\eta^{(1)}(\eta^{(1)}-\eta^{(2)}),\end{aligned}$$

where $n\in\{0,1,2,\dots\}$. These results have been found previously in [17].

4.2 Meixner of second kind multiple orthogonal polynomials

In this case we have:

$$\theta(k)=k, \qquad \sigma^{(1)}(k)=\eta(k+b_1), \qquad \sigma^{(2)}(k)=\eta(k+b_2).$$

The weights are:

$$w^{(1)}(k) = (b_1)_k \frac{\eta^k}{k!}, \qquad\qquad w^{(2)}(k) = (b_2)_k \frac{\eta^k}{k!},$$

if $b_1, b_2, \eta > 0$, according with Lemma 2.4, it is an AT system.

The Laguerre-Freud matrix has four non-trivial diagonals:

$$\Psi = (\Lambda^\top)^2 \psi^{(-2)} + \cdots + \psi^{(1)} \Lambda.$$

The main diagonal and the first super diagonal can be obtained by means of $\Psi = \Pi^{-1}\theta(T)$:

$$\psi^{(1)} = I, \qquad\qquad \psi^{(0)} = \alpha - \mathfrak{a}_+ D.$$

Both subdiagonals are obtained through $\Psi = \sigma^{(1)}(T_1^\top)\Pi_1^\top + \sigma^{(2)}(T_2^\top)\Pi_2^\top$:

$$\psi^{(0)} = \eta(\alpha + \mathfrak{a}_+^2 D_1^{[1]} I^{(1)} + \mathfrak{a}_+^2 D_2^{[1]} I^{(2)} + b_1 I^{(1)} + b_2 I^{(2)}), \quad \psi^{(-1)} = \eta\beta, \quad \psi^{(-2)} = \eta\gamma.$$

Using Compatibility I we can obtain three non trivial equations from first superdiagonal, first subdiagonals and the main diagonal, respectively:

$$\begin{aligned}
\mathfrak{a}_-\alpha - \alpha &= \frac{1}{1-\eta}(I + \eta(I^{(2)} + (b_1 - b_2)(I^{(2)} - I^{(1)})),\\
\mathfrak{a}_+\gamma - \gamma &= \frac{\eta}{\eta - 1}\beta(I^{(1)} + (b_1 - b_2)(I^{(1)} - I^{(2)})),\\
\mathfrak{a}_+\beta - \beta &= \frac{1}{\eta - 1}(\alpha - \mathfrak{a}_+ D).
\end{aligned}$$

Equating both expressions for $\psi^{(0)}$ we obtain expressions for α:

$$\alpha_{2n} = \frac{1}{1-\eta}2n + \frac{\eta}{1-\eta}(n + b_1), \quad \alpha_{2n+1} = \frac{1}{1-\eta}(2n+1) + \frac{\eta}{1-\eta}(n + b_2)$$

From compatibility conditions we obtain:

$$\begin{aligned}
\beta_{2n} &= \frac{\eta}{(1-\eta)^2}(3n^2 + n(b_1 + b_2 - 2)),\\
\beta_{2n+1} &= \frac{\eta}{(1-\eta)^2}(3n^2 + n(b_1 + b_2 + 1) + b_1),\\
\gamma_{2n} &= \frac{\eta^2}{(1-\eta)^3} n(n + b_1 - 1)(n + b_1 - b_2),\\
\gamma_{2n+1} &= \frac{\eta^2}{(1-\eta)^3} n(n + b_2 - 1)(n + b_2 - b_1).
\end{aligned}$$

As we can see, results for α_{2n}, α_{2n+1}, β_{2n}, β_{2n+1}, γ_{2n} and γ_{2n+1} agree with results obtained in [17].

4.3 Generalized Charlier multiple orthogonal polynomials

In this case we have that:

$$\theta(k) = k(k+c), \qquad \sigma^{(1)}(k) = \eta^{(1)}, \qquad \sigma^{(2)}(k) = \eta^{(2)}.$$

The weights are

$$w^{(1)}(k) = \frac{1}{(c+1)_k}\frac{{\eta^{(1)}}^k}{k!}, \qquad w^{(2)}(k) = \frac{1}{(c+1)_k}\frac{{\eta^{(2)}}^k}{k!},$$

are for $c > -1, \eta^{(1)}, \eta^{(2)} > 0$, according to the Lemma 2.4, AT systems. Due to the $\sigma^{(1)}, \sigma^{(2)}$ and θ polynomials degrees, Laguerre-Freud matrix has three non-trivial diagonals:

$$\Psi = \psi^{(0)} + \psi^{(1)}\Lambda + \psi^{(2)}\Lambda^2.$$

By means of $\Psi = \Pi^{-1}\theta(T)$ we can obtain expressions for first and second subdiagonals and through $\Psi = \Pi_1^\top\sigma^{(1)}(T_1^\top) + \Pi_2^\top\sigma^{(2)}(T_2^\top)$ the main diagonal is obtained:

$$\psi^{(0)} = \eta^{(1)}I^{(1)} + \eta^{(2)}I^{(2)}, \qquad \psi^{(1)} = \mathfrak{a}_-\alpha + \alpha + c - \mathfrak{a}_+D, \qquad \psi^{(2)} = I.$$

From Compatibility I non-trivial equations for first superdiagonal, main diagonal and first subdiagonal are obtained:

$$\begin{aligned}
\mathfrak{a}_-\beta - \mathfrak{a}_+\beta &= (\alpha + I - \mathfrak{a}_-\alpha)(\mathfrak{a}_-\alpha + \alpha + c - \mathfrak{a}_+D)\\
&\quad + (\eta^{(1)} - \eta^{(2)})(I^{(2)} - I^{(1)}),\\
\eta^{(1)}I^{(1)} + \eta^{(2)}I^{(2)} &= \gamma - \mathfrak{a}_+^2\gamma + \beta(\mathfrak{a}_-\alpha + \alpha + c - \mathfrak{a}_+D)\\
&\quad - \mathfrak{a}_+\beta(\alpha + \mathfrak{a}_+\alpha + c - \mathfrak{a}_+^2D),\\
\beta(\eta^{(1)} - \eta^{(2)})(I^{(1)} - I^{(2)}) &= \gamma(\mathfrak{a}_-^2\alpha + \mathfrak{a}_-\alpha + c - D)\\
&\quad - \mathfrak{a}_+\gamma(\alpha + \mathfrak{a}_+\alpha + c - \mathfrak{a}_+^2D).
\end{aligned}$$

These equations can be written element by element respectively as

$$\begin{aligned}
&\beta_{n+2} - \beta_n = (\alpha_n + 1 - \alpha_{n+1})(\alpha_{n+1} + \alpha_n + c - n) + (-1)^{n+1}(\eta^{(1)} - \eta^{(2)}),\\
&\frac{\eta^{(1)} + \eta^{(2)}}{2} + (-1)^n\frac{\eta^{(1)} - \eta^{(2)}}{2} = \gamma_{n+2} - \gamma_n + \beta_{n+1}(\alpha_{n+1} + \alpha_n + c - n)\\
&\quad - \beta_n(\alpha_n + \alpha_{n-1} + c - n + 1),\\
&(-1)^n\beta_{n+1}(\eta^{(1)} - \eta^{(2)}) = \gamma_{n+2}(\alpha_{n+2} + \alpha_{n+1} + c - n - 1)
\end{aligned}$$

$$- \gamma_{n+1}(\alpha_n + \alpha_{n-1} + c - n + 1).$$

From Compatibility II we obtain three non-trivial equations from first superdiagonal, main diagonal and first subdiagonal, respectively:

$$\begin{aligned}
&\mathfrak{a}_-\beta - \mathfrak{a}_+\beta = (\alpha + I - \mathfrak{a}_-\alpha)(\mathfrak{a}_-\alpha + \alpha + c - \mathfrak{a}_+ D) + (\eta^{(1)} - \eta^{(2)})(I^{(2)} - I^{(1)}),\\
&\gamma - \mathfrak{a}_+^2\gamma = \eta^{(1)} I^{(1)} + \eta^{(2)} I^{(2)} + \mathfrak{a}_+\beta(\alpha + \mathfrak{a}_+\alpha + c - \mathfrak{a}_+^2 D)\\
&\quad - \beta(\mathfrak{a}_-\alpha + \alpha + c - \mathfrak{a}_+ D),\\
&\beta(\eta^{(1)} - \eta^{(2)})(I^{(1)} - I^{(2)}) = \gamma(\mathfrak{a}_-^2\alpha + \mathfrak{a}_-\alpha + c - D) - \mathfrak{a}_+\gamma(\alpha + \mathfrak{a}_+\alpha + c - \mathfrak{a}_+^2 D).
\end{aligned}$$

Proposition 4.1 (Laguerre–Freud equations) *Laguerre–Freud Equations* (68) *for generalized Charlier multiple orthogonal polynomials are*

$$\begin{aligned}
\alpha_{n+2} &= n + 1 - c - \alpha_{n+1}\\
&\quad + \gamma_{n+2}^{-1}\Big((-1)^n\beta_{n+1}(\eta^{(1)} - \eta^{(2)}) + \gamma_{n+1}(\alpha_n + \alpha_{n-1} + c - n + 1)\Big),\\
\beta_{n+2} &= \beta_n + (\alpha_n + 1 - \alpha_{n+1})(\alpha_{n+1} + \alpha_n + c - n) + (-1)^{n+1}(\eta^{(1)} - \eta^{(2)}),\\
\gamma_{n+2} &= \gamma_n + \beta_n(\alpha_n + \alpha_{n-1} + c - n + 1) - \beta_{n+1}(\alpha_{n+1} + \alpha_n + c - n)\\
&\quad + \frac{\eta^{(1)} + \eta^{(2)}}{2} + (-1)^n \frac{\eta^{(1)} - \eta^{(2)}}{2}.
\end{aligned}$$

4.4 Generalized Meixner of second kind multiple orthogonal polynomials

In this case we have

$$\theta(k) = k(k + c), \qquad \sigma^{(1)}(k) = \eta(k + b_1), \qquad \sigma^{(2)}(k) = \eta(k + b_2).$$

The weights are:

$$w^{(1)}(k) = \frac{(b_1)_k}{(c+1)_k}\frac{\eta^k}{k!}, \qquad w^{(2)}(k) = \frac{(b_2)_k}{(c+1)_k}\frac{\eta^k}{k!},$$

if $b_1, b_2, \eta > 0$ and $c > -1$, according with Lemma 2.4, it is an AT system.

In this case Laguerre–Freud matrix has five non-null diagonals, it is:

$$\Psi = (\Lambda^{\top})^2\psi^{(-2)} + \cdots + \psi^{(2)}\Lambda^2.$$

Second and first superdiagonals are obtained using $\Psi = \Pi^{-1}\theta(T)$:

$$\begin{aligned}
&\psi^{(2)} = I, \quad \psi^{(1)} = \alpha + \mathfrak{a}_-\alpha + c - \mathfrak{a}_+ D,\\
&\psi^{(0)} = \mathfrak{a}_+\beta + \beta + \alpha(\alpha + c) - \mathfrak{a}_+ D(\mathfrak{a}_+\alpha + \alpha + c) + \mathfrak{a}_+^2\Pi^{[-2]}.
\end{aligned}$$

Using $\Psi=\sigma^{(1)}(T_1^\intercal)\Pi_1^\intercal+\sigma^{(2)}(T_2^\intercal)\Pi_2^\intercal$:

$$\psi^{(-2)}=\eta\gamma,\qquad \psi^{(-1)}=\eta\beta,\qquad \psi^{(0)}=\eta(\alpha+I^{(1)}(\mathfrak{a}_+^2D_1^{[1]}+b_1)+I^{(2)}(\mathfrak{a}_+^2D_2^{[1]}+b_2)).$$

From Compatibility I we get three non-trivial equations from the first subdiagonal, the main diagonal and the first superdiagonal, respectively:

$$\begin{aligned}
&\eta(\mathfrak{a}_+\gamma-\gamma)+\gamma(\mathfrak{a}_-\alpha+\mathfrak{a}_-^2\alpha+c-D)-\mathfrak{a}_+\gamma(\mathfrak{a}_+\alpha+\alpha+c-\mathfrak{a}_+^2D)\\
&\quad=\eta\beta\left(I^{(1)}+(b_1-b_2)(I^{(1)}-I^{(2)})\right),\\
&\qquad\gamma-\mathfrak{a}_+^2\gamma=\eta\left(\alpha+\beta-\mathfrak{a}_+\beta+I^{(1)}(\mathfrak{a}_+^2D_1^{[1]}+b_1)+I^{(2)}(\mathfrak{a}_+^2D_2^{[1]}+b_2)\right)\\
&\qquad+\mathfrak{a}_+\beta(\mathfrak{a}_+\alpha+\alpha+c-\mathfrak{a}_+^2D)\\
&\qquad-\beta(\alpha+\mathfrak{a}_-\alpha+c-\mathfrak{a}_+D),\\
&\qquad\mathfrak{a}_-\beta-\mathfrak{a}_+\beta+(\mathfrak{a}_-\alpha-\alpha)(\alpha-\mathfrak{a}_-\alpha+c-\mathfrak{a}_+D)\\
&\qquad+\eta\left(\alpha-\mathfrak{a}_-\alpha+(b_1-b_2)(I^{(1)}-I^{(2)})-I^{(2)}\right)=\alpha+\mathfrak{a}_-\alpha+c-\mathfrak{a}_+D.
\end{aligned}$$

Equations above can be expressed as

$$\begin{aligned}
&\eta(\gamma_{n+1}-\gamma_{n+2})+\gamma_{n+2}(\alpha_{n+1}+\alpha_{n+2}+c-(n+1))-\gamma_{n+1}(\alpha_n+\alpha_{n+1}+c-n)\\
&\quad=\eta\beta_{n+1}\left(\frac{1+(-1)^n}{2}+(-1)^n(b_1-b_2)\right),\\
&\qquad\gamma_{n+2}-\gamma_n+\beta_{n+1}(\alpha_n+\alpha_{n+1}+c-n)-\beta_n(\alpha_{n-1}\\
&\quad+\alpha_n+c-(n-1))+\eta(\beta_n-\beta_{n+1})\\
&\quad=\eta\left(\alpha_n+\frac{n-1}{2}+\left(\frac{1}{2}+b_2\right)\frac{1-(-1)^n}{2}+\frac{b_1}{2}(1+(-1)^n)\right),\\
&\qquad\alpha_n+\alpha_{n+1}+c-n=\beta_{n+2}-\beta_n+(\alpha_{n+1}-\alpha_n)(\alpha_n-\alpha_{n+1}+c-n)\\
&\qquad+\eta\left(\alpha_n-\alpha_{n+1}+(-1)^n(b_1-b_2)+\frac{(-1)^n-1}{2}\right),
\end{aligned}$$

respectively.

