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1 Introduction

We consider the public good location problem, in which a society of voters uses a voting rule to choose from a set of alternatives according to voters' preferences when those

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preferences over alternatives are single-peaked. If the social choice is supposed to reflect an optimal compromise among the conflicting interests of the voters, it is necessary that the voters have incentives to reveal the true information about their preferences. That is, the voting rule should satisfy strategy-proofness. The seminal work by Moulin (1980) shows that when voters are constrained to disclose their preferred alternative –their peaks–, and the social choice consists of a unique alternative; the family of voting rules that satisfy strategy-proofness is a generalization of selecting the median from the reported voters' peaks (min-max rules). Despite the inherent appeal of min-max or median rules, they may fail to achieve a satisfactory social choice proposal. Think of a society with 100 voters, 50 voters vote for alternative a; 50 voters vote for b. A median voter rule would inevitably select either a or b. However, an alternative approach involving the preliminary selection of several alternatives as a first shortlisting, followed by a final choice deferred to an eventual screening process after acquiring additional information, may appear more appropriate.

This observation directs our focus towards Social Choice Correspondences (SCCs), voting rules that, from each profile of voters' preferences over alternatives, select a finite set of alternatives. Since SCCs admit the selection of multiple alternatives, voters' preferences over alternatives do not contain enough information to compare the possible outcomes of the social choice and determine the incentives to misrepresent preferences. Therefore, it becomes necessary to extend voters' preferences over alternatives to preferences over sets of alternatives. In line with the interpretation of sets of alternatives as shortlistings pending a final resolution of the social choice, we assume that voters consider sets of alternatives as the basis for lotteries. In a similar vein to Barberà *et al.* (2001), voters consider a probability assessment over alternatives that assigns positive probability to each initial alternative, and conditional on a set being selected, the probability attached to each alternative in the selected set is consistent with Bayesian updating from the initial probability assessment. With this interpretation of sets, we characterize the family of SCCs that satisfy strategy-proofness for each arbitrary initial probability assessment λ and unanimity (Theorem 1).[1] We show that every SCC that satisfies strategy-proofness and unanimity – a λ-consistent union min-max SCC– consists of the union of up to two single-valued min-max rules. In fact, when the initial probability assessment is uniform, then every union of two min-max rules satisfies those properties. However, if the initial probability assessment is not uniform, λ-consistent union min-max SCCs consistently select either the same alternative, or adjacent alternatives in the real line, or alternatives contained in a segment of the real line for which the initial probability assessment is uniform.

To obtain our results, we propose a constructive approach that may be of independent interest. Our proofs exploit the relation between SCCs and probabilistic decision schemes (PDSs), –mappings from preferences over alternatives to lotteries over alternatives–. Ex-

[1] A voting rule satisfies unanimity if whenever all voters agree on a single alternative as the most preferred alternative, that alternative is uniquely selected.

tending an argument proposed by Feldman (1980), we associate to each SCC a specific PDS that retains the properties of the SCC. Exploiting the characterization of PDS satisfying *strategy-proofness* and *unanimity* under single–peaked preferences by Ehlers *et al.* (2002), we are able to deduce the structure of the original SCCs. This approach is applicable to alternative social choice settings. For instance, we extend the analysis to multi-dimensional convex spaces of alternatives. In this setting, from the results on PDSs by Dutta *et al.* (2002), we demonstrate that under uniform initial probability assessments, only SCCs concentrating the social decision in the hands of up to two voters satisfy both *strategy-proofness* and *unanimity.*

A substantial body of literature has examined both multivalued social choice rules and the domain of single-peaked preferences as mechanisms to circumvent the negative implications of the Gibbard–Satterthwaite Theorem (Gibbard, 1973; Satterthwaite, 1975). For an in-depth overview, we direct the reader to the comprehensive survey by Barberà (2011). The most closely related works are Ingalagavi and Sadhukhan (2023) and Rodríguez-Álvarez (2017). These works also analyze multivalued social choice rules under single-peaked preferences over alternatives when voters' preferences over sets are consistent with sets as the basis of a random device and present results in line with ours. Both papers consider a more general framework than ours proposed by Barberà *et al.* (2001). They analyze rules that directly use voters' preferences over sets as the input for the social choice instead of SCCs. Ingalagavi and Sadhukhan (2023) considers two domains of admissible preferences over sets that are obtained by Bayesian updating from either a uniform probability assessment or from any probability assessment. Ingalagavi and Sadhukhan (2023) focuses on rules that only use the information about the most preferred set of each voter to determine the social choice. Since that information is contained in voters' preferences over singleton sets, Ingalagavi and Sadhukhan (2023)'s framework coincides with ours for specific instances of the domains of preferences analyzed here. Finally, Rodríguez-Álvarez (2017) considers general preferences over sets but focuses on rules that select either singleton sets or sets consisting of pairs of adjacent alternatives.[2] Other relevant papers that address multivalued social choice rules that satisfy *strategy-proofness* under single-peaked preferences, under constraints on feasible sets and domains of preferences over sets different than the restrictions analyzed here. For instance, Miyagawa (1998, 2001), and Heo (2013) consider the problem of locating two public facilities in a line when each voter may attend exactly one location. Alternatively, Klaus and Storcken (2002); Klaus and Protopapas (2020); Bhattacharya and Khare (2024); Rodríguez-Álvarez (2025) consider the choice of sets of alternatives that form intervals in an ordered alternative space. Regarding probabilistic decision schemes, the seminal paper by Gibbard (1977) proposes PDSs as a way to overcome the negative implications of the Gibbard–Satterthwaite Theorem in unconstrained domains of preferences. Ehlers *et al.* (2002) analyzes single-peaked one-dimensional domains. Peters *et al.* (2014) studies the conditions under which strategy-proofness is solely achieved through

[2]We postpone the detailed comparison of the frameworks and domain restrictions between our model and theirs to Section 5.

probabilistic combinations of deterministic rules that satisfy strategy-proofness. Dutta *et al.* (2002) focuses on general multi-dimensional convex preferences.

After this brief summary, in Section 2, we present notation and definitions. In Section 3, we present the standard single-peaked framework and our characterization results. In Section 4, we analyze the relation between SCCs and PDSs, provide an outline of the proof of the main result. In Section 5, we discuss the extension to multi-dimensional alternative spaces and compare our framework with Ingalagavi and Sadhukhan (2023) and Rodríguez-Álvarez (2017). In Section 6, we complete the proofs.

2 Notation and Basic Definitions

Consider a society formed by a finite set of voters $N = \{1, \ldots, n\}$ that must choose from a set of alternatives A. We defer additional structure of the alternative space to subsequent sections. Let $\mathcal{A}$ denote the set of all finite non-empty subsets of A. A ***preference*** (over alternatives) is a complete, reflexive, and transitive binary relation over A. Each voter $i \in N$ is equipped with a ***preference*** R_i over A. As usual, for each x, $y \in A$, $x\ R_i\ y$ means x is weakly preferred to y, and $x\ P_i\ y$ means x is strictly preferred to y. The ***peak of*** R_i is defined as $p(R_i) = \{x \in A : \text{ for each } y \in A,\ x\ R_i\ y\}$. In this paper, we admit indifferences among alternatives, but we focus on preferences with singleton peaks, $\#p(R_i) = 1$.[3] Let $\mathcal{R}$ denote the set of all preferences relations over A with singleton peaks. A ***preference profile*** is a n-tuple of preferences $R = (R_i)_{i \in N}$. A ***preference domain*** $\tilde{\mathcal{R}}$ is a subset of $\mathcal{R}$. For each $\tilde{\mathcal{R}} \subseteq \mathcal{R}$, $\tilde{\mathcal{R}}^N \equiv \times_{i \in N} \tilde{\mathcal{R}}$ denotes the set of all preference profiles with preferences in $\tilde{\mathcal{R}}$. For each $i \in N$, $S \subset N$, and each $R \in \mathcal{R}^N$, $R_S = (R_i)_{i \in S}$ denotes the restriction of the profile R to the voters in S, and R_{-i} denotes the restriction of the profile$R$ to the voters in $N \setminus \{i\}$.

A ***preference over sets*** is a complete, reflexive, and transitive binary relation over $\mathcal{A}$. Each voter i is also equipped with a preference over sets, $\succsim_i$. Let $\mathcal{D}$ be the set of all preferences over sets. For each $\succsim_i \in \mathcal{D}$, $\succ_i$ denotes its strict component. We denote by $\succsim = (\succsim_1, \ldots, \succsim_n) \in \mathcal{D}^N$ a profile of voters' preferences over sets. For each voter i and each profile of preferences over sets $\succsim$, we denote the restriction of $\succsim$ to the voters $N \setminus \{i\}$ by $\succsim_{-i}$. For each $\succsim_i \in \mathcal{D}$, the ***top-set of*** $\succsim_i$, denoted by $\tau(\succsim_i)$, is the set of most preferred sets according to $\succsim_i$, $\tau(\succsim_i) = \{X \in \mathcal{A} : \text{ for each } Y \in \mathcal{A},\ X \succsim_i Y\}$.

Voters' preferences over the set of alternatives are naturally derived from voters' preferences over alternatives. That is, the way in which a voter compares sets of alternatives respects preferences over singleton sets. For each $R_i \in \mathcal{R}$, we say that $\succsim_i \in \mathcal{D}$ is R_i-***consistent*** if, for each $c, c' \in A$, $c\ R_i\ c'$ if and only if $\{c\} \succsim_i \{c'\}$. An ***extension of preferences*** over alternatives to preferences over sets E is a correspondence that selects,

[3] For each set X, $\#X$ denotes its cardinality. Abusing notation, we drop brackets when referring to singleton sets of alternatives whenever there is no room for confusion.

for each preference over alternatives a non-empty set of R_i-consistent preferences over sets. Hence, for each $R_i \in \mathcal{R}$, $E(R_i) \subset \mathcal{D}$ denotes the set of R_i-consistent elements of $\mathcal{D}$ under the extension E.

We explore alternative preference extensions, interpreting sets as the basis for lotteries. Building upon the framework proposed by Barberà *et al.* (2001), we assume that preferences over sets are consistent with conditional expected utility maximization. Under this framework, voters initially assign a prior probability to each alternative in A, and upon selecting a set, they update these probabilities according to the Bayes' rule.

A ***probability assessment*** is a function $\lambda : A \to (0,1)$ with $\sum_{x\in A} \lambda(x) = 1$, that assigns a positive prior probability to each alternative. A ***utility function*** u_i is a function from $u_i : A \to \mathbb{R}$. A utility function u_i ***represents*** the preference $R_i \in \mathcal{R}$ if, for each $x, y \in A$, $u_i(x) \geq u_i(y)$ if and only if $x \; R_i \; y$.

For each probability assessment λ, a preference over sets $\succsim_i$ is λ-***Conditional Expected Utility Consistent (***λ***-CEUC) with*** $R_i \in \mathcal{R}$, $\succsim_i \in E^{\lambda}(R_i)$, if for each $X, Y \in \mathcal{A}$, there exists a utility function u_i that represents R_i, such that $X \succsim_i Y$ if and only if

$$\frac{1}{\sum_{x\in X} \lambda(x)} \sum_{x\in X} \lambda(x) u_i(x) \geq \frac{1}{\sum_{y\in Y} \lambda(y)} \sum_{y\in Y} \lambda(y) u_i(y).$$

We are particularly interested in probability assignments that assign equal probability within specific subsets of alternatives in A.

