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On a problem of Lions concerning real interpolation spaces. The quasi-Banach case

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We prove that, under a mild condition on a couple (A_0, A_1) of quasi-Banach spaces, all real interpolation spaces $(A_0, A_1)_{\theta,p}$ with $0 < \theta < 1$ and $0 < p \leq \infty$ are different from each other. In the Banach case and for $1 \leq p \leq \infty$ this was shown by Janson, Nilsson, Peetre and Zafran, thus solving an old problem posed by J.-L. Lions. Moreover, we give an application to certain spaces which are important objects in Operator Theory and which consist of bounded linear operators whose approximation numbers belong to Lorentz sequence spaces.

1. Introduction

The problem of determining whether a given family of interpolation spaces really depends on its parameters was proposed long ago by J.-L. Lions. In the case of the real method, the problem was solved by Janson, Nilsson, Peetre and Zafran, see [8, Section 3, Theorem 1]. They proved that if $0 < \theta, \eta < 1$, $1 \leq p, q \leq \infty$ and (A_0, A_1) is a Banach couple such that $A_0 \cap A_1$ is not closed in $A_0 + A_1$, then $(A_0, A_1)_{\theta,p} = (A_0, A_1)_{\eta,q}$ implies that $\theta = \eta$ and $p = q$. Previous partial solutions to Lions's problem were obtained by Stafney [23] (for the complex method) and by Bergh and Löfström [3, Exercise 21, p. 82].

The approach in [8] is based on deep results of M. Lévy [13,14] who showed that any real interpolation space $(A_0, A_1)_{\theta,p}$ with $0 < \theta < 1$ and $1 \leq p \leq \infty$ contains a subspace isomorphic to ℓ_p, provided that $A_0 \cap A_1$ is not closed in $A_0 + A_1$. It is natural to consider this condition because, for any Banach couple (A_0, A_1), if $A_0 \cap A_1$ is closed in $A_0 + A_1$ then $(A_0, A_1)_{\theta,p} = A_0 \cap A_1$ for all parameters.

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The real method can also be defined for couples of *quasi-Banach* spaces, and not just for $1 \le p \le \infty$ but also for $0 < p < 1$ (see [12,18,3,26]). Then $(A_0, A_1)_{\theta,p}$ is a quasi-Banach space. However, the proof of [8] does not work in this more general setting. The reason is that it makes crucial use of duality properties of $A_0 \cap A_1$, $A_0 + A_1$ and $(A_0, A_1)_{\theta,p}$, see [13, Proposition] and [14, Theoreme II.1]. In the quasi-Banach case, however, duality arguments are no longer available. Therefore one has to find a different technique to obtain a similar result for couples of quasi-Banach spaces.

Quasi-Banach spaces arise quite naturally in Operator Theory. For example, for $0 < p < \infty$ and $0 < q \le \infty$, consider the space $\mathscr{L}^{(a)}_{p,q}(X, Y)$ of all bounded linear operators T from a Banach space X into a Banach space Y whose approximation numbers $(a_n(T))$ belong to the Lorentz sequence space $\ell_{p,q}$. These spaces of operators are quasi-Banach spaces with respect to appropriate quasi-norms, see [19,20,10]. Except in trivial cases, they do not possess an equivalent norm.

Recently Prof. Albrecht Pietsch asked one of the present authors whether or not the result by Janson, Nilsson, Peetre and Zafran [8] remains true in the quasi-Banach setting. Such a more general result might be interesting for potential applications in Operator Theory. In the present paper, motivated by this question, we prove such a result, see Theorem 3.5.

We work with general quasi-Banach couples and $0 < p \le \infty$. Our technique avoids the use of duality arguments, instead our proof is based on computations with Peetre's K-functional. Apart from this modification we follow Lévy's strategy to a certain extent. First we show for quasi-Banach Gagliardo couples (A_0, A_1) (i.e. those for which A_j coincides with the Gagliardo completion $A_j^\sim$ for $j = 0, 1$), that if $(A_0, A_1)_{\theta,p}$ is closed in $A_0 + A_1$, for at least one $0 < \theta < 1$ and at least one $0 < p < \infty$, then $A_0 \cap A_1$ is also closed in $A_0 + A_1$. Moreover we prove that $(A_0, A_1)_{\theta,p}$ contains a subspace isomorphic to ℓ_p, provided that (A_0, A_1) is a Gagliardo couple such that $A_0 \cap A_1$ is not closed in $A_0 + A_1$. As a consequence, we deduce that all spaces of the family $(A_0, A_1)_{\theta,p}$ with $0 < \theta < 1$ and $0 < p \le \infty$ are different from each other, thus providing a positive answer to Lions's question in the quasi-Banach setting.

Moreover we give some applications to the spaces $\mathscr{L}^{(a)}_{p,q}(X, Y)$. We show that, for arbitrary infinite-dimensional Banach spaces X and Y, all spaces $\mathscr{L}^{(a)}_{p,q}(X, Y)$ with $0 < p < \infty$ and $0 < q \le \infty$ are different from each other.

Previous partial results on Lions's problem for quasi-Banach couples with $A_1 \hookrightarrow A_0$ are due to Almira and Fernández-Martínez [1]. An extension of Lévy's theorem to quasi-Banach couples was obtained by López Molina [15,16], but his assumptions are rather restrictive. He showed that if there exists a sequence (x_n) in $(A_0, A_1)_{\theta,p}$ which converges to 0 in $A_0 + A_1$ but satisfies $\inf_{n\in\mathbb{N}} \|x_n\|_{(A_0,A_1)_{\theta,p}} > 0$, then $(A_0, A_1)_{\theta,p}$ contains a subspace isomorphic to ℓ_p, where $0 < p < \infty$. Assuming in addition that $A_0 + A_1$ has a separating dual and the sequence (x_n) satisfies a certain extra condition, he proved that this subspace is complemented.

The plan of the paper is as follows. In Section 2 we review some basic concepts from interpolation theory. The solution to Lions's problem in the quasi-Banach setting is given in Section 3, and in Section 4 we apply this to the Lorentz approximation spaces $\mathscr{L}^{(a)}_{p,q}(X, Y)$.

We dedicate this paper to the memory of Jaak Peetre (1935-2019), who, among his many other notable achievements, was one of the founders of the theory of interpolation spaces. We are among many mathematicians who were inspired by his profound and prolific research and his contagious enthusiasm, and by his very wide and deep knowledge of many fields. He was also our dear friend.

2. Preliminaries

First let us recall some basic facts concerning quasi-Banach spaces. As general references for this subject and, more generally, for topological vector spaces we recommend [22] and [11].

Let $(A, \|.\|)$ be a quasi-normed space, and let $c_A \ge 1$ be its quasi-triangle constant. Then $c_A = 2^{1/r-1}$ for some $0 < r \le 1$, and by the Aoki-Rolewicz theorem (see [11, Sect. 15.10], [19, Sect. 6.2] or [10, Proposition 1.c.5]) there is an equivalent quasi-norm $\|.\|_\star$ such that

$$\|x\|_\star \le \|x\| \le 2^{1/r}\|x\|_\star \qquad \text{and} \qquad \|x+y\|_\star^r \le \|x\|_\star^r + \|y\|_\star^r$$

for all $x, y \in A$. We say that $\|.\|_\star$ is an r-norm. A complete r-normed space is called an r-Banach space. Note that an r-norm is also an s-norm for every $0 < s < r$. Important examples of quasi-Banach spaces are the Lorentz sequence spaces $\ell_{p,q}$, where $0 < p < \infty$ and $0 < q \le \infty$. For details about these spaces we refer to [26,19,20,10].

