Gravitational collapse in $f(R)$ theories

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We study the gravitational collapse in modified gravitational theories. In particular, we analyze a general $f(R)$ model with uniformly collapsing cloud of self-gravitating dust particles. This analysis shares analogies with the formation of large-scale structures in the early Universe and with the formation of stars in a molecular cloud experiencing gravitational collapse. In the same way, this investigation can be used as a first approximation to the modification that stellar objects can suffer in these modified theories of gravity. We study concrete examples, and find that the analysis of gravitational collapse is an important tool to constrain models that present late-time cosmological acceleration.

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INTRODUCTION

In the general study of astrophysical weak gravitational fields, relativistic effects tend to be ignored. However, there are clear examples of stellar objects in which these effects may have important consequences, such as neutron stars, white dwarfs, supermassive stars or black holes. Indeed, it becomes necessary to consider observationally consistent gravitational theories to study these objects. General Relativity (GR) has been the most widely used theory but other gravitational theories may be studied for a better understanding of the features and properties of such objects and to compare their predictions with experimental results.

The gravitational collapse for a spherically symmetric stellar object has been extensively studied in the GR framework (see [1] and references therein). By assuming the metric of the space-time to be spherically symmetric and that the collapsing fluid is pressureless, the found metric interior to the object turns to be Robertson-Walker type with a parameter playing the role of spatial curvature and proportional to initial density. The time lapse and the size of the object are given by a cycloid parametric equation with an angle parameter ψ. Further results are that the time when the object gets zero size is finite and inversely proportional to the square root of the initial density. Finally, the redshift seen by an external observer is nevertheless infinite when time approaches the collapse time.

In spite of the fact that GR has been one of the most successful theories of the twentieth century, it does not give a satisfactory explanation to some of the latest cosmological and astrophysical observations with usual matter sources. In the first place, a dark energy contribution needs to be considered to provide cosmological acceleration whereas the baryonic matter content has to be supplemented by a dark matter (DM) component to give a satisfactory description of large scale structures, rotational speeds of galaxies, orbital velocities of galaxies in clusters, gravitational lensing of background objects by galaxy clusters, such as the Bullet Cluster, and the temperature distribution of hot gas in galaxies and clusters of galaxies. All these evidences have revealed the interest to study alternative cosmological theories. This extra DM component is required to account for about 20% of the energy content of our Universe. Although there are many possible origins for this component [2], DM is usually assumed to be in the form of thermal relics that naturally freeze-out with the right abundance in many extensions of the standard model of particles [3]. Future experiments will be able to discriminate among the large number of candidates and models, such as direct and indirect detection designed explicitly for their search [4], or even at high energy colliders, where they could be produced [5].

A larger number of possibilities can be found in the literature to generating the present accelerated expansion of the Universe [6]. One of these methods consists of modifying Einstein's gravity itself [7, 8] without invoking the presence of any exotic dark energy among the cosmological components. In this context, functions of the scalar curvature when included in the gravitational action give rise to the so-called $f(R)$ theories of modified gravity [9]. They amount to modifying the l.h.s. of the corresponding equations of motion and provide a geometrical origin to the accelerated cosmological expansion.

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Although such theories are able to describe the accelerated expansion on cosmological scales correctly [10], they typically give rise to strong effects on smaller scales. In any case, viable models can be constructed to be compatible with local gravity tests and other cosmological constraints [11].

The study of alternative theories of gravitation requires establishing methods able to confirm or discard their validity by studying the cosmological evolution, the growing of cosmological perturbation and, at astrophysical level, the existence of objects predicted by GR such as black holes [12] or dust clouds forming compact structures. It is well-known that $f(R)$ gravity theories may mimic any cosmological evolution by choosing adequate $f(R)$ models, in particular that of ΛCDM [10]. This is the so-called *degeneracy problem* that some modified gravity theories present: accordingly, the exclusive use of observations such as high-redshift Hubble diagrams from SNIa [13], baryon acoustic oscillations [14] or CMB shift factor [15], based on different distance measurements which are sensitive only to the expansion history, cannot settle the question of the nature of dark energy [16] since identical results may be explained by several theories. Nevertheless, it has been proved that $f(R)$ theories - even mimicking the standard cosmological expansion - provide different results from ΛCDM if the scalar cosmological perturbations are studied [17]. Consequently, the power spectra would be distinguishable from that predicted by ΛCDM [18].

It is therefore of particular interest, to establish the predictions of $f(R)$ theories concerning the gravitational collapse, and in particular collapse times, for different astrophysical objects. Collapse properties may be either exclusive for Einstein's gravity or intrinsic to any covariant gravitation theory. On the other hand, obtained results may be shed some light about the models viability and be useful to discard models in disagreement with expected physical results.

In [19] the authors studied gravitational collapse of a spherically symmetric perfect fluid in $f(R)$ gravity. By proceeding in a similar way to [1], the object mass was deduced from the junction conditions for interior and exterior metric tensors. Finally, they concluded that $f(R_0)$ (constant scalar curvature term) slows down the collapse of matter and plays the role of a cosmological constant. Authors in [20] paid attention to the curvature singularity appearing in the star collapse process in $f(R)$ theories. This singularity was claimed to be generated in viable $f(R)$ gravity and can be avoided by adding a R^α term. They also studied exponential gravity and the time scale of the singularity appearance in that model. It was shown that in case of star collapse, this time scale is much shorter than the age of the universe. Analogous studies were carried on by [21] claiming that in this class of theories, explosive phenomena in a finite time may appear in systems with time dependent increasing mass density.

