



ELSEVIER

Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

Physics Letters B 586 (2004) 117–124

PHYSICS LETTERS B

www.elsevier.com/locate/physletb

A new family of \mathcal{N} -fold supersymmetry: type B

Artemio González-López^a, Toshiaki Tanaka^b

^a *Departamento de Física Teórica II, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain*

^b *Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*

Received 15 July 2003; received in revised form 24 October 2003; accepted 29 October 2003

Editor: G.F. Giudice

Abstract

We construct a new family of \mathcal{N} -fold supersymmetric systems which is referred to as “type B”. A higher-derivative representation of the \mathcal{N} -fold supercharge for this new family is given by a deformation of the type A \mathcal{N} -fold supercharge. By utilizing the same method as in the $\mathfrak{sl}(2)$ construction of type A \mathcal{N} -fold supersymmetry, we show that this family includes two of the quasi-solvable models of Post–Turbiner type.

© 2004 Elsevier B.V. Open access under [CC BY license](https://creativecommons.org/licenses/by/4.0/).

PACS: 03.65.Ca; 03.65.Fd; 03.65.Ge; 11.30.Pb

Keywords: Quantum mechanics; Quasi-solvability; \mathcal{N} -fold supersymmetry; Intertwining relation

1. Introduction

Progress in the field of quasi-solvability in quantum systems (see [1–4] and references therein) has recently reached a new stage due to the discovery of the intimate relation between quasi-solvability and \mathcal{N} -fold supersymmetry. An idea essentially equivalent to \mathcal{N} -fold supersymmetry was introduced for the first time in Ref. [5] as an extension of ordinary supersymmetric quantum mechanics [6,7], and investigated in various related contexts [8–21]. The connection of \mathcal{N} -fold supersymmetry with quasi-solvability was first uncovered in Ref. [22] in an unexpected way, through the analysis of the large order behavior of the perturbation

series. After similar relations were found in several different contexts [23–27], the equivalence between quasi-solvability and \mathcal{N} -fold supersymmetry was finally proved in Ref. [28].

Up to now, virtually all the \mathcal{N} -fold supersymmetric models explicitly constructed for *arbitrary* \mathcal{N} belonged to the so-called *type A* class, introduced in Ref. [29]. From the viewpoint of the connection with quasi-solvability, it was shown in Refs. [30,31] that type A \mathcal{N} -fold supersymmetric systems are essentially equivalent to the well-known quasi-solvable models constructed from the $\mathfrak{sl}(2)$ generators [4,32,33]. Other recent developments in this respect can be found in Refs. [34–37]. In this Letter we construct a new type of \mathcal{N} -fold supersymmetric models which is a deformation of (and hence different from) the type A class. We also show that the new \mathcal{N} -fold supersymmetric models are related to some of the quasi-solvable models asso-

E-mail addresses: artemio@fis.ucm.es (A. González-López), totanaka@yukawa.kyoto-u.ac.jp (T. Tanaka).

ciated to the spaces of monomials classified by Post and Turbiner [38].

The Letter is organized as follows. In the next section we briefly summarize \mathcal{N} -fold supersymmetry and quasi-solvability. In Section 3 we introduce the type B \mathcal{N} -fold supercharge and construct an \mathcal{N} -fold supersymmetric system with respect to it by the same method used in the $\mathfrak{sl}(2)$ construction of type A \mathcal{N} -fold supersymmetry [30,31]. In Section 4 several examples of the type B \mathcal{N} -fold supersymmetric models constructed in Section 3 are presented. The results obtained in this Letter and some open problems they give rise to are discussed in the last section.

2. \mathcal{N} -fold supersymmetry

First of all, we briefly review \mathcal{N} -fold supersymmetry in one-dimensional quantum mechanics. To this end, we introduce a bosonic coordinate q and fermionic coordinates ψ and ψ^\dagger satisfying

$$\{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0, \quad \{\psi, \psi^\dagger\} = 1. \quad (2.1)$$

The Hamiltonian $\mathbf{H}_{\mathcal{N}}$ is given by

$$\mathbf{H}_{\mathcal{N}} = H_{\mathcal{N}}^- \psi \psi^\dagger + H_{\mathcal{N}}^+ \psi^\dagger \psi, \quad (2.2)$$

where the components $H_{\mathcal{N}}^\pm$ are ordinary scalar Hamiltonians,

$$H_{\mathcal{N}}^\pm = \frac{1}{2} p^2 + V_{\mathcal{N}}^\pm(q), \quad (2.3)$$

with $p = -i d/dq$. \mathcal{N} -fold supercharges $Q_{\mathcal{N}}^\pm$ are introduced by

$$Q_{\mathcal{N}}^- = P_{\mathcal{N}}^- \psi^\dagger, \quad Q_{\mathcal{N}}^+ = P_{\mathcal{N}}^+ \psi, \quad (2.4)$$

where the components $P_{\mathcal{N}}^\pm$ are defined by

$$P_{\mathcal{N}}^- = P_{\mathcal{N}}, \quad P_{\mathcal{N}}^+ = (-1)^{\mathcal{N}} P_{\mathcal{N}}^t \quad (2.5)$$

in terms of an \mathcal{N} th-order linear differential operator $P_{\mathcal{N}}$ of the form

$$\begin{aligned} P_{\mathcal{N}} &= p^{\mathcal{N}} - i w_{\mathcal{N}-1}(q) p^{\mathcal{N}-1} + \dots \\ &\quad + (-i)^{\mathcal{N}-1} w_1(q) p + (-i)^{\mathcal{N}} w_0(q) \\ &= (-i)^{\mathcal{N}} \left(\frac{d^{\mathcal{N}}}{dq^{\mathcal{N}}} + w_{\mathcal{N}-1}(q) \frac{d^{\mathcal{N}-1}}{dq^{\mathcal{N}-1}} + \dots \right. \\ &\quad \left. + w_1(q) \frac{d}{dq} + w_0(q) \right). \end{aligned} \quad (2.6)$$