Next we recall some facts from interpolation theory. (More information can be found in the monographs [3,26,2,4].) We say that (A_0, A_1) is a quasi-Banach couple, if A_0 and A_1 are quasi-Banach spaces which are continuously embedded into the same Hausdorff topological vector space. If A_0, A_1 are both r-Banach spaces, then we say that (A_0, A_1) is an r-Banach couple, and if $r = 1$, we call it a Banach couple.
Peetre's K-functional of a (quasi-)Banach couple (A_0, A_1) is defined by

$$K(t,a) := \inf\Big\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\Big\} \quad , \quad t > 0, a \in A_0 + A_1\,.$$

The quasi-norms of the sum $A_0 + A_1$, respectively the intersection $A_0 \cap A_1$, are given by

$$\|a\|_{A_0+A_1} := K(1,a) \quad \text{respectively} \quad \|a\|_{A_0\cap A_1} := \max\{\|a\|_{A_0}, \|a\|_{A_1}\}\,.$$

Moreover, for each $t > 0$, the functional $K(t,.)$ is an equivalent quasi-norm on $A_0 + A_1$. For $0 < r < 1$, we shall also use the functional

$$K_r(t,a) := \inf\Big\{\big(\|a_0\|_{A_0}^r + t^r\|a_1\|_{A_1}^r\big)^{1/r} : a = a_0 + a_1, a_j \in A_j\Big\}.$$

For each $t > 0$, $K_r(t,.)$ is a quasi-norm on $A_0 + A_1$. If $\|.\|_{A_j}$ is an r-norm on A_j for $j = 0, 1$, then $K_r(t,.)$ is also an r-norm which is equivalent to $K(1,.)$. It satisfies

$$K(t,a) \le K_r(t,a) \le 2^{1/r-1}K(t,a) \qquad \text{for all } t > 0 \text{ and } a \in A_0 + A_1\,. \tag{2.1}$$

Let (A_0, A_1) be a quasi-Banach couple, $0 < \theta < 1$ and $0 < p \le \infty$. Then the *real interpolation space* $(A_0, A_1)_{\theta,p}$ consists of all elements $a \in A_0 + A_1$ for which the quasi-norm

$$\|a\|_{\theta,p} := \begin{cases} \Big(\int_0^\infty \big(t^{-\theta}K(t,a)\big)^p \frac{dt}{t}\Big)^{1/p} & \text{if } p < \infty \\ \sup\limits_{t>0} t^{-\theta}K(t,a) & \text{if } p = \infty \end{cases} \tag{2.2}$$

is finite. $(A_0, A_1)_{\theta,p}$ is a quasi-Banach space with respect to this quasi-norm. In view of (2.1) it is clear that we obtain an equivalent quasi-norm on $(A_0, A_1)_{\theta,p}$ if we replace $K(t,a)$ in (2.2) by $K_r(t,a)$. Furthermore, we have a variant of (2.2) where the function $t \mapsto K(t,a)$ is replaced by a sequence. If we put

$$j_m(a) := 2^{-\theta m}K_r(2^m, a) \quad \text{for } m \in \mathbb{Z} \text{ and } a \in A_0 + A_1$$

and set

$$\|a\|_{\theta,p;r} := \begin{cases} \Big(\sum\limits_{m\in\mathbb{Z}} j_m(a)^p\Big)^{1/p} & \text{if } p < \infty \\ \sup\limits_{m\in\mathbb{Z}} j_m(a) & \text{if } p = \infty\,, \end{cases} \tag{2.3}$$

then $\|.\|_{\theta,p;r}$ is quasi-norm on $(A_0, A_1)_{\theta,p}$ which is equivalent to the original quasi-norm $\|.\|_{\theta,p}$. Moreover, if A_0 and A_1 are both r-Banach spaces for some $0 < r \le 1$ and $p \ge r$, then it is not hard to check that $\|.\|_{\theta,p;r}$ is also an r-norm. Later on we will often work with this r-norm.

The *Gagliardo completion* $A_0^\sim$ of A_0, respectively $A_1^\sim$ of A_1 consists of all $a \in A_0 + A_1$ having a finite quasi-norm

$$\|a\|_{A_0^\sim} = \sup_{t>0} K(t,a), \quad \text{respectively} \quad \|a\|_{A_1^\sim} = \sup_{t>0} \frac{K(t,a)}{t}$$

see [7,2–4]. We always have continuous embeddings $A_0 \hookrightarrow A_0^\sim$ and $A_1 \hookrightarrow A_1^\sim$. Since $K(t,a)$ is non-decreasing in t and $K(t,a)/t$ non-increasing, the quasi-norms satisfy

$$\|a\|_{A_0^\sim} = \lim_{t\to\infty} K(t,a) \qquad \text{and} \qquad \|a\|_{A_1^\sim} = \lim_{t\to 0} \frac{K(t,a)}{t}\,.$$

Note that if we take $p = \infty$ and extend the definition (2.2) of the real interpolation spaces $(A_0, A_1)_{\theta,p}$ by allowing θ to also take the values 0 and 1, then the Gagliardo completions can be identified as

$$A_0^\sim = (A_0, A_1)_{0,\infty} \quad \text{and} \quad A_1^\sim = (A_0, A_1)_{1,\infty}$$

with equality of quasi-norms, $\|.\|_{A_j^\sim} = \|.\|_{j,\infty}$ for $j = 0, 1$. A quasi-Banach couple (A_0, A_1) is called a *Gagliardo couple*, if $A_0^\sim = A_0$ and $A_1^\sim = A_1$. This is a rather mild condition which is satisfied in many concrete cases. A specific example will be studied in Section 4.

For non-negative real functions $\varphi : (0,\infty) \to [0,\infty)$ and $\psi : (0,\infty) \to [0,\infty)$ we write $\varphi(t) \lesssim \psi(t)$ if there is a constant $c > 0$ such that $\varphi(t) \le c\psi(t)$ for all $t > 0$, and $\varphi(t) \sim \psi(t)$ if $\varphi(t) \lesssim \psi(t) \lesssim \varphi(t)$.

3. Lions's problem in the quasi-Banach setting

We begin with a preliminary result.