Reference [22] includes a complete study of neutron stars in $f(R)$ theories. The most relevant result in this investigation suggests that $f(R)$ theory allows stars in equilibrium with arbitrary baryon number, no matter how large they are. Very recently authors in [23] studied collapse of charged black holes by using the double-null formalism.

Charged black holes in f(R) gravity can have a new type of singularity due to higher curvature corrections, the so-called f(R)-induced singularity, although it is highly model-dependent.

The present work has been arranged as follows: in section II, $f(R)$ modified gravity theories will be introduced. Gravitational collapse in $f(R)$ theories will be presented in section III. After performing some calculations, the evolution equation for the object scale factor will be obtained. This equation will be used throughout the following sections. Section IV is then dedicated to achieve solutions for the modified equations in three qualitatively different $f(R)$ models, which try to illustrate the broad phenomenology of the subject. This is therefore the aim of this section: to study gravitational collapse by calculating the evolution of the object scale factor in particular $f(R)$ models. Finally, the conclusions based upon the presented results will be analyzed in detail in section V.

II. $f(R)$ THEORIES OF GRAVITY

With the aim of proposing and alternative theory to GR, a possible modification consists of adding a function of the scalar curvature, $f(R)$, to the Einstein-Hilbert (EH) Lagrangian. Therefore the gravitational action becomes [1]

$$S_G = \frac{1}{16\pi G} \int \mathrm{d}^4x \sqrt{|g|}\,(R + f(R)). \tag{1}$$

By performing variations with respect to the metric, the modified Einstein equations turn out to be

$$(1 + f_R)R_{\mu\nu} - \frac{1}{2}(R + f(R))g_{\mu\nu} + \mathcal{D}_{\mu\nu}f_R = -8\pi G\,T_{\mu\nu}, \tag{2}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the matter content, $f_R \equiv \mathrm{d}f(R)/\mathrm{d}R$ and $\mathcal{D}_{\mu\nu} \equiv \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box$ with $\Box \equiv \nabla_\alpha\nabla^\alpha$ and ∇ is the usual covariant derivative.

These equations may be written *à la Einstein* by isolating on the l.h.s. the Einstein tensor and the $f(R)$

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[1] In the present work we employ the natural units system in which $\hbar = c = 1$. Note also that our definition for the Riemann tensor is $R^\sigma_{\mu\nu\kappa} = \partial_\kappa\Gamma^\sigma_{\mu\nu} - \partial_\nu\Gamma^\sigma_{\mu\kappa} + \Gamma^\sigma_{\kappa\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\mu\kappa}$.

contribution on the r.h.s. as follows

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{(1+f_R)}\left[-8\pi G T_{\mu\nu} \right.$$
$$\left. - \mathcal{D}_{\mu\nu}f_R + \frac{1}{2}\left(f(R) - Rf_R\right)g_{\mu\nu}\right] \quad (3)$$

We can also find the expression for the scalar curvature by contracting (2) with $g^{\mu\nu}$ which gives:

$$(1 - f_R)R + 2f(R) + 3\Box f_R = 8\pi G T. \quad (4)$$

Note that, unlike GR where R and T are related algebraically, for a general $f(R)$ those two quantities are dynamically related. In the homogeneous and isotropic case, the scalar curvature in $f(R)$ theories becomes

$$R = \frac{8\pi G T - 2f(R) - 3\ddot{f}_R}{(1 - f_R)} \quad (5)$$

where dot means the derivative with respect to cosmic time.

III. GRAVITATIONAL COLLAPSE IN $f(R)$

In the case of our investigation, we introduce the spherically symmetric metric

$$ds^2 = dt^2 - U(r,t)dr^2 - V(r,t)(d\theta^2 + \sin^2\theta d\phi^2) \quad (6)$$

If the collapsing object is approximated to be pressureless $p \simeq 0$, the components of the energy-momentum tensor can be expressed as follows

$$T_{\mu\nu} = \rho u_\mu u_\nu \;;\; T^t_{\ t} = \rho \;;\; T^i_{\ i} = 0 \text{ if } i = r, \theta, \phi. \quad (7)$$

We may further simplify the collapse model by considering ρ independent from the position. Therefore, we can search -as is actually the usual approach in the GR case- a separable solution for this metric as follows

$$U(r,t) = A_1^2(t)h(r), \quad V(r,t) = A_2^2(t)r^2, \quad (8)$$

where a previous reparametrization of the radial coordinate is required. When $f(R)$ modified tensorial equations are studied in the homogeneous and isotropic case - in which $f(R)$ does not depend on the position-, the trace component provides

$$\left(\frac{\dot{A}_2}{A_2} - \frac{\dot{A}_1}{A_1}\right)\frac{g'}{g} = 0 \Rightarrow \frac{\dot{A}_2}{A_2} = \frac{\dot{A}_1}{A_1}. \quad (9)$$

