Chaos and 1/f noise in nuclear spectra

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Abstract. Many complex systems in nature and in human society exhibit time fluctuations characterized by a power spectrum $S(f)$ which is a power function of the frequency f. Examples with this behavior are the Sun spot activity, the human heartbeat, the DNA sequence, or Bach's First Brandenburg Concert. In this work, we show that the energy spectrum fluctuations of quantum systems can be formally considered as a discrete time series, with energy playing the role of time. Because of this analogy, the fluctuations of quantum energy spectra can be studied using traditional methods of time series, like calculating the Fourier transform and studying the power spectrum. We present the results for paradigmatic quantum chaotic systems like atomic nuclei (by means of large scale shell-model calculations) and the predictions of random matrix theory. We have found a surprising general property of quantum systems: The energy spectra of chaotic quantum systems are characterized by $1/f$ noise, while regular quantum systems exhibit $1/f^2$ noise. Some other interesting applications of this time series analogy are a test of the existence of quantum chaos remnants in the nuclear masses, and the study of the order to chaos transition in semiclassical systems. In this case, it is found that the energy level spectrum exhibits $1/f^\alpha$ noise with the exponent changing smoothly from $\alpha = 2$ in regular systems to $\alpha = 1$ in chaotic systems.

Keywords: Quantum Chaos, Nuclei, Time Series
PACS: 05.40.-a, 05.45.Tp, 05.45.Mt, 05.45.Pq

INTRODUCTION

One of the most ubiquitous features of complex physical systems is the appearance of so–called $1/f^\alpha$ noise in fluctuating physical variables, meaning that the Fourier power spectrum $S(f)$ behaves as $1/f^\alpha$ in terms of the frequency f. Examples of such systems are electronic devices, sun spot activity, human heartbeat and the DNA sequence, but there are also quite different cases, like the music of J. S. Bach [1, 2]. Why this type of fluctuations are so very ubiquitous is not yet well understood. But surprisingly or not, we shall show in this talk that to the long list of known systems with $1/f^\alpha$ noise we can further add all the Hamiltonian quantum systems with purely regular or purely chaotic motion.

It is well known that the spectral fluctuations of a quantum system whose classical analogue is fully integrable are well described by Poisson statistics, i. e. the successive energy levels are not correlated. By contrast, the fluctuation properties of generic quantum systems that are fully chaotic in the classical limit, coincide with those of random matrix theory (RMT) [3].

Here we present a recently discovered, very different approach to quantum chaos

CP831, *Frontiers in Nuclear Structure, Astrophysics, and Reactions: FINUSTAR*
edited by S. V. Harissopulos, P. Demetriou, and R. Julin

[4], which is based on traditional methods of time series analysis. The essential feature of chaotic energy spectra in quantum systems is the existence of level repulsion and correlations. To study these correlations, we can consider the energy spectrum as a discrete signal, and the sequence of energy levels as a time series where the energy plays the role of time. We shall see that examination of the power spectrum of energy level fluctuations reveals very accurate power laws for completely regular or completely chaotic Hamiltonian quantum systems. It turns out that chaotic systems have $1/f$ noise, in contrast to the $1/f^2$ Brown noise of regular systems.

OUTLINE OF THE POWER SPECTRUM APPROACH

Generally, two suitable statistics are used to study the fluctuation properties of the unfolded energy levels ε_i of quantum systems. If s is the spacing between two consecutive energy levels, the nearest neighbor spacing distribution $P(s)$ gives information on the short range correlations among the energy levels. The $\Delta_3(L)$ statistic makes it possible to study correlations of length L: as we change the L value we obtain information on the level correlations at all scales [3]. By contrast, in this paper we characterize the spectral fluctuations by the statistic δ_n [5] defined by

$$\delta_n = \sum_{i=1}^{n} (s_i - \langle s \rangle) = \sum_{i=1}^{n} (s_i - 1), \tag{1}$$

where $s_i = \varepsilon_{i+1} - \varepsilon_i$, N is the size of the series, and the index n runs from 1 to $N-1$. The statistic δ_n represents the deviation of the unfolded excitation energy from its mean value n.