In Eq. (2.5), the superscript t denotes the transposition of operators defined through a *real* inner product by $(A^t \phi, \psi) = (\phi, A \psi)$. For example, $p^t = -p$ on a suitable space. Note that when all the functions w_k appearing in Eq. (2.6) are real-valued, the operator $P_{\mathcal{N}}^+$ defined by Eq. (2.5) is identical with the adjoint of $P_{\mathcal{N}}$: $P_{\mathcal{N}}^+ = P_{\mathcal{N}}^\dagger$. Hence the above definition is essentially equivalent to the ones in previous articles [28–31,35]. The system (2.2) is said to be *\mathcal{N} -fold supersymmetric* if the following algebra holds:

$$\{Q_{\mathcal{N}}^-, Q_{\mathcal{N}}^-\} = \{Q_{\mathcal{N}}^+, Q_{\mathcal{N}}^+\} = 0, \quad (2.7)$$

$$[Q_{\mathcal{N}}^-, \mathbf{H}_{\mathcal{N}}] = [Q_{\mathcal{N}}^+, \mathbf{H}_{\mathcal{N}}] = 0. \quad (2.8)$$

The former relation holds automatically due to the nilpotency of ψ and ψ^\dagger , while the latter is equivalent to the following intertwining relations:

$$P_{\mathcal{N}}^- H_{\mathcal{N}}^- - H_{\mathcal{N}}^+ P_{\mathcal{N}}^- = 0, \quad P_{\mathcal{N}}^+ H_{\mathcal{N}}^+ - H_{\mathcal{N}}^- P_{\mathcal{N}}^+ = 0. \quad (2.9)$$

Therefore, the relations (2.9) give the condition for the system $\mathbf{H}_{\mathcal{N}}$ to be \mathcal{N} -fold supersymmetric. Note that the Hamiltonians (2.3) are always symmetric under the transposition (on suitable spaces) even when they are not Hermitian, and thus each of the relations in Eq. (2.9) actually implies the other. Note also that, due to the transposition symmetry of the Hamiltonian, it was proved in Ref. [37] that the anti-commutator of $Q_{\mathcal{N}}^-$ and $Q_{\mathcal{N}}^+$ defined by Eqs. (2.4) and (2.5) can be always expressed as a polynomial of \mathcal{N} th degree in $\mathbf{H}_{\mathcal{N}}$.

The \mathcal{N} -fold supersymmetric models defined above have several significant properties similar to those of the ordinary supersymmetric models. One of the most notable ones is quasi-solvability [28,30,31]. A differential operator H of a single variable q is said to be *quasi-solvable* with respect to a given \mathcal{N} th-order linear differential operator $P_{\mathcal{N}}$ of the form (2.6) if it leaves invariant $\ker P_{\mathcal{N}}$, namely,

$$P_{\mathcal{N}} H \mathcal{V}_{\mathcal{N}} = 0, \quad \mathcal{V}_{\mathcal{N}} = \ker P_{\mathcal{N}}. \quad (2.10)$$

Then it can be easily shown [28] that an \mathcal{N} -fold supersymmetric system satisfying Eq. (2.9) can always be constructed from a quasi-solvable Hamiltonian H by setting $H_{\mathcal{N}}^- = H$, $H_{\mathcal{N}}^+ = H + w'_{\mathcal{N}-1}(q)$ and $P_{\mathcal{N}}^\pm$ as in Eq. (2.5). The converse is also true. Indeed, from the intertwining relation (2.9) we find that all the \mathcal{N} -fold supersymmetric systems are quasi-solvable,

the quasi-solvability condition (2.10) holding, respectively, for $H = H_{\mathcal{N}}^{\pm}$ and $P_{\mathcal{N}} = P_{\mathcal{N}}^{\pm}$. When a system is quasi-solvable but the elements of $\mathcal{V}_{\mathcal{N}}$ are not known explicitly, the system is said to be *weakly* quasi-solvable [31]. More rigorous and sophisticated definitions of quasi-solvability and related concepts can be found in Ref. [39].

3. Type B \mathcal{N} -fold supersymmetry

Recall [29] that the type A \mathcal{N} -fold supercharge is defined by an \mathcal{N} th-order linear differential operator $P_{\mathcal{N}}$ of the form

$$\begin{aligned}
 P_{\mathcal{N}} &= \prod_{k=-(\mathcal{N}-1)/2}^{(\mathcal{N}-1)/2} (p - iW + ikE) \\
 &\equiv (-i)^{\mathcal{N}} \left(\frac{d}{dq} + W - \frac{\mathcal{N}-1}{2} E \right) \\
 &\quad \times \left(\frac{d}{dq} + W - \frac{\mathcal{N}-3}{2} E \right) \\
 &\quad \times \cdots \times \left(\frac{d}{dq} + W + \frac{\mathcal{N}-3}{2} E \right) \\
 &\quad \times \left(\frac{d}{dq} + W + \frac{\mathcal{N}-1}{2} E \right), \tag{3.1}
 \end{aligned}$$

