Existence, uniqueness and attractivity properties of positive complete trajectories for non-autonomous reaction-diffusion problems.*

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1 Introduction

Logistic equations have been widely used in the mathematical biology literature in the modeling of the spread, over a given spatial domain, of all sorts of biological quantities (e.g. population species densities, genes), [11, 12], [5], [13]. Their success in this field is due to the ability of these equations in describing natural features of biological systems as self–production of species (such as birth and death rates) and medium limitations, or competition for resources. At the same time, spatial heterogeneities of biological characteristics of each model can be easily taken into account.

A typical basic model includes diffusion, boundary conditions and a nonlinear space-dependent reaction term.

At the next level of complexity, when considering seasonal or environmental influences, one must include time as an explicit variable in the problem. In other words the equations become non autonomous. In this context a widely used assumption, which may be taken as realistic in many real situations, is that of periodic time dependence in the equations; see [11] and [8]. However, non periodic (e.g. almost periodic) situations appear as relevant in realistic applications.

In each of these situations a crucial question becomes that of determining ranges of parameters and typical models in which some important solutions exist. Namely, those reflecting the coexistence or, on the other hand, extinction of some of the species involved. Of course the space-time structures described by these important solutions are of capital importance in the applications. An important remark is that, due to the intrinsic nature of the quantities modeled, (population densities for example), in the applications one is only interested in considering nonnegative solutions of the equations. It is remarkable however that the dynamics of positive solutions is typically much simpler than these of arbitrary solutions, [2].

For example, in the case of autonomous problems, conditions are known implying the existence of a unique globally asymptotically positive steady state which completely describes the

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long time behavior of such models, [1]. An analogous situation is also know to occur for time–periodic problems, in which a nontrivial positive globally asymptotic periodic solution exist, [8]. Also for some almost–periodic problems the same sort of behavior have been shown to occur, [17]; see the next section for a more detailed description and references.

In this paper we approach the problem of describing the dynamics of scalar time–dependent logistic–like equations (or some more general related models) with no prescribed behavior in time (e.g. periodic or almost–periodic). For a wide class of these type of problems we show that the same type of behavior for time–independent or periodic problems is observed in such systems. That is, we give conditions guaranteeing the existence of a unique positive special solution that describes the asymptotic dynamics of the model.

Observe that as time appears explicitly in the equations, to construct solutions and to describe their behavior, one must necessarily take into account the initial time at which processes start. Then describing what is the relevant dynamics or, in other words, when transient behaviors have ceased to reveal themselves, requires a suitable concept. For this goal here, we make use here of two concepts of "asymptotic behavior", or "attraction".

First, we use the concept of "pull-back attraction". These is better explained with an example; see the next section and the references for a more technical description. At present time, i.e. today, we see in nature (say, in living animal species) the result of long term evolution processes; that is, processes that started long time ago and have evolved in time leading to the states that we see at this time. This means that at present time, t, the relevant dynamics, that is, the one that has gone through any transient behavior, is the one that we observe today but started long time ago. These states are relevant today because they have been able to survive natural selection and evolution. These states are the important states of today, or, in more mathematical terms, they are the pull-back attractor at present time. Therefore, in this approach at each time there exist a set of important states (from the dynamics point of view) and these collections of states over all times is the relevant set of states describing the dynamics of the system in the pull-back sense, i.e. the pullback attractor, [3].

Of course at any given initial time some processes may start to evolve and they may eventually reach in the far away future some states that will be then describing the "forward in time" evolution of the system, , i.e. the forward attractor, [7], [18].

Then, our results imply that the unique positive special solution mentioned before, $\varphi(x,t)$, has the remarkable property of describing the asymptotic behavior of the system both in the pull-back and forwards senses. That is, at any "present" time $t \in I\!R$, the state $\varphi(\cdot,t)$ is the pull-back attractor of the system. On the other hand as time increases, that is as $t \to \infty$, any solution starting at any finite time t_0 will eventually get arbitrarily close to the state $\varphi(\cdot,t)$.

2 Statements of the results

Our goal in this paper is to prove, under suitable conditions, the existence, of a unique bounded, positive, nondegenerate, complete trajectory for the nonautonomous reaction diffusion model problem

$$\left\{ \begin{array}{rcll} u_t - \Delta u &=& f(t,x,u), & \text{in } \Omega \quad t > s \\ u &=& 0 & \text{on } \partial\Omega \\ u(s) &=& u_0 & \end{array} \right. \tag{2.1}$$

whose solutions are denoted $u(t,s;u_0)$, where $\Omega \subset I\!R^N$ is a bounded smooth domain and $f : I\!R \times \Omega \times I\!R \to I\!R$ is suitable smooth and satisfies $f(t,x,0) \geq 0$ and

$$\frac{f(t,x,u)}{u} \qquad \text{is nonincreasing for} \qquad u \geq 0. \tag{2.2}$$

For more general problems see Section 8.

Note that by a complete trajectory we mean a solution of (2.1) defined for all times, in the sense that it is a continuous function $u(x,t)$ such that for each $s \in I\!R$ the solution of (2.1) with initial data $u_0(x) = u(x,s)$ is given by $u(x,t)$ for each $t > s$. Also, non–degenerate at $\pm\infty$ means that as $t \to \pm\infty$, $u(x,t)$ remains away from zero in some sense to be made precise below. Finally, by bounded we mean that $u(x,t)$ remains bounded in $C^1(\overline{\Omega})$.

We will also show that this unique positive complete trajectory, $\varphi(x,t)$, captures the asymptotic dynamics of all positive solutions of (2.1) both in the pullback and in the forward sense as we now explain. First, this solution attracts the dynamics of all positive solution of (2.1) in a pullback sense. This means that at present time, t, the relevant dynamics, that is, the one that has gone through any transient behavior, is the one that we observe today but started long time ago, i.e. as $s \to -\infty$. More precisely, we will show that for each $t \in I\!R$,

$$\lim_{s\to-\infty} u(t,s;u_0) = \varphi(t) \tag{2.3}$$

uniformly for u_0 in any bounded set of initial data in $X = C(\overline{\Omega})$.

Of course, today some processes may start, and we will show that they will approach in the future, i.e. as $t \to \infty$, to the state described by the solution $\varphi(t)$. This is the meaning of forward attraction, which can be written as

$$\lim_{t\to\infty} (u(t,s;u_0) - \varphi(t)) = 0 \tag{2.4}$$

for all $s \in I\!R$ and uniformly for u_0 in any bounded set of initial data in $X = C(\overline{\Omega})$.

Observe that pullback attraction refers to the concept "we see today what started long time ago, and has lost all transient behavior meanwhile". On the other hand the notion of forward attraction refers to the concept "what we will ultimately see in the future".

Note that our results will be obtained without any specific behavior in time of nonautonomous terms in (2.1) (e.g. periodic, or almost–periodic).

On f we will assume suitable regularity conditions in order to have that for each $u_0 \in X = C(\overline{\Omega})$, (2.1) has a unique global solution for $t > s$ that we denote $u(t,s;u_0) = U(t,s)u_0$ where $U(t,s)$ is the evolution operator associated to (2.1) which satisfies $U(t,s) \in C(X,X)$ and

i) $U(t,t) = I$ for all $t \in I\!R$,

ii) $U(t,s)U(s,r)u = U(t,r)u$ for all $r \leq s \leq t$, $u \in X$, and

iii) $t \mapsto U(t,r)u$ is continuous in X for $t > r$

In such a case $U(t,s)$ is moreover order preserving and smoothes the solutions in the sense that $U(t,s)$ is continuous and bounded from $X = C(\overline{\Omega})$ into $C^1_0(\overline{\Omega})$, that is, the subset of $C^1(\overline{\Omega})$ of functions that vanish on $\partial\Omega$. For example, all these holds provided $f(t,x,u)$ is continuous in $I\!R \times \Omega \times I\!R$, locally Lipschitz on u and locally Hölder in t, uniformly respect to $x \in \Omega$, as in [14].

On the other hand, for applications it is very useful to be able to include some singular term in the equations. Following this idea, we will assume that f has a decomposition of the from

$$f(t,x,u) = g(x,t) + m(x,t)u + f_0(t,x,u) \tag{2.5}$$

with f_0 continuous in $\mathbb{R}\times\Omega\times\mathbb{R}$, locally Lipschitz on u and locally Hölder in t uniformly respect to $x \in \Omega$, and

$$f_0(x,0) = 0 \qquad \frac{\partial}{\partial u} f_0(x,0) = 0; \tag{2.6}$$

g is a suitable regular function, say locally Hölder in time with values in $L^\infty(\Omega)$, (to simplify the arguments; having values in $L^q(\Omega)$ for certain $q > N$ would suffice); and $m \in C^\alpha(\mathbb{R}, L^p(\Omega))$ for certain $p > N/2$ and $0 < \alpha \leq 1$.

Under these assumptions the following result holds.

Theorem 2.1 *Let Ω be a bounded domain of $\mathbb{R}^N$. Suppose that f satisfies (2.5) and (2.6). Then, for any $u_0 \in L^\infty(\Omega)$ there exists a local solution of the problem $u \in C((s,T); C(\overline{\Omega}))$, for certain $T > s$. This solution is given by the variation of constants formula*

$$u(t,s;u_0) = T(t,s)u_0 + \int_s^t T(t,r)(g(r) + f_0(\cdot, u(t,r)))\,dr, \quad s < t \leq T, \tag{2.7}$$

where $T(t,s)$ is the evolution operator associated to with $A(t) = -\Delta + m(x,t)I$, with Dirichlet boundary conditions.

Observe that, concerning the asymptotic behavior of positive solutions of (2.1), several results are already known.

In the autonomous case, i.e. when f is independent of t, the uniqueness of positive equilibrium, $u_E(x)$, for the elliptic stationary problem

$$\left\{ \begin{array}{rcll} -\Delta u &=& f(x,u) & \text{in} \quad \Omega \\ u &=& 0 & \text{on} \quad \partial\Omega \end{array} \right. \tag{2.8}$$

with $\dfrac{f(x,u)}{u}$ decreasing goes back to [1]. Moreover this stationary solution is globally asymptotically stable for the corresponding autonomous evolution problem, see also [6], [15] and [16].

On the other hand, in the periodic case, i.e. when $f(t+T,x,u) = f(t,x,u)$, the uniqueness of positive solutions of (2.1) under assumption (2.2) was proved in [8], where it was also shown that this solution is globally asymptotically stable. For almost periodic dependence in time similar results have been shown in [17]. More recently, and with no specified dependence in time, the case $f(t,x,u) = \lambda u - b(x,t)u^\rho$ has been considered in [9] under the restriction that

$$0 < a_0 \leq b(x,t) \leq A_0, \qquad \text{with} \qquad A_0 \leq \rho a_0.$$

All these result will fall into the general result that we state below and the result in [9] will be proved without the restriction above.

Note that the results in [1] for the autonomous problem (2.8) relies on the variational structure of the equations that allows to use suitable energy arguments and a suitable description of some associated linear eigenvalue problems. With this, one is able to show that the associated semigroup has a unique positive globally stable equilibria. On the other hand the results in [8] for the periodic problem, makes essential use of the strong positivity and compactness properties inherited by the associated Poincaré map. With this, it can be shown that, under suitable structure conditions, the Poincaré map has a unique positive, globally stable, fixed point. Finally, the results in [17] for the almost periodic case, use as an essential tool the techniques of skew–product flows in which the original problem is embedded. Using properties of the extended semigroup the existence of a unique positive attracting almost periodic solution is obtained. Of course, neither of these techniques can be used for a general problem, with no specified time–dependence.

Now we make precise some conditions that we use below.

Definition 2.2
i) A positive function with values in $X = C(\overline{\Omega})$ and vanishing on Ω, is non–degenerate at ∞ (respectively $-\infty$) if there exists $t_0 \in I\!R$ such that u is defined in $[t_0, \infty)$ (respectively $(-\infty, t_0]$) and there exists a $C^1(\overline{\Omega})$ function $\varphi_0(x) > 0$ in Ω, vanishing on $\partial\Omega$, such that

$$u(x,t) \geq \varphi_0(x) \qquad \text{for all} \quad t \geq t_0 \tag{2.9}$$

(respectively for all $t \leq t_0$).
ii) A positive function with values in $X = C(\overline{\Omega})$ and vanishing on Ω, is bounded above at ∞ (respectively $-\infty$) if there exists $t_0 \in I\!R$ such that u is defined in $[t_0, \infty)$ (respectively $(-\infty, t_0]$) and there exists a $C^1(\overline{\Omega})$ function $\varphi_1(x) > 0$ in Ω, vanishing on $\partial\Omega$, such that

$$u(x,t) \leq \varphi_1(x) \qquad \text{for all} \quad t \geq t_0 \tag{2.10}$$

(respectively for all $t \leq t_0$).

Note that to have (2.10) satisfied it is enough to have that $0 \leq u(t)$ is bounded in $C^1(\overline{\Omega})$ for all $t \geq t_0$ (respectively for all $t \leq t_0$).