From (9), we deduce that A_1 and A_2 are proportional, in other words, $A_1(t) = C(r)A_2(t)$. So, if we choose $A_1(t) = A_2(t) \equiv A(t)$, the dependence in the radial coordinate is reabsorbed by $h(r)$. Hence:

$$U(r,t) = A^2(t)h(r), \quad V(r,t) = A^2(t)r^2. \quad (10)$$

Components tt, rr and $\theta\theta$ for the modified tensorial equations may be written respectively in terms of the functions $A(t)$ and $h(r)$ as follows

$$3\frac{\ddot{A}}{A} = \frac{1}{(1+f_R)}\left[-8\pi G\rho + 3\frac{\dot{A}}{A}\dot{f}_R + \frac{1}{2}\left(R + f(R)\right)\right], \quad (11)$$

$$A\ddot{A} + 2\dot{A}^2 + \frac{h'}{rh^2} = \frac{A^2}{(1+f_R)}\left[\ddot{f}_R + 2\frac{\dot{A}}{A}\dot{f}_R \right.$$
$$\left. + \frac{1}{2}\left(R + f(R)\right)\right], \quad (12)$$

$$A\ddot{A} + 2\dot{A}^2 + \frac{1}{r^2} - \frac{1}{hr^2} + \frac{h'}{2rh^2}$$
$$= \frac{A^2}{(1+f_R)}\left[\ddot{f}_R + 2\frac{\dot{A}}{A}\dot{f}_R + \frac{1}{2}\left(R + f(R)\right)\right]. \quad (13)$$

Let us point out two important aspects of equations (12) and (13): firstly, terms on the r.h.s of both equations are equal. Secondly, the term on the l.h.s. exclusively depends on r, whereas the term on the r.h.s. only depends on t in both equations, so that they must be constants[2]. Therefore we may equal l.h.s. of both equations to provide

$$\frac{1}{r}\frac{h'}{h^2} = \frac{1}{r^2} - \frac{1}{hr^2} + \frac{1}{2r}\frac{h'}{h^2} \equiv 2k, \quad (14)$$

where we have equaled both equations (multiplied by a factor A^2) to a constant $-2k$. The resulting solution is $h(r) = (1 - kr^2)^{-1}$. Once we have calculated $h(r)$, the resulting metric can be expressed as follows:

$$ds^2 = dt^2 - A^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \quad (15)$$

which is formally the same as the one obtained in the GR case [1]. Expression (14) for k may be substituted in either expression (12) or (13) yielding:

$$-\frac{\ddot{A}}{A} - 2\left(\frac{\dot{A}}{A}\right)^2 - \frac{2k}{A^2} = \frac{1}{(1+f_R)}\left[-\ddot{f}_R - 2\frac{\dot{A}}{A}\dot{f}_R \right.$$
$$\left. - \frac{1}{2}\left(R + f(R)\right)\right]. \quad (16)$$

Taking into account $\rho(t) = \rho(t = 0)/A(t)^3$ (given by the energy motion equation for dust matter) and the results

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[2] This fact is also satisfied in the GR case and allows the simplification of the calculus.

in (16), equation (11) becomes:

$$\dot{A}^2 = -k + \frac{1}{(1+f_R)}\left[\frac{4}{3}\pi G\rho(0)A^{-1} + \frac{1}{2}A^2\ddot{f}_R\right.$$
$$\left. + \frac{1}{2}A\dot{A}\dot{f}_R + \frac{A^2}{6}\left(R + f(R)\right)\right]. \tag{17}$$

Furthermore, provided that the fluid is assumed to be at rest for $t = 0$, initial conditions $\dot{A}(t = 0) = 0$ and $A(t = 0) = 1$ hold. This last condition means that the scale factor of the object at initial time is normalized

to unity. In order to simplify the notation we define $R(t = 0) \equiv R_0$ and $\rho(t = 0) \equiv \rho_0$. Therefore, evaluation of (17) at $t = 0$ allows to recast k as follows

$$k = \frac{1}{(1+f_R(R_0))}\left[\frac{4\pi G}{3}\rho_0 + \frac{1}{2}\ddot{f}_R(R_0) + \frac{1}{6}(R_0 + f(R_0))\right] \tag{18}$$

Once k has been expressed in terms of different quantities initial values, equations (5) and (18) may be inserted in (17) to provide

$$\dot{A}^2 = -\frac{1}{6(1-f_R^2(R_0))}\left[8\pi G\rho_0\left(2 - f_R(R_0)\right) - f(R_0)(1 + f_R(R_0)) - 3\ddot{f}_R(R_0)f_R(R_0)\right] + \frac{1}{(1-f_R^2)}\frac{8\pi G}{3}\rho_0 A^{-1}$$
$$-\frac{1}{6(1-f_R^2)}\left[8\pi G\rho_0 A^{-1}f_R + 3A^2\ddot{f}_Rf_R - 3A\dot{A}\dot{f}_R(1 - f_R) + A^2f(R)(1 + f_R)\right]. \tag{19}$$

The previous expression will be solved perturbatively to first order in perturbations for different $f(R)$ models. Let us remind at this stage that the zeroth order solution of GR is given by the parametric equations of a cycloid [1]:

$$t = \frac{\psi + \sin\psi}{2\sqrt{k}}, \quad A_G = \frac{1}{2}(1 + \cos\psi). \tag{20}$$

Expression (20) clearly implies that a sphere with initial density ρ_0 and negligible pressure will collapse from rest to a state of infinite proper energy density in a finite time that we will denote T_G. This time is obtained for the first value of ψ such as $A_G = 0$, i.e. for $\psi = \pi$. It means

$$T_G = \left(\frac{\pi + \sin\pi}{2\sqrt{k}}\right) = \frac{\pi}{2\sqrt{k}} = \frac{\pi}{2}\left(\frac{3}{8\pi G\rho_0}\right)^{1/2}. \tag{21}$$