For each $X \subseteq A$, the probability assessment λ is X***-uniform*** if for each $x, x' \in X$, $\lambda(x) = \lambda(x')$. As a special case, the uniform probability assessment by $\bar{\lambda}$ assigns equal probability to each initial alternative, for each $x \in A$, $\bar{\lambda}(x) = \frac{1}{\#A}$. The uniform probability assessment $\bar{\lambda}$ defines a specific extension.

A preference over sets $\succsim_i$ is ***Conditional Expected Utility Consistent with Equal Probabilities (CEUCEP) with*** $R_i \in \mathcal{R}$, $\succsim_i \in E^{EP}(R_i)$, if for each $X, Y \in \mathcal{A}$, there exists a utility function representing R_i, u_i such that $X \succsim_i Y$ if and only if

$$\frac{1}{\#X} \sum_{x\in X} u_i(x) \geq \frac{1}{\#Y} \sum_{y\in Y} u_i(y).$$

We now consider a larger domain of preferences over sets consisting of all preferences consistent with conditional expected utility for some probability assessment.

A preference over sets $\succsim_i$ is (unrestricted) ***Conditional Expected Utility Consistent (CEUC) with*** $R_i \in \mathcal{R}$, $\succsim_i \in E^{UC}(R_i)$, if there is a probability assessment λ such that $\succsim_i \in E^{\lambda}(R_i)$.

For every probability assessment λ and each $R_i \in \mathcal{R}$, we have $E^{\lambda}(R_i) \subset E^{UC}(R_i)$. Moreover, if $\lambda \neq \bar{\lambda}$, the definition of λ-CEUC preferences requires that the set of alternatives A be finite. However, since we only consider preferences over finite subsets, CEUCEP preferences remain well-defined when the set A is infinite.

We consider social choice rules that select a finite set of alternatives for each preference profile. Hence, these rules act as preliminary screening devices that narrow the social choice set before a final decision is made. We first focus on rules that take as input the voters' preferences over alternatives to determine the final selected set. Given a preference domain $\widetilde{\mathcal{R}} \subseteq \mathcal{R}$, a ***social choice correspondence (SCC)*** defined on the domain $\widetilde{\mathcal{R}}$ is a mapping

$$f : \widetilde{\mathcal{R}}^N \to \mathcal{A}.$$

SCCs are based solely on the information contained in the voters' preferences over alternatives.

Before introducing specific structure for preferences over alternatives and their extensions, we propose desirable properties for SCCs. We next define two fundamental properties of SCCs: *strategy-proofness* and *unanimity.*

Let f be an SCC defined on a domain $\tilde{\mathcal{R}} \subseteq \mathcal{R}$ and let E be an extension.

Strategy-Proofness (for SCCs) under extension E***.*** For each $i \in N$, each $R \in \tilde{\mathcal{R}}^N$, $R_i' \in \tilde{\mathcal{R}}$, and each $\succsim_i \in E(R_i)$, $f(R) \succsim_i f(R_i', R_{-i})$.

Unanimity (for SCCs). For each $R \in \tilde{\mathcal{R}}^N$ and $x \in A$, if for each $i \in N$ $p(R_i) = x$, then $f(R) = \{x\}$.

Strategy-proofness implies that voters have no incentive to misrepresent their preferences. The existence of such incentives depends on the domain of the SCC, the preference extension used by voters, and the structure of the SCCs. *Unanimity* ensures that if all voters share the same peak alternative, the social choice reflects that consensus. In the next section, we examine which SCCs satisfy these properties under different domains and extensions.

3 One-Dimensional Single-Peaked Preferences

In this section, we present our main results under a specific assumption regarding the structure of the set of alternatives, which motivates a natural restriction on voters' preferences. Alternatives can be ordered on the one-dimensional real line, and voters' preferences over alternatives are single-peaked.

Let $A = \{a, a+1, \ldots, b-1, b\}$ be a finite subset of integers with at least two elements.[4] For each $x, x' \in A$ such that $x \leq x'$, $[x, x']$ denotes the ***closed interval of integers*** from x to x', $[x, x'] = \{x, x+1, \ldots, x'\}$. We interpret alternatives as locations on the one-dimensional line which are naturally ordered according to the less-than-or-equal binary relation $\leq$. Hence, each $x \in A$ satisfies $a \leq x \leq b$. With this interpretation, we consider the following restriction on voters' preferences. A preference $R_i \in \mathcal{R}$ is ***single-peaked*** for each $c, c' \in A$, $c < c' < p(R_i)$ or $p(R_i) < c' < c$ implies $c \; P_i \; c'$. Let $\mathcal{S}$ denote the set of all single-peaked preferences over A.

We begin by presenting the family of single-valued SCCs that satisfy *strategy-proofness* and *unanimity* under single-peaked preferences over alternatives. Moulin (1980) introduces the following family of single-valued SCCs that generalizes the concept of the median voter rule.

An SCC f is a ***min-max single-valued SCC*** if there is a family of parameters $\{\alpha_S\}_{S \subseteq N}$, $\alpha_S \in A$ with $\alpha_S \leq \alpha_T$ whenever $T \subseteq S$, such that for each $R \in \mathcal{S}^N$, $\alpha_\varnothing = b$, $\alpha_N = a$, and

$$f(R) = \inf_{S \subseteq N} \left\{ \sup_{i \in S} \{p(R_i), \alpha_S\} \right\}.$$

Moulin (1980) characterizes min-max single-valued SCCs as the class of single-valued SCCs that satisfy *strategy-proofness*, *unanimity*, and only use the information contained in the peaks of the agents to determine the social choice. Barberà *et al.* (1993) shows that this final condition is directly implied by *strategy-proofness* and *unanimity* under single-peaked preferences.

It is important to note that for a min-max single-valued SCC f with parameters $\{\alpha_S\}_{S \subseteq N}$, for each $S \subseteq N$, α_S and each $R \in \mathcal{S}^N$ such that for each $i \in S$, $p(R_i) = a$, and each $j \in N \setminus S$, $p(R_j) = b$, $f(R) = \alpha_S$. Thus, each min-max single-valued SCC f is uniquely characterized by the associated family of parameters $\{\alpha_S\}_{S \subseteq N}$ with $\alpha_S \leq \alpha_T$ for $T \subseteq S$.[5] The definition of different families of parameters $\{\alpha_S\}_{S \subseteq N}$ allows for substantial flexibility on the construction of min-max single-valued SCCs.

An SCC f is a ***dictatorial SCC*** if there is $i \in N$ such that for each $R \in \mathcal{S}^N$, $f_i(R) = \{p(R_i)\}$. For each $i \in N$, we denote by f_i the dictatorship that for each $R \in \mathcal{S}^N$ selects the unique peak of voter i's preference.

[4] This assumption can be replaced by assuming that S is a segment of real line, or the compactified real line under CEUCEP extension.

[5] This unique representation depends on the restriction $\alpha_S \leq \alpha_T$ for $T \subseteq S$. A single-valued SCC f that does not satisfy this condition is properly defined as SCC and satisfies *strategy-proofness* for every extension and *unanimity*. However if for some coalitions $T \subseteq S$, $\alpha_S > \alpha_T$, the coalition S becomes ineffective in the determination of the social choice since for each $R \in \mathcal{S}^N$, $\sup_{i \in S}\{p(R_i), \alpha_S\} > \sup_{i \in T}\{p(R_i), \alpha_T\}$. Hence, the SCC f is equivalent to the min-max single-valued SCC with family of parameters $\{\alpha'_S\}_{S \subseteq N}$ such that whenever $T \subseteq S$ and $\alpha_S > \alpha_T$, $\alpha'_S = \alpha'_T$ and $\alpha'_S = \alpha_S$ otherwise.

Observe that f_i is the min-max single-valued SCC with parameters $\{\alpha_S\}_{S\subseteq N}$ such that for each $S \subseteq N$, if $i \in S$, then $\alpha_S = a$, if $i \notin S$, then $\alpha_S = b$.

Let $l \in \{1,\ldots,n\}$. An SCC f^l is a ***l-th positional voter SCC*** if for each $R \in \mathcal{S}^N$, $f^l(R) = \inf\{x \in A: \#\{i : p(R_i) \leq x\} \geq l\}$.

For each $l \in \{1,\ldots,n\}$, f^l is the min-max single-valued SCC with parameters $\{\alpha_S\}_{S\subseteq N}$ such that for each $S \subseteq N$, if $\#S < l$, then $\alpha_S = b$, and if $\#S \geq l$, then $\alpha_S = a$. For n odd, $f^{\frac{n+1}{2}}$ is the SCC that selects the median from the peaks of the voters' preferences. Alternatively, f^1 (and f^n) select the alternative corresponding to the minimum (respectively, the maximum) of all peaks.

Every min-max single-valued SCC f satisfies *strategy-proofness* under every extension and *unanimity*. If at some preference profile $R \in \mathcal{S}^N$, for some $i \in N$ $f(R) < p(R_i)$, implies the existence of a coalition $S \subseteq N$ with $i \notin S$ with $f(R) = \sup_{j\in S}\{p(R_j), \alpha_S\} < p(R_i)$. Hence, for each $R_i' \in \mathcal{S}$, $f(R_i', R_{-i}) \leq f(R)$ and by single-peakedness of preferences over alternatives, $f(R)\ R_i\ f(R_i', R_{-i})$. A parallel argument applies if $p(R_i) < f(R)$. Finally, *unanimity* follows from $\alpha_N = a$ and for each $T \subseteq S$, $\alpha_S \leq \alpha_T$. For each $R \in \mathcal{S}^N$ such that all voters share the same peak, i.e., $p(R_i) = p(R_j)$ for all $i, j \in N$, $\sup\{a, p(R_i)\} = p(R_i)$, and therefore $\inf_{S\subseteq N}\{\sup_{i\in S}\{p(R_i), a_S\}\} = p(R_i)$.

In the following Remark 1 we formalize the notion of uncompromisingness introduced by Border and Jordan (1983). Every min-max single-valued SCCs satisfies this property, and we use it extensively henceforth.

Remark 1. ***(Uncompromisingness)*** *Let f be a min-max single-valued SCC. For each $i \in N$, $R \in \mathcal{S}^N$, and $R_i' \in \mathcal{S}$:*

i) If $f(R) \geq p(R_i)$, then $p(R_i') \leq f(R)$ implies $f(R_i', R_{-i}) = f(R)$, and $p(R_i') > f(R)$ implies $f(R_i', R_{-i}) \in [f(R), p(R_i')]$.

ii) If $f(R) \leq p(R_i)$, then $p(R_i') \geq f(R)$ implies $f(R_i', R_{-i}) = f(R)$, and $p(R_i') < f(R)$ implies $f(R_i', R_{-i}) \in [p(R_i'), f(R)]$.

Extending the notion of min-max single-valued SCCs to the multivalued setting is straightforward. We consider SCCs that select the union of the alternatives selected by two min-max SCCs.

An SCC f is a ***union min-max SCC*** if there exist two min-max single-valued SCCs g and h such that for each $R \in \mathcal{R}^N$, $f(R) = g(R) \cup h(R)$.

Union min-max SCCs are characterized by the two associated families of parameters that define each of the single-valued union min-max SCCs that form them. We present now an extreme example of union min-max SCCs. Bidictatorial SCCs are union min-max SCCs that combine two dictatorial SCCs.

