ON THE LIMIT OF SOLUTIONS OF $u_t = \Delta u^m$ AS $m \to \infty$

Abstract. We consider the Cauchy problem

$$u_t = \Delta u^m, \ u(0) = f$$

where $f \in L^1(\mathbb{R}^N)$, $N \geq 1$ and $f \geq 0$ a.e., and prove that as $m \to \infty$, the corresponding solutions $u_m(t)$ converge in $L^1$, uniformly for $t$ in a compact set in $]0, \infty[$, to the solution of a suitable limit problem.

We also show similar results for the Cauchy-Dirichlet and Cauchy-Neumann boundary value problems for (1) in bounded domains.

Key words and phrases. Porous medium equation, singular limit, mesa problem, asymptotic behavior.

Let $f \in L^1(\mathbb{R}^N)$, $f \geq 0$ be given and consider the problem

$$u_t = \Delta u^m \text{ on } ]0, \infty[ \times \mathbb{R}^N, \ u(0,.) = f \text{ on } \mathbb{R}^N$$

It is well known (see for instance [1]) that for any $m > 1$, there exists a unique "strong solution" of (1), that is a function $u(t)(x) = u(t, x)$ satisfying

$u \in C([0, \infty[, L^1(\mathbb{R}^N)) \cap C([0, \infty[ \times \mathbb{R}^N)$, $u \geq 0$ on $]0, \infty[ \times \mathbb{R}^N$, $u(0,.) = f$ on $\mathbb{R}^N$,

for any $\tau > 0$, $u \in L^\infty(]0, \tau[ \times \mathbb{R}^N)$, $u_t, \Delta u^m \in L^\infty(]0, \tau[ \times \mathbb{R}^N)$, $u_t = \Delta u^m$ a.e. on $]0, \infty[ \times \mathbb{R}^N$.

We note $u_m$ the solution of (1) and prove the following
THEOREM 1. As $m \to \infty$,

$$u_m(t) \to u = f + \Delta w \text{ in } L^1(\mathbb{R}^N)$$

uniformly for $t$ in a compact set in $]0, \infty[$, where $w$ is the solution of the variational inequality

(2) $w \in L^1(\mathbb{R}^N)$, $\Delta w \in L^1(\mathbb{R}^N)$, $0 \leq f + \Delta w \leq 1$, $w \geq 0$, $w(f + \Delta w - 1) = 0$ a.e.

Existence and uniqueness of a solution $w$ of (2) follows by the results in [3]: indeed the problem may be rewritten under the form

(3) $u, w \in L^1(\mathbb{R}^N)^+$, $u - \Delta w = f$ in $D(\mathbb{R}^N)$, $u \in \beta(w)$ a.e.

where $\beta$ is the sign graph.

If $w \in W^{2,1}_{\text{loc}}(\mathbb{R}^N)$, which is the case if $N = 1$ or $f \in L^{1+\epsilon}_{\text{loc}}(\mathbb{R}^N)$ with some $\epsilon > 0$, then $\Delta w = 0$ a.e. on $\{w = 0\}$ so that

(4) $u = \chi_{\Sigma} + f \chi_{\mathbb{R}^N \setminus \Sigma}$ with $\Sigma = \mathbb{R}^N \setminus \{w > 0\}$.

The fact that for $m$ large the solution of the porous medium equation develops "mesas" on the set of noncoincidence of the solution of the variational inequality (2), and tends to $f$ on the complementary set, has been noticed in [8]. In [7], it has been proved that for $f$ bounded and satisfying strong geometric assumptions, $u_m(t) \to u$ given by (4) in the weak-* topology of $L^\infty(\mathbb{R}^N)$ as $m \to \infty$, uniformly for $t$ in a compact set in $]0, \infty[$. In [9], Theorem 1 has been proved in the cases $N = 1$ and $N \geq 2$ with $f$ radially symmetric.

We also consider equation (1) on a bounded open set $\Omega$ in $\mathbb{R}^N$ with Dirichlet or Neumann boundary conditions, and prove the results corresponding to Theorem 1; for the Cauchy-Dirichlet boundary value problem, such result has been shown in [9] in the case $N = 1$.