In order to study the modification to the gravitational collapse in $f(R)$ theories, we will expand A around A_G and $f(R)$ around the scalar curvature in GR ($R = R_G$):

$$A = A_G + g(\psi), \tag{22}$$

$$f(R) \simeq f(R_G) + f'(R_G)(R - R_G). \tag{23}$$

The presence of a function $f(R)$ in the gravitational Lagrangian will represent a correction of first order with respect to the usual EH Lagrangian. Hence, $g(\psi)$ as defined in (22) will be also first order at least. By substituting the series expansions (22) and (23) in expression (19) until first order in ε, we find that equation (19) becomes

$$\text{tg}\left(\frac{\psi}{2}\right)g' = -\frac{1}{2}\cos^{-2}\left(\frac{\psi}{2}\right)g + \frac{1}{12k}\cos^2\left(\frac{\psi}{2}\right)(f(R_{G0}) + 3kf_R(R_{G0})) - \frac{1}{4}f_R(R_G)$$
$$+\frac{1}{4\sqrt{k}}\sin\left(\frac{\psi}{2}\right)\cos^3\left(\frac{\psi}{2}\right)\dot{f}_R(R_G) - \frac{1}{12k}\cos^6\left(\frac{\psi}{2}\right)f(R_G), \tag{24}$$

where we have cancelled out the GR exact solution and only kept first order perturbed terms. Equation (24) will provide $g(\psi)$ evolution for different $f(R)$ models to be considered in the next section.

IV. $f(R)$ THEORY RESULTS

In this section we shall consider three illustrative $f(R)$ models and study the gravitational collapse process for collapsing dust. The models under consideration are

Model 1: $f(R) = \varepsilon R^2$

This function has been proposed both as a viable inflation candidate [24] and as a dark matter model [25]. In this last reference, the ε parameter definition reads

$$\varepsilon = \frac{1}{6m_0^2}, \tag{25}$$

and the minimum value allowed for m_0 is computed as $m_0 = 2.7 \times 10^{-12}$ GeV at 95 % confidence level, i.e. $\varepsilon \leq 2.3 \times 10^{22}\,\text{GeV}^{-2}$. On the other hand, $\varepsilon > 0$ is needed to ensure the stability of the model, since in the opposite case, a tachyon is present in the theory. These constraints are in agreement with [26].

After some algebra, for this model, equation (24) can be written as follows:

$$g'(\psi) + g(\psi)\csc(\psi) + \frac{9}{8}k\varepsilon\left(\sin(\psi) + 2\tan\left(\frac{\psi}{2}\right)\right.$$
$$- \left. 4\tan^4\left(\frac{\psi}{2}\right)\csc(\psi)\right) = 0. \tag{26}$$

The homogeneous equation associated to (26) presents the solution $g_{hom}(\psi) \propto \cot(\psi/2)$, which diverges at $\psi = 0$. Therefore, its contribution will be ignored in the upcoming analysis. The analytical full solution of (26) becomes

$$g(\psi) = c_1 \cot\left(\frac{\psi}{2}\right) - \frac{9}{128}k\varepsilon\cot\left(\frac{\psi}{2}\right)\left\{-16\left(\psi + \sin(\psi)\right) + \frac{64}{5}\tan\left(\frac{\psi}{2}\right)\left[4 - \sec^2\left(\frac{\psi}{2}\right)\left(\sec^2\left(\frac{\psi}{2}\right) - 2\right)\right]\right\}, \tag{27}$$

This analytical solution given by (27) can be compared with the GR one by plotting them together as shown in Figure 1. As we see in this Figure, in the first stage of the collapse, the correction is negative, what implies that we have a larger contraction. On the contrary, very close to $\psi = \pi$, where the solution can be approximated as

$$g(\psi) \simeq \frac{72k\varepsilon}{5(\psi - \pi)^4}. \tag{28}$$

the sign of the modification changes and the total collapse is avoided. Exactly at this moment, the perturbation leaves the linear regime and a more complete analysis is required. It is interesting to estimate when the linear approach fails and an important modification is expected.

By using the collapse time parametrization (20) in the $\psi \to \pi$ limit, one gets

$$t = \frac{\psi + \sin(\psi)}{2\sqrt{k}} \simeq \frac{\pi + 1/6(\Delta\psi)^3}{2\sqrt{k}}, \tag{29}$$

or written in terms of the relative variation:

$$\frac{\Delta t}{t_{GR}} \simeq \frac{(\Delta\psi)^3}{6\pi}. \tag{30}$$

We are interested in estimating the region of the parameter space of the model, where the modified collapse, and the result for ψ_C (ψ value for the collapse) is significantly different from the one predicted by GR ($\psi_{C,GR} \equiv \pi$). With this purpose, we can estimate the values for which A_G is of the same order of its correction. As one can see in Figure 1, this deviation is more important close to the final stage of the collapse, for $\psi \sim \pi$. In

this region, A_G can be approximated by:

$$A_G = \frac{1}{2}\left(1 + \cos\psi\right) \simeq \frac{(\psi - \pi)^2}{4}. \tag{31}$$