An SCC f is ***bidictatorial*** if there are $i, j \in N$ such that for each $R \in \mathcal{R}^N$, $f(R) = p(R_i) \cup p(R_j)$.

With Remark 1, we can check that every union min-max SCC satisfies *strategy-proofness* under the E^{EP} extension and *unanimity.* By changing her preference, a voter may only take the infimum and/or the supremum of the selected set further away from her peak. Under CEUCEP preferences, the voter assigns probability $\frac{1}{2}$ to each selected alternative (unless the infimum and the supremum selected alternatives coincide). Hence, since neither the infimum nor the supremum are closer to the original peak of the voter, the voter does not prefer the chosen set after changing her preferences according to any CEUCEP preference consistent with the original preference. However, for arbitrary probability assessment $\lambda \neq \bar{\lambda}$, union min-max SCC may violate *strategy-proofness* under E^{λ}.

Example 1. *Let $N = \{i, j\}$ and $A = \{-1, 0, 1\}$. Let f be the bidictatorial SCC defined by $f(R) = f_i(R) \cup f_j(R)$ for all $R \in \mathcal{S}^N$. Let $R \in \mathcal{S}^N$, $R_i' \in \mathcal{S}$ be such that:*

$$\begin{array}{ccccc} \{0\} & P_i & \{-1\} & P_i & \{1\}, \\ \{1\} & P_j & \{0\} & P_j & \{-1\}, \\ \{-1\} & P_i' & \{0\} & P_i' & \{1\}. \end{array}$$

Note that $f(R) = \{0, 1\}$ and $f(R_i', R_j) = \{-1, 1\}$. Let $\varepsilon \in \left(0, \frac{1}{4}\right)$. Let the probability assessment λ be such that $\lambda(0) = \varepsilon$, $\lambda(-1) = \lambda(1) = \frac{1}{2}(1 - \varepsilon)$. Let the utility function u_i represents R_i and be such that $u_j(-1) \in (0.8, 1)$, $u_j(0) = 1$, and $u_j(1) = 0$. Note that

$$\frac{1}{2}\left(u_i(-1) + u_i(1)\right) = \frac{1}{2}u_i(-1) > \frac{1}{\frac{1}{2}(1+\varepsilon)}\left(\varepsilon u_i(0) + \frac{1}{2}(1-\varepsilon)u_i(1)\right) = \frac{2\varepsilon}{1+\varepsilon}.$$

Hence, there is $\succsim_i \in E^{\lambda}(R_i)$ such that $f(R_i, R_j') \succ_i f(R)$ and f does not satisfy strategy-proofness under E^{λ}.

The logic behind Example 1 is straightforward. Given an initial probability assessment λ and a preference profile for the remaining voters, whenever the peak reported by a voter may determine one of the alternatives selected by the union min-max SCC, the voter faces a trade-off between selecting her peak and determining the (conditional) probability of other alternatives being selected. She may have incentives not to report and select her peak alternative if by reporting another alternative she may decrease the (conditional) probability of selecting a less preferred alternative. This selection–probability trade-off is not present under the uniform probability assessment, and the voter will have incentives to report her true preferences.

Next, we consider arbitrary probability assessments λ. We present now further notation and auxiliary results that will prove helpful to understand the structure of union min-max SCCs that preserve *strategy-proofness* under E^λ.

Let f be a union min-max SCC defined by the pair of family of parameters $\{\alpha_S\}_{S\subseteq N}$ and $\{\beta_S\}_{S\subseteq N}$. We say that f is ***regular*** if whenever $\alpha_S < \beta_S$ for some $S \subseteq N$, then $\alpha_{S'} \leq \beta_{S'}$, for each $S' \subseteq N$.

We denote by $\underline{f}$ the ***infimum of*** f ***SCC*** and by $\bar{f}$ be the ***supremum of*** f ***SCC***. That is, for each $R \in \mathcal{S}^N$:

$$\underline{f}(R) \equiv \inf\{x : x \in f(R)\} \quad \text{and} \quad \bar{f}(R) \equiv \sup\{x : x \in f(R)\}.$$

Clearly, for each union min-max SCC f and each $R \in \mathcal{S}^N$, $f(R) = \underline{f}(R) \cup \bar{f}(R)$. Our next auxiliary result, Lemma 1, shows that $\underline{f}$ and $\bar{f}$ are both min-max single-valued SCCs.

Lemma 1. *Let g and h be two min-max single-valued SCCs with respective family of parameters $\{\alpha_S\}_{S\subseteq N}$ and $\{\beta_S\}_{S\subseteq N}$, and f be the union min-max SCC such that for each $R \in \mathcal{S}^N$, $f(R) = g(R) \cup h(R)$. The single-valued SCCs $\underline{f}$ and $\bar{f}$ are min-max single-valued SCCs and for each each $R \in \mathcal{S}^N$:*

$$\bar{f}(R) = \inf_{S\subseteq N}\left\{\sup_{i\in S}\{p(R_i), \inf\{\alpha_S, \beta_S\}\}\right\},$$
$$\underline{f}(R) = \inf_{S\subseteq N}\left\{\sup_{i\in S}\{p(R_i), \sup\{\alpha_S, \beta_S\}\}\right\}.$$

Lemma 1 shows that different pairs of min-max single-valued SCCs with their respective pairs of families of parameters may generate the same union min-max SCC.

Example 2. *Let $N = \{i, j\}$ and $A = \{-1, 0, 1\}$. Let the union min-max SCCs f, g be such that for each $R \in \mathcal{S}^N$, $f(R) = f_i(R) \cup f_j(R)$ and $g(R) = f^1(R) \cup f^2(R)$. That is, f is a bidictatorial SCCs and g is the union of the 1^{st} and 2^{nd} positional voter SCCs. Let $\{\alpha^i_S\}_{S\subseteq N}$, $\{\alpha^j_S\}_{S\subseteq N}$, $\{\alpha^1_S\}_{S\subseteq N}$, and $\{\alpha^2_S\}_{S\subseteq N}$ be the families of parameters that define f_i, f_j, f^1, and f^2 respectively. Note that for each $S \subseteq N$:*

$$\alpha^1_S = \inf\{\alpha^i_S, \alpha^j_S\},$$
$$\alpha^2_S = \sup\{\alpha^i_S, \alpha^j_S\}.$$

and $f^1 = \underline{f}$ and $f^2 = \bar{f}$. For each $R \in \mathcal{S}^N$, $f(R) = g(R)$. Hence, f is not a regular union min-max SCC but it is equivalent to g, which is regular.

Lemma 1 implies that for every union min-max SCC f there is a regular union min-max SCC f' such that for each $R \in \mathcal{S}^N$, $f(R) = \underline{f}(R) \cup \bar{f}(R) = f'(R)$. Without loss of generality, we now restrict attention to regular union min-max SCCs.

Next, we present an important auxiliary result that we use to define two specific families of union min-max SCCs.

Lemma 2. *Let f be a regular union min-max SCC defined by the families of parameters $\{\alpha_S\}_{S\subseteq N}$ and $\{\beta_S\}_{S\subseteq N}$ such that $\alpha_S \leq \beta_S$ for each $S \subseteq N$. For each $R \in \mathcal{S}^N$, either*

i) there is $S \subseteq N$ such that $[\underline{f}(R), \bar{f}(R)] \subseteq [\alpha_S, \beta_S]$, or

ii) for some $i \in N$, $f(R) = \underline{f}(R) = \bar{f}(R) = \{p(R_i)\}$.

For a probability assessment λ, a regular union min-max SCC defined by the families of parameters $\{\alpha_S\}_{S\subseteq N}$ and $\{\beta_S\}_{S\subseteq N}$ such that for each $S \subseteq N$, $\alpha_S \leq \beta_S$ is a λ***-consistent union min-max SCC*** if for each $S \subseteq N$ one of the following holds:

i) $|\alpha_S - \beta_S| \leq 1$: or

ii) λ is $[\alpha_S, \beta_S]$-uniform.

A regular union min-max SCC defined by the families of parameters $\{\alpha_S\}_{S\subseteq N}$ and $\{\beta_S\}_{S\subseteq N}$ with $\alpha_S \leq \beta_S$ for each $S \subseteq N$ is an ***adjacent union min-max SCC*** if for each $S \subseteq N$, $|\alpha_S - \beta_S| \leq 1$.

From Lemma 2, every λ-consistent union min-max SCC always selects either a singleton set, or sets containing a pair of adjacent alternatives, or sets whose elements are contained in a closed interval of integers for which λ is uniform. Under the uniform probability assessment $\bar{\lambda}$, every regular union min-max SCC automatically satisfies the conditions of a λ-consistent union min-max SCC. Moreover, for each λ, every adjacent union min-max SCC qualifies as a λ-consistent union min-max SCC. Lemma 2 directly implies the following characterization of adjacent union min-max SCCs.

Remark 2. *A regular union min-max SCC f satisfies $|\bar{f}(R) - \underline{f}(R)| \leq 1$ for each $R \in \mathcal{S}^N$ if and only if f is an adjacent union min-max SCC.*

For every arbitrary probability assessment λ, we can check that every adjacent union min-max SCC satisfies *strategy-proofness* under E^λ, due to the absence of any trade-off between selection and induced probabilities over alternatives that we presented in Example 1. Under an adjacent union min-max, the peak alternative of a voter is either larger or equal than the supremum of the selected alternatives, or smaller or equal than the infimum of the selected alternatives, since by Remark 2, it is not possible that $\underline{f}(R) < p(R_i) < \bar{f}(R)$.

Assume that for some $R \in \mathcal{S}^N$, $p(R_i) < \underline{f}(R) \neq \bar{f}(R)$, and for some $R_i' \in \mathcal{S}$, $f(R_i', R_{-i}) \neq f(R)$. By Remark 1, $\underline{f}(R_i', R_{-i}) \geq \underline{f}(R)$ and $\bar{f}(R_i', R_{-i}) \geq \bar{f}(R)$. If $\underline{f}(R_i', R_{-i}) > \underline{f}(R)$, then $\bar{f}(R_i', R_{-i}) > \bar{f}(R)$. Hence $\bar{f}(R)\ P_i\ \bar{f}(R_i', R_{-i})$ and $\underline{f}(R)\ P_i\ \underline{f}(R_i', R_{-i})$ and for each λ, and each $\succsim_i \in E^\lambda(R_i)$, $f(R) \succ_i f(R_i', R_{-i})$. The same reasoning applies to all remaining configurations. Combining this insight with the absence of selection-probability trade-off for uniform probability assessments leads to the following Proposition 1.

Proposition 1. *For each probability assessment λ, every λ-consistent union min-max SCC satisfies strategy-proofness under E^λ.*

For a given probability assessment λ, our main result, Theorem 1, characterizes all SCCs that satisfy *strategy-proofness* under E^λ and *unanimity.*

Theorem 1. *For each probability assessment λ, an SCC f satisfies strategy-proofness under E^λ and unanimity if and only if f is a λ-consistent union min-max SCC.*

To establish the necessity side of Theorem 1, we propose a constructive proof that exploits the relation between SCCs and PDSs. This relation between SCCs and PDSs is of independent interest and can be applied to different social choice problems with different preference domain restrictions. We present all the details and a sketch of the proof in the next section.