Lemma 3.1. *Let (A_0, A_1) be an r-Banach couple for some $0 < r \le 1$. Suppose that there is a constant $C \ge 1$ such that, for some fixed $\theta \in (0,1)$,*

$$K_r(t,a) \le C\, t^\theta\, \|a\|_{A_0+A_1} \quad \textit{for all } t > 0 \textit{ and all } a \in A_0 \cap A_1\,. \tag{3.1}$$

Then

$$\|a\|_{A_0^\sim \cap A_1^\sim} \le C^{\lambda(\theta)}\, \|a\|_{A_0+A_1} \quad \textit{for all } a \in A_0 \cap A_1\,,$$

where

$$\lambda(\theta) := \max\left\{\exp\left(\frac{\theta}{1-\theta}\right), \exp\left(\frac{1-\theta}{\theta}\right)\right\}.$$

Proof. Step 1. First we estimate $\|a\|_{A_0^\sim}$ for $a \in A_0 \cap A_1$. We use (3.1), but only for $t \ge 1$. Given a particular $a \in A_0 \cap A_1$, consider any decomposition $a = a_0 + a_1$ with $a_j \in A_j$. Of course, since $a \in A_0 \cap A_1$, we in fact have that a_0 and a_1 are both in $A_0 \cap A_1$. The functional $K_r(t,.)$ is an r-norm on $A_0 + A_1$. Therefore, by applying (3.1) to a_1, it follows that for all $t \ge 1$ the inequality

$$\begin{aligned} K_r(t,a)^r &\le K_r(t,a_0)^r + K_r(t,a_1)^r \\ &\le \|a_0\|_{A_0}^r + (C\, t^\theta)^r\, \|a_1\|_{A_0+A_1}^r \\ &\le \|a_0\|_{A_0}^r + (C\, t^\theta)^r\, \|a_1\|_{A_1}^r \end{aligned}$$

holds. We now take the infimum over all decompositions of a. Note that $C\,t^{\theta} \geq 1$. Hence, using (3.1) once again, we get for all $t \geq 1$ and indeed for all $a \in A_0 \cap A_1$ that

$$K_r(t,a) \leq K_r(C\,t^{\theta},a) \leq C\,(C\,t^{\theta})^{\theta}\,\|a\|_{A_0+A_1} = C^{1+\theta}\,t^{\theta^2}\,\|a\|_{A_0+A_1}\,. \tag{3.2}$$

Thus we have shown that condition (3.1) (restricted to $t \geq 1$) with parameter $\theta \in (0,1)$ and constant $C \geq 1$ implies the same condition with parameter θ^2 and constant $C^{1+\theta}$. Repeating these arguments, but starting now with (3.2) instead of (3.1), we obtain that

$$K_r(t,a) \leq C^{(1+\theta)(1+\theta^2)}\,t^{\theta^4}\,\|a\|_{A_0+A_1} \quad \text{for all } t \geq 1 \text{ and all } a \in A_0 \cap A_1\,.$$

By induction we get for all $n \in \mathbb{N}$, all $t \geq 1$ and all $a \in A_0 \cap A_1$ that

$$K_r(t,a) \leq C_n\,t^{\theta^{2^n}}\,\|a\|_{A_0+A_1} \qquad \text{with } C_n = C^{p(n)} \text{ and } p(n) = \prod_{k=0}^{n-1}(1+\theta^{2^k})\,. \tag{3.3}$$

Using the trivial estimate $1 + x \leq e^x$ we can bound the exponents $p(n)$ as follows:

$$p(n) = \prod_{k=0}^{n-1}(1+\theta^{2^k}) \leq \exp\left(\sum_{k=0}^{\infty}\theta^{2^k}\right) \leq \exp\left(\sum_{k=1}^{\infty}\theta^{k}\right) = \exp\left(\frac{\theta}{1-\theta}\right) \leq \lambda(\theta)\,.$$

This shows that $\sup_{n\in\mathbb{N}} C_n \leq C^{\lambda(\theta)}$. Moreover, we have $\lim_{n\to\infty} \theta^{2^n} = 0$. Therefore, for any fixed $t \geq 1$, we can take the limit as $n \to \infty$ in (3.3) and obtain that

$$K_r(t,a) \leq C^{\lambda(\theta)}\,\|a\|_{A_0+A_1} \qquad \text{for all } t \geq 1 \text{ and all } a \in A_0 \cap A_1\,.$$

Since $K(t,a) \leq K_r(t,a)$, this finally implies that, for all $a \in A_0 \cap A_1$,

$$\|a\|_{A_0^{\sim}} = \lim_{t\to\infty} K(t,a) \leq \lim_{t\to\infty} K_r(t,a) \leq C^{\lambda(\theta)}\,\|a\|_{A_0+A_1}\,. \tag{3.4}$$

Step 2. Now we estimate $\|a\|_{A_1^{\sim}}$. Here we use (3.1) only for $t \in (0,1)$, but otherwise we proceed similarly to our reasoning in the first step. For convenience of notation we set $\alpha = 1 - \theta$. By (3.1) we have

$$K_r(t,a) \leq C\,t^{1-\alpha}\,\|a\|_{A_0+A_1} \qquad \text{for all } t \in (0,1) \text{ and all } a \in A_0 \cap A_1\,. \tag{3.5}$$

For a fixed $a \in A_0 \cap A_1$ we again consider any decomposition $a = a_0 + a_1$ with $a_j \in A_j$. Then (3.5) implies for all $t \in (0,1)$ that

$$K_r(t,a)^r \leq K_r(t,a_0)^r + K_r(t,a_1)^r \leq (C\,t^{1-\alpha})^r\,\|a_0\|_{A_0}^r + t^r\,\|a_1\|_{A_1}^r\,,$$

or, equivalently,

$$K_r(t,a) \leq C\,t^{1-\alpha}\left(\|a_0\|_{A_0}^r + \left(\frac{t^{\alpha}}{C}\right)^r\,\|a_1\|_{A_1}^r\right)^{1/r}\,.$$

Now we pass to the infimum over all such decompositions of a. Note that $t^{\alpha}/C < 1$, so we can again use (3.5), and we obtain that

$$K_r(t,a) \leq C\,t^{1-\alpha}\cdot C\left(\frac{t^{\alpha}}{C}\right)^{1-\alpha}\,\|a\|_{A_0+A_1} = C^{1+\alpha}\,t^{1-\alpha^2}\,\|a\|_{A_0+A_1}\,.$$

By iteration we obtain that, for all $n \in \mathbb{N}$ and all $t \in (0,1)$ and all $a \in A_0 \cap A_1$,

$$K_r(t,a) \le C_n^* \, t^{1-\alpha^{2^n}} \, \|a\|_{A_0+A_1} \,, \qquad \text{with } C_n^* = C^{p^*(n)} \text{ and } p^*(n) = \prod_{k=0}^{n-1} (1+\alpha^{2^k}) \,.$$

The same reasoning as in Step 1 shows that the exponents can be bounded by

$$p^*(n) = \prod_{k=0}^{n-1} (1+\alpha^{2^k}) \le \exp\left(\frac{\alpha}{1-\alpha}\right) = \exp\left(\frac{1-\theta}{\theta}\right) \le \lambda(\theta) \,,$$

whence $\sup\limits_{n\in\mathbb{N}} C_n^* \le C^{\lambda(\theta)}$, and therefore

$$t^{-1} K_r(t,a) \le C^{\lambda(\theta)} \, t^{-\alpha^{2^n}} \, \|a\|_{A_0+A_1} \,.$$

Taking first the limit as $n \to \infty$ for fixed $t \in (0,1)$, and then the limit as $t \to 0$, we obtain that

$$\|a\|_{A_1^\sim} = \lim_{t\to 0} t^{-1} K(t,a) \le \lim_{t\to 0} t^{-1} K_r(t,a) \le C^{\lambda(\theta)} \|a\|_{A_0+A_1} \quad \text{for all } a \in A_0 \cap A_1 \,.$$

Combining this with (3.4), we arrive at the desired conclusion

$$\|a\|_{A_0^\sim \cap A_1^\sim} = \max\left\{\|a\|_{A_0^\sim}, \|a\|_{A_1^\sim}\right\} \le C^{\lambda(\theta)} \|a\|_{A_0+A_1} \quad \text{for all } a \in A_0 \cap A_1 \,. \quad \square$$

Based on this lemma we can now prove the following theorem, which will be one of the main ingredients for our solution of Lions's problem.