We can use these approximations in the limit $\psi \to \pi$ to determine the intersection between the particular solution of (26) and the GR solution: A_G. This calculation will help establishing the validity regime of the perturbative approach. Therefore, by imposing $|g(\psi)| = |A_G|$, with $g(\psi)$ given by equation (28), we obtain:

$$\psi = \pi - \left(\frac{288|k\varepsilon|}{5}\right)^{1/6}. \tag{32}$$

At this point, it is necessary to clarify the physical value for k in order to discuss if the departure from linearity is important. k is the initial condition given by equation (18) that depends on the matter density, the initial curvature and the particular $f(R)$ model. In our analysis we are interested in studying the modification to the gravitational collapse in GR and for this reason we will assume the same value of k than as given in GR. This implies that the entire modification has a dynamical origin and it does not come from a change in the initial conditions. Therefore, we will assume that k only depends on the matter density:

$$k = \frac{8\pi G}{3}\rho_0. \tag{33}$$

For the most physically interesting values of k and ε, for which we have studied gravitational collapse of a dust matter cloud, the value of ψ is quite close to $\psi = \pi$ and
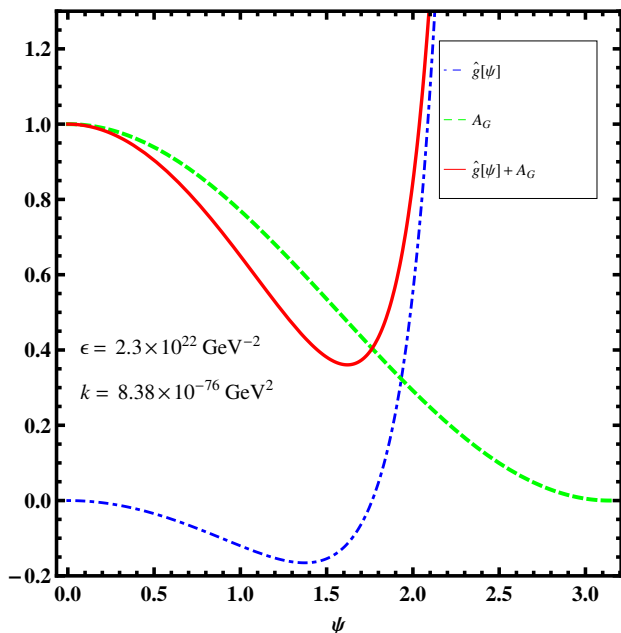
Figure 1: Comparison between the solution given by (27) setting $c_1 = 0$ and the GR case for the model 1: $f(R) = \varepsilon R^2$. The plotted k value is fixed by Eq. (33), where the density is $\rho_{SF} \simeq 1.5 \times 10^{-38}$ GeV^4 $\simeq 3.5 \times 10^{-18}$ kg/m^3, i.e., the matter density in the early Universe at redshift $z \simeq 1100$ marking the decoupling of matter and radiation and the beginning of structure formation (SF). The modification is extraordinarily small and has been increased 52 orders of magnitude to make it observable: $\hat{g}(\psi) = 10^{52} g(\psi)$.

therefore the asymptotic approach to obtain (32) is fully justified. The results are summarized in Figure 2, were the non-linear regime is shown for different values of ϵ and initial densities. For example, it is interesting to check the behavior for the matter density in the early Universe at redshift $z \simeq 1100$, which marks the decoupling of matter and radiation and the beginning of structure formation (SF): $\rho_{SF} \simeq 1.5 \times 10^{-38}$ GeV$^4 \simeq 3.5 \times 10^{-18}$ kg/m^3. In this particular case, the calculated root presents a slight difference with respect to the solution for GR. In particular, ψ_C differs from $\psi_{C,\,GR} \equiv \pi$ in the ninth significant figure.

Although this modification is not detectable for this first model as we deduce from the last considerations, it is interesting to stress that the relative modification is higher for denser media since the correction increases with ρ_0 as $\Delta t / t_{GR} \propto \sqrt{\rho_0}$. This behavior is significantly different with respect to other models as we will see in the

following sections. In Figure 2, we have represented some relevant density values as well as the non linear regime region. The loss of the linear regime takes place at high densities since the correction is directly proportional to ρ.
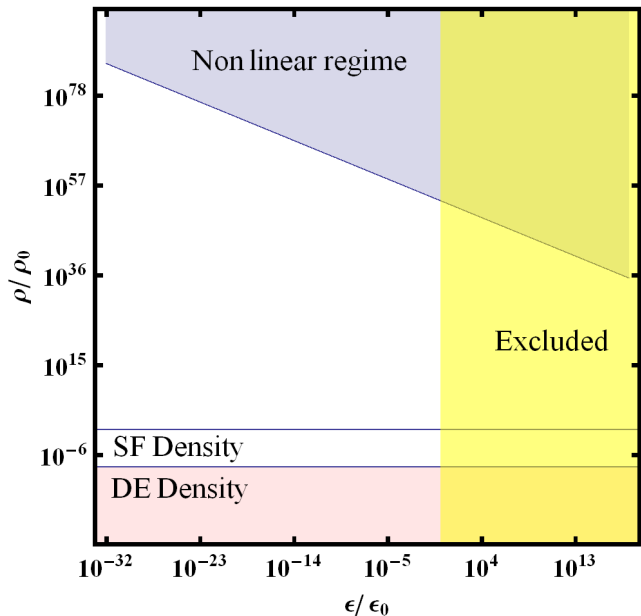

Figure 2: Validity of the perturbative regime for model 1, showing different relevant regions: In blue we show the region where our linear approach loses its validity. The excluded region is depicted in yellow and determined by the condition $\varepsilon \leq 2.3 \times 10^{22}$ GeV^{-2}. Finally, the density marking the beginning of structure formation (SF) and the dark energy (DE) density ($\rho_{DE} \simeq 2.8 \times 10^{-47}$ GeV^4) have also been plotted for reference.