From Compatibility II we obtain

$$\begin{aligned}
&\eta(\mathfrak{a}_+\gamma-\gamma)+\gamma(\mathfrak{a}_-\alpha+\mathfrak{a}_-^2\alpha+c-D)-\mathfrak{a}_+\gamma(\mathfrak{a}_+\alpha+\alpha+c-\mathfrak{a}_+^2D)\\
&\quad=\eta\beta(I^{(1)}+(b_1-b_2)(I^{(1)}-I^{(2)})),\\
&\qquad\gamma-\mathfrak{a}_+^2\gamma=\eta[\alpha+\beta-\mathfrak{a}_+\beta+I^{(1)}(\mathfrak{a}_+^2D_1^{[1]}+b_1)+I^{(2)}(\mathfrak{a}_+^2D_2^{[1]}+b_2)]\\
&\qquad+\mathfrak{a}_+\beta(\mathfrak{a}_+\alpha+\alpha+c-\mathfrak{a}_+^2D)\\
&\qquad-\beta(\alpha+\mathfrak{a}_-\alpha+c-\mathfrak{a}_+D)\\
&\qquad\eta(I+\mathfrak{a}_-^2\alpha-\alpha)=T_-^2\beta+\mathfrak{a}_-\beta-\beta-\mathfrak{a}_+\beta+\mathfrak{a}_-^2\alpha(\mathfrak{a}_-^2\alpha+c)-\alpha(\alpha+c)\\
&\qquad-\mathfrak{a}_-D(\mathfrak{a}_-^2\alpha+\mathfrak{a}_-\alpha)+\mathfrak{a}_+D(\alpha+\mathfrak{a}_+\alpha)\\
&\qquad-2cI+\pi^{[-2]}-\mathfrak{a}_+^2\pi^{[-2]}.
\end{aligned}$$

as we can see that first and second equations appear in compatibility I.

Proposition 4.2 (Laguerre–Freud equations) *Laguerre–Freud equations* (68) *for generalized Meixner II multiple orthogonal polynomials are*

$$
\begin{aligned}
\alpha_{n+2} &= n+1-c-\alpha_{n+1}+\gamma_{n+2}^{-1}\left(\gamma_{n+1}\left(\alpha_n+\alpha_{n+1}+c-n\right)+\eta(\gamma_{n+2}-\gamma_{n+1})\right.\\
&\quad\left.+\eta\beta_{n+1}\left(\frac{1+(-1)^n}{2}+(-1)^n(b_1-b_2)\right)\right),\\
\beta_{n+2} &= \beta-n+\alpha_n+\alpha_{n+1}+c-n-(\alpha_{n+1}-\alpha_n)(\alpha-n-\alpha_{n+1}+c-n)\\
&\quad-\eta\Big(\alpha_n-\alpha_{n+1}+(-1)^n(b_1-b_2)+\frac{(-1)^n-1}{2}\Big),\\
\gamma_{n+2} &= \gamma_n+\beta_n(\alpha_{n-1}+\alpha_n+c-n+1)-\beta_{n+1}(\alpha_n+\alpha_{n+1}+c-n)-\eta(\beta_n-\beta_{n+1})\\
&\quad+\eta\left(\alpha_n+\frac{n-1}{2}+\left(\frac{1}{2}+b_2\right)\frac{1-(-1)^n}{2}+\frac{b_1}{2}(1+(-1)^n)\right).
\end{aligned}
$$

Conclusions and outlook

Adler and van Moerbeke extensively utilized the Gauss–Borel factorization of the moment matrix in their studies of integrable systems and orthogonal polynomials [1–3]. We have also employed these ideas in various contexts, including CMV orthogonal polynomials, matrix orthogonal polynomials, multiple orthogonal polynomials, and multivariate orthogonal polynomials [5, 7–16]. For a comprehensive overview, refer to [53].

In a recent work [55], we applied this approach to investigate the implications of the Pearson equation on the moment matrix and Jacobi matrices. To describe these implications, we introduced a new banded matrix called the Laguerre-Freud structure matrix, which encodes the Laguerre–Freud relations for the recurrence coefficients. We also discovered that the contiguous relations satisfied generalized hypergeometric functions, determining the moments of the weight described by the squared norms of the orthogonal polynomials. This led to a discrete Toda hierarchy known as the Nijhoff–Capel equation [59].

In [54], we further examined the role of Christoffel and Geronimus transformations in describing the aforementioned contiguous relations and used Geronimus–Christoffel transformations to characterize shifts in the spectral independent variable of the orthogonal polynomials. Building on this program, we delved deeper into the study of discrete semiclassical cases, discovering Laguerre–Freud relations for the recursion coefficients of three types of discrete orthogonal polynomials: generalized Charlier, generalized Meixner, [37] and generalized Hahn of type I [36].

In [35], we completed this program by obtaining Laguerre-Freud structure matrices and equations for three families of hypergeometric discrete orthogonal polynomials.

In this paper, we extend the previous lines of research to multiple orthogonal polynomials on the step line with two weights. We derive hypergeometric τ-function representations for these multiple orthogonal polynomials and introduce a third-order integrable nonlinear Toda-type equation for these τ-functions. Additionally, we discover systems of nonlinear discrete Nijhoff–Capel Toda-type equations satisfied by these objects. Finally, we apply the Laguerre–Freud matrix structure and compatibilities to derive Laguerre–Freud equations for the multiple generalized Charlier and multiple generalized Meixner of the second type.

As a future outlook, we highlight the following five lines of research:

(i) Finding larger families of hypergeometric AT systems.
(ii) Analyzing the case of p weights with $p > 2$.
(iii) Studying the connections between the topics discussed in this paper and the transformations presented in [21, 22] and quadrilateral lattices [31, 56].
(iv) Exploring the relationship with multiple versions of discrete and continuous Painlevé equations involving the coefficients α_n, β_n, and γ_n. The results presented in this paper constitute progress in this direction, as identifying this connection is a nontrivial problem that can be further investigated in future works.
(v) Discussing higher-order integrable flows and finding applications to multicomponent KP-type equations.

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Conflict of interest The authors declare no conflict of interest.

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TODA AND LAGUERRE–FREUD EQUATIONS FOR MULTIPLE DISCRETE ORTHOGONAL POLYNOMIALS WITH AN ARBITRARY NUMBER OF WEIGHTS

ITSASO FERNÁNDEZ-IRISARRI[1] AND MANUEL MAÑAS[2]

Abstract. In this paper, we extend our investigation into semiclassical multiple discrete orthogonal polynomials by considering an arbitrary number of weights. We derive multiple versions of the Toda equations and the Laguerre-Freud equations for the multiple generalized Charlier and multiple generalized Meixner II families.

1. Introduction

Discrete multiple orthogonal polynomials, τ-functions, generalized hypergeometric series, Pearson equations, Laguerre–Freud equations, and Toda type equations are intricate mathematical concepts that are deeply intertwined and have broad applications across various fields of mathematics and physics. They are fundamental tools for understanding complex systems, integrable models, and specialized functions.

In our prior work, cf. [26], we focused on the scenario involving two weights. Now, in this exposition, we venture into a deeper investigation of discrete multiple orthogonal polynomials encompassing an arbitrary number of weights. We aim to unveil the intricate mathematical landscape surrounding these polynomials, uncovering their symbiotic connections with τ-functions and generalized hypergeometric series. Additionally, we explore their fascinating applications in Laguerre–Freud equations and various instances of Toda equations.

Orthogonal polynomials, see [33, 15, 5], an indispensable cornerstone of Applied Mathematics and Theoretical Physics, hold sway over a vast array of disciplines, finding utility in approximation theory, numerical analysis, and beyond. Within this expansive realm, multiple orthogonal polynomials and discrete orthogonal polynomials emerge as particularly intriguing subclasses.

Originating in 1895, Pearson's pioneering work [45] in curve fitting gave birth to a renowned family of frequency curves encapsulated by the differential equation $\frac{f'(x)}{f(x)}=\frac{p_1(x)}{p_2(x)}$. Here, f denotes the probability density function, and p_i represents polynomials of degree at most i, for $i=1,2$. Pearson equations, an enduring subject of study, find their nexus with orthogonal polynomials through their solutions, which often align with or closely resemble certain families of orthogonal polynomials. Central to this connection is the weight function that underpins the orthogonality properties of these polynomials. These equations have garnered considerable attention owing to their intimate connection with special functions, including classical orthogonal polynomials like Legendre, Hermite, and Laguerre polynomials, as well as their extensions. Our approach in this study involves employing the Gauss-Borel method for orthogonal polynomials, extensively discussed in a review paper [40]. Building upon our prior research, which explored the application of this technique to various domains, such as discrete hypergeometric orthogonal polynomials, Toda equations, τ-functions, and Laguerre–Freud equations [41, 23, 24, 25], we delve deeper into its implications in this paper.

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Multiple orthogonal polynomials [44, 33, 42, 2] emerge when the orthogonality conditions involve multiple weight functions. In contrast with classical orthogonal polynomials, which rely on a single weight function, multiple orthogonal polynomials possess a more intricate structure due to the presence of several weight functions. These polynomials have been extensively studied and find applications in various fields, including simultaneous approximation, random matrix theory, and statistical mechanics. For insights into the utilization of the Pearson equation in multiple orthogonality, refer to [18, 8] and also [9].

In contrast, discrete orthogonal polynomials [33, 43] arise when orthogonality conditions are defined over a discrete set of points. These polynomials are employed in diverse areas such as combinatorial optimization, signal processing, and coding theory. They serve as valuable tools for analyzing and solving problems in discrete settings, where the underlying structure often comprises a finite or countable set of points. Discrete Pearson equations play a crucial role in the classification of classical discrete orthogonal polynomials [43]. When the weight satisfies a discrete Pearson equation, we encounter semiclassical discrete orthogonal polynomials, as discussed in [21, 22, 19, 20]. Furthermore, multiple discrete orthogonal polynomials have been examined in [4].

The realm of integrable discrete equations [32] stands as an enthralling and consequential domain within Mathematical Physics, shedding light on the dynamics of discrete systems endowed with extraordinary attributes. These equations boast a complex algebraic and geometric framework, affording them soliton solutions, conservation laws, and integrability properties. Integrability in discrete equations transcends mere solvability, encompassing a wealth of conserved quantities and symmetries [1]. This integrability property paves the way for the development of robust mathematical tools to analyze and comprehend their behavior.

A distinctive hallmark of integrable discrete equations lies in their intimate connection with orthogonal polynomials. These polynomials assume a pivotal role in crafting discrete equations with distinct characteristics. The nexus between discrete equations and orthogonal polynomials offers profound insights into their solutions, recursion relations, and symmetry properties.

The τ-function, as elucidated by Harnad [31], stands as a linchpin in the domain of integrable systems, seamlessly bridging the spectral theory of linear operators with the dynamics of soliton equations. Its significance reverberates across various mathematical realms, including algebraic geometry, representation theory, and special functions.

Semiclassical orthogonal polynomials and their associated functionals, supported on complex curves, have been subject to extensive scrutiny in the context of isomonodromic τ functions and matrix models by Bertola, Eynard, and Harnad [7].