The paper is organized as follows:

1. Proof of Theorem 1.
2. The Cauchy-Dirichlet boundary value problem.
SECTION 1. Proof of Theorem 1

We first recall that the map $f \rightarrow u_m(t)$ is a contraction in $L^1(\mathbb{R}^N)$ for any $m > 1$ and $t \geq 0$; a similar result holds for the map $f \rightarrow u$. Therefore, as it was noticed in [9], it is enough to prove the Theorem assuming that $f$ is bounded and compactly supported. Namely, we will assume throughout this Section that

$$0 \leq f \leq M \text{ a.e. on } \{|x| < R_0\}, \quad f = 0 \text{ a.e. on } \{|x| > R_0\}.$$  

By the maximum principle we have

$$0 \leq u_m(t) \leq M \text{ a.e. for any } t \geq 0 \text{ and } m > 1.$$  

Fix now $T > 0$ and $m_0 > 1$. It follows from Lemma 2.1 in [9] that there exists $R$, depending on $N, M, R_0, T$ and $m_0$, such that

$$u_m(t) = 0 \text{ on } \{|x| > R\} \text{ for any } t \in [0, T] \text{ and } m \geq m_0.$$  

By the translation invariance and the $L^1$-contractivity of the maps $f \rightarrow u_m(t)$, we have that for any $t \geq 0$ and $m > 1$

$$\int |u_m(t, x + y) - u_m(t, x)| \, dx \leq \int |f(x + y) - f(x)| \, dx \quad \text{for any } y \in \mathbb{R}^N.$$  

Therefore, as in [9], it follows from (6)-(8) that

$$\{u_m(t); \ t \in [0, T], \ m \geq m_0\} \text{ is precompact in } L^1(\mathbb{R}^N).$$

We now recall the following one-side estimate (see [1]) for the solution $u = u_m$ of (1)

$$u_t = \Delta u^m \geq -u/(m - 1 + 2/N) t \text{ a.e. .}$$  

Since $\Delta u(t)^m \in L^1(\mathbb{R}^N)$ and $\int \Delta u(t)^m = 0$ a.e. $t > 0$, one then has

$$\|u_t(t)\|_{L^1} = \|\Delta u(t)^m\|_{L^1} = 2\|\Delta u(t)^m\|_{L^1} \leq 2\|u(t)\|_{L^1}/\left( m - 1 + \frac{2}{N} \right) t \leq 2\|f\|_{L^1}/\left( m - 1 + \frac{2}{N} \right) t \quad \text{a.e. } t > 0.$$  

From (6), (7) and (10), it follows that

$$(u_m(t, x))^m \leq ME(x)/\left( m - 1 + \frac{2}{N} \right) t \quad \text{on } [0, T] \times \mathbb{R}^N \text{ for } m \geq m_0.$$
where $E \in \mathcal{C}(\mathbb{R}^N)$ is the solution of

$$E = 0 \text{ on } \{|x| \geq R\}, \quad -\Delta E = 1 \text{ in } \mathcal{D}'(\{|x| < R\}) .$$

In particular for $0 < \tau < T$, we have

$$\text{(13)} \quad (u_m)^m \to 0 \text{ uniformly on } [\tau, T] \times \mathbb{R}^N \text{ as } m \to \infty.$$ 

Thanks to (6) we have $(u_m)^m \in \mathcal{C}([0, \infty[ , L^1(\mathbb{R}^N))$ and we may define, for $t > 0$ and $m > 1$

$$\text{(14)} \quad w_m(t) = \int_0^t (u_m(s))^m ds ,$$

which satisfies

$$\text{(15)} \quad u_m(t) - \Delta w_m(t) = f \text{ in } \mathcal{D}'(\mathbb{R}^N) .$$

If for a subsequence $m_k \to \infty$ we have $u_{m_k}(1) \to u$ in $L^1(\mathbb{R}^N)$, then by (11)

$$\text{(16)} \quad u_{m_k}(t) \to u \text{ in } L^1(\mathbb{R}^N) \text{ uniformly for } t \in [\tau, T] .$$

whereas, by (7) and (15)

$$\text{(17)} \quad w_{m_k}(1) \to w \text{ in } L^1(\mathbb{R}^N) .$$

with

$$\text{(18)} \quad u - \Delta w = f \text{ in } \mathcal{D}'(\mathbb{R}^N) , \quad w \geq 0 \text{ a.e. on } \mathbb{R}^N .$$

and using (13)

$$\text{(19)} \quad 0 \leq u \leq 1 \text{ a.e. on } \mathbb{R}^N .$$

We claim that

$$\text{(20)} \quad w = 0 \text{ a.e. on } \{u < 1\} .$$

This will end the proof the Theorem 1.

