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Strategy-Proof Interval Selection

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We investigate interval selection problems in which a society chooses an interval from a discrete ordered set. We consider a domain of interval monotonic preferences, where each voter has an interval of acceptable integers, benefits from enlarging the selected interval by including acceptable integers, and is worse off when unacceptable integers are included. We study how different institutional frameworks affect the design of strategy-proof interval choice functions. In particular, we distinguish between unrestricted interval selection and the expansion of a commonly accepted status quo interval. We show that the latter transforms interval selection into a separable multidimensional choice problem, whereas unrestricted interval selection remains intrinsically multidimensional. This institutional distinction leads to fundamentally different characterizations of strategy-proof interval choice functions.

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Strategy-Proof Interval Selection *

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July 16, 2026

Abstract

We investigate interval selection problems in which a society chooses an interval from a discrete ordered set. We consider a domain of interval-monotonic preferences, where each voter has an interval of acceptable integers, benefits from enlarging the selected interval by including acceptable integers, and is worse off when unacceptable integers are included. We study how different institutional frameworks affect the design of strategy-proof interval choice functions. In particular, we distinguish between unrestricted interval selection and the expansion of a commonly accepted status quo interval. We show that the latter transforms interval selection into a separable multidimensional choice problem, whereas unrestricted interval selection remains intrinsically multidimensional. This institutional distinction leads to fundamentally different characterizations of strategy-proof interval choice functions.

Keywords: Strategy-proofness, Interval Choice Function, Interval-monotonic Preferences.
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1 Introduction

Consider the following social choice scenarios:

- The City Council determines the working hours at construction sites.

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- A Central Bank Committee selects the band of fluctuation for the exchange rate for which no adjustment in monetary policy is required.

- The Board of Trustees of a college establishes the minimum and maximum enrollment required to offer a course.

All these problems share a common characteristic—they involve a society of voters selecting an interval of admissible values. Each voter has a most-preferred interval, and adding elements from this interval to the socially chosen interval increases the voter's welfare.

In this paper, we propose a social choice and implementation approach to interval selection problems. We analyze interval choice functions that select an interval for each profile of voters' preferences. These mappings provide a recommendation for society that represents an optimal compromise among the voters' conflicting preferences. As voters' preferences are not directly observable and must be elicited, we are interested in interval choice functions that satisfy *strategy-proofness* and incentivize voters to report their preferences truthfully. This ensures that society can use the relevant information to determine the "optimal" social outcome. Moreover, truthfully revealing one's preferences is a dominant strategy for every voter.

It is well known that *strategy-proofness* is hard to satisfy. In the abstract model of social choice, Gibbard (1973) and Satterthwaite (1975) show that—provided there are more than two alternatives at stake—every strategy-proof social choice function is dictatorial. That is, the preferences of a single voter always determine the social choice outcome. Reasonable strategy-proof social choice functions exist, however, when appropriate restrictions are imposed on voters' preferences. Such domain restrictions arise naturally in interval selection problems. They reflect situations in which voters distinguish between acceptable and unacceptable values of the underlying variable. We consider a discrete problem where intervals are defined as the union of integers that represent minimum size intervals. In this context, we study ***interval-monotonic preferences*** over intervals. Each voter classifies integers as either acceptable or unacceptable, subject to the restriction that the set of acceptable integers is non-empty and forms an interval of integers. A voter is better off whenever acceptable integers are added to the chosen interval and worse off whenever unacceptable integers are added.[1] A natural subdomain of interval-monotonic preferences is the set of additively representable preferences, where voters attach positive utility to acceptable integers and negative utility to unacceptable integers, and voters compare intervals according to the sum of the utilities attached to their components.

[1] Interval-monotonic preferences over intervals capture the idea of the separable domain of preferences in Barberà *et al.* (1991), with the additional restriction that the set of acceptable integers forms an integer interval.

Within this framework, we investigate how institutional restrictions affect the possibility of designing interval choice functions that satisfy *strategy-proofness*. We first consider a restricted environment in which society starts from an initial status quo interval that may only be expanded. We first focus on the subdomain of preferences where each voter's interval of acceptable integers either intersects or is adjacent to the status quo interval. Under this subdomain of preferences and additively representable preferences, preferences over intervals induce binary relations over the possible inferior and superior endpoints of the selected interval. These induced preferences are single-peaked, with peaks determined by the limits of each voter's most-preferred interval. By the decomposability result of Le Breton and Sen (1999), every interval choice function satisfying *strategy-proofness* whose range consists of all intervals containing the status quo is characterized by two one-dimensional rules determined by the marginal preference relations, one selecting each endpoint of the interval. Applying the classical characterization by Moulin (1980), each endpoint is determined by a generalized median voter scheme (Theorem 1). The previous characterization relies critically on the induced single-peaked structure that only arises when there are no unacceptable integers between the status quo and voters' most-preferred interval. For the domain of additively representable preferences, although the status quo restriction is maintained, the single-peaked structure is lost and generalized median voter schemes no longer characterize all strategy-proof rules. We show that only interval choice functions whose inferior and superior endpoints are selected according to the preferences of (possibly different) voters satisfy *strategy-proofness*. On the full domain of interval-monotonic preferences, complementarities prevent the independent determination of the two endpoints. Consequently, the characterization further collapses to dictatorship (Theorem 2). Finally, we consider the unrestricted interval selection problem under the full domain of interval-monotonic preferences. Without the status quo restriction, interval selection can no longer be decomposed into two one-dimensional problems. Only dictatorial interval choice functions satisfy *strategy-proofness* and have full range (Theorem 3). Thus, despite the substantial domain restriction, the Gibbard–Satterthwaite impossibility essentially persists in interval selection problems.

This paper belongs to the vast literature that studies either multivalued social choice rules or domain restrictions as ways of avoiding the negative conclusions of the Gibbard–Satterthwaite theorem. We refer the reader to Barberà (2011) for a comprehensive survey. There are, however, relatively few papers that combine multivalued social choice rules with domain restrictions such as single-peaked preferences. Miyagawa (1998, 2001); Ehlers (2002); Heo (2013) consider the problem of locating two public facilities in a line when each voter may attend one facility. Ingalagavi and Sadhukhan (2023); Rodríguez-Álvarez (2017, 2026) study multivalued so-

cial choice rules that may select restricted ranges of intervals under single-peaked preferences. They assume that alternatives are mutually exclusive and voters' preferences are consistent with expected utility maximization. Klaus and Storcken (2002); Klaus and Protopapas (2020) study explicit interval selection with single-peaked preferences using maximin and maximax preferences. Bhattacharya and Khare (2024) considers the selection of integer intervals of fixed length when preferences over singleton alternatives are single-peaked, and preferences over intervals are responsive, a stronger restriction than interval-monotonic preferences. Restricting attention to intervals of fixed length implies that responsive preferences over intervals are one-dimensional single-peaked with respect to the location of the inferior endpoints of the chosen intervals. By contrast, we allow intervals of variable size, consider different domain restrictions, and analyze several institutional frameworks. Block (2010); Farfel and Conitzer (2011) study strategy-proof choice of intervals when voters have multidimensional single-peaked preferences in the product space of inferior and superior endpoints of intervals. In a similar preference domain, Miller (2025) interprets intervals as the limits of tolerance for a given real-valued variable. More recently, LiCalzi and Yasar (2026) study the aggregation of vocabularies, modeled as interval partitions of a linearly ordered set. Although their object of choice differs from ours, both papers consider interval structures over ordered alternatives.

The paper most closely related to ours is Caramuta (2010), who studies interval selection under a domain of preferences closely related to interval-monotonic preferences. In his framework, the interval selection problem is interpreted as the selection of sets of ordered alternatives under the feasibility constraint that the selected set must form an interval. The main difference between the two frameworks is that we impose the additional natural restriction that the set of acceptable integers must itself form an interval. Consequently, our dictatorial results are obtained on a strictly smaller preference domain. Moreover, Caramuta (2010) admits the empty interval as a feasible outcome. His model is therefore a particular case of the general framework for selecting finite subsets of alternatives under separable preferences and feasibility constraints studied by Barberà *et al.* (1991); Barberà *et al.* (1997, 2005). Methodologically, our approach is also substantially different. Rather than relying on a finite set of admissible intervals, our analysis exploits the decomposition of multidimensional strategy-proof social choice functions (Le Breton and Sen, 1999), and the option-set approach developed by Barberà and Peleg (1990), both of which apply to countably infinite sets of alternatives. Finally, unlike Caramuta (2010), we also study the restricted problem of status quo expansion. In this environment, the relevant restricted domain of interval-monotonic preferences induces one-dimensional single-peaked preferences over interval endpoints, leading to positive characterization results.

The remainder of the paper is organized as follows. In Section 2, we introduce the notation and present the model. In Section 3, we present our characterization results in the restricted framework of status quo expansion (Subsection 3.1) and in the unrestricted interval selection framework (Subsection 3.2). In Section 4, we discuss possible extensions to finite sets of alternatives and continuous variables. Finally, in Section 5, we collect all proofs.

2 Notation and Basic Definitions

Let N be a finite society of at least two voters ($|N| \geq 2$). Voters are interested in selecting a closed interval of integers. Let $\mathbb{Z}$ be the set of integers and let $\mathcal{I} = \left\{(z, z') \in \mathbb{Z}^2 \,:\, z \leq z'\right\}$ denote the set of all non-empty closed intervals of integers. Each integer represents an interval on the real line of possible values of a relevant variable. Thus, the set of integers induces a partition of the real line into intervals. Each interval in $\mathcal{I}$ then corresponds to a union of adjacent elements of this partition.[2] For each $x \in \mathcal{I}$, let $\underline{x}, \bar{x} \in \mathbb{Z}$ denote respectively the inferior and superior endpoints of x, with $\underline{x} = \inf x$ and $\bar{x} = \sup x$ (which coincide with the minimum and maximum elements of x). We write $x = [\underline{x}, \bar{x}]$. We identify each $z \in \mathbb{Z}$ with the interval $[z, z]$, and denote it by $[z]$. The empty set represents the outcome in which no interval is selected.

A ***preference*** is a complete, antisymmetric, and transitive binary relation on $\mathcal{I} \cup \{\varnothing\}$. Each voter $i \in N$ has a preference relation $\succsim_i$ over $\mathcal{I} \cup \{\varnothing\}$. Let $\succ_i$ and $\sim_i$ denote respectively the associated strict preference and indifference relations respectively. We denote by $\mathcal{D}$ the domain of all preferences on $\mathcal{I} \cup \{\varnothing\}$. For each $i \in N$ and each preference $\succsim_i \in \mathcal{D}$, $b(\succsim_i)$ denotes the unique ***most-preferred elemen***t. For each set $\mathcal{X} \subset \mathcal{I}$, $b(\succsim_i, \mathcal{X})$ is the ***most-preferred element of*** $\succsim_i$ ***in*** $\mathcal{X}$. A preference profile is an n-tuple $\succsim = (\succsim_1, \ldots, \succsim_n) \in \mathcal{D}^N$. For each voter $i \in N$ and each preference profile $\succsim$, $\succsim_{-i}$ denotes the preference profile of the voters in $N \setminus \{i\}$. Let $\widetilde{\mathcal{D}}^i \subseteq \mathcal{D}$ denote a domain of admissible preferences for voter $i \in N$, and $\widetilde{\mathcal{D}}^N = \times_{i \in N} \widetilde{\mathcal{D}}^i$ be the corresponding profile space. We assume that admissible preference domains are symmetric across voters. Thus, for each $i \in N$, $\widetilde{\mathcal{D}}^i = \widetilde{\mathcal{D}}$.

Because voters are never indifferent between any interval and the empty set, each voter partitions the set of integers into ***acceptable integers***

$$\{z \in \mathbb{Z} : [z] \succ_i \varnothing\},$$

[2] Our analysis also applies to finite partitions of the real line. All results extend directly whenever the partition contains at least four elements. We discuss the case of finite sets of elements in Section 4.

and ***unacceptable integers***

$$\left\{z' \in \mathbb{Z} : \varnothing \succ_i [z']\right\}.$$

Interpreting integer intervals as ranges of admissible values for a real-valued variable naturally motivates a structured restriction on the domain of preferences. We impose two natural conditions consistent with this interpretation. First, for each voter, the set of acceptable integers forms an interval in $\mathbb{Z}$. Second, preferences are monotonic with respect to the inclusion of integers. The key feature of the domain is that the marginal effect of adding an integer depends only on whether that integer is acceptable. A voter is better off whenever an acceptable integer is added to the chosen interval and worse off whenever an unacceptable integer is added. Consequently, the most-preferred interval coincides with the set of all integers the voter considers acceptable. The definition of interval-monotonic preferences formalizes these two conditions.

A preference $\succsim_i \in \mathcal{D}$ is an ***interval-monotonic*** preference if

i) the set of acceptable integers is a non-empty integer interval:

$$\{z \in \mathbb{Z} : [z] \succ_i \varnothing\} \in \mathcal{I}, \text{ and}$$

ii) for each $x \in \mathcal{I}$ and each $z \in \mathbb{Z}$ such that $z \notin x$ and $[z] \cup x \in \mathcal{I}$ (that is, $[z] \cup x$ is itself an interval):

$$[z] \succ_i \varnothing \text{ if and only if } [z] \cup x \succ_i x.$$

Let $\mathcal{D}^M$ be the set of all interval-monotonic preferences.[3]

The following remark shows that these two conditions imply the existence of a unique most-preferred interval.

Remark 1. *For each $\succsim_i \in \mathcal{D}^M$, the unique most-preferred interval coincides with the set of acceptable integers: $b(\succsim_i) = \{z \in \mathbb{Z} : [z] \succ_i \varnothing\}$.*

An important subdomain of Interval-monotonic preferences is the class of additively representable preferences. An interval-monotonic preference $\succsim_i \in \mathcal{D}^M$ is ***additively representable*** if there exists a function $u : \mathbb{Z} \cup \{\varnothing\} \to \mathbb{R}$ such that $u(\varnothing) = 0$, and for each $x, y \in \mathcal{I} \cup \{\varnothing\}$, $x \succsim_i y$ if and only if $\sum_{z \in x} u(z) \geq \sum_{z \in y} u(z)$. Let $\mathcal{D}^A$ denote the set of all additively representable interval-monotonic preferences. Clearly, $\mathcal{D}^A \subsetneq \mathcal{D}^M$.

[3] Condition ii) is the separability condition introduced by Barberà *et al.* (1991).

Given a domain of admissible preferences $\widetilde{\mathcal{D}} \subseteq \mathcal{D}$, an ***interval choice function*** is a mapping $f : \widetilde{\mathcal{D}}^N \to \mathcal{I}$ that selects a non-empty interval for every profile of voters' preferences.

Our definition of interval choice functions explicitly rules out the possibility of empty social choice outcomes. Although voters compare intervals with the empty outcome in order to express acceptability, the interval choice functions considered in this paper always select a non-empty interval. Thus, empty social outcomes are ruled out by definition.

For each interval choice function $f : \widetilde{\mathcal{D}}^N \to \mathcal{I}$, r_f denotes its ***range***,

$$r_f = \{x \in \mathcal{I} : f(\succsim) = x \text{ for some } \succsim \in \widetilde{\mathcal{D}}^N\}.$$

We conclude this section by introducing the standard properties used to evaluate interval choice functions.

Strategy-proofness. For each $i \in N$, each $\succsim \in \widetilde{\mathcal{D}}^N$, and each $\succsim_i' \in \widetilde{\mathcal{D}}$, $f(\succsim) \succsim_i f(\succsim_i', \succsim_{-i})$.

Since we study different institutional frameworks, the relevant range of the interval choice function depends on the feasible set of intervals.

Ontoness. $r_f = \mathcal{I}$.

Unanimity (on the range). For each $\succsim \in \widetilde{\mathcal{D}}^N$ and each $x \in \mathcal{I}$, $b(\succsim_i, r_f) = x$ for each $i \in N$ implies $f(\succsim) = x$.

Strategy-proofness requires that no voter can benefit from misrepresenting her preferences. *Ontoness* requires that every interval belongs to the range of the social choice function. *Unanimity* requires that whenever voters have the same most-preferred interval in the range of the rule, that interval is selected. The following well-known lemma shows that *strategy-proofness* implies *unanimity* (on the range) in domains of strict preferences.

Lemma 1. *If an interval choice function $f : \widetilde{\mathcal{D}}^N \to \mathcal{I}$ satisfies* strategy-proofness, *then f satisfies* unanimity *(on the range).*

Lemma 1 implies that, under *strategy-proofness*, whether an interval choice function satisfies *unanimity* is completely determined by its range.

3 Results

This section presents our main characterizations of interval choice functions satisfying *strategy-proofness*. We distinguish between two institutional environments. The first assumes the existence of a predetermined status quo interval that may only be expanded through collective choice. This setting captures situations in which previously accepted values or categories cannot be removed, so that social choice concerns only the possible enlargement of the current interval. The second places no initial restriction on feasible outcomes, allowing any interval to be selected. Comparing these two environments reveals the role played by feasibility constraints in determining the structure of interval choice functions that satisfy *strategy-proofness*.

3.1 A Restricted Framework: Extending a Status Quo Interval

We begin by considering a natural restricted environment in which the range of socially admissible values of a variable is determined by a status quo interval $q \in \mathcal{I}$. The interval q serves as the default outcome, and collective decisions may only expand it, never contract it.

Let $\mathcal{I}(q)$ denote the set of all intervals containing the status quo interval q:

$$\mathcal{I}(q) = \left\{x \in \mathcal{I} : \underline{x} \leq \underline{q},\ \bar{x} \geq \bar{q}\right\}.$$

The set $\mathcal{I}(q)$ can be identified with $(-\infty, \underline{q}] \times [\bar{q}, \infty) \subset \mathbb{Z} \times \mathbb{Z}$. Geometrically, it is an orthant with vertex at $(\underline{q}, \bar{q})$ (Figure 1).

A natural and relevant restriction on preferences over multidimensional outcome sets such as $\mathcal{I}(q)$ is single-peakedness. In our interval selection framework, single-peakedness requires that, given a preference $\succsim_i \in \mathcal{D}$ and two intervals $x, y \in \mathcal{I}(q)$, if x is unambiguously closer than y to the most-preferred interval $b(\succsim_i, \mathcal{I}(q))$, then $x \succ_i y$. To formalize this notion, we introduce additional notation.

For each $x, y \in \mathcal{I}$, let the ***minimal box containing*** x ***and*** y be

$$\mathcal{B}(x, y) = \left\{x' \in \mathcal{I} :\ \min\{\underline{x}, \underline{y}\} \leq \underline{x}' \leq \max\{\underline{x}, \underline{y}\}; \min\{\bar{x}, \bar{y}\} \leq \bar{x}' \leq \max\{\bar{x}, \bar{y}\}\right\}.$$

The set $\mathcal{B}(x, y)$ consists of all intervals lying coordinatewise between x and y.

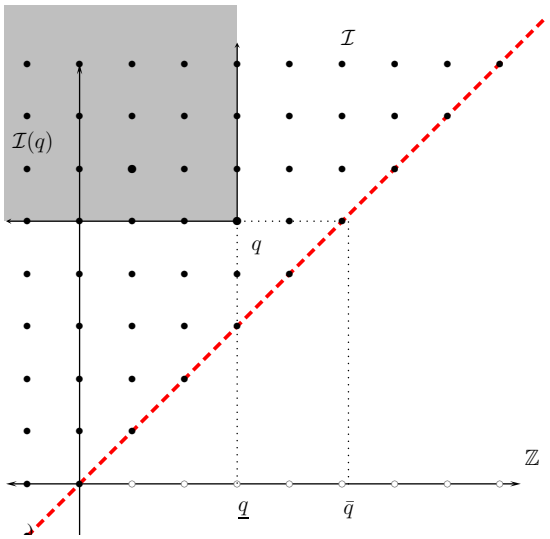


Figure 1: $\mathcal{I}$ and $\mathcal{I}(q)$.

A preference $\succsim_i \in \mathcal{D}$ is ***intervalwise single-peaked on a set*** $\mathcal{X} \subseteq \mathcal{I}$ if for each pair of distinct intervals $x, y \in \mathcal{X}$,

$$x \in \mathcal{B}(b(\succsim_i, \mathcal{X}), y) \quad \Longrightarrow \quad x \succ_i y.$$

Intervalwise single-peakedness requires that preferences evolve monotonically toward the most-preferred interval along each coordinate dimension.