In the sphere of discrete multiple orthogonal polynomials, the τ-function assumes a paramount role, encapsulating the intrinsic integrable structure and offering insights into the dynamics and symmetries of the corresponding discrete systems. Its exploration unveils profound connections between diverse mathematical entities and unveils hidden patterns lurking within.

Generalized hypergeometric series, a focal point of this research, represent a class of special functions prevalent across diverse mathematical landscapes [14, 3]. Renowned for their exceptional properties, they have been subject to extensive inquiry owing to their connections with combinatorics, number theory, and mathematical physics. These series play a pivotal role in expressing solutions of differential equations, evaluating integrals, and comprehending the behavior of various mathematical models.

Crucially, generalized hypergeometric series occupy a central position in the Askey scheme, a comprehensive framework that unifies essential families of orthogonal functions through a chain of limits [34]. The Askey scheme, as extended to encompass multiple orthogonality in [6], provides a structured approach to understanding various families of special functions.

Recently, we have made strides in advancing the multiple Askey scheme by establishing hypergeometric expressions for Hahn type I multiple orthogonal polynomials and their progeny within this scheme [10], see

[11] for the corresponding bidiagonal factorization, and [12, 13] for the explicit expressions for type I and II classical multiple orthogonal polynomials and ts recursion coefficients.

Laguerre–Freud equations form a family of differential equations that surface in the realm of orthogonal polynomials associated with exponential weights. These equations encode nonlinear connections among the recursion coefficients within the three-term recurrence relation for orthogonal polynomials. In essence, they entail a set of nonlinear equations governing the coefficients $\{\beta_n, \gamma_n\}$ within the recurrence relation $zP_n(z) = P_{n+1}(z)+\beta_n P_n(z)+\gamma_n P_{n-1}(z)$ characterizing the orthogonal polynomial sequence. These equations manifest as $\gamma_{n+1} = \mathcal{G}(n, \gamma_n, \gamma_{n-1}, \dots, \beta_n, \beta_{n-1}, \dots)$ and $\beta_{n+1} = \mathcal{B}(n, \gamma_{n+1}, \gamma_n, \dots, \beta_n, \beta_{n-1}, \dots)$.

Named Laguerre–Freud equations by Magnus [36, 37, 38, 39] with reference to [35, 30], these equations have garnered attention in various contexts. Numerous studies delve into Laguerre–Freud relations concerning generalized Charlier, generalized Meixner, and type I generalized Hahn cases [46, 16, 27, 28, 29, 19].

Laguerre–Freud equations share a profound connection with Painlevé equations, a family of nonlinear ordinary differential equations initially introduced by Painlevé in the early 20th century. These equations have garnered considerable attention owing to their integrability properties and the intricate mathematical structures underpinning their solutions.

The intertwining of orthogonal polynomials with Painlevé equations stems from the fact that specific families of orthogonal polynomials can be linked to solutions of particular Painlevé equations. This association offers profound insights into the solvability and analytic properties of Painlevé equations, enabling the construction of explicit solutions through the theory of orthogonal polynomials. For a comprehensive exploration of this interplay between these domains, refer to [47, 16, 17].

Expanding upon the insights gleaned from prior investigations [41, 24, 25, 23], this paper delves into the interrelations among discrete multiple orthogonal polynomials, τ-functions, generalized hypergeometric series, and specific equations such as Laguerre–Freud equations and Toda type equations.

In contrast, Toda type equations represent nonlinear partial differential-difference equations of paramount importance in mathematical physics and integrable systems [31, 32]. Unraveling the nexus between these equations and the aforementioned mathematical constructs offers profound insights into the dynamics of discrete systems, the behavior of special functions, and the intricacies of integrable models.

In this paper, our goal is to offer a comprehensive exploration of discrete multiple orthogonal polynomials with an arbitrary number of weights, τ-functions, generalized hypergeometric series, Laguerre–Freud equations, and Toda type equations. Moreover, we will investigate the intricate interplay between these entities, uncovering their connections and mutual dependencies. Specifically, we will examine how discrete multiple orthogonal polynomials intertwine with τ-functions and generalized hypergeometric series, unraveling the underlying mathematical relationships.

Furthermore, we will delve into the practical implications of these mathematical concepts by exploring their applications in Laguerre–Freud equations and Toda type equations. By examining both continuous and discrete variants, we aim to showcase their significance in the realm of mathematical physics and integrable systems, illustrating their versatility and utility in diverse contexts.

The layout of the paper is as follows. We continue this section with a basic introduction to discrete multiple orthogonal polynomials and their associated tau functions. In the second section, we delve into Pearson equations for multiple orthogonal polynomials, generalized hypergeometric series, and the corresponding hypergeometric discrete multiple orthogonal polynomials. We discuss their properties, including the Laguerre–Freud matrix that models the shift in the spectral variable, as well as contiguity relations and their consequences.

In the third section, we present two of the main findings discussed in this paper. In Theorem 3.9, we present nonlinear partial differential equations of the Toda type, for which the hypergeometric tau functions provide solutions. Additionally, we discuss a completely discrete system, extending the completely discrete Nikhoff–Capel Toda equation, and its solutions in terms of the hypergeometric tau functions.

Finally, in the fourth section, we present explicit Laguerre–Freud equations for the first time for the multiple versions of the generalized Charlier and generalized Meixner II orthogonal polynomials.

1.1. **Discrete multiple orthogonality on the step line with an arbitrary number of weights.** Let's delve into the fundamentals of discrete multiple orthogonality concisely.

We start by defining two sets of vectors of monomials in the variable x:

$$X = \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \end{bmatrix}, \quad X^{(a)} = \begin{bmatrix} e_a \\ xe_a \\ x^2 e_a \\ \vdots \end{bmatrix},$$

where $a \in \{1, \dots, p\}$ and $\{e_a\}_{a=1}^{p}$ represents the canonical basis in $\mathbb{R}^p$.

Next, we introduce the moment matrix:

$$\mathscr{M} := \sum_{k=0}^{\infty} X(k)\big(X^{(1)}(k)w^{(1)}(k) + \dots + X^{(q)}(k)w^{(q)}(k)\big)^{\top},$$

where functions $w^{(a)}$, $a \in \{1, \dots, p\}$, denote p weights supported on the homogeneous lattice $\mathbb{N}_0$.

Now, let's discuss some key matrix components:

i) ***Projection Matrices:*** These are semi-infinite matrices defined as

$$I^{(a)} := \operatorname{diag}(0, \dots, 1, 0, \dots, 0, 1, \dots),$$

where $a \in \{1, \dots, p\}$, and non-zero diagonal entries occur in positions $a + ip$ for $i \in \mathbb{N}$.

ii) ***Truncated Projection Matrices:*** These are defined as

$$\tilde{I}^{(a)} := \operatorname{diag}(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^{p\times p},$$

with the non-zero element at $(\tilde{I}^{(a)})_{a,a}$, for $a \in \{1, \dots, p\}$.

iii) ***Shift Matrices:*** These are given by

$$\Lambda := \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

and $\Lambda^{(a)} := \Lambda^q I^{(a)}$, $a \in \{1, \dots, p\}$.

We observe the following equations hold true:

$$\Lambda X(x) = xX(x), \quad \Lambda^{(a)} X^{(b)}(x) = xX^{(a)}(x)\delta_{a,b}.$$

The moment matrix $\mathscr{M}$ is a p-Hankel matrix, satisfying:

$$\Lambda\mathscr{M} = \mathscr{M}(\Lambda^{\top})^p.$$

We then consider the Gauss-Borel factorization for $\mathscr{M}$:

$$\mathscr{M} = S^{-1} H \tilde{S}^{-\top},$$

where S and $\tilde{S}$ are lower unitriangular matrices and H is diagonal.

These matrices S and $\tilde{S}$ can be expressed by adding diagonal matrices:

$$\begin{aligned} S &= I + \Lambda^{\top} S^{[1]} + (\Lambda^{\top})^2 S^{[2]} + \cdots, & S^{-1} &= I + \Lambda^{\top} S^{[-1]} + (\Lambda^{\top})^2 S^{[-2]} + \cdots, \\ \tilde{S} &= I + \Lambda^{\top} \tilde{S}^{[1]} + (\Lambda^{\top})^2 \tilde{S}^{[2]} + \cdots, & \tilde{S}^{-1} &= I + \Lambda^{\top} \tilde{S}^{[-1]} + (\Lambda^{\top})^2 \tilde{S}^{[-2]} + \cdots. \end{aligned}$$

Additionally, we define the lowering and rising shift operators, $\mathfrak{a}_-$ and $\mathfrak{a}_+$, for diagonal matrices D as:

$$\mathfrak{a}_- \operatorname{diag}(m_0, m_1, \dots) = \operatorname{diag}(m_1, m_2, \dots), \quad \mathfrak{a}_+ \operatorname{diag}(m_0, m_1, \dots) = \operatorname{diag}(0, m_0, \dots).$$

Lastly, we define multiple orthogonal polynomials:

i) ***Type I:***

$$A^{(n)} := H^{-1} \tilde{S} X^{(n)},$$

where $n \in \{1, \dots, p\}$.

ii) ***Type II:***

$$B := SX.$$

For perfect systems of weights, the degrees of these polynomials are:

$$\deg B_n = n, \quad \deg A_n^{(a)} = \left\lceil \frac{n+2-a}{p} \right\rceil - 1, \quad a \in \{1, \dots, p\},$$

where $\lceil x \rceil$ represents the smallest integer greater than or equal to x. Multiple orthogonality relations are expressed as:

$$\sum_{a=1}^{p} \sum_{k=0}^{\infty} A_n^{(a)}(k) w^{(a)}(k) k^m = 0, \quad k \in \{0, \dots, \deg B_{n-1}\},$$

$$\sum_{k=0}^{\infty} B_n(k) w^{(a)}(k) k^m = 0, \quad k \in \{0, \dots, \deg A_{n-1}^{(a)}\}, \quad a \in \{1, \dots, p\}.$$

Let's introduce the τ-functions:

i) τ functions are defined departing from truncations of the moment matrix $M^{[n]}$ as:

$$\tau_n := \det \mathscr{M}^{[n]}.$$

ii) τ-associated functions τ_n^j with $j \in \{1, \dots, n-1\}$, are the matrix determinants $\mathscr{M}^{[n,j]}$ obtained from $\mathscr{M}^{[n]}$ by removing the $(n-j)$-th row $\begin{bmatrix} M_{n-j,0} & \dots & M_{n-j,n-1} \end{bmatrix}$ and adding, as the last row, the row $\begin{bmatrix} M_{n,0} & \dots & M_{n,n-1} \end{bmatrix}$.

Type I polynomials can be expressed as:

$$A_n^{(a)}(x) = \frac{1}{\tau_{n+1}} \left| \begin{array}{ccc|c} & & & \mathscr{M}_{0,n} \\ & \mathscr{M}^{[n]} & & \vdots \\ & & & \mathscr{M}_{n-1,n} \\ \hline X_0^{(a)} & \cdots\cdots\cdots & X_{n-1}^{(a)} & X_n^{(a)} \end{array} \right|, \quad n \in \mathbb{N}_0, \quad a \in \{1, \dots, p\}.$$

Similarly, type II polynomials can be expressed as:

$$B_n(x) = \frac{1}{\tau_n} \left| \begin{array}{ccc|c} & & & 1 \\ & \mathscr{M}^{[n]} & & \vdots \\ & & & x^{n-1} \\ \hline \mathscr{M}_{n,0} & \cdots\cdots\cdots & \mathscr{M}_{n,n-1} & x^n \end{array} \right|, \quad n \in \mathbb{N}_0.$$

These multiple orthogonal polynomials satisfy recurrence relations. Considering the multi-Hankel structure of the moment matrix $\mathscr{M}$, the banded Hessenberg matrix T can be constructed in terms of S and $\tilde{S}$ as follows:

$$T := S \Lambda S^{-1} = H \tilde{S}^{-\top} (\Lambda^\top)^p \tilde{S}^\top H^{-1}.$$

Thus, we have a $p+2$ terms recurrence relation that can be written as:

$$TB(x) = xB(x), \quad T^\top A^{(a)}(x) = xA^{(a)}(x).$$

The banded Hessenberg matrix has $p+2$ non-zero diagonals:

$$T=\Lambda+\alpha^{(0)}+\Lambda^{\top}\alpha^{(1)}+\cdots+(\Lambda^{\top})^{q}\alpha^{(q)},$$

where expressions for α matrices depending on $\tilde{S}$ or S are as follows:

$$\begin{aligned}
\alpha^{(0)}&=\mathfrak{a}_{+}S^{[1]}-S^{[1]},\\
\alpha^{(1)}&=\mathfrak{a}_{+}S^{[2]}-S^{[2]}+S^{[1]}(\mathfrak{a}_{-}S^{[1]}-S^{[1]}),\\
\alpha^{(a)}&=\mathfrak{a}_{+}S^{[a+1]}+S^{[-a-1]}+\sum_{i=1}^{a}\mathfrak{a}_{-}^{a}S^{[a-i]}S^{[-i]},\\
&=H^{-1}(\mathfrak{a}_{-}^{a}H)\Bigg(\mathfrak{a}_{+}^{p-a}\tilde{S}^{[p-a]}+\mathfrak{a}_{-}^{a}\tilde{S}^{[-(p-a)]}+\sum_{j=1}^{q-a-1}(\mathfrak{a}_{+}^{j}\tilde{S}^{[j]})\mathfrak{a}_{-}^{a}\tilde{S}^{[-(p-a-j)]}\Bigg),\\
&\ \ \vdots\\
\alpha^{(p-1)}&=H^{-1}(\mathfrak{a}_{-}^{p-1}H)(\mathfrak{a}_{+}\tilde{S}^{[1]}-\mathfrak{a}_{-}^{q-1}\tilde{S}^{[1]}),\\
\alpha^{(p)}&=(\mathfrak{a}_{-}^{p}H)H^{-1}.
\end{aligned}$$