In order to prove (20), we first remark that according to (10), the map $t \to t^{(n+1+\frac{2}{N})^{-1}} u(t)$ is nondecreasing so that

$$\text{(21)} \quad u_m(t) \leq t^{-1/(m-1+\left(\frac{2}{N}\right))} u_m(1) \text{ for any } 0 < t \leq 1 .$$
By (6)

$$u_m(t)^m \leq Mu_m(t)^{m-1}. \quad (22)$$

so that by the definition (14) of $w_m(t)$ and (21)

$$w_m(1) \leq Mu_m(1)^{m-1}(1 + N(m - 1)/2). \quad (23)$$

Property (20) is now clear: we may assume $u_m(1) \to u$ a.e., such that, a.e. $x \in \{u < 1\}$ we will have for $k$ large, $u_m(1)(x) \leq \delta < 1$ and then, using (23), $w_m(1)(x) \to 0$ as $k \to \infty$.

**REMARK 1.** Under assumption (5), we have justified the definition (14) of $w_m(t)$, and actually proved that

$$w_m(t) \to w \text{ in } W^{2,p}(\mathbb{R}^N) \text{ for any } 1 \leq p < \infty, \text{ as } m \to \infty. \quad (24)$$

uniformly for $t$ in a compact set in $]0, \infty[$.

Actually, according to (15), one has that for any $f \in L^1(\mathbb{R}^N)$, $w_m(t)$ is well defined and converges to $w$ in $W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ for any $1 \leq p < N/(N - 1)$.

**SECTION 2. The Cauchy-Dirichlet boundary value problem**

In this section $\Omega$ will be an open set in $\mathbb{R}^N$ and $f \in L^1(\Omega), f \geq 0$. We consider now the problem

$$u_t = \Delta u^m \text{ on } ]0, \infty[ \times \Omega, \quad u(0,.) = f \text{ on } \Omega, \quad u = 0 \text{ on } ]0, \infty[ \times \partial \Omega. \quad (25)$$

For simplicity we will assume $\Omega$ bounded with smooth boundary $\partial \Omega$, although the results which follow can be easily extended to a general open set $\Omega$.

Using for instance the results of [2], it follows that for $m > 1$ there exists a unique "strong solution" of (25) satisfying

$$u \in C([0, \infty[, L^1(\Omega)) \cap C([0, \infty[ \times \Omega), \quad u \geq 0 \text{ on } ]0, \infty[ \times \Omega, \quad u = 0 \text{ on } ]0, \infty[ \times \partial \Omega,$n

$u(0,.) = f \text{ on } \Omega$; for any $\tau > 0$, $u_t, \Delta u^m \in L^\infty(\tau, \infty[, L^1(\Omega))$ and

$$u_t = \Delta u^m \text{ a.e. on } ]0, \infty[ \times \Omega.$$
We shall denote by $u_m$ the strong solution of (25).

On the other hand there is existence and uniqueness of a solution of the variational inequality

\[(26) \quad w \in W^{1,1}_0(\Omega), \quad 0 \leq f + \Delta w \leq 1 \text{ in } \mathcal{D}'(\Omega), \quad w \geq 0, \quad w(f + \Delta w - 1) = 0 \text{ a.e. on } \Omega.\]

This follows by the result of [6].

We have the following

**THEOREM 2.** With the notations of this section, as $m \to \infty$

\[u_m(t) \to u = f + \Delta w \text{ in } L^1(\Omega)\]

uniformly for $t$ in a compact set in $]0,\infty[.$

To show this we will adapt the proof of Section 1. Using the contraction property in $L^1(\Omega)$ of the maps $f \to u_m(t)$ and $f \to u$, we may assume $f$ bounded, namely $0 \leq f \leq M$, such that (6) still holds.