The existence of a status quo interval naturally restricts the set of admissible preferences. If collective decisions may only expand the status quo interval, then an agent's most-preferred interval must either contain the status quo interval or be sufficiently close to it so that their union is itself an interval. This motivates the following domain restriction.

Let $\mathcal{D}^M(q)$ denote the domain of interval-monotonic preferences such that

$$b(\succsim_i) \cup q \in \mathcal{I},$$

and let $\mathcal{D}^A(q) = \mathcal{D}^A \cap \mathcal{D}^M(q)$. The domain $\mathcal{D}^M(q)$ consists of preferences for which either every element of the status quo interval is acceptable, or the interval of acceptable integers intersects or is adjacent to the status quo interval.

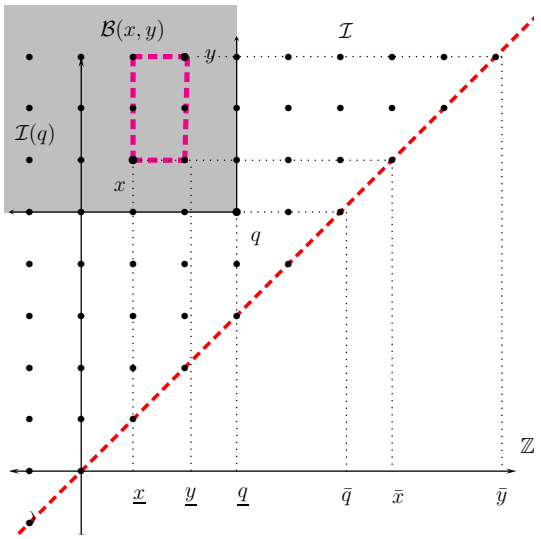


Figure 2: Intervalwise single-peakedness, $x = b(\succsim_i) = b(\succsim_i, \mathcal{I}(q))$, for each $x' \in \mathcal{B}(x,y)$, $x' \succ_i y$.

Remark 2. *For each $q \in \mathcal{I}$ and each $\succsim_i \in \mathcal{D}^M(q)$, the most-preferred interval in $\mathcal{I}(q)$ is obtained by taking the union of the voter's most-preferred interval with the status quo interval:*

$$b(\succsim_i, \mathcal{I}(q)) = b(\succsim_i) \cup q.$$

In fact, for each $y \in \mathcal{I}(q)$, every interval in the minimal box $\mathcal{B}(y, b(\succsim_i, \mathcal{I}(q)))$ is obtained from y by adding acceptable integers, removing unacceptable integers, or both. Therefore, by monotonicity with respect to the inclusion of acceptable integers and the exclusion of unacceptable ones, every interval in $\mathcal{B}(y, b(\succsim_i, \mathcal{I}(q)))$ is strictly preferred to y (Figure 2).

Remark 3. *Let $q \in \mathcal{I}$, for each $\succsim_i \in \mathcal{D}^M(q)$ is intervalwise single-peaked on $\mathcal{I}(q)$.*

The previous property may fail for preferences outside the domain $\mathcal{D}^M(q)$. The following example shows that intervalwise single-peakedness need not hold when there are unacceptable integers between the status quo interval and a voter's most-preferred interval.

Example 1. *Let $q = [0]$, and let $\succsim_i \in \mathcal{D}^A$ be represented by a utility function $u_i : \mathbb{Z} \to \mathbb{R}$ such that $u_i(-2) = 1$, for each $z \in [-1,1]$, $u_i(z) = -0.25$ and, for each $z' \notin [-2,1]$, $u_i(z') < -2$. Then*

$$b(\succsim_i) = [-2] \quad \textit{and} \quad b(\succsim_i, \mathcal{I}(q)) = [-2,0].$$

Let $y = [0,1]$. Observe that $[-1,1] \in \mathcal{B}([-2,0],[0,1])$, but $[0,1] \succ_i [-1,1]$. Hence, $\succsim_i$ is not intervalwise single-peaked on $\mathcal{I}(q)$. (Figure 3).

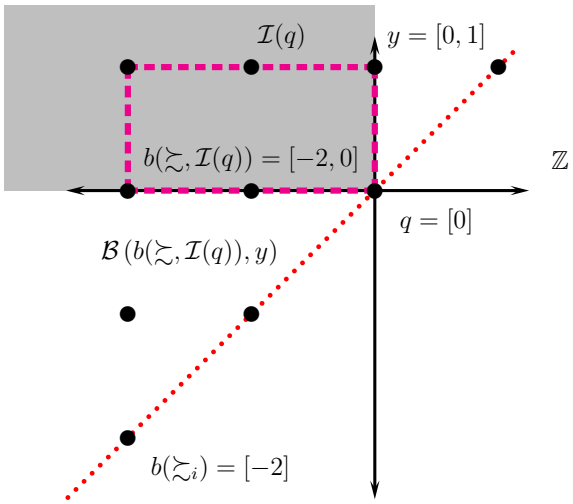


Figure 3: Example 1, $b(\succsim_i, \mathcal{I}(q)) = [-2,0]$, $y = [0,1]$, $[-1,1] \in \mathcal{B}([-2,0],[0,1])$ but $y \succ_i [-1,1]$.

Intervalwise single-peakedness on $\mathcal{I}(q)$ implies that, once one endpoint of an interval is fixed, preferences over the remaining endpoint exhibit a one-dimensional single-peaked structure. This decomposition suggests studying interval choice functions that determine the inferior and superior endpoints of the selected interval separately. Since strategy-proof social choice functions on one-dimensional single-peaked domains are characterized by generalized median voter schemes (Moulin, 1980; Barberà *et al.*, 1993), it is natural to extend this class of rules to the present interval selection framework.

For each preference $\succsim_i \in \mathcal{D}$ and each $q \in \mathcal{I}$, let $\underline{\pi}_q(\succsim_i)$ and $\bar{\pi}_q(\succsim_i)$ denote respectively the inferior and superior endpoints of the most-preferred feasible interval $b(\succsim_i, \mathcal{I}(q))$.

Let $q \in \mathcal{I}$ and $\widetilde{\mathcal{D}} \subseteq \mathcal{D}$. An interval choice function $f : \widetilde{\mathcal{D}}^N \to \mathcal{I}(q)$ is a ***min-max extreme points function*** if there exist two families of parameters $\{\alpha_S\}_{S\subseteq N}$ and $\{\beta_S\}_{S\subseteq N}$ such that:

i) for each $S, S' \subseteq N$, $S \subseteq S'$ implies $\alpha_S \leq \alpha_{S'}$ and $\beta_S \leq \beta_{S'}$;

ii)
$$\alpha_S \leq \underline{q} \quad \text{for all } S \subseteq N, \qquad \alpha_N = -\infty, \qquad \alpha_\varnothing = \underline{q};$$

iii)
$$\beta_S \geq \bar{q} \quad \text{for all } S \subseteq N, \qquad \beta_N = \bar{q}, \qquad \beta_\varnothing = +\infty;$$

iv) for each preference profile $\succsim \in \widetilde{\mathcal{D}}^N$,

$$\begin{aligned}\inf f(\succsim) &= \inf_{S\subseteq N}\left\{\sup_{i\in S}\{\underline{\pi}_q(\succsim_i), \alpha_S\}\right\}, \quad \text{and}\\ \sup f(\succsim) &= \inf_{S\subseteq N}\left\{\sup_{i\in S}\{\bar{\pi}_q(\succsim_i), \beta_S\}\right\}.\end{aligned}$$

Min-max extreme points functions encompass a broad class of strategy-proof aggregation procedures. Different specifications of the parameters $\{\alpha_S\}_{S\subseteq N}$ and $\{\beta_S\}_{S\subseteq N}$ give rise to familiar classes of interval choice functions, including generalized median-type procedures and asymmetric schemes reflecting institutional constraints or veto power. The following examples illustrate the flexibility of this class of interval choice functions.

For each $q \in \mathcal{I}$ and each $\succsim \in \mathcal{D}^N$, let $\underline{\pi}_q^1(\succsim) = \inf_{i\in N}\{\underline{\pi}_q(\succsim_i)\}$, $\underline{\pi}_q^n(\succsim) = \sup_{i\in N}\{\underline{\pi}_q(\succsim_i)\}$, $\bar{\pi}_q^1(\succsim) = \inf_{i\in N}\{\bar{\pi}_q(\succsim_i)\}$, $\bar{\pi}_q^n(\succsim) = \sup_{i\in N}\{\bar{\pi}_q(\succsim_i)\}$. Thus, $\underline{\pi}_q^1(\succsim)$ and $\underline{\pi}_q^n(\succsim)$ denote, respectively, the smallest and largest inferior endpoints among the voters' most-preferred feasible intervals. Analogously, $\bar{\pi}_q^1(\succsim)$ and $\bar{\pi}_q^n(\succsim)$ denote the smallest and largest superior endpoints.

Example 2. *Let $q \in \mathcal{I}$ and define the interval choice functions $f^{minc} : \mathcal{D}^N \to \mathcal{I}(q)$ and $f^{mexp} : \mathcal{D}^N \to \mathcal{I}(q)$ in such a way that for each $\succsim \in \mathcal{D}^N$, $f^{minc}(\succsim) = \left[\underline{\pi}_q^1(\succsim), \bar{\pi}_q^n(\succsim)\right]$, and $f^{mexp}(\succsim) = \left[\underline{\pi}_q^n(\succsim), \bar{\pi}_q^1(\succsim)\right]$.*

The interval choice function f^{minc} selects the minimal interval (with respect to inclusion) containing the most-preferred feasible intervals of all voters. In contrast, f^{mexp} selects the minimal expansion of the status quo interval q contained in the most-preferred feasible interval of all voters.

It is straightforward to verify that f^{minc} is a min-max extreme points function with parameter families $\{\alpha_S\}_{S\subseteq N}$ and $\{\beta_S\}_{S\subseteq N}$ such that

$$\alpha_S = -\infty \quad \textit{for each } S \neq \varnothing, \qquad \alpha_\varnothing = \underline{q}, \quad \textit{and}$$

$$\beta_S = \bar{q} \quad \textit{for each } S \neq \varnothing, \qquad \beta_\varnothing = +\infty.$$

Analogously, f^{mexp} is a min-max extreme points function with parameter families $\{\alpha'_S\}_{S\subseteq N}$ and $\{\beta'_S\}_{S\subseteq N}$ such that

$$\alpha'_N = -\infty, \qquad \alpha'_S = \underline{q} \quad \textit{for each } S \neq N, \textit{ and}$$

$$\beta'_N = \bar{q}, \qquad \beta'_S = +\infty \quad \textit{for each } S \neq N.$$

Min-max extreme points functions also allow for asymmetric aggregation procedures. For instance, some voters may determine one or both endpoints of the selected interval.

An interval choice function $f : \widetilde{\mathcal{D}}^N \to \mathcal{I}(q)$ is an ***endpoint dictatorial function*** if there exist voters $i, j \in N$ such that, for each preference profile $\succsim \in \widetilde{\mathcal{D}}^N$,

$$f(\succsim) = [\underline{\pi}_q(\succsim_i), \bar{\pi}_q(\succsim_j)].$$

An interval choice function is ***dictatorial*** if there exists a voter $i \in N$, called the ***dictator***, such that for each preference profile $\succsim \in \widetilde{\mathcal{D}}^N$, $f(\succsim) = b(\succsim_i, r_f)$.

Note that every endpoint dictatorial function is a min-max extreme points function with parameter families satisfying:

$$\alpha_S = \begin{cases} -\infty & \text{if } i \in S, \\ \underline{q} & \text{otherwise,} \end{cases}$$

and

$$\beta_S = \begin{cases} \bar{q} & \text{if } j \in S, \\ +\infty & \text{otherwise.} \end{cases}$$

Every dictatorial function is an endpoint dictatorial function.

Our first result characterizes min-max extreme points functions.

Theorem 1. *Let $q \in \mathcal{I}$, and let $\widetilde{\mathcal{D}} \subset \mathcal{D}$ be such that $\mathcal{D}^A(q) \subseteq \widetilde{\mathcal{D}} \subseteq \mathcal{D}^M(q)$. Let $f : \widetilde{\mathcal{D}}^N \to \mathcal{I}(q)$ be an interval choice function such that $r_f = \mathcal{I}(q)$. Then f satisfies* strategy-proofness *if and only if f is a min-max extreme points function.*

The sufficiency is immediate. For every $q \in \mathcal{I}$, every min-max extreme points function satisfies *strategy-proofness* on $\mathcal{D}^M(q)$ and has range $\mathcal{I}(q)$. Indeed, a unilateral misreport can only induce the inclusion of unacceptable integers or the exclusion of acceptable integers from the selected interval.

The converse implication is more involved. Its proof exploits the fact that interval choice can be viewed as a two-dimensional choice problem, where the inferior and superior endpoints are selected separately. We first restrict attention to the subdomain $\mathcal{D}^A(q)$ of additively representable preferences. On this domain, preferences are separable and therefore induce unique marginal preference relations over the sets of feasible inferior and superior endpoints. Moreover, these marginal preferences are one-dimensional single-peaked, with peaks given by the

endpoints of $b(\succsim_i, \mathcal{I}(q))$. Hence, $\mathcal{D}^A(q)$ satisfies the richness assumptions of Le Breton and Sen (1999), and Le Breton and Sen (1999, Theorem 4.1) implies that every strategy-proof and unanimous interval choice function can be decomposed into two independent one-dimensional social choice functions. Since these functions are defined on single-peaked domains, they are generalized median voter schemes (Moulin, 1980; Barberà *et al.*, 1993), and therefore induce a min-max extreme points function. Finally, we extend the characterization from $\mathcal{D}^A(q)$ to the whole domain $\mathcal{D}^M(q)$ by showing that *strategy-proofness* implies that the outcome depends only on each voter's most-preferred feasible interval.

The characterization obtained in Theorem 1 relies crucially on the restriction to the domain $\mathcal{D}^M(q)$. Outside this domain, a voter's most-preferred interval may neither intersect nor be adjacent to the status quo interval. Consequently, the most-preferred feasible interval may contain unacceptable integers or fail to contain all acceptable ones. The following Remark 4 describes the structure of the most-preferred feasible interval for every additively representable preference.

For two intervals $x, y \in \mathcal{I}$, let $ch(x, y) \in \mathcal{I}$ denote the convex hull of x and y, that is the smallest interval containing both x and y.

Remark 4. *For each* $\succsim_i \in \mathcal{D}^A$*:*

- *If* $\succsim_i \in \mathcal{D}^A(q)$*, then* $b(\succsim_i, \mathcal{I}(q)) = ch(b(\succsim_i), q) = b(\succsim_i) \cup q$*.*
- *If* $\succsim_i \notin \mathcal{D}^A(q)$*, then either* $b(\succsim_i, \mathcal{I}(q)) = ch(b(\succsim_i), q)$*, or* $b(\succsim_i, \mathcal{I}(q)) = q$*.*

When a voter's most-preferred interval is separated from the status quo interval by unacceptable integers, intervalwise single-peakedness on the feasible set may fail (Example 1). Moreover, even under the restriction of additively representable preferences, it is not possible to directly appeal to the decomposability result by Le Breton and Sen (1999). Additively representable preferences in $\mathcal{D}^A$ are separable and associated marginal preference relations over inferior and superior endpoints are well defined. However, $\mathcal{D}^A$ is not sufficiently rich for the direct application of the decomposition theorem. Specifically, there are pairs of admissible associated marginal preferences over the inferior and superior endpoints of intervals that cannot be induced by any specific preference in $\mathcal{D}^A$. Nevertheless, *strategy-proofness* continues to impose strong restrictions on admissible interval choice functions. Although the decomposition theorem of Le Breton and Sen (1999) cannot be applied directly, a multidimensional decomposition still emerges from *strategy-proofness*. However, the class of admissible interval choice functions

collapses dramatically. In the proof we analyze specific subdomains of additively representable preferences in which voters care only about one endpoint of the interval. On such subdomains, the induced preferences over the relevant endpoint contain all single-peaked and single-dipped preferences. By single-dipped preferences, we mean one-dimensional preference relations having a unique worst endpoint, so that moving the endpoint in either direction increase the voter's welfare. *Strategy-proofness* then forces a dictatorial determination of that endpoint on preference profiles in the subdomain. This result then can be extended to the full domain of additively representable interval-monotonic preferences. Our next result shows that for additively representable preferences only extreme dictatorial functions survive. Surprisingly, the possibility result cannot be extended to interval-monotonic preferences. Since interval-monotonic preferences admit complementarities among integers, the choice made by the voter that determines the inferior endpoint may depend on the choice made by the voter that determines the superior endpoint. This fact implies that only dictatorial functions survive on the full domain of interval-monotonic preferences.

Theorem 2. *Let $q \in \mathcal{I}$, and let f be an interval choice function with range $r_f = \mathcal{I}(q)$.*

i) If $f : (\mathcal{D}^A)^N \to \mathcal{I}(q)$, then f satisfies strategy-proofness *if and only if it is an endpoint dictatorial function.*

ii) If $f : (\mathcal{D}^M)^N \to \mathcal{I}(q)$, then f satisfies strategy-proofness *if and only if it is dictatorial.*

3.2 The Unrestricted Framework

When the range of an interval choice function is unrestricted, the feasible set $\mathcal{I}$ no longer has the orthant structure that underlies the analysis of the previous subsection. As a consequence, the multidimensional decomposition exploited in the restricted framework is no longer available, and the characterization of strategy-proof interval choice functions requires a different argument.

Two features of the unrestricted framework are particularly important. First, allowing any interval in $\mathcal{I}$ to be selected enlarges the set of feasible outcomes from the orthant $\mathcal{I}(q)$ to the whole set $\mathcal{I}$. One might therefore expect that the additional flexibility would allow for a larger class of strategy-proof interval choice functions. Surprisingly, the opposite is true: *strategy-proofness* imposes substantially stronger restrictions. Second, both min-max extreme points functions and endpoint dictatorial functions are intrinsically tied to the existence of a status

quo interval. One might attempt to extend these notions by replacing the endpoints of the most-preferred feasible interval with the endpoints of each voter's interval of acceptable integers. However, such an extension is not well defined. Indeed, if voters i and j determine respectively the inferior and superior endpoints of the selected interval, there may exist a profile $\succsim \in (\mathcal{D}^A)^N$ such that

$$\inf b(\succsim_i) > \sup b(\succsim_j).$$

In this case, the endpoints selected by the two decisive voters do not define a feasible interval. Hence, the endpoint-by-endpoint decomposition underlying min-max extreme points and endpoint dictatorial functions is no longer available in the unrestricted framework.

Even when a min-max extreme points function is well defined on the unrestricted framework, it need not satisfy *strategy-proofness*.

Example 3. *Let $N = \{i, j\}$ and consider the interval choice function $f^{ext} : (\mathcal{D}^A)^N \to \mathcal{I}$ such that for each $\succsim \in (\mathcal{D}^A)^N$:*

$$\inf f(\succsim) = \inf_{i \in N}\{\inf b(\succsim_i)\},$$
$$\sup f(\succsim) = \sup_{i \in N}\{\sup b(\succsim_i)\}.$$

Consider $\succsim \in (\mathcal{D}^A)^N$ such that $b(\succsim_i) = [-1]$, $b(\succsim_j) = [1]$, and $\succsim_i$ is represented by the utility function u_i such that for some $\alpha \in \mathbb{R}_{++}$, $u_i(-1) = \alpha$, $u_i(0) = -4\alpha$, $u_i(1) = -\alpha$, and for each $z \notin [-1, 1]$, $u_i(z) < -10\alpha$. Let $\succsim_i' \in \mathcal{D}^A$ be such that $b(\succsim_i') = [1]$. Then $f^{ext}(\succsim_i', \succsim_j) = [1] \succ_i f^{ext}(\succsim) = [-1, 1]$, showing that f^{ext} violates strategy-proofness. *Observe that $[-1, 1] \in \mathcal{B}([1], b(\succsim_i))$ but $[1] \succ_i [-1, 1]$. Hence, although $\succsim_i \in \mathcal{D}^A$, it is not intervalwise single-peaked on $\mathcal{I}$ (Figure 4).*

Example 3 illustrates why the implications of intervalwise single-peakedness established in Section 3.1 cannot be extended to general interval-monotonic preferences. If no status quo interval is imposed, a voter may be willing to exclude some acceptable integers from the selected interval whenever doing so also allows the exclusion of sufficiently unacceptable integers. Consequently, the monotonicity arguments underlying the characterization of min-max extreme points functions no longer guarantee *strategy-proofness*. In fact, only dictatorial interval choice functions satisfy *strategy-proofness* and *ontoness*.