Theorem 3.2. *Let (A_0, A_1) be a Gagliardo couple of quasi-Banach spaces, and let $0 < \theta < 1$ and $0 < p \le \infty$. Then the following conditions are equivalent.*

(i) *$A_0 \cap A_1$ is closed in $A_0 + A_1$ with respect to the quasi-norm of $A_0 + A_1$.*
(ii) *$(A_0, A_1)_{\theta,p}$ is closed in $A_0 + A_1$ with respect to the quasi-norm of $A_0 + A_1$.*

Proof. (i) $\Longrightarrow$ (ii): By the equivalence theorem [3, Theorem 3.11.3] we know that $A_0 \cap A_1$ is dense in $(A_0, A_1)_{\theta,p}$ with respect to the quasi-norm of $A_0 + A_1$. Therefore, if $A_0 \cap A_1$ is closed in $A_0 + A_1$ with respect to the quasi-norm of $A_0 + A_1$, then we have $(A_0, A_1)_{\theta,p} = A_0 \cap A_1$. Consequently $(A_0, A_1)_{\theta,p}$ is closed in $A_0 + A_1$ with respect to the quasi-norm of $A_0 + A_1$.
(ii) $\Longrightarrow$ (i): By the Aoki-Rolewicz theorem we can assume without loss of generality that the quasi-norms in A_0 and A_1 are both r-norms for some $0 < r \le 1$. If $(A_0, A_1)_{\theta,p}$ is closed in $A_0 + A_1$, then it follows from [22, Corollary 2.12] that there is a constant $M > 0$ such that

$$\|a\|_{\theta,p} \le M \, \|a\|_{A_0+A_1} \quad \text{for all } a \in (A_0, A_1)_{\theta,p} \,.$$

Due to the continuous embedding $(A_0, A_1)_{\theta,p} \hookrightarrow (A_0, A_1)_{\theta,\infty}$ and the equivalence $K(t,.) \sim K_r(t,.)$ there is another constant $C > 0$ such that

$$\sup_{t>0} t^{-\theta} K_r(t,a) \le 2\|a\|_{\theta,\infty;r} \le C \, \|a\|_{A_0+A_1} \quad \text{for all } a \in A_0 \cap A_1 \,.$$

Note that then $C \ge 1$, which follows from

$$\|a\|_{A_0+A_1} = K(1,a) \leq K_r(1,a) \leq C\|a\|_{A_0+A_1}\,.$$

By assumption (A_0, A_1) is a Gagliardo couple, and hence $A_0 \cap A_1 = A_0^{\sim} \cap A_1^{\sim}$. Therefore the norm one embedding $A_0 \cap A_1 \hookrightarrow A_0^{\sim} \cap A_1^{\sim}$ is bijective, and by [22, Corollary 2.12] its inverse is continuous. This shows that the quasi-norms $\|.\|_{A_0\cap A_1}$ and $\|.\|_{A_0^{\sim}\cap A_1^{\sim}}$ are equivalent. By Lemma 3.1 we have

$$\|a\|_{A_0\cap A_1} \sim \|a\|_{A_0^{\sim}\cap A_1^{\sim}} \leq C^{\lambda(\theta)}\|a\|_{A_0+A_1} \quad \text{for all } a \in A_0 \cap A_1\,,$$

whence $A_0 \cap A_1$ is closed in $A_0 + A_1$ with respect to the quasi-norm of $A_0 + A_1$. □

In the Banach case, Theorem 3.2 corresponds to the Proposition in [13] which Lévy proved with the help of duality properties of $A_0 \cap A_1$, $A_0 + A_1$ and $(A_0, A_1)_{\theta,p}$. We have had to find a different proof from Lévy's, since in the quasi-Banach case duality arguments are no longer available.

Our next goal is to generalize another deep result of Lévy (see [14, Ch. 2, Thm. 1] or [13, Theoreme]) from the Banach case to the quasi-Banach setting. Theorem 3.3 below will be another important ingredient of our solution to Lions's problem for quasi-Banach spaces.

Theorem 3.3. *Let (A_0, A_1) be a quasi-Banach couple, let $0 < \theta < 1$ and $0 < p < \infty$. If $(A_0, A_1)_{\theta,p}$ is not closed in $A_0 + A_1$ with respect to the quasi-norm of $A_0 + A_1$, then $(A_0, A_1)_{\theta,p}$ contains a subspace isomorphic to ℓ_p.*

Proof. By the Aoki-Rolewicz theorem we can again assume that the quasi-norms of A_0 and A_1 are both r-norms for some $0 < r \leq 1$. Since every r-norm is an s-norm for any $s \leq r$, we can furthermore assume that $r \leq p$. In this case, as we have pointed out in Section 2,

$$\|a\|_{\theta,p;r} := \Big(\sum_{m\in\mathbb{Z}} j_m(a)^p\Big)^{1/p} \quad \text{with } \ j_m(a) := 2^{\theta m} K_r(2^m, a)$$

is an r-norm on $(A_0, A_1)_{\theta,p}$ which is equivalent to the original quasi-norm of $(A_0, A_1)_{\theta,p}$. In what follows we shall always equip $(A_0, A_1)_{\theta,p}$ with the r-norm $\|a\|_{\theta,p;r}$. By our assumption this r-norm is not equivalent to the quasi-norm $\|.\|_{A_0+A_1}$. In particular this implies that $\dim(A_0, A_1)_{\theta,p} = \infty$, since on finite-dimensional spaces all quasi-norms are equivalent, which follows from [22, Corollary 2.12]. This observation ensures that the recursive construction described below never stops.
Given any $0 < \varepsilon < 1$, one can inductively construct a sequence $(x_n)_{n\in\mathbb{N}}$ of vectors in $(A_0, A_1)_{\theta,p}$ and an increasing sequence $(N_n)_{n\in\mathbb{N}}$ of natural numbers satisfying the following conditions:

$$\begin{aligned}
&\text{(a)} \quad \|x_n\|_{\theta,p;r} = 1 && \text{for all } n \in \mathbb{N}\\
&\text{(b)} \quad \Big(\sum_{|m|>N_n} j_m(x_n)^p\Big)^{1/p} \leq 2^{-n/r}\varepsilon && \text{for all } n \in \mathbb{N}\\
&\text{(c)} \quad \Big(\sum_{|m|\leq N_{n-1}} j_m(x_n)^p\Big)^{1/p} \leq 2^{-n/r}\varepsilon && \text{for all } n \geq 2
\end{aligned} \tag{3.6}$$

Let us now describe this construction in some detail. First take any $x_1 \in (A_0, A_1)_{\theta,p}$ with $\|x_1\|_{\theta,p;r} = \Big(\sum_{m\in\mathbb{Z}} j_m(x_1)^p\Big)^{1/p} = 1$ and select $N_1 \in \mathbb{N}$ such that $\Big(\sum_{|m|>N_1} j_m(x_1)^p\Big)^{1/p} \leq 2^{-1/r}\varepsilon$. Hence, for $n = 1$ the conditions (a) and (b) are satisfied, and condition (c) is not needed.