On the nonlinear term, we will assume the following two nondegeneracy conditions. First, if $u(x,t)$ is a positive non–degenerate curve at ∞ (respectively at $-\infty$) and bounded above, we have

$$q(x,t) := \frac{f(t,x,u(x,t))}{u(x,t)} \geq q_0(x) \qquad \text{for all} \quad t \geq t_0 \tag{2.11}$$

(respectively for all $t \leq t_0$) for every $x \in \Omega$ and for some $q_0 \in L^p(\Omega)$ for some $p > N/2$.

Moreover we assume that if $0 \leq u_1(x,t) \leq u_2(x,t)$ are positive non–degenerate curves at ∞ (respectively at $-\infty$) and bounded above, such that for some set of positive measure $A \subset \Omega$ and some $\alpha > 0$ we have

$$\liminf_{t\to\infty} (u_2(x,t) - u_1(x,t)) \geq \alpha > 0 \quad a.e. \quad x \in A \tag{2.12}$$

then the potential

$$P(x,t) := \frac{f(t,x,u_1(x,t))}{u_1(x,t)} - \frac{f(t,x,u_2(x,t))}{u_2(x,t)} \geq 0,$$

see (2.2), satisfies that there exists some set of positive measure, that we still denote $A \subset \Omega$, and a positive number, that we still denote $\alpha > 0$, such that

$$\liminf_{t\to\infty} P(x,t) \geq \alpha > 0 \quad a.e. \quad x \in A \tag{2.13}$$

(with a completely analogous condition as $t \to -\infty$).

Finally, we assume the following mild compactness assumption. Namely, if $u(x,t)$ is a positive, bounded above solution of (2.1) (respectively, a positive bounded above complete trajectory) then, for some $t_0 \in I\!R$,

$$\{u(\cdot,t), \quad t \geq t_0\} \quad \text{is precompact in} \quad L^1(\Omega) \tag{2.14}$$

(respectively for $t \leq t_0$). Note, as it will used below, that if the solution is bounded above for $t \geq t_0$, then it goes to zero in $L^1(\Omega)$, as $t \to \infty$, iff it converges to zero a.e. $x \in \Omega$. Also, if u is precompact in $L^1(\Omega)$ for $t \geq t_0$, then it goes to zero in $L^1(\Omega)$, as $t \to \infty$, iff $\liminf_{t\to\infty} u(x,t) = 0$ a.e. $x \in \Omega$. The same holds for $t \leq t_0$.

With these notations we state now the main results we prove in this paper. Concerning uniqueness of complete positive trajectories, we have

Theorem 2.3 *Under assumption (2.2), the nondegeneracy conditions (2.11) and (2.13) and the compactness assumption (2.14) as* $t \to -\infty$, *if* $0 \leq u_1(t) \leq u_2(t)$ *are positive, nondegenerate at* $-\infty$, *bounded above, complete trajectories of (2.1), then* $u_1 = u_2$.

Note that without some sort of nondegeneracy of the solution, the uniqueness result above, Theorem 2.3, can not be true. Indeed assume for the autonomous case (2.8) the positive solution, $u_E(x)$, exist, $f(x,0) = 0$ and $u = 0$ is unstable in the sense that the first eigenvalue of the linearized eigenvalue problem

$$\begin{cases} -\Delta z = \partial_u f(x,0)z + \lambda z & \text{in} \quad \Omega \\ z = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{2.15}$$

satisfies $\lambda_1 < 0$. Then the associated evolution problem (2.1) has an evolution operator $U(t,s)u_0 = T(t-s)u_0$ where $T(t)$ is a nonlinear semigroup. Hence if there exists an heteroclinic connection for $T(t)$ between the unstable equilibria $u = 0$ and the stable one u_E, then (2.1) would have a complete positive solution, $u_1(t)$, that satisfies $u_1(t) \to 0$, as $t \to -\infty$, and $u_1(t) \to u_E$, as $t \to \infty$, besides the complete positive nondegenerate solution $u_2(t) = u_E$ (they are indeed infinitely many such solutions, since all time shifts of $u_1(t)$ also connect both equilibria).

Concerning the asymptotic behavior forward in time of solutions of (2.1), we have

Theorem 2.4 *Under assumption (2.2), the nondegeneracy conditions (2.11) and (2.13) and the compactness assumption (2.14) as* $t \to \infty$, *if* $0 \leq u_1(t) \leq u_2(t)$ *are positive, nondegenerate at* ∞, *bounded above, solutions of (2.1), then*

$$\lim_{t\to\infty} (u_2(x,t) - u_1(x,t)) = 0 \qquad \textit{uniformly in} \quad x \in \Omega.$$

Finally, concerning existence of positive complete trajectories we have the following result which is stated in a less general form than stated after in the paper. Note that assumption (2.2) is not used here.

Theorem 2.5 *Assume* $f(t,x,0) \geq 0$ *and*

$$f(t,x,u) \leq C(x,t)u + D(x,t) \qquad \textit{for} \quad u \geq 0$$

such that the evolution operator associated to $\Delta + C(t,x)$ *is exponentially stable with exponent* $\beta > 0$ *and*

$$D \in L^\infty(I\!R, L^r(\Omega)) \qquad \textit{for} \quad r > N/2.$$

Then there exist a maximal complete trajectory of (2.1)

$$0 \leq \varphi_M \in C_b(I\!R, C_0(\overline{\Omega}))$$

which bounds the dynamics of the positive solutions of (2.1) uniformly in the pullback sense , i.e., for all $t \in I\!R$ we have

$$0 \leq \liminf_{s \to -\infty} u(t, s, x; u_0) \leq \limsup_{s \to -\infty} u(t, s, x; u_0) \leq \varphi_M(x, t)$$

uniformly in $x \in \overline{\Omega}$ and for u_0 in a bounded set of initial data. Moreover, φ_M is globally asymptotically stable from above in pullback sense, i.e., for all $v \in C_b(I\!R, X)$, $v \geq \varphi_M$ we have for all $t \in I\!R$,

$$\lim_{s \to -\infty} u(t, s; v(s)) = \varphi_M(t).$$

Finally $\varphi_M(t)$ is T–periodic if $f(t + T, x, u) = f(t, x, u)$.

Note that we have made use above of the following definition

Definition 2.6 *If X is a Banach space and $T(t, s) \in \mathcal{L}(X)$, we say that the evolution operator $T(t, s)$ is* exponentially stable *of exponent $\beta > 0$ if for some $M > 0$*

$$||T(t, s)||_{\mathcal{L}(X)} \leq M \mathrm{e}^{-\beta(t-s)} \quad \textit{for all } t > s.$$

See also Theorem 5.6 giving conditions for the existence of a minimal positive, complete, nondegenerate bounded trajectory, which is stable from below in the pullback sense. From here and the uniqueness of Theorem 2.3 we will obtain that the pullback attractor is given by such solution. Moreover Theorem 2.4 will prove that this special solution also describes the forward asymptotic behavior of positive solutions of (2.1).

The paper is organized as follows. In Section 3 we prove the uniqueness result, Theorem 2.3. In Section 4 we show the results on the asymptotic forward behavior and prove in particular Theorem 2.4. In Section 5 we prove a more general version of Theorem 2.5 and give conditions for the complete solutions of (2.1) to be bounded and/or nondegenerate at $\pm\infty$. In Section 6 we discuss the case of asymptotic autonomous or asymptotic periodic problems. In Section 7 we illustrate how our previous results apply to the important case of logistic equations. Finally in Section 8 we show how our results apply to problems like (2.1), with more general diffusion operator and different boundary conditions.

An interesting concluding remark is that most of our results are based on suitable perturbation argument of linear equations. For such problems several interesting results are obtained in the forthcoming sections.

3 Uniqueness of positive complete nondegenerate solutions

In this section we prove Theorem 2.3. For this observe that if $u(x, t)$ is a complete, positive, nondegenerate solution, CPNDS, at $-\infty$, of (2.1), then it is also a CPNDS at $-\infty$ of the linear problem

$$\left\{ \begin{array}{rcll} z_t - \Delta z &=& q(x, t) z, & \text{in } \Omega \\ z &=& 0 & \text{on } \partial\Omega \end{array} \right. \tag{3.1}$$

with $q(x, t) := \dfrac{f(t, x, u(x, t))}{u(x, t)}$.

Note that from the results in [14] if $q \in C^{\alpha}(I\!R, L^p(\Omega))$ for some $0 < \alpha \leq 1$ and $p > N/2$, then (3.1) defines an order preserving evolution operator $T_q(t,s)$ in $\mathcal{L}(Y)$ where $Y = L^r(\Omega)$, with $1 \leq r < \infty$ or $Y = C(\overline{\Omega})$.

Hence, we have

Proposition 3.1 *With the assumptions above assume that (3.1) has a CPND solution, $z(x,t)$, which is bounded above at $-\infty$. Then for $Y = L^r(\Omega)$, with $1 \leq r < \infty$ or $Y = C(\overline{\Omega})$ and for each $s \leq t \leq t_0$ we have*

$$C_0(t_0) \leq \|T_q(t,s)\|_{\mathcal{L}(Y)} \leq C_1(t_0) \tag{3.2}$$

where t_0 is as in Definition 2.2.

Proof First note that for each $u_0 \in C^1(\overline{\Omega})$, vanishing on $\partial\Omega$, there exists $\lambda = \lambda(u_0)$ such that $|u_0(x)| \leq \lambda\varphi_0(x)$ in Ω, where φ_0 is as in Definition 2.2. Then by comparison, we have for each $s \leq t \leq t_0$

$$|T_q(t,s)u_0(x)| \leq \lambda T_q(t,s)\varphi_0(x) \leq \lambda T_q(t,s)z(s)(x) = \lambda z(x,t)$$

and then

$$\|T_q(t,s)u_0\|_Y \leq \lambda\|z(t)\|_Y \leq \lambda \sup_{t\leq t_0} \|z(t)\|_Y.$$

Hence, $T_q(t,s)$ is pointwise bounded in a dense subset of Y and hence, from the Uniform Boundedness Principle we get the upper bound on $\|T_q(t,s)\|_{\mathcal{L}(Y)}$.

On the other hand, we have for $s \leq t \leq t_0$

$$0 < \varphi_0(x) \leq z(x,t) = T_q(t,s)z(s)(x)$$

and then

$$\|\varphi_0\|_Y \leq \|T_q(t,s)\|_{\mathcal{L}(Y)}\|z(s)\|_Y$$

and we get the lower bound on $\|T_q(t,s)\|_{\mathcal{L}(Y)}$. □

As a consequence we get for (2.1)

Corollary 3.2 *Under assumption (2.2), assume $0 \leq u_1(t) \leq u_2(t)$ are CPNDS at $-\infty$ of (2.1) and bounded above.*

Then if

$$\lim_{t\to-\infty} (u_2(t) - u_1(t)) = 0 \qquad in \quad L^1(\Omega)$$

then $u_1(t) = u_2(t)$ for all $t \in I\!R$.

Proof If we denote $q_i(x,t) := \dfrac{f(t,x,u_i(x,t))}{u_i(x,t)}$, for $i = 1,2$, then, from (2.2), $q_1(x,t) \geq q_2(x,t)$ and we have

$$(u_1)_t - \Delta u_1 = q_1(x,t)u_1 \geq q_2(x,t)u_1.$$

In particular $u_1(t) \geq T_{q_2}(t,s)u_1(s)$ and since $u_2(t) = T_{q_2}(t,s)u_2(s)$, we get

$$u_1(t) - u_2(t) \geq T_{q_2}(t,s)(u_1(s) - u_2(s)).$$

Then from Proposition 3.1 and the assumption on u_1, u_2, taking limits as $s \to -\infty$, we get $u_1(t) - u_2(t) \geq 0$ for $t \leq t_0$. Hence $u_1(t) = u_2(t)$ for all $t \leq t_0$. Also from here equality for $t \geq t_0$ follows.□

Hence, in what follows we can assume $0 \leq u_1(t) \leq u_2(t)$ are CPNDS at $-\infty$ of (2.1) and bounded above such that

$$0 \leq u_2(t) - u_1(t) \not\to 0 \qquad \text{in} \quad L^1(\Omega)$$

as $t \to -\infty$. Note that, from the Lebesgue's dominated convergence theorem, since the solutions are bounded above, the condition above is equivalent to

$$0 \leq u_2(x,t) - u_1(x,t) \not\to 0 \qquad \text{a.e.} \quad x \in \Omega,$$

as $t \to -\infty$. Furthermore, from the compactness assumption (2.14) as $t \to -\infty$, then we can assume that for some set of positive measure $A \subset \Omega$ and some $\alpha > 0$ we have

$$\liminf_{t \to -\infty} (u_2(x,t) - u_1(x,t)) \geq \alpha > 0.$$

Indeed, in the opposite case we would have $\liminf_{t\to-\infty} (u_2(x,t) - u_1(x,t)) = 0$ a.e. $x \in \Omega$ and then the compactness assumption (2.14) as $t \to -\infty$ implies that, along subsequences if necessary $\lim_{t\to-\infty} (u_2(x,t) - u_1(x,t)) = 0$ a.e. $x \in \Omega$ and then convergence in $L^1(\Omega)$ follows, which is a contradiction.