Theorem 3. *Let $\mathcal{D}^A \subseteq \widetilde{\mathcal{D}} \subseteq \mathcal{D}^M$. An interval choice function $f : \widetilde{\mathcal{D}}^N \to \mathcal{I}$ satisfies* strategy-proofness *and* ontoness *if and only if f is dictatorial.*

Extending the intuition of Example 3 to the general domain of additively representable preferences requires arguments that go beyond a direct application of the Gibbard–Satterthwaite

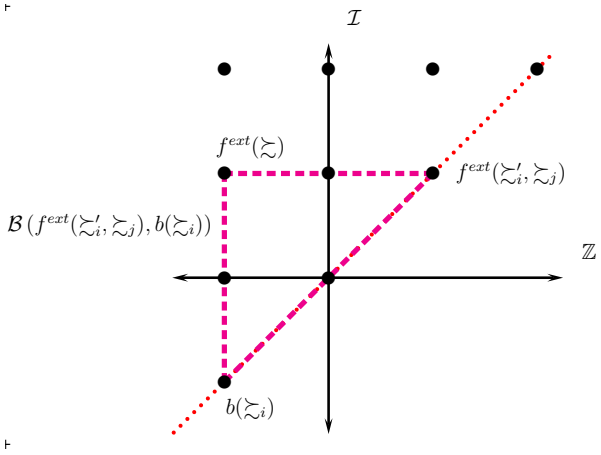


Figure 4: Example 3. The manipulation by voter i changes the outcome from $f^{ext}(\succsim) = [-1, 1]$ to $f^{ext}(\succsim_i', \succsim_j) = [1]$, even though $f^{ext}(\succsim) \in \mathcal{B}\big(f^{ext}(\succsim_i', \succsim_j), b(\succsim_i)\big)$.

Theorem.[4] Although interval-monotonic preferences impose no restrictions on how voters rank singleton intervals, they impose restrictions on the ranking of larger intervals. In particular, for every preference $\succsim_i \in \mathcal{D}^M$ and every integer z such that $[z] \succ_i \varnothing$, monotonicity implies that

$$[z, z+1] \succ_i [z+1].$$

Hence, preference orderings over intervals cannot be arbitrary. Consequently, standard dictatorship theorems do not apply directly.

Since the feasible set $\mathcal{I}$ cannot be represented as a multidimensional product space, the arguments used in the proofs of Theorems 1 and 2 are no longer available. Instead, we follow the option-set approach developed by Barberà and Peleg (1990). For each voter $i \in N$ and each profile of preferences $\succsim_{-i}$ reported by the remaining voters, the option set of voter i is the set of all intervals that can be selected by the interval choice function when voter i varies her reported preference. We begin by restricting attention to the domain of additively representable preferences. The proof starts with two-voter societies and analyzes the structure of voters' option sets. It first considers profiles in which each voter regards a unique singleton interval as acceptable. In this case, we show that all decision-making power is concentrated in a single voter. We then extend this decisiveness result to arbitrary additively representable preferences. An induction argument on the number of voters establishes dictatorship for all societies. Finally, we show

[4] As shown by the positive results in Theorem 1, the domain of interval-monotonic preferences does not satisfy standard richness domain conditions used in dictatorial results, such as the linked domain condition of Aswal *et al.* (2003).

that the characterization extends immediately from additively representable preferences to the full domain of interval-monotonic preferences.

4 Discussion

We conclude by considering two extensions of the analysis.

Throughout the paper, intervals are subsets of the set of integers and may contain an arbitrary finite number of elements. Suppose instead that the set of admissible values is a finite ordered set and that feasible alternatives are unions of consecutive categories. The interval selection problem can then be formulated as the choice of intervals in a finite ordered set. This setting is particularly relevant when admissible values represent ordered classifications rather than numerical magnitudes. The definitions of interval-monotonic and additively representable preferences extend directly to this setting. Most of our results continue to hold, with only minor qualifications depending on the number of available categories. The characterization in Theorem 1 remains valid. However, if the status quo interval can be expanded by at most one category in each direction, then $\mathcal{D}^A = \mathcal{D}^A(q)$. In this case, the unrestricted domain considered in Theorem 2 coincides with the restricted domain of Theorem 1. Hence, Theorem 2 requires the status quo interval not to lie too close to the boundaries of the set of admissible categories. Finally, the dictatorship result in Theorem 3 extends provided that the set of admissible categories contains at least four elements. When there are only three categories, non-dictatorial interval choice functions satisfying *strategy-proofness* and *ontoness* do exist. Nevertheless, these rules still concentrate all effective decision-making power in a single voter.

Example 4. *Let* $\mathcal{I}^* = \{[-1], [0], [1], [-1, 0], [0, 1], [-1, 1]\}$*, and let* $\widehat{\mathcal{D}}^A$ *denote the set of all additively representable preferences on* $\mathcal{I}^*$*. Let* $f^* : (\widehat{\mathcal{D}}^A)^{\{i,j\}} \to \mathcal{I}^*$ *be such that for each* $\succsim \in (\widehat{\mathcal{D}}^A)^{\{i,j\}}$*:*

$$f^*(\succsim) = \begin{cases} b(\succsim_i) \cup [0] & \text{if } [0] \subseteq b(\succsim_j), \\ b(\succsim_i) & \text{otherwise.} \end{cases}$$

It is immediate to verify that f^* *satisfies* strategy-proofness *and* unanimity*, and hence* ontoness*. Moreover,* f^* *is not dictatorial.*

We conclude by discussing an extension of our analysis to a continuous framework. Although our model is formulated on the set of integers, the ideas underlying the proofs of Theorems 1 and 3 extend naturally to interval selection problems on the real line. In this setting,

admissible alternatives are closed intervals whose length (according to the Lebesgue measure) is bounded below by a strictly positive constant. This lower bound plays a technical role by ensuring that every interval choice function that satisfies *strategy-proofness* has closed range. The analogue of additive separability can be formulated by requiring that voters' preferences are representable by Riemann integrable utility functions whose values are negative outside the voter's most-preferred interval. The continuous framework raises different technical difficulties for Theorems 1 and 3. In the extension of Theorem 1, since the arguments of Le Breton and Sen (1999) do not apply to domains with indifferences, we rely on the results in the literature on decomposable strategy-proof social choice functions initiated by Border and Jordan (1983); Chilchinisky and Heal (1997). We first focus on the subdomain of quadratic separable preferences, for which Border and Jordan (1983) show that every interval choice function satisfying *strategy-proofness* and *ontoness* depends only on the voters' sets of most-preferred intervals. Using the decomposition theorem of Weymark (1999), we then show that every interval choice function that satisfies *strategy-proofness* and *ontoness* is decomposable into two rules that select the endpoints of the interval. For the counterpart of Theorem 3, the continuous framework implies that the set of social choice alternatives is a metric space and we can apply the option sets approach developed by Barberà and Peleg (1990). As in Barberà and Peleg (1990); Le Breton and Weymark (1999), proving that the range of every interval choice function that satisfies *strategy-proofness* is closed is crucial. The assumption that admissible intervals have a strictly positive minimal length is precisely what guarantees this closedness property.

5 Proofs

5.1 Preliminary results

We present first some auxiliary results that are useful in the proofs of our characterizations. We start with the proof of Lemma 1.

Proof of Lemma 1. Let $x \in r_f$. Then there exists $\succsim^x \in \widetilde{\mathcal{D}}^N$ such that $f(\succsim^x) = x$. Suppose, to the contrary, that there exists $\succsim \in \widetilde{\mathcal{D}}^N$ such that for each $i \in N$, $b(\succsim_i, r_f) = x$, but $f(\succsim) \neq x$. Fix $i \in N$. By *strategy-proofness*, $f(\succsim) \succsim_i f(\succsim_i^x, \succsim_{-i})$. Since $f(\succsim_i^x, \succsim_{-i}) \in r_f$ and $x \succ_i f(\succsim)$, we have $f(\succsim_i^x, \succsim_{-i}) \neq x$. Repeating this argument successively for each voter, we obtain $f(\succsim^x) \neq x$, a contradiction. □

In our proofs, we work extensively with the domain of additively representable preferences.

We first establish some auxiliary results that will be used throughout the proofs of our main results. The first lemma guarantees the existence of specific additively representable preferences.

Lemma 2. *Let $z, z' \in \mathbb{Z}$, $z \neq z'$.*

i) If $z \notin [z'-1, z'+1]$, then there is $\succsim_i \in \mathcal{D}^A$ such that for each $x \in \mathcal{I} \setminus \{[z], [z']\}$:

$$[z] \succ_i [z'] \succ_i x.$$

ii) If $z \in [z'-1, z'+1]$, then there is $\succsim_i \in \mathcal{D}^A$ such that for each $x \in \mathcal{I} \setminus \{[z], [z'], [z] \cup [z']\}$:

$$[z] \succ_i [z] \cup [z'] \succ_i [z'] \succ_i x.$$

iii) There is $\succsim_i \in \mathcal{D}^A$ such that for each $x, x' \in \mathcal{I}$, $z \in x$, if $z \notin x'$, then $x \succ_i x'$.

iv) There is $\succsim_i \in \mathcal{D}^A$ such that for each $y, y' \in \mathcal{I}$, if $z \notin y$ and $z \in y'$, then $y \succ_i y'$.

v) For each $x \in \mathcal{I}$ with $z \in x$, there is $\succsim_i \in \mathcal{D}^A$ such that $b(\succsim_i) = [z]$ and for each $y, y' \in \mathcal{I}$, if $y \subseteq x$ and y' contains an integer outside x, then $y \succ_i y'$.

Proof. i) Consider $\succsim_i \in \mathcal{D}^A$ represented by a utility function u_i such that, for some $\alpha \in \mathbb{R}_{++}$, $u_i(z) = \alpha$, $u_i(z') = -\alpha$, and for each $z'' \in \mathbb{Z} \setminus \{z, z'\}$, $u_i(z'') < -2\alpha$. Note that $[z] \cup [z'] \notin \mathcal{I}$. Hence, for each $x \in \mathcal{I} \setminus \{[z], [z']\}$:

$$u_i(z) > u_i(z') > \sum_{z'' \in x} u_i(z'').$$

ii) Consider $\succsim_i in \mathcal{D}^A$ represented by the same utility function as in part i) except that $[z] \cup [z'] \in \mathcal{I}$. Hence, for each $x \in \mathcal{I} \setminus \{[z], [z'], [z] \cup [z']\}$:

$$u_i(z) > u_i(z) + u_i(z') > u_i(z') > \sum_{z'' \in x} u_i(z'').$$

iii) Consider $\succsim_i \in \mathcal{D}^A$ represented by a utility function u_i such that, for some $\alpha \in (0,1)$, $u_i(z) = \alpha$, for each $z'' < z$, $|u_i(z'')| < \frac{1}{6(1-\alpha)}(\alpha^{z-z''})$ and for each $z'' > z$, $|u_i(z'')| < \frac{1}{\alpha}(\alpha^{z''-z})$. Note that $u_i(z) > 2\sum_{z'' \in \mathbb{Z} \setminus \{z\}} u_i(z'')$. Hence, if $z \in x$ and $z \notin x'$, then

$$\sum_{\zeta \in x} u_i(\zeta) > \sum_{\zeta \in x'} u_i(\zeta).$$

iv) This follows by reversing the construction in part iii).

v) Consider $\succsim_i \in \mathcal{D}^A$ represented by a utility function u_i such that, for some $\alpha \in \mathbb{R}_{++}$, $u_i(z) = \alpha$, for each $z' \in x \setminus \{z\}$, $-\frac{\alpha}{\bar{x}-\underline{x}+1} < u_i(z') < 0$, and for each $z'' \notin x$, $u_i(z'') < -2\sum_{\zeta \in x} |u_i(\zeta)|$. Hence, for each $y \subseteq x$ and each $y' \in \mathcal{I}$ containing an integer outside x,

$$\sum_{\zeta \in y} u_i(\zeta) > -\alpha > -3\alpha > \sum_{\zeta' \in y'} u_i(\zeta').$$

□

The next two lemmata establish restrictions on the endpoints of the interval selected by interval choice functions satisfying *strategy-proofness* and *ontoness*. The first result concerns the unrestricted framework, while the second presents its counterpart for the restricted framework.

For each $\succsim \in \mathcal{D}^N$, let $\underline{\pi}^1(\succsim) = \inf_{i \in N} \{\underline{\pi}(\succsim_i)\}$, $\underline{\pi}^n(\succsim) = \sup_{i \in N} \{\underline{\pi}(\succsim_i)\}$, $\bar{\pi}^1(\succsim) = \inf_{i \in N} \{\bar{\pi}(\succsim_i)\}$, and $\bar{\pi}^n(\succsim) = \sup_{i \in N} \{\bar{\pi}(\succsim_i)\}$.

Lemma 3. *Let $\mathcal{D}^A \subseteq \widetilde{\mathcal{D}} \subseteq \mathcal{D}^M$ and $f : \widetilde{\mathcal{D}}^N \to \mathcal{I}$. If f satisfies* strategy-proofness *and* ontoness, *then for each $\succsim \in \widetilde{\mathcal{D}}^N$:*

$$\underline{\pi}^1(\succsim) \leq \inf f(\succsim) \leq \underline{\pi}^n(\succsim),$$
$$\bar{\pi}^1(\succsim) \leq \sup f(\succsim) \leq \bar{\pi}^n(\succsim),$$

Proof. Suppose, to the contrary, that there exists $\succsim \in \widetilde{\mathcal{D}}^N$ such that $\inf f(\succsim) < \underline{\pi}^1(\succsim)$. Consider the interval $\left[\underline{\pi}^1(\succsim), \sup f(\succsim)\right] \in \mathcal{I}$. Since $\succsim \in (\mathcal{D}^M)^N$, for each $i \in N$, $[\underline{\pi}^1(\succsim), \sup f(\succsim)] \succ_i f(\succsim)$.

Let $\succsim' \in (\mathcal{D}^A)^N$ be such that, for every $i \in N$, $[\underline{\pi}^1(\succsim), \sup f(\succsim)]$ is her most-preferred interval. Moreover, intervals extending beyond $\sup f(\succsim)$ are extremely undesirable, and among the intervals with upper endpoint $\sup f(\succsim)$, those containing $\underline{\pi}^1(\succsim)$ are strictly preferred. More precisely,

- $b(\succsim_i') = [\underline{\pi}^1(\succsim), \sup f(\succsim)]$.

- For each $x, y \in \mathcal{I}$ such that $x \subseteq [\underline{\pi}^1(\succsim), \sup f(\succsim)]$, if $\bar{y} > \sup f(\succsim)$, then $x \succ_i' y$-

- For each $x', y' \in \mathcal{I}$ such that $\sup\{\bar{x}', \bar{y}'\} \leq \sup f(\succsim)$, if $\bar{y}' < \bar{x}' = \sup f(\succsim)$ then $x' \succ_i' y'$.

- For each $x'', y'' \in \mathcal{I}$, with $\bar{x}'' = \bar{y}'' = \sup f(\succsim)$. $\underline{\pi}^1(\succsim) \in x''$, if $\underline{\pi}^1(\succsim) \notin y''$ then $x'' \succ_i' y''$.

The existence of such a profile $\succsim' \in (\mathcal{D}^A)^N$ is guaranteed because, for each $i \in N$, there exists $\succsim_i'$ represented by a utility function u_i such that:

- For each $z \in \mathbb{Z}$ such that $z > \sup f(\succsim)$, $u_i(z) < -\sum_{z' \in [\underline{\pi}^1(\succsim), \sup f(\succsim)]} u_i(z') + \sum_{z' < \underline{\pi}^1(\succsim)} u_i(z')$,

- For each $z' \in \mathbb{Z}$, such that $z' < \underline{\pi}^1(\succsim)$, $u_i(z') = -(\frac{1}{2})^{(\underline{\pi}^1(\succsim) - 1 - z')}$.

- $\sum_{z'' \in [\underline{\pi}^1(\succsim)+1, \sup f(\succsim)-1]} u_i(z'') < \frac{1}{2}$.

- $u_i(\underline{\pi}^1(\succsim)) > \frac{1}{2} + 2$.

- $u_i(\sup f(\succsim)) > 2(\sum_{z \in [\underline{\pi}^1(\succsim), \sup f(\succsim)-1]} u_i(z) - \sum_{z' < \underline{\pi}^1(\succsim)} u_i(z'))$.

The geometric decay below the inferior endpoint ensures that the total disutility generated by integers smaller than $\underline{\pi}^1(\succsim)$ remains bounded. Let $i \in N$. By *strategy-proofness*, $f(\succsim_i', \succsim_{-i}) \succsim_i' f(\succsim)$. Thus, $\sup f(\succsim_i', \succsim_{-i}) = \sup f(\succsim)$, $[\underline{\pi}^1(\succsim), \sup f(\succsim)] \subseteq f(\succsim_i', \succsim_{-i})$, and $f(\succsim_i', \succsim_{-i}) \subseteq f(\succsim)$. Since $\underline{\pi}^1(\succsim) \leq \underline{\pi}(\succsim_i)$, and by *strategy-proofness* $f(\succsim) \succsim_i f(\succsim_i', \succsim_{-i})$, $\inf f(\succsim_i', \succsim_{-i}) < \underline{\pi}^1(\succsim)$, and $f(\succsim_i', \succsim_{-i}) \neq [\underline{\pi}^1(\succsim), \sup f(\succsim)]$. Repeating the argument successively for the remaining voters, we obtain that $\inf f(\succsim') < \underline{\pi}^1(\succsim)$, $\sup f(\succsim') = \sup f(\succsim)$, and $f(\succsim') \neq [\underline{\pi}^1(\succsim), \sup f(\succsim)]$, which contradicts Lemma 1.

The proof of the inequality $\inf f(\succsim) \leq \underline{\pi}^n(\succsim)$ follows by a parallel argument. Suppose, to the contrary, that $\underline{\pi}^n(\succsim) < \inf f(\succsim)$. Consider the interval $[\underline{\pi}^n(\succsim), \sup f(\succsim)] \in \mathcal{I}$. Since $\succsim \in (\mathcal{D}^M)^N$, for each $i \in N$, $[\underline{\pi}^n(\succsim), \sup f(\succsim)] \succ_i f(\succsim)$. Let $\succsim'' \in (\mathcal{D}^A)^N$ be such that for each voter, $[\underline{\pi}^n(\succsim), \sup f(\succsim)]$ is the most-preferred interval. Intervals extending beyond $\sup f(\succsim)$ are strictly worse than intervals contained in $[\underline{\pi}^n(\succsim), \sup f(\succsim)]$, and among intervals with superior endpoint $\sup f(\succsim)$, intervals excluding $\underline{\pi}^n(\succsim)$ are strictly worse than those containing it. More precisely:

- $b(\succsim_i'') = [\underline{\pi}^n(\succsim), \sup f(\succsim)]$,

- for each $x, y \in \mathcal{I}$ such that $x \subseteq [\underline{\pi}^n(\succsim), \sup f(\succsim)]$, $\bar{y} > \sup f(\succsim)$, $x \succ_i'' y$,

- for each $x', y' \in \mathcal{I}$, with $\bar{x}' = \bar{y}' = \sup f(\succsim)$, $\underline{y} < \underline{\pi}^n(\succsim) \leq \underline{x}$ implies $x' \succ_i'' y'$.

The existence of such a profile $\succsim'' \in (\mathcal{D}^A)^N$ is guaranteed since for each $i \in N$ there exists $\succsim_i''$ represented by an utility function u_i such that:

- For each $z \in \mathbb{Z}$ such that $z > \sup f(\succsim)$, $u_i(z) < -\sum_{z' \in [\underline{\pi}^n(\succsim), \sup f(\succsim)]} u_i(z')$,

- For each $z' \in \mathbb{Z}$, such that $z' < \underline{\pi}^n(\succsim)$, $u_i(z') = -2(\frac{1}{2})^{(\underline{\pi}^n(\succsim) - 1 - z')}$.

- $\sum_{z'' \in [\underline{\pi}^n(\succsim), \sup f(\succsim)-1]} u_i(z'') < \frac{1}{2}$.

- $u_i(\sup f(\succsim)) > 2(\sum_{z\in[\underline{\pi}^n(\succsim),\sup f(\succsim)-1]} u_i(z) - \sum_{z'<\underline{\pi}^n(\succsim)} u_i(z'))$.