where $W(q)$ and $E(q)$ are arbitrary (smooth) functions. Consider next what is perhaps the simplest deformation of the operator (3.1), namely

$$\begin{aligned}
 P_{\mathcal{N}} &= \left(p - iW + iF + i \frac{\mathcal{N}-1}{2} E \right) \\
 &\quad \times \prod_{k=-(\mathcal{N}-1)/2}^{(\mathcal{N}-3)/2} (p - iW + ikE) \\
 &= (-i)^{\mathcal{N}} \left(\frac{d}{dq} + W - F - \frac{\mathcal{N}-1}{2} E \right) \\
 &\quad \times \left(\frac{d}{dq} + W - \frac{\mathcal{N}-3}{2} E \right) \\
 &\quad \times \cdots \times \left(\frac{d}{dq} + W + \frac{\mathcal{N}-3}{2} E \right) \\
 &\quad \times \left(\frac{d}{dq} + W + \frac{\mathcal{N}-1}{2} E \right). \tag{3.2}
 \end{aligned}$$

The \mathcal{N} th-order differential operator (3.2), which depends on an additional function $F(q)$, clearly reduces

to the type A supercharge (3.1) when F vanishes identically. In this Letter we shall show that if the functions E and F are related by the equation

$$F'(q) - E(q)F(q) + F(q)^2 = 0, \tag{3.3}$$

the operator (3.2) defines a new type of \mathcal{N} -fold supersymmetry. Indeed, in this section we shall construct two Hamiltonians $H_{\mathcal{N}}^{\pm}$ that are quasi-solvable with respect to the \mathcal{N} -fold supercharges $P_{\mathcal{N}}^{\pm}$ defined by Eq. (2.5) in terms of the “type B” operator (3.2).

To construct an \mathcal{N} -fold supersymmetric model two different approaches can be followed, namely the so-called *analytic* and *algebraic* methods [31]. We shall restrict ourselves in what follows to the latter approach. As in the case of type A \mathcal{N} -fold supersymmetry [30,31], to achieve our aim it is convenient to make a suitable *gauge* transformation and change of variable. Indeed, using the following gauge potentials

$$\mathcal{W}_{\mathcal{N}}^{\pm}(q) = \frac{\mathcal{N}-1}{2} \int dq E(q) \mp \int dq W(q), \tag{3.4}$$

the type B \mathcal{N} -fold supercharges $P_{\mathcal{N}}^{\pm}$ are transformed into

$$\begin{aligned}
 \tilde{P}_{\mathcal{N}}^{-} &= i^{\mathcal{N}} e^{\mathcal{W}_{\mathcal{N}}^{-}} P_{\mathcal{N}}^{-} e^{-\mathcal{W}_{\mathcal{N}}^{-}} \\
 &= (h')^{\mathcal{N}} \left(\frac{d}{dh} - \frac{1}{h} \right) \frac{d^{\mathcal{N}-1}}{dh^{\mathcal{N}-1}}, \tag{3.5a}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{P}_{\mathcal{N}}^{+} &= i^{\mathcal{N}} e^{\mathcal{W}_{\mathcal{N}}^{+}} P_{\mathcal{N}}^{+} e^{-\mathcal{W}_{\mathcal{N}}^{+}} \\
 &= (h')^{\mathcal{N}} \frac{d^{\mathcal{N}-1}}{dh^{\mathcal{N}-1}} \left(\frac{d}{dh} + \frac{1}{h} \right), \tag{3.5b}
 \end{aligned}$$

where $h(q)$ is a solution of the following differential equation:

$$h'(q) - F(q)h(q) = 0. \tag{3.6}$$

As in Eqs. (3.5), we shall hereafter attach tildes (bars) to operators, vectors and vector spaces to indicate that they are quantities gauge-transformed with the gauge potential $\mathcal{W}_{\mathcal{N}}^{-}$ ($\mathcal{W}_{\mathcal{N}}^{+}$), respectively. From Eqs. (3.3) and (3.6), the relation between $h(q)$ and $E(q)$ reads

$$h''(q) - E(q)h'(q) = 0, \tag{3.7}$$

which in turn is the same relation as the one for the type A case employed in Refs. [29–31]. From Eqs. (3.5), we obtain gauge-transformed solvable subspaces as

$$\tilde{\mathcal{V}}_{\mathcal{N}}^{-} = \ker \tilde{P}_{\mathcal{N}}^{-} = \text{span}\{1, h, \dots, h^{\mathcal{N}-2}, h^{\mathcal{N}}\}, \tag{3.8}$$

$$\bar{\mathcal{V}}_{\mathcal{N}}^+ = \ker \bar{P}_{\mathcal{N}}^+ = \text{span}\{h^{-1}, h, h^2, \dots, h^{\mathcal{N}-1}\}. \quad (3.9)$$

Let us begin with the construction of $\tilde{H}_{\mathcal{N}}^-$. The operator $\tilde{H}_{\mathcal{N}}^-$ should be constructed so that it is quasi-solvable with respect to $\tilde{P}_{\mathcal{N}}^-$. This is achieved by finding a second-order differential operator $\tilde{H}_{\mathcal{N}}^-$ satisfying

$$\left(\frac{d}{dh} - \frac{1}{h}\right) \frac{d^{\mathcal{N}-1}}{dh^{\mathcal{N}-1}} \tilde{H}_{\mathcal{N}}^- h^k = 0, \quad \forall k = 0, 1, \dots, \mathcal{N} - 2, \mathcal{N}. \quad (3.10)$$