Then the equation for $u_2(t)$ can be written as

$$(u_2)_t - \Delta u_2 = q_2(x,t)u_2 = (q_1(x,t) - P(x,t))u_2$$

where

$$P(x,t) = (q_1 - q_2)(x,t) \geq 0$$

satisfies (2.13) as $t \to -\infty$. Also, from assumption (2.11) we have $q_1(x,t) \geq q_0(x)$ for all $t \leq t_0$, for every $x \in \Omega$ and for some $q_0 \in L^p(\Omega)$ for some $p > N/2$.

Therefore Theorem 2.3 will be proved as soon as we prove Proposition 3.3 below, since it implies that u_2 is not a bounded CPNDS of (2.1), which is a contradiction. Note that the result below implies that if a linear problem has a CPNDS bounded above at $-\infty$, then a sustained perturbation destroys this solution.

Proposition 3.3 *Assume for some $t_0 \in I\!R$*

$$q(x,t) \geq q_0(x) \qquad \textit{for all} \quad t \leq t_0$$

for every $x \in \Omega$ and for some $q_0 \in L^p(\Omega)$ for some $p > N/2$, and

$$\begin{cases} z_t - \Delta z &= q(x,t)z, \quad \textit{in } \Omega \\ z &= 0 \qquad\qquad \textit{on } \partial\Omega \end{cases}$$

has a CPNDS at $-\infty$ bounded above. Consider a time dependent linear perturbation $P(x,t) \geq 0$. Then

i) If for some set of positive measure $A \subset \Omega$ and some $\alpha > 0$ we have

$$\liminf_{t\to-\infty} P(x,t) \geq \alpha > 0 \quad \textit{a.e.} \quad x \in A$$

then the perturbed problem

$$\begin{cases} v_t - \Delta v + P(x,t)v &= q(x,t)v, \quad \textit{in } \Omega \\ v &= 0 \qquad\qquad \textit{on } \partial\Omega \end{cases} \tag{3.3}$$

does not have a CPNDS and bounded at $-\infty$.
ii) If $P(x,t) \geq \varphi_0(x)$ for some positive function in Ω, which may vanish on $\partial\Omega$, then $v = 0$ is the unique complete nonnegative and bounded at $-\infty$ solution of (3.3).

Proof Note first that since $P \geq 0$ then any complete solution of (3.3) is bounded above.
i) Assume on the contrary that there exists a CPNDS and bounded at $-\infty$ of (3.3), v. Then for $s < t \leq t_0$ we have from the variations of constants formula

$$0 \leq v(t) = T_q(t,s)v(s) - \int_s^t T_q(t,r)P(\cdot,r)v(r)\,dr \leq T_q(t,s)v(s). \tag{3.4}$$

Denote $J(s) = \int_s^t T_q(t,r)P(\cdot,r)v(r)\,dr \geq 0$ which increases as $s \downarrow -\infty$, since the integrand is nonnegative. Then from (3.4), Proposition 3.1 and the boundedness of v, we have that $J(s)$ is bounded in $L^p(\Omega)$ for $1 \leq p \leq \infty$, as $s \to -\infty$.

Therefore, by the monotone convergence theorem we have

$$0 \leq \int_{-\infty}^t T_q(t,r)P(\cdot,r)v(r)\,dr \in L^1(\Omega)$$

which is equivalent to

$$\int_{-\infty}^t \|T_q(t,r)P(\cdot,r)v(r)\|_{L^1(\Omega)}\,dr < \infty.$$

In particular, there exists a sequence $r_n \to -\infty$ such that, as $n \to \infty$,

$$\|T_q(t,r_n)P(\cdot,r_n)v(r_n)\|_{L^1(\Omega)} \to 0.$$

Since v is nondegenerate at $-\infty$ and the assumption on $P(x,t)$, we get

$$0 \leq \int_\Omega T_q(t,r_n)(\mathcal{X}_A\varphi_0)(x)\,dx \to 0 \tag{3.5}$$

where $\mathcal{X}_A$ denotes the characteristic function and φ_0 is as in Definition 2.2.

But from Lemma 3.5 below we have that there exists a $C^1(\overline{\Omega})$ function $\varphi_2(x) > 0$ in Ω, vanishing on $\partial\Omega$, such that

$$T_q(r_n+1,r_n)(\mathcal{X}_A\varphi_0)(x) \geq \varphi_2(x) \qquad \text{for all} \quad n.$$

Hence we can write, for $t > r_n + 1$,

$$T_q(t,r_n)(\mathcal{X}_A\varphi_0) = T_q(t,r_n+1)T_q(r_n+1,r_n)(\mathcal{X}_A\varphi_0) \geq T_q(t,r_n+1)\varphi_2 \geq 0$$

and the left hand side above converges to zero in $L^1(\Omega)$ as $n \to \infty$.

But since z is a CPNDS at $-\infty$ bounded above, then there exists $\lambda > 0$ and φ_1 as in Definition 2.2 such that for all $n \in I\!N$, $z(r_n+1) \leq \varphi_1 \leq \lambda\varphi_2$ and then

$$0 < \varphi_0 \leq z(t) = T_q(t,r_n+1)z(r_n+1) \leq \lambda T_q(t,r_n+1)\varphi_2 \to 0 \quad \text{in} \quad L^1(\Omega)$$

as $n \to \infty$, which is a contradiction.
ii) Assume on the contrary that (3.3) has a bounded CP solution and there exists a set of positive measure $A \subset \Omega$ and some $\alpha > 0$ we have

$$\liminf_{t \to -\infty} v(x,t) \geq \alpha > 0.$$

Then using the assumption on P and arguing as in i) above we get (3.5) and we reach a contradiction as before. Hence, we can assume $\liminf_{t \to -\infty} v(x,t) = 0$ a.e $x \in \Omega$. But denoting $U(t,s)$ the evolution operator associated to (3.3), we can write $v(t) = U(t,t-1)v(t-1)$ and then from the boundedness of v and smoothing estimates we get that $v(t)$ is precompact in $L^1(\Omega)$ as $t \to -\infty$. But then using sequences if necessary we get that v must converge to zero in $L^1(\Omega)$ as $t \to -\infty$. But then from the second inequality in (3.4) and Proposition 3.1 we get

$$\|v(t)\| \leq C\|v(s)\| \to 0 \qquad \text{as} \quad s \to -\infty$$

i.e. $v(t) = 0$.□

Remark 3.4 *Note that in fact the last part of the proof of point i) of the Proposition above shows that for any nonnegative nontrivial initial data u_0 and $t \in I\!R$ the curve $\{T_q(t,s)u_0,\ s < t\}$ is nondegenerate at $-\infty$. From this a contradiction is obtained with (3.5).*

Now we prove the following nondegeneracy result used above.

Lemma 3.5 *Assume that for some unbounded interval $I \subset I\!R$ we have*

$$q(x,t) \geq q_0(x) \qquad \textit{for all} \quad t \in I$$

for every $x \in \Omega$ and for some $q_0 \in L^p(\Omega)$ for some $p > N/2$.

Then for any $0 \leq \xi \in L^1(\Omega)$ there exists a $C^1(\overline{\Omega})$ function $\varphi(x) > 0$ in Ω, vanishing on $\partial\Omega$, such that

$$T_q(s+1,s)\xi \geq \varphi \tag{3.6}$$

for all s such that $s, s+1 \in I$.

Proof Just note that from comparison

$$T_q(s+1,s)\xi \geq T_{q_0}(s+1,s)\xi = S_{q_0}(1)\xi = \varphi$$

where $S_{q_0}(t)$ denotes the semigroup associated to the autonomous linear equation

$$\begin{cases} z_t - \Delta z &= q_0(x)z, & \text{in } \Omega \\ z &= 0 & \text{on } \partial\Omega. \end{cases}$$

Now observe that smoothing implies that id $\xi \in L^1(\Omega)$ then $S_{q_0}(1)\xi \in C^1(\overline{\Omega})$.□

Note that the assumption in the Lemma reads $q(x,t) = q^+(x,t) - q^-(x,t) \geq q_0(x)$, which gives some restriction on the size of q^- and this is also necessary to get the result. Indeed if $q^-(x,t)$ can become very large in some subset of Ω as $t \in (s,s+1) \subset I$ and $|s| \to \infty$, it is easy to see that $T_q(s+1,s)\xi$ must approach zero on this set and the conclusion of the lemma fails.

4 Forward behavior of positive solutions

In this section we discuss the forward asymptotic behavior in time of positive solutions of (2.1). In particular we show that under assumption (2.2) all bounded positive solutions of (2.1) have the same asymptotic behavior as $t \to \infty$, i.e. we prove Theorem 2.4.

We start with the following result on linear equations.

Proposition 4.1 *Let $T_q(t,s)$ be the evolution operator associated to equation (3.1).*
i) Assume for some $s \in I\!R$ there exist a positive bounded solution of (3.1), $z(t)$, for $t > s$.

Then all solutions of (3.1) are bounded for $t > s$ and for all $t \geq t_0 \geq s$ we have

$$\|T_q(t,t_0)\|_{\mathcal{L}(Y)} \leq C_1(t_0)$$

for $Y = L^r(\Omega)$, with $1 \leq r < \infty$ or $Y = C(\overline{\Omega})$.

In particular, either all positive solutions of (3.1) are bounded or unbounded for $t > s$.
ii) Assume moreover the solution in i) is such that $z(t) \to 0$ in $L^\infty(\Omega)$ as $t \to \infty$. Then all solutions of (3.1) converge to zero uniformly in Ω as $t \to \infty$.
iii) If $z(t)$ for $t > s$ is a positive bounded solution of (3.1), then $z(t) \to 0$ in $L^\infty(\Omega)$ as $t \to \infty$ if and only if

$$z(x,t_k) \to 0 \quad a.e. \quad x \in \Omega$$

for some subsequence $t_k \to \infty$.
iv) Assume for some $s \in I\!R$ there exists a positive non degenerate solution, PND at ∞ of (3.1), z, in the sense of Definition 2.2.

Then all nontrivial solutions of (3.1) are PND at ∞ and for any $t \geq t_0 \geq s$ we have

$$C_0(t_0) \leq \|T_q(t,t_0)\|_{\mathcal{L}(Y)}$$

for $Y = L^r(\Omega)$, with $1 \leq r < \infty$ or $Y = C(\overline{\Omega})$.

Proof
i) and ii). It is clear that, for any $t > s$, if $0 \leq v_0 \leq z(s)$ then $0 \leq v(t) = T_q(t,s)v_0 \leq T_q(t,s)z(s) = z(t)$ and then v is also bounded or tends to zero as $t \to \infty$.

On the other hand, we can always assume $v_0 \in C^1_0(\overline{\Omega})$, since otherwise we shift the initial time to the right and use the smoothing effect of the evolution operator. Then for any $t_0 \geq s$ there exists $0 < \lambda = \lambda(v_0,t_0)$, such that $|v_0(x)| \leq \lambda z(x,t_0)$ and then

$$|v(x,t)| = |T_q(t,t_0)v_0(x)| \leq \lambda T_q(t,t_0)z(t_0)(x) = \lambda z(x,t)$$

and then

$$\|v(t)\|_Y \leq \lambda \|z(t)\|_Y$$

which is bounded or converges to zero as $t \to \infty$. Moreover

$$\|T_q(t,t_0)v_0\|_Y \leq \lambda \|z(t)\|_Y \leq \lambda \sup_{t\geq t_0} \|z(t)\|_Y.$$

Hence, $T_q(t,t_0)$ is pointwise bounded in a dense subset of Y and hence, from the Uniform Boundedness Principle we get the upper bound on $\|T_q(t,t_0)\|_{\mathcal{L}(Y)}$.

iii) Since z is bounded in $L^\infty(\Omega)$ for $t > s$, then if $z(x,t_k) \to 0$ a.e. $x \in \Omega$ for some subsequence $t_k \to \infty$ then Lebesgue's theorem implies that $\|z(t_k)\|_{L^1(\Omega)} \to 0$. Then from the results in [14] we get that

$$\|z(t)\|_{L^\infty(\Omega)} \leq \begin{cases} \dfrac{C}{(t-t_k)^{N/2}}\|z(t_k)\|_{L^1(\Omega)} & \text{for } t_k < t < t_k + 1 \\ C\|z(t_k)\|_{L^1(\Omega)} & \text{for } t > t_k + 1 \end{cases} \tag{4.1}$$

for some C independent of k and then taking $t > t_k + 1$ the result follows.
iv) Since $0 \leq \varphi_0 \leq z(t) = T_q(t,t_0)z(t_0)$ then we have

$$\|\varphi_0\|_Y \leq \|T_q(t,t_0)\|_{\mathcal{L}(Y)}\|z(t_0)\|_Y$$

and we get the estimate.

On the other hand let now $v_0 \geq 0$ and observe that for $t > s+1$ we have $v(t) = T_q(t, s+1)T_q(s+1,s)v_0$ and $w_0 = T_q(s+1,s)v_0 \in C^1_0(\overline{\Omega})$ is positive in Ω. Then there exists $\delta > 0$ such that $w_0 \geq \delta z(s+1)$ and then

$$v(t) = T_q(t,s)w_0 \geq \delta T_q(t,s+1)z(s+1) = \delta z(t) \geq \delta\varphi_0$$

Hence v is a PNDS at ∞.□

Now we consider nonnegative solutions of the nonlinear problem (2.1) under assumption (2.2). Then we have the following result that states that all the statements of the Proposition 4.1 above remain true for the nonnegative solutions of (2.1). More precisely

Corollary 4.2 *Assume (2.2).*
i) Assume for some $s \in I\!R$ there exist a positive bounded solution of (2.1), $u(t)$, for $t > s$. Then all solutions of (2.1) are bounded for $t > s$.