As direct corollaries of Theorem 1, we obtain the characterizations for E^{EP} and E^{UC} extensions. Lemma 1 implies that every union min-max SCC is equivalent to a $\bar{\lambda}$-consistent union min-max SCC. On the other hand, there are probability assessments that assign different probability to each initial alternative, and for each probability assessment λ every adjacent union min-max SCC is a λ-consistent union min-max SCC.

Corollary 1. *An SCC f satisfies strategy-proofness under E^{EP} and unanimity if and only if f is a union min-max SCC.*

Corollary 2. *An SCC f satisfies strategy-proofness under E^{UC} and unanimity if and only if f is an adjacent union min-max SCC.*

4 Outline of the Proofs: Probabilistic Decision Schemes

In this section, we present the probabilistic social choice framework and the results in Ehlers *et al.* (2002) that allow for a direct proof of Theorem 1. We need to introduce first the notation and the corresponding definitions of *strategy-proofness* and *unanimity* for PDSs.

A ***probability distribution*** D ***on*** A is a mapping $D : \mathcal{A} \cup \{\varnothing\} \to [0, 1]$ such that $D(\{\varnothing\}) = 0$, $D(A) = 1$, and for each $X, X' \subseteq \mathcal{A} \cup \{\varnothing\}$ such that $X \cap X' = \varnothing$, $D(X \cup X') = D(X) + D(X')$. Let Δ be the set of all probability distributions on A. Note that every probability assessment is a probability distribution, but probability distributions may assign null probability to some alternatives.

Let $\widetilde{\mathcal{R}} \subseteq \mathcal{R}$. A ***probabilistic decision scheme (PDS)*** on $\tilde{\mathcal{R}}$ is a mapping Φ that selects, for each profile of voters' preferences over alternatives, a a probability distribution on A,

$$\Phi : \widetilde{\mathcal{R}}^N \to \Delta.$$

Strategy-proofness (for PDSs). For each $i \in N$, each $R \in \widetilde{\mathcal{R}}^N$, each $R'_i \in \tilde{\mathcal{R}}$, and each u_i representing R_i,

$$\sum_{x \in A} \Phi(R)(\{x\}) u_i(x) \geq \sum_{x \in A} \Phi(R'_i, R_{-i})(\{x\}) u_i(x).$$

Unanimity (for PDSs). For each $R \in \tilde{\mathcal{R}}^N$ and $x \in A$, if for each $i \in N$ $p(R_i) = x$, then $\Phi(R)(\{x\}) = 1$.

Strategy-proofness for PDSs is based on first-order stochastic dominance condition. For every utility function representing a voter's preference, her expected utility is maximal when she reports her true preference.

To obtain the characterization for SCCs we explore the relationship between SCCs and PDSs. Extending the argument firstly introduced by Feldman (1980) for the E^{EP} extension, for each probability assessment λ and each SCC f defined on a domain $\tilde{\mathcal{R}}$, we can define the ***PDS associated with*** λ ***and*** f, $\Phi_{(\lambda,f)}$, in the following way. For each $R \in \tilde{\mathcal{R}}^N$ and each $x \in A$:

$$\Phi_{(\lambda,f)}(R)(\{x\}) = \begin{cases} \frac{\lambda(x)}{\sum_{x' \in f(R)} \lambda(x')} & \text{if } x \in f(R), \\ 0 & \text{otherwise.} \end{cases}$$

The following Lemma 3 shows that for each probability assessment λ, *strategy-proofness* under E^λ and *unanimity* are transferred to the associated PDS $\Phi_{(\lambda,f)}$.

Lemma 3. *For each probability assessment λ, if an SCC f satisfies strategy-proofness under E^{λ} and unanimity then the associated PDS $\Phi_{(\lambda,f)}$ satisfies strategy-proofness and unanimity.*

Ehlers *et al.* (2002) characterizes the family of PDS that satisfy *strategy-proofness* and *unanimity* when voters' preferences over alternatives are single-peaked. By Lemma 3, the result for PDSs can be used to characterize SCCs that satisfy *strategy-proofness* under E^{λ} and *unanimity.* We introduce now additional notation to present the results in Ehlers *et al.* (2002) that we use in our proofs.

For each $S \subset N$, let D_S be a probability distribution on A. We call $\mathcal{F} = \{D_S\}_{S\subseteq N}$ a ***collection of fixed probabilistic ballots*** if for each $S, T \subseteq N$, and each $x \in A \setminus \{a\}$,

$$D_{S\cup T}([a,x]) - D_S([a,x]) \geq 0.$$

In every collection of fixed probabilistic ballots, the probability distribution assigned to larger (with respect to inclusion) coalitions first-degree stochastically dominate those of smaller coalitions. Given $X \in \mathcal{A}$, let $\mathbb{1}_X$ denote the indicator function of set X. For each $R \in \mathcal{S}^N$, let $p(R) = \{x \in A: \text{ for some } i \in N, p(R_i) = x\}$, $p^1(R) < p^2(R) < \ldots < p^k(R)$ for $k < n$ denote the different peaks at R such that for each $t \in \{1, \ldots, k-1\}$, $p^t(R) < p^{t+1}(R)$ and $\cup_{t=1}^{k} p^t(R) = p(R)$. Let $S_t(R)$ denote the set of voters whose peaks at profile R are less than or equal to $p^t(R)$. Thus, $S_1(R) \subsetneq S_2(R) \subsetneq \ldots \subsetneq S_k(R) = N$. Let $S_0(R) = \varnothing$, $p^0(R) = a$ and $p^{k+1}(R) = b$.

Probabilistic fixed–ballots rule. Let $\mathcal{F} = \{D_S\}_{S\subseteq N}$ be a collection of fixed probabilistic ballots with $D_{\varnothing}(\{b\}) = 1$ and $D_N(\{a\}) = 1$, the ***fixed probabilistic ballots rule based on*** $\mathcal{F}$, $\Phi^{\mathcal{F}}$ is defined as follows. For each $R \in \mathcal{S}^N$, and each $X \in \mathcal{A}$,

$$\begin{aligned}\Phi^{\mathcal{F}}(R)(X) = \; & \textstyle\sum_{l=0}^{k} D_{S_l(R)}(X \cap \left[p^l(R)+1, p^{l+1}(R)-1\right])+ \\ & + \textstyle\sum_{l=1}^{k} \mathbb{1}_X(p^l(R))(D_{S_l(R)}(\left[a, p^l(R)\right]) - D_{S_{l-1}}(R)(\left[a, p^l(R)-1\right])).\end{aligned}$$

In the next remarks, we provide the intuition behind the definition of probabilistic fixed-ballot rules..

Remark 3. *Let $\Phi^{\mathcal{F}}$ be probabilistic fixed ballot rule with associated collection of fixed probabilstic ballots $\mathcal{F} = \{D_S\}_{S\subseteq N}$ with $D_{\varnothing}(\{b\}) = 1$ and $D_N(\{a\}) = 1$ For each $R \in \mathcal{S}^N$,*

- *$\Phi^{\mathcal{F}}(R)$ coincides with the distribution $D_{\varnothing}$ on the interval $[a, p^1(R) - 1]$.*
- *For each $l \in \{1, k-1\}$ $\Phi^{\mathcal{F}}(R)$ is equal to $D_{S_l(R)}$ on $[p^l(R) + 1, p^{l+1}(R) - 1]$ (whenever the interval is properly defined).*

- *On the interval* $[p^k+1, b]$*,* $\Phi^{\mathcal{F}}(R)$ *is equal to* D_N*.*
- *For each* $l \in \{1, \ldots, k\}$*,* $\Phi^{\mathcal{F}}(R)(\{p^l(R)\}) = D_{S_l(R)}([a, p^l(R)]) - D_{S_{l-1}(R)}([a, p^l(R) - 1])$*.*

Remark 4. *Let* $\Phi^{\mathcal{F}}$ *be a probabilistic fixed-ballot rule with a collection of fixed probability ballots* $\mathcal{F} = \{D_S\}_{S \subseteq N}$*. For each* $S \subseteq N$ *and each* $R \in \mathcal{S}^N$ *such that for each* $i \in S$*,* $p(R_i) = a$ *and for each* $j \in N \setminus S$*,* $p(R_j) = b$*,* $\Phi^{\mathcal{F}}(R) = D_S$*.*

Theorem 4.1 and Proposition 5.2 in Ehlers *et al.* (2002) immediately imply the following characterization.

Proposition 2. *A PDS* $\Phi : \mathcal{S}^N \to \Delta$ *satisfies strategy-proofness and unanimity if and only if* Φ *is a probabilistic fixed-ballots rule.*

With Proposition 2 at hand, the proof of Theorem 1 follows from successive application of Remarks 3 and 4. Note that by Lemma 3 and Proposition 2, for every probability assessment λ, if f is an SCC that satisfies *strategy-proofness* under E^λ and *unanimity*, then $\Phi_{(f,\lambda)}$ is a probabilistic fixed ballot rule. This immediately implies that f only takes into account the information contained in the peaks of the voters to determine the social choice. The crucial step of the proofs relies on demonstrating that the collection of probabilistic fixed ballots associated to $\Phi_{(\lambda,f)}$, $\mathcal{F} = \{D_S\}_{S \subseteq N}$ is such that for each $S \subseteq N$, D_S assigns positive probability to at most two distinct alternatives (Lemma 5). This is the only way for the associated probabilistic fixed-ballot rule to assign positive probability to multiple alternatives consistently with Bayesian updating from a prior probability assessment. Hence, the collection of probabilistic fixed ballots is characterized by two sets of parameters $\{\alpha_S\}_{S \subseteq N}$ and $\{\beta_S\}_{S \subseteq N}$, with $\alpha_S \leq \beta_S$ for each $S \subseteq N$ that define a regular union min-max SCCs (Lemma 6). Then, it only remains to check that the conditions that define λ-consistent union min-max SCCs are necessary for union min-max SCCs to satisfy *strategy-proofness* under E^λ (Lemma 7). Since by Proposition 1, every λ-consistent union min-max SCC satisfies *strategy-proofness* under E^λ, the characterization result follows.

5 Discussion

We have explored the possibility of constructing strategy-proof SCCs in the single-peaked domain of preferences over alternatives when preferences over sets are consistent with Bayesian updating from an arbitrary probability assessment. We have characterized the structure of the parameters that define union min-max SCCs that provide incentives for the voters to report their true preferences. To obtain these results, we have analyzed the relationship between SCCs and PDSs and exploited the characterization by Ehlers *et al.* (2002) of PDSs that satisfy *strategy-proofness*. In this concluding section, we replicate the approach to alternative social choice problems under different domains of preferences, where existing results on PDS that satisfy *strategy-proofness* immediately translate to the SCC framework. Finally, we compare our approach with the most related works in the literature (Ingalagavi and Sadhukhan , 2023) and Rodríguez-Álvarez (2017).

5.1 Multi-Dimensional Convex Preferences in $\mathbb{R}^k$

In this subsection we use the relation between SCC and PDS to explore the implications of *strategy-proofness* under E^{EP} in general single-peaked domains of preferences.

From this point onward, we assume that A is a non-empty convex subset of some Euclidean space $\mathbb{R}^k$. The dimension of A, denoted by $dim(A)$, represents the dimension of the smallest affine subset of $\mathbb{R}^k$ containing A. We can conceptualize A as a societal budget set available to society, indicating how resources are allocated across different alternative uses. Since in this section, the set of alternatives is infinite and uncountable, we exclusively consider CEUCEP preferences.

