



Almost continuous Sierpiński–Zygmund functions under different set-theoretical assumptions

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Abstract

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is: *almost continuous* in the sense of Stallings, $f \in AC$, if each open set $G \subset \mathbb{R}^2$ containing the graph of f contains also the graph of a continuous function $g: \mathbb{R} \rightarrow \mathbb{R}$; *Sierpiński–Zygmund*, $f \in SZ$ (or, more generally, $f \in SZ(\text{Bor})$), provided its restriction $f \upharpoonright M$ is discontinuous (not Borel, respectively) for any $M \subset \mathbb{R}$ of cardinality continuum. It is known that an example of a Sierpiński–Zygmund almost continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ (i.e., an $f \in SZ \cap AC$) cannot be constructed in ZFC; however, an $f \in SZ \cap AC$ exists under the additional set-theoretical assumption $\text{cov}(\mathcal{M}) = \mathfrak{c}$, that is, that \mathbb{R} cannot be covered by less than \mathfrak{c} -many meager sets. The primary purpose of this paper is to show that the existence of an $f \in SZ \cap AC$ is also consistent with ZFC plus the negation of $\text{cov}(\mathcal{M}) = \mathfrak{c}$. More precisely, we show that it is consistent with $\text{ZFC} + \text{cov}(\mathcal{M}) < \mathfrak{c}$ (follows from the assumption that $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$) that there is an $f \in SZ(\text{Bor}) \cap AC$ and that such a map may have even stronger properties expressed in the language of Darboux-like functions. We also examine, assuming either $\text{cov}(\mathcal{M}) = \mathfrak{c}$ or $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$, the lineability and the additivity coefficient of the class of all almost continuous Sierpiński–Zygmund functions. Several open problems are also stated.

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1 Introduction

We use standard notations. In particular, $|X|$ denotes the cardinality of the