Assume now that $x_1, \ldots, x_n$ and $N_1, \ldots, N_n$ are already found. Since all j_m are r-norms on A_0+A_1, it follows that $\Big(\sum_{|m|\le N_n} j_m(a)^p\Big)^{1/p}$ is also an r-norm on A_0+A_1, and clearly this r-norm is equivalent to the quasi-norm $\|a\|_{A_0+A_1}$. Therefore, due to our assumption, it cannot be equivalent to $\|.\|_{\theta,p;r}$ on $(A_0,A_1)_{\theta,p}$. Hence one can find an element $x_{n+1} \in (A_0,A_1)_{\theta,p}$ of quasi-norm one such that $\Big(\sum_{|m|\le N_n} j_m(x_{n+1})^p\Big)^{1/p} \le 2^{-(n+1)/r}\varepsilon$. Since $\|x_{n+1}\|_{\theta,p;r} = 1$, we can select $N_{n+1} > N_n$ such that $\Big(\sum_{|m|>N_{n+1}} j_m(x_{n+1})^p\Big)^{1/p} \le 2^{-(n+1)/r}\varepsilon$. Then (a), (b) and (c) hold also for x_{n+1} and N_{n+1}, and repeating these arguments we can recursively construct sequences $(x_n)_{n\in\mathbb{N}}$ and $(N_n)_{n\in\mathbb{N}}$ satisfying all required conditions.

Our next aim is to show that (for sufficiently small $\varepsilon > 0$) the sequence $(x_n)_{n\in\mathbb{N}} \subseteq (A_0,A_1)_{\theta,p}$ is equivalent to the unit vector basis in ℓ_p. More precisely, we will prove that there are constants $0 < M_0 \le M_1 < \infty$ such that for all scalar sequences (α_n) with at most finitely many non-zero entries the inequalities

$$M_0\Big(\sum_{n=1}^{\infty}|\alpha_n|^p\Big)^{1/p} \le \Big\|\sum_{n=1}^{\infty}\alpha_n x_n\Big\|_{\theta,p;r} \le M_1\Big(\sum_{n=1}^{\infty}|\alpha_n|^p\Big)^{1/p} \tag{3.7}$$

hold. Certainly the equivalence constants M_0 and M_1 will depend on ε and r, and maybe also on the parameters θ and p. For later use we shall keep track of this dependence.

Let $E_m := A_0 + A_1$ equipped with the r-norm j_m, $m \in \mathbb{Z}$, and define

$$F_p := \Big(\bigoplus_{m\in\mathbb{Z}} E_m\Big)_p = \Big\{X = (X_m)_{m\in\mathbb{Z}} : X_m \in E_m, \Big(\sum_{m\in\mathbb{Z}} j_m(X_m)^p\Big)^{1/p} < \infty\Big\}.$$

Then

$$\|X\|_{F_p} := \Big(\sum_{m\in\mathbb{Z}} j_m(X_m)^p\Big)^{1/p}$$

is an r-norm on F_p, and the interpolation space $(A_0,A_1)_{\theta,p}$ is just the diagonal in F_p, if we identify $x \in (A_0,A_1)_{\theta,p}$ with the vector $X = (X_m) \in F_p$, where $X_m = x$ for all $m \in \mathbb{Z}$. Moreover, we have equality of the corresponding r-norms,

$$\|x\|_{\theta,p;r} = \|(\ldots, x, x, x, \ldots)\|_{F_p} \quad \text{for all } x \in (A_0,A_1)_{\theta,p}\,.$$

In the definitions below we use the notation

$$I_1 := \{m \in \mathbb{Z} : |m| \le N_1\} \quad \text{and} \quad I_n := \{m \in \mathbb{Z} : N_{n-1} < |m| \le N_n\} \quad \text{for } n \ge 2\,.$$

With this notation, conditions (b) and (c) imply for all $n \in \mathbb{N}$

$$\Big(\sum_{m\notin I_n} j_m(x_n)^p\Big)^{1/p} \le 2^{1/p}\cdot 2^{-n/r}\varepsilon \le 2^{-(n-1)/r}\varepsilon\,. \tag{3.8}$$

For any given scalar sequence $\alpha = (\alpha_n)$ with at most finitely many non-zero entries we define now elements $X = (X_m), Y = (Y_m), Z^n = (Z^n_m) \in F_p$, $n \in \mathbb{N}$, by

$$X_m := \sum_{n=1}^{\infty}\alpha_n x_n \quad \text{for all } m \in \mathbb{Z}\,,$$

$$Y_m := \quad \alpha_n x_n \qquad \text{if } m \in I_n\,,$$
$$Z^n_m := \begin{cases} 0 & \text{if } m \in I_n \\ \alpha_n x_n & \text{if } m \notin I_n \end{cases}$$

Then we have

$$X = Y + \sum_{n=1}^{\infty} Z^n \qquad \text{and} \qquad \|X\|_{F_p} = \Big\| \sum_{n=1}^{\infty} \alpha_n x_n \Big\|_{\theta,p;r}\,, \tag{3.9}$$

and by definition of Z^n and (3.8) we get

$$\big\|Z^n\big\|_{F_p} = |\alpha_n| \Big(\sum_{m \notin I_n} j_m(x_n)^p \Big)^{1/p} \le 2^{-(n-1)/r}\, \|\alpha\|_p\, \varepsilon\,.$$

By the r-triangle inequality this implies

$$\Big\| \sum_{n=1}^{\infty} Z^n \Big\|_{F_p}^r \le \sum_{n=1}^{\infty} \big\|Z^n\big\|_{F_p}^r \le \sum_{n=1}^{\infty} 2^{-(n-1)} \|\alpha\|_p^r\, \varepsilon^r = 2\, \|\alpha\|_p^r\, \varepsilon^r\,. \tag{3.10}$$

The quasi-norm of Y is given by

$$\|Y\|_{F_p}^p = \sum_{m\in\mathbb{Z}} j_m(Y_m)^p = \sum_{n=1}^{\infty} |\alpha_n|^p \sum_{m \in I_n} j_m(x_n)^p\,,$$

and from (3.6) and (3.8) we see that

$$\sum_{m \in I_n} j_m(x_n)^p \le 1 \qquad \text{and} \qquad \sum_{m \in I_n} j_m(x_n)^p \ge 1 - 2^{-(n-1)p/r} \varepsilon^p \ge 1 - \varepsilon^p\,.$$

This leads to the bounds

$$\|Y\|_{F_p}^p \le \sum_{n=1}^{\infty} |\alpha_n|^p \qquad \text{and} \qquad \|Y\|_{F_p}^p \ge (1-\varepsilon^p) \sum_{n=1}^{\infty} |\alpha_n|^p\,,$$

or, equivalently,

$$\|Y\|_{F_p}^r \le \|\alpha\|_p^r \qquad \text{and} \qquad \|Y\|_{F_p}^r \ge (1-\varepsilon^p)^{r/p} \|\alpha\|_p^r \ge (1-\varepsilon^r) \|\alpha\|_p^r\,. \tag{3.11}$$