Using the arguments of the previous paragraph, we obtain that $f(\succsim_i'', \succsim_{-i}) \subsetneq [\underline{\pi}^n(\succsim), \sup f(\succsim)]$, and $\sup f(\succsim) \in f(\succsim_i'', \succsim_{-i})$. Repeating the argument successively for the remaining voters, we obtain that $\underline{\pi}^n(\succsim) < \inf f(\succsim'')$, $\sup f(\succsim'') = \sup f(\succsim)$, and $f(\succsim'') \neq [\underline{\pi}^n(\succsim), \sup f(\succsim)]$, which contradicts Lemma 1.

The remaining inequalities follow from parallel arguments. □

Lemma 4. *Let $f : (\mathcal{D}^A)^N \to \mathcal{I}(q)$. If f satisfies* strategy-proofness *and $r_f = \mathcal{I}(q)$, then for each $\succsim \in (\mathcal{D}^A)^N$:*

$$\inf_{i\in N}\{\inf ch(b(\succsim_i), q)\} \leq \inf f(\succsim),$$
$$\sup_{i\in N}\{\sup ch(b(\succsim_i), q)\} \geq \sup f(\succsim).$$

Proof. The proof follows the same arguments as the proof of Lemma 3. Suppose, to the contrary, that there is $\succsim \in (\mathcal{D}^A)^N$ such that $\inf f(\succsim) < \inf_{i\in N}\{\inf ch(b(\succsim_i), q)\}$. Note that, for each $\succsim_i \in \mathcal{D}^A$, $\inf ch(b(\succsim_i), q) \leq \underline{q}$. Hence, $\inf f(\succsim) < \underline{q}$. Define the interval $x^* \in \mathcal{I}(q)$ by

$$x^* = [\inf_{i\in N}\{\inf ch(b(\succsim_i), q)\}, \sup f(\succsim)].$$

Since $\succsim \in (\mathcal{D}^A)^N \subseteq (\mathcal{D}^M)^N$, and by Remark 4, for each $i \in N$, $x^* \succ_i f(\succsim)$, the arguments in the proof of Lemma3 apply without modification, yielding the desired contradiction. The proof of the second inequality is symmetric. □

5.2 Proof of Theorem 1

To prove the results in Section 3.1, we first restrict attention to the domain of additively representable preferences. The key step in the proof of Theorem 1 is to relate $\mathcal{D}^A(q)$ to a multidimensional single-peaked domain. To this end, we introduce additional notation and definitions connecting our interval selection framework to the decomposition approach for voting rules on multidimensional spaces developed by Le Breton and Sen (1999).

Let $q \in \mathcal{I}$ and $\succsim_i \in \mathcal{D}$. The preference $\succsim_i$ is ***separable*** at q if, for each $a, b \leq \underline{q}$ and each $c, d \geq \bar{q}$:

$$[a, c] \succsim_i [b, c] \iff [a, d] \succsim_i [b, d], \text{ and}$$
$$[a, c] \succsim_i [a, d] \iff [b, c] \succsim_i [b, d].$$

Lemma 5. *For each $q \in \mathcal{I}$, each $\succsim_i \in \mathcal{D}^A$ is separable at q.*

Proof. Let $q = [\underline{q}, \bar{q}] \in \mathcal{I}$. Let $a, b \leq \underline{q}$ and $c, d \geq \bar{q}$. Consider $\succsim_i \in \mathcal{D}^A$, represented by a utility function $u_i : \mathbb{Z} \cup \{\varnothing\} \to \mathbb{R}$. Suppose that $[a, c] \succsim_i [b, c]$. Then,

$$\sum_{z' \in [a,c]} u_i(z') \geq \sum_{z'' \in [b,c]} u_i(z'') \Rightarrow \sum_{z' \in [a,\underline{q}]} u_i(z') \geq \sum_{z'' \in [b,\underline{q}]} u_i(z'') \Rightarrow \sum_{z' \in [a,d]} u_i(z') \geq \sum_{z'' \in [b,d]} u_i(z''),$$

which implies that $[a, d] \succsim_i [b, d]$. The same argument establishes the second separability condition, while the converse implications follow symmetrically. □

Consider a subset of the integers $X \subseteq \mathbb{Z}$ naturally ordered by the relation $\leq$. A preference relation on X is a complete, antisymmetric, and transitive binary relation on X. For each preference R_i, let P_i denote its strict component. Let $\mathcal{R}_X$ denote the set of all preference relations on X. A domain of preferences on X is a subset $\widetilde{\mathcal{R}}(X) \subseteq \mathcal{R}_X$. The corresponding profile spaces are denoted by $(\mathcal{R}_X)^N$ and $(\widetilde{\mathcal{R}}(X))^N$, respectively.

A preference $R_i \in \mathcal{R}_X$ is ***single-peaked*** with respect to $\leq$ if there exists a unique alternative $\tau(R_i) \in X$ such that:

(i) for each $x \in X \setminus \{\tau(R_i)\}$, $\tau(R_i)\ P_i\ x$;

(ii) for each $y, y' \in X \setminus \{\tau(R_i)\}$,

$$y' < y < \tau(R_i) \quad \text{or} \quad \tau(R_i) < y < y' \quad \Longrightarrow \quad y\ P_i\ y'.$$

Let $\mathcal{R}_X^{SP}$ denote the domain of all single-peaked preferences on X with respect to $\leq$.

Each $\succsim_i \in \mathcal{D}^A$ induces two marginal strict preference relations, $P_q^-(\succsim_i)$, $P_q^+(\succsim_i)$ on the sets $\{z \in \mathbb{Z} : z \leq \underline{q}\} = (-\infty, \underline{q}]$ and $\{z' \in \mathbb{Z} : z' \geq \bar{q}\} = [\bar{q}, \infty)$ respectively. By Lemma 5, these relations are well defined and satisfy the following properties:

- For each $z, z' \leq \underline{q}$, $z\ P_q^-(\succsim_i)\ z'$ if and only if $[z, \bar{q}] \succsim_i [z', \bar{q}]$.

- For each $z, z' \geq \bar{q}$, $z\ P_q^+(\succsim_i)\ z'$ if and only if $[\underline{q}, z] \succsim_i [\underline{q}, z']$.

Let $\widetilde{\mathcal{D}} \subseteq \mathcal{D}$ be a domain of separable preferences at q.

Let

$$\mathcal{R}^-_{(-\infty,\underline{q}]}(\widetilde{\mathcal{D}}) = \left\{ R_i \in \mathcal{R}_{(-\infty,\underline{q}]} : \exists\ \succsim_i \in \widetilde{\mathcal{D}} \text{ such that } R_i = P_q^-(\succsim_i) \right\},$$

and

$$\mathcal{R}^+_{[\bar{q},\infty)}(\widetilde{\mathcal{D}}) = \left\{ R_i \in \mathcal{R}_{[\bar{q},\infty)} : \exists\ \succsim_i \in \widetilde{\mathcal{D}} \text{ such that } R_i = P_q^+(\succsim_i) \right\}.$$

That is, $\mathcal{R}^{-}_{(-\infty,\underline{q}]}(\widetilde{\mathcal{D}})$ and $\mathcal{R}^{+}_{[\bar{q},\infty)}(\widetilde{\mathcal{D}})$ are the domains of marginal preference relations over inferior and superior endpoints induced by the preferences in $\widetilde{\mathcal{D}}$.

Lemma 6 shows that each preference in $\mathcal{D}^A(q)$ induces two one-dimensional marginal preference relations that are single-peaked, one over inferior endpoints and one over superior endpoints.

Lemma 6. *For each $\succsim_i \in \mathcal{D}^A(q)$, $P^-_q(\succsim_i) \in \mathcal{R}^{SP}_{(-\infty,\underline{q}]}$ and $P^+_q(\succsim_i) \in \mathcal{R}^{SP}_{[\bar{q},\infty)}$. Moreover, $\tau(P^-_q(\succsim_i)) = \underline{\pi}_q(\succsim_i)$ and $\tau(P^+_q(\succsim_i)) = \bar{\pi}_q(\succsim_i)$.*

Proof. Let $\succsim_i \in \mathcal{D}^A(q)$. By Remark 4,

$$b(\succsim_i, \mathcal{I}(q)) = [\underline{\pi}_q(\succsim_i), \bar{\pi}_q(\succsim_i)].$$

Since preferences in $\mathcal{D}^A(q)$ are monotonic with respect to the inclusion of acceptable integers and the exclusion of unacceptable ones, moving an inferior endpoint away from $\underline{\pi}_q(\succsim_i)$ or a superior endpoint away from $\bar{\pi}_q(\succsim_i)$ strictly decreases utility. Hence the marginal preference relations $P^-_q(\succsim_i)$ and $P^+_q(\succsim_i)$ are single-peaked, with peaks $\underline{\pi}_q(\succsim_i)$ and $\bar{\pi}_q(\succsim_i)$, respectively. □

Lemma 7 shows that every single-peaked preference on $(-\infty, \underline{q}]$ and every single-peaked preference on $[\bar{q}, \infty)$ can be obtained as the marginal preference relation over inferior and superior endpoints, respectively, induced by some preference in $\mathcal{D}^A(q)$.

Lemma 7. $\mathcal{R}^{-}_{(-\infty,\underline{q}]}(\mathcal{D}^A(q)) = \mathcal{R}^{SP}_{(-\infty,\underline{q}]}$ *and* $\mathcal{R}^{+}_{[\bar{q},\infty)}(\mathcal{D}^A(q)) = \mathcal{R}^{SP}_{[\bar{q},\infty)}$

Proof. Consider $R_i \in \mathcal{R}^{SP}_{(-\infty,\underline{q}]}$. Let $\tau(R_i) \leq \underline{q}$ be the peak of R_i. We consider two cases. Assume first that $\tau(R_i) = \underline{q}$. Then there exists $\succsim_i \in \mathcal{D}^A(q)$ represented by a utility function $u_i : \mathbb{Z} \to \mathbb{R}$ such that $u_i(z) < 0$ for every $z < \underline{q}$. Hence,

$$z' < z \leq \underline{q} \quad \Longrightarrow \quad [z, \bar{q}] \succ_i [z', \bar{q}].$$

Consider now the case $\tau(R_i) < \underline{q}$. We next establish that there exists $\succsim_i \in \mathcal{D}^A(q)$ such that for each $z, z' \leq \underline{q}$, $z \; P_i \; z'$ if and only if $[z, \bar{q}] \succ_i [z', \bar{q}]$ by constructing a utility function $u'_i : \mathbb{Z} \to \mathbb{R}$ representing $\succsim_i$. The construction is based on the fact that R_i is single-peaked with peak $\tau(R_i)$. Assign utility 1 to each integer in $[\tau(R_i), \underline{q} - 1]$. For integers smaller than $\tau(R_i)$, assign negative utilities recursively so that, for every $z < \tau(R_i)$, the total utility assigned to the interval $[z, \bar{q}]$ is chosen so that the ranking of intervals $\{[z, \bar{q}] : z \leq \underline{q}\}$ coincides with the ranking induced by R_i. Since the set of integers below $\tau(R_i)$ is countable, the assigned negative utilities may be chosen

to be summably small, which guarantees the existence of such a construction. Thus, there exists an additive utility representation u_i' representing $\succsim_i$ such that for every $z, z' \leq \underline{q}$,

$$zP_iz' \iff \sum_{\zeta\in[z,\bar{q}]} u_i'(\zeta) > \sum_{\zeta\in[z',\bar{q}]} u_i'(\zeta).$$

Hence, $P_q^-(\succsim_i) = R_i$. The proof for $\mathcal{R}^{SP}_{[\bar{q},\infty)}$ is analogous. □

We now introduce the counterparts of Properties A and B of Le Breton and Sen (1999). Since interval selection is represented by two coordinates (the inferior and superior endpoints), we state these properties in their equivalent two-dimensional form.

Property A requires that every admissible inferior endpoint $z \leq \underline{q}$ can be the peak of an induced marginal preference over inferior endpoints. Likewise, every admissible superior endpoint $z' \geq \bar{q}$ can arise as the peak of an induced marginal preference over superior endpoints. Property B requires that, for every pair of admissible marginal preferences, the domain contains two preferences. A preference that ranks intervals lexicographically according to the inferior endpoint (the inferior endpoint determines the ranking whenever the inferior endpoints differ) and a preference that ranks intervals lexicographically according to the superior endpoint (the superior endpoint determines the ranking whenever the superior endpoints differ). Together, these properties ensure that the domain contains sufficiently many separable preferences to identify independently the social aggregation of each endpoint.

Property A. For each $z \in (-\infty, \underline{q}]$ and $z' \in [\bar{q}, \infty)$, there exist $R_i \in \mathcal{R}^-_{(-\infty,\underline{q}]}(\widetilde{\mathcal{D}})$ and $R_i' \in \mathcal{R}^+_{[\bar{q},\infty)}(\widetilde{\mathcal{D}})$ such that

$$\tau(R_i) = z \qquad \text{and} \qquad \tau(R_i') = z'.$$

Property B. For each $(R_i, R_i') \in \mathcal{R}^-_{(-\infty,\underline{q}]}(\widetilde{\mathcal{D}}) \times \mathcal{R}^+_{[\bar{q},\infty)}(\widetilde{\mathcal{D}})$, there exist $\succsim_i, \succsim_i' \in \widetilde{\mathcal{D}}$ such that

$$R_i = P_q^-(\succsim_i) = P_q^-(\succsim_i'), \qquad R_i' = P_q^+(\succsim_i) = P_q^+(\succsim_i'),$$

and:

i) For each $x, y \in \mathcal{I}(q)$, if $\underline{x} \neq \underline{y}$, then

$$x \succ_i y \iff \underline{x}\, R_i\, \underline{y}.$$

ii) For each $x, y \in \mathcal{I}(q)$, if $\bar{x} \neq \bar{y}$, then

$$x \succ_i' y \quad \Longleftrightarrow \quad \bar{x}\, R_i'\, \bar{y}.$$

Following Le Breton and Sen (1999), a domain of separable preferences $\widetilde{\mathcal{D}}$ is ***rich*** if it satisfies Property A and Property B.

The following Lemma8 establishes that $\mathcal{D}^A(q)$ satisfies Properties A and B and is therefore a rich domain of separable preferences.

Lemma 8. *The domain $\mathcal{D}^A(q)$ is rich.*

Proof. We verify separately Properties A and B.

Property A follows directly from Lemma 7. Indeed, every single-peaked marginal preference over inferior endpoints and every single-peaked marginal preference over superior endpoints are induced by some preference in $\mathcal{D}^A(q)$. In particular, every feasible inferior endpoint and every feasible superior endpoint arise as the peak of an induced marginal preference. Hence, $\mathcal{D}^A(q)$ satisfies Property A.

To verify Property B, let $(R_i, R_i') \in \mathcal{R}^-_{(-\infty,\underline{q}]}(\mathcal{D}^A(q)) \times \mathcal{R}^+_{[\bar{q},\infty)}(\mathcal{D}^A(q))$. By the construction employed in the proof of Lemma 7, there exist additive utility representations u_i^- and u_i^+ inducing respectively R_i on $(-\infty, \underline{q}]$ and R_i' on $[\bar{q}, +\infty)$. Moreover, by defining $u_i : \mathbb{Z} \to \mathbb{R}$ such that for each $z \leq \underline{q}$, $u_i(z) = u_i^-(z)$ and for each $z' \geq \bar{q}$, $u_i(z') = u_i^+(z')$, there is $\succsim_i \in \mathcal{D}^A(q)$ such that $R_i = P_q^-(\succsim_i)$ and $R_i' = P_q^+(\succsim_i)$.

Since $(-\infty, \underline{q}]$ and $[\bar{q}, \infty)$ are both countable, u_i^- and u_i^+ may be chosen so that $\sum_{z \leq \underline{q}} |u_i^-(z)| < \infty$ and $\sum_{z \geq \bar{q}} |u_i^+(z)| < \infty$. Choose $\varepsilon > 0$ sufficiently small and define an additive utility function u by

$$\tilde{u}_i(z) = \begin{cases} u_i^-(z), & z \leq \underline{q}, \\ \varepsilon u_i^+(z), & z \geq \bar{q}. \end{cases}$$

Let $\widetilde{\succsim}_i$ denote the preference represented by $\tilde{u}_i$. Since ε may be chosen arbitrarily small, any utility difference generated by the superior endpoint is dominated by the utility difference generated by the inferior endpoint. Consequently, for every pair of intervals $x, y \in \mathcal{I}(q)$ with $\underline{x} \neq \underline{y}$,

$$x \widetilde{\succ}_i y \Longleftrightarrow \underline{x}\, P_i\, \underline{y}.$$

Moreover, $P_q^-(\widetilde{\succsim}_i) = R_i$ and $P_q^+(\widetilde{\succsim}_i) = R_i'$.

Similarly, choose $\delta > 0$ sufficiently small and define

$$\hat{u}_i(z) = \begin{cases} \delta u_i^-(z), & z \leq \underline{q}, \\ u_i^+(z), & z \geq \bar{q}. \end{cases}$$

Let $\widehat{\succsim}_i'$ be the preference represented by $\hat{u}_i$. Then the contribution of the inferior endpoint becomes negligible relative to that of the superior endpoint, and therefore, for every $x, y \in \mathcal{I}(q)$ with $\bar{x} \neq \bar{y}$,

$$x \widehat{\succ}_i y \iff \bar{x} \; P_i' \; \bar{y}.$$

Furthermore, $P_q^-(\widehat{\succsim}_i') = R_i$ and $P_q^+(\widehat{\succsim}_i') = R_i'$.

Thus, for every pair of marginal preferences (R_i, R_i'), the domain contains both the inferior endpoint and the superior endpoint lexicographic extensions required by Property B. Hence, $\mathcal{D}^A(q)$ satisfies Property B,. Since Property A also holds, $\mathcal{D}^A(q)$ is rich. □

Now we present the notions of social choice functions and their associated properties to arbitrary sets of alternatives.

Let X be a set of alternatives and let $\widetilde{\mathcal{R}}_X \subseteq \mathcal{R}_X$ be a domain of preferences on X. A ***social choice function*** on $\widetilde{\mathcal{R}}_X$ is a mapping

$$\varphi : (\widetilde{\mathcal{R}}_X)^N \to X.$$

- A social choice function φ satisfies ***strategy-proofness*** if for each $i \in N$, each $R \in (\widetilde{\mathcal{R}}_X)^N$, and each $R_i' \in \widetilde{\mathcal{R}}_X$, $\varphi(R) \; R_i \; \varphi(R_{-i}, R_i')$.

- A social choice function φ satisfies ***unanimity*** if for each $R \in (\widetilde{\mathcal{R}}_X)^N$ and each $x \in X$, $x = \tau(R_i)$ for each $i \in N$ implies $\varphi(R) = x$.

Next, we introduce the family of generalized median voter schemes, proposed by Moulin (1980), which play a crucial role in domains of one-dimensional single-peaked preferences.

Let $X \subseteq \mathbb{Z}$. Let $\widetilde{\mathcal{R}}_X \subseteq \mathcal{R}_X$ and $\varphi : (\widetilde{\mathcal{R}}_X)^N \to X$ The social choice function φ is a ***generalized median voter scheme*** if there is a family of parameters $\{\alpha_S\}_{S \subseteq N}$, $\alpha_S \in X$ with $\alpha_S \leq \alpha_T$ whenever $T \subseteq S$, and for each $R \in (\widetilde{\mathcal{R}})_X^N$,

$$\varphi(R) = \inf_{S \subseteq N} \left\{ \sup_{i \in S} \{\tau(R_i), \alpha_S\} \right\}.$$

A social choice function φ is ***dictatorial*** if there exists a voter $i \in N$, called the ***dictator***, such that for each $R \in (\widetilde{\mathcal{R}}_X)^N$,

$$\varphi(R) = \tau(R_i).$$

A dictatorial social choice function is a generalized median voter scheme with family of parameters $\{\alpha_S\}_{S\subseteq N}$, $\alpha_S \in X$ such that if $i \in S$, $\alpha_S = \inf X$ and $\alpha_S = \sup X$ otherwise.

Remark 5. *A generalized median voter scheme satisfies* unanimity *if and only if* $\alpha_\varnothing = \sup X$, $\alpha_N = \inf X$.

We now present the key step in the proof. Applying the decomposition result of Le Breton and Sen (1999, Theorem 4.1), we obtain Proposition 1, which establishes the necessity part of Theorem 1 on the domain $\mathcal{D}^A(q)$.