For $\mathcal{N} \geq 3$, there are six linearly independent differential operators of order not greater than two solving Eqs. (3.10), namely:

$$J_{--} = \frac{d^2}{dh^2}, \quad (3.11a)$$

$$J_{0-} = h \frac{d^2}{dh^2} - (\mathcal{N} - 1) \frac{d}{dh}, \quad (3.11b)$$

$$J_0 = h \frac{d}{dh}, \quad (3.11c)$$

$$J_{00} = h^2 \frac{d^2}{dh^2}, \quad (3.11d)$$

$$J_{+0} = h^3 \frac{d^2}{dh^2} - (2\mathcal{N} - 3)h^2 \frac{d}{dh} + \mathcal{N}(\mathcal{N} - 2)h, \quad (3.11e)$$

$$J_{++} = h^4 \frac{d^2}{dh^2} - 2(\mathcal{N} - 2)h^3 \frac{d}{dh} + \mathcal{N}(\mathcal{N} - 3)h^2. \quad (3.11f)$$

Therefore, the general solution of (3.10) for $\mathcal{N} \geq 3$ can be expressed as

$$\tilde{H}_{\mathcal{N}}^- = - \sum_{\substack{i,j=+,0,- \\ i \geq j}} a_{ij}^{(-)} J_{ij} + b_0^{(-)} J_0 - C^{(-)}, \quad (3.12)$$

where $a_{ij}^{(-)}$, $b_0^{(-)}$ and $C^{(-)}$ are constants. Substituting Eqs. (3.11) into Eq. (3.12) we obtain

$$\tilde{H}_{\mathcal{N}}^- = -A_4^-(h) \frac{d^2}{dh^2} + A_3^-(h) \frac{d}{dh} - A_2^-(h), \quad (3.13)$$

with

$$A_4^-(h) = a_{++}^{(-)} h^4 + a_{+0}^{(-)} h^3 + a_{00}^{(-)} h^2 + a_{0-}^{(-)} h + a_{--}^{(-)}, \quad (3.14a)$$

$$A_3^-(h) = 2(\mathcal{N} - 2)a_{++}^{(-)} h^3 + (2\mathcal{N} - 3)a_{+0}^{(-)} h^2 + b_0^{(-)} h + (\mathcal{N} - 1)a_{0-}^{(-)}, \quad (3.14b)$$

$$A_2^-(h) = \mathcal{N}(\mathcal{N} - 3)a_{++}^{(-)} h^2 + \mathcal{N}(\mathcal{N} - 2)a_{+0}^{(-)} h + C^{(-)}. \quad (3.14c)$$

On the other hand, the partner operator $\tilde{H}_{\mathcal{N}}^+$ should be constructed so that it is quasi-solvable with respect to $\tilde{P}_{\mathcal{N}}^+$. This is achieved by finding a second-order differential operator $\tilde{H}_{\mathcal{N}}^+$ such that

$$\frac{d^{\mathcal{N}-1}}{dh^{\mathcal{N}-1}} \left(\frac{d}{dh} + \frac{1}{h}\right) \tilde{H}_{\mathcal{N}}^+ h^k = 0, \quad \forall k = -1, 1, 2, \dots, \mathcal{N} - 1. \quad (3.15)$$

For $\mathcal{N} \geq 3$, Eqs. (3.15) are solved by the following six linearly independent differential operators of order less than or equal to two:

$$K_{--} = \frac{d^2}{dh^2} - \frac{2}{h^2}, \quad (3.16a)$$

$$K_{0-} = h \frac{d^2}{dh^2} + \frac{d}{dh} - \frac{1}{h}, \quad (3.16b)$$

$$K_0 = h \frac{d}{dh}, \quad (3.16c)$$

$$K_{00} = h^2 \frac{d^2}{dh^2}, \quad (3.16d)$$

$$K_{+0} = h^3 \frac{d^2}{dh^2} - (\mathcal{N} - 3)h^2 \frac{d}{dh} - (\mathcal{N} - 1)h, \quad (3.16e)$$

$$K_{++} = h^4 \frac{d^2}{dh^2} - 2(\mathcal{N} - 2)h^3 \frac{d}{dh} + (\mathcal{N} - 1)(\mathcal{N} - 2)h^2. \quad (3.16f)$$

Therefore, the general solution of (3.15) for $\mathcal{N} \geq 3$ can be expressed as

$$\tilde{H}_{\mathcal{N}}^+ = - \sum_{\substack{i,j=+,0,- \\ i \geq j}} a_{ij}^{(+)} K_{ij} + b_0^{(+)} K_0 - C^{(+)}, \quad (3.17)$$

where $a_{ij}^{(+)}$, $b_0^{(+)}$ and $C^{(+)}$ are constants. Substituting Eqs. (3.16) into Eq. (3.17) we have

$$\tilde{H}_{\mathcal{N}}^+ = -A_4^+(h) \frac{d^2}{dh^2} + A_3^+(h) \frac{d}{dh} - A_2^+(h), \quad (3.18)$$

with

$$A_4^+(h) = a_{++}^{(+)} h^4 + a_{+0}^{(+)} h^3 + a_{00}^{(+)} h^2 + a_{0-}^{(+)} h + a_{--}^{(+)}, \quad (3.19a)$$

$$A_3^+(h) = 2(\mathcal{N} - 2)a_{++}^{(+)} h^3 + (\mathcal{N} - 3)a_{+0}^{(+)} h^2 + b_0^{(+)} h - a_{0-}^{(+)}, \quad (3.19b)$$

$$A_2^+(h) = (\mathcal{N} - 1)(\mathcal{N} - 2)a_{++}^{(+)}h^2 - (\mathcal{N} - 1)a_{+0}^{(+)}h + C^{(+)} - \frac{a_{0-}^{(+)}}{h} - \frac{2a_{--}^{(+)}}{h^2}. \quad (3.19c)$$