The last inequality follows from $p \ge r$, whence $1 = \big(\varepsilon^p + (1-\varepsilon^p)\big)^{1/p} \le \big(\varepsilon^r + (1-\varepsilon^p)^{r/p}\big)^{1/r}$. The r-triangle inequality in F_p and the bounds (3.10) and (3.11) imply the estimates

$$\|X\|_{F_p}^r \le \|Y\|_{F_p}^r + \sum_{n=1}^{\infty} \|Z^n\|_{F_p}^r \le (1+2\varepsilon^r) \|\alpha\|_p^r \quad \text{and}$$
$$\|X\|_{F_p}^r \ge \|Y\|_{F_p}^r - \sum_{n=1}^{\infty} \|Z^n\|_{F_p}^r \ge (1-3\varepsilon^r) \|\alpha\|_p^r\,.$$

That means, taking (3.9) in to account, we have proved (3.7) with equivalence constants

$$M_0 = M_0(\varepsilon, r) := (1-3\varepsilon^r)^{1/r} \qquad \text{and} \qquad M_1 = M_1(\varepsilon, r) := (1+2\varepsilon^r)^{1/r}\,. \tag{3.12}$$

Of course we need $M_0 > 0$. This holds whenever $0 < \varepsilon < 3^{-1/r}$, and then the sequence $(x_n)_{n\in\mathbb{N}}$ in $(A_0, A_1)_{\theta,p}$ (equipped with the quasi-norm $\|.\|_{\theta,p;r}$) is indeed equivalent to the unit vector basis in ℓ_p. $\square$

Remark 3.4. (1) Let us point out that the equivalence constants in (3.12) are independent on the parameters θ, p of the interpolation space, as long as $p \geq r$. This observation will be important in the proof of our next theorem, which is the positive solution to Lions's problem in the quasi-Banach setting.
(2) Since $\lim_{\varepsilon\to 0} M_j(\varepsilon, r) = 1$ for $j = 0, 1$, one can even achieve that for any given $\delta \in (0, 1)$ the subspace of $(A_0, A_1)_{\theta,p}$ spanned by the sequence (x_n) is $(1 + \delta)$-isomorphic to ℓ_p, i.e. we have

$$(1-\delta)\Big(\sum_{n=1}^{\infty} |\alpha_n|^p\Big)^{1/p} \leq \Big\| \sum_{n=1}^{\infty} \alpha_n x_n \Big\|_{\theta,p;r} \leq (1+\delta)\Big(\sum_{n=1}^{\infty} |\alpha_n|^p\Big)^{1/p}.$$

In the Banach case $r = 1$ and for parameters $1 \leq p < \infty$ this is due to Lévy [13,14]. She also showed that these subspaces are complemented, which was used in an essential way in the solution to Lions's problem in [8]. We do not know whether the corresponding subspaces in the quasi-Banach setting are complemented, but we shall give a different proof which does not rely on complementedness of these subspaces.

Now we are ready to prove our main theorem which provides the positive solution to Lions's problem for quasi-Banach couples and the extended range of parameters $0 < p, q \leq \infty$. However we shall need an additional assumption, namely that the couple is a Gagliardo couple. This is a rather mild condition which is satisfied in many concrete cases. As an example which might be of interest for potential applications we shall consider in Section 4 the spaces $\mathscr{L}^{(a)}_{p,q}(X, Y)$ of operators defined by approximation numbers.

Theorem 3.5. *Let (A_0, A_1) be a quasi-Banach Gagliardo couple such that*

$$A_0 \cap A_1 \textit{ is not closed in } A_0 + A_1 \textit{ with respect to the quasi-norm } \|.\|_{A_0+A_1}.$$

Then, for $0 < \theta, \eta < 1$ and $0 < p, q \leq \infty$, we have that

$$(A_0, A_1)_{\theta,p} \neq (A_0, A_1)_{\eta,q} \qquad \textit{unless} \qquad \theta = \eta \textit{ and } p = q.$$

Proof. According to Theorem 3.2 none of the interpolation spaces $(A_0, A_1)_{\theta,p}$ with $0 < \theta < 1$ and $0 < p \leq \infty$ can be closed in $A_0 + A_1$ with respect to the quasi-norm $\|.\|_{A_0+A_1}$. Hence we can apply Theorem 3.3. We proceed by contradiction and assume that $(A_0, A_1)_{\theta,p} = (A_0, A_1)_{\eta,q}$ for some $(\theta, p) \neq (\eta, q)$.
If $\underline{\theta = \eta}$, then $p \neq q$, say $p < q$. In this case, all spaces $(A_0, A_1)_{\theta,s}$ with $p \leq s \leq q$ coincide, which is due to the embeddings

$$(A_0, A_1)_{\theta,p} \hookrightarrow (A_0, A_1)_{\theta,s} \hookrightarrow (A_0, A_1)_{\theta,q} = (A_0, A_1)_{\theta,p}.$$

If $\underline{\theta \neq \eta}$, then it follows from the reiteration theorem (see [3, Theorem 3.11.5] or [10, 2.c.4]) that the spaces $(A_0, A_1)_{\frac{\theta+\eta}{2},s}$ do not depend on s, $0 < s \leq \infty$. Therefore our assumption implies that

$$X := (A_0, A_1)_{\theta,p} = (A_0, A_1)_{\theta,q} \quad \text{for some } 0 < \theta < 1 \text{ and } 0 < p < q < \infty.$$

Moreover, we can again assume without loss of generality that A_0 and A_1 are both r-normed for some $0 < r \leq \min\{p, 1\}$. Then the two r-norms $\|.\|_{\theta,p;r}$ and $\|.\|_{\theta,q;r}$ on X must be equivalent, that means there is a constant $C \geq 1$ such that

$$\|x\|_{\theta,q;r} \leq \|x\|_{\theta,p;r} \leq C\,\|x\|_{\theta,q;r} \qquad \text{for all } x \in X. \tag{3.13}$$

(The first inequality is due to $p < q$.) Now we apply Theorem 3.3. We choose $0 < \varepsilon < 1$ small enough such that we get $M_0(\varepsilon, r) \geq 1/2$ for the lower equivalence constant (see (3.12)), and construct a sequence (x_n) in $(A_0, A_1)_{\theta,p}$ and an increasing sequence (N_n) of natural numbers satisfying conditions (a)–(c) from (3.6). Then (x_n) is equivalent to the unit vector basis in ℓ_p, in particular we have

$$\Big\| \sum_{n=1}^{N} x_n \Big\|_{\theta,p;r} \geq M_0(\varepsilon, r)\, N^{1/p} \geq \frac{N^{1/p}}{2} \qquad \text{for all } N \in \mathbb{N}\,. \tag{3.14}$$

Now we set $y_n := \frac{x_n}{\|x_n\|_{\theta,q;r}}$ and study the properties of the sequence (y_n) in $(A_0, A_1)_{\theta,q}$. By (3.13) we have for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$

$$j_m(y_n) = \frac{1}{\|x_n\|_{\theta,q;r}}\, j_m(x_n) \leq \frac{C}{\|x_n\|_{\theta,p;r}}\, j_m(x_n) = C\, j_m(x_n)\,.$$