The onceside estimate (10), with the constant \(\left( m - 1 + \frac{2}{N} \right)\) there is no more true, but as it is proved in [4] for general homogeneous evolution equation, we have for $u = u_m$

\[(27) \quad u_t = \Delta u^m \geq -u/(m - 1) t \text{ a.e.}.\]

and also

\[(28) \quad \|u_t(t)\|_{L^1} = \|\Delta u(t)^m\|_{L^1} \leq 2\|f\|_{L^1}/(m - 1) t \text{ a.e. } t > 0.\]

From (27) and (6), we deduce

\[(29) \quad (u_m(t, x))^m \leq ME(x)/(m - 1) t \quad \text{on } ]0, T[ \times \Omega \text{ for } m > 1.\]

where $E$ is now the solution of the Dirichlet problem on $\Omega$

\[-\Delta E = 1 \text{ on } \Omega, \quad E = 0 \text{ on } \partial \Omega\]

and it follows that for $0 < r < T$

\[(30) \quad (u_m)^m \to 0 \text{ uniformly on } [r, T] \times \Omega \text{ as } m \to \infty.\]
We may define again $w_m(t)$ by (14), and we have

$$u_m(t) - \Delta w_m(t) = f \text{ on } \Omega, \ w_m(t) = 0 \text{ on } \partial\Omega.$$  

If for a subsequence $m_k \to \infty$ we have $u_{m_k}(1) \to u$ in $L^1(\Omega)$, then we will have by (30), (29) and (31)

$$0 \leq u \leq 1 \text{ a.e. on } \Omega$$
$$u_{m_k}(t) \to u \text{ in } L^1(\Omega) \text{ uniformly for } t \in [\tau, T]$$
$$w_{m_k}(1) \to w \text{ in } L^1(\Omega)$$

where $w \geq 0$ is the solution of

$$u - \Delta w = f \text{ on } \Omega, \ w = 0 \text{ on } \partial\Omega.$$  

The proof of (20) can be done as in Section 1, with slight modifications: according to (27), the map $t \to t^{1/(m-1)}u(t)$ is nondecreasing, so that replacing (22) by $u_m(t)^m \leq M^2u_m(t)^{m-2}$, we will have in place of (23)

$$w_m(1) \leq M^2u_m(1)^{m-2}(m-1).$$

which gives also (20) exactly in the same way.

In other words, according to these remarks, the proof of Theorem 2 reduces to showing that

$$\{u_m(t); \ t \in [0, T], m \geq m_0\} \text{ is precompact in } L^1(\Omega).$$

According to (6), it is actually enough to prove that

$$\{u_m(t); \ t \in [0, T], m \geq m_0\} \text{ is precompact in } L^1_{\text{loc}}(\Omega).$$

To prove this, fix $\rho \in D(\Omega), \ \rho \geq 0$. Let $u = u_m$, and for $y \in \mathbb{R}^N$ with $\text{supp}(\rho + y)$ contained in $\Omega$, let $v(t, x) = \rho(x)[u(t, x + y) - u(t, x)]$. By Kato's inequality, we have

$$v_t \leq \rho \Delta w \text{ in } D'(0, \infty \times \Omega) \text{ with } w(t, x) = |u(t, x + y)^m - u(t, x)^m|$$

and then integrating

$$\int \rho(x)[u(t, x + y) - u(t, x)]dx \leq \int \rho(x)[f(x + y) - f(x)]dx + R|y|$$
with
\[ R = |y|^{-1} \int_0^t \int \Delta \rho(x) w(s, x) \, dx \, ds \leq \| \Delta \rho \|_{L^\infty(\Omega)} \| \text{grad } u_m \|_{L^1(0, t; L^\infty(\Omega))}. \]

Therefore (33) will follow from

**Lemma 1.** For \( T > 0 \) and \( m_0 > 1 \), there exists \( C \) such that
\[ \| \text{grad } (u_m)^m \|_{L^1(0, T; L^\infty(\Omega))} \leq C \quad \text{for } m \geq m_0. \]

**Proof of lemma 1.**

Set \( u = u_m \). For any \( t \geq 0 \), let \( v(t) \) be the solution of
\[- \Delta v(t) = u(t) \text{ on } \Omega, \quad v(t) = 0 \text{ on } \partial \Omega.\]