For each $\succsim \in (\mathcal{D}^A)^N$, let $P_q^-(\succsim) = (P_i)_{i\in N} \in (\mathcal{R}^{SP}_{(-\infty,\underline{q}]})^N$ denote the profile of marginal preference relations induced by $\succsim$, where $P_i = P_q^-(\succsim_i)$ for each $i \in N$. Define $P_q^+(\succsim) = (P'_i)_{i\in N} \in (\mathcal{R}^{SP}_{[\bar{q},\infty)})^N$ analogously.

Proposition 1. *For each $q \in \mathcal{I}$, if the interval choice function $f : (\mathcal{D}^A(q))^N \to \mathcal{I}(q)$ satisfies* strategy-proofness *and $r_f = \mathcal{I}(q)$, then f is a min-max extreme points function.*

Proof. We prove the result in two steps.

Step 1. f admits a decomposition into two social choice functions $\varphi^- : (\mathcal{R}^{SP}_{(-\infty,\underline{q}]})^N \to (-\infty, \underline{q}]$, $\varphi^+ : (\mathcal{R}^{SP}_{[\bar{q},\infty)})^N \to [\bar{q}, \infty)$ that satisfy *strategy-proofness* and

$$\inf f(\succsim) = \varphi^-(P_q^-(\succsim)), \quad \sup f(\succsim) = \varphi^+(P_q^+(\succsim)).$$

By Lemma 5, $\mathcal{D}^A(q)$ is a domain of separable preferences. By Lemma 6 and Lemma 7, $\mathcal{R}^-_{(-\infty,\underline{q}]}(\mathcal{D}^A(q)) = \mathcal{R}^{SP}_{(-\infty,\underline{q}]}$ and $\mathcal{R}^+_{[\bar{q},\infty)}(\mathcal{D}^A(q)) = \mathcal{R}^{SP}_{[\bar{q},\infty)}$. By Lemma 8, $\mathcal{D}^A(q)$ is a rich domain.

By Le Breton and Sen (1999, Theorem 4.1), f admits a decomposition into two social choice functions $\varphi^- : (\mathcal{R}^{SP}_{(-\infty,\underline{q}]})^N \to (-\infty, \underline{q}]$ and $\varphi^+ : (\mathcal{R}^{SP}_{[\bar{q},\infty)})^N \to [\bar{q}, \infty)$, such that for each $\succsim \in (\mathcal{D}^A(q))^N$:

$$\inf f(\succsim) = \varphi^-(P^-_q(\succsim)) \text{ and } \sup f(\succsim) = \varphi^+(P^+_q(\succsim)).$$

Moreover, both φ^- and φ^+ satisfy *strategy-proofness*.

It remains to characterize the social choice functions φ^- and φ^+ obtained in Step 1.

Step 2. φ^- and φ^+ are generalized median voter schemes with associated parameter families $\{\alpha_S\}_{S\subseteq N}$ and $\{\beta_S\}_{S\subseteq N}$, respectively, satisfying $\alpha_N = -\infty$, $\alpha_\varnothing = \underline{q}$, $\beta_N = \bar{q}$, and $\beta_\varnothing = \infty$.

By Lemma 1, and since f satisfies *unanimity*, and every feasible inferior and superior endpoint may arise as the endpoint of a unanimous profile,

$$r_{\varphi^-} = (-\infty, \underline{q}] \text{ and } r_{\varphi^+} = [\bar{q}, \infty).$$

Adapting the proof of Lemma1 to one-dimensional social choice functions, φ^- and φ^+ satisfy *unanimity*. By Barberà *et al.* (1993, Theorem 1), φ^- and φ^+ depend only on the peaks of the marginal preference relations.[5] By Moulin (1980, Proposition 3), φ^- and φ^+ are generalized median voter scheme with associated parameters families $\{\alpha_S\}_{S\subseteq N}$ and $\{\beta_S\}_{S\subseteq N}$, respectively, satisfying . Since φ^- and φ^+ satisfy *unanimity*, by Remark 5, $\alpha_N = -\infty$, $\alpha_\varnothing = \underline{q}$, $\beta_N = \bar{q}$, and $\beta_\varnothing = \infty$.

To conclude, by Steps 1 and 2, f is a min-max extreme points function. □

We now complete the proof of Theorem 1.

Proof of Theorem 1. We first prove that every min-max extreme points function satisfies *strategy-proofness* and *unanimity* on $\mathcal{D}^M(q)$. Let f be a min-max extreme points function defined on $\widetilde{\mathcal{D}} \subseteq \mathcal{D}^M(q)$. Since f is a min-max extreme points function, it decomposes into two generalized median voter schemes selecting the inferior and superior endpoints, respectively. Moreover, each component depends only on the corresponding endpoint of every voter's most-preferred feasible interval. To prove *strategy-proofness*, it is enough to show that no voter can manipulate

[5]The framework in Barberà *et al.* (1993) only considers finite sets of alternatives in the range of the social choice function, but the arguments in the proof of Theorem 1 apply without modifications for finite as well as countable ranges insofar each element of the range correspond to the unique most-preferred element for some preference in the domain.

either endpoint independently. We first consider the inferior endpoint. Assume that for some preference profile $\succsim \in (\mathcal{D}^M(q))^N$, for some $i \in N$ $\inf f(\succsim) < \underline{\pi}_q(\succsim_i)$. By the definition of a generalized median voter scheme, there exists a coalition $S \subseteq N$ with $i \notin S$ such that

$$\inf f(\succsim) = \sup_{j \in S}\{\underline{\pi}_q(\succsim_j), \alpha_S\} < \underline{\pi}_q(\succsim_i).$$

Hence, for each $\succsim_i' \in \mathcal{D}^M(q)$, $\inf f(\succsim_i', \succsim_{-i}) \leq \inf f(\succsim)$. The case $\underline{\pi}_q(\succsim_i) < \inf f(\succsim)$ is analogous. Applying the same reasoning to the superior endpoint, neither endpoint can be moved farther away from $b(\succsim_i, \mathcal{I}(q))$ through manipulation. Therefore, we obtain that for each $i \in N$, each $\succsim \in (\mathcal{D}^M(q))^N$, and each $\succsim_i' \in \mathcal{D}^M(q)$, $f(\succsim) \in \mathcal{B}(f(\succsim_i', \succsim_{-i}), b(\succsim_i, \mathcal{I}(q)))$. Since $\succsim_i$ is intervalwise single-peaked on $\mathcal{I}(q)$,, $f(\succsim) \succsim_i f(\succsim_i', \succsim_{-i})$ which proves *strategy-proofness*. It remains to prove that $r_f = \mathcal{I}(q)$. Consider $x \in \mathcal{I}(q)$ and $\succsim \in \mathcal{M}(q)^N$ such that for each $i \in N$, $b(\succsim_i, \mathcal{I}(q)) = x$. For each non-empty $S \subseteq N$, , $\sup_{i \in S}\{\underline{\pi}_q(\succsim_i), \alpha_S\} = \underline{x}$. Since $\alpha_\varnothing = \underline{q}$, $\inf f(\succsim) = \inf_{S \subseteq N}\{\sup_{i \in S}\{\underline{\pi}_q(\succsim_i),\ \alpha_S\}\} = \underline{x}$. The parallel argument shows that $\sup f(\succsim) = \inf_{S \subseteq N}\{\sup_{i \in S}\{\bar{\pi}_q(\succsim_i), \beta_S\}\} = \bar{x}$, and $f(\succsim) = x$. Since every interval in $\mathcal{I}(q)$ may arise as the unanimous most-preferred feasible interval, *unanimity* implies that $r_f = \mathcal{I}(q)$.

Conversely, let $\mathcal{D}^A(q) \subseteq \widetilde{\mathcal{D}} \subseteq \mathcal{D}^M(q)$ and let $f : \widetilde{\mathcal{D}}^N \to \mathcal{I}(q)$ satisfy *strategy-proofness* and $r_f = \mathcal{I}(q)$. The result follows for $\mathcal{D}^A(q)$ from Proposition 1. Define the auxiliary interval choice function $f^A : (\mathcal{D}^A(q))^N \to \mathcal{I}(q)$ by $f^A(\succsim) = f(\succsim)$ for each $\succsim \in (\mathcal{D}^A(q))^N$. That is, f^A is the restriction of f to $(\mathcal{D}^A(q))^N$. Since f satisfies *strategy-proofness*, so does f^A. Moreover, since $r_f = \mathcal{I}(q)$, by Lemma 1, f satisfies *unanimity*, and so does f^A. Therefore, $r_{f^A} = \mathcal{I}(q)$. Hence, by Proposition1, f^A is a min-max extreme points function.

To conclude, it remains to show that the value of f depends only on the voters' most-preferred feasible intervals. That is, for each $\succsim, \succsim' \in \widetilde{\mathcal{D}}^N$ if $b(\succsim_i, \mathcal{I}(q)) = b(\succsim_i', \mathcal{I}(q))$ for each $i \in N$, then $f(\succsim) = f(\succsim')$.. Suppose, to the contrary, that there exist $i \in N$, $\succsim \in (\mathcal{D}^A(q))^N$, and $\succsim_i' \in \widetilde{\mathcal{D}}$ with $b(\succsim_i, \mathcal{I}(q)) = b(\succsim_i', \mathcal{I}(q))$ such that $f(\succsim) \neq f(\succsim_i', \succsim_{-i})$. Note that since $b(\succsim_i, \mathcal{I}(q)) = b(\succsim_i) \cup q = b(\succsim_i', \mathcal{I}(q)) = b(\succsim_i') \cup q$, $b(\succsim_i) \setminus q = b(\succsim_i') \setminus q$. Moreover, $q \subseteq f(\succsim)$, and $q \subseteq f(\succsim_i', \succsim_{-i})$. By *strategy-proofness*,

$$f(\succsim_i', \succsim_{-i}) \notin \mathcal{B}(f(\succsim), b(\succsim_i, \mathcal{I}(q))) \qquad \text{and} \qquad f(\succsim) \notin \mathcal{B}(f(\succsim_i', \succsim_{-i}), b(\succsim_i, \mathcal{I}(q))).$$

Hence, by the definition of the minimal boxes $\mathcal{B}(f(\succsim), b(\succsim_i, \mathcal{I}(q)))$ and $\mathcal{B}(f(\succsim_i', \succsim_{-i}), b(\succsim_i, \mathcal{I}(q)))$, exactly one of the four following cases applies. These are precisely the four possible ways in which two outcomes can simultaneously lie outside each other's minimal box with respect to the same most-preferred feasible interval.

Case 1 : There are $z, z' \in \mathbb{Z}$, $z \in b(\succsim_i)$, $z' \notin b(\succsim_i)$, $\{z, z'\} \subseteq f(\succsim_i', \succsim_{-i})$.

Case 2 : There are $z, z' \in \mathbb{Z}$, $z, z' \in b(\succsim_i)$, $z \in f(\succsim'_i, \succsim_{-i}) \setminus f(\succsim)$, $z' \in f(\succsim) \setminus f(\succsim'_i, \succsim_{-i})$.

Case 3 : There are $z, z' \in \mathbb{Z}$, $z, z' \notin b(\succsim_i)$, $z \in f(\succsim'_i, \succsim_{-i}) \setminus f(\succsim)$, $z' \in f(\succsim) \setminus f(\succsim'_i, \succsim_{-i})$.

Case 4 : There are $z, z' \in \mathbb{Z}$, $z, z' \notin b(\succsim_i)$, $\{z, z'\} \subseteq f(\succsim) \setminus f(\succsim'_i, \succsim_{-i})$.

Consider Case 1. By Lemma 2.iii), there exists $\succsim''_i \in \mathcal{D}^A(q)$ such that $b(\succsim''_i) = b(\succsim_i)$, for each x, x', if $z \in x$ and $z \notin x'$, then $x \succ''_i x'$. Since $\succsim''_i \in \mathcal{D}^A(q)$, we have that $f(\succsim''_i, \succsim_{-i}) = f(\succsim)$, which implies $f(\succsim'_i, \succsim_{-i}) \succ''_i f(\succsim''_i, \succsim_{-i})$, contradicting *strategy-proofness*. Using Lemma 2.iii)–iv) as appropriate, the remaining cases follow from parallel arguments. Repeating the argument successively for each voter, we conclude that for each $\succsim \in (\widetilde{\mathcal{D}})^N$ and each $\succsim' \in (\mathcal{D}^A(q))^N$, if $b(\succsim_i, \mathcal{I}(q)) = b(\succsim'_i, \mathcal{I}(q))$ for each $i \in N$, then $f(\succsim) = f(\succsim') = f^A(\succsim')$. Hence, $f = f^A$ on $\widetilde{\mathcal{D}}^N$. Since f^A is a min-max extreme points function, so is f. □

5.3 Proof of Theorem 2

Before turning to the proof of Theorem 2, it is worth highlighting that the decomposability arguments used in the proof of Theorem 1 cannot be applied to interval choice functions defined in the unrestricted domain of additively representable preferences $\mathcal{D}^A$.

By Lemma 5, every preference $\succsim_i \in \mathcal{D}^A$ is separable at q. Hence the marginal preferences $P_q^-(\succsim_i)$ on $(-\infty, \underline{q}]$ and $P_q^+(\succsim_i)$ on $[\bar{q}, +\infty)$ are well defined. However, the domain $\mathcal{D}^A$ is not rich. Let $q = [0]$ and let $\succsim_i, \succsim'_i \in \mathcal{D}^A$ be such that $b(\succsim_i) = [\underline{q} - 2]$ and $b(\succsim'_i) = [\bar{q} + 2]$. Then, $P_q^-(\succsim_i) \in \mathcal{R}^-_{(-\infty, \underline{q}]}(\mathcal{D}^A)$ and $P_q^+(\succsim'_i) \in \mathcal{R}^+_{[\bar{q}, \infty)}(\mathcal{D}^A)$. However, there is no preference $\succsim''_i \in \mathcal{D}^A$ such that $P_q^-(\succsim''_i) = P_q^-(\succsim_i)$ and $P_q^+(\succsim''_i) = P_q^+(\succsim'_i)$.

To see this, note that any such preference would have to regard both $\underline{q} - 2$ and $\bar{q} + 2$ as acceptable integers. Since acceptable integers must form an interval, $\succsim''_i$ would necessarily regard every integer in $[\underline{q} - 2, \bar{q} + 2]$ as acceptable, contradicting the fact that both $\succsim_i$ and $\succsim'_i$ have singleton most-preferred intervals. Therefore, Property B fails: not every admissible pair of marginal preferences can be combined into a preference in $\mathcal{D}^A$. Consequently, the decomposition result of Le Breton and Sen (1999) cannot be invoked. Instead, our proof proceeds by considering suitable subdomains of $\mathcal{D}^A$ in which one endpoint is fixed.

Proposition 2. *Let $q \in \mathcal{I}$ and let $f : (\mathcal{D}^A)^N \to \mathcal{I}(q)$ be an interval choice function with $r_f = \mathcal{I}(q)$. If f satisfies* strategy-proofness, *then f is an endpoint dictatorial function.*

Proof. The proof proceeds in fourth stages. First, we identify domains on which one endpoint is fixed by *strategy-proofness* (Steps 1–2). Second, we introduce auxiliary domains whose induced marginal preferences consist of all single-peaked and single-dipped preferences, allowing us to apply the dictatorship result of Achuthankutty and Roy (2018) (Steps 3–4). Third, we extend this dictatorship result to increasingly larger preference domains (Steps 5–10). Fourth, we combine the two endpoint dictatorships to obtain the characterization first on $\mathcal{D}^A(q)$ (Step 11), and then on the full domain $\mathcal{D}^A$ (Step 12).

Let $\mathcal{D}^{A-} \subset \mathcal{D}^A$ be the domain of additively representable interval-monotonic preferences such that $\sup b(\succsim_i) \leq \bar{q}$ and let $\mathcal{D}^{A+} \subset \mathcal{D}^A$ be the domain of additively representable interval-monotonic preferences such that $\inf b(\succsim_i) \geq \underline{q}$.

Step 1. For each $\succsim \in (\mathcal{D}^{A-})^N$ and each $\succsim' \in (\mathcal{D}^{A+})^N$, $\sup f(\succsim) = \bar{q}$ and $\inf f(\succsim') = \underline{q}$.

This follows from Remark 4 and Lemma 4.

Next, we focus on the domain $\mathcal{D}^{A-}$.

Step 2. For each $\succsim, \succsim' \in (\mathcal{D}^{A-})^N$, if for each $i \in N$, $P_q^-(\succsim_i) = P_q^-(\succsim_i')$, then $f(\succsim) = f(\succsim')$.

By Step 1, $\sup f(\succsim) = \sup f(\succsim') = \bar{q}$. By *strategy-proofness*, we have that $f(\succsim) \succsim_i f(\succsim_i', \succsim_{-i})$ and $f(\succsim_i', \succsim_{-i}) \succsim_i' f(\succsim)$. Observe that $\sup f(\succsim) = \sup f(\succsim')$, preferences in $\mathcal{D}^A$ are strict, and $P_q^-(\succsim_i) = P_q^-(\succsim_i')$. Hence, by Lemma 5 and because the $\succsim_i$ and $\succsim_i'$ induce the same ordering on intervals with superior endpoint $\bar{q}$, both incentive constraint inequalities can hold only if $f(\succsim) = f(\succsim_i', \succsim_{-i})$. Repeating this argument successively for the remaining voters, we obtain $f(\succsim) = f(\succsim')$.

For each $k < \underline{q} - 1$, let $\mathcal{D}_k^{A-} \subset \mathcal{D}^{A-}$ be the domain of preferences in $\mathcal{D}^{A-}$ such that, for each $x, y \in \mathcal{I}$, if $(k-1) \notin x$, $(k-1) \in y$, then $x \succ_i y$. By Lemma 2.iii), $\mathcal{D}_k^{A-} \neq \varnothing$.

Next, define the subdomain $\mathcal{D}_k^{S-} \subset \mathcal{D}_k^{A-}$ as the domain of preferences in $\mathcal{D}_k^{A-}$ such that either $\succsim_i \in \mathcal{D}^A(q)$ or $\inf b(\succsim_i) = k$.

For each $X \subseteq \mathbb{Z}$, a preference relation $P_i \in \mathcal{R}_X$ is single-dipped on X, if there is a unique integer $d(P_i) \in X$ such that for each $z \in X \setminus \{d(P_i)\}$, $z \; P_i \; d(P_i)$, and for each $z, z' \in X \setminus \{d(P_i)\}$, if $z < z' < d(P_i)$ then $z \; P_i \; z'$, and if $d(P_i) < z' < z$, then $z \; P_i \; z'$. Let $\mathcal{R}_X^{SD}$ denote the set of all single-dipped preferences on X.

Step 3. For each $k < \underline{q}$, $\mathcal{R}^{-}_{[k,\underline{q}]}(\mathcal{D}^{S-}_k) = \mathcal{R}^{SP}_{[k,\underline{q}]} \cup \mathcal{R}^{SD}_{[k,\underline{q}]}$.

For each $\succsim_i \in \mathcal{D}^{S-}_k \cap \succsim_i \in \mathcal{D}^{A}(q)$, by Lemma 6, $P^{-}_q(\succsim_i) \in \mathcal{R}^{SP}_{[k,\underline{q}]}$. Using the same argument of the proof of Lemma 6, for each $\succsim_i \in \mathcal{D}^{S-}_k \setminus \mathcal{D}^{A}(q)$, $P^{-}_q(\succsim_i) \in \mathcal{R}^{SD}_{[k,\underline{q}]}$, and $d(P^{-}_q(\succsim_i)) = \sup b(\succsim_i)+1$. Hence, $\mathcal{R}^{-}_{[k,\underline{q}]}(\mathcal{D}^{S-}_k) \subseteq \mathcal{R}^{SP}_{[k,\underline{q}]} \cup \mathcal{R}^{SD}_{[k,\underline{q}]}$. Conversely, replicating the construction in the proof of Lemma 7, every preference in $\mathcal{R}^{SP}_{[k,\underline{q}]} \cup \mathcal{R}^{SD}_{[k,\underline{q}]}$ arises as the marginal preference of some $\succsim_i \in \mathcal{D}^{S-}_k$.

Step 4. Let $k < \underline{q} - 1$. There is $i \in N$ such that for each $\succsim \in (\mathcal{D}^{S-}_k)^N$, $\inf f(\succsim) = \inf b(\succsim_i, \mathcal{I}(q))$.