Partial recurrence matrices $T^{(n)}$ are given as follows

$$T^{(a)}=H^{-1}\tilde{S}\Lambda_{a}\tilde{S}^{-1}H.$$

Note that

$$\sum_{a=1}^{q}T^{(a)}=T^{\top}.$$

For this discrete scenario we require also of the Pascal matrices $L^{\pm}$ are defined as matrices with entries are

$$L^{+}_{n,m}=\begin{cases}\binom{n}{m}, & n\geq m,\\ 0, & n<m,\end{cases}\qquad\qquad L^{-}_{n,m}=\begin{cases}(-1)^{n+m}\binom{n}{m}, & n\geq m,\\ 0 & n<m.\end{cases}$$

Note that $L^{+}L^{-}=I$. Moreover, we can readily check that

$$X(x+1)=L^{+}X(x),\qquad\qquad X(x-1)=L^{-}X(x).$$

Pascal matrices can be expanded as follows

$$L^{\pm}=I\pm\Lambda^{\top}D+(\Lambda^{\top})^{2}D^{[2]}+\cdots,$$

where

$$D:=\operatorname{diag}(1,2,3,\dots),\quad D^{[n]}:=\frac{1}{n}\operatorname{diag}(n^{(n)},(n+1)^{(n)},\dots)$$

and $x^{(n)}=x(x-1)\dots(x-n+1)$.

Partial Pascal matrices $L^{(n)\pm}$ are defined departing from the truncated projectors as follows and they have the following nonzero entries:

$$L^{\pm(a)}_{2n+a-1,2m+a-1}=(\pm1)^{n+m}\begin{pmatrix}n\\m\end{pmatrix},\quad a\in\{1,\dots,p\},\quad n,m\in\mathbb{N}_{0},\quad n\geq m$$

Partial Pascal matrices satisfy

$$X^{(a)}(x+1)=L^{(a)+}X^{(a)}(x),\quad X^{(a)}(x-1)=L^{(a)-}X^{(a)}(x).$$

Partial Pascal matrices can be expanded as:

$$L^{(a)\pm}=I^{(a)}\pm(\Lambda^{\top})^{q}D_{a}+(\Lambda^{\top})^{2q}D_{a}^{[2]}+\dots.$$

where we have

$$D_a^{[n]} := \frac{1}{n}\operatorname{diag}\left(n^{(n)}e_a, (n+1)^{(n)}e_a, \dots\right), \quad a \in \{1,\dots,p\}.$$

The dressed Pascal matrices can be defined from Pascal matrices as follows

$$\Pi^+ = SL^+S^{-1}, \quad \Pi^- = SL^-S^{-1}$$

and satisfy

$$B(x+1) = \Pi^+B(x), \quad B(x-1) = \Pi^-B(x).$$

Partial dressed Pascal matrices are defined as follows

$$\Pi^{(a)+} = H^{-1}\tilde{S}L^{(a)+}\tilde{S}^{-1}H, \quad \Pi^{(a)-} = H^{-1}\tilde{S}L^{(a)-}\tilde{S}^{-1}H,$$

so that the following relations hold

$$\Pi^{(a)+}A^{(a)}(x) = A^{(a)}(x+1), \quad \Pi^{(a)-}A^{(a)}(x) = A^{(a)}(x-1).$$

In this paper, generalized hypergeometric series are used, and they are given by

$${}_MF_N\left(\begin{matrix} b_1,\dots,b_M \\ \\ c_1,\dots,c_N\end{matrix};\eta\right) = {}_MF_N(b_1,\dots,b_M;c_1,\dots,c_N;\eta) := \sum_{k=0}^{\infty}\frac{(b_1)_k\dots(b_M)_k}{(c_1)_k\dots(c_N)_k}\frac{\eta^k}{k!}$$

where $(a)_k$ is known as the Pochhammer symbol and is defined as

$$(a)_k := a(a+1)\dots(a+k-1), \quad (a)_0 = 1.$$

2. Discrete Multiple Orthogonal Polynomials and Pearson equation

2.1. **Pearson equations**. From hereon we will assume that all the weights adhere to the Pearson equations, represented as:

$$\theta(x+1)w^{(a)}(x+1) = \sigma_n(x)w^{(a)}(x), \tag{1}$$

where $\theta(x)$ and $\sigma_a(x)$ are polynomials and $a \in \{1,\dots,p\}$.

First, we give an important relation fulfilled by the moment matrix $\mathscr{M}$:

Theorem 2.1. *The moment matrix satisfies the following equation:*

$$\theta(\Lambda)\mathscr{M} = L^+\mathscr{M}\left(\sum_{a=1}^{p}L^{(a)+}\sigma_a(\Lambda^{(n)})\right)^\top. \tag{2}$$

Proof. Beginning with $\theta(\Lambda)\mathscr{M}$, we expand it as follows:

$$\begin{aligned}
\theta(\Lambda)\mathscr{M} &= \sum_{k=1}^{\infty}\theta(k)X(k)\left(\sum_{a=1}^{p}w^{(a)}(k)X^{(a)}(k)\right)^\top \\
&= \sum_{k=0}^{\infty}\theta(k+1)X(k+1)\left(\sum_{a=1}^{p}w^{(a)}(k+1)X^{(a)}(k+1)\right)^\top \\
&= \sum_{k=0}^{\infty}X(k+1)\left(\sum_{a=1}^{p}\sigma_a(k)w^{(a)}(k)X^{(a)}(k+1)\right)^\top \\
&= \sum_{k=0}^{\infty}L^+X(k)\left(\sum_{a=1}^{p}\sigma_a(k)w^{(a)}(k)X^{(a)}(k)^\top(L^{(a)+})\right)^\top \\
&= L^+\mathscr{M}\left(\sum_{a=1}^{p}L^{(a)+}\sigma_a(\Lambda^{(a)})\right)^\top.
\end{aligned}$$

□

Then, we provide its counterpart for the Hessenberg recurrence matrix:

Proposition 2.2. *The following equation holds for the Hessenberg matrix T if the weights satisfy the Pearson equations:*

$$\Pi^{-}\theta(T)=\sum_{a=1}^{p}\sigma_a(T^{(a)})^{\top}(\Pi^{(a)+})^{\top}.$$

Proof. Utilizing equation (2) and the Gauss-Borel factorization for the moment matrix, we have

$$\theta(\Lambda)S^{-1}H\tilde{S}^{-\top}=L^{+}S^{-1}H\tilde{S}^{-\top}\left(\sum_{a=1}^{p}L^{(a)+}\sigma_a(\Lambda^{(a)})\right)^{\top}.$$

If we multiply both sides by S on the right and by $\tilde{S}^{\top}H^{-1}$ on the left, we obtain:

$$S\theta(\Lambda)S^{-1}=SL^{+}S^{-1}H\tilde{S}^{-\top}\sum_{a=1}^{p}\sigma_a(\Lambda^{(a)})^{\top}\tilde{S}^{\top}H^{-1}H\tilde{S}^{-\top}(L^{(a)+})^{\top}\left(H\tilde{S}^{-\top}\right)^{-1}.$$

Finally, we get

$$\theta(T)=\Pi^{+}\sum_{a=1}^{p}\sigma_a(T^{(a)})^{\top}(\Pi^{(a)+})^{\top}.$$

□

2.2. **Relation with generalized hypergeometric series.** If we write the polynomials θ and σ_n as

$$\theta(k)=k(k+c_1-1)\cdots(k+c_N-1),\quad \sigma_a(k)=\eta^{(a)}(k+b_1^{(a)})\cdots(z+b_{M^{(a)}}^{(a)}),$$

with $M^{(a)},N\in\mathbb{N}_0$ and $a\in\{1,\dots,p\}$, the weights that satisfy the Pearson equation are:

$$w^{(a)}(k)=\frac{(b_1^{(m)})_k\cdots(b_{M^{(a)}}^{(a)})_k}{(c_1)_k\dots(c_N)_k}\frac{(\eta^{(a)})^k}{k!},$$

where $a\in\{1,\dots,p\}$. We define the following operators:

$$\vartheta^{(a)}:=\eta^{(a)}\frac{\partial}{\partial\eta^{(a)}},\quad \vartheta:=\sum_{a=1}^{q}\vartheta^{(a)}.$$

Theorem 2.3. *The moment matrix can be written in terms of hypergeometric functions as follows*

$$\mathscr{M}=\begin{bmatrix}\rho_0^{(1)} & \rho_0^{(2)} & \cdots\cdots & \rho_0^{(p)} & \rho_1^{(1)} & \cdots\cdots & \rho_1^{(p)}\cdots\cdots\\ \rho_1^{(1)} & \rho_1^{(2)} & \cdots\cdots & \rho_1^{(p)} & \rho_2^{(1)} & \cdots\cdots & \rho_2^{(p)}\cdots\cdots\\ \vdots & \vdots & & \vdots & \vdots & & \vdots\end{bmatrix},$$

where the moments are

$$\rho_0^{(a)}:={}_{M_a}F_N(b_1^{(a)},\dots,b_{M^{(a)}}^{(a)};c_1,\dots,c_N;\eta^{(a)}),\qquad \rho_m^{(a)}=(\vartheta^{(a)})^m\rho_0^{(a)},\tag{3}$$

and $a\in\{1,\dots,p\}$.

Proof. It follows from the definition of the moment matrix. □

Definition 2.4. *We introduce the notation*

$$\vec{f} = \big[f_1 \cdot\cdots\cdot f_p\big], \qquad \vec{f}^{[0]} = \emptyset, \qquad \vec{f}^{[r]} = \big[f_1 \cdot\cdots\cdot f_r\big], \quad r \in \{1, \dots, p-1\}.$$

The extended Wronskians are the following determinants:

$$\mathscr{W}_{pn+r}[\vec{f}] := \left|\begin{array}{c|c|c|c|c|c} \vec{f} & \vartheta\vec{f} & \vartheta^2\vec{f} & \cdots\cdots\cdots & \vartheta^{n-1}\vec{f} & \vartheta^n\vec{f}^{[r]} \\ \hline \vartheta\vec{f} & \vartheta^2\vec{f} & \vartheta^3\vec{f} & \cdots\cdots\cdots & \vartheta^n\vec{f} & \vartheta^{n+1}\vec{f}^{[r]} \\ \hline \vdots & \vdots & \vdots & & \vdots & \vdots \\ \hline \vartheta^{pn-1}\vec{f} & \vartheta^{pn}\vec{f} & \vartheta^{pn+1}\vec{f} & \cdots\cdots\cdots & \vartheta^{(p+1)n-2}\vec{f} & \vartheta^{(p+1)n-1}\vec{f}^{[r]} \\ \hline \vdots & \vdots & \vdots & & \vdots & \vdots \\ \hline \vartheta^{pn+r-1}\vec{f} & \vartheta^{pn+r}\vec{f} & \vartheta^{pnr+1}\vec{f} & \cdots\cdots\cdots & \vartheta^{(p+1)n+r-2}\vec{f} & \vartheta^{(p+1)n+r-1}\vec{f}^{[r]} \end{array}\right|,$$

Proposition 2.5. *The τ function is the extended Wronskian of a hipergeometric series:*

$$\tau_n = \mathscr{W}_n\big[{}_{M_1}F_N(b_1^{(1)}, \dots, b_{M^{(1)}}^{(1)}; c_1, \dots, c_N; \eta^{(1)}), \dots, {}_{M_p}F_N(b_1^{(p)}, \dots, b_{M^{(p)}}^{(p)}; c_1, \dots, c_N; \eta^{(p)})\big].$$

Proof. It follows from Theorem 2.3. □

Proposition 2.6. *The following equation holds*

$$\vartheta^{(a)}\mathscr{M} = \mathscr{M}(\Lambda^{(a)})^\top.$$

Proof. It follows from (3). □

2.3. **Laguerre–Freud Matrix**. Next, we demonstrate that when a Pearson equation is fulfilled, there is a banded semi-infinite matrix that connects the shifted multiple orthogonal polynomials in the homogeneous lattice with the non-shifted ones; i.e., it models translations in the lattice.