A ***strictly convex continuous preference*** R_i over $A \subseteq \mathbb{R}^k$ is a transitive and complete binary relation on A with the following properties:

- There is a unique $p(R_i) \in A$ with $p(R_i)\ R_i\ a$ for all $a \in A$
- For all $a, b \in A$ with $a\ R_i\ b$ and $a \neq b$ and all $\lambda \in \mathbb{R}_{++}$ with $0 < \lambda < 1$, $[\lambda a + (1-\lambda)b]\ P_i\ b$.
- For every $a \in A$ the sets $\{b \in A :\ b\ R_i\ a\}$ and $\{b \in A :\ a\ R_i\ b\}$ are closed.

We denote by $\mathcal{M}$ the domain of strictly convex continuous preferences over the space A. Note that in one-dimensional convex spaces of alternatives, the definition of strictly convex continuous preferences coincides with that of single-peaked continuous preferences.

While the domain of strict convex continuous preferences serves as a natural extension of the concept of single-peakedness to higher-dimensional spaces of alternatives,[6] the implications of *strategy-proofness* are more restrictive. This aspect has been emphasized for single-valued social choice rules in Zhou (1991). In multi-dimensional spaces with convex preferences, the only single-valued SCCs that satisfy *strategy-proofness* and *unanimity* concentrate decision power in a single voter. For PDS, Dutta *et al.* (2002) shows that only convex combinations of dictatorial single-valued SCCs satisfy *strategy-proofness* and *unanimity.* In a similar vein to our analysis for one-dimensional single-peaked preferences, we show that similar implications hold for SCCs, but under Bayesian updating and the uniform probability assessment, the decision-making power is concentrated in up to two voters.

[6] There are other possible generalizations of single-peaked preferences to multi-dimensional spaces. For instance, Barberà *et al.* (1993) considers separable single-peaked preferences in a multi-dimensional finite grid. However, we are not aware of any results on PDSs that satisfy *strategy-proofness* in that domain and our direct approach is not applicable. We leave this interesting case for further research

Theorem 2. *Let $dim(A) \geq 2$. An SCC $f : \mathcal{M}^N \to \mathcal{A}$ satisfies strategy-proofness under E^{EP} and unanimity if and only if f is either dictatorial or bidictatorial.*

5.2 Relation to the Literature: Social Choice Correspondences and Social Choice Functions over Sets

We conclude by contrasting our framework with Ingalagavi and Sadhukhan (2023) and Rodríguez-Álvarez (2017). Following Barberà *et al.* (2001), these works examine multivalued social choice processes in which voters hold single-peaked preferences over alternatives but voters directly report their preferences over sets of alternatives.

Given a domain of preferences over sets $\widetilde{\mathcal{D}} \subseteq \mathcal{D}$, a ***social choice function over sets (SCF)*** on the domain $\widetilde{\mathcal{D}}$ is a mapping

$$\varphi : \widetilde{\mathcal{D}}^N \to \mathcal{A}.$$

SCFs take as input the information on preferences regarding the possible outcomes of the social choice, namely preferences over sets. An SCC is an SCF subject to the following constraint: for any two profiles of preferences over sets that are identical in terms of how each voter compares singleton sets, the outcome of the SCF remains unchanged. Hence, both Ingalagavi and Sadhukhan (2023) and Rodríguez-Álvarez (2017) analyze a more general framework than the one studied here. We introduce now the appropriate definitions of *strategy-proofness* and *unanimity* for SCFs. Consider an arbitrary domain $\widetilde{\mathcal{D}}$ and a SCF $\varphi : \widetilde{\mathcal{D}}^N \to \mathcal{A}$.

Strategy-Proofness (for SCFs). For each $i \in N$, each $\succsim \in \widetilde{\mathcal{D}}^N$, and each $\succsim'_i \in \widetilde{\mathcal{D}}$, $\varphi(\succsim) \succsim_i \varphi(\succsim'_i, \succsim_{-i})$.

Unanimity (for SCFs). For each $\succsim \in \widetilde{\mathcal{D}}^N$ and $X \in \mathcal{A}$, if for each $i \in N$, $\tau(\succsim_i) = X$, then $\varphi(\succsim) = X$.

The definition of *strategy-proofness* for SCFs is natural, as it solely refers to the domain on which the SCF is defined. The input of the SCF inherently encompasses all the pertinent information necessary to assess the voters' incentives to misreport. This stands in contrast to the definition of *strategy-proofness* for SCCs for any arbitrary extension which requires the additional information that is not included in the informational input reported by the voters –the extension– to determine the profitability of misreporting.

For each probability assessment λ, Let $\mathcal{D}_\lambda \subseteq \mathcal{D}$ be the domain of preferences over sets such that $\succsim_i \in \mathcal{D}_\lambda$ if there is $R_i \in \mathcal{S}$ such that $\succsim_i \in E^\lambda(R_i)$. Let $\mathcal{D}_{EP} \subseteq \mathcal{D}$ denote the domain of preferences over sets such that $\succsim_i \in \mathcal{D}_{EP}$ if there is $R_i \in \mathcal{S}$ such that $\succsim_i \in E^{EP}(R_i)$. Let $\mathcal{D}_{UC} \subseteq \mathcal{D}$ be the domain of preferences over sets such that $\succsim_i \in \mathcal{D}_{UC}$ if there is $R_i \in \mathcal{S}$ such that $\succsim_i \in E^{UC}(R_i)$.

Ingalagavi and Sadhukhan (2023) analyzes SCFs when voters preferences over singleton sets are single-peaked and preferences over sets are either on the domain $\mathcal{D}_{EP}$ or on the domain $\mathcal{D}_{UC}$. In addition to *unanimity*, Ingalagavi and Sadhukhan (2023) considers the following property that restricts the informational content of voters' preferences processed by a SCF to determine the social choice.

Tops-only. For each $\succsim, \succsim' \in \widetilde{\mathcal{D}}^N$, and for each $i \in N$; if $\tau(\succsim_i) = \tau(\succsim_i')$ then $\varphi(\succsim) = \varphi(\succsim')$.

Since for each $R_i \in \mathcal{S}$, and for each probability assessment λ and each $\succsim_i, \succsim_i' \in E^\lambda(R_i)$, $\tau(\succsim_i) = \tau(\succsim_i') = \{p(R_i)\}$, we can immediately state the following result.

Lemma 4. *Let $\varphi : \mathcal{D}_\lambda^N \to \mathcal{A}$ satisfy tops–only, then for each $\succsim, \succsim' \in \mathcal{D}_\lambda^N$ such that for each $i \in N$ there is $R_i \in \mathcal{S}$ with $\{\succsim_i, \succsim_i'\} \subset E^\lambda(R_i)$, $\varphi(\succsim) = \varphi(\succsim')$.*

Lemma 4 implies that for each λ, every SCF on $\mathcal{D}_\lambda$ that satisfies *tops–only* is essentially behaves as an SCC. This is because all the information that uses the SCF is derived from the preferences of voters' top-sets, which, in turn, depends only on how the voters compare singleton sets. For each SCF that satisfies *tops–only*, we can construct an SCC that selects an identical set of alternatives for every profile of preferences where each voter's peak coincide with the respective top-set. Accordingly, we derive the following Theorem 3 that parallels Theorem 1.

Theorem 3. *An SCF $\varphi : \mathcal{D}_\lambda^N \to \mathcal{A}$ satisfies strategy-proofness, unanimity, and tops–only if and only if φ is a λ-consistent union min-max.*

From Theorem 3, we obtain the main results in Ingalagavi and Sadhukhan (2023) that are paralleling Corollaries 1 and 2.

Corollary 3 (Theorem 1, Ingalagavi and Sadhukhan (2023))**.** *An SCF $\varphi : \mathcal{D}_{EP}^N \to \mathcal{A}$ satisfies strategy-proofness, unanimity, and tops–only if and only if φ is a union min-max SCC.*

Corollary 4 (Theorem 2, Ingalagavi and Sadhukhan (2023))**.** *An SCF $\varphi : \mathcal{D}_{UC}^N \to \mathcal{A}$ satisfies strategy-proofness, unanimity, and tops–only if and only if φ is an adjacent union min-max SCC.*

From Lemma 3 and Proposition 2, every SCC that satisfies *strategy-proofness* under either E^{EP} or E^{UC} and *unanimity*, only uses the information on the peaks alternatives of the voters. Hence, Corollary 1 and 2 can be obtained from the main results in Ingalagavi and Sadhukhan (2023). Ingalagavi and Sadhukhan (2023, Example 4) demonstrates that if *tops–only* is omitted there exist SCFs that satisfy *strategy-proofness* in $\mathcal{D}_{UC}$ and *unanimity*, and yet are not union min-max SCCs. To sum up, Ingalagavi and Sadhukhan (2023) proposes a more general framework and introduces additional properties, while in this work we explore more general domains of preferences over sets that are not analyzed in Ingalagavi and Sadhukhan (2023).

Alternatively, Rodríguez-Álvarez (2017) considers SCFs that operate within domains of preferences over sets consistent with single-peaked preferences over singleton alternatives and arbitrary extensions, but admit only singleton or adjacent doubleton sets as feasible outcomes. Under this restriction on the range of the SCF, Rodríguez-Álvarez (2017, Theorem 1) demonstrates that SCFs satisfying *strategy-proofness* and *unanimity* also satisfy *tops–only*. Consequently, every SCF that satisfies *strategy-proofness* and *unanimity* is equivalent to an adjacent union min-max SCC. Since Rodríguez-Álvarez (2017) analyzes different domains of preferences over sets and SCCs with restricted ranges, Theorem 1 and 3 and Rodríguez-Álvarez (2017, Theorem 1) are logically distinct and independent.

6 Proofs

Proof of Lemma 1. For each $R \in \mathcal{S}^N$:

$$\begin{aligned}
g(R) \cup h(R) &= \inf_{S\subseteq N}\{\sup_{i\in S}\{p(R_i),\alpha_S\}\} \cup \inf_{S\subseteq N}\{\sup_{i\in S}\{p(R_i),\beta_S\}\} \\
&= \inf\{\inf_{S\subseteq N}\{\sup_{i\in S}\{p(R_i),\alpha_S\}\}, \inf_{S\subseteq N}\{\sup_{i\in S}\{p(R_i),\beta_S\}\}\} \\
&\qquad \cup \\
&\quad\ \sup\{\inf_{S\subseteq N}\{\sup_{i\in S}\{p(R_i),\alpha_S\}\}, \inf_{S\subseteq N}\{\sup_{i\in S}\{p(R_i),\beta_S\}\}\} \\
&= \inf_{S\subseteq N}\{\sup_{i\in S}\{p(R_i),\inf\{\alpha_S,\beta_S\}\}\} \\
&\qquad \cup \\
&\quad\ \inf_{S\subseteq N}\{\sup_{i\in S}\{p(R_i),\sup\{\alpha_S,\beta_S\}\}\}.
\end{aligned}$$

For each $R \in \mathcal{R}^N$ and each $S \subseteq N$, $\sup_{i\in S}\{p(R_i),\inf\{\alpha_S,\beta_S\}\} \leq \sup_{i\in S}\{p(R_i),\sup\{\alpha_S,\beta_S\}\}$, and

$$\inf_{S\subseteq N}\left\{\sup_{i\in S}\{p(R_i),\inf\{\alpha_S,\beta_S\}\}\right\} \leq \inf_{S\subseteq N}\left\{\sup_{i\in S}\{p(R_i),\sup\{\alpha_S,\beta_S\}\}\right\}.$$

Therefore, $\underline{f}(R) = \inf\{g(R)\cup h(R)\} = \inf_{S\subseteq N}\{\sup_{i\in S}\{p(R_i),\inf\{\alpha_S,\beta_S\}\}\}$ and $\bar{f}(R) = \sup\{g(R)\cup h(R)\} = \inf_{S\subseteq N}\{\sup_{i\in S}\{p(R_i),\sup\{\alpha_S,\beta_S\}\}\}$.