In particular, either all positive solutions of (2.1) are bounded or unbounded for $t > s$.
ii) Assume moreover the solution in i) is such that $u(t) \to 0$ in $L^\infty(\Omega)$ as $t \to \infty$. Then all solutions of (2.1) converge to zero uniformly in Ω as $t \to \infty$.
iii) If $u_1(t), u_2(t)$ for $t > s$ are positive bounded solutions of (2.1) and

$$u_1(x,t_k) - u_2(x,t_k) \to 0 \quad a.e. \quad x \in \Omega$$

for some subsequence $t_k \to \infty$ then

$$u_1(x,t) - u_2(x,t) \to 0 \quad \text{uniformly in} \quad x \in \Omega \quad \text{as} \quad t \to \infty.$$

iv) Assume for some $s \in I\!R$ there exists a positive non degenerate solution, PND at ∞ of (2.1), u, in the sense of Definition 2.2.

Then all nontrivial solutions of (2.1) are PND at ∞.

Proof
i) and ii). If for $t > s$, $u(t)$ is a positive bounded solution of (2.1) then if $v(t)$ is another solution such that $v(t) \leq u(t)$ then the result is obvious. On the other hand, if $v(t) \geq u(t)$ then

$$v_t - \Delta v = \frac{f(t,x,v)}{v}v \leq \frac{f(t,x,u)}{u}v = q(x,t)v.$$

Hence

$$0 \leq v(t) \leq T_q(t,s)v(s) \tag{4.2}$$

and $T_q(t,s)$ satisfies the assumptions i) or ii) respectively of Proposition 4.1 (with $z(t) = u(t)$) and the result follows.
iii) Note that we can always assume $u_1(t) \leq u_2(t)$ and then from (4.2) applied to $v = u_2$ and $q(x,t) = \dfrac{f(t,x,u_1)}{u_1}$ and using $u_1(t) = T_q(t,t_k)u_1(t_k)$ we get

$$0 \leq u_2(t) - u_1(t) \leq T_q(t,t_k)(u_2(t_k) - u_1(t_k))$$

and using (4.1) we get the result.
iv) If $u(t)$ for $t > s$ is a PND solution of (2.1) and $v(t)$ is another solution such that $v(t) \geq u(t)$ then it is clear that it is also a PND solution. On the other hand if $0 \leq v(t) \leq u(t)$ arguing as above we get now

$$v_t - \Delta v = \frac{f(t,x,v)}{v} v \geq \frac{f(t,x,u)}{u} v = q(x,t)v.$$

Hence $v(t) \geq T_q(t,s)v(s)$ and the result follows from the Proposition. □

Therefore, we assume hereafter that (2.1) has a bounded PND solution and hence all nontrivial solutions are bounded PND. Then, assuming the the nondegeneracy conditions (2.11) and (2.13) and the compactness assumptions (2.14) as $t \to \infty$, we are in a position to prove the main result of this section, Theorem 2.4.

Proof of Theorem 2.4

Assume $0 \leq u_1(t) \leq u_2(t)$ are positive, nondegenerate at ∞, bounded above, solutions of (2.1). From Corollary 4.2 iii), if $\lim_{t\to\infty}(u_2(x,t) - u_1(x,t)) = 0$ a.e. $x \in \Omega$ then the result follows. Hence, from the compactness assumption (2.14) we can assume that for some set of positive measure $A \subset \Omega$ and some $\alpha > 0$ we have

$$\liminf_{t\to\infty} (u_2(x,t) - u_1(x,t)) \geq \alpha > 0.$$

Then the equation for $u_2(t)$ can be written as

$$(u_2)_t - \Delta u_2 = q_2(x,t)u_2 = (q_1(x,t) - P(x,t))u_2$$

where,

$$P(x,t) = (q_1 - q_2)(x,t) \geq 0$$

satisfies assumption (2.13).

Also, from assumption (2.11) we have $q_1(x,t) \geq q_0(x)$ for all $t \geq t_0$, for every $x \in \Omega$ and for some $q_0 \in L^p(\Omega)$ for some $p > N/2$.

Then from Proposition 4.3 below we obtain that $u_2(t)$ is not a PND solution which is a contradiction.□

Now we state and prove the Proposition used in the proof above which is interesting by itself. Note that this result is the analogous one, but at ∞, of Proposition 3.3 and states that sustained perturbations destroy the existence of nondegenerate solutions.

Proposition 4.3 *Assume for some* $t_0 \in I\!R$

$$q(x,t) \geq q_0(x) \qquad \textit{for all} \quad t \geq t_0$$

for every $x \in \Omega$ and for some $q_0 \in L^p(\Omega)$ for some $p > N/2$, and

$$\begin{cases} z_t - \Delta z &= q(x,t)z, \quad \text{in } \Omega \\ z &= 0 \qquad\qquad \text{on } \partial\Omega \end{cases}$$

has a PND solution bounded above for $t > s$. Consider a time dependent perturbation $P(x,t) \geq 0$. Then

i) If for some set of positive measure $A \subset \Omega$ and some $\alpha > 0$ we have

$$\liminf_{t\to\infty} P(x,t) \geq \alpha > 0 \qquad a.e. \quad x \in A.$$

then the perturbed problem

$$\begin{cases} v_t - \Delta v + P(x,t)v &= q(x,t)v, \quad \text{in } \Omega \\ v &= 0 \qquad\qquad \text{on } \partial\Omega \end{cases} \tag{4.3}$$

does not have a PND solution at ∞.

ii) If $P(x,t) \geq \varphi_0(x)$ for some positive function in Ω, which may vanish on $\partial\Omega$, then all solutions of (4.3) decay to zero uniformly in Ω, as $t \to \infty$.

Proof Note that since $P \geq 0$ then all solutions of (4.3) are bounded above.

i) Assume on the contrary that there exist a PNDS of (4.3), v (which must be necessarily bounded). Then for $t \geq t_0 > s$ we have from the variations of constants formula

$$0 \leq v(t) = T_q(t,t_0)v(t_0) - \int_{t_0}^t T_q(t,r)P(\cdot,r)v(r)\,dr \leq T_q(t,t_0)v(t_0). \tag{4.4}$$

Denoting $J(t) = \int_{t_0}^t T_q(t,r)P(\cdot,r)v(r)\,dr \geq 0$ we have, from (4.4) and the assumptions, that $J(t)$ is bounded in $L^p(\Omega)$ for $1 \leq p \leq \infty$, as $t \to \infty$.

In particular,

$$0 \leq \int_{t_0}^t \int_\Omega T_q(t,r)P(x,r)v(x,r)\,dxdr \leq C.$$

Since v is nondegenerate at ∞ and the assumption on $P(x,t)$, we have that for some set of positive measure $A \subset \Omega$ we have, if t_0 is sufficiently large,

$$0 \leq \int_{t_0}^{t-1} \int_\Omega T_q(t,r)(\mathcal{X}_A\varphi_0)(x)\,dxdr \leq \int_{t_0}^t \int_\Omega T_q(t,r)(\mathcal{X}_A\varphi_0)(x)\,dxdr \leq C \tag{4.5}$$

where $\mathcal{X}_A$ denotes the characteristic function and φ_0 is as in Definition 2.2.

But note that we can write, for $t > r+1 \geq t_0+1$,

$$T_q(t,r)(\mathcal{X}_A\varphi_0)(x) = T_q(t,r+1)T_q(r+1,r)(\mathcal{X}_A\varphi_0)(x)$$

and from Lemma 3.5 we have

$$T_q(r+1,r)(\mathcal{X}_A\varphi_0)(x) \geq \varphi_2(x)$$

for some $C^1(\overline{\Omega})$ function $\varphi_2(x) > 0$ in Ω, vanishing on $\partial\Omega$. Now, since z is a PND solution bounded above there exists φ_1, as in Definition 2.2, and some $\delta > 0$ independent of r such that, $\varphi_2(x) \geq \delta\varphi_1(x) \geq \delta z(x,r+1)$ which implies that

$$T_q(t,r)(\mathcal{X}_A\varphi_0)(x) \geq \delta T_q(t,r+1)z(x,r+1) = \delta z(x,t) \geq \delta\varphi(x)$$

for some $C^1(\overline{\Omega})$ function $\varphi(x) > 0$ in Ω, vanishing on $\partial\Omega$. Hence we get

$$0 \leq \int_{t_0}^{t-1} \int_\Omega \varphi(x)\, dxdr \leq C$$

which is a contradiction.
ii) Assume on the contrary that v is a nonnegative solution of (4.3) (which must necessarily be bounded) and there exists a set of positive measure $A \subset \Omega$ and some $\alpha > 0$ we have

$$\liminf_{t\to\infty} v(x,t) \geq \alpha > 0.$$

Then using the assumption on P and arguing as in i) above we get (4.5) and we reach a contradiction as before.

Hence, we can assume $\liminf_{t\to\infty} v(x,t) = 0$ a.e $x \in \Omega$. Then denoting $U(t,s)$ the evolution operator associated to (4.3), we can write $v(t) = U(t,t_0)v(t_0)$ and then from smoothing estimates we get that $v(t)$ is precompact in $L^1(\Omega)$ as $t \to \infty$. But then using sequences if necessary we get that v must converge to zero in $L^1(\Omega)$ as $t \to \infty$. But then from the second inequality in (4.4) and (4.1) we get, for $t > t_0 + 1$,

$$\|v(t)\|_{L^\infty(\Omega)} \leq C\|v(t_0)\|_{L^1(\Omega)} \to 0 \qquad \text{as} \quad t_0 \to \infty.\square$$

5 Existence of complete positive nondegenerate trajectories

In this section we address the question of existence of complete nonnegative or positive trajectories for (2.1). We also give conditions guaranteeing that the complete trajectories are nondegenerate and bounded above at $\pm\infty$. In particular we prove Theorem 2.5.

We start with the case in which the asymptotic dynamics of (2.1) is trivial and no nonnegative complete trajectories exist at all. In fact as a consequence of the arguments above, we have

Theorem 5.1 *Assume $f(t,x,0) = 0$, (2.2) is satisfied and*

$$m(x,t) = \liminf_{u\to 0^+} \frac{f(t,x,u)}{u}$$

is such that the linear evolution operator $T_m(t,s)$, as in (3.1), is exponentially stable.

Then all nonnegative solutions of (2.1) satisfy $||u(t,s;u_0)||_{L^\infty(\Omega)} \to 0$ as $t \to +\infty$ or $s \to -\infty$ uniformly for u_0 in bounded sets of $X = C(\overline{\Omega})$.

In particular $u(t) = 0$ is the unique complete nonnegative solution and attracts all solutions of (2.1) both in the pullback and forward senses.

Proof If $0 \leq u(x,t)$ is a nonnegative solution of (2.1), then we have

$$u_t - \Delta u = \frac{f(t,x,u)}{u} u \leq m(x,t)u$$

and then $0 \leq u(t) \leq T_m(t,s)u_0$ and the result follows.$\square$

Now , let $\beta > 0$, we define $\mathcal{D} = \mathcal{D}_\beta(I\!R, X)$, the "basin of attraction", consisting of families of bounded sets in X with at most exponential growth less than β at $-\infty$, i.e., families of bounded sets of the form $\{B(t)\}_t$ such that for all $t_0 \in I\!R$ there exists $M_1(t_0) \geq 1$ with

$$e^{\gamma t}||B(t)||_X = \sup_{b\in B(t)} e^{\gamma t}||b||_X \leq M_1(t_0) \quad \text{for all } t \leq t_0,$$

for certain constant $\beta > \gamma > 0$.

Now, quoting the results in [14] we get the following result that guarantees the existence of a complete nonnegative trajectory. Moreover this trajectory is maximal among all complete nonnegative trajectories and moreover is stable from above in the pullback sense. See [14] for further details. Note that we do not use assumption (2.2) here.

Theorem 5.2 *Assume $f(t,x,0) \geq 0$ and*

$$f(t,x,u) \leq C(x,t)u + D(x,t) \qquad \text{for} \quad u \geq 0, \tag{5.1}$$

such that the evolution operator $T_C(t,s)$ as in (3.1), is exponentially stable with exponent $\beta > 0$. Then under each one of the following assumptions on D there exists a maximal complete trajectory $\varphi_M \in \mathcal{D}_\beta = \mathcal{D}_\beta(I\!R, C_0(\overline{\Omega}))$ in the sense that any other complete nonnegative trajectory for (2.1) in $\mathcal{D}_\beta$, ψ, satisfies $0 \leq \psi(t) \leq \varphi_M(t)$ for all $t \in I\!R$.

i) $D \in \mathcal{D}_\beta(I\!R, L^r(\Omega))$ with $N/2 < r \leq \infty$; in such a case $\varphi_M \in \mathcal{D}_\beta(I\!R, C_0(\overline{\Omega}))$, or

ii) for any $T < \infty$, $D \in L^\sigma((-\infty,T), L^r(\Omega))$ with $1 < \sigma < \infty$ and $N\sigma'/2 < r \leq \infty$, or $N/2 < r \leq \infty$ if $\sigma = \infty$, or $r = \infty$ if $\sigma = 1$; in such a case $\varphi_M \in L^\infty((-\infty,T), C_0(\overline{\Omega})) \subset \mathcal{D}_\beta(I\!R, C_0(\overline{\Omega}))$.