Theorem 2.7. *Suppose the weights adhere to Pearson equations (1), where θ and $\sigma_1, \dots, \sigma_q$ are polynomials satisfying $\deg\theta = N$ and $\max\{\deg\sigma_1, \dots, \deg\sigma_q\} = M$. Then, we establish the following relationship:*

$$\Pi^-\theta(T) = \sum_{a=1}^{p} \sigma_a(T^{(a)})^\top(\Pi^{(a)+})^\top =: \Psi. \tag{4}$$

This matrix Ψ emerges as the Laguerre–Freud matrix. It exhibits a banded structure, characterized by pM subdiagonals and N superdiagonals, detailed as follows:

$$\Psi = (\Lambda^\top)^{pM}\psi^{(-pM)} + \cdots + \psi^{(N)}\Lambda^N. \tag{5}$$

Proof. The relation (4) follows from the symmetry equation (2) and the Gauss–Borel factorization. The equation (5) follows from the degree of θ and maximun of the degrees of σ, that departing from (4) gives us the number of non-zero superdiagonals and non-zero subdiagonals, respectively. □

Theorem 2.8. *For $a \in \{1, \dots, p\}$, the following equations are fulfilled*

$$\theta(x)B(x-1) = \Psi B(x), \qquad \Psi^\top A^{(a)}(x) = \sigma_a(x)A^{(a)}(x+1). \tag{6}$$

Proof. For the type I polynomials we can write:

$$\Psi^\top A^{(a)}(x) = \sum_{a=1}^{q} \Pi^{(a)+}\sigma_a(T^{(a)})A^{(a)}(x) = \Pi^{(a)+}\sigma_a(x)A^{(a)}(x) = \sigma_a(x)A^{(a)}(x+1).$$

For the type II polynomials we have that

$$\Psi B(x) = \Pi^-\theta(T)B(x) = \Pi^-\theta(x)B(x) = \theta(x)B(x-1).$$

□

The Laguerre–Freud and the recurrence matrices are closely connected:

Proposition 2.9. *The following compatibility formula holds*
$$[\Psi, T] = \Psi.$$

Proof. Using (6) we obtain the following two equations:
$$T\Psi B(x) = T\theta(x)B(x-1) = \theta(x)(x-1)B(x-1)$$
and
$$\Psi T B(x) = \Psi x B(x) = x\theta(x)B(x-1),$$
with both equations above we have that
$$[\Psi, T]B(x) = \theta(x)B(x-1) = \Psi B(x).$$
□

2.4. **Contiguous hypergeometric relations and connection matrices.** Generalized hypergeometric series fulfill:
$$\left(\eta\frac{d}{d\eta} + b_i\right){}_MF_N(b_1,\dots,b_M;c_1,\dots,c_N;\eta) = b_i\,{}_MF_N(b_1,\dots,b_i+1,\dots,b_M;c_1,\dots,C_N;\eta)$$
and
$$\left(\eta\frac{d}{d\eta} + c_j - 1\right){}_MF_N(b_1,\dots,b_M;c_1,\dots,c_N;\eta) = b_i\,{}_MF_N(b_1,\dots,b_M;c_1,\dots,c_j-1,\dots,C_N;\eta)$$
for $i \in \{1,\dots,M\}$ and $j \in \{1,\dots,N\}$.

We will use the following notation: ${}_i\Theta^{(a)}$ corresponds to the shift $b_i^{(a)} \to b_i^{(a)}+1$ and Θ_j corresponds to the shift $c_j \to c_j - 1$.

These relations imply for the moment matrix:

Theorem 2.10. *For $a \in \{1,\dots,p\}$, the following equations for the moment matrix are fulfilled*
$$\mathscr{M}(\Lambda^{(a)})^\top + b_i^{(a)}\mathscr{M} = b_i^{(a)}\,{}_i\Theta^{(a)}\mathscr{M}, \tag{7}$$
$$\mathscr{M}(\Lambda^q)^\top + (c_j-1)\mathscr{M} = (c_j-1)\Theta_j\mathscr{M}. \tag{8}$$

Proof. The equations (7) and (8) are derived from the contiguous hypergeometric relations for generalized hypergeometric functions. To prove (7), we first analyze how the operator $(\vartheta_{\eta^{(a)}} + b_i^{(a)})$ acts on different entries of the moment matrix. There are two scenarios: one where it acts on ρ^a_m and another where it acts on ρ^k_m with $k \neq a$.

Firstly, we have:
$$(\vartheta_{\eta_a} + b_i^{(a)})\rho^a_m = (\vartheta_{\eta_a} + b_i^{(a)})(\vartheta^m_{\eta_a}\rho^a_0) = \vartheta^m_{\eta_a}(\vartheta_{\eta_a} + b_i^{(a)})\rho^a_0 = \vartheta^m_{\eta_a}(b_i^{(a)}\,{}_i\Theta^{(a)}\rho^a_0) = b_i^{(a)}\,{}_i\Theta^{(a)}\rho^a_m,$$
and
$$(\vartheta_{\eta_a} + b_i^{(a)})\rho^k_m = b_i^{(a)}\rho^k_m = b_i^{(a)}\,{}_i\Theta^{(a)}\rho^k_m.$$

From these results, we see how the operator acts on the moment matrix:
$$(\vartheta_{\eta_a} + b_i^{(a)})\mathscr{M} = \mathscr{M}(\Lambda^{(a)})^\top + b_i^{(a)}\mathscr{M} = b_i^{(a)}\,{}_i\Theta^{(a)}\mathscr{M}.$$

To prove (8), we examine how the operator $(\vartheta_{\eta_a} + c_j - 1)$ acts on different elements of the moment matrix:
$$(\vartheta_{\eta_a} + c_j - 1)\rho^a_m = (c_j-1)\Theta_j\rho^a_m, \quad \text{and} \quad (\vartheta_{\eta_a} + c_j - 1)\rho^k_m = (c_j-1)\Theta_j\rho^k_m.$$

Thus, we can write:
$$\left(\sum_{i=1}^p \vartheta_{\eta_i} + c_j - 1\right)\rho^a_m = (c_j-1)\Theta_j\rho^a_m,$$

and when acting on the moment matrix, we obtain:

$$\left(\sum_{i=1}^{q}\vartheta_{\eta_i}+c_j-1\right)\mathscr{M}=\mathscr{M}\left(\sum_{l=1}^{q}\Lambda_l^\top\right)+(c_j-1)\mathscr{M}=(c_j-1)\Theta_j\mathscr{M}.$$

□

Definition 2.11. *For $i\in\{1,\dots,M^{(a)}\}$, $j\in\{1,\dots,N\}$ and $a\in\{1,\dots,p\}$, the connection matrices are defined as*

$$\begin{aligned}
{}_i\omega^{(a)}&=({}_i\Theta^{(a)}H)^{-1}({}_i\Theta^{(a)}\tilde{S})(\Lambda^{(a)}+b_i^{(a)})\tilde{S}^{-1}H,\\
{}_i\Omega^{(a)}&=S({}_i\Theta^{(a)}S)^{-1},\\
\omega_i&=(\Theta_iH)^{-1}(\Theta_i\tilde{S})(\Lambda^{(q)}+c_j-1)\tilde{S}^{-1}H,\\
\Omega_i&=S(\Theta_iS)^{-1}.
\end{aligned}$$

Theorem 2.12. *The connection matrices fulfill the following relations*

$$({}_i\omega^{(a)})^\top=b_i^{(a)}{}_i\Omega^{(a)},\tag{9}$$

$$\omega_j^\top=(c_j-1)\Omega_j.\tag{10}$$

Proof. Starting from (7) and using the Gauss–Borel factorization for the moment matrix, we can prove (9). We have:

$$S^{-1}H\tilde{S}^{-\top}(\Lambda^{(a)})^\top+b_i^{(a)}S^{-1}H\tilde{S}^{-\top}=b_i^{(a)}{}_i\Theta^{(a)}(S^{-1}H\tilde{S}^{-\top}).$$

Multiplying the equation above by S on the right and by $\tilde{S}^\top H^{-1}$ on the left, we obtain:

$$\tilde{S}^\top H^{-1}S^{-1}H\tilde{S}^{-\top}(\Lambda^{(a)})^\top+b_i^{(a)}\tilde{S}^\top H^{-1}S^{-1}H\tilde{S}^{-\top}=b_i^{(a)}\tilde{S}^\top H^{-1}{}_i\Theta^{(a)}(S^{-1}H\tilde{S}^{-\top}).$$

This simplifies to:

$$(H\tilde{S}^{-\top})((\Lambda^{(a)})^\top+b_i^{(a)})=b_i^{(a)}{}_i\Theta^{(a)}.$$

Thus,

$$(H\tilde{S}^{-\top})(\Lambda^{(a)})^\top+b_i^{(a)}H\tilde{S}^{-\top}=b_i^{(a)}{}_i\Theta^{(a)}H\tilde{S}^{-\top}.$$

This confirms (9).

Similarly, equation (10) can be proved starting from (8) using an analogous process:

$$\left(\sum_{i=1}^{q}\vartheta_{\eta_i}+c_j-1\right)\mathscr{M}=\mathscr{M}\left(\sum_{l=1}^{q}\Lambda_l^\top\right)+(c_j-1)\mathscr{M}=(c_j-1)\Theta_j\mathscr{M}.$$

Following a similar factorization and multiplication approach as above, we will obtain:

$$\left(H\tilde{S}^{-\top}\right)\left(\sum_{i=1}^{q}\Lambda_i^\top+c_j-1\right)=(c_j-1)\Theta_j\left(H\tilde{S}^{-\top}\right).$$

□

Proposition 2.13. *Connection matrices have a triangular structure with p non-zero subdiagonals:*

$${}_i\omega^{(a)}=b_i^{(a)}I+({}_i\omega^{(a)})^{[1]}\Lambda+\cdots+({}_i\omega^{(a)})^{[p]}\Lambda^p,\tag{11}$$

$${}_i\Omega^{(a)}=I+\Lambda^\top({}_i\Omega^{(a)})^{[1]}+\cdots+(\Lambda^\top)^p({}_i\Omega^{(a)})^{[p]},\tag{12}$$

with

$$({}_i\omega^{(a)})^{[p]}=I^{(a)}\mathfrak{a}_-^pH_i\Theta^{(a)}H^{-1},\quad({}_i\Omega^{(a)})^{[m]}=\sum_{l=0}^{m}\mathfrak{a}_-^{m-l}S^{[l]}{}_i\Theta^{(a)}S^{[l-m]},\quad({}_i\omega^{(a)})^{[m]}=b_i^{(a)}({}_i\Omega^{(a)})^{[m]}.$$

Proof. Equations (11) and (12) can be proved starting from (9) and using the definitions of the connection matrices. We have:

$$({}_i\omega^{(a)})^\top = H\left(I - \tilde{S}^{[1]}\Lambda + \tilde{S}^{[2]}\Lambda^2 - \dots\right]\left[(\Lambda^q)^\top + b_i^{(a)}\right]{}_i\Theta^{(a)}\left[I + \tilde{S}^{[1]}\Lambda + \tilde{S}^{[2]}\Lambda^2 + \dots\right]H^{-1}.$$

Expanding this, we get:

$$({}_i\omega^{(a)})^\top = H\left[(\Lambda^\top)^p I^{(a)}\mathfrak{a}_-^p H_i\Theta^{(a)}H^{-1} + \dots\right].$$

For the second connection matrix:

$${}_i\Omega^{(a)} = I + \dots + (\Lambda^\top)^m\left[\sum_{l=0}^{m}\mathfrak{a}_-^{m-l}S^{[l]}{}_i\Theta^{(a)}S^{[l-m]}\right].$$

Therefore, the expanded forms show that:

$${}_i\omega^{(a)} = b_i^{(a)}I + ({}_i\omega^{(a)})^{[1]}\Lambda + \dots + ({}_i\omega^{(a)})^{[p]}\Lambda^p,$$

where

$$({}_i\omega^{(a)})^{[p]} = I^{(a)}\mathfrak{a}_-^p H_i\Theta^{(a)}H^{-1},$$

and

$${}_i\Omega^{(a)} = I + \Lambda^\top({}_i\Omega^{(a)})^{[1]} + \dots + (\Lambda^\top)^p({}_i\Omega^{(a)})^{[p]},$$

where

$$({}_i\Omega^{(a)})^{[m]} = \sum_{l=0}^{m}\mathfrak{a}_-^{m-l}S^{[l]}{}_i\Theta^{(a)}S^{[l-m]}.$$

Additionally, the relationship between these matrices is given by:

$$({}_i\omega^{(a)})^{[m]} = b_i^{(a)}({}_i\Omega^{(a)})^{[m]}.$$

□

The type II and type I multiple orthogonal polynomials satisfy:

Proposition 2.14. *The following equations hold*

$$i\omega^{(a)}A^{(a)}(z) = (z + b_i^{(a)})({}_i\Theta^{(a)}A^{(a)}(z)) \tag{13}$$

$${}_i\Omega^{(a)}{}_i\Theta^{(a)}B(z) = B(z) \tag{14}$$

$$\omega_i A^{(a)}(z) = (z + c_i - 1)(\Theta_i A^{(a)}(z) \tag{15}$$

$$\Omega_i\Theta_i B(z) = B(z). \tag{16}$$

Proof. Relations (13), (14), (15), and (16) are proven by applying the connection matrices to the vectors of polynomials. As an example, the proof of (13) is shown below:

$$({}_i\Theta^{(a)}H^{-1}\tilde{S})(\Lambda^{(a)} + b_i^{(a)})\tilde{S}^{-1}H(H^{-1}\tilde{S}X^a(z)) = {}_i\Theta^{(a)}(H^{-1}\tilde{S})(z + b_i^{(a)})X^{(a)}(z) = (z + b_i^{(a)}){}_i\Theta^{(a)}A^{(a)}(z).$$

□

3. Multiple Toda systems and hypergeometric τ functions

In this section the multiple Toda equations and the multiple Toda system for the recurrence coefficients will be studied. In particular, for the case of three weights Toda equations and the differential-difference equation solved by *tau* function will be studied.