The function δ_n has a formal similarity with a time series. For example, we may compare the energy level spectrum with the diffusion process of a particle. The analogy is clear if the index i of the nearest level spacings is considered as a discrete time, and the spacing fluctuation $s_i - \langle s \rangle$ as the analogue of the particle displacement d_i from the collision at time i to the next collision. Therefore, we can analyze its fluctuations with numerical techniques normally used in the study of complex systems, like the calculation of the power spectrum $S(k)$. For the δ_n function it is given by

$$S(k) = \left| \hat{\delta}_k \right|^2, \quad \hat{\delta}_k = \frac{1}{N} \sum_{n=0}^{N-1} \delta_n \exp\left(\frac{2\pi i k n}{N}\right), \quad k = 0, \cdots, N-1 \tag{2}$$

where $\hat{\delta}_k$ is the Fourier transform of δ_n, k is the frequency index related to the actual frequency $f = k/N$.

As an example of a very chaotic system, we take the atomic nucleus at high excitation energy, where the level density is very large. To obtain the energy spectrum, shell-model calculations for selected nuclei are performed, using realistic interactions that reproduce well experimental data of nuclei in the appropriate mass region. The Hamiltonian matrices for different angular momenta, parity and isospin are fully diagonalized using the shell-model code Nathan [6], and careful global unfolding is performed. Then, sets of 256 consecutive levels of the same $J^{\pi}T$, from the high level density region, are selected.

To characterize the statistical properties of the δ_n signal, we calculate an ensemble average of its power spectrum, in order to reduce statistical fluctuations and clarify its main trend. The average $\langle S(k)\rangle$ is calculated with 25 sets.

Fig. 1 shows the results for a typical stable *sd* shell nucleus, ^{24}Mg, with matrix dimensionalities up to about 2000, using the effective W interaction [7]; and for a very exotic nucleus, ^{34}Na, with dimensions up to about 5000, in the *sd* proton and pf neutron shells, using a realistic interaction [8]. Clearly, the power spectrum of δ_n follows closely a power law. We may assume the simple functional form

$$\langle S(k)\rangle \sim \frac{1}{k^{\alpha}}. \tag{3}$$

A least squares fit to the data of Fig. 1 gives $\alpha = 1.11 \pm 0.03$ for ^{34}Na, and $\alpha = 1.06 \pm 0.05$ for ^{24}Mg. These results rise the question of whether there is a general relationship between quantum chaos and the power spectrum of the δ_n fluctuations of the system.

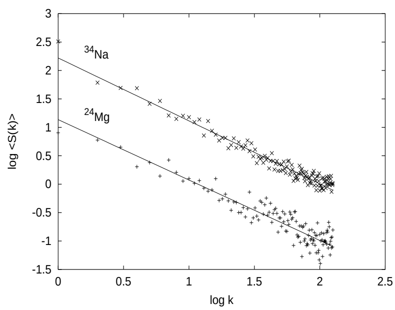

FIGURE 1. Average power spectrum of the δ_n function for ^{24}Mg and ^{34}Na, using 25 sets of 256 levels from the high level density region. The plots are displaced to avoid overlapping.

RMT DERIVATION OF THE $1/f$ NOISE IN QUANTUM CHAOS

Probably, the simplest and most reliable way to clarify this issue is to study the predictions of RMT for $\langle S(k)\rangle$. Random matrix theory plays a predominant role in the description of chaotic quantum systems [3]. It deals with three basic Hamiltonian matrix ensembles: The Gaussian orthogonal ensemble (GOE) of N-dimensional matrices, the Gaussian unitary ensemble (GUE), and the Gaussian symplectic ensemble (GSE), which apply to different systems, depending on the integer or half-integer spin, the time-reversal and the rotational symmetries of the system. To describe the spectral fluctuations of quantum integrable systems, we introduce here the ensemble of diagonal matrices whose elements are random Gaussian variables, and call it the *Gaussian diagonal ensemble* (GDE).