Finally, let $S,T \subseteq N$ with $S \subseteq T$. Since g and h are min-max single-valued SCCs, $\alpha_S \leq \alpha_T$ and $\beta_S \leq \beta_T$ and we have:

$$\begin{aligned}\inf\{\alpha_S,\beta_S\} \leq \alpha_S \leq \alpha_T,\\ \inf\{\alpha_S,\beta_S\} \leq \beta_S \leq \beta_T,\end{aligned}$$

which implies $\inf\{\alpha_S,\beta_S\} \leq \inf\{\alpha_T,\beta_T\}$, and $\underline{f}$ is a min-max SCC with family of parameters $\{\inf\{\alpha_S,\beta_S\}\}_{S\subseteq N}$. Analogously,

$$\begin{aligned}\alpha_S \leq \alpha_T \leq \sup\{\alpha_T,\beta_T\},\\ \beta_S \leq \beta_T \leq \sup\{\alpha_T,\beta_T\}.\end{aligned}$$

Therefore, $\sup\{\alpha_S,\beta_S\} \leq \sup\{\alpha_T,\beta_T\}$, and $\bar{f}$ is a min-max SCC with family of parameters $\{\sup\{\alpha_S,\beta_S\}\}_{S\subseteq N}$. □

Proof of Lemma 2. Let $R \in \mathcal{S}^N$, and

$$\mathcal{C} = \left\{ S' \subseteq N \,\middle|\, \sup_{i\in S'}\{p(R_i),\alpha_{S'}\} = \inf_{S\subseteq N}\{\sup_{i\in S}\{p(R_i),\alpha_S\}\} \right\}.$$

Let $S^* \in \mathcal{C}$ be maximal with respect to inclusion and such that there is no $S \in \mathcal{C}$ such that $\beta_S \leq \beta_{S^*}$. Note that since $S^* \in \mathcal{C}$, $\sup_{i\in S^*}\{p(R_i),\alpha_{S^*}\} = \underline{f}(R)$, and for each $S \subseteq N$, $\sup_{i\in S^*}\{p(R_i),\alpha_{S^*}\} \leq \sup_{i\in S}\{p(R_i),\alpha_S\}$. Since for each $S \subseteq N$, $\alpha_S \leq \beta_S$, for each $S \subseteq N$, $\sup_{i\in S^*}\{p(R_i),\alpha_{S^*}\} \leq \sup_{i\in S}\{p(R_i),\beta_S\}$. Let $p^* = \sup_{i\in S^*}\{p(R_i)\}$. Consider two mutually exclusive cases:

Case 1): $p^* \leq \beta_{S^*}$. Since $S^* \in \mathcal{C}$, $\alpha_{S^*} \leq \sup\{p^*,\alpha_{S^*}\} = \sup_{i\in S^*}\{p(R_i),\alpha_{S^*}\} = \underline{f}(R)$. Since $p^* \leq \beta_{S^*}$, $\sup_{i\in S^*}\{p(R_i),\beta_{S^*}\} = \beta_{S^*}$, and $\inf_{S\subseteq N}\{\sup_{i\in S}\{p(R_i),\beta_S\}\} \leq \sup_{i\in S^*}\{p(R_i),\beta_{S^*}\} = \beta_{S^*}$. Hence, $\bar{f}(R) \leq \beta_{S^*}$, and $\alpha_{S^*} \leq \underline{f}(R) \leq \bar{f}(R) \leq \beta_{S^*}$.

Case 2): $p^* > \beta_{S^*}$. Since $\alpha_{S^*} \leq \beta_{S^*}$, $\sup_{i\in S^*}\{p(R_i),\alpha_{S^*}\} = \sup_{i\in S^*}\{p(R_i),\beta_{S^*}\} = p^*$. Since $S^* \in \mathcal{C}$, $\underline{f}(R) = \inf_{S\subseteq N}\{\sup_{i\in S}\{p(R_i),\alpha_S\}\} = p^*$. Thus, for each $S \subseteq N$, $p^* \leq \sup_{i\in S}\{p(R_i),\alpha_S\}$. Since for each $S \subseteq N$, $\alpha_S \leq \beta_S$, $p^* \leq \sup_{i\in S}\{p(R_i),\beta_S\}$. Thus, since $p^* = \sup_{i\in S^*}\{p(R_i),\beta_{S^*}\}$, we have $p^* = \inf_{S\subseteq N}\{\sup_{i\in S}\{p(R_i),\beta_S\}\} = \bar{f}(R)$, and $p^* = \underline{f}(R) = \bar{f}(R)$.

Therefore, for each $R \in \mathcal{S}^N$, either there is $S^* \subseteq N$ such that $\alpha_{S^*} \leq \underline{f}(R) \leq \bar{f}(R) \leq \beta_{S^*}$, or there is $i \in N$, such that $\underline{f}(R) = \bar{f}(R) = p(R_i)$. □

Proof of Proposition 1. Let $i \in N$, $R \in \mathcal{S}^N$, and $R'_i \in \mathcal{S}$, we check that for each $\succsim_i \in E^\lambda(R_i)$, $f(R) \succsim_i f(R'_i, R_{-i})$. In this proof, since there is no room for confusion, we abuse notation and drop the brackets from singleton sets to keep notation simple. We consider three exclusive cases:

Case 1): $\bar{f}(R) \leq p(R_i)$: We divide this case in two subcases.

Subcase 1.i): $|\underline{f}(R) - \bar{f}(R)| > 1$. Note that by Lemma 2, λ is $[\underline{f}(R), \bar{f}(R)]$-uniform. Assume first that $p(R_i') \geq \bar{f}(R)$. By Remark 1, $\underline{f}(R_i', R_{-i}) = \underline{f}(R)$, and $\bar{f}(R_i', R_{-i}) = \bar{f}(R)$, and $f(R_i', R_{-i}) = f(R)$, and trivially for each $\succsim_i \in E^\lambda(R_i)$, $f(R) \succsim_i f(R_i', R_{-i})$. Assume next that $p(R_i') \in [\underline{f}(R), \bar{f}(R) - 1]$. By Remark 1, $\underline{f}(R_i', R_{-i}) = \underline{f}(R)$ and $\bar{f}(R_i', R_{-i}) \in [p(R_i'), \bar{f}(R)]$. Since $R_i \in \mathcal{S}$, $\bar{f}(R)\ R_i\ \bar{f}(R_i', R_{-i})$, and for each u_i representing R_i:

$$\frac{1}{2}\left(u_i(\underline{f}(R)) + u_i(\bar{f}(R))\right) \geq \frac{1}{2}\left(u_i(\underline{f}(R)) + u_i(\bar{f}(R_i', R_{-i}))\right).$$

Since λ is $[\underline{f}(R), \bar{f}(R)]$-uniform, for each $\succsim_i \in E^\lambda(R_i)$, $f(R) \succsim_i f(R_i', R_{-i})$. Finally, assume that $p(R_i') < \underline{f}(R)$. By Remark 1, $\underline{f}(R_i', R_{-i}) \in [p(R_i'), \underline{f}(R)]$ and $\bar{f}(R_i', R_{-i}) \in [p(R_i'), \bar{f}(R)]$. We consider two possibilities. If $\bar{f}(R_i', R_{-i}) > \underline{f}(R)$, then $\underline{f}(R)\ R_i\ \underline{f}(R_i', R_{-i})$ and $\bar{f}(R)\ R_i\ \bar{f}(R_i', R_{-i})$. Since λ is $[\underline{f}(R_i', R_{-i}), \bar{f}(R_i', R_{-i})]$-uniform, for each u_i representing R_i:

$$\frac{1}{2}\left(u_i(\underline{f}(R)) + u_i(\bar{f}(R))\right) \geq \frac{1}{2}\left(u_i(\underline{f}(R_i', R_{-i})) + u_i(\bar{f}(R_i', R_{-i}))\right).$$

Then, for each $\succsim_i \in E^\lambda(R_i)$, $f(R) \succsim_i f(R_i', R_{-i})$. To conclude, if $\bar{f}(R_i', R_{-i}) \leq \underline{f}(R)$, then $\bar{f}(R)\ P_i\ \underline{f}(R)\ R_i\ \bar{f}(R_i', R_{-i})\ R_i\ \underline{f}(R_i', R_{-i})$. For each u_i representing R_i:

$$\begin{aligned} &\tfrac{1}{2}\left(u_i(\underline{f}(R)) + u_i(\bar{f}(R))\right) > u_i(\underline{f}(R)), \\ &u_i(\underline{f}(R)) \geq \tfrac{\lambda(\underline{f}(R_i', R_{-i}) u_i(\underline{f}(R_i', R_{-i})) + \lambda(\bar{f}(R_i', R_{-i}) u_i(\bar{f}(R_i', R_{-i}))}{\lambda(\underline{f}(R_i', R_{-i}) + \lambda(\bar{f}(R_i', R_{-i})}, \end{aligned}$$

and for each $\succsim_i \in E^\lambda(R_i)$, $f(R) \succsim_i f(R_i', R_{-i})$.

Subcase 1.ii): $|\underline{f}(R) - \bar{f}(R)| \leq 1$. Assume first that $p(R_i') \geq \bar{f}(R)$. By Remark 1, $f(R) = f(R_i', R_{-i})$, and for each $\succsim_i \in E^\lambda(R_i)$, $f(R) \succsim_i f(R_i', R_{-i})$. Assume next that $p(R_i') < \bar{f}(R)$. By Remark 1, $\underline{f}(R_i', R_{-i}) \in [p(R_i'), \underline{f}(R)]$, $\bar{f}(R_i', R_{-i}) \in [p(R_i'), \bar{f}(R)]$. We have two possibilities. First, if $f(R) = f(R_i', R_{-i})$, then, for each $\succsim_i \in E^\lambda(R_i)$, $f(R) \succsim_i f(R_i', R_{-i})$. Next, if $f(R) \neq f(R_i', R_{-i})$, then $\bar{f}(R_i', R_{-i}) = \bar{f}(R)$ and $\underline{f}(R_i', R_{-i}) < \underline{f}(R)$. Since f is a λ-consistent union min-max SCC, λ is $[\underline{f}(R_i', R_{-i}), \bar{f}(R)]$-uniform, and $\underline{f}(R)\ P_i\ \underline{f}(R_i', R_{-i})$, for each u_i representing R_i:

$$\frac{1}{2}\left(u_i(\underline{f}(R)) + u_i(\bar{f}(R))\right) \geq \frac{1}{2}\left(u_i(\underline{f}(R_i', R_{-i})) + u_i(\bar{f}(R))\right),$$

and for each $\succsim_i \in E^\lambda(R_i)$, $f(R) \succsim_i f(R_i', R_{-i})$. To conclude, assume that $f(R) \neq f(R_i', R_{-i})$ and $\bar{f}(R_i', R_{-i}) < \bar{f}(R)$, for each u_i representing R_i:

$$\begin{aligned} &\tfrac{\lambda(\underline{f}(R)) u_i(\underline{f}(R)) + \lambda(\bar{f}(R)) u_i(\bar{f}(R))}{\lambda(\underline{f}(R)) + \lambda(\bar{f}(R))} > u_i(\underline{f}(R)), \\ &u_i(\underline{f}(R)) \geq \tfrac{\lambda(\underline{f}(R_i', R_{-i})) u_i(\underline{f}(R_i', R_{-i})) + \lambda(\bar{f}(R_i', R_{-i})) u_i(\bar{f}(R_i', R_{-i}))}{\lambda(\underline{f}(R_i', R_{-i})) + \lambda(\bar{f}(R_i', R_{-i}))}, \end{aligned}$$

and for each $\succsim_i \in E^\lambda(R_i)$, $f(R) \succsim_i f(R_i', R_{-i})$.