We have
\[ v \in C^1(\Omega \times [0, T]), \quad v_t = -u_m. \]
such that, taking integrating over \( \Omega \times [0, T] \), we obtain
\[ \int_0^T \int \nabla v(t) \, dx \, dt = - \int \int (|\nabla v(t)|^2) \, dx \, dt \leq \int |\nabla v(0)|^2 \]
and then
\[ \int_0^T \int u_m \, dx \, dt \leq C. \] (35)

Using now (27) we have that for any \( t > 0 \)
\[ (m - 1)t \int |\nabla u_m(t)|^2 \leq \int u_m^2(t) \leq \| u(t) \|_{L^{m+1}} \left( \int u_m^2(t) \right)^{m/(m+1)} \]
Then using Holder and the fact that \( \| u(t) \|_{L^{m+1}} \leq \| f \|_{L^{m+1}} \), we deduce that
\[ (m - 1) \left( \int \int |\nabla u_m| \right)^2 \leq |\Omega| \| f \|_{L^{m+1}} \left( \int u_m^2 \right)^{m/(m+1)} \left( \int dt/t^{(m+1)/(m+2)} \right)^{(m+2)/(m+1)} \]
and
\[ \left( \int \int |\nabla u_m| \right)^2 \leq M|\Omega|^{(m+2)/(m+1)}T^{1/m+1} C^{m/(m+1)}(m+2)^{(m+2)/(m+1)}(m-1)^{-1}, \]
whence the Lemma follows.
SECTION 3. The Cauchy-Neumann boundary value problem

In this section we consider the problem

\[ u_t = \Delta u^m \text{ on } ]0, \infty[, \quad u(0,.) = f \text{ on } \Omega, \quad \partial u^m / \partial n = 0 \text{ on } ]0, \infty[ \times \partial \Omega. \]

where \( \Omega \) is a bounded connected open set with smooth boundary \( \partial \Omega \), and \( f \in L^1(\Omega), \ f \geq 0 \). Using again the results of [2], for any \( m > 1 \) there exists a unique "strong solution" of (36) satisfying

\[ u \in C([0, \infty[ , L^1(\Omega)) \cap C(]0, \infty[, [x \Omega]_0, \quad u \geq 0 \text{ on } ]0, \infty[, \quad u(0,.) = f \text{ on } \Omega, \]
\[ u_t \in L^\infty(\Omega), \quad u \in L^\infty(\Omega) \text{ for any } \tau > 0, \quad u^m \in L^\infty_{\text{loc}}(\Omega), \quad \partial u^m / \partial n = 0 \text{ a.e. on } ]0, \infty[ \times \partial \Omega. \]

We denote now by \( u_m \) this solution of (36).

On the other hand, consider the variational inequality

\[ w \in W^{1,1}(\Omega), \quad 0 \leq f + \Delta w \leq 1 \text{ in } D'(\Omega), \quad w \geq 0, \quad w(f + \Delta w - 1) = 0 \text{ a.e. on } \Omega \]
\[ \int \rho \Delta w = - \int \text{grad } \rho \text{ grad } w \quad \text{for any } \rho \in C^1(\overline{\Omega}). \]

According to the results in [5], (37) has a solution if and only if

\[ \int f = |\Omega|^{-1} \int f \leq 1. \]

Moreover,

if \( \int f < 1 \), then the solution \( w \) of (37) is unique

if \( \int f = 1 \), for any solution \( w \) of (37), \( f + \Delta w = 1 \) a.e. on \( \Omega \).

We have

THEOREM 3. With the notations of this section,

i) if \( \int f \geq 1 \), then \( u_m(t) \to f \text{ in } L^1(\Omega) \text{ as } m \to \infty, \text{ uniformly for } t \text{ in a compact set in } ]0, \infty[. \)
ii) if \( \int f < 1 \), then \( u_m(t) \to u = f + \Delta w \text{ in } L^1(\Omega) \text{ as } m \to \infty, \text{ uniformly for } t \text{ in a compact set in } ]0, \infty[. \)
To prove this Theorem we may assume again that \( f \) is bounded and then that (6) holds. According to the results in [4], (27) and (28), still are true in this case. In particular, it is enough to prove that the conclusion is satisfied at \( t = 1 \): it will then hold uniformly for \( t \) in a compact set in \([0, \infty]\).