By the properties (a)–(c) of the sequence (x_n) in $(A_0, A_1)_{\theta,p}$ this implies for all $n \in \mathbb{N}$

(a*) $\|y_n\|_{\theta,q;r} = 1$

(b*) $$\Big(\sum_{|m|>N_n} j_m(y_n)^q \Big)^{1/q} \leq C \Big(\sum_{|m|>N_n} j_m(x_n)^p \Big)^{1/p} \leq 2^{-n/r}\, C\, \varepsilon$$

(c*) $$\Big(\sum_{|m|\leq N_{n-1}} j_m(y_n)^q \Big)^{1/q} \leq C \Big(\sum_{|m|\leq N_{n-1}} j_m(x_n)^p \Big)^{1/p} \leq 2^{-n/r}\, C\, \varepsilon\,,$$

where the first inequalities in (b*) and (c*) are due to $p < q$. The properties (a*)–(c^*) of the sequence (y_n) in $(A_0, A_1)_{\theta,q}$ are in complete analogy to the properties (a)–(c) of the sequence (x_n) in $(A_0, A_1)_{\theta,p}$, just replacing the parameters p and ε in (a)–(c) by q and $C\varepsilon$. We set now $M := M_1(C\varepsilon, r)$, see (3.12), and $\alpha_n := \|x_n\|_{\theta,q;r}$, whence $\alpha_n \leq 1$ and $x_n = \alpha_n\, y_n$. The proof of Theorem 3.3 applied to (y_n) yields (in particular) the upper estimate

$$\begin{aligned} \Big\| \sum_{n=1}^{N} x_n \Big\|_{\theta,p;r} &\leq C \Big\| \sum_{n=1}^{N} x_n \Big\|_{\theta,q;r} = C \Big\| \sum_{n=1}^{N} \alpha_n\, y_n \Big\|_{\theta,q;r} \\ &\leq C\, M \Big(\sum_{n=1}^{N} |\alpha_n|^q \Big)^{1/q} \leq C\, M\, N^{1/q} \end{aligned}$$

for all $N \in \mathbb{N}$. Together with (3.14) and letting $N \to \infty$ this implies the contradiction

$$1 \leq \lim_{N\to\infty} 2\, C\, M\, N^{1/q-1/p} = 0\,.$$

Therefore the assumption $(A_0, A_1)_{\theta,p} = (A_0, A_1)_{\theta,q}$ was wrong and the proof is finished. □

4. An application to Lorentz operator classes

In this final section we apply our previous results to spaces of operators defined by the behaviour of their approximation numbers.

Let X, Y be Banach spaces, and let $\mathscr{L}(X, Y)$ be the space of all bounded linear operators from X into Y. The n-th $a$pproximation number of an operator $T \in \mathscr{L}(X, Y)$ is defined by

$$a_n(T) = \inf\{\|T - A\| : A \in \mathscr{L}(X, Y) \text{ with rank } A < n\} \quad , \quad n \in \mathbb{N}\,.$$

For $0 < p < \infty$ and $0 < q \leq \infty$ the space $\mathscr{L}^{(a)}_{p,q}(X,Y)$ consists of all operators $T \in \mathscr{L}(X,Y)$ for which the sequence $(a_n(T))$ of its approximation numbers belongs to the Lorentz sequence space $\ell_{p,q}$. The space $\mathscr{L}^{(a)}_{p,q}(X,Y)$ is complete with respect to the quasi-norm

$$\|T\|_{p,q} := \|(a_n(T))\|_{\ell_{p,q}} = \begin{cases} \left(\sum\limits_{n=1}^{\infty} \left(n^{1/p-1/q} a_n(T)\right)^q \right)^{1/q} & \text{if } q < \infty \\ \sup\limits_{n\in\mathbb{N}} n^{1/p} a_n(T) & \text{if } q = \infty\,, \end{cases}$$

see [25,20,10]. If $p = q$, then we write simply $\mathscr{L}^{(a)}_{p}(X,Y)$ and $\|T\|_p$. Note that these quasi-norms are (for general infinite-dimensional Banach spaces X, Y) not equivalent to a norm, not even for $p, q \geq 1$. Therefore it is in this context necessary to work in the quasi-Banach setting.

Let $0 < p_0 < p_1 < \infty$, $0 < \theta < 1$ and $0 < q \leq \infty$. Then the continuous embedding $\mathscr{L}^{(a)}_{p_0}(X,Y) \hookrightarrow \mathscr{L}^{(a)}_{p_1}(X,Y)$ shows that $\big(\mathscr{L}^{(a)}_{p_0}(X,Y), \mathscr{L}^{(a)}_{p_1}(X,Y)\big)$ is a quasi-Banach couple. According to [9, Corollary 1], we have with equivalence of quasi-norms

$$\big(\mathscr{L}^{(a)}_{p_0}(X,Y), \mathscr{L}^{(a)}_{p_1}(X,Y)\big)_{\theta,q} = \mathscr{L}^{(a)}_{p,q}(X,Y) \quad \text{where} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\,. \tag{4.1}$$

In the following two lemmata we show that the couple $\big(\mathscr{L}^{(a)}_{p_0}(X,Y), \mathscr{L}^{(a)}_{p_1}(X,Y)\big)$ satisfies the assumptions of Theorem 3.5. Subsequently, we write $\lfloor\cdot\rfloor$ for the greatest integer function.

Lemma 4.1. *Let X, Y be Banach spaces and $0 < p_0 < p_1 < \infty$. Then $\big(\mathscr{L}^{(a)}_{p_0}(X,Y), \mathscr{L}^{(a)}_{p_1}(X,Y)\big)$ is a Gagliardo couple.*

Proof. Put $A_0 = \mathscr{L}^{(a)}_{p_0}(X,Y)$ and $A_1 = (\mathscr{L}^{(a)}_{p_1}(X,Y)$. Due to the embedding $A_0 \hookrightarrow A_1$ we have $A_1^{\sim} = A_1$. Moreover, $A_0 \hookrightarrow A_0^{\sim}$. To prove the reverse embedding, we use the result of König [9, Proposition 1] concerning Peetre's K-functional for the couple (A_0, A_1),

$$K(t,T) \sim \left(\sum_{n=1}^{\lfloor t^r \rfloor} a_n(T)^{p_0} \right)^{1/p_0} + t \left(\sum_{n=\lfloor t^r \rfloor}^{\infty} a_n(T)^{p_1} \right)^{1/p_1} \quad \text{where} \quad 1/r = 1/p_0 - 1/p_1\,.$$

Let $T \in A_0^{\sim}$. Then we have

$$\|T\|_{A_0^{\sim}} = \sup_{t>0} K(t,T) \gtrsim \sup_{t>0} \Big(\sum_{n=1}^{\lfloor t^r \rfloor} a_n(T)^{p_0} \Big)^{1/p_0} = \left(\sum_{n=1}^{\infty} a_n(T)^{p_0} \right)^{1/p_0} = \|T\|_{A_0}\,.$$

Since $\|T\|_{A_0}$ is finite, it follows that $T \in A_0$ and consequently we have $A_0^{\sim} = A_0$. □