Model 2: $f(R) = \varepsilon R^{-1}$

We will continue our analysis with the $f(R)$ model proposed in reference [27] as a dark energy candidate. This possibility is currently excluded, but this model is a simple example that help to understand the gravitational collapse modifications in models that provide late-time acceleration.

For this model, equation (24) becomes:

$$g'(\psi) + g(\psi) \csc(\psi) = \frac{\varepsilon}{6k^2} \sin\left(\frac{\psi}{2}\right) \cos^{13}\left(\frac{\psi}{2}\right) , \quad (34)$$

whose full solution is

$$g(\psi) = c_1 \cot\left(\frac{\psi}{2}\right) + \frac{1}{6}\frac{\varepsilon}{k^2}\left(\frac{33\psi}{2048} + \frac{165\sin(\psi)}{8192} - \frac{11\sin(2\psi)}{8192} - \frac{121\sin(3\psi)}{24576} - \frac{25\sin(4\psi)}{8192}\right.$$
$$\left. - \frac{43\sin(5\psi)}{40960} - \frac{5\sin(6\psi)}{24576} - \frac{\sin(7\psi)}{57344}\right)\cot\left(\frac{\psi}{2}\right).$$
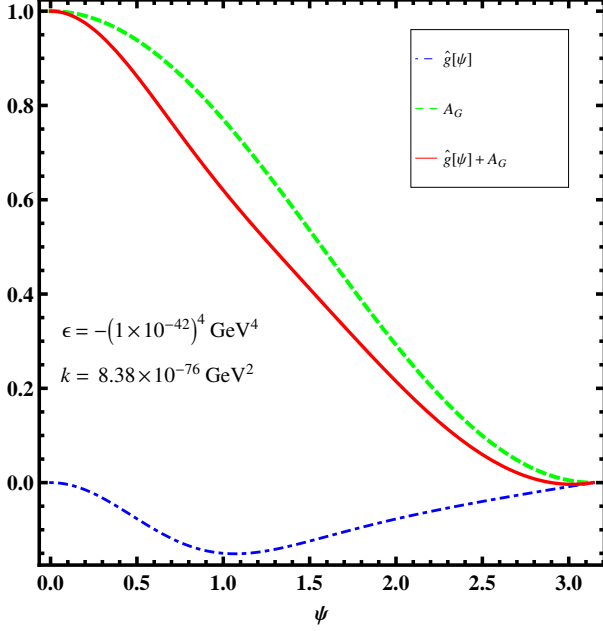
$$(35)$$

Figure 3: Analogous representation to the one shown in Fig. 1, which includes the GR solution, the modification given by (35) and the sum of the two. In this figure $\hat{g}(\psi) = 10^{19}g(\psi)$ in order to make the modification observable.

Figure 4: Analogous to Figure 2 for model 2. The limit of the region depicted in yellow shows the proposed value $\varepsilon = -\mu^4$, and $\mu = 10^{-42}$ GeV according to [27].

In Figure 3, it is possible to see the behavior of the modification for $\varepsilon = -\mu^4$, and $\mu = 10^{-42}$ GeV as it was the value originally proposed in reference [27].

The series expansion of (35) around $\psi = \pi$ reads, in this case:

$$g(\psi) \simeq -\frac{11\pi\varepsilon(\psi - \pi)}{8192k^2}.$$

$$(36)$$

Once again the intersection between the particular solution of (34) and the GR solution A_G can be determined in the $\psi \to \pi$ limit, with help of equation (31). $|g(\psi)| = |A_G|$ implies

$$\psi \simeq \pi - \frac{11\pi}{2048}\frac{|\varepsilon|}{k^2}.$$

$$(37)$$

As it can be seen in Figure 3, the difference between the modified ψ_C and $\psi_{C,\text{GR}}$ is not distinguishable for $\varepsilon = -\mu^4$ if density is higher than the standard dark energy density. The same result is found for $\varepsilon > -\mu^4$
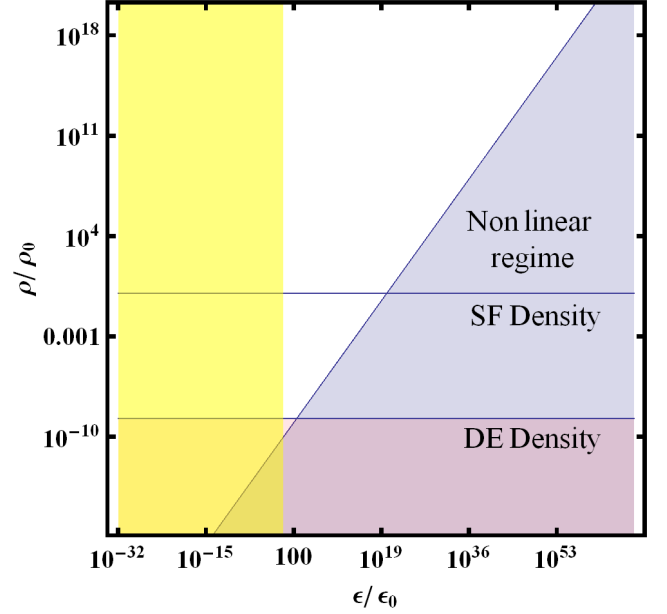
($|\varepsilon| < \mu^4$ and negative). The situation changes for $\varepsilon < -\mu^4$ ($|\varepsilon| > \mu^4$ and positive). This behavior can be observed in Figure 4, where the validity of the linear regime is shown to decrease for higher values of $|\varepsilon|$ and lower densities. As we will see in the following example, this is a general property of $f(R)$ models that provide accelerated cosmologies, at least, for densities higher than the vacuum energy. Results in Figure 4 can be understood by estimating the correction of the collapsing time in the linear regime as it is determined by equation (30). This $f(R)$ model provides a relative difference for the collapsing time in GR value given by