For each $\succsim \in (\mathcal{D}^{S-}_k)^N$, $f(\succsim) \subseteq [k, \bar{q}]$ and, by Lemma 4, $\sup f(\succsim) = \bar{q}$. Hence, the problem reduces to selecting the inferior endpoint. Moreover, $\mathcal{D}^{S-}_k \subseteq \mathcal{D}^{A-}$. Therefore, by Step 2, for each $\succsim, \succsim' \in (\mathcal{D}^{S-}_k)^N$ such that $P^{-}_q(\succsim_i) = P^{-}_q(\succsim'_i)$ for every $i \in N$, we have $f(\succsim) = f(\succsim')$.

Define the social choice function $\varphi^{-}_k : (\mathcal{R}^{-}_{[k,\underline{q}]}(\mathcal{D}^{S-}_k))^N \to [k, \underline{q}]$, by letting, for each $P \in (\mathcal{R}^{-}_{[k,\underline{q}]}(\mathcal{D}^{S-}_k))^N$, $\varphi^{-}_k(P) = \inf f(\succsim)$ where $\succsim \in (\mathcal{D}^{S-}_k)^N$ satisfies $P_i = P^{-}_q(\succsim_i)$ for each $i \in N$. By Step 2, φ^{-}_k is well defined. Since f satisfies *strategy-proofness*, φ^{-}_k satisfies *strategy-proofness*. By Lemma 1, the range of φ^{-}_k is $[k, \underline{q}]$ which contains at least three elements. Hence, since the domain consists of all single-peaked and all single-dipped preferences on $[k, \underline{q}]$, by Achuthankutty and Roy (2018, Corollary 1), φ^{-}_k is dictatorial and there is $i \in N$ such that for each $P \in (\mathcal{R}^{-}_{[k,\underline{q}]}(\mathcal{D}^{S-}_k))^N$, $\varphi^{-}_k(P) = \tau(P_i)$, and therefore for each $\succsim \in (\mathcal{D}^{S-}_k)^N$, $\inf f(\succsim) = \inf b(\succsim_i, \mathcal{I}(q))$.

For each $k \leq \underline{q} - 1$, let $i_k \in N$ denote the voter such that for each $\succsim \in (\mathcal{D}^{S-}_k)^N$, $\inf f(\succsim) = \inf b(\succsim_i, \mathcal{I}(q))$. Next we extend the result to arbitrary preference profiles in $\mathcal{D}^{A-}_k$.

Step 5. For each $k < \underline{q} - 1$, and each $\succsim \in (\mathcal{D}^{A-}_k)^N$, $\inf f(\succsim) = \inf b(\succsim_{i_k}, \mathcal{I}(q))$.

Let $\succsim \in (\mathcal{D}^{A-}_k)^N$. Let $z = \inf b(\succsim_{i_k}, \mathcal{I}(q))$. Let $\succsim' \in (\mathcal{D}^{S-}_k)^N$ be such that $b(\succsim_{i_k}, \mathcal{I}(q)) = b(\succsim'_{i_k}, \mathcal{I}(q))$, and for each $j \neq i_k$, $P^{-}_q(\succsim'_j) \in \mathcal{R}^{SD}_{[k,\underline{q}]}$ with $d(P^{-}_q(\succsim'_j)) = z$. By Step 4, $\inf f(\succsim') = z$, and by Step 1, $\sup f(\succsim') = \bar{q}$. Hence, $f(\succsim') = [z, \bar{q}]$. By *strategy-proofness*, $f(\succsim') \succsim'_j f(\succsim_j, \succsim'_{-j})$. Since $d(P^{-}_q(\succsim'_j)) = z$, $f(\succsim_j, \succsim'_{-j}) = [z, \bar{q}]$. Repeating the argument successively for the remaining voters $j \neq i_k$, yields $f(\succsim'_{i_k}, \succsim_{-i_k}) = f(\succsim') = b(\succsim_{i_k}, \mathcal{I}(q)$. Finally, by *strategy-proofness*, $f(\succsim) \succsim_{i_k} f(\succsim'_{i_k}, \succsim_{-i_k})$ implies that $f(\succsim) = b(\succsim_{i_k}, \mathcal{I}(q))$.

Next, we prove that for each $k < \underline{q} - 1$, the same voter determines the inferior endpoint for each domain $\mathcal{D}^{A-}_k$.

Step 6. There is $i \in N$ such that for each $k < \underline{q} - 1$, and each $\succsim \in (\mathcal{D}_k^{A-})^N$,

$$\inf f(\succsim) = \inf b(\succsim_i, \mathcal{I}(q)).$$

Let $k < \underline{q} - 1$. Consider the voters i_k, i_{k-1}. Suppose, to the contrary, that $i_k \neq i_{k-1}$. Let $\succsim \in (\mathcal{D}_k^{A-})^N$ be such that $\inf b(\succsim_{i_k}, \mathcal{I}(q)) \neq \inf b(\succsim_{i_{k-1}}, \mathcal{I}(q))$. By Step 5, $f(\succsim) = [\inf b(\succsim_i, \mathcal{I}(q)), \bar{q}]$. Consider $\succsim' \in (\mathcal{D}_{(}^{A-} k-1))^N$ such that for each $j \in N$ and each $x, y \subseteq [k-1, \bar{q}]$, if $(k-1) \notin x$ and $(k-1) \in y$, then $x \succ_j' y$, and for each $x', y' \subseteq [k, \bar{q}]$. $x' \succsim_j' y'$ if and only if $x' \succsim_j y'$. By Step5, $f(\succsim') = [\inf b(\succsim_{i_{k-1}}, \mathcal{I}(q)), \bar{q}] \neq f(\succsim)$. Consider $j \in N$. By Lemma 4, $f(\succsim_j', \succsim_{-j}) \subseteq [k, \bar{q}]$. By *strategy-proofness*, $f(\succsim) \succsim_j f(\succsim_j', \succsim_{-j})$ and $f(\succsim_j', \succsim_{-j}) \succsim_j' f(\succsim)$. Since $\succsim_j$ and $\succsim_j'$ coincide on intervals contained in $[k, \bar{q}]$, both incentive constraints can hold only if $f(\succsim) = f(\succsim_j', \succsim_{-j})$. Repeating the argument successively for the remaining voters, we obtain that $f(\succsim) = f(\succsim')$, a contradiction. This contradiction shows that $i_k = i_{k-1}$. Since k was arbitrary, the same voter is the dictator for every domain $\mathcal{D}_k^{A-}$.

From now on, let $i^* \in N$ be the voter such that for each $k < \underline{q} - 1$, for each $\succsim \in (\mathcal{D}_k^{A-})^N$, $f(\succsim) = [\inf b(\succsim_{i^*}, \mathcal{I}(q)), \bar{q}]$. Next we extend the result to arbitrary preference profiles in $\mathcal{D}^{A-}$.

Step 7. For each $\succsim \in (\mathcal{D}^{A-})^N$, $f(\succsim) = [\inf b(\succsim_{i^*}, \mathcal{I}(q)), \bar{q}]$.

Let $\succsim \in (\mathcal{D}^{A-})^N$. Let $z = \inf_{i \in N}\{\inf b(\succsim_i)\}$. By Remark 4, $z \leq \inf_{i \in N}\{\inf b(\succsim_i), \mathcal{I}(q))\}$. By Lemma 4, $z \leq \inf f(\succsim)$. Let $\succsim' \in (\mathcal{D}_z^{A-})^N$, be such that $b(\succsim_{i^*}', \mathcal{I}(q)) = b(\succsim_{i^*}, \mathcal{I}(q))$ and for each $j \neq i^*$, $P_q^-(\succsim_j') \in \mathcal{R}_{[z,\underline{q}]}^{SD}$ and $d(P_q^-(\succsim_j')) = \inf b(\succsim_{i^*}, \mathcal{I}(q))$. By Step 6, $f(\succsim') = [\inf b(\succsim_{i^*}, \mathcal{I}(q)), \bar{q}]$. Repeating the voter-by-voter replacement argument used in the proof of Step 5, we conclude that $f(\succsim) = [\inf b(\succsim_{i^*}, \mathcal{I}(q)), \bar{q}]$.

Next, we extend the previous result to arbitrary preference profiles in which voter i^* does not regard any integer greater than $\bar{q}$ as acceptable. We show that, under this condition, voter i^* determines the inferior endpoint of the chosen interval.

Step 8. For each $\succsim \in (\mathcal{D}^{A})^N$, if $\succsim_{i^*} \in \mathcal{D}^{A-}$, then $\inf f(\succsim) = \inf b(\succsim_{i^*}, \mathcal{I}(q))$.

Let $\succsim \in (\mathcal{D}^{A})^N$ be such that $\succsim_{i^*} \in \mathcal{D}^{A-}$. We first show that $\inf b(\succsim_{i^*}, \mathcal{I}(q)) \in f(\succsim)$. This is immediate if $\inf b(\succsim_{i^*}, \mathcal{I}(q)) = \underline{q}$. Thus, we consider the case $\inf b(\succsim_{i^*}, \mathcal{I}(q)) \neq \underline{q}$. By Lemma 2.iii), there exists $\succsim' \in (\mathcal{D}^{A-})^N$ such that $\succsim_{i^*}' = \succsim_{i^*}$, and for each $j \neq i^*$, for each $x, y \in \mathcal{I}(q)$, if $\inf b(\succsim_{i^*}, \mathcal{I}(q)) \notin x$ and $\inf b(\succsim_{i^*}, \mathcal{I}(q)) \in y$, then $x \succ_j' y$. By Step 7, $\inf f(\succsim') = \inf b(\succsim_{i^*}, \mathcal{I}(q))$.

Let $j \neq i^*$. By *strategy-proofness*, $f(\succsim') \succsim'_j f(\succsim_j, \succsim'_{-j})$. Hence, $\inf b(\succsim_{i^*}, \mathcal{I}(q)) \in f(\succsim_j, \succsim'_{-j})$. Repeating the argument successively for the remaining voters $j' \neq i^*$, $\inf b(\succsim_{i^*}, \mathcal{I}(q)) \in f(\succsim)$. It remains to show that no smaller integer belongs to $f(\succsim)$. By Lemma 2.iv), there exists $\succsim'' \in (\mathcal{D}^{A-})^N$ such that $\succsim''_{i^*} = \succsim_{i^*}$ and for each $j \neq i^*$, for each $x, y \in \mathcal{I}(q)$, $\inf b(\succsim_{i^*}, \mathcal{I}(q)) - 1 \in x$, $\inf b(\succsim_{i^*}, \mathcal{I}(q)) - 1 \notin y$, implies $x \succ''_j y$. By Step 7, $\inf f(\succsim'') = \inf b(\succsim_{i^*}, \mathcal{I}(q))$. Let $j \neq i^*$. By *strategy-proofness*, $f(\succsim'') \succsim''_j f(\succsim_j, \succsim''_{-j})$. Since $\inf b(\succsim_{i^*}, \mathcal{I}(q)) - 1 \notin f(\succsim'')$, $\inf b(\succsim_{i^*}, \mathcal{I}(q)) - 1 \notin f(\succsim_j, \succsim''_{-j})$. Repeating the argument successively for the remaining voters $j' \neq i^*$, we obtain that $\inf f(\succsim) = \inf b(\succsim_{i^*}, \mathcal{I}(q))$.

Applying symmetric arguments to the domain $\mathcal{D}^{A+}$, we obtain the following dual result.

Step 9. There is $j^* \in N$ such that for each $\succsim \in (\mathcal{D}^A)^N$, if $\succsim_{j^*} \in \mathcal{D}^{A+}$, then

$$\sup f(\succsim) = \sup b(\succsim_{j^*}, \mathcal{I}(q)).$$

Steps 8 and 9 immediately imply the corresponding results for preference domains in which each endpoint dictator has lexicographic concerns for the endpoint that she determines.

Let $\mathcal{D}^{A^*-} \subset \mathcal{D}^A$ be the domain of additively representable interval-monotonic preferences such that for each $x, y \in \mathcal{I}(q)$, $x \succ_i y$ if and only if $[\underline{x}, \bar{q}] \succ_i [\underline{y}, \bar{q}]$ or $\underline{x} = \underline{y}$ and $[\underline{q}, \bar{x}] \succ_i [\underline{q}, \bar{y}]$. Symmetrically, let $\mathcal{D}^{A^*+} \subset \mathcal{D}^A$ be the domain of additively representable interval-monotonic preferences such that for each $x, y \in \mathcal{I}(q)$, $x \succ_i y$ if and only if $[\underline{q}, \bar{x}] \succ_i [\underline{q}, \bar{y}]$ or $\bar{x} = \bar{y}$ and $[\underline{x}, \bar{q}] \succ_i [\underline{y}, \bar{q}]$. Thus, preferences in $\mathcal{D}^{A^*-}$ compare intervals lexicographically, giving priority to the inferior endpoint and using the superior endpoint only to break ties. Symmetrically, preferences in $\mathcal{D}^{A^*+}$ give priority to the superior endpoint and use the inferior endpoint only to break ties.

Step 10. For each $\succsim \in (\mathcal{D}^A)^N$, if $\succsim_{i^*} \in \mathcal{D}^{A^*-}$, then $\inf f(\succsim) = \inf b(\succsim_{i^*}, \mathcal{I}(q))$. Symmetrically, if $\succsim_{j^*} \in \mathcal{D}^{A^*+}$, then $\sup f(\succsim) = \sup b(\succsim_{j^*}, \mathcal{I}(q))$.

Let $\succsim \in (\mathcal{D}^A)^N$ be such that $\succsim_{i^*} \in \mathcal{D}^{A^*-}$. Suppose, to the contrary, that we have $\inf f(\succsim) \neq \inf b(\succsim_{i^*}, \mathcal{I}(q))$. Let $\succsim'_{i^*} \in \mathcal{D}^{A^-}$ such that $\inf b(\succsim'_{i^*}, \mathcal{I}(q)) = \inf b(\succsim_{i^*}, \mathcal{I}(q))$. By Step 8, we have $\inf f(\succsim'_{i^*}, \succsim_{-i^*}) = \inf b(\succsim_{i^*}, \mathcal{I}(q))$. Since $\succsim_{i^*} \in \mathcal{D}^{A^*-}$, preferences are lexicographic with priority given to the inferior endpoint. Therefore, $f(\succsim'_{i^*}, \succsim_{-i^*}) \succ_i f(\succsim)$, contradicting *strategy-proofness*. The argument for $\succsim_{j^*} \in \mathcal{D}^{A^*+}$ is entirely symmetric.

In the final stage of the proof, we show that the endpoint dictators identified for the domains $\mathcal{D}^{A-}$ and $\mathcal{D}^{A+}$ continue to determine the corresponding endpoints at profiles in $\mathcal{D}^{A}(q)$.

Step 11. For each $\succsim \in (\mathcal{D}^{A})^{N}$, if $\succsim_{i^*}, \succsim_{j^*} \in \mathcal{D}^{A}(q)$, then $f(\succsim) = [\inf b(\succsim_{i^*}, \mathcal{I}(q)), \sup b(\succsim_{j^*}, \mathcal{I}(q))]$.

By Step 8 and Step 9, it only remains to consider profiles $\succsim \in (\mathcal{D}^{A}(q))^{N}$ such that $\succsim_{i^*} \in \mathcal{D}^{A}(q) \setminus \mathcal{D}^{A-}$ and $\succsim_{j^*} \in \mathcal{D}^{A}(q) \setminus \mathcal{D}^{A+}$. Let $\succsim^* \in \mathcal{D}^{A}$. Define the auxiliary interval choice function $f^{\succsim^*} : (\mathcal{D}^{A}(q))^{\{i^*, j^*\}} \to \mathcal{I}(q)$, by for each $\succsim_{i^*}, \succsim_{j^*} \in \mathcal{D}^{A}(q)$, $f^{\succsim^*}(\succsim_{i^*}, \succsim_{j^*}) = f(\succsim')$, where $\succsim'_{i^*} = \succsim_{i^*}$, $\succsim'_{j^*} = \succsim_{j^*}$, and $\succsim'_{j} = \succsim^*_{j}$ for each $j \notin \{i^*, j^*\}$. By f's *strategy-proofness*, $f^{\succsim^*}$ satisfies *strategy-proofness*. Since, for each $x \in \mathcal{I}(q)$ there exist $\succsim_{i^*} \in \mathcal{D}^{A^*-}$ and $\succsim_{j^*} \in \mathcal{D}^{A^*+}$ such that $\underline{x} = \inf b(\succsim_{i^*}, \mathcal{I}(q))$ and $\bar{x} = \sup b(\succsim_{j^*}, \mathcal{I}(q))$, by Step 10, we have $r_{f^{\succsim^*}} = \mathcal{I}(q)$. By Theorem 1, $f^{\succsim^*}$ is a min-max extreme points function. Hence, its value depends only on the inferior and superior endpoints of the voters i^* and j^*'s most-preferred intervals. Suppose, to the contrary, that there exists $\succsim_{i^*}, \succsim_{j^*} \in \mathcal{D}^{A}(q)$ such that $\inf f^{\succsim^*}(\succsim_{i^*}, \succsim_{j^*}) \neq \inf b(\succsim_{i^*}, \mathcal{I}(q))$. Let $\succsim'_{i^*} \in \mathcal{D}^{A}(q) \cap \mathcal{D}^{A^*-}$ be such that $b(\succsim'_{i^*}, \mathcal{I}(q)) = b(\succsim_{i^*}, \mathcal{I}(q))$. Since $f^{\succsim^*}$ is a min-max extreme points function, $f^{\succsim^*}(\succsim'_{i^*}, \succsim_{j^*}) = f^{\succsim^*}(\succsim_{i^*}, \succsim_{j^*})$, and $\inf f^{\succsim^*}(\succsim''_{i^*}, \succsim_{j^*}) \neq \inf b(\succsim''_{i^*}, \mathcal{I}(q))$, which contradicts Step 10. Hence, $f^{\succsim^*}$ is an endpoint dictatorial function in which i^* determines the inferior endpoint and j^* determines the superior endpoint. The argument also covers the case $i^* = j^*$, in which the same voter determines both endpoints. Since the choice of $\succsim^*$ is arbitrary, the conclusion follows.

Step 12. For each $\succsim \in (\mathcal{D}^{A})^{N}$, $f(\succsim) = [\inf b(\succsim_{i^*}, \mathcal{I}(q)), \sup b(\succsim_{j^*}, \mathcal{I}(q))]$.

To conclude, it remains to determine the value of f at profiles where the endpoint dictators i^* and j^* report arbitrary preferences in $\mathcal{D}^{A}$.

Case 1: $i^* = j^*$**.** Consider an arbitrary profile $\succsim \in (\mathcal{D}^{A})^{N}$. There exists $\succsim'_{i^*} \in \mathcal{D}^{A}(q)$ such that $b(\succsim'_{i^*}, \mathcal{I}(q)) = b(\succsim_{i^*}, \mathcal{I}(q))$. By Step 11, $f(\succsim'_{i^*}, \succsim_{-i^*}) = b(\succsim_{i^*}, \mathcal{I}(q))$. By *strategy-proofness*, $f(\succsim) \succsim_{i^*} f(\succsim'_{i^*}, \succsim_{-i^*})$. Hence, $f(\succsim) = b(\succsim_{i^*}, \mathcal{I}(q))$.