Definition 3.1. *For $a \in \{1,\dots,p\}$, the consider the strictly lower triangular matrices:*

$$\phi^{(a)} := (\vartheta^{(a)} S)S^{-1}, \quad \tilde{\phi}^{(a)} := (\vartheta^{(a)}\tilde{S})\tilde{S}^{-1}.$$

Differential properties of the multiple orthogonal polynomials can be described in terms of these matrices:

Lemma 3.2. *For $a \in \{1,\dots,p\}$, the multiple orthogonal polynomials of type II and type I satisfy:*

$$\vartheta^{(a)} B = \phi^{(a)} B, \quad \vartheta^{(a)}(HA^{(a)}) = \tilde{\phi}^{(a)} HA^{(a)}.$$

Proof. First, if we act with $\vartheta^{(a)}$ over B we have the following equalities:

$$\vartheta^{(a)} B = (\vartheta^{(a)} S)X = (\vartheta^{(a)} S)S^{-1}SX = (\vartheta^{(a)} S)S^{-1}B.$$

Second, we can see that

$$\vartheta^{(a)}(HA^{(a)}) = (\vartheta^{(a)}\tilde{S})X^{(a)} = \tilde{\phi}^{(a)} HA^{(a)}.$$

□

The interplay between these matrices follow from the Gauss–Borel factorization:

Proposition 3.3. *For $a \in \{1,\dots,p\}$, the following equations are fulfilled*

$$\vartheta^{(a)} H - \phi^{(a)} H - H(\tilde{\phi}^{(a)})^\top = (T^{(a)})^\top H, \tag{17}$$

Proof. Departing from the Gauss–Borel factorization for the moment matrix we can see that

$$\vartheta^{(a)}(S^{-1}H\tilde{S}^{-}\top) = -S^{-1}(\vartheta^{(a)} S)S^{-1}H\tilde{S}^{-\top} + S^{-1}(\vartheta^{(a)} H)\tilde{S}^{-\top} - S^{-1}H\tilde{S}^{-\top}(\vartheta^{(a)}\tilde{S})^\top\tilde{S}^{-\top} = S^{-1}H\tilde{S}^{-\top}(\Lambda^{(a)})^\top.$$

Consequently, we find:

$$\vartheta^{(a)} H - (\vartheta^{(a)} S)S^{-1}H - H\tilde{S}^{-\top}(\vartheta^{(a)}\tilde{S})^\top = H\tilde{S}^{-\top}(\Lambda^{(a)})^\top\tilde{S}^\top H^{-1}H = (T^{(a)})^\top H.$$

□

For the complete derivation ϑ we have the following associated strictly lower matrices:

Definition 3.4. *The following matrices are defined*

$$\phi := \sum_{a=1}^{p}\phi^{(a)}, \qquad \tilde{\phi} := \sum_{a=1}^{p}\tilde{\phi}^{(a)}.$$

Proposition 3.5. *The following equations are fulfilled*

$$(\vartheta H)H^{-1} = \alpha^{(0)}, \tag{18}$$

$$-\phi = \sum_{a=1}^{p}(\Lambda^\top)^a\alpha^{(a)}, \tag{19}$$

$$-\tilde{\phi} = \Lambda^\top\mathfrak{a}_- HH^{-1}. \tag{20}$$

Proof. Adding equations (17) for $a \in \{1,\dots,p\}$ we obtain

$$\vartheta H - \phi H - H(\phi)^\top = (T)^\top H.$$

Identifying diagonal, subdiagonal and superdiagonal parts in both sides of the equation we obtain equations (18), (19) and (20). □

For the subdiagonal in the lower connection factors in the Gauss–Borel factorization we have:

Proposition 3.6. *The following equations are satisfied*

$$\vartheta\tilde{S}^{[1]} = -\mathfrak{a}_- HH^{-1}, \tag{21}$$

$$\vartheta\tilde{S}^{(a)} = -\mathfrak{a}_-^a H\mathfrak{a}_-^{a-1}H^{-1}\tilde{S}^{[a-1]}, \tag{22}$$

$$\vartheta S^{[1]} = -\alpha^{(1)}, \tag{23}$$

$$\vartheta S^{[a]} = -\alpha^{(a)} - \sum_{i=1}^{a-1}\mathfrak{a}_-\alpha^{(a-i)}S^{[i]}. \tag{24}$$

Proof. Equations (21) and (22) can be proved equating diagonal by diagonal (20). Analogously, equations (23) and (24) can be proved from (19). □

Proposition 3.7. *The following equation are fulfilled*

$$\vartheta\tilde{S}^{[-a]} = \mathfrak{a}_-\tilde{S}^{[1-a]}\mathfrak{a}_- HH^{-1}, \quad a \in \{1, \dots, q\}.$$

Proof. Departing from $\vartheta(\tilde{S}\tilde{S}^{-1}) = \vartheta(I) = 0$ we have that $0 = (\vartheta\tilde{S})\tilde{S}^{-1} + \tilde{S}(\vartheta\tilde{S}^{-1})$, so:

$$\tilde{S}\vartheta\tilde{S}^{-1} = -\Lambda^\top\mathfrak{a}_- HH^{-1}$$

and we obtain that

$$\vartheta\tilde{S}^{-1} = -\tilde{S}^{-1}\Lambda^\top\mathfrak{a}_- HH^{-1}.$$

□

Proposition 3.8. *For $a \in \{1, \dots, p\}$, the recursion coefficients can be expressed as follows*

$$\alpha^{(0)} = (\vartheta H)H^{-1}, \tag{25}$$

$$\alpha^{(a)} = -[\vartheta S^{[a]} + \sum_{j=1}^{a-1}(\vartheta\mathfrak{a}_-^j S^{[a-j]})S^{[-j]}]. \tag{26}$$

Proof. It is proven splitting Equation (19) by diagonals. □

We now derive a differential system for the functions

$$q_n := \log H_n, \quad f_n^{(m)} := S_n^{[m]},$$

where $m \in \{1, \dots, p\}$

Theorem 3.9. *Functions $q_n = \log H_n$, $f_n^{(m)} = S_n^{[m]}$, where $m \in \{1, \dots, p\}$ solve the following system:*

$$\vartheta q_n = f_{n-1}^{(1)} - f_n^{(1)}, \tag{27}$$

$$f_{n-1}^{(m+1)} + f_n^{(-m-1)} + \sum_{j=1}^{m} f_{n+j}^{(m-j)} f_n^{(-j)} = -[\vartheta f_n^{(m)} + \sum_{j=1}^{m-1}(\vartheta f_{m+j}^{(m-j)})f_m^{(-j)}], \tag{28}$$

$$e^{q_{n+p}-q_n} = -[\vartheta f_n^{(p)} + \sum_{j=1}^{p-1}(\vartheta f_{n+j}^{(p-j)})f_n^{(-j)}]. \tag{29}$$

Proof. Equations (27) and (28) are obtained from (25) and (26) for $m \in \{1, \dots, p-2\}$ and equating them to expressions of the coefficients α depending on S matrices.

Equation (29) is obtained from (28) for $m = p-1$ and (28) for $m = p$: we derive the first equation, isolate $\vartheta S^{[p]}$ in the second equation and substitute in order to eliminate $S^{[p]}$. □

As an example of the result above we have the case for three weights. Now, we have that $q_n = \log H_n$, $f_n = S_n^{[1]}$, $g_n = S_n^{[2]}$ and $k_n = S_n^{[3]}$.
We have four equations:

$$\vartheta q_n = f_{n-1} - f_n,$$
$$\vartheta f_n = g_n - g_{n-1} + f_n(f_n - f_{n+1}), \tag{30}$$
$$\vartheta g_n - (\vartheta f_{n+1}) f_n = k_n - g_{n+1} f_n - f_{n+2} g_n + f_n f_{n+1} f_{n+2} - k_{n-1} + f_n f_{n+2} + g_n - f_n f_{n+1}, \tag{31}$$
$$e^{q_{n+3}-q_n} = -\vartheta k_n + \vartheta g_{n+1} f_n + \vartheta f_{n+2}(g_n - f_n f_{n+1}). \tag{32}$$

The variable k_n can be omited if we derivate equation (31) and obtain ϑk_n from (32), so the resultant expression is:

(33)
$$\vartheta^2 g_n - (\vartheta^2 f_{n+1}) f_n - (\vartheta f_{n+1}) \vartheta f_n = e^{q_{n+2}-q_{n-1}} - e^{q_{n+3}-q_n} - (\vartheta f_n) g_{n+1} - f_{n+2} \vartheta g_n + (\vartheta f_n) f_{n+1} f_{n+2} + f_n (\vartheta f_{n+1}) f_{n+2}$$
$$- (\vartheta g_n) f_{n-1} + (\vartheta f_{n+1})(f_{n-1} f_n - g_{n-1}) + (\vartheta f_n) f_{n+2} + \vartheta g_n + f_n \vartheta f_{n+2} - f_{n+1} \vartheta f_n - f_n \vartheta f_{n+1}.$$

In terms of τ-functions we have that:

$$q_n = \log H_n = \log \tau_{n+1} - \log \tau_n, \tag{34}$$
$$f_n = -\frac{\vartheta \tau_{n+1}}{\tau_{n+1}}, \tag{35}$$
$$g_n = \frac{\tau^2_{n+2}}{\tau_{n+2}}.$$

If (30) is written in terms of τ-functions, the following result is obtained:

$$g_n - g_{n-1} = \frac{\tau^2_{n+2}}{\tau_n + 2} - \frac{\tau^2_{n+1}}{\tau_{n+1}} = \frac{(\vartheta \tau_{n+1}) \vartheta \tau_{n+2}}{\tau_{n+1} \tau_{n+2}} - \frac{\vartheta^2 \tau_{n+1}}{\tau_{n+1}},$$

so we have that

$$g_n = g_0 + \sum_{j=1}^{n} \left(\frac{(\vartheta \tau_{j+1}) \vartheta \tau_{j+2}}{\tau_{j+1} \tau_{j+2}} - \frac{\vartheta^2 \tau_{j+1}}{\tau_{j+1}} \right). \tag{36}$$

Let us take a derivation on Equation (36):
(37)
$$\vartheta g_n = \vartheta g_0 + \sum_{j=1}^{n} \left(\frac{(\vartheta^2 \tau_{j+1}) \vartheta \tau_{j+2} + (\vartheta \tau_{j+1}) \vartheta^2 \tau_{j+2}}{\tau_{j+1} \tau_{j+2}} - \frac{(\vartheta \tau_{j+1})^2 \vartheta \tau_{j+2}}{\tau_{j+2} (\tau_{j+1})^2} - \frac{(\vartheta \tau_{j+2})^2 \vartheta \tau_{j+1}}{\tau_{j+1} (\tau_{j+2})^2} - \frac{\vartheta^3 \tau_{j+1}}{\tau_{j+1}} + \frac{(\vartheta^2 \tau_{j+1}) \vartheta \tau_{j+1}}{\tau_{j+1}} \right).$$

Remark 3.10. By substituting the expressions from (34), (35), (36), and (37) into equation (33), we obtain a fourth-order differential equation (with discrete integrations) for the τ-functions.

Proposition 3.11. *For $n \in \{0, \dots, p-1\}$, we have the following Toda type system*

$$\vartheta \alpha^{(0)} = \alpha^{(1)} - \mathfrak{a}_+ \alpha^{(1)},$$
$$\vartheta \alpha^{(n)} = \alpha^{(n+1)} - \mathfrak{a}_+ \alpha^{(n+1)} + \alpha^{(n)} (\mathfrak{a}_-^n \alpha^{(0)} - \alpha^{(0)}),$$
$$\vartheta \alpha^{(p)} = \alpha^{(p)} (\mathfrak{a}_-^p \alpha^{(0)} - \alpha^{(0)}),$$

Proof. This can be proved by applying the operator ϑ to Equations (25) and (26), and utilizing Equation (18). ◻

Proposition 3.12. *The following Lax equation is satisfied:*

$$\vartheta T = [\phi, T] = [T_+, T],$$

where $T_- = \sum_{n=1}^{p} (\Lambda^\top)^n \alpha^{(n)}$ *and* $T_+ \alpha^{(0)} + \Lambda$.