$$\frac{\Delta t}{t_{GR}} \simeq -\frac{11^3\varepsilon^3\pi^2}{3k^62^{34}}.$$

$$(38)$$

We observe that the correction is more negligible for denser objects. This unexpected fact can be understood since GR modification to the scale factor is proportional to $g \propto \varepsilon/k^2$ whereas $k \propto \rho_0$. According to this dependence, a stellar object with a higher density will suffer

a less important modification and vice versa. The relative time modification is lower for denser media since the correction decreases with ρ_0 as $\Delta t/t_{GR} \propto \rho_0^{-6}$.

Model 3: $f(R) = \lambda R_0 \left[\left(1 + \frac{R^2}{R_0^2} \right)^{-n} - 1 \right]$

The last $f(R)$ model to be analyzed in the present work is the well-known Starobinsky model proposed in Reference [28]. For this model, $n, \lambda > 0$ and R_0 is considered

[3] It is important to remark that R_0 in this context is a parameter of the model and not the initial scalar curvature.

to be of the order of the presently observed effective cosmological constant[3]. With such parameter choice, this model is a viable dark energy candidate. The relation between λ and R_0 in vacuum is given by $H_0^2 = \lambda R_0/6$ according to [28], where H_0 is the present Hubble parameter (see [29] for recent WMAP data) and the proposed value for $\lambda = 0.69$.

For the sake of simplicity, let us choose $n = 1$. In this case, the equation (24) may be rewritten as follows:

$$-g'(\psi) - g(\psi)\csc(\psi) - \frac{9k\lambda R_0 \sin^3(\psi)\csc^4\left(\frac{\psi}{2}\right)}{32\left(9k^2 + R_0^2\right)^2}\left(R_0^2 + 3k^2\right) + \frac{72k\lambda R_0 \sin^4\left(\frac{\psi}{2}\right)\csc(\psi)}{\left(R_0^2 + 9k^2\sec^{12}\left(\frac{\psi}{2}\right)\right)^2}\left(R_0^2 + 3k^2\sec^{12}\left(\frac{\psi}{2}\right)\right)$$
$$-\frac{9k\lambda R_0^3 \tan\left(\frac{\psi}{2}\right)\sec^4\left(\frac{\psi}{2}\right)\left(R_0^2 - 27k^2\sec^{12}\left(\frac{\psi}{2}\right)\right)}{2\left(9k^2\sec^{12}\left(\frac{\psi}{2}\right) + R_0^2\right)^3} = 0. \tag{39}$$

Unlike the other two cases, we are not able to find analytical solution for equation (39). Thus, specific values for k and λ parameters and R_0 are required to find a numerical solution. This solution is plotted in Figure 5. In any case, equation (39) can be studied in the asymptotic limits $\psi \to 0$ and $\psi \to \pi$. Thus, the corresponding series expansion of (39) in the $\psi \to 0$ becomes

$$- g'(\psi) - \frac{g(\psi)}{\psi} - \frac{9k\lambda R_0 \left(3k^2 - R_0^2\right)\psi}{4\left(9k^2 + R_0^2\right)^2} = 0, \tag{40}$$

whose analytical solution is

$$g(\psi) = \frac{c_1}{\psi} + \frac{3k\lambda R_0 \left(R_0^2 - 3k^2\right)\psi^2}{4\left(9k^2 + R_0^2\right)^2}. \tag{41}$$

Since the homogeneous equation does not depend on the $f(R)$ model, the condition $c_1 = 0$ is also necessary in order to have a finite solution. When the considered asymptotic limit is $\psi \to \pi$, equation (39) approximately becomes

$$- g'(\psi) + \frac{g(\psi)}{\psi - \pi} + \frac{9k\lambda R_0(\psi - \pi)^3\left(3k^2 + R_0^2\right)}{32\left(9k^2 + R_0^2\right)^2} = 0, \tag{42}$$

whose analytical solution is

$$g(\psi) = c_1(\psi - \pi) + \frac{3k\lambda R_0(\psi - \pi)^4\left(3k^2 + 2R_0^2\right)}{32\left(9k^2 + R_0^2\right)^2}. \tag{43}$$

This asymptotic limit of the linear correction depends on a higher power of $(\psi - \pi)$ than the GR solution given by eq. (31). This fact implies that we cannot estimate the validity of the linear regime by using the $\psi \to \pi$ as in the previous cases. The modification is more important at intermediates values of ψ, as it can be observed in Figure 5. The numerical results are showed in Figure 6. In a similar way to the second model, for $\lambda < \lambda_0$ the modification of the collapse is always linear and not important. The situation is different for $\lambda > \lambda_0$, where the collapse is severely modified at densities closer to the vacuum one. We have checked numerically that denser environments are less affected by this gravitational model.