If the operators (3.13) and (3.18) are gauge-transformed back with the gauge potentials $\mathcal{W}_{\mathcal{N}}^-$ and $\mathcal{W}_{\mathcal{N}}^+$, respectively, they are not in general Schrödinger operators of the form Eq. (2.3). The operators $H_{\mathcal{N}}^{\pm}$ assume the canonical form (2.3) if and only if the following conditions are fulfilled:

$$\frac{1}{2}(h')^2 = A_4^{\pm}(h) \equiv P(h) = a_4h^4 + a_3h^3 + a_2h^2 + a_1h + a_0, \quad (3.20)$$

$$A_3^{\pm}(h) = \frac{\mathcal{N} - 2}{2}P'(h) \mp Wh'. \quad (3.21)$$

If the above conditions are satisfied, we have

$$H_{\mathcal{N}}^{\pm} = e^{-\mathcal{W}_{\mathcal{N}}^{\pm}} \tilde{H}_{\mathcal{N}}^{\pm} e^{\mathcal{W}_{\mathcal{N}}^{\pm}} = -\frac{1}{2} \frac{d^2}{dq^2} + V_{\mathcal{N}}^{\pm}(q), \quad (3.22)$$

where the potentials $V_{\mathcal{N}}^{\pm}(q)$ are given by

$$V_{\mathcal{N}}^{\pm}(q) = \frac{1}{2} \left[\left(\frac{d\mathcal{W}_{\mathcal{N}}^{\pm}(q)}{dq} \right)^2 - \frac{d^2\mathcal{W}_{\mathcal{N}}^{\pm}(q)}{dq^2} \right] - A_2^{\pm}(h(q)). \quad (3.23)$$

From the second condition (3.21) we obtain

$$-Wh' = -\frac{\mathcal{N}}{2}a_3h^2 + b_1h - \frac{\mathcal{N}}{2}a_1 \equiv Q(h), \quad (3.24)$$

where the constant b_1 is given by

$$b_0^{(\pm)} = (\mathcal{N} - 2)a_2 \pm b_1. \quad (3.25)$$

The constants $a_{ij}^{(-)}$ and $b_0^{(-)}$ in $\tilde{H}_{\mathcal{N}}^-$ are related with the corresponding constants $a_{ij}^{(+)}$ and $b_0^{(+)}$ in $\tilde{H}_{\mathcal{N}}^+$ by Eqs. (3.20) and (3.25). To establish the relation between $C^{(-)}$ and $C^{(+)}$, we will invoke the identity

$$H_{\mathcal{N}}^+ - H_{\mathcal{N}}^- = w'_{\mathcal{N}-1}(q),$$

where $w_{\mathcal{N}-1}(q)$ is defined in Eq. (2.6). For the \mathcal{N} -fold supercharge of type B (3.2) we have

$$w_{\mathcal{N}-1}(q) = \mathcal{N}W(q) - F(q).$$

Thus the condition for $H_{\mathcal{N}}^-$ and $H_{\mathcal{N}}^+$ to form an \mathcal{N} -fold supersymmetric pair reads

$$H_{\mathcal{N}}^+ - H_{\mathcal{N}}^- = V_{\mathcal{N}}^+(q) - V_{\mathcal{N}}^-(q) = \mathcal{N}W'(q) - F'(q). \quad (3.26)$$

On the other hand, from Eqs. (3.4) and (3.23) we have

$$V_{\mathcal{N}}^+ - V_{\mathcal{N}}^- = W' - (\mathcal{N} - 1)EW - A_2^+(h) + A_2^-(h). \quad (3.27)$$

In order to rearrange the r.h.s. of Eq. (3.27), the following relations derived from Eqs. (3.6), (3.7), (3.20) and (3.24) are useful:

$$Q'(h) = -W' - EW, \quad P'(h) = EFh. \quad (3.28)$$

With the aid of the above relations, we finally obtain

$$V_{\mathcal{N}}^+ - V_{\mathcal{N}}^- = \mathcal{N}W' - F' + (\mathcal{N} - 1)b_1 - C^{(+)} + C^{(-)}. \quad (3.29)$$

Therefore, the condition for \mathcal{N} -fold supersymmetry (3.26) holds when

$$C^{(+)} - C^{(-)} = (\mathcal{N} - 1)b_1. \quad (3.30)$$

In order to fix $C^{(\pm)}$, we write $A_2^{\pm}(h)$ as follows:

$$A_2^{\pm}(h) = A_{21}(h) \pm A_{22}(h). \quad (3.31)$$

From Eqs. (3.14c), (3.19c) and (3.30) we have

$$A_{22}(h) = \frac{P'(h)}{2h} - \frac{P(h)}{h^2} + \frac{\mathcal{N} - 1}{2}Q'(h), \quad (3.32a)$$

$$A_{21}(h) = \frac{(\mathcal{N} - 1)(\mathcal{N} - 2)}{12}P''(h) - \frac{P(h)}{h^2} - \frac{Q(h)}{\mathcal{N}h} + R, \quad (3.32b)$$

where the constant R is given by

$$R = -\frac{(\mathcal{N} + 1)(\mathcal{N} - 4)}{6}a_2 + \frac{b_1}{\mathcal{N}} + \frac{1}{2}(C^{(+)} + C^{(-)}). \quad (3.33)$$