Furthermore, in each of the cases above, the order intervals $I(t) = [0, \varphi_M(t)]$ are forward invariant and attract the dynamics of the positive solutions of (2.1) uniformly in the pullback sense, i.e., for all $t \in I\!R$ we have

$$0 \leq \liminf_{s\to-\infty} u(t,s,x;v_s) \leq \limsup_{s\to-\infty} u(t,s,x;v_s) \leq \varphi_M(x,t) \tag{5.2}$$

uniformly in $x \in \overline{\Omega}$ for all $0 \leq \{v_s\}$ in $\{B(s)\}_s \in \mathcal{D} = \mathcal{D}_\beta$.

Moreover, $\varphi_M(t)$ is globally asymptotically stable from above in pullback sense, i.e., for all $v \in \mathcal{D}_\beta(I\!R, C_0(\overline{\Omega}))$, $v \geq \varphi_M$ we have

$$\lim_{s\to-\infty} u(t,s;v(s)) = \varphi_M(t).$$

Finally $\varphi_M(t)$ is T–periodic if $f(t+T,x,u) = f(t,x,u)$. □

Note that in general the maximal solution $\varphi_M(t)$ constructed above might not be bounded as t goes to $\pm\infty$. See [10] and [14]. Note that with the notations in [14] one can say that as a consequence of the Theorem, there exists a pullback attractor for nonnegative solutions of (2.1) with basin of attraction $\mathcal{D}_\beta$, denoted by $\mathcal{A} = \{\mathcal{A}(t)\}_t$, and

$$\mathcal{A}(t) \subset I(t) = [0, \varphi_M(t)] \quad \text{for all } t \in I\!R.$$

Moreover, $\varphi_M(t) \in \mathcal{A}(t)$ for all $t \in I\!R$.

Now we give conditions to have that a complete trajectory for (2.1) is bounded above at $\pm\infty$ in the sense of Definition 2.2.

Proposition 5.3 *i) Assume that a complete trajectory for (2.1), $u(t)$, satisfies*

$$u \in L^\infty((-\infty,T), C_0(\overline{\Omega}))$$

for some $T \in I\!R$ and $f(\cdot, u(\cdot))$ remains in a bounded set of $L^p(\Omega)$ for some $p > N$. Then u is bounded above and precompact in $L^1(\Omega)$ at $-\infty$.

ii) Assume that a trajectory for (2.1), $u(t)$, satisfies

$$u \in L^{\infty}((T,\infty), C_0(\overline{\Omega}))$$

for some $T \in I\!R$ and $f(\cdot, u(\cdot))$ remains in a bounded set of $L^p(\Omega)$ for some $p > N$. Then u is bounded above and precompact in $L^1(\Omega)$ at ∞.

Proof As observed in the introduction to prove that u is bounded above it is enough to prove that it is bounded in $C^1(\overline{\Omega})$. Then we have
i) Note that for any $t < T$ we have $u(t) = U(t,s)u(s) = U(t,t-1)u(t-1)$ and $u(t-1)$ remains in a bounded set of $C_0(\overline{\Omega})$. Therefore it is enough to show that $U(t,t-1)$ transforms a bounded set in $C_0(\overline{\Omega})$ into a bounded set of $C^1(\overline{\Omega})$ independent of t. For this note that if $\eta \in B \subset C_0(\overline{\Omega})$ is a bounded set then from the variation of constants formula for (2.1) we have

$$U(t,t-1)\eta = \mathrm{e}^{\Delta}\eta + \int_{t-1}^{t} \mathrm{e}^{\Delta(t-\tau)} f(\tau, U(\tau,t-1)\eta)\, \mathrm{d}\tau. \tag{5.3}$$

Since $f(\tau, U(\tau,t-1)\eta)$ remains in a bounded set of $L^p(\Omega)$ for some $p > N$ independent of t, the smoothing estimates for the heat semigroup give the result.
ii) Now for any $t > s > T$ we have

$$u(t) = \mathrm{e}^{\Delta(t-s)}u(s) + \int_{s}^{t} \mathrm{e}^{\Delta(t-\tau)} f(\tau, u(\tau))\, \mathrm{d}\tau. \tag{5.4}$$

Again $f(\tau, u(\tau))$ remains in a bounded set of $L^p(\Omega)$ for some $p > N$ independent of t and the exponential bounds on the heat semigroup give the result. □

In particular, using this and the results in [14], we have

Corollary 5.4 *Under the conditions of Theorem 5.2 and the assumptions on f of Proposition 5.3*
i) Assume that D satisfies for any $T < \infty$, $D \in L^{\sigma}((-\infty,T), L^r(\Omega))$ with $1 < \sigma < \infty$ and $N\sigma'/2 < r \leq \infty$, or $N/2 < r \leq \infty$ if $\sigma = \infty$, or $r = \infty$ if $\sigma = 1$.

Then $\varphi_M \in L^{\infty}((-\infty,T), C_0(\overline{\Omega})) \subset \mathcal{D}_{\beta}(I\!R, C_0(\overline{\Omega}))$ is bounded above at $-\infty$.
ii) Assume D satisfies either one of the assumptions in cases i) or ii) of Theorem 5.2 and moreover $D \in L^{\sigma}((T,\infty), L^r(\Omega))$, for each $T > -\infty$, with σ and r as in case i) above.

Then $\varphi_M \in \mathcal{D}_{\beta}(I\!R, C_0(\overline{\Omega})) \cap L^{\infty}((T,\infty), C_0(\overline{\Omega}))$ and is bounded above at ∞.

Remark 5.5 *Notice that Corollary 5.6 in [14] implies that in any of the cases of Corollary 5.4 above, if $1 \leq \sigma < \infty$ then*

$$\varphi_M(t) \to 0$$

as t goes to $-\infty$ or ∞. The same holds if $\sigma = \infty$ and $\|D(t)\|_{L^r(\Omega)} \to 0$ as $t \to \pm\infty$. Such cases are not included in Theorem 5.1.

Therefore we will assume henceforth that $\sigma = \infty$ and $N/2 < r \leq \infty$, that is,

$$D \in L^{\infty}((-\infty,T), L^r(\Omega)) \quad \text{or} \quad D \in L^{\infty}((T,\infty), L^r(\Omega)) \tag{5.5}$$

and $D(t)$ does not go to zero in $L^r(\Omega)$ as $t \to \pm\infty$.

Now, we turn our attention to the question of nondegeneracy of positive solutions of (2.1) at $\pm\infty$. For this, in what follows we will make use of some properties of solutions of the autonomous nonlinear equation

$$\begin{cases} w_t - \Delta w &= f_0(x,w) \quad \text{in} \quad \Omega \\ w &= 0 \quad\quad\quad\;\; \text{on} \quad \partial\Omega \\ w(0) &= u_0 \end{cases} \tag{5.6}$$

whose solutions are denoted $w(t;u_0)$, assuming $f_0(x,0) \geq 0$ and

$$\frac{f_0(x,u)}{u} \quad \text{is nonincreasing for} \quad u \geq 0. \tag{5.7}$$

Namely, under suitable conditions on f_0, this problem has a unique positive equilibrium, $u_E^0(x) > 0$ in Ω, (see Section 7 and [16], [19], [15] and [1]). Moreover $u_E^0(x)$ is globally asymptotically stable, i.e., for any bounded set B of positive functions away from zero, we have

$$w(t;u_0) \to u_E^0 \quad \text{as} \quad t \to \infty \tag{5.8}$$

uniformly for all $0 \leq u_0 \in B$.

Then concerning non–degeneracy of solutions at $-\infty$ we have the following result. Note that this results states not only the non–degeneracy but the asymptotic stability from below in a pullback sense of some minimal positive solution of (2.1). Also note that assumption (2.2) is not used here.

Theorem 5.6 *Assume the conditions in Theorem 5.2 hold and that for* $t \leq t_0 \leq \infty$

$$f_0(x,u) \leq f(t,x,u) \quad \text{for all} \quad x \in \Omega, \; u \geq 0 \tag{5.9}$$

for some f_0 *satisfying (5.7) and (5.8).*

Then the maximal complete trajectory $\varphi_M(t)$ *constructed in Theorem 5.2 is a CPNDS at* $-\infty$.

Moreover there exists a minimal CPNDS at $-\infty$ *that we denote by* $\varphi_m \leq \varphi_M$.

Furthermore the order intervals, $I^+(t) = [\varphi_m(t), \varphi_M(t)]$ *are forward invariant for (2.1), i.e.,*

$$U(t,s)I^+(s) \subset I^+(t) \quad \text{for all} \quad s < t$$

and attracts the dynamics of the system uniformly in the pullback sense, i.e. for all $t \in \mathbb{R}$ *we have*

$$0 \leq \varphi_m(t) \leq \liminf_{s\to-\infty} u(t,s;u_0) \leq \limsup_{s\to-\infty} u(t,s;u_0) \leq \varphi_M(t) \tag{5.10}$$

uniformly for $u_0 > 0$ *in bounded sets of positive initial data bounded away from zero.*

Even more, $\varphi_m(t)$ *is asymptotically stable from below in the pullback sense for* $t \in \mathbb{R}$ *that is, for any nondegenerate* $v \in C_b(\mathbb{R}, X)$ *at* $-\infty$ *such that* $v(s) \leq \varphi_m(s)$ *we have*

$$\lim_{s\to-\infty} u(t,x;v(s)) = \varphi_m(t).$$

Finally, if $t_0 = \infty$ *then* φ_m *and* φ_M *are CPNDS at* ∞.

Proof Note that from comparison, we have for any nonnegative initial data and $s < t \leq t_0$

$$0 \leq w(t-s;u_0) \leq u(t,s;u_0)$$

where w is the solution of the problem (5.6).

Consider now $u_E^0(x)$ the only positive equilibrium solution of (5.6) and notice that $u_E^0(x)$ can be considered as a subtrajectory for (2.1) since, by comparison

$$0 \leq u_E^0(x) = w(r-s;u_E^0(x)) \leq U(r,s)u_E^0(x) \tag{5.11}$$

for all $s \leq r < t_0$. Thus, letting the evolution operator $U(t,r)$ act in (5.11) we have,

$$U(t,r)u_E^0(x) \leq U(t,r)U(r,s)u_E^0(x) = U(t,s)u_E^0(x) \tag{5.12}$$

for all $s \leq r \leq t \leq t_0$. Thus, we have that $0 \leq u(t,s;u_E^0) = U(t,s)u_E^0$ is increasing as s goes to $-\infty$.

Now since hypotheses in Theorem 5.2 hold, we have, in particular, the existence of a maximal non-negative complete trajectory $\varphi_M(t)$. So, taking limits in (5.11) as $s \to -\infty$ we have from (5.2)

$$0 \leq u_E^0 \leq \liminf_{s\to-\infty} u(t,s;u_E^0) \leq \varphi_M(t) \qquad \text{for} \quad t \leq t_0. \tag{5.13}$$

In particular φ_M is non-degenerate at $-\infty$.

Moreover, we have that $u(t,s;u_E^0) \geq 0$ is bounded from above and increasing as $s \to -\infty$ so $u(t,s;u_E^0) \to \varphi_m(t)$ in $X = C(\overline{\Omega})$ as s goes to $-\infty$ for some $\varphi_m(t)$ (see [10] and [14]). By (5.13), φ_m satisfies

$$0 \leq u_E^0 \leq \varphi_m(t) \leq \varphi_M(t) \qquad \text{for} \quad t \leq t_0.$$

Also, by the continuity of $U(t,s)$, $t > s$, following the same argument as in the proof of Theorem 7.1 in [14] it follows that $\varphi_m(t)$ is a complete trajectory for (7.1) and it is clearly non-degenerate at $-\infty$. From here we also get $0 \leq \varphi_m(t) \leq \varphi_M(t)$ for all $t \in \mathbb{R}$.

We now prove (5.10). First, we assume $t \leq t_0$. Let B a bounded subset of positive initial data bounded away from zero, i.e.,

$$\text{dist}_X(B,\{0\}) > 0.$$

Set $s < r < t \leq t_0$. Then, for all $u_0 \in B$ we have

$$w(r-s;u_0) \leq u(r,s;u_0) \quad \text{for all} \quad s < r < t_0 \tag{5.14}$$

and acting with the evolution operator on both sides we have for $s < r < t < t_0$

$$U(t,r)w(r-s;u_0) \leq U(t,r)u(r,s;u_0) = u(t,s;u_0).$$

Taking first, limit as s goes to $-\infty$ and using the asymptotic stability of u_E^0 for (5.6) (see (5.8)) plus the continuity of U, we have

$$0 \leq u_E^0 \leq U(t,r)u_E^0 \leq \liminf_{s\to-\infty} u(t,s;u_0).$$

Taking now the limit as r goes to $-\infty$ we have

$$0 \leq u_E^0 \leq \varphi_m(t) \leq \liminf_{s\to-\infty} u(t,s;u_0) \tag{5.15}$$

where the limit is uniform for u_0 in B. This and (5.2) gives (5.10).