We denote by \( G \) the Green operator in \( L^1(\Omega) \) associated to the Neumann problem for the Laplacian: for \( w \in L^1(\Omega) \), \( v = Gw \) is the unique solution of the problem

\[
 v \in W^{1,1}(\Omega), \quad \int v = 0, \quad \int \rho \left( w - \int w \right) = \int \text{grad} \rho \text{grad} v \quad \text{for any } \rho \in C^1(\overline{\Omega}).
\]

It is clear that \( G \) is a bounded (actually compact) linear operator from \( L^1(\Omega) \) into \( W^{1,1}(\Omega) \) (see [6]).

Finally, we set \( I = \int f \). We then have

\[
\int u_m(t) = I \quad \text{for any } m > 1, \quad t > 0.
\]

**Proof of part i): case \( I \geq 1 \).**

We note for simplicity \( u_m = u_m(1) \); using (40), we have \( \int |u_m - I| = 2 \int (I - u_m)_+ \), where \( r_+ = \sup(r, 0) \), and then it is enough to prove that

\[
\int (I - u_m)_+ \to 0 \quad \text{as } m \to \infty.
\]

Using (28), since \( (u_m)^m - \int (u_m)^m = G(-\Delta (U_m)^m) \), we see that

\[
\varepsilon_m = |(u_m)^m - \int (u_m)^m| \to 0 \quad \text{in } W^{1,1}(\Omega) \quad \text{as } m \to \infty.
\]

Now, by convexity, we have

\[
\int (u_m)^m \geq \left( \int u_m \right)^m = l_m^m \geq 1
\]
so that

\[
(u_m)^m \geq (1 - \varepsilon_m)_+.\]
and
\[ \varepsilon_m \geq I^m - (u_m)^m \geq m(u_m)^{m-1}(I - u_m). \]

Then
\[ I - u_m \leq \varepsilon_m (1 - \varepsilon_m)^{1-1/m} \]
and, thanks to (42), (41) holds by Lebesgue dominated convergence theorem.

**Proof of part ii): case \( I < 1 \).**

We will prove

**LEMMA 2.** With the notations of this Section 3, if \( I < 1 \) then for \( T > 0 \) there exists \( C \) such that
\[ \|(u_m)^{m+1}\|_{L^1(0,T;\mathbb{R}^n)} \leq C \quad \text{for } m > 1. \]

Using Lemma 2, and repeating the proof of Lemma 1, one sees that for \( m_0 > 1 \), there exists \( C \) such that
\[ \|\text{grad } (u_m)^m\|_{L^1(0,T;\mathbb{R}^n)} \leq C \quad \text{for } m \geq m_0 \]
and then (33) holds also in this case.

Another consequence of Lemma 2 is that
\[ \liminf_{m \to \infty} (u_m)^{m+1} < \infty \quad \text{a.e. on } [0,T]\times\Omega \]
which we will use instead of (30) to obtain, thanks to (28), that if for a subsequence \( m_k \to \infty \) we have \( u_{m_k}(1) \to u \) in \( L^1(\Omega) \), then we will have \( 0 \leq u \leq 1 \) a.e. on \( \Omega \).

The proof of Theorem 3 in this case will follow then exactly as that of Theorem 2. To end up, we give the

**Proof of lemma 2.**

Let \( u = u_m \), \( v(t) = Gu(t) \). We have that
\[ v_t(t) = \int u^m(t) - u^m(t), \quad u(t) = I - \Delta v(t) \]
and then
\[ \int u^{m+1}(t) = \int u^m(t) \int u(t) - \int u(t) v(t) = \]
\[ = I \int u^m(t) - I \int v(t) - \int \text{grad } v(t) \text{ grad } v(t). \]

Using the convexity inequality \((m + 1)u^m \leq mu^{m+1} + 1\), we obtain
\[ (m + 1 - mI) \int \int u^{m+1} \leq IT + (m + 1) \{ I \int v(0) + 1/2 \int |\text{grad } v(0)|^2 \} \]
whence the result.

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