Case 2): $\underline{f}(R) < p(R_i) < \bar{f}(R)$**.** In this case, $|\underline{f}(R), \bar{f}(R)| > 1$, and λ is $[\underline{f}(R), \bar{f}(R)]$-uniform. Assume first that, $p(R_i') \in [\underline{f}(R), \bar{f}(R)]$. By Remark 1, $f(R) = f(R_i', R_{-i})$, and for each $\succsim_i \in E^\lambda(R_i)$, $f(R) \succsim_i f(R_i', R_{-i})$. Next, assume next that $p(R_i') > \bar{f}(R)]$ and $f(R) \neq f(R_i', R_{-i})$. By Remark 1, $\underline{f}(R_i', R_{-i}) = \underline{f}(R)$, and $\bar{f}(R_i', R_{-i}) \in [\bar{f}(R)+1, p(R_i')]$. Since f is λ-consistent union min-max λ is $[\underline{f}(R_i', R_{-i}), \bar{f}(R_i', R_{-i})]$-uniform. Hence, as $\bar{f}(R)\ P_i\ \bar{f}(R_i', R_{-i})$, for each u_i representing R_i,

$$\frac{1}{2}\left(u_i(\underline{f}(R)) + u_i(\bar{f}(R))\right) \geq \frac{1}{2}\left(u_i(\underline{f}(R)) + u_i(\bar{f}(R_i', R_{-i}))\right),$$

and for each $\succsim_i \in E^\lambda(R_i)$, $f(R) \succsim_i f(R_i', R_{-i})$. Finally, assume that $p(R_i') < \underline{f}(R)]$ and $f(R) \neq f(R_i', R_{-i})$. By Remark 1, $\bar{f}(R_i', R_{-i}) = \bar{f}(R)$, and $\underline{f}(R_i', R_{-i}) \in [p(R_i'), \underline{f}(R)-1]$. Since f is λ-consistent union min-max λ is $[\underline{f}(R_i', R_{-i}), \bar{f}(R_i', R_{-i})]$-uniform. Hence, as $\underline{f}(R)\ P_i\ \underline{f}(R_i', R_{-i})$, for each u_i representing R_i,

$$\frac{1}{2}\left(u_i(\underline{f}(R)) + u_i(\bar{f}(R))\right) \geq \frac{1}{2}\left(u_i(\underline{f}(R_i', R_{-i})) + u_i(\bar{f}(R))\right),$$

and for each $\succsim_i \in E^\lambda(R_i)$, $f(R) \succsim_i f(R_i', R_{-i})$.

Case 3): $p(R_i) \leq \underline{f}(R)$**.** This case follows from arguments that replicate the analysis of Case 1).

□

Proof of Lemma 3. Let $i \in N$, R, $R' \in \tilde{\mathcal{R}}^N$, and $R_i' \in \tilde{\mathcal{R}}$ be such that $R' = (R_i', R_{-i})$, and let u_i represent R_i. Since f satisfies *strategy-proofness* under E^λ,

$$\frac{1}{\sum_{x\in f(R)} \lambda(x)} \sum_{x\in f(R)} \lambda(x) u_i(x) \geq \frac{1}{\sum_{x'\in f(R')} \lambda(x')} \sum_{x'\in f(R')} \lambda(x') u_i(x'),$$

Since for each $x \notin f(R)$ $\Phi_{(\lambda,f)}(R)(\{x\}) = 0$, and for each $x' \notin f(R')$, $\Phi_{(\lambda,f)}(R')(\{x'\}) = 0$,

$$\begin{aligned}\textstyle\sum_{x\in A} \Phi_{(\lambda,f)}(R)(\{x\}) u_i(x) &= \tfrac{1}{\sum_{x\in f(R)} \lambda(x)} \textstyle\sum_{x\in f(R)} \lambda(x) u_i(x),\\ \textstyle\sum_{x'\in A} \Phi_{(\lambda,f)}(R)(\{x'\}) u_i(x') &= \tfrac{1}{\sum_{x'\in f(R')} \lambda(x')} \textstyle\sum_{x'\in f(R')} \lambda(x') u_i(x'),\end{aligned}$$

and

$$\sum_{x\in A} \Phi_{(\lambda,f)}(R)(\{x\}) u_i(x) \geq \sum_{x'\in A} \Phi_{(\lambda,f)}(R)(\{x'\}) u_i(x'),$$

which proves that $\Phi_{(\lambda,f)}$ satisfies *strategy-proofness*.

To conclude, let $R \in \mathcal{R}^N$ and $x \in A$ be such that, for each $i \in N$, $p(R_i) = \{x\}$. Since f satisfies *unanimity*, $f(R) = \{x\}$ and $\Phi_{(\lambda,f)}(R)(\{x\}) = 1$, which proves that $\Phi_{(\lambda,f)}$ satisfies *unanimity*. □

Before we proceed with the proof of Theorem 1, we present a series of Lemmata.

Lemma 5. *Let λ be an probability assessment and f be an SCC that satisfies strategy-proofness under E^{λ} and unanimity, and let $\Phi_(\lambda, f)$ be the associated PDS to λ and f with collection of fixed probabilistic ballots $\mathcal{F} = \{D_S\}_{S\in 2^N}$. For each $S \subset N$, there is at most two alternatives such that $D_S(\{x\}) \neq 0$.*

Proof of Lemma 5. Note that since f is an SCC that satisfies *strategy-proofness* under E^{λ} and *unanimity*, by Lemma 3, $\Phi_{(\lambda,f)}$ satisfies *strategy-proofness* and *unanimity*. By Proposition 2, $\Phi_{(\lambda,f)}$ is a probabilistic fixed–ballots rule with collection of fixed probabilistic ballots $\mathcal{F} = \{D_S\}_{S\in 2^N}$. Assume, to the contrary, that there are $S \subset N$ and $x, x', x'' \in A$ such that $x < x' < x''$, and for each $y \in \{x, x', x''\}$, $D_S([a, y]) - D_S([a, y-1]) > 0$. Let $R \in \mathcal{S}^N$ be such that for each $i \in S$, $p(R_i) = a$ and for each $j \in N \setminus S$, $p(R_j) = b$. By Remark 4, $\Phi_{(\lambda,f)}(R) = D_S$ and $f(R) = \{y \in A : D_S([a, y]) - D_S([a, y-1]) > 0\}$. Let $R' \in \mathcal{S}^N$ be such that for each $i \in S$, $p(R'_i) = x'$, and for each $j \in N \setminus S$, $R'_j = R_j$. By Remark 3, for each $y < x'$, $\Phi_{(\lambda,f)}(R')([a, y]) = 0$, $\Phi_{(\lambda,f)}(R')(\{x'\}) = \sum_{y\in f(R), y\leq x'} \Phi_{(\lambda,f)}(R)(\{y\})$, and for each $y' > x'$, $\Phi_{(\lambda,f)}(R')(\{y'\}) = \Phi_{(\lambda,f)}(R)(\{y'\})$. Thus, $x \notin f(R')$, and $f(R') \subset f(R)$. By the definition of $\Phi_{(\lambda,f)}$

$$\Phi_{(\lambda,f)}(R)(\{x''\}) = \frac{\lambda(x'')}{\sum_{y\in f(R)} \lambda(y)}, \quad \text{and} \quad \Phi_{(\lambda,f)}(R')(\{x''\}) = \frac{\lambda(x'')}{\sum_{y'\in f(R')} \lambda(y')}.$$

However, $\sum_{y\in f(R)} \lambda(y) \neq \sum_{y'\in f(R')} \lambda(y')$, which contradicts the equality $\Phi_{(\lambda,f)}(R)(\{x''\}) = \Phi_{(\lambda,f)}(R')(\{x''\})$. □

Lemma 6. *For each probability assessment λ, if an SCC f satisfies strategy-proofness under E^{λ} and unanimity, then f is a regular union min-max SCC.*

Proof of Lemma 6. By Lemma 3 and Proposition 2, the associated PDS, $\Phi_{(\lambda,f)}$ is a probabilistic fixed ballot rule. Let $\mathcal{F} = \{D_S\}_{S\subseteq N}$ be the collection of fixed probability ballots that defines $\Phi_{(\lambda,f)}$. From Lemma 5 and Remark 3:

- For each $S \subseteq N$, there are at most two alternatives $\alpha_S, \beta_S \in A$ with $\alpha_S \leq \beta_S$, such that for each $x \notin \{\alpha_S, \beta_S\}$, $D_S(\{x\}) = 0$. We define:

$$\alpha_S = \inf\{x \in A : D_S(x) \neq 0\} = \inf\{x \in A : \ D_S[a, x] \neq 1\},$$
$$\beta_S = \sup\{x \in A : D_S(x) \neq 0\} = \inf\{x \in A : \ D_S[a, x] = 1\}.$$

- For each $S, T \subseteq N$, and each $x \in A$, $D_{S\cup T}([a, x] - D_S([a, x]) \geq 0$. Hence, for each S, S' with $S' \subseteq S$, $\alpha_S \leq \alpha_{S'}$ and $\beta_S \leq \beta_{S'}$.

These observations imply that $\mathcal{F}$ defines two families of parameters $\{\alpha_S\}_{S\subseteq N}$ and $\{\beta_S\}_{S\subseteq N}$ such that for each $S \subseteq N$, $\alpha_S \leq \beta_S$. We check now, that the families of parameters $\{\alpha_S\}_{S\subseteq N}$ and $\{\beta_S\}_{S\subseteq N}$ induce a union min-max SCC.

Let $R \in \mathcal{S}^N$ and let $\{p^1(R), \dots, p^k(R)\}$ be the list of ordered peaks, and for each $t = 1, \dots, k$, let $S_t(R) = \{i \in N : \ p(R_i) \leq p^t(R)\}$. Define

$$x^* = \inf\{x \in A : \ \Phi_{(\lambda,f)}(R)([a,x]) \neq 0\}.$$

By Remark 3 there is $t^* \leq k$ such that $x^* \in \{p^{t^*}, \alpha_{S_{t^*}(R)}\}$. Note that for each $t' < t^*$, $S_{t'}(R) \subset S_{t^*}(R)$, $\sup_{i \in S_{t'}(R)}\{p(R_i)\} \leq \sup_{j \in S_{t^*}(R)}\{p(R_i)\} = p^{t^*}(R)$. By the definition of x^*, $\alpha_{S_{t^*}(R)} \geq \alpha_{S_{t'}(R)}$. Moreover, for each $S'' \subseteq N$ with $S'' \cap (N \setminus \{S_{t^*}(R)\})$, $\sup_{i \in S''(R)}\{p(R_i)\} > \sup_{j \in S_{t^*}(R)}\{p(R_i)\} = p^{t^*}(R)$. By Remark 3, $x^* = \sup_{i \in S_{t^*}(R)}\{p(R_i), \alpha_{S_{l^*}(R)}\}$. Note that for each $S \subsetneq S_{t^*}(R)$, $\sup_{i \in S_{t^*}(R)}(\{p(R_i)\}) \geq \sup_{j \in S}\{p(R_j)\}$, and $\alpha_{S_{t^*}(R)} \leq x^* \leq \alpha_S$, while for each $S' \subseteq N$ with $S' \cap (N \setminus \{S_{l^*}(R)\})$, $\sup_{i \in S'}\{p(R_i)\} > \sup_{j \in S_{t^*}(R)}\{p(R_i)\} = p^{t^*}(R)$. Thus, $x^* = \inf_{S \subseteq N}\{\sup_{i \in S}\{p(R_i), \alpha_S\}\}$. Since $\Phi_{(\lambda,f)}(R)(\{x^*\}) \neq 0$, $x^* \in f(R)$ and for each $y < x^*$, we have that $\Phi_{(\lambda,f)}(R)(\{y\}) = 0$ and $y \notin f(R)$. Since for each $t' > t^*$, $\alpha_{S_{t'}(R)} \leq \alpha_{S_{t^*}(R)}$, by Remark 3 and the fact that for each $S \subseteq N$, for each $x \notin \{\alpha_S, \beta_S\}$, $D_S(\{x\}) = 0$, then if for some $\bar{x} > x^*$, $\Phi_{(\lambda,f)}(R)(\{\bar{x}\}) \neq 0$, then there is $\bar{t} \geq t^*$ such that $\bar{x} \in \{p^{\bar{t}}(R), \beta_{S^{\bar{t}}(R)}\}$. The argument we used for defining x^* applies to show that $\bar{x} = \inf_{S \subseteq N}\{\sup_{i \in S}\{p(R_i), \beta_S\}\} \in f(R)$, $\bar{x} = \inf\{x \in A : \ \Phi_{(\lambda,f)}(R)([a,x]) = 1\}$. Hence, since for each $S \subseteq N$, $\beta_S = \inf\{x \in A : \ D_S[a,x] = 1\}$, for each $y > \inf_{S \subseteq N}\{\sup_{i \in S}\{p(R_i), \beta_S\}\}$, $y \notin f(R)$. This implies that $f(R) = \inf_{S \subseteq N}\{\sup_{i \in S}\{p(R_i), \alpha_S\}\} \cup \inf_{S \subseteq N}\{\sup_{i \in S}\{p(R_i), \beta_S\}\}$, and the families $\{\alpha_S\}_{S \subseteq N}$ and $\{\beta_S\}_{S \subseteq N}$ define a regular union min-max SCC. $\square$

Lemma 7. *Let λ be a probability assessment and f a union min-max SCC that satisfies strategy-proofness under E^λ, then for each $S \subseteq N$:*

- *either $|\alpha_S - \beta_S| \leq 1$,*
- *or for each $x \in [\alpha_S, \beta_S]$ $\lambda(x) = \lambda(\alpha_S) = \lambda(\beta_S)$.*

Proof of Lemma 7. Assume that there is $S \subseteq N$ such that there exists $x^* \in A$, $\alpha_S < x^* < \beta_S$. Let $R \in \mathcal{S}^N$ be such that for each $i \in S$, $p(R_i) = a$, and for each $j \in N \setminus \{S\}$, $p(R_j) = b$. Since f is a union min-max SCC, $f(R) = \alpha_S \cup \beta_S$. Let $R' \in \mathcal{S}^N$ be such that for each $i \in S$, $p(R'_i) = x^*$, for each $y \in [\alpha_S, x^*]$ and each $y' \notin [\alpha_S, x^*]$, $y \ P'_i \ y'$, and for each $j \in N \setminus S$, $R'_j = R_j$. Let $i \in S$. Consider now the profile (R'_i, R_{-i}). By Remark 1, either $f(R'_i, R_{-i}) = f(R)$ or there is $y \in [\alpha_S, x^*]$ such that $f(R'_i, R_{-i}) = y \cup \beta_S$. Assume that $\lambda(y) > \lambda(\alpha_S)$. Let u_i be a utility function that represents R_i such that $u_i(\alpha_S) = 1$, $u_i(\beta_S) = 0$, and for $u_i(y) = \varepsilon \in \left(\frac{\lambda(\alpha_S)}{\lambda(y)} \frac{\lambda(y)+\lambda(\beta_S)}{\lambda(\alpha_S)+\lambda(\beta_S)}, 1\right)$, then

$$\frac{\lambda(\alpha_S)u_i(\alpha_S) + \lambda(\beta_S)u_i(\beta_S)}{\lambda(\alpha_S) + \lambda(\beta_S)} = \frac{\lambda(\alpha_S)}{\lambda(\alpha_S) + \lambda(\beta_S)} < \frac{\lambda(y)\varepsilon}{\lambda(y) + \lambda(\beta_S)} = \frac{\lambda(y)u_i(y) + \lambda(\beta_S)u_i(\beta_S)}{\lambda(y) + \lambda(\beta_S)}.$$

This implies that there is $\succsim_i \in E^\lambda(R_i)$ such that $f(R'_i, R_{-i}) \succ_i f(R)$ which contradicts E^λ-*strategy-proofness*. Assume that $\lambda(y) < \lambda(\alpha_S)$. Let u'_i be a utility function that represents R'_i such that $u_i(y) = 1$, $u_i(\beta_S) = 0$, and for $u'_i(\alpha_S) = \varepsilon' \in \left(\frac{\lambda(y)}{\lambda(\alpha_S)} \frac{\lambda(\alpha_S)+\lambda(\beta_S)}{\lambda(y)+\lambda(\beta_S}, 1\right)$, then

$$\frac{\lambda(\alpha_S)u'_i(\alpha_S) + \lambda(\beta_S)u'_i(\beta_S)}{\lambda(\alpha_S) + \lambda(\beta_S)} = \frac{\lambda(\alpha_S)\varepsilon'}{\lambda(\alpha_S) + \lambda(\beta_S)} > \frac{\lambda(y)}{\lambda(y) + \lambda(\beta_S)} = \frac{\lambda(y)u'_i(y) + \lambda(\beta_S)u'_i(\beta_S)}{\lambda(y) + \lambda(\beta_S)}.$$

This implies that there is $\succsim_i \in E^\lambda(R'_i)$ such that $f(R) \succ'_i f(R'_i, R_{-i})$ which contradicts E^λ-*strategy-proofness*. Hence, either $f(R'_i, R_{-i}) = f(R)$ or $f(R'_i, R_{-i}) = y \cup \beta_S$ for some $y \in [\alpha_S, x^*]$ with $\lambda(y) = \lambda(\alpha_S)$. We can repeat the argument as many times as necessary with the remaining members of S and reach the conclusion that either $f(R') = f(R)$ which is a contradiction, since by the definition of union min-max SCC $f(R') = \{x^*, \beta_S\}$, or $\lambda(\alpha_S) = \lambda(x^*)$. Repeating the argument by changing the preferences in the group $N \setminus S$, we can check that if $|\alpha_S - \beta_S| > 1$, then for each $x, x' \in [\alpha_S, \beta_S]$, $\lambda(x) = \lambda(x')$. □

Proof of Theorem 1. By Proposition 1, for every probability assessment λ, every λ-consistent union min-max SCC satisfies *strategy-proofness* under E^λ. Since every λ-consistent union min-max SCC is a union min-max SCC, and since every min-max single-valued SCC satisfies *unanimity*, every λ-consistent union min-max SCC satisfies *unanimity*.

To prove the necessity side, let λ be a probability assessment and f an SCC that satisfies *strategy-proofness* under E^λ and *unanimity*. By Lemma 3, the associated PDS $\Phi_{(\lambda,f)}$ satisfies *strategy-proofness* and *unanimity* for PDS. By Proposition 2, $\Phi_{(\lambda,f)}$ is a probabilistic fixed ballot PDS. By Lemmata 5, 6, and 7, f is a λ-consistent union min-max SCC. □

Proof of Theorem 2. The sufficiency side is obvious and we focus on necessity. Let $f : \mathcal{M}^N \to \mathcal{A}$ satisfy *strategy-proofness* according to E^{EP} and *unanimity*. Let Φ_f be the auxiliary PDS associated with f defined in the following way. For each $R \in \tilde{\mathcal{M}}^N$ and each $x \in A$:

$$\Phi_f(R)(\{x\}) = \begin{cases} \frac{1}{\#f(R)} & \text{if } x \in f(R), \\ 0 & \text{otherwise.} \end{cases}$$

With the same arguments of Lemma 3, Φ_f satisfies *strategy-proofness* under E^{EP} and *unanimity*. By Theorem 2.1 in Dutta *et al.* (2002), Φ_f is a random dictatorship, that is there are nonnegative numbers $\{\mu_1, \ldots, \mu_n\}$ with $\mu_i \geq 0$ for each $i \in N$ and $\sum_{i \in N} \mu_i = 1$ such that for each $R \in \mathcal{M}^N$ and every $X \subset A$, $\Phi_f(R)(X) = \sum_{i \in N} \mu_i \mathbb{1}_X(p(R_i))$.

To complete the proof, we only need to show that there cannot be more than two voters for which $\mu_i \neq 0$, and therefore at most two alternatives can be selected by f. First, we prove that for each $i, j \in N$ such that $\mu_i > 0$ and $\mu_j > 0$, $\mu(i) = \mu(j)$. Consider $R \in \mathcal{M}^N$ such that for each $i, j \in N$, $p(R_i) \neq p(R_j)$. Then,

$$f(R) = \{x \in A : \text{ there exists } i \in N, \text{ such that } x = p(R_i) \text{ and } \mu_i \neq 0\}.$$

By the definition of Φ_f, for each $x, x' \in f(R)$, $\Phi_f(R)(\{x\}) = \Phi_f(R)(\{x'\})$ and $\Phi_f(R)(\{p(R_i)\}) = \mu_i$, and for each $i, j \in N$ such that $\mu_i \neq 0$ and $\mu_j \neq 0$, $\mu_i = \mu_j$.

Finally, assume to the contrary that there exist at least three voters $\{i, j, k\}$ such that for each $i' \in \{i, j, k\}$, $\mu_{i'} > 0$. Let $R \in \mathcal{M}^N$ be such that for $p(R_i) \neq p(R_j)$ and for all $l \neq i$, $p(R_l) = p(R_j)$. Then, $f(R) = \{p(R_i), p(R_j)\}$, and by the definition of Φ_f, $\Phi_f(R)(p(R_i)) = \Phi_f(R)(p(R_j)) = \frac{1}{2}$. However, $\Phi_f(R)(p(R_i)) = \mu_i$, and $\Phi_f(R)(p(R_j)) = \mu_j + \mu_k$, which is a contradiction since $\mu_i = \mu_j = \mu_k > 0$. Hence, either there is one voter i with $\mu_i = 1$ or there are two voters j, k with $\mu_j = \mu_k = \frac{1}{2}$. Thus, f is either dictatorial or bidictatorial. □

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