Assume now that $t > t_0$. Then, acting with the evolution operator on both sides of (5.14) we have for $r < \tau < t_0 < t$,

$$U(t,\tau)U(\tau,r)w(r-s;u_0) \leq U(t,\tau)U(\tau,r)u(r,s;u_0) = u(t,s;u_0).$$

Again, taking limit as $s \to -\infty$ as above we have

$$U(t,\tau)U(\tau,r)u_E^0 \leq \liminf_{s\to-\infty} u(t,s;u_0)$$

and taking now limit as $r \to -\infty$ we have

$$U(t,\tau)\varphi_m(\tau) \leq \liminf_{s\to-\infty} u(t,s;u_0).$$

Finally, using that φ_m is a complete trajectory we have

$$0 \leq \varphi_m(t) \leq \liminf_{s\to-\infty} u(t,s;u_0)$$

where the limit is uniform for u_0 in B.

We prove now that $\varphi_m(t)$ is asymptotically stable from below in the pullback sense for each $t \in I\!R$. Now, let $v \in C_b(I\!R, X)$ be a nondegenerate function at $-\infty$ such that $0 \leq v(s) \leq \varphi_m(s)$ for all $s \in I\!R$. Then, for $s < t$,

$$u(t,s;v(s)) \leq u(t,s;\varphi_m(s)) = \varphi_m(t).$$

Taking now limits as $s \to -\infty$ and using (5.15) we have

$$\varphi_m(t) \leq \liminf_{s\to-\infty} u(t,s;v(s)) \leq \limsup_{s\to-\infty} u(t,s;v(s)) \leq \varphi_m(t).$$

Therefore, φ_m is asymptotically stable from below in the pullback sense for $t \in I\!R$.

Finally, we show that $\varphi_m(t)$ is the minimal positive complete trajectory nongenerate at $-\infty$. For this, let ψ another positive complete trajectory nondegenerate at $-\infty$. Now, let $v \in C_b(I\!R, X)$ be a nondegenerate function at $-\infty$ such that $0 \leq v(s) \leq \psi(s)$ for all $s \in I\!R$. Then, by the comparison principle, we have for $t \in I\!R$,

$$u(t,s;v(s)) \leq U(t,s)\psi(s) = \psi(t).$$

Taking now limits as $s \to -\infty$ we have , from the results above,

$$\varphi_m(t) \leq \liminf_{s\to-\infty} u(t,s;v(s)) \leq \psi(t) \qquad \text{for} \quad t \in I\!R.$$

Hence $\varphi_m(t)$ is minimal.

The forward invariance of $I^+(t)$ follows from the comparison principle and the fact that $\varphi_m(t)$ and $\varphi_M(t)$ are complete trajectories. □

Remark 5.7 *Observe that the Theorem above implies that (2.1) is pullback permanent as defined in [10].*

Again note that with the notation in [14] we have that, in particular, there exists a pullback attractor for positive solutions of (2.1), $\{\mathcal{A}_+(t)\}_{t\in\mathbb{R}}$, which satisfies

$$\mathcal{A}_+(t) \subset [\varphi_m(t), \varphi_M(t)] \quad \textit{for all} \quad t \in \mathbb{R}$$

and $\varphi_m(t), \varphi_M(t) \in \mathcal{A}_+(t)$ for all $t \in \mathbb{R}$.

Also observe that if $t_0 = \infty$ in the Theorem above, then any non-trivial complete trajectory $\psi(t)$ in the order interval $[0, \varphi_m(t)]$ must satisfy $\psi(t) \to 0$ as $t \to -\infty$. That is, it must connect 0 at $t = -\infty$ with $\varphi_m(t)$ at $t = \infty$.

Remark 5.8 *The results above show that if the uniqueness result Theorem 2.3 applies, then there exists a unique positive complete trajectory for (2.1), i.e., $\varphi_m(t) = \varphi_M(t)$ and then it is globally asymptotically stable in the pullback sense.*

If in addition, if the asymptotic result in Theorem 2.4 holds true, then $\varphi_m(t) = \varphi_M(t)$ also describes the forward dynamics of (2.1).

Now we give some other conditions for the solutions of (2.1) to be PND at ∞.

Proposition 5.9 *Assume that for $t \geq t_0$*

$$f_0(x, u) \leq f(t, x, u) \quad \textit{for all} \quad x \in \Omega, \ u \geq 0 \tag{5.16}$$

for some f_0 satisfying (5.7) and (5.8).

Then for every $s \geq t_0$ and every nonnegative initial data the solution of (2.1), $u(t, s; u_0)$, is PND at ∞.

Proof Just note that from comparison we have

$$w(t - s; u_0) \leq u(t, s; u_0)$$

and the result follows from (5.8).□

Note that the arguments above suggest another type of conditions to obtain that complete trajectories are bounded above. These conditions are complementary of those in Corollary 5.4.

Proposition 5.10 *Assume that for $t \leq t_0 \leq \infty$ (or $t \geq t_0$ respectively) we have*

$$f(t, x, u) \leq f_1(x, u) \quad \textit{for all} \quad x \in \Omega, \ u \geq 0 \tag{5.17}$$

for some f_1 satisfying (5.7) and (5.8).

Then any complete nonnegative solution of (2.1) is bounded above at $-\infty$ (or ∞ respectively).

Proof Note that from comparison, we have for any nonnegative initial data and $s < t \leq t_0$

$$0 \leq u(t, s; u_0) \leq w(t - s; u_0)$$

where w is the solution of the problem (5.6) with nonlinear term $f_1(x, u)$. Then property (5.8) gives the result.□

Remark 5.11 *Note that combining the arguments above we have that if, for all $t \in \mathbb{R}$, we have*

$$f_0(x, u) \leq f(t, x, u) \leq f_1(x, u) \quad \textit{for all} \quad x \in \Omega, \ u \geq 0$$

with f_0 and f_1 satisfying (5.7) and (5.8) then we can obtain the existence of two extremal complete trajectories $\varphi_m \leq \varphi_M$ satisfying (5.10), which are non degenerate and bounded above at $\pm\infty$. These ideas have been used in [9].

6 Asymptotically autonomous and asymptotically periodic problems

From the results above it is clear that it is interesting to determine the behavior at ∞ of complete trajectories. An specially important case is this in which the forward dynamics of (2.1) becomes asymptotically autonomous or periodic as we now discuss.

Hence, suppose that the nonlinear nonautonomous problem

$$\left\{\begin{array}{rcll} u_t - \Delta u &=& f(t,x,u) & \text{in } \Omega, \quad t > s \\ u &=& 0 & \text{on } \partial\Omega \\ u(s) &=& u_0 & \end{array}\right. \tag{6.1}$$

has a positive nondegenerate solution bounded above at ∞ that we denote $u(x,t)$; see Theorem 2.4.

In addition, assume (6.1) is asymptotically autonomous in the sense that for each $\varepsilon > 0$ there exists $t_0 \in I\!R$ such that for any $M > 0$ and $t \geq t_0$

$$f_0(x,u) \leq f(t,x,u) \leq f_1(x,u) \quad \text{for all} \quad x \in \Omega, \ 0 \leq u \leq M \tag{6.2}$$

for some f_0 and f_1 satisfying (5.7) and (5.8) and

$$\sup_{0 \leq u \leq M} \|f_0(\cdot,u) - f_1(\cdot,u)\|_{L^r(\Omega)} \leq \varepsilon, \tag{6.3}$$

for some $r > N/2$. In particular the limit

$$\lim_{t\to\infty} f(t,\cdot,u) = f_\infty(\cdot,u)$$

exist in $L^r(\Omega)$ uniformly in compact sets of $u \geq 0$. Moreover, f_∞ satisfies (5.7) and (5.8).

Then, we have the following result

Theorem 6.1 *Under the assumptions above, all positive solutions of (6.1) converge uniformly in $x \in \Omega$, as $t \to \infty$, to $u_\infty(x)$ which is given by the unique positive solution of*

$$\left\{\begin{array}{rcll} -\Delta w &=& f_\infty(x,w) & \textit{in} \quad \Omega \\ w &=& 0 & \textit{on} \quad \partial\Omega \end{array}\right. \tag{6.4}$$

Proof Denote $u_E^i(x)$ the the unique positive solution of (5.6) with nonlinear term $f_i(x,u)$. Then from the arguments in the previous section, for sufficiently large $t >> t_0$ we may assume

$$u_E^0(x) \leq u(x,t) \leq u_E^1(x).$$

Now we prove that $u_E^0(x)$ and $u_E^1(x)$ are uniformly close in Ω.

For this we first show that each of these equilibria is linearly asymptotically stable for the corresponding autonomous evolution problem. In fact the linearized equation around, say u_E^0, is given by

$$\eta_t - \Delta\eta = \partial_u f_0(x, u_E^0(x))\eta = \Big(- P(x) + \frac{f_0(x,u_E^0(x))}{u_E^0(x)}\Big)\eta$$

with Dirichlet boundary conditions, where

$$P(x)=\frac{f_0(x,u_E^0(x))}{u_E^0(x)}-\partial_u f_0(x,u_E^0(x))\geq 0$$

is not identically zero. Then 0 is the first eigenvalue of $-\Delta-q(x)$, with $q(x)=\dfrac{f_0(x,u_E^0(x))}{u_E^0(x)}$, since u_E^0 is a positive eigenfunction. Hence, the first eigenvalue of $-\Delta-q(x)+P(x)$, i.e. the linearized elliptic operator, is positive. Therefore, the linearized equation is exponentially stable.

With this, denote $z(x)=u_E^1(x)-u_E^0(x)\geq 0$ and then we have

$$\begin{cases} -\Delta z=f_1(x,u_E^1(x))-f_0(x,u_E^0(x))=C(x)z+D(x), & \text{in } \Omega\\ z=0 & \text{on } \partial\Omega\end{cases}$$

with

$$D(x)=(f_1(x,u_E^1(x))-f_0(x,u_E^1(x)))\geq 0 \quad \text{and} \quad C(x)=\partial_u f_0(x,\xi(x))$$

for some $u_E^0(x)\leq\xi(x)\leq u_E^1(x)$. Hence $\partial_u f_0(x,\xi(x))\leq\partial_u f_0(x,u_E^0(x))$ and thus the first eigenvalue of $-\Delta-C(x)$ is still positive. All these imply that

$$\|z\|_{L^\infty(\Omega)}\leq K\|D\|_{L^r(\Omega)}\leq K\varepsilon$$

because of (6.3).

The rest follows easily.□

Now we can use a very similar argument for the asymptotically periodic case as follows. Indeed in this case we will assume that (6.1) is asymptotically periodic in the sense that, there exists some $T>0$ such that for each $\varepsilon>0$ there exists $t_0\in\mathbb{R}$ such that for any $M>0$ and $t\geq t_0$

$$f_0(t,x,u)\leq f(t,x,u)\leq f_1(t,x,u) \quad \text{for all} \quad x\in\Omega,\ 0\leq u\leq M \tag{6.5}$$

for some T–periodic function f_0 and f_1 having a unique PND T-periodic solution $u_P^i(x,t)$, which are globally asymptotically stable for positive solutions, e.g. [8], and

$$\int_{t_0}^{t_0+T}\Big[\sup_{0\leq u\leq M}\|f_0(t,\cdot,u)-f_1(t,\cdot,u)\|_{L^r(\Omega)}\Big]^\sigma dt\leq\varepsilon^\sigma, \tag{6.6}$$

for some $1<\sigma\leq\infty$ and $r>N\sigma'/2$, or $\sigma=1$ and $r=\infty$. In particular the limit

$$\lim_{t_0\to\infty}(f(\cdot,\cdot,u)-f_\infty(\cdot,\cdot,u))=0$$

exist in $L^\sigma((t_0,t_0+T),L^r(\Omega))$ uniformly in compact sets of $u\geq 0$ for some T–periodic function $f_\infty(t,x,u)$.

Moreover, we will assume that for any positive T–periodic function $\phi(x,t)$ we have

$$\frac{f_0(t,x,\phi(x,t))}{\phi(x,t)}-\partial_u f_0(t,x,\phi(x,t))\geq\varphi_0(x), \quad \text{for all} \quad x\in\Omega, \tag{6.7}$$

for some positive function in Ω, which may vanish on $\partial\Omega$. Then, we have the following result

Theorem 6.2 *Under the assumptions above, all positive solutions of (6.1) converge uniformly in $x \in \Omega$, as $t \to \infty$, to $u_\infty(x,t)$ which is given by the unique positive T–periodic solution of*

$$\left\{ \begin{array}{rcll} w_t - \Delta w &=& f_\infty(t,x,w) & \text{in } \Omega \\ w &=& 0 & \text{on } \partial\Omega \end{array} \right. \tag{6.8}$$

Proof Denote $u_P^i(x,t)$ the the unique positive periodic solution of (6.8) with nonlinear term $f_i(t,x,u)$. Then from the arguments in the previous section, for sufficiently large t_0 and $t > t_0$ we may assume

$$u_P^0(x,t) \le u(x,t) \le u_P^1(x,t).$$

Now we prove that $u_P^0(x,t)$ and $u_P^1(x,t)$ are uniformly close in Ω.