We have recently shown that the power spectrum of δ_n for these ensembles can be written as [9]

$$
\begin{aligned}
\langle S(k)\rangle_\beta = {} & \frac{N^2}{4\pi^2}\left\{\frac{K\left(\frac{k}{N}\right)-1}{k^2}+\frac{K\left(1-\frac{k}{N}\right)-1}{(N-k)^2}\right\} \\
& + \frac{1}{4\sin^2\left(\frac{\pi k}{N}\right)}+\Delta, \quad k=1,2,\cdots,N-1,\ N\gg 1,
\end{aligned}
\tag{4}
$$

where β is the repulsion parameter of RMT ensembles, that takes the values $\beta=0$ for GDE, $\beta=1$ for GOE, $\beta=2$ for GUE, and $\beta=4$ for GSE [5]; Δ is a constant equal to $-1/12$ for GOE, GUE and GSE, and zero for GDE; and $K(\tau)$ is the spectral form factor, defined in terms of the fluctuating part of the energy level density $\widetilde{\rho}(\varepsilon)$ as

$$
K(\tau)=\left\langle\left|\int d\varepsilon\widetilde{\rho}(\varepsilon)e^{-i2\pi\varepsilon\tau}\right|^2\right\rangle, \tag{5}
$$

This equation, together with the appropriate values of $K(\tau)$, gives explicit expressions of $\langle S(k)\rangle$ for specific ensembles. When $k\ll N$ the first term of eq. (4) becomes dominant and we can write

$$
\langle S(k)\rangle_\beta = \begin{cases} \dfrac{N}{2\beta\pi^2 k}, & \text{for chaotic systems,} \\ \dfrac{N^2}{4\pi^2 k^2}, & \text{for integrable systems.} \end{cases} \tag{6}
$$

These expressions show that, for small frequencies, the excitation energy fluctuations exhibit $1/f$ noise in chaotic systems and $1/f^2$ noise in integrable systems. Numerical calculations show that these power laws are approximately valid through almost the whole frequency domain, due to partial cancellation of higher order terms. Only near $k=N/2$ the effect of these terms becomes appreciable.

CONCLUSIONS

Summarizing, we have seen that for quantum systems the δ_n function can be considered as a time series, where the level order index n plays the role of a discrete time. The power spectrum $\langle S(k)\rangle$ of δ_n has been studied for energy spectra of atomic nuclei as representative examples of chaotic quantum systems, and a neat power law behavior $\langle S(k)\rangle\sim 1/k$ has been found. We have also derived a theoretical expression for $\langle S(k)\rangle_\beta$ in random matrix theory. To a very good approximation we find power laws $\langle S(k)\rangle_\beta\sim 1/k^\alpha$ in all cases. For integrable spectra ($\beta=0$), we get $\alpha=2$, as expected for independent random variables. On the other hand, for chaotic spectra we obtain $\alpha=1$.

Finally, we would like to point out some other interesting applications of the power spectrum approach in quantum systems. It has been recently used to shed some light on the existence of quantum chaos remnants in the nuclear masses [10]. In Ref. [11], the authors realize that the series of nuclear masses behave in a way like the δ_n statistic. A measure of correlations was given using the $1/f^\alpha$ noise of different mass series along the nuclear chart. They found different exponents according to the mass formula used to define the average behavior of the nuclear masses. In the case of the phenomenological mass formula of Möller *et al.* [12] they found an exponent close to one, showing indeed that there is some chaos in these mass fluctuations. On the other hand, the fluctuations from the microscopic formula by Zuker and Duflo [13] lead to an exponent larger than one and closer to those found for integrable systems. This result showed that important many-body correlations were included in this latter mass formula.

Another important line of work is in progress. The analogy with a time series has been used to study the transition from regular to chaotic motion in the Robnik billiard [14], in the coupled quartic oscillator and in the kicked quantum top [15]. In all cases, neat power laws $\langle S(k)\rangle \sim 1/k^\alpha$ emerge and the frequency exponent changes smoothly from $\alpha = 2$ for a regular system with Poisson level fluctuations, to $\alpha = 1$ for a chaotic system with RMT-like level fluctuations. Therefore, the value of α reflects the underlying classical dynamics, namely, regular, chaotic or mixed. These results suggest that $1/f^\alpha$ noise in the level fluctuations is an intrinsic characteristic of quantum systems.

ACKNOWLEDGMENTS

This work is supported in part by Spanish Government grants BFM2003-04147-C02 and FTN2003-08337-C04-04.

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