In this case, $C^{(\pm)}$ are determined from Eqs. (3.30) and (3.33) as

$$C^{(\pm)} = \frac{(\mathcal{N} + 1)(\mathcal{N} - 4)}{6}a_2 \pm \frac{\mathcal{N}^2 - \mathcal{N} \mp 2}{2\mathcal{N}}b_1 + R. \quad (3.34)$$

Summarizing the results obtained so far, the gauge-transformed operators of the type B \mathcal{N} -fold supersymmetric Hamiltonians are given by

$$\begin{aligned} \tilde{H}_{\mathcal{N}}^{\pm} = & -P(h) \frac{d^2}{dh^2} + \left[\frac{\mathcal{N} - 2}{2}P'(h) \pm Q(h) \right] \frac{d}{dh} \\ & - \left\{ \frac{(\mathcal{N} - 1)(\mathcal{N} - 2)}{12}P''(h) - \frac{P(h)}{h^2} - \frac{Q(h)}{\mathcal{N}h} \right. \\ & \left. \pm \left[\frac{P'(h)}{2h} - \frac{P(h)}{h^2} + \frac{\mathcal{N} - 1}{2}Q'(h) \right] + R \right\}. \end{aligned} \quad (3.35)$$

The type B potentials $V_{\mathcal{N}}^{\pm}$ are calculated by substituting Eqs. (3.4) and (3.22) into Eq. (3.23). In terms of h we have

$$V_{\mathcal{N}}^{\pm}(h) = -\frac{1}{12P(h)} \left\{ (\mathcal{N}^2 - 1) \left[P(h)P''(h) - \frac{3}{4}(P'(h))^2 \right] - 3Q(h)^2 \right\} + \frac{P(h)}{h^2} + \frac{Q(h)}{\mathcal{N}h} \pm \left[\mathcal{N} \frac{P'(h)Q(h) - 2P(h)Q'(h)}{4P(h)} - \frac{P'(h)}{2h} + \frac{P(h)}{h^2} \right] - R, \quad (3.36)$$

while in terms of q

$$V_{\mathcal{N}}^{\pm}(q) = \frac{1}{2}W(q)^2 - \frac{1}{\mathcal{N}}F(q)W(q) + \frac{1}{2}F(q)^2 - \frac{\mathcal{N}^2 - 1}{24}[2E'(q) - E(q)^2] \pm \frac{1}{2}[\mathcal{N}W'(q) - F'(q)] - R. \quad (3.37)$$

We note again that the potential (3.37) reduces to the type A form [31,35] if we set $F(q) = 0$.

4. Examples

In this section we shall exhibit some examples of the \mathcal{N} -fold supersymmetric models of type B constructed in the previous section. As the first example, we choose $P(h) = 2(h - h_0)$. In this case $Q(h) = b_1h - \mathcal{N}$ from Eq. (3.24). Using Eqs. (3.6), (3.7), (3.20) and (3.24) we obtain

$$h(q) = q^2 + h_0, \quad E(q) = \frac{1}{q}, \quad F(q) = \frac{2q}{q^2 + h_0}, \quad (4.1a)$$

$$W(q) = -\frac{b_1}{2}q + \frac{\mathcal{N} - b_1h_0}{2q}. \quad (4.1b)$$

The pair of potentials (3.37) reads

$$V_{\mathcal{N}}^{\pm}(q) = \frac{b_1^2}{8}q^2 + \frac{(\mathcal{N} \mp \mathcal{N} - b_1h_0 - 1)(\mathcal{N} \mp \mathcal{N} - b_1h_0 + 1)}{8q^2}$$

$$+ (1 \pm 1) \frac{q^2 - h_0}{(q^2 + h_0)^2} - \frac{b_1}{4}(\mathcal{N} \pm \mathcal{N} - b_1h_0) + \frac{b_1}{\mathcal{N}} - R. \quad (4.2)$$

In the next example, we choose $P(h) = v^2(h - h_0)^2/2$. In this case, $Q(h) = b_1h + \mathcal{N}v^2h_0/2$ from Eq. (3.24). Employing again Eqs. (3.6), (3.7), (3.20) and (3.24) we obtain

$$h(q) = e^{\nu q} + h_0, \quad E(q) = v, \quad F(q) = \frac{v}{1 + h_0e^{-\nu q}}, \quad (4.3a)$$

$$W(q) = -\frac{(2b_1 + \mathcal{N}v^2)h_0}{2v}e^{-\nu q} - \frac{b_1}{v}. \quad (4.3b)$$