Case 2: $i^* \neq j^*$**.** Consider an arbitrary profile $\succsim \in (\mathcal{D}^A)^N$. Unlike in the previous steps, we cannot simply replace the preferences of i^* and j^* by preferences in $\mathcal{D}^A(q)$ and then restore them one at a time. Once one of these voters reports an arbitrary preference in $\mathcal{D}^A$, the previous steps no longer guarantee that only the corresponding endpoint may change; a priori, both endpoints of the chosen interval could be affected. We therefore prove that any such change would contradict *strategy-proofness*. Let $\succsim' \in \mathcal{D}^A$ such that $\succsim'_{i^*} \in \mathcal{D}^A(q) \cap \mathcal{D}^{A-}$, $\inf b(\succsim'_{i^*}, \mathcal{I}(q)) = \inf b(\succsim_{i^*}, \mathcal{I}(q))$, $\succsim'_{j^*} \in \mathcal{D}^A(q) \cap \mathcal{D}^{A+}$, with $\sup b(\succsim'_{j^*}, \mathcal{I}(q)) = \sup b(\succsim_{j^*}, \mathcal{I}(q))$, and $\succsim'_i = \succsim_i$ for each $i \notin \{i^*, j^*\}$. By Steps 8 and 11, $f(\succsim') = [\inf b(\succsim_{i^*}, \mathcal{I}(q)), \sup b(\succsim_{j^*}, \mathcal{I}(q))]$. By Lemma 5, for each $z \geq \bar{q}$, $[\inf b(\succsim_{i^*}, \mathcal{I}(q)), \sup b(\succsim_{j^*}, \mathcal{I}(q))] \succsim_{j^*} [\inf b(\succsim_{i^*}, \mathcal{I}(q)), z]$. By Step 8, we have that $\inf f(\succsim_{j^*}, \succsim'_{-j^*}) = \inf b(\succsim_{i^*}, \mathcal{I}(q))$. By *strategy-proofness*, $f(\succsim_{j^*}, \succsim'_{-j^*}) \succsim_{j^*} f(\succsim')$. Hence, $f(\succsim_{j^*}, \succsim'_{-j^*}) = f(\succsim')$. We next show that restoring voter j^*'s original preference does not change the superior endpoint. Suppose, to the contrary, that $\sup f(\succsim) \neq \sup b(\succsim_{j^*}, \mathcal{I}(q))$. Let $\succsim'' \in \mathcal{D}^A$, such that $\succsim''_{i^*} \in \mathcal{D}^A(q) \cap \mathcal{D}^{A^*+}$, $\sup b(\succsim''_{i^*}, \mathcal{I}(q)) = \sup f(\succsim)$, and $\succsim''_i = \succsim'_i$ for each $i \neq i^*$. By Step 11, $f(\succsim'') = [\underline{q}, \sup f(\succsim')]$. By *strategy-proofness*, $f(\succsim_{j^*}, \succsim''_{-j^*}) \succsim_j f(\succsim'')$. Assume now that $\inf f(\succsim_{j^*}, \succsim''_{-j^*}) \neq \underline{q}$. Observe that there exists $\succsim'''_{j^*} \in \mathcal{D}^A(q) \cap \mathcal{D}^{A^*-}$, such that

$$\inf b(\succsim'''_{j^*}, \mathcal{I}(q)) = \inf f(\succsim_{j^*}, \succsim''_{-j^*}).$$

By Step 11, $f(\succsim'''_{j^*}, \succsim''_{-j^*}) = [\underline{q}, \bar{q}]$. However, since $\succsim'''_{j^*} \in \mathcal{D}^{A^*-}$, $f(\succsim_{j^*}, \succsim''_{-j^*}) \succ'''_{j^*} f(\succsim'''_{j^*}, \succsim''_{-j^*})$, which contradicts *strategy-proofness*. Hence, $\inf f(\succsim_{j^*}, \succsim''_{-j^*}) = \underline{q}$. Since by *strategy-proofness*, $f(\succsim_{j^*}, \succsim''_{-j^*}) \succsim_{j^*} f(\succsim'') = [\underline{q}, \sup b(\succsim_{j^*}, \mathcal{I}(q))]$. By Lemma 5, for each $z \geq \bar{q}$,

$$[\underline{q}, \sup b(\succsim_{j^*}, \mathcal{I}(q))] \succsim_{j^*} [\underline{q}, z],$$

and $f(\succsim_{j^*}, \succsim''_{-j^*}) = f(\succsim'')$. Finally, since $\succsim''_{i^*} \in \mathcal{D}^{A^*+}$ and $\sup f(\succsim) = b(\succsim''_{i^*}, \mathcal{I}(q))$, $f(\succsim) \succ''_{i^*} f(\succsim''_{i^*}, \succsim_{-i^*})$, which contradicts *strategy-proofness*. Hence, $\sup f(\succsim) = \sup b(\succsim_{j^*}, \mathcal{I}(q))$. Having established that the superior endpoint is fixed, it only remains to determine the inferior endpoint. Since for each $z' \leq \underline{q}$, by Lemma 5,

$$f(\succsim_{j^*}, \succsim'_{-j^*}) = [\inf b(\succsim_{i^*}, \mathcal{I}(q)), \sup b(\succsim_{j^*}, \mathcal{I}(q))] \succsim_{i^*} [z', \sup b(\succsim_{j^*}, \mathcal{I}(q))].$$

By *strategy-proofness*, $f(\succsim) \succsim_{i^*} f(\succsim_{j^*}, \succsim'_{-j^*})$ and $\sup f(\succsim) = \sup b(\succsim_{j^*}, \mathcal{I}(q))$ imply that $f(\succsim) = [\inf b(\succsim_{i^*}, \mathcal{I}(q)), \sup b(\succsim_{j^*}, \mathcal{I}(q))]$. □

Proof of Theorem 2. It is immediate to check that endpoint dictatorial interval choice functions satisfy *strategy-proofness* on $\mathcal{D}^A$, and dictatorial interval choice functions satisfy *strategy-proofness* on $\mathcal{D}^M$. We therefore prove the converse statements. By Proposition 2, every interval choice

function with domain $\mathcal{D}^A$ that satisfies *strategy-proofness*and $r_f = \mathcal{I}(q)$ is an endpoint dictatorial interval choice function, so it remains to prove that every interval choice function defined on $\mathcal{D}^M$ that satisfies *strategy-proofness* and has full range is dictatorial.

Let $f : (\mathcal{D}^M)^N \to \mathcal{I}(q)$ be an interval choice function that satisfies *strategy-proofness* and $r_f = \mathcal{I}(q)$. Define the auxiliary $\hat{f} : (\mathcal{D}^A)^N \to \mathcal{I}(q)$ as the restriction of f to $(\mathcal{D}^A)^N$. Since f satisfies *strategy-proofness*, $\hat{f}$ satisfies *strategy-proofness*, and by Lemma 1, $\hat{f}$ satisfies *unanimity*, and therefore $r_{\hat{f}} = \mathcal{I}(q)$. By Proposition refPr.unr, $\hat{f}$ is an endpoint dictatorial interval choice function. There exist $i^*, j^* \in N$ such that for each $\succsim \in (\mathcal{D}^A)^N$, $\hat{f}(\succsim) = [\inf b(\succsim_{i^*}, \mathcal{I}(q)), \sup b(\succsim_{j^*}, \mathcal{I}(q))]$.

Suppose, to the contrary, that $i^* \neq j^*$. Replicating the arguments in Step 8 of the proof of Proposition 2, we obtain that for each $\succsim \in (\mathcal{D}^M)^N$, if $\succsim_{I^*} \in \mathcal{D}^A$, $\inf f(\succsim) = \inf b(\succsim_{i^*}, \mathcal{I}(q))$. Analogously, for each $\succsim \in (\mathcal{D}^M)^N$, if $\succsim_{j^*} \in \mathcal{D}^A$, $\sup f(\succsim) = \sup b(\succsim_{j^*}, \mathcal{I}(q))$.

Consider $\succsim \in (\mathcal{D}^A)^N$ such that $b(\succsim_{i^*}) = [\underline{q}-2]$, $b(\succsim_{i^*}, \mathcal{I}(q)) = q$, $b(\succsim_{j^*}, \mathcal{I}(q)) = q$, and for each $x, x' \in \mathcal{I}(q)$, if $(\underline{q}-2) \notin x$, $(\underline{q}-2) \in x'$ then $x \succ_{j^*} x'$. Since $\succsim \in (\mathcal{D}^A)^N$, $f(\succsim) = q$. Consider $\succsim' \in (\mathcal{D}^M)^N$ such that $\succsim'_{i^*} \in \mathcal{D}^M \setminus \mathcal{D}^A$, $b(\succsim'_{i^*}) = b(\succsim'_{i^*}, \mathcal{I}(q)) = [\underline{q}-2, \bar{q}]$, but $[\underline{q}, \bar{q}+1] \succ'_{i^*} [\underline{q}-2, \bar{q}+1]$, and for each $i \neq i^*$, $\succsim'_i = \succsim_i$. Note that there exists $\succsim^*_{i^*} \in \mathcal{D}^A$ such that $b(\succsim^*_{i^*}, \mathcal{I}(q)) = [\underline{q}-2, \bar{q}]$. By the argument of the previous paragraph and $f(\succsim^*_{i^*}, \succsim'_{-i^*}) = [\underline{q}-2, \bar{q}]$. By *strategy-proofness*, $f(\succsim') \succsim'_{i^*} f(\succsim^*_{i^*}, \succsim'_{-i^*})$, which implies $f(\succsim') = [\underline{q}-2, \bar{q}]$. Next, consider $\succsim'' \in (\mathcal{D}^M)^N$ such that $b(\succsim''_{j^*}, \mathcal{I}(q)) = [\underline{q}, \bar{q}+1]$, for each $y, y' \in \mathcal{I}(q)$, if $(\bar{q}-2) \notin y$ and $(\bar{q}-2) \in y'$, then $y \succ''_{j^*} y'$, and for each $i \neq j^*$, $\succsim''_i = \succsim'_i$. By *strategy-proofness*, $f(\succsim') \succsim'_{j^*} f(\succsim''_{j^*}, \succsim'_{j^*}) = f(\succsim'')$. Since $(\underline{q}-2) \in f(\succsim')$, $(\underline{q}-2) \in f(\succsim'')$. By the argument of the previous paragraph, there exists $\succsim^*_{j^*} \in \mathcal{D}^A$ such that $f(\succsim^*_{j^*}, \succsim''_{-j^*}) = [\underline{q}-2, \bar{q}+1]$. By *strategy-proofness*, $f(\succsim'') \succsim''_{j^*} f(\succsim^*_{j^*}, \succsim''_{-j^*}) = [\underline{q}-2, \bar{q}+1]$. Hence, $f(\succsim'') = [\underline{q}-2, \bar{q}+1]$. Let $\succsim^+ \in (\mathcal{D}^M)^N$ such that $\succsim^+_{i^*} \in \mathcal{D}^A$, $b(\succsim^+_{i^*}, \mathcal{I}(q)) = [\underline{q}, \bar{q}+1]$, and $\succsim^+_i = \succsim''_i$ for each $i \neq i^*$. By the argument of the previous paragraph, $\inf f(\succsim^+) = \underline{q}$. Since there exists $\succsim'_{j^*} \in \mathcal{D}^A$ with $\sup b(\succsim'_{j^*}, \mathcal{I}(q)) = (\bar{q}+1)$, by *strategy-proofness*, $\sup f(\succsim^+) = \sup b(\succsim^*_{j^*}, \mathcal{I}(q)) = (\bar{q}+1)$. Thus, $f(\succsim^+) = [\underline{q}, \bar{q}+1]$. However, $f(\succsim^+) \succ''_{i^*} f(\succsim''_{i^*}, \succsim^+_{-i^*}) = f(\succsim'') = [\underline{q}-2, \bar{q}+1]$, contradicting *strategy-proofness*. Hence, $i^* = j^*$ and $\hat{f}$ is a dictatorial interval choice function.

Finally, the arguments used in the proof of Proposition 2, Step 12, Case 1, imply that if $\hat{f}$ is dictatorial, then f is also dictatorial. □

5.4 Proof of Theorem 3

The proof of Theorem 3 requires a different approach from that of Theorem 1. In the unrestricted framework, the range of the interval choice function is $\mathcal{I}$, which is not a product space. Consequently, the decomposition arguments based on the independent aggregation of the inferior and superior endpoints are no longer applicable. Instead, we follow the option sets approach introduced by Barberà and Peleg (1990).

Let $\widetilde{\mathcal{D}} \subseteq \mathcal{D}$ and let $f : \widetilde{\mathcal{D}}^N \to \mathcal{I}$. For each voter $i \in N$ and each preference profile $\succsim_{-i} \in \widetilde{\mathcal{D}}^{N\setminus\{i\}}$, the ***option set*** of voter i at $\succsim_{-i}$ is the set of intervals that can be selected by f when the preferences of all voters except i are fixed at $\succsim_{-i}$. Formally,

$$o_i(\succsim_{-i}) = \left\{ x \in \mathcal{I} : \text{ there exists } \succsim_i \in \widetilde{\mathcal{D}} \text{ such that } f(\succsim_i, \succsim_{-i}) = x \right\}.$$

As in the proof of the previous characterization theorems, we first restrict attention to the domain of additively representable preferences. Once the result has been established on this domain, its extension to the full domain of interval-monotonic preferences follows by straightforward arguments.

Proposition 3. *Let $f : (\mathcal{D}^A)^N \to \mathcal{I}$. If f satisfies* strategy-proofness *and* ontoness, *then f is dictatorial.*

Proof. We proceed in several steps. We first consider a two-voter society and analyze the structure of option sets. After establishing two preliminary observations (Steps 1–2), we focus on the subdomain of additively representable preferences in which voters regard a unique singleton interval as acceptable, and on the singleton intervals contained in each voter's option set. We show that, for each preference in this subdomain, a voter's option set either contains every singleton interval or contains only the uniquely acceptable singleton interval of the other voter. The first possibility implies that the voter can force any singleton outcome and is therefore a dictator on the subdomain in which voters report only singleton intervals as acceptable (Steps 3–7). We then show that, for every profile of additively representable preferences, the option set of the other voter consists only of the dictator's most-preferred interval. Hence, the same voter is in fact a dictator on the whole domain (Steps 8–9). Finally, we extend the result to societies with an arbitrary number of voters by an induction argument that repeatedly applies the reasoning developed in Step 8 of the proof of Proposition 2.

We begin with the two-voter case. Let $N = \{i, j\}$ and let f satisfy *strategy-proofness* and *ontoness*. By Lemma 1, f also satisfies *unanimity*.

Step 1. For each $\succsim_j \in \mathcal{D}^A$, $b(\succsim_j) \in o_i(\succsim_j)$.

Indeed, let $\succsim_i \in \mathcal{D}^A$ be such that $b(\succsim_i) = b(\succsim_j)$. Since f satisfies *unanimity*,

$$f(\succsim_i, \succsim_j) = b(\succsim_j),$$

which proves that $b(\succsim_j) \in o_i(\succsim_j)$.

Step 2. For each $\succsim \in (\mathcal{D}^A)^N$, $f(\succsim) = b(\succsim_i, o_i(\succsim_j))$.

Suppose, to the contrary, that there exist $\succsim \in (\mathcal{D}^A)^N$ and $x \in o_i(\succsim_j)$, such that $x \succ_i f(\succsim)$. Since $x \in o_i(\succsim_j)$, there exists $\succsim_i' \in \mathcal{D}^A$ such that $f(\succsim_i', \succsim_j) = x$. Hence, $f(\succsim_i', \succsim_j) \succ_i f(\succsim)$, which contradicts *strategy-proofness*.

We first investigate option sets on the subdomain in which voters regard a unique integer as acceptable. Let

$$\mathcal{I}_1 = \{x \in \mathcal{I} : \underline{x} = \bar{x}\}$$

denote the set of singleton intervals, and let $\mathcal{D}_1^A \subseteq \mathcal{D}^A$ be the set of preferences whose unique most-preferred interval is a singleton, that is,

$$\mathcal{D}_1^A = \{\succsim_i \in \mathcal{D}^A : b(\succsim_i) \in \mathcal{I}_1\}.$$

Step 3. For each $\succsim_i, \succsim_i' \in \mathcal{D}_1^A$, if $b(\succsim_i) = b(\succsim_i')$ then $o_j(\succsim_i) \cap \mathcal{I}_1 = o_j(\succsim_i') \cap \mathcal{I}_1$.

Suppose, to the contrary, that there exist $x \in \mathcal{I}_1$, $\succsim_i, \succsim_i' \in \mathcal{D}_1^A$ such that $x = b(\succsim_i) = b(\succsim_i')$, and there exists $x' \in \mathcal{I}_1$, $x' \neq x$ such that $x' \in o_j(\succsim_i') \setminus o_j(\succsim_i)$. Since $x, x' \in \mathcal{I}_1$, we have $\underline{x} = \bar{x}$, and $\underline{x}' = \bar{x}'$. We distinguish two cases.

Case 1: $\bar{x}' \notin [\underline{x} - 1, \bar{x} + 1]$. By Lemma 2.i), there exists $\succsim_j \in \mathcal{D}_1^A$ such that for each $x'' \in \mathcal{I} \setminus \{x, x'\}$ $x' \succ_j x \succ_j x''$. By Step 2, $f(\succsim_i, \succsim_j) = x'$, and $f(\succsim_i', \succsim_j) = x$, and $f(\succsim_i', \succsim_j) \succ_i f(\succsim_i, \succsim_j)$, contradicting *strategy-proofness*.

Case 2: $\bar{x}' \in [\underline{x}-1, \bar{x}+1]$. Without loss of generality, let $\bar{x}' = \bar{x}+1$. By Lemma 2.ii), there exists $\succsim_j' \in \mathcal{D}_1^A$ such that for each $x'' \in \mathcal{I} \setminus \{x, x', [\bar{x}, \bar{x}+1]\}$, $x' \succ_j [\bar{x}, \bar{x}+1] \succ_j x \succ_j x''$. By Step 2, $f(\succsim_i, \succsim_j') = x'$ and $f(\succsim_i', \succsim_j') = [\bar{x}, \bar{x}+1]$. For each $\succsim_i^* \in \mathcal{D}_1^A$ with $b(\succsim_i^*) = x$, $x \succ_i^* [\bar{x}, \bar{x}+1] \succ_i^* x'$. Hence, $f(\succsim_i', \succsim_j') \succ_i f(\succsim_i, \succsim_j')$, contradicting *strategy-proofness*.

Step 4. For each $\succsim_i \in \mathcal{D}_1^A$, either $o_j(\succsim_i) \cap \mathcal{I}_1 = \{b(\succsim_i)\}$ or $o_j(\succsim_i) \cap \mathcal{I}_1 = \mathcal{I}_1$.

Suppose, to the contrary, that there exist $\succsim_i \in \mathcal{D}_1^A$ and $x, x', x'' \in \mathcal{I}_1$ such that $x = b(\succsim_i)$, $x' \in o_j(\succsim_i)$, $x'' \notin o_j(\succsim_i)$. By Step 3, without loss of generality, we may assume that $\succsim_i$ is such that $x'' \succ_i x'$. We distinguish two cases.

Case 1: $\bar{x}' \notin [\underline{x}''-1, \bar{x}''+1]$. By Lemma 2.i) there exists $\succsim_j \in \mathcal{D}_1^A$ such that $x'' = b(\succsim_j)$, $x'' \succ_j x' \succ_j x^*$ for each $x^* \in \mathcal{I} \setminus \{x', x''\}$. By Step 2, $f(\succsim_i, \succsim_j) = b(\succsim_i, o_j(\succsim_i))$. Since $x'' \notin o_j(\succsim_i)$, $f(\succsim_i, \succsim_j) = x'$. Let $\succsim_i'$ be such that $b(\succsim_i') = b(\succsim_j) = x''$. By *unanimity*, $f(\succsim_i', \succsim_j) = x''$. Hence, $f(\succsim_i', \succsim_j) \succ_i f(\succsim_i, \succsim_j)$ contradicting *strategy-proofness*.

Case 2: $\bar{x}' \in [\underline{x}''-1, \bar{x}''+1]$. Assume without loss of generality that $\bar{x}'' = \bar{x}'+1$. Then, $[\bar{x}', \bar{x}''] \in \mathcal{I}$. By Lemma 2.ii), there exists $\succsim_j \in \mathcal{D}_1^A$ such that $x'' = b(\succsim_j)$, $x'' \succ_j [\bar{x}', \bar{x}''] \succ_j x' \succ_j x^*$ for each $x^* \in \mathcal{I} \setminus \{x', x'', [\bar{x}', \bar{x}'']\}$. By Step 2, $f(\succsim_i, \succsim_j) = b(\succsim_i, o_j(\succsim_i))$, and since $x'' \notin o_j(\succsim_i)$, either $f(\succsim_i, \succsim_j) = x'$ or $f(\succsim_i, \succsim_j) = [\bar{x}', \bar{x}'']$. Let $\succsim_i'$ be such that $b(\succsim_i') = b(\succsim_j) = x''$. By *unanimity*, $f(\succsim_i', \succsim_j) = x''$. Since $b(\succsim_i) = x \in \mathcal{I}_1$, both x' and x'' are unacceptable according to $\succsim_i$, and therefore $x' \succ_i [\bar{x}', \bar{x}'']$. Hence, $f(\succsim_i', \succsim_j) \succ_i f(\succsim_i, \succsim_j)$, contradicting *strategy-proofness*.

While Theorem 3 is stated for the family of finite intervals of the countably infinite set of integers, its proof applies equally to finite sets of at least four categories. Step 5 is the only part of the proof that requires the existence of at least four categories. For completeness, we present an argument that does not rely on the countable infinitude of the set of integers and therefore also applies to finite problems with exactly four categories. Example 4 shows that the conclusion of Theorem 3 no longer holds when there are only three categories.