V. CONCLUSIONS

In this work we have studied the gravitational collapse in $f(R)$ gravity theories. These theories provide corrections to the field equations that modify the evolution of gravitational collapse with respect to the usual General Relativity results. In this context, viable $f(R)$ models must provide similar results for the collapse times to the values obtained in General Relativity. In addition, collapse times must be much shorter than the age of the universe and long enough to allow matter cluster.

The analyzed $f(R)$ models present both important different quantitative and qualitative behaviors when compared with General Relativity collapses. In fact, all of
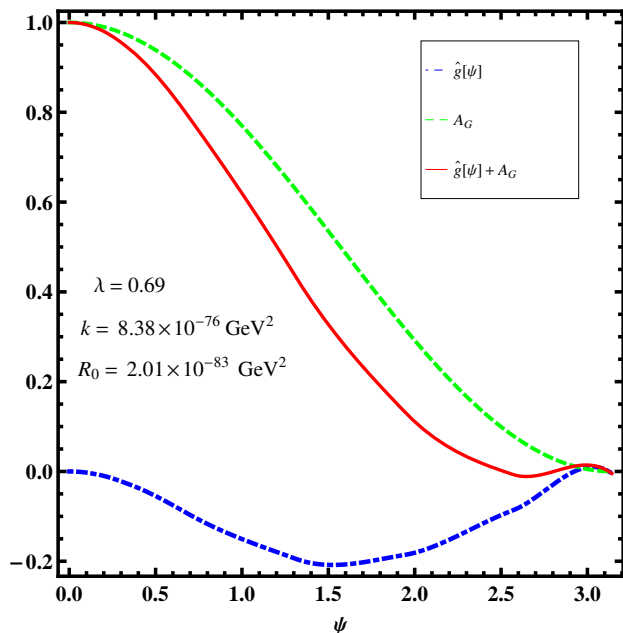
Figure 5: The plotted lines are analogous to the ones in Fig. 1 and Fig. 3 but with the solution given by (39). In this figure $\hat{g}(\psi) = 10^9 g(\psi)$ in order to make the modification observable.

Figure 6: Analogous to Figure 2 for studied model 3. The limit of the region depicted in yellow shows the proposed value $\lambda = 0.69$ [28].

them show a collapsing initial epoch with higher contraction than in General Relativity. This result is expected since $f(R)$ theories modify the gravitational interaction by the addition of a new scalar mediator. It is well-known that a scalar force is always attractive and can only reduce the time of gravitational collapse. This result is interesting since observations of structures at high redshift introduce some tension with the standard ΛCDM model [30], and the tendency of $f(R)$ models to increase the gravitational attraction at early times can alleviate this problem.

Although this general behavior is shared by the three models analyzed throughout this investigation, they present significant differences when the modifications to the General Relativity collapse leave the linear regime. On the one hand, the R^2 model has a modification that increases with the density of the collapse object: $(\Delta t_c/t_{GR}) \propto \sqrt{\rho}$. The opposite behavior is found for the R^{-1} model, where this modification decreases with density as $(\Delta t_c/t_{GR}) \propto \rho^{-6}$. Finally, a similar situation is reproduced numerically in the Starobinsky model. The departure from the linear collapse is able to exclude interesting parameters regions of these models that support late-time acceleration as seen in Figures 2, 4 and 6.

Another relevant question is related to the physics of stellar objects when analyzed in the $f(R)$ modified gravity theories frame. Although we cannot use straightforwardly the results of this analysis due to the fundamental role that pressure plays in the stability of these objects,
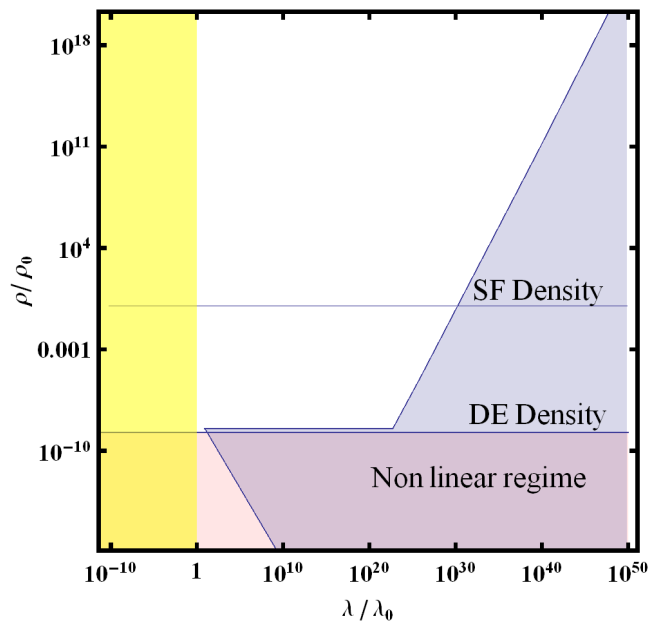
we may get an idea of the importance of the corrections. Inside these objects, pressure is the same order of magnitude as density, and it is expected to introduce an important modification into star evolution and dynamics. Therefore, it is enough to take into account the typical value of the density of a neutron star, approximately $10^{-3}\,\text{GeV}^4$ to estimate if correction will be important. Although this value is 35 orders of magnitude larger than the dust density used above, the results do not change dramatically. A direct extrapolation suggests that we may expect even more negligible modifications for $f(R)$ models that present dark energy scenarios (models 2 and 3). In models with higher powers of the scalar curvature, the correction to General Relativity will be more important but still negligible (as for model 1).

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