The pair of potentials (3.37) now reads

$$V_{\mathcal{N}}^{\pm}(q) = \frac{(2b_1 + \mathcal{N}v^2)^2h_0^2}{8v^2}e^{-2\nu q} + \frac{(2b_1 + \mathcal{N}v^2)(2b_1 \pm \mathcal{N}v^2)h_0}{4v^2}e^{-\nu q} - (1 \pm 1) \frac{v^2h_0e^{-\nu q}}{2(1 + h_0e^{-\nu q})^2} + \frac{b_1^2}{2v^2} + \frac{b_1}{\mathcal{N}} + \frac{\mathcal{N}^2 + 11}{24}v^2 - R. \quad (4.4)$$

In our final example we take

$$P(h) = \frac{v^2}{2}(1 - h^2)(1 - k^2h^2), \quad \text{so that } Q(h) = b_1h \text{ and } h(q) = \text{sn}(\nu q), \quad E(q) = -\frac{\nu \text{sn}(\nu q)(k'^2 + 2k^2 \text{cn}^2(\nu q))}{\text{cn}(\nu q) \text{dn}(\nu q)}, \quad (4.5a)$$

$$F(q) = \frac{\nu \text{cn}(\nu q) \text{dn}(\nu q)}{\text{sn}(\nu q)}, \quad W(q) = -\frac{b_1}{v} \frac{\text{sn}(\nu q)}{\text{cn}(\nu q) \text{dn}(\nu q)}. \quad (4.5b)$$

In the latter formulas sn, cn, and dn denote the Jacobi elliptic functions of modulus k (with $0 \leq k \leq 1$), $k' = \sqrt{1 - k^2}$ is the complementary modulus, and ν is a positive constant. The pair of \mathcal{N} -fold supersymmetric

potentials (3.37) constructed from this choice of $P(h)$ is

$$\begin{aligned}
 V_{\mathcal{N}}^{\pm}(q) &= \frac{1}{2}(1 \mp 1)v^2k^2 \operatorname{sn}^2(vq) + \frac{1}{2}(1 \pm 1)\frac{v^2}{\operatorname{sn}^2(vq)} \\
 &+ \frac{(2b_1 - k'^2v^2(\pm\mathcal{N} - 1))(2b_1 - k'^2v^2(\pm\mathcal{N} + 1))}{8v^2k'^2 \operatorname{cn}^2(vq)} \\
 &- \frac{(2b_1 + k'^2v^2(\pm\mathcal{N} - 1))(2b_1 + k'^2v^2(\pm\mathcal{N} + 1))}{8v^2k'^2 \operatorname{dn}^2(vq)} \\
 &+ \frac{v^2}{12}(1 + k^2)(\mathcal{N}^2 - 7) + \frac{b_1}{\mathcal{N}}\left(1 \pm \frac{\mathcal{N}^2}{2}\right) - R.
 \end{aligned} \tag{4.6}$$

5. Discussion

In this Letter, we have constructed a new family of \mathcal{N} -fold supersymmetry in which a higher-derivative representation of the \mathcal{N} -fold supercharge is given by Eq. (3.2) with the constraint (3.3). In view of quasi-solvability, it turns out that the gauged Hamiltonians obtained here correspond to quasi-solvable operators of the type investigated by Post and Turbiner [38]. More precisely, $\tilde{H}_{\mathcal{N}}^-$ in Eq. (3.12) is identical with the case C operator in [38], while $\tilde{H}_{\mathcal{N}}^+$ in Eq. (3.17) without K_{--} is equivalent to the case D operator in [38]. The reason why the operator K_{--} does not appear in Ref. [38] is that the authors considered only differential operators with polynomial coefficients. From our point of view, however, it is evident that the operator K_{--} is indispensable for $\tilde{H}_{\mathcal{N}}^+$ to be the \mathcal{N} -fold supersymmetric partner of $\tilde{H}_{\mathcal{N}}^-$. Indeed, without K_{--} the number of independent parameters in $\tilde{H}_{\mathcal{N}}^+$ differs from the one in $\tilde{H}_{\mathcal{N}}^-$. This is clearly impossible, since any differential operator $\tilde{H}(h)$ leaving the space (3.9) invariant is equivalent to an operator $\tilde{H}(h)$ preserving the space (3.8) under the transformation $\tilde{H}(h) = h^{\mathcal{N}-1}\tilde{H}(h^{-1})h^{-\mathcal{N}+1}$. An interesting fact is that the \mathcal{N} -fold supersymmetric systems of type B characterized by the \mathcal{N} -fold supercharge (3.2) and by the potentials (3.37) connect, by the formal limit $F(q) \rightarrow 0$, the quasi-solvable models of Post–Turbiner type [38] with the $\mathfrak{sl}(2)$ quasi-solvable models [32], which are essentially equivalent to the type A \mathcal{N} -fold supersymmetric systems.

The algebraic construction with the constraint (3.3) presented in this article turns out to be especially

useful when the solvable subspace can be gauge-transformed into a space of monomial type. With the use of this method it is possible to construct, as an application, the most general \mathcal{N} -fold supersymmetry whose solvable sector is spanned by monomials [40].

On the other hand, a direct calculation of the intertwining relation (2.9) with the type B \mathcal{N} -fold supercharge indicates that the condition (3.3) imposed in this article may be only sufficient but not necessary for the existence of type B \mathcal{N} -fold supersymmetry. This suggests that the class of type B potentials may be wider than the quasi-solvable models of Post–Turbiner type obtained here. One of the reasons for this is that the framework of \mathcal{N} -fold supersymmetry makes sense even when the solvable subspace has no known analytic expression, that is, when the Hamiltonian is weakly quasi-solvable. Therefore, it may also be possible that, once we have a system of \mathcal{N} -fold supersymmetry constructed from an \mathcal{N} -dimensional vector space $\tilde{\mathcal{V}}_{\mathcal{N}}$ with the above procedure, it turns out that this system can be extended in such a way that the solvable sector is no longer given by the starting vector space $\tilde{\mathcal{V}}_{\mathcal{N}}$. Work on these and related issues is in progress and will be reported elsewhere.