Step 5. If there exists $\succsim_i \in \mathcal{D}_1^A$ such that $o_j(\succsim_i) \cap \mathcal{I}_1 = \mathcal{I}_1$, then for each $\succsim_i' \in \mathcal{D}_1^A$, $o_j(\succsim_i') \cap \mathcal{I}_1 = \mathcal{I}_1$.

Suppose, to the contrary, that there exist $\succsim_i, \succsim_i' \in \mathcal{D}_1^A$ such that

$$o_j(\succsim_i) \cap \mathcal{I}_1 = \mathcal{I}_1 \qquad \text{and} \qquad o_j(\succsim_i') \cap \mathcal{I}_1 = \{b(\succsim_i')\}.$$

By Step 3, $b(\succsim_i) \neq b(\succsim_i')$. Let $b(\succsim_i) = x$ and $b(\succsim_i') = x'$. By Lemma 2.i) there exists $\succsim_j \in \mathcal{D}_1^A$ such that $b(\succsim_j) = x'' \notin \{x, x'\}$, $\bar{x}'' \notin [\bar{x} - 1, \bar{x} + 1]$, and for each $y \in \mathcal{I} \setminus \{x', x''\}$, $x'' \succ_j x' \succ_j y$. By Step 2, $f(\succsim_i, \succsim_j) = x''$ and $f(\succsim_i', \succsim_j) = x'$. By Lemma 2.i) there exists $\succsim_i^* \in \mathcal{D}_1^A$ such that $b(\succsim_i^*) = b(\succsim_i)$, and for each $y' \in \mathcal{I}_1 \setminus \{x, x'\}$, $x \succ_i^* x' \succ_i^* y'$. By Step 3, $o_j(\succsim_i^*) \cap \mathcal{I}_1 = \mathcal{I}_1$, and by Step 2, $f(\succsim_i^*, \succsim_j) = x''$. Hence, $f(\succsim_i', \succsim_j) \succ_i^* f(\succsim_i^*, \succsim_j)$, contradicting *strategy-proofness*.

Step 6. There is $i \in N$ such that for each $\succsim_i \in \mathcal{D}_1^A$, $o_j(\succsim_i) \cap \mathcal{I}_1 = \{b(\succsim_i)\}$.

By Steps 4 and 5, for each $\succsim_i \in \mathcal{D}_1^A$, either $o_j(\succsim_i) \cap \mathcal{I}_1 = \mathcal{I}_1$ or $o_j(\succsim_i) \cap \mathcal{I}_1 = \{b(\succsim_i)\}$. Likewise, for each $\succsim_j \in \mathcal{D}_1^A$, either $o_i(\succsim_j) \cap \mathcal{I}_1 = \mathcal{I}_1$ or $o_i(\succsim_i) \cap \mathcal{I}_1 = \{b(\succsim_j)\}$, Suppose, to the contrary that each $\succsim_i, \succsim_j \in \mathcal{D}_1^A$, $o_j(\succsim_i) \cap \mathcal{I}_1 = o_i(\succsim_j) \cap \mathcal{I}_1 = \mathcal{I}_1$. Let $\succsim_i, \succsim_j \in \mathcal{D}_1^A$, such that $b(\succsim_i) \neq b(\succsim_j)$. By Step 2, $f(\succsim_i, \succsim_j) = b(\succsim_i)$ and also $f(\succsim_i, \succsim_j) = b(\succsim_j)$, a contradiction.

We next show that if, for one voter, the option set induced by every preference in $\mathcal{D}_1^A$ contains only the singleton most-preferred by the other voter, then the option set of the other voter necessarily contains every singleton interval.

Step 7. If for each $\succsim_i \in \mathcal{D}_1^A$, $o_j(\succsim_i) \cap \mathcal{I}_1 = \{b(\succsim_i)\}$, then for each $\succsim_j \in \mathcal{D}_1^A$, $o_i(\succsim_j) \cap \mathcal{I}_1 = \mathcal{I}_1$.

By Step 6, we assume without loss of generality that for each $\succsim_i \in \mathcal{D}_1^A$, $o_j(\succsim_i) \cap \mathcal{I}_1 = \{b(\succsim_i)\}$. Suppose, to the contrary, that for each $\succsim_i, \succsim_j \in \mathcal{D}_1^A$, $o_i(\succsim_j) \cap \mathcal{I}_1 = \{b(\succsim_j)\}$ and $o_j(\succsim_i) \cap \mathcal{I}_1 = \{b(\succsim_i)\}$. Let $\succsim_i, \succsim_j \in \mathcal{D}_1^A$ with $b(\succsim_i) = x$, $b(\succsim_j) = x'$, and without loss of generality assume that $\bar{x} < \bar{x}' - 1$. By Step 3 and Lemma 2.i), we may assume without loss of generality that for each $x'' \in \mathcal{I} \setminus \{x, x'\}$, $x \succ_i x' \succ_i x''$ and $x' \succ_j x \succ_j x''$. Since $x' \in o_i(\succsim_j)$ and $x \notin o_i(\succsim_j)$, by Step 2, $f(\succsim_i, \succsim_j) = x'$. However, the symmetric argument for voter$j$ yields $f(\succsim_i, \succsim_j) = x$, a contradiction.

Up to this point, we have proved that if one voter reports a preference whose most-preferred interval is a singleton, then the singleton intervals in the option set of the other voter reduce to that singleton. Moreover, whenever the other voter reports a preference whose most-preferred interval is a singleton, the first voter has every singleton interval in her option set. Therefore, for every preference profile $\succsim \in (\mathcal{D}_1^A)^{\{i,j\}}$, there exists $i^* \in \{i, j\}$ such that $[f(\succsim_i, \succsim_j) = b(\succsim_{i^*})$. Hence, i^* is a dictator on the domain $\mathcal{D}_1^A$. In the next step, we show that i^* is in fact a dictator on the whole domain $\mathcal{D}^A$.

Step 8. For each $\succsim_{i^*} \in \mathcal{D}^A$, $o_j(\succsim_{i^*}) = \{b(\succsim_{i^*})\}$.

Suppose, to the contrary, that there exists $\succsim_{i^*} \in \mathcal{D}^A$ such that $o_j(\succsim_{i^*}) \neq \{b(\succsim_{i^*})\}$. Let $b(\succsim_{i^*}) = [\underline{b}, \bar{b}]$. Hence, there exists $x \in o_j(\succsim_{i^*})$ such that $x \neq b(\succsim_{i^*})$. Without loss of generality, assume that x is minimal with respect to inclusion; that is, there is no $x' \in o_j(\succsim_{i^*})$ such that $x' \subset x$. We distinguish two cases.

Case 1. $x \subset b(\succsim_i^*)$.

Notice first that $x \subset b(\succsim_i^*)$ implies that $b(\succsim_i^*) \notin \mathcal{I}_1$ and either $\underline{b} \notin x$ or $\bar{b} \notin x$. Assume that $\underline{b} \notin x$. A parallel argument applies when $\bar{b} \notin x$. By Lemma 2.v), there exists $\succsim_j \in \mathcal{D}_1^A$ such that $b(\succsim_j) = \underline{b}$ and for each $y, y' \in \mathcal{I}$, if $y \subseteq [\underline{b}, \bar{b}]$ and y' contains an integer outside $[\underline{b}, \bar{b}]$, then $y \succ_j y'$. Since x is minimal with respect to inclusion, by Step 2, $f(\succsim_{i^*}, \succsim_j) = x$. Let $\succsim'_{i^*} \in \mathcal{D}_1^A$ be such that $b(\succsim'_{i^*}) = b(\succsim_{i^*})$ and for each $y, y' \subseteq [\underline{b}, \bar{b}]$, if $\bar{b} \in y$ and $\bar{b} \notin y'$, then $y \succ'_{i^*} y'$. Note that $\succsim'_{i^*}$ may be represented by a utility function u'_{i^*} such that $u'_{i^*}(\bar{b}) > \sum_{z \in [\underline{b}, \bar{b}-1]} u'_{i^*}(z)$. By *strategy-proofness*, $f(\succsim'_{i^*}, \succsim_j) \neq b(\succsim_{i^*})$ and by Lemma 3, $f(\succsim'_{i^*}, \succsim_j) \subset b(\succsim_{i^*})$. Let $\succsim''_{i^*} \in \mathcal{D}_1^A$ be such that $b(\succsim''_{i^*}) = [\bar{b}]$. By Steps 4 and 5, $o_{i^*}(\succsim_j) \cap \mathcal{I}_1 = \mathcal{I}_1$. By Step 2, $f(\succsim''_{i^*}, \succsim_j) = [\bar{b}]$. By *strategy-proofness*, $f(\succsim'_{i^*}, \succsim_j) \succsim'_{i^*} f(\succsim''_{i^*}, \succsim_j)$, which implies that $\bar{b} \in f(\succsim'_{i^*}, \succsim_j)$ and since $f(\succsim'_{i^*}, \succsim_j) \neq b(\succsim_{i^*})$, $\underline{b} \notin f(\succsim'_{i^*}, \succsim_j)$. Finally, let $\succsim'_j \in \mathcal{D}^A$ be such that $b(\succsim'_j) = b(\succsim_{i^*})$. Since, by Lemma 1, f satisfies *unanimity*, $f(\succsim'_{i^*}, \succsim'_j) = b(\succsim_{i^*})$, $\underline{b} \in f(\succsim'_{i^*}, \succsim'_j)$, and therefore $f(\succsim'_{i^*}, \succsim'_j) \succ_j f(\succsim'_{i^*}, \succsim_j)$, contradicting *strategy-proofness*.

Case 2. For some $z \in x$, $z \notin b(\succsim_{i^*})$.

We consider two subcases.

Subcase 2.1. $x \cap b(\succsim_{i^*}) = \varnothing$. Let $z^* \in x$. By Lemma 2.v), there exists $\succsim_j \in \mathcal{D}_1^A$, such that $b(\succsim_j) = [z^*]$, and for each $y, y' \in \mathcal{I}$, if $y \subseteq x$ and y' contains an integer outside x, then $y \succ_j y'$. By Step 2, since x is minimal with respect to inclusion, $f(\succsim_{i^*}, \succsim_j) = x$. Note that since $\succsim_j \in \mathcal{D}_1^A$, by Step 7, $o_{i^*}(\succsim_j) \cap \mathcal{I}_1 = \mathcal{I}_1$. For every $z \in [\underline{b}, \bar{b}]$, we have $[z] \in o_{i^*}(\succsim_j)$ and $[z] \succ_{i^*} x = f(\succsim_{i^*}, \succsim_j)$. Hence, by Step 2, $f(\succsim_{i^*}, \succsim_j) = b(\succsim_{i^*}, o_{i^*}(\succsim_j)) \neq x$, a contradiction.

Subcase 2.2. $x \cap b(\succsim_{i^*}) \neq \varnothing$. Without loss of generality, assume that $\underline{x} < \underline{b}$. Assume first that $[\underline{b}] \neq b(\succsim_{i^*})$. Hence, $\succsim_{i^*} \notin \mathcal{D}_1^A$. By Case 1, for each $z \in x$ with $z < \underline{b}$, $[z] \notin o_j(\succsim_{i^*})$. By Lemma 2.v), there exists $\succsim_j \in \mathcal{D}_1^A$ such that $b(\succsim_j) = [\underline{b}]$ and for each $y, y' \in \mathcal{I}$, if $y \subseteq x$ and y' contains an integer outside x, then $y \succ_j y'$. Since x is minimal with respect to

inclusion, $f(\succsim_{i^*}, \succsim_j) = x$. However, by Lemma 3, $f(\succsim_{i^*}, \succsim_j) \subseteq [\underline{b}, \bar{b}]$, a contradiction. Finally, assume that $[\underline{b}] = b(\succsim_{i^*})$ and $\succsim_{i^*} \in \mathcal{D}_1^A$. By Lemma 2.v), there exists $\succsim'_j \in \mathcal{D}_1^A$ such that $b(\succsim'_j) = [\underline{x}]$, and such that for each $y, y' \in \mathcal{I}$, if $y \subseteq x$ and y' contains an integer outside x, then $y \succ'_j y'$. Since x is minimal with respect to inclusion, and by Step 2, $f(\succsim_{i^*}, \succsim'_j) = x$. However, since $\succsim'_j \in \mathcal{D}_1^A$, by Step 7, $o_{i^*}(\succsim'_j) \cap \mathcal{I}_1 = \mathcal{I}_1$, $b(\succsim_{i^*}) \in o_{i^*}(\succsim'_j)$. Therefore, by Step 2, $f(\succsim_{i^*}, \succsim'_j) = b(\succsim_{i^*})$, a contradiction.

Step 9. f is dictatorial. That is, for each $\succsim \in (\mathcal{D}^A)^{\{i,j\}}$, $f(\succsim) = b(\succsim_{i^*})$.

By Step 2, for each $\succsim \in (\mathcal{D}^A)^{\{i,j\}}$, $f(\succsim) = b(\succsim_j, o_j(\succsim_{i^*}))$. By Steps 1 and 8, $o_j(\succsim_{i^*}) = \{b(\succsim_{i^*})\}$. Therefore, $f(\succsim) = b(\succsim_{i^*})$.

We have proved that if $|N| = 2$, every $f : (\mathcal{D}^A)^N \to \mathcal{I}$ that satisfies *strategy-proofness* and *ontoness* is dictatorial. We now extend the result to arbitrary finite societies of voters by induction.

Induction Step. Assume that for each society N with cardinality $2 \leq | N | \leq n$, where $n \geq 2$, every interval choice function $f : (\mathcal{D}^A)^N \to \mathcal{I}$ that satisfies *strategy-proofness* and *ontoness* is dictatorial. $|N| = n + 1$.

Let $| N | = n+1$ and $i, j \in N$. Define the two-voter interval choice function $\bar{f} : (\mathcal{D}^A)^{\{i,j\}} \to \mathcal{I}$ as follows. For each $\succsim \in (\mathcal{D}^A)^{\{i,j\}}$, $\bar{f}(\succsim_i, \succsim_j) = f(\succsim_i, \succsim'_{-i})$ where $\succsim'_{-i}$ is the profile in $(\mathcal{D}^A)^{N\setminus\{i\}}$ defined by $\succsim'_{j'} = \succsim_j$ for every $j' \neq i$. Since f satisfies *strategy-proofness*, for each $\succsim \in (\mathcal{D}^A)^{\{i,j\}}$, and each $\succsim'_i, \succsim'_j \in \mathcal{D}^A$, $\bar{f}(\succsim_i, \succsim_j) \succsim_i \bar{f}(\succsim'_i, \succsim_j)$. Moreover, repeated applications of *strategy-proofness* to the voters in $N \setminus \{i\}$ yield $\bar{f}(\succsim_i, \succsim_j) \succsim_j \bar{f}(\succsim_i, \succsim'_j)$. Hence, $\bar{f}$ satisfies *strategy-proofness*. Since f satisfies *strategy-proofness* and *ontoness*, by Lemma 1, f also satisfies *unanimity*. Hence, $\bar{f}$ satisfies *unanimity* and *ontoness*. By the induction hypothesis, $\bar{f}$ is also dictatorial. We next show that f is dictatorial.

Assume that j is the dictator for $\bar{f}$. Then, for each $\succsim_i, \succsim'_i$, $\bar{f}(\succsim_i, \succsim_j) = \bar{f}(\succsim'_i, \succsim_j)$. Define the $(n-1)$-voter interval choice function $\tilde{f}_{\succsim_i} : (\mathcal{D}^A)^{N\setminus\{i\}} \to \mathcal{I}$ by $\tilde{f}_{\succsim_i}(\succsim_{-i}) = f(\succsim_i, \succsim_{-i})$ for each $\succsim_{-i} \in (\mathcal{D}^A)^{N\setminus\{i\}}$. Since f satisfies *strategy-proofness* and *ontoness*, $\tilde{f}_{\succsim_i}$ also satisfies *strategy-proofness* and *ontoness*. Hence, by the induction hypothesis, $\tilde{f}_{\succsim_i}$ is dictatorial. We next show that the dictator of $\tilde{f}_{\succsim_i}$ is independent of $\succsim_i$. Suppose, to the contrary, that there exist $j, j' \in N \setminus \{i\}$ and $\succsim_i, \succsim'_i \in \mathcal{D}^A$ such that j is a dictator of $\tilde{f}_{\succsim_i}$ and j' is a dictator of $\tilde{f}_{\succsim'_i}$. Since $\succsim_i \neq \succsim'_i$, there exist $x, x' \in \mathcal{I}$ such that $x \succ_i x'$ and $x' \succ'_i x$. Let $\succsim^* \in (\mathcal{D}^A)^N$ be such that $\succsim^*_i = \succsim_i$, $b(\succsim^*_j) = x'$,

$b(\succsim^*_{j'}) = x$, and let the preferences of the remaining voters be arbitrary. Since j is the dictator of $\tilde{f}_{\succsim_i}$, $f(\succsim^*) = x'$. Similarly, since j' is the dictator of $\tilde{f}_{\succsim'_i}$, $f(\succsim'_i, \succsim^*_{-i}) = x$. However, this implies that $f(\succsim'_i, \succsim^*_{-i}) \succ^*_i f(\succsim^*)$, contradicting *strategy-proofness*.

Finally, assume that i is the dictator of $\bar{f}$. The remainder of the proof replicates the argument in Step 8 of the proof of Proposition 2. Consider an arbitrary profile $\succsim \in (\mathcal{D}^A)^N$. Let $x = b(\succsim_i)$. By Lemma 2.iv) , there exists a profile $\succsim' \in (\mathcal{D}^A)^N$ such that $\succsim'_i = \succsim_i$, and for each $j, j' \in N \setminus \{i\}$, $\succsim'_j = \succsim'_{j'}$, $b(\succsim'_j) \cap x = \varnothing$, and such that for each $y, y' \in \mathcal{I}$, $\underline{x} \notin y$, $\underline{x} \in y'$ implies that $y \succ'_j y'$. Since i is the dictator of $\bar{f}$, $f(\succsim') = \bar{f}(\succsim'_i, \succsim'_j) = b(\succsim'_i) = x$ and $\underline{x} \in f(\succsim')$. Let $j \neq i$. By *strategy-proofness*, $f(\succsim') \succ'_j f(\succsim_j, \succsim'_{-j})$, and $\underline{x} \in f(\succsim_j, \succsim'_{-j})$. Repeating the same argument for each remaining voters $j' \neq i$, we obtain that $\underline{x} \in f(\succsim)$. With an analogous reasoning, by Lemma 2.iii), there exists $\succsim'' \in (\mathcal{D}^A)^N$ such that $\succsim''_i = \succsim_i$, for each $j, j' \in N \setminus \{i\}$, $\succsim''_j = \succsim''_{j'}$, $b(\succsim''_j) = [\underline{x} - 1]$ and such that for each $y, y' \in \mathcal{I}$, $\underline{x} - 1 \in y$, $\underline{x} - 1 \notin y'$ implies that $y \succ''_j y'$. Applying the same argument, we obtain that $(\underline{x} - 1) \notin f(\succsim)$, and therefore $\inf f(\succsim) = \underline{x}$. The parallel argument proves that $\sup f(\succsim) = \bar{x}$. Therefore, for each $\succsim \in (\mathcal{D}^A)^N$, $f(\succsim) = b(\succsim_i)$. □

Proof of Theorem 3. It is immediate that every dictatorial interval choice function satisfies *strategy-proofness* on $\mathcal{D}^M$ and *ontoness*. To check the converse result, consider an interval choice function $f : (\mathcal{D}^M)^N \to \mathcal{I}$. Define the auxiliary interval choice function $f^A : (\mathcal{D}^A)^N \to \mathcal{I}$ by $f^A(\succsim) = f(\succsim)$ for each $\succsim \in (\mathcal{D}^A)^N$. Since f satisfies *strategy-proofness* and $\mathcal{D}^A \subset \mathcal{D}^M$, f^A satisfies *strategy-proofness*. Since. by Lemma 1, f satisfies *unanimity*, f^A also satisfies *unanimity*, and therefore *ontoness*. By Proposition 3, f^A is dictatorial. Hence, there is $i \in N$ such that for each $\succsim \in (\mathcal{D}^A)^N$, $f^A(\succsim) = b(\succsim_i)$. The arguments used in the induction step of the proof of Proposition 3 extend the dictatorship from $(\mathcal{D}^A)^N$ to $(\mathcal{D}^M)^N$. Hence, for each $\succsim \in (\mathcal{D}^M)^N$, $f(\succsim) = b(\succsim_i)$. □

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