Proof. Departing from $T = S\Lambda S^{-1}$ we have that

$$\vartheta T = (\vartheta S)\Lambda S^{-1} - S\Lambda S^{-1}(\vartheta S)S^{-1} = \phi T - T\phi = [\phi, T],$$

and attending to Equation (19) we see that $\phi = -T_- = T_+ - T$ so $[\phi, T] = [T_+, T]$. □

Proposition 3.13. *The following compatibility conditions hold:*

$$\vartheta \Psi = [\phi, \Psi], \tag{38}$$

$$\vartheta \Psi^\top = [\mu, \Psi^\top] + \Psi^\top, \tag{39}$$

where $\mu^{(n)} := (\vartheta_n H^{-1})H + H^{-1}\tilde{\phi}^{(n)} H$ *and* $\mu = \sum_{i=1}^{p} \mu^{(i)}$.

Proof. Starting from equations

$$\theta(z)B(z-1) = \Psi B(z), \quad \vartheta B(z) = \phi B(z),$$

we find:

$$\begin{aligned}
\vartheta(\theta(z)B(z-1)) &= \vartheta(\Psi B(z)) = \vartheta\Psi B(z) + \Psi\vartheta B(z) = (\vartheta\Psi + \Psi\phi)B(z),\\
\vartheta(\theta(z)B(z-1)) &= \theta(z)\phi B(z-1) = \phi\Psi B(z).
\end{aligned}$$

Equating both equations, we have:

$$\phi\Psi = \vartheta\Psi + \Psi\phi,$$

and isolating the term with the derivative of Ψ, we obtain (38).

To derive (39), we first prove that $\vartheta^{(m)} A^{(n)} = \mu^{(m)} A^{(n)}$:

$$\begin{aligned}
\vartheta^{(m)} A^{(n)} &= \vartheta^{(m)}(H^{-1}\tilde{S}X^{(n)})\\
&= ((\vartheta^{(m)} H^{-1})HH^{-1}\tilde{S} + H^{-1}(\vartheta^{(m)}\tilde{S})(H^{-1}\tilde{S})^{-1}(H^{-1}\tilde{S}))X^{(n)}\\
&= ((\vartheta^{(m)} H^{-1})H + H^{-1}\tilde{\phi}^{(m)} H)A^{(n)}\\
&= \mu^{(m)} A^{(n)}.
\end{aligned}$$

From this equation, we get $\sum_{i=1}^{p} \vartheta^{(i)} A^{(n)} = \sum_{i=1}^{q} \mu^{(i)} A^{(n)} \to \vartheta A^{(n)} = \mu A^{(n)}$.

Now, starting from equations

$$\sigma_n(z)A^{(n)}(z+1) = \Psi^\top A^{(n)}(z), \quad \vartheta A^{(n)} = \mu A^{(n)},$$

we find:

$$\begin{aligned}
\vartheta(\sigma_n(z)A^{(n)}(z+1)) &= \vartheta(\Psi^\top A^{(n)}) = (\vartheta\Psi^\top)A^{(n)} + \Psi^\top\vartheta A^{(n)} = (\vartheta\Psi^\top + \Psi^\top\mu)A^{(n)},\\
\vartheta(\sigma_n(z)A^{(n)}(z+1)) &= \sigma_n(z)(\mu A^{(n)}(z+1) + A^{(n)}(z+1)) = (\mu + I)\Psi^\top)A^{(n)}(z),
\end{aligned}$$

so equating both equations and simplifying, we obtain:

$$\vartheta\Psi^\top = [\mu, \Psi^\top] + \Psi^\top.$$

□

Multiple Nijhoff-Capel discrete Toda equations. We study now the compatibility between the shifts of the $p+1$ families of parameters for the hypergeometric series:

$$\{b_i^{(n)}\}_{i=1}^{M^{(a)}}, \{c_i\}_{i=1}^{N},$$

where $a \in \{1,\dots,p\}$. For a function u_n it is considered the following notation:

i) $\hat{u}$ means a shift in one and only one parameter in the family of parameters $\{b^{(a)}\}$, i. e. $b_r^{(a)} \to b_r^{(a)}+1$ for only one given $r \in \{1,\dots,M^{(a)}\}$, the corresponding connection matrix is denoted by $\hat{\Omega}^{(r)}$, we denote $\hat{d} := b_r^{(a)}$,

ii) $\check{u}$ means a shift in one and only one parameter in the family of parameters $\{b^{(a)}\}$ with $m \neq m$, i. e. $b_s^{(a)} \to b_s^{(a)}+1$ for only one given $q \in \{1,\dots,M^{(a)}\}$, the corresponding connection matrix is denoted by $\check{\Omega}^{(r)}$, we denote $\check{d} := b_p^{(m)}$,

iii) $\tilde{u}$ means a shift in one and only one parameter in the family of parameters $\{c\}$, i. e. $c_s \to c_s - 1$ for only one given $s \in \{1,\dots,N\}$, the corresponding connection matrix is denoted by $\tilde{\Omega}^{(s)}$, we denote $\tilde{d} := c_s - 1$,

iv) $\bar{u}$ to denote the shift $n \to n+q$.

Lemma 3.14. *Connection matrices fulfill the following compatibility conditions:*

$$\Omega^{(s)}\tilde{\Omega}^{(r)} = \Omega^{(r)}\hat{\Omega}^{(s)}, \tag{40a}$$

$$\Omega^{(s)}\tilde{\Omega}^{(q)} = \Omega^{(q)}\check{\Omega}^{(s)}, \tag{40b}$$

$$\Omega^{(r)}\hat{\Omega}^{(q)} = \Omega^{(q)}\check{\Omega}^{(r)}. \tag{40c}$$

Proof. In the one hand, connection formulas can be written as follows

$$\Omega^{(r)}\hat{B} = B, \quad \Omega^{(s)}\tilde{B} = B,$$

so that $\tilde{\Omega}^{(r)}\tilde{\hat{B}} = \tilde{B}$ and, consequently, $\Omega^{(s)}\tilde{\Omega}^{(r)}\tilde{\hat{B}} = \Omega^{(s)}\tilde{B} = B$. On the other hand, we have $\hat{\Omega}^{(s)}\hat{\tilde{B}} = \hat{B}$ so that $\Omega^{(r)}\hat{\Omega}^{(s)}\hat{\tilde{B}} = \Omega^{(r)}\hat{B} = B$. Hence, the compatibility $\tilde{\hat{B}} = \hat{\tilde{B}}$ leads to (40a). Relations (40b) and (40c) are proven analogously. □

Lemma 3.15. *The following equations are satisfied*

$$\hat{T}^{\top}\omega^{(r)} = \omega^{(r)}T^{\top} \tag{41a}$$

$$\check{T}^{\top}\omega^{(q)} = \omega^{(q)}T^{\top} \tag{41b}$$

$$\tilde{T}^{\top}\omega^{(s)} = \omega^{(s)}T^{\top} \tag{41c}$$

holds.

Proof. All Equations in (41) are proven similarly. Let us prove (41a). From the eigenvalue equation for $T^{\top}$ and $A^{(a)}$, $A^{(a)\top}T = zA^{(a)\top}$, and Equation (13) for this case, $\hat{A}^{(a)}(z) = \frac{\omega^{(r)}}{z+\hat{d}}A^{(a)}(z)$, it can be seen that:

$$\hat{A}^{(a)\top}\hat{T} = z\hat{A}^{(a)\top},$$

so that:

$$\hat{T}^{\top}\frac{\omega^{(r)}}{z+\hat{d}}A^{(a)}(z) = z\frac{\omega^{(r)}}{z+\hat{d}}A^{(a)}(z) = \frac{\omega^{(r)}}{z+\hat{d}}T^{\top}A^{(a)}.$$

Hence, if we denote $M = (\hat{T}^{\top}\omega^{(r)} - \hat{T}^{\top}\omega^{(r)}T)H^{-1}\tilde{S}$, we find $MX^{(a)} = 0$, so that $M = 0$. Since $H^{-1}\tilde{S}$ is a lower triangular invertible matrix, we conclude the result. Finally, we obtain:

$$\hat{T}^{\top}\omega^{(r)} = \omega^{(r)}T^{\top}.$$

For equations (41b) and (41c), similar proofs can be carried out. □

As an example, for three weights, $p=3$, we find tridiagonal connection matrices:

$$\begin{aligned}\Omega^{(s)}&=(\Lambda^\top)^3(\Omega^{(s)})^{[3]}+(\Lambda^\top)^2(\Omega^{(s)})^{[2]}+\Lambda^\top(\Omega^{(s)})^{[1]}+I,\\ \Omega^{(r)}&=(\Lambda^\top)^3(\Omega^{(r)})^{[3]}+(\Lambda^\top)^2(\Omega^{(r)})^{[2]}+\Lambda^\top(\Omega^{(r)})^{[1]}+I,\\ \Omega^{(q)}&=(\Lambda^\top)^3(\Omega^{(q)})^{[3]}+(\Lambda^\top)^2(\Omega^{(q)})^{[2]}+\Lambda^\top(\Omega^{(q)})^{[1]}+I,\end{aligned}$$

where the diagonals are given by

$$(\Omega^{(s)})^{[3]}=\frac{\mathfrak{a}_-^3H}{\tilde{d}\tilde{H}},\qquad (\Omega^{(s)})^{[2]}=S^{[2]}-\tilde{S}^{[2]}+\tilde{S}^{[1]}(\mathfrak{a}_-\tilde{S}^{[1]}-\mathfrak{a}_-S^{[1]}),\qquad (\Omega^{[s]})^{[1]}=S^{[1]}-\tilde{S}^{[1]},$$

$$(\Omega^{(r)})^{[3]}=p_n\frac{\mathfrak{a}_-^3H}{\hat{d}\hat{H}},\qquad (\Omega^{(r)})^{[2]}=S^{[2]}-\hat{S}^{[2]}+\hat{S}^{[1]}(\mathfrak{a}_-\hat{S}^{[1]}-\mathfrak{a}_-S^{[1]}),\qquad (\Omega^{[r]})^{[1]}=S^{[1]}-\hat{S}^{[1]},$$

$$(\Omega^{(q)})^{[3]}=p_m\frac{\mathfrak{a}_-^3H}{\check{d}\check{H}},\qquad (\Omega^{(q)})^{[2]}=S^{[2]}-\check{S}^{[2]}+\check{S}^{[1]}(\mathfrak{a}_-\check{S}^{[1]}-\mathfrak{a}_-S^{[1]}),\qquad (\Omega^{[q]})^{[1]}=S^{[1]}-\check{S}^{[1]}.$$

4. Laguerre–Freud Equations by Examples

Our objective is to determine the Laguerre–Freud matrix for various specific cases and derive the coefficients of the polynomials or the Laguerre–Freud equations for these coefficients.

In the case of p weights, our extension to this multiple setting of the Laguerre–Freud equations for the coefficients of the T matrix are defined as:

$$\begin{aligned}\alpha_n^{(0)}&=\mathscr{F}_n^{(0)}(\alpha_n^{(p)},\alpha_n^{(p-1)},\dots,\alpha_n^{(1)},\alpha_{n-1}^{(p)},\dots),\\ \alpha_n^{(m)}&=\mathscr{F}_n^{(m)}(\alpha_n^{(p)},\dots,\alpha_n^{(m+1)},\alpha_{n-1}^{(p)},\dots,\alpha_{n-1}^{(0)},\dots),\\ \alpha_n^{(p)}&=\mathscr{F}_n^{(p)}(\alpha_{n-1}^{(p)},\dots,\alpha_{n-1}^{(0)},\alpha_{n-2}^{(p)},\dots),\end{aligned}$$

with $m\in\{1,\dots,p-1\}$.

4.1. **Multiple Charlier case**.

In this scenario, the weights are given by $w^{(a)}(k)=\frac{(\eta^{(a)})^k}{k!}$ with $a\in\{1,\dots,p\}$, and the polynomials of the Pearson equations are $\theta=k$ and $\sigma^{(a)}(k)=\eta^{(a)}$, where $a\in\{1,\dots,p\}$.