Acknowledgements

This work was partially supported by the DGI under grant No. BFM2002-02646 (A.G.-L.) and by a JSPS research fellowship (T.T.). One of us (T.T.) would like to thank A.A. Andrianov and M. Sato for useful discussions, as well as all the members of the Departamento de Física Teórica II, Universidad Complutense, for their kind hospitality during his stay.

References

- [1] A.V. Turbiner, A.G. Ushveridze, Phys. Lett. A 126 (1987) 181.
- [2] M.A. Shifman, Int. J. Mod. Phys. A 4 (1989) 2897.
- [3] A.G. Ushveridze, Quasi-Exactly Solvable Models in Quantum Mechanics, IOP, Bristol, 1994.
- [4] A. González-López, N. Kamran, P.J. Olver, Contemp. Math. 160 (1994) 113.
- [5] A.A. Andrianov, M.V. Ioffe, V.P. Spiridonov, Phys. Lett. A 174 (1993) 273.
- [6] E. Witten, Nucl. Phys. B 188 (1981) 513.
- [7] E. Witten, Nucl. Phys. B 202 (1982) 253.

- [8] A.A. Andrianov, M.V. Ioffe, F. Cannata, J.-P. Dedonder, *Int. J. Mod. Phys. A* 10 (1995) 2683.
- [9] A.A. Andrianov, M.V. Ioffe, D.N. Nishnianidze, *Phys. Lett. A* 201 (1995) 103.
- [10] A.A. Andrianov, M.V. Ioffe, D.N. Nishnianidze, *Theor. Math. Phys.* 104 (1995) 1129.
- [11] V.G. Bagrov, B.F. Samsonov, *Theor. Math. Phys.* 104 (1995) 1051.
- [12] B.F. Samsonov, *Mod. Phys. Lett. A* 11 (1996) 1563.
- [13] V.G. Bagrov, B.F. Samsonov, *Phys. Part. Nucl.* 28 (1997) 374.
- [14] B.F. Samsonov, *Phys. Lett. A* 263 (1999) 274.
- [15] D.J. Fernández C., *Int. J. Mod. Phys. A* 12 (1997) 171.
- [16] J.O. Rosas-Ortiz, *J. Phys. A* 31 (1998) 10163.
- [17] B. Bagchi, A. Ganguly, D. Bhaumik, A. Mitra, *Mod. Phys. Lett. A* 14 (1999) 27.
- [18] D.J. Fernández C., V. Hussin, *J. Phys. A* 32 (1999) 3603.
- [19] D.J. Fernández C., J. Negro, L.M. Nieto, *Phys. Lett. A* 275 (2000) 338.
- [20] M. Plyushchay, *Int. J. Mod. Phys. A* 15 (2000) 3679.
- [21] S. Klishevich, M. Plyushchay, *Mod. Phys. Lett. A* 14 (1999) 2739.
- [22] H. Aoyama, H. Kikuchi, I. Okouchi, M. Sato, S. Wada, *Nucl. Phys. B* 553 (1999) 644.
- [23] H. Aoyama, M. Sato, T. Tanaka, M. Yamamoto, *Phys. Lett. B* 498 (2001) 117.
- [24] S.M. Klishevich, M.S. Plyushchay, *Nucl. Phys. B* 606 (2001) 583.
- [25] S.M. Klishevich, M.S. Plyushchay, *Nucl. Phys. B* 616 (2001) 403.
- [26] P. Dorey, C. Dunning, R. Tateo, *J. Phys. A* 34 (2001) 5679.
- [27] P. Dorey, C. Dunning, R. Tateo, *J. Phys. A* 34 (2001) L391.
- [28] H. Aoyama, M. Sato, T. Tanaka, *Nucl. Phys. B* 619 (2001) 105.
- [29] H. Aoyama, M. Sato, T. Tanaka, *Phys. Lett. B* 503 (2001) 423.
- [30] H. Aoyama, N. Nakayama, M. Sato, T. Tanaka, *Phys. Lett. B* 519 (2001) 260.
- [31] T. Tanaka, *Nucl. Phys. B* 662 (2003) 413.
- [32] A.V. Turbiner, *Commun. Math. Phys.* 118 (1988) 467.
- [33] A. González-López, N. Kamran, P.J. Olver, *Commun. Math. Phys.* 153 (1993) 117.
- [34] R. Sasaki, K. Takasaki, *J. Phys. A* 34 (2001) 9533.
- [35] M. Sato, T. Tanaka, *J. Math. Phys.* 43 (2002) 3484.
- [36] F. Cannata, M.V. Ioffe, D.N. Nishnianidze, *J. Phys. A* 35 (2002) 1389.
- [37] A.A. Andrianov, A.V. Sokolov, *Nucl. Phys. B* 660 (2003) 25.
- [38] G. Post, A. Turbiner, *Russ. J. Math. Phys.* 3 (1995) 113.
- [39] T. Tanaka, *Ann. Phys.* 309 (2004) 239, hep-th/0306174.
- [40] A. González-López, T. Tanaka, in preparation.