Lemma 4.2. *Let X, Y be infinite-dimensional Banach spaces, and let $0 < p_0 < p_1 < \infty$ such that $1/p_0 - 1/p_1 > 1$. Then $\mathscr{L}^{(a)}_{p_0}(X,Y)$ is not closed in $\mathscr{L}^{(a)}_{p_1}(X,Y)$. In particular, $\mathscr{L}^{(a)}_{p_0}(X,Y)$ is strictly smaller than $\mathscr{L}^{(a)}_{p_1}(X,Y)$.*

Proof. If $\mathscr{L}^{(a)}_{p_0}(X,Y)$ would be closed in $\mathscr{L}^{(a)}_{p_1}(X,Y)$, then there would exist a constant $M > 0$ such that

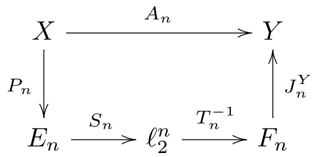
$$\|T\|_{p_0} \leq M\|T\|_{p_1} \quad \text{for all } T \in \mathscr{L}^{(a)}_{p_0}(X,Y)\,. \tag{4.2}$$

We will show that this is impossible. As a consequence of Dvoretzky's famous theorem on almost spherical sections of convex bodies [5] (for improved versions see [17,24,6] and [21, Chapter 4]), the Banach space X

contains for every $n \in \mathbb{N}$ an n-dimensional subspace E_n with $d(E_n, \ell_2^n) \le 2$, where d denotes the Banach-Mazur distance. That means there are invertible operators $S_n \in \mathscr{L}(E_n, \ell_2^n)$ such that $\|S_n\|\|S_n^{-1}\| \le 2$. By homogeneity we can assume that $\|S_n\| = \|S_n^{-1}\|$. Similarly, for the space Y there are subspaces F_n and operators $T_n \in \mathscr{L}(F_n, \ell_2^n)$ such that $\|T_n\|\|T_n^{-1}\| \le 2$ and $\|T_n\| = \|T_n^{-1}\|$. It is well known (see e.g. [10, Theorem 4.b.6]) that every n-dimensional subspace of a Banach space is $\sqrt{n}$-complemented. Hence there are projections

$P_n \in \mathscr{L}(X, X)$ such that $P_n^2 = P_n$, $P_n(X) = E_n$ and $\|P_n\| \le \sqrt{n}$ and
$Q_n \in \mathscr{L}(Y, Y)$ such that $Q_n^2 = Q_n$, $Q_n(Y) = F_n$ and $\|Q_n\| \le \sqrt{n}$.

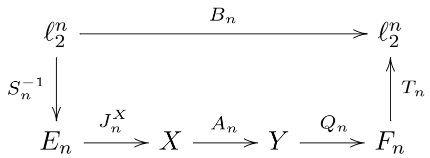
Moreover we shall use the embeddings J_n^X from E_n into X and J_n^Y from F_n into Y. Now we define operators $A_n \in \mathscr{L}(X, Y)$ by the following composition



Since rank $A_n = n$, we have $a_k(A_n) = 0$ for $k > n$, and for $k \le n$ we use the trivial estimate

$$a_k(A_n) \le \|A_n\| \le \|P_n\|\|S_n\|\|T_n^{-1}\|\|J_n^Y\| \le 2\sqrt{n}\,.$$

In order to estimate the approximation numbers of A_n from below we consider the operators B_n that are given by the following composition.



Using that $S_n P_n J_n^X S_n^{-1} = \mathrm{id}_{\ell_2^n}$ and $T_n Q_n J_n^Y T_n^{-1} = \mathrm{id}_{\ell_2^n}$ we get

$$B_n = T_n Q_n A_n J_n^X S_n^{-1} = (T_n Q_n J_n^Y T_n^{-1})(S_n P_n J_n^X S_n^{-1}) = \mathrm{id}_{\ell_2^n}\,.$$

Hence, by [20, Section 2.3], $a_n(B_n) = 1$ for $k = 1, \dots, n$, and we obtain

$$1 = a_k(B_n) \le \|S_n^{-1}\|\,\|J_n^X\|\,a_k(A_n)\,\|Q_n\|\,\|T_n\| \le 2\sqrt{n}\,a_k(A_n)\,.$$

Therefore

$$\frac{1}{2\sqrt{n}} \le a_k(A_n) \le 2\sqrt{n} \quad \text{for } k = 1, \dots, n \qquad \text{and} \qquad a_k(A_n) = 0 \quad \text{for } k > n\,.$$

It follows that

$$\|A_n\|_{p_1} = \Big(\sum_{k=1}^{n} a_k(A_n)^{p_1}\Big)^{1/p_1} \le 2\,n^{1/p_1 + 1/2} \quad \text{and}$$

$$\|A_n\|_{p_0} = \Big(\sum_{k=1}^{n} a_k(A_n)^{p_0}\Big)^{1/p_0} \ge \tfrac{1}{2}\,n^{1/p_0 - 1/2}\,.$$

This implies

$$\frac{\|A_n\|_{p_0}}{\|A_n\|_{p_1}} \geq \frac{1}{4}\, n^{1/p_0 - 1/p_1 - 1} \underset{n\to\infty}{\longrightarrow} \infty\,,$$

which contradicts (4.2). The proof is complete. □

Now we are prepared to prove the main result of this section.

Theorem 4.3. *Let X and Y be infinite-dimensional Banach spaces. Then all spaces $\mathscr{L}^{(a)}_{p,q}(X,Y)$ are different for $0 < p < \infty$ and $0 < q \leq \infty$.*

Proof. Let $(r_0, q_0) \neq (r_1, q_1)$ be any two different pairs of indices with $0 < r_0, r_1 < \infty$ and $0 < q_0, q_1 \leq \infty$. We want to show that $\mathscr{L}^{(a)}_{r_0,q_0}(X,Y)) \neq \mathscr{L}^{(a)}_{r_1,q_1}(X,Y)$. To this end we choose first p_0 and p_1 such that $0 < p_0 < \min(r_0, r_1) \leq \max(r_0, r_1) < p_1 < \infty$ and $1/p_0 - 1/p_1 > 1$, and then we determine $0 < \theta_0, \theta_1 < 1$ such that $1/r_j = (1-\theta_j)/p_0 + \theta_j/p_1$ for $j = 0, 1$. By Lemma 4.1 $\big(\mathscr{L}^{(a)}_{p_0}(X,Y), \mathscr{L}^{(a)}_{p_1}(X,Y)\big)$ is a Gagliardo couple, and by Lemma 4.2 $\mathscr{L}^{(a)}_{p_0}(X,Y)$ is not closed in $\mathscr{L}^{(a)}_{p_1}(X,Y)$. Clearly we have $(\theta_0, q_0) \neq (\theta_1, q_1)$, hence we can apply Theorem 3.5, and taking also König's interpolation formula (4.1) into account we obtain the desired result

$$\begin{aligned} \mathscr{L}^{(a)}_{r_0,q_0}(X,Y) &= \big(\mathscr{L}^{(a)}_{p_0}(X,Y), \mathscr{L}^{(a)}_{p_1}(X,Y)\big)_{\theta_0,q_0} \\ &\neq \big(\mathscr{L}^{(a)}_{p_0}(X,Y), \mathscr{L}^{(a)}_{p_1}(X,Y)\big)_{\theta_1,q_1} = \mathscr{L}^{(a)}_{r_1,q_1}(X,Y)\,. \quad \square \end{aligned}$$

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