Since $\deg\theta=1$, we have one non-zero superdiagonal and zero non-null subdiagonal. This results in the following structure for the Laguerre–Freud matrix:

$$\Psi=\psi^{(0)}+\psi^{(1)}\Lambda.$$

By applying $\Psi=\Pi^-\theta(T)$, we can obtain:

$$\psi^{(0)}=\alpha^{(0)}-\mathfrak{a}_+D,\quad \psi^{(1)}=I;$$

and using $\Psi=\sum_{a=1}^p\eta^{(a)}(\Pi^{(a)+})^\top$, we have:

$$\psi^{(0)}=\sum_{a=1}^p\eta^{(a)}I^{(a)},\quad \psi^{(1)}=\sum_{a=1}^p\eta^{(a)}H\mathfrak{a}_-H^{-1}(I^{(a)}-I^{(a-1)})\tilde{S}^{[1]}.$$

By equating both expressions for $\psi^{(0)}$, we can derive that:

$$\alpha_a^{(0)}=a+\eta^{(a+1)},\quad a\in\{0,1,\dots,p-1\}.$$

If we calculate the compatibility $[\Psi,T]=\Psi$ we obtain the following non-trivial equations:

$$\begin{aligned}\mathfrak{a}_-^p\alpha^{(0)}-\alpha^{(0)}&=pI,\\ \alpha^{(a+1)}-\mathfrak{a}_+\alpha^{(a+1)}&=\alpha^{(a)}(\alpha^{(0)}-\mathfrak{a}_-^a\alpha^{(0)}+aI),\end{aligned}$$

$$\alpha^{(1)} - \mathfrak{a}_+\alpha^{(1)} = \sum_{a=1}^{p} \eta^{(a)} I^{(a)},$$
$$\mathfrak{a}_-\alpha^{(0)} - \alpha^{(0)} = \sum_{a=1}^{p} \eta^{(a)} (\mathfrak{a}_- I^{(a)} - I^{(a)}) + I,$$

where $a \in \{1, \dots, p-1\}$. So we can write the coefficients as

$$\begin{aligned}
\alpha^{(0)}_{pm+a} &= pm + a + \eta^{(a+1)}, && a \in \{0, \dots, p-1\},\\
\alpha^{(1)}_{pm+a} &= \sum_{b=1}^{a} \eta^{(b)} + m \sum_{b=1}^{p} \eta^{(b)}, && a \in \{1, 2, \dots, p-1\},\\
\alpha^{(l+1)}_{pm+a} &= \alpha^{(l+1)}_{pm+a-1} + \alpha^{(l)}_{pm+a-1}(\eta^{(a-l)} - \eta^{(a)}), && a \in \{l+1, \dots, p-1\},
\end{aligned}$$

where $m \in \mathbb{N}_0$.

4.2. **Multiple generalized Charlier case.** In this case, the weights are given by

$$w^{(a)}(k) = \frac{1}{(c+1)_k} \frac{(\eta^{(a)})^k}{k!}, \tag{42}$$

and the polynomials from the Pearson equations are

$$\theta(k) = k(k+c), \quad \sigma^{(a)}(k) = \eta^{(a)}.$$

Due to the degrees of these polynomials, the Laguerre–Freud matrix Ψ has the form:

$$\Psi = \psi^{(0)} + \psi^{(1)}\Lambda + \psi^{(2)}\Lambda^2.$$

Now, using the relation $\Psi = \Pi^{-}\theta(T)$, we find:

$$\psi^{(1)} = \alpha^{(0)} + \mathfrak{a}_-\alpha^{(0)} + c - \mathfrak{a}_+ D, \quad \psi^{(2)} = I,$$

where I is the identity matrix.

From the equation $\Psi = \sum_{a=1}^{p} \eta^{(a)} (\pi^{(a)+})^\top$, it follows that:

$$\psi^{(0)} = \sum_{a=1}^{p} \eta^{(a)} I^{(a)}.$$

From compatibility $[\Psi, T] = \Psi$ we can obtain

Proposition 4.1. *The following Laguerre–Freud equations hold for the generalized multiple Charlier weights* (42)

$$\begin{aligned}
\alpha^{(0)}_{m+p} &= m + p - 1 - c - \alpha^{(0)}_{m+p-1} + \frac{\alpha^{(p)}_{m+p-1}}{\alpha^{(p)}_{m+p}}(\alpha^{(0)}_{m-1} + \alpha^{(0)}_{m} + c - m) + \frac{\alpha^{(p-1)}_{m+p-1}}{\alpha^{(p)}_{m+p}}(r(m+1) - r(m)),\\
\alpha^{(1)}_{m+2} &= \alpha^{(1)}_{m} + \alpha^{(0)}_{m+1} + \alpha^{(0)}_{m} + c - m + (\alpha^{(0)}_{m} - \alpha^{(0)}_{m+1})(\alpha^{(0)}_{m} + \alpha^{(0)}_{m+1} + c - m) + r(m+2) - r(m+1),\\
\alpha^{(2)}_{m+2} &= \alpha^{(2)}_{m} + \alpha^{(1)}_{m}(\alpha^{(0)}_{m} + \alpha^{(0)}_{m-1} + c - m + 1) - \alpha^{(1)}_{m+1}(\alpha^{(0)}_{m} + \alpha^{(0)}_{m+1} + c - m) + r(m+1),\\
\alpha^{(n+2)}_{n+2+m} &= \alpha^{(n+2)}_{n+m} - \alpha^{(n+1)}_{m+n+1}(\alpha^{(0)}_{a+m} + \alpha^{(0)}_{a+m+1} + c - (a+m)) + \alpha^{(a+1)}_{a+m}(\alpha^{(0)}_{m-1} + \alpha^{(0)}_{m} + c - (m-1))\\
&\quad + \alpha^{(n)}_{m+a}(r(m+1) - r(a+m+1)),
\end{aligned}$$

where $a \in \{1, \dots, p-2\}$, and $r(n) = n - p\lfloor \frac{n}{p} \rfloor$, $r(n) \in \{1, \dots, p\}$. Here $\lfloor x \rfloor$, is the largest integer that is equal or smaller than x.

4.3. **Multiple Meixner II case.** For this case, the weights are given by

$$w^{(a)}(k) = (b_a)_k \frac{\eta^k}{k!},$$

and the polynomials from the Pearson equations are

$$\theta(k) = k, \quad \sigma^{(a)}(k) = \eta(k + b_a),$$

where $a \in \{1, \dots, p\}$.

Considering the degrees of these polynomials, the Laguerre–Freud matrix has the structure:

$$\Psi = (\Lambda^\top)^p \psi^{(-p)} + \cdots + \psi^{(1)}\Lambda.$$

By applying $\Psi = \Pi^- \theta(T)$, we obtain:

$$\psi^{(1)} = I, \quad \psi^{(0)} = \alpha^{(0)} - \mathfrak{a}_+ D.$$

Using the relation $\Psi = \sum_{a=1}^p \sigma_a (T^{(a)})^\top (\pi^{(a)+})^\top$, we find:

$$\psi^{(0)} = \eta \left(\alpha^{(0)} + \sum_{a=1}^p I^{(a)} (b_a + (\mathfrak{a}_+)^p D^{(a)}) \right), \quad \psi^{(-m)} = \eta \alpha^{(m)}, \quad m \in \{1, \dots, p\}.$$

From the compatibility condition $[\Psi, T] = \Psi$, we obtain the following non-trivial equations for the coefficient matrices:

$$\begin{aligned}
\mathfrak{a}_- \alpha^{(0)} - \alpha^{(0)} &= \frac{1}{1-\eta} \left(I + \sum_{a=1}^p \eta \left(I^{(a+1)} (b_a + \mathfrak{a}_+^{p-1} D^{(a)}) - I^{(a)} (b_a + \mathfrak{a}_+^p D^{(a)}) \right) \right), \\
\alpha^{(1)} - \mathfrak{a}_+ \alpha^{(1)} &= \frac{1}{1-\eta} (\alpha^{(0)} - \mathfrak{a}_+ D), \\
\alpha^{(l+1)} - \mathfrak{a}_+ \alpha^{(l+1)} &= \frac{\eta}{1-\eta} \left(\alpha^{(l)} \left(I + \sum_{a=1}^p I^{(a)} (b_a + \mathfrak{a}_+^p D^{(a)}) \right) - \sum_{a=1}^p I^{(a+l)} (b_a + \mathfrak{a}_+^{p-l} D^{(a)}) \right),
\end{aligned}$$

where $l \in \{1, \dots, p-1\}$.

Equating both expressions for $\psi^{(0)}$ we can obtain $\alpha^{(0)}$ entrywise:

$$\alpha^{(0)}_{pa+m} = \frac{1}{1-\eta} (pa + m + \eta(b_{m+1} + a))$$

Departing from compatibility equations and using the expression for the $\alpha^{(0)}$ elements we can obtain:

$$\begin{aligned}
\alpha^{(1)}_{pa+m} &= \frac{1}{1-\eta} \left(\frac{1}{1-\eta} \left((p+\eta) \frac{a(a+1)}{2} + \frac{m(m+1)}{2} + \eta \sum_{j=0}^m b_{j+1} \right) - \frac{(pa+m-1)(pa+m)}{2} \right), \\
\alpha^{(l+1)}_{pa+m+l+1} &= \alpha^{(l+1)}_{pa+m+l} + \frac{\eta}{1-\eta} \alpha^{(l)}_{pa+m+l} (1 + b_{m+1} + a).
\end{aligned}$$

4.4. **Multiple generalized Meixner II case.** In this case, the weights are given by:

$$w^{(a)}(k) = \frac{(b_a)_k}{(c+1)_k} \frac{\eta^k}{k!}, \tag{43}$$

and the polynomials from the Pearson equations are:

$$\theta(k) = k(k+c) \quad \text{and} \quad \sigma^{(a)}(k) = \eta(k + b_a),$$

where $a \in \{1, \dots, p\}$.

To find the Laguerre–Freud equation we proceed as follows. Considering the degrees of the polynomials defining the Pearson equations, the Laguerre–Freud matrix has the structure:

$$\Psi = (\Lambda^\top)^p \psi^{(-p)} + \cdots + \psi^{(2)}\Lambda^2.$$

Applying $\Psi = \Pi^- \theta(T)$, we obtain:

$$\psi^{(2)} = I, \quad \psi^{(1)} = \mathfrak{a}_- \alpha^{(0)} + \alpha^{(0)} + c - \mathfrak{a}_+ D.$$

Using the relation $\Psi = \sum_{a=1}^p \sigma_a (T^{(a)})^\top (\pi^{(a)+})^\top$, we find:

$$\psi^{(0)} = \eta\left(\alpha^{(0)} + \sum_{a=1}^{p} I^{(a)}(b_a + \mathfrak{a}_+^p D^{(a)})\right), \quad \psi^{(-m)} = \eta\alpha^{(m)}, \quad m \in \{1,\dots,p\}.$$

From the compatibility condition $[\Psi, T] = \Psi$, we find

Proposition 4.2. *The following Laguerre–Freud equations are satisfied for the generalized multiple Meixner weights given in* (43)

$$\begin{aligned}
\alpha^{(0)}_{(a+1)p+m} = {}& -\alpha^{(0)}_{(a+1)p+m-1} + a + p - 1 - c + (\alpha^{(p)}_{(a+1)p+m})^{-1}(\eta(\alpha^{(p-1)}_{(a+1)p+m-1}(b_{m+1} + a + 1) + \alpha^{(p)}_{(a+1)p+m} \\
& - \alpha^{(p)}_{(a+1)p+m-1}) + \alpha^{(p)}_{(a+1)p+m-1}(\alpha^{(0)}_{pa+m} + \alpha^{(0)}_{pa+m-1} + c - (a-1))),\\
\alpha^{(1)}_{pa+m+2} = {}& \alpha^{(1)}_{pa+m} + \eta(\alpha^{(0)}_{pa+m+1} - \alpha^{(0)}_{pa+m} - b_m - a) + (\alpha^{(0)}_{pa+m} - \alpha^{(0)}_{pa+m+1} + 1)(\alpha^{(0)}_{pa+m+1} + \alpha^{(0)}_{pa+m} + c - (m + pa)),\\
\alpha^{(l+2)}_{pa+m+l+2} = {}& \alpha^{(l+2)}_{pa+m+l} + \eta(\alpha^{(l+1)}_{pa+m+l+1} - \alpha^{(l+1)}_{pa+m+l}) - \alpha^{(l+1)}_{pa+m+l+1}(\alpha^{(0)}_{pa+m+l+1} + \alpha^{(0)}_{pa+m+l} + c - (a + l)) \\
& + \alpha^{(l+1)}_{pa+m+l}(\alpha^{(0)}_{pa+m} + \alpha^{(0)}_{pa+m-1} + c - (a-1)),\\
\alpha^{(l+2)}_{pa+m+l+2} = {}& \alpha^{(l+2)}_{pa+m+l} + \eta(\alpha^{(l)}_{pa+m+l}(b_{m+1} + a + 1) + \alpha^{(l+1)}_{pa+m+l+1} - \alpha^{(l+1)}_{pa+m+l}) + \alpha^{(l+1)}_{pa+m+l+1}(\alpha^{(0)}_{pa+m+l+1} + c - (a + l) \\
& + \alpha^{(0)}_{pa+m+l}) + \alpha^{(l+1)}_{pa+m+l}(\alpha^{(0)}_{pa+m} + \alpha^{(0)}_{pa+m-1} + c - (a-1)).
\end{aligned}$$

Conclusion and Outlook

In this paper, we continue our study of semiclassical multiple discrete orthogonal polynomials, now considering an arbitrary number of weights. We have derived multiple versions of the Toda equations and the Laguerre–Freud equations for the multiple generalized Charlier and multiple generalized Meixner II families.

The investigation of mixed multiple discrete orthogonal polynomials, along with the corresponding Toda and Laguerre-Freud equations, presents an intriguing area of research. Additionally, the exploration of Pearson equations in the multiple scenario for the unit circle appears to be both unexplored and promising.

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