For this we first show that u_P^0 is linearly asymptotically stable for the corresponding evolution problem. In fact the linearized equation around, say u_P^0, is given by

$$\eta_t - \Delta\eta = \partial_u f_0(t,x,u_P^0(x))\eta = \Big(- P(x,t) + \frac{f_0(t,x,u_P^0(x,t))}{u_P^0(x,t)} \Big)\eta$$

with Dirichlet boundary conditions, where

$$P(x,t) = \frac{f_0(t,x,u_P^0(x,t))}{u_P^0(x,t)} - \partial_u f_0(t,x,u_P^0(x,t)) \ge \varphi_0(x)$$

for some positive function in Ω, which may vanish on $\partial\Omega$. Then Proposition 4.3 ii), with $q(x,t) = \dfrac{f_0(t,x,u_P^0(x,t))}{u_P^0(x,t)}$, implies that all solution of the linearized equation converge to zero as $t \to \infty$. Since this is a periodic problem, this, in turn implies that the linearized equation is exponentially stable.

With this, denote $z(x,t) = u_P^1(x,t) - u_P^0(x,t) \ge 0$ and then we have

$$z_t - \Delta z = f_1(t,x,u_P^1(x,t)) - f_0(t,x,u_P^0(x,t)) = C(x,t)z + D(x,t)$$

with

$$D(x,t) = (f_1(t,x,u_P^1(x,t)) - f_0(t,x,u_P^1(x,t))) \ge 0 \quad \text{and} \quad C(x,t) = \partial_u f_0(t,x,\xi(x,t))$$

for some $u_P^0(x,t) \le \xi(x,t) \le u_P^1(x,t)$. Hence $\partial_u f_0(t,x,\xi(x,t)) \le \partial_u f_0(t,x,u_P^0(x,t))$ and thus the evolution operator defined by of $-\Delta - C(x,t)$ is still exponentially stable, see [14]. All these implies that

$$\|z\|_{L^\infty((t_0,t_0+T),L^\infty(\Omega))} \le K\|D\|_{L^\sigma((t_0,t_0+T),L^r(\Omega))} \le K\varepsilon$$

for some $K = K(T)$, because of (6.6), see [14].

The rest follows easily.□

7 A model example: logistic equations

We will now show how our techniques can be applied to some important classes of problems defined by the non-autonomous logistic equation

$$\left\{ \begin{array}{rcll} u_t - \Delta u &=& f(t,x,u), & \text{in } \Omega \quad t > s \\ u &=& 0 & \text{on } \partial\Omega \\ u(s) &=& u_0 & \end{array} \right. \tag{7.1}$$

with nonlinearity

$$f(t,x,u) = m(t,x)u - n(t,x)u^{\rho}, \qquad \rho \geq 2 \tag{7.2}$$

where $m \in C^{\alpha}(I\!R, L^p(\Omega))$ for certain $p > N/2$ and $0 < \alpha \leq 1$ and $n \geq 0$ is continuous and locally Hölder in t, not identically zero. All these examples satisfy (2.2).

Observe that for these problems, we have for a function satisfying $\varphi_0(x) \leq u(x,t) \leq \varphi_1(x)$ for all $x \in \Omega$ and t in some time interval I, for some $C^1(\overline{\Omega})$ functions $\varphi_i(x) > 0$ in Ω, vanishing on $\partial\Omega$,

$$q(x,t) = \frac{f(t,x,u(x,t))}{u(x,t)} = m(x,t) - n(x,t)u^{\rho-1}(x,t) \geq m(x,t) - n(x,t)\varphi_1^{\rho-1}(x).$$

Therefore to have condition (2.11) satisfied we require that $m(x,t)-n(x,t)\varphi_1^{\rho-1}(x) \geq q_0(x)$.This is satisfied for example if

$$m(x,t) \geq M(x), \quad n(x,t) \leq N(x), \quad x \in \Omega, \quad t \geq t_0$$

(respectively for all $t \leq t_0$).

On the other hand, for functions satisfying $\varphi_0(x) \leq u_1(x,t) \leq u_2(t,x) \leq \varphi_1(x)$ for all $x \in \Omega$ and t in some time interval I, for some $C^1(\overline{\Omega})$ functions $\varphi_i(x) > 0$ in Ω, vanishing on $\partial\Omega$, we have,

$$P(x,t) = \frac{f(t,x,u_1(x,t))}{u_1(x,t)} - \frac{f(t,x,u_2(x,t))}{u_2(x,t)} = n(x,t)(u_2^{\rho-1}(x,t) - u_1^{\rho-1}(x,t))$$

then

$$P(x,t) \geq (\rho-1)n(x,t)\varphi_0^{\rho-2}(x)(u_2(x,t) - u_1(x,t)).$$

Therefore to have condition (2.13) satisfied we require that for every subset $\Omega_0 \subset \Omega$ there exists some set of positive measure $A \subset \Omega_0$ and some $\alpha > 0$ such that $\liminf_{t\to\infty} n(x,t) \geq \alpha > 0$, a.e. $x \in A$, i.e.

$$\liminf_{t\to\infty} n(x,t) > 0 \quad a.e. \quad x \in \Omega$$

(respectively as $t \to -\infty$). This is satisfied in particular if $n(x,t) \geq N(x) > 0$ for every $x \in \Omega$ and all t in an unbounded interval.

With this all the results about uniqueness in Section 3 and forward behavior in Section 4 apply to (7.1).

As for the existence results in Section 5, note that Theorem 5.1 applies if the evolution operator associated to $\Delta + m(t,x)$ is exponentially stable.

On the other hand, note that for (7.2), assume that there exists a decomposition $m(t,x) = m_1(t,x) + m_2(t,x)$ with $m_2(t,x) \geq 0$ and

$$m_1 \in C^{\alpha}(I\!R, L^p(\Omega)) \quad \text{with} \quad 0 < \alpha \leq 1 \quad \text{and some} \quad p > N/2.$$

such that the evolution operator associated to $\Delta + m_1(t,x)$, $T_{m_1}(t,s)$, is exponentially stable with exponent $\beta > 0$. Then assumption, using Young's inequality, (5.1) is satisfied, with

$$C(x,t) = m_1(x,t) \quad \text{and} \quad D(x,t) = D_0\left(\frac{m_2(x,t)}{n^{\frac{1}{\rho}}(x,t)}\right)^{\rho'} \tag{7.3}$$

for some constant D_0. Then Theorem 5.2 applies.

To show how these conditions are satisfied, we start with the case in which the coefficient $n(x,t)$ either never vanishes or vanishes "slowly" and in a "small set". In this case the nonlinear dissipative character of the equation is acting everywhere at every time and the existence of a bounded complete trajectory follows.

Proposition 7.1 *Assume either $n(x,t) \geq \gamma > 0$ in $\overline{\Omega} \times I\!R$ or, more generally,*

$$\frac{1}{n} \in L^\infty(I\!R, L^s(\Omega)), \quad s > \frac{N}{2(\rho-1)}.$$

Then for any $m \in L^\infty(I\!R, L^p(\Omega))$ with $p > N/2$ there exists C and D satisfying (5.1) such that the evolution operator associated to $\Delta + C$, $T_C(t,s)$, decays exponentially and $D \in L^\infty(I\!R, L^r(\Omega))$ for some $r > N/2$.

Therefore Theorem 5.2 applies and we obtain a complete trajectory $0 \leq \varphi_M \in L^\infty(I\!R, C_0(\overline{\Omega}))$.

Proof We choose

$$m_1(x,t) = m(x,t) - \lambda \quad \text{and} \quad m_2(x,t) = \lambda$$

with λ large enough such that the evolution operator associated to $\Delta + C$ decays exponentially, see [14]. In such a case, in (7.3), we get

$$0 \leq D(x,t) = D_0 \left[\frac{m_2(x,t)}{n^{1/\rho}(x,t)}\right]^{\rho'} \leq \frac{D_0 \lambda^{\rho'}}{n(x,t)^{\rho'/\rho}} = \frac{D_0 \lambda^{\rho'}}{n(x,t)^{\frac{1}{\rho-1}}} \in L^\infty(I\!R, L^r(\Omega))$$

for some $r > N/2$, by the assumption on $n(x,t)$.□

Now we turn into the case in which $n(x,t)$ vanishes "fast" at some points and/or in a "large set". In such a case on such set the equation is linear and then $m(x,t)$ somehow must prevent solutions from becoming very large. For this we introduce some notations.

Denote $Q = \Omega \times I\!R$, $Q_0 = \{(x,t) \in Q : \ n(x,t) = 0\}$ and assume Q_γ is a neighborhood of Q_0 such that $n(x,t) \geq \gamma > 0$ for all $(x,t) \in Q \setminus \overline{Q}_\gamma$.

Denote then

$$\Omega_\gamma(t) = \{x \in \Omega, \ (x,t) \in Q_\gamma\}$$

and

$$\lambda_1(t) = \lambda_1(t)(-\Delta - m(\cdot,t)) = \lambda_1^{\Omega_\gamma(t)}(-\Delta - m(\cdot,t))$$

the first eigenvalue of the operator $-\Delta - m(\cdot,t)$ in $\Omega_\gamma(t)$ with Dirichlet boundary conditions.

Finally we assume that the parametric family of sets $\Omega_\gamma(t)$ and potentials $m(\cdot,t)$ is "uniformly regular" for $t \to \pm\infty$ in the following sense.

Definition 7.2 *Assume $\Omega \subset I\!R^N$ is a bounded set. A family of proper subsets of Ω, $\{\Omega_\epsilon\}_\epsilon$ and a family of potentials $\{V_\epsilon(x)\}_\epsilon$ defined in Ω_ϵ, is "uniformly regular" iff for any given $\varphi \in H_0^1(\Omega)$, we have for some constant $C > 0$ independent of ϵ,*
i)

$$\|\varphi\|^2_{H^{1/2}(\partial\Omega_\epsilon)} \leq C\|\varphi\|^2_{H^1(\Omega\setminus\overline{\Omega_\epsilon})}, \quad \|\varphi\|^2_{H^{1/2}(\partial\Omega_\epsilon)} \leq C\|\varphi\|^2_{H^1(\Omega_\epsilon)}$$

ii) For each ϵ, the first eigenvalue of $-\Delta + V_\epsilon$ in Ω_ϵ, with Dirichlet boundary conditions, is positive *iii) The unique solution of the elliptic problem*

$$\left\{ \begin{array}{rcll} -\Delta\xi + V_\epsilon \xi &=& 0 & in \quad \Omega_\epsilon \\ \xi &=& \varphi & on \quad \partial\Omega_\epsilon. \end{array} \right.$$

satisfy

$$\|\frac{\partial \xi}{\partial n}\|^2_{H^{-1/2}(\partial\Omega_\epsilon)} \leq C\|\varphi\|^2_{H^{1/2}(\partial\Omega_\epsilon)}.$$

Note that the above definition is meant to control the constant in some trace inequalities, uniformly in the parameter. For example, if $V_\epsilon = \lambda > 0$ and the boundaries $\partial\Omega_\epsilon$ are uniformly C^1 hypersurfaces, the above definition is satisfied.

Proposition 7.3 *With the notations above, assume the family of sets $\Omega_\gamma(t)$ and potentials $m(\cdot, t)$ is "uniformly regular" for $t \to \pm\infty$ and*

$$\liminf_{t\to\pm\infty} \lambda_1(t) > 0.$$

Assume also m is such that its positive part satisfies

$$m^+(x,t) = m_0^+(x,t) + m_1^+(x,t), \quad x \in Q \setminus \overline{Q_\gamma}$$

where, after extending by zero to Q_γ, we have the smallness condition

$$0 \leq m_0^+ \in \left\{ \begin{array}{ll} L^1(I\!R, L^\infty(\Omega)) & \\ L^\sigma(I\!R, L^p(\Omega)) & \textit{with } 1 < \sigma < \infty \textit{ and } p > \frac{N\sigma'}{2} \\ L^\infty(I\!R, L^p(\Omega)) & p > N/2, \textit{ where } \|m_0^+\|_{L^\infty(L^p)} \textit{ is small} \end{array} \right.$$

and the regularity condition $0 \leq m_1^+ \in L^\infty(I\!R, L^s(Q \setminus \overline{Q_\gamma}))$ with $s > \rho' N/2$.

Then there exist C and D satisfying (5.1) and such that the evolution operator $T_C(t,s)$ is exponentially stable and $D \in L^\infty(I\!R, L^r(\Omega))$ for some $r > N/2$.

Therefore Theorem 5.2 applies and we obtain a complete trajectory $0 \leq \varphi_M \in L^\infty(I\!R, C_0(\overline{\Omega}))$.

Proof Note that if $(x,t) \in Q_\gamma$ then

$$f(t,x,u) \leq m(x,t)u^2$$

and we take $C(x,t) = m(x,t)$ and $D(x,t) = 0$ for $(x,t) \in Q_\gamma$.

On the other hand, if $(x,t) \in Q \setminus \overline{Q_\gamma}$ then, for sufficiently large A, we write

$$m(x,t) = (m_0^+(x,t) - m^-(x,t) - A) + (m_1^+(x,t) + A) = m_1(x,t) + m_2(x,t), \quad (x,t) \in Q \setminus \overline{Q_\gamma}.$$

Then we chose $C(x,t) = m_1(x,t)$ for $(x,t) \in Q \setminus \overline{Q_\gamma}$.

By Lemma 7.6, that we prove below and the smallness assumption on m_0^+ we have that the evolution operator $T_C(t,s)$, with Dirichlet boundary conditions, is exponentially stable, see Corollary 4.6 in [14].

Arguing as in (7.3) we get, for $(x,t) \in Q \setminus \overline{Q_\gamma}$,

$$D(x,t) = D_0 \left[\frac{m_2(x,t)}{n^{1/\rho}(x,t)}\right]^{\rho'} \leq D_0 \gamma^{-\rho'/\rho} m_2^{\rho'}(x,t).$$

Hence, from the regularity assumption on m_1^+, we get that $D \in L^\infty(I\!R, L^r(\Omega))$, with $r > N/2$ and (5.1) is satisfied.□

Remark 7.4 *Observe that Proposition 7.3 does not assume any regularity on the set Q_0. Observe that if Q_0 is a regular set and $\Omega_0(t) = \{x \in \Omega,\ (x,t) \in Q_0\}$*

$$\lambda_1^{\Omega_0(t)}(-\Delta - m(\cdot, t)) > 0$$

then for a sufficiently small neighborhood of $\Omega_0(t)$, $\Omega_\gamma(t)$ we have

$$\lambda_1^{\Omega_\gamma(t)}(-\Delta - m(\cdot, t)) > 0$$

and $n(x,t) \geq \gamma > 0$ for all $x \in \Omega \setminus \overline{\Omega_\gamma(t)}$.

Then we may take $Q_\gamma = \cup_{t \in \mathbb{R}} \Omega_\gamma(t)$ to have the assumptions of Proposition 7.1 satisfied.

Then, by a straight application of Proposition 5.3 we get

Corollary 7.5 *In either case of Propositions 7.1 or 7.3, assume furthermore that $m, n \in L^\infty(\mathbb{R}, L^p(\Omega))$ with $p > N$.*

Then φ_M is bounded above and precompact in $L^1(\Omega)$ at both $\pm\infty$.

Now we prove the lemma used above.

Lemma 7.6 *With the notations above assume the parametric family of sets $\Omega_\gamma(t) = \{x \in \Omega,\ (x,t) \in Q_\gamma\}$ and potentials $m(\cdot, t)$ is uniformly regular for $t \to \pm\infty$ and*

$$\liminf_{t \to \pm\infty} \lambda_1(t) > 0.$$

Then for sufficiently large A, taking

$$C_0(x,t) = \begin{cases} m(x,t) & \text{for} \quad (x,t) \in Q_\gamma \\ -A & \text{for} \quad (x,t) \in Q \setminus \overline{Q_\gamma} \end{cases}$$

then the evolution operator $T_{C_0}(t,s)$, with Dirichlet boundary conditions, is exponentially stable.

Proof The proof follows closely the proof of Lemma 6.3 in [15], hence we just point out the main steps and differences. First, note that for fixed t such that $|t| >> 1$, dropping momentarily the dependence on t, we have

$$\lambda_1(-\Delta - C_0) = \inf_{\varphi \in H_0^1(\Omega)} \frac{J(\varphi)}{\int_\Omega \varphi^2}$$

where

$$J(\varphi) = \int_\Omega |\nabla \varphi|^2 - \int_\Omega C_0 \varphi^2.$$

Now, given $\varphi \in H_0^1(\Omega)$ we define $P(\varphi) = \xi \in H_0^1(\Omega_\gamma)$ where $\Omega_\gamma = \Omega_\gamma(t)$ and ξ satisfies

$$\left\{ \begin{array}{rcll} -\Delta\xi - m\xi & = & 0 & \text{in} \quad \Omega_\gamma \\ \xi & = & \varphi & \text{on} \quad \partial\Omega_\gamma. \end{array} \right. \tag{7.4}$$

Observe that $\varphi_{|\partial\Omega_\gamma} \in H^{1/2}(\partial\Omega_\gamma)$ and $\lambda_1^{\Omega_\gamma}(-\Delta - m) > 0$ and therefore the elliptic problem is well set.

Denote now $\eta = \varphi - \xi \in H_0^1(\Omega_\gamma)$ which satisfies

$$\begin{cases} -\Delta\eta - m\eta &= -\Delta\varphi - m\varphi \quad \text{in} \quad \Omega_\gamma \\ \eta &= 0 \qquad\qquad\quad\ \text{on} \quad \partial\Omega_\gamma. \end{cases}$$

We still denote by η its extension zero to the whole Ω.

Therefore, a simple computation gives

$$\begin{aligned} J(\varphi) = \int_\Omega |\nabla\varphi|^2 - \int_\Omega C_0\varphi^2 \quad &= \quad \int_{\Omega_\gamma} \left[|\nabla\eta|^2 - m\eta^2\right] + \int_{\Omega\setminus\overline{\Omega_\gamma}} \left[|\nabla\varphi|^2 + A\varphi^2\right] \\ &\quad +2\int_{\Omega_\gamma} [\nabla\eta\nabla\xi - m\eta\xi] + \int_{\Omega_\gamma} \left[|\nabla\xi|^2 - m\xi^2\right]. \end{aligned} \tag{7.5}$$

But, since $\eta \in H_0^1(\Omega_\gamma)$,

$$\int_{\Omega_\gamma} \left[|\nabla\eta|^2 - m\eta^2\right] \geq \lambda_1^{\Omega_\gamma}(-\Delta - m)\int_\Omega \eta^2.$$

On the other hand, from (7.4), multiplying by ξ and integrating by parts

$$\int_{\Omega_\gamma} \left[|\nabla\xi|^2 - m\xi^2\right] = \int_{\partial\Omega_\gamma} \frac{\partial\xi}{\partial n}\varphi.$$

From the definition of ξ and the uniform regularity of the sets $\Omega_\gamma(t)$ we get

$$\left|\int_{\partial\Omega_\gamma} \frac{\partial\xi}{\partial n}\varphi\right| \leq C\|\varphi\|^2_{H^{1/2}(\partial\Omega_\gamma)}$$

with some constant independent of $t \in I\!R$.

From here, given $\varepsilon > 0$

$$\int_{\partial\Omega_\gamma} \frac{\partial\xi}{\partial n}\varphi \geq -\varepsilon\int_{\Omega\setminus\overline{\Omega_\gamma}} |\nabla\varphi|^2 - C_\varepsilon\int_{\Omega\setminus\overline{\Omega_\gamma}} |\varphi|^2.$$

From this

$$\int_{\Omega\setminus\overline{\Omega_\gamma}} \left[|\nabla\varphi|^2 + A\varphi^2\right] + \int_{\partial\Omega_\gamma} \frac{\partial\xi}{\partial n}\varphi \geq (1-\varepsilon)\int_{\Omega\setminus\overline{\Omega_\gamma}} |\nabla\varphi|^2 + (A - C_\varepsilon)\int_{\Omega\setminus\overline{\Omega_\gamma}} \varphi^2$$

and taking $\varepsilon = 1/2$ we have for sufficiently large A

$$\int_{\Omega\setminus\overline{\Omega_\gamma}} \left[|\nabla\varphi|^2 + A\varphi^2\right] + \int_{\partial\Omega_\gamma} \frac{\partial\xi}{\partial n}\varphi \geq \delta\|\varphi\|^2_{H^1(\Omega\setminus\overline{\Omega_\gamma})}$$

for some positive constant $\delta > 0$ independent of $t \in I\!R$.

On the other hand, since $\eta \in H_0^1(\Omega_\gamma)$ and ξ is a solution of problem (7.4) we have

$$\int_{\Omega_\gamma} [\nabla\eta\nabla\xi - m\eta\xi] = 0.$$

Hence, in (7.5) we get

$$J(\varphi) \geq \lambda_1^{\Omega_\gamma}\int_{\Omega_\gamma} \eta^2 + \delta\|\varphi\|^2_{H^1(\Omega\setminus\overline{\Omega_\gamma})} \tag{7.6}$$

Now note that from (7.4) we have

$$\|\varphi\|^2_{H^1(\Omega\setminus\overline{\Omega_\gamma})} \geq a\Big(\int_{\Omega\setminus\overline{\Omega_\gamma}} |\varphi|^2 + \|\xi\|^2_{H^1(\Omega_\gamma)}\Big)$$

for some $a > 0$ independent of $t \in I\!R$. Hence, using the assumption on $\lambda_1^{\Omega_\gamma}(t)$, for $|t| >> 1$, in (7.6) we get

$$J(\varphi) \geq \beta \int_\Omega |\varphi|^2$$

for some $\beta > 0$ independent of $t \in I\!R$.

Finally, taking $\varphi = \varphi_1(t)$, the first eigenfunction of $-\Delta - C_0(t)$, with Dirichlet boundary conditions, we conclude that

$$\lambda_1(-\Delta - C_0(t)) = \beta > 0.$$

for $|t| >> 1$, and then Lemma 4.3 in [14] concludes the result.□

On the other hand (5.9) and (5.16) are satisfied for

$$f_0(x,u) = M(x)u - N(x)u^\rho$$

provided $m(t,x) \geq M(x)$ for some $M \in L^p(\Omega)$, $p > N/2$ and $0 \leq n(t,x) \leq N(x)$ for all t in a suitable unbounded interval. On the other hand (5.17) is satisfied with $f_1 = f_0$ as above, provided $m(t,x) \leq M(x)$ for some $M \in L^p(\Omega)$, $p > N/2$ and $0 \leq N(x) \leq n(t,x)$.

From the results in [15] (see also, [16], [19] and [1]), the corresponding problem (5.6) satisfies (5.8) provided that the zero solution is unstable for

$$\left\{\begin{array}{rcl} v_t - \Delta v &=& M(x)v \\ v &=& 0 \\ v(0) &=& v_0 \end{array}\right. \tag{7.7}$$

and either

i) $N(x) \geq \gamma > 0$ in $\overline{\Omega}$ or $1/N(x) \in L^s(\Omega)$ with $s > \frac{N}{2\rho}$

ii) if $\Omega_0 = \{x \in \Omega : \ N(x) = 0\}$, let Ω_δ be a neighborhood of Ω_0 such that $N(x) \geq \delta > 0$ for all $x \in \Omega\setminus\overline{\Omega}_\delta$ and the first eigenvalue of $-\Delta - M$ with Dirichlet boundary conditions, $\lambda_1^{\Omega_\delta}(-\Delta - M)$, is positive.

Now we consider the case of asymptotically autonomous case for (7.2). For given $t_0 \in I\!R$, define

$$M_0(x) = \inf_{t\geq t_0} m(x,t), \quad M_1(x) = \sup_{t\geq t_0} m(x,t),$$
$$N_0(x) = \sup_{t\geq t_0} n(x,t), \quad N_1(x) = \inf_{t\geq t_0} n(x,t)$$

and

$$f_0(x,u) = M_0(x)u - N_0(x)u^\rho, \quad \text{and} \quad f_1(x,u) = M_1(x)u - N_1(x)u^\rho.$$

Then (6.2) is satisfied and (6.3) as soon as, as $t \to \infty$ one has

$$m(x,t) \to m_\infty(x) \quad \text{in} \quad L^r(\Omega)$$

and

$$n(x,t) \to n_\infty(x) \quad \text{in} \quad L^\infty(\Omega).$$

A completely analogous analysis can be carried out for the asymptotically periodic case. Details are left to the reader.

8 Final remarks

Note that the analysis in the previous sections, extends to more general equations of the form

$$\left\{\begin{array}{rcll} u_t + Au & = & f(t,x,u), & \text{in } \Omega \qquad t > s \\ u & = & 0 & \text{on } \partial\Omega \\ u(s) & = & u_0 & \end{array}\right. \tag{8.1}$$

where A represents a general elliptic differential operator, of the form

$$Au = -\mathrm{div}(a(x)\nabla u) + c(x)u$$

or even

$$Au = -\sum_{i,j=1}^{N} a_{ij}(x)\partial_i\partial_j u + \sum_{i=1}^{N} a_i(x)\partial_i u + a(x)u$$

with suitable smooth coefficients.

As for other boundary conditions, note that if we consider either Neumann or Robin type boundary conditions, the arguments become a little more simple. In fact in these cases solutions of (8.1) become then strictly positive in the whole Ω. Hence in the definition of nondegeneracy, Definition 2.2, the functions φ_0 and φ_1 can be taken as positive constants. This simplifies a lot some of the proofs given above.

Finally note that, for the case of Neumann boundary conditions in (2.1), when $f = f(t,u)$ that is, the nonlinear term is space independent, then in particular the results in previous sections apply to the ODE

$$\dot{u}(t) = f(t,u).$$

Also, in the case of Neumann boundary conditions it is often the case in which $f(t,x,c) = 0$ for all (t,x) and some $c > 0$, [17]. Then it is possible to restrict the analysis to solutions taking values in $[0,c]$.

Then the definition of nondegenerate solutions would require $0 < \varphi_0(x) \leq u(x,t) \leq \varphi_1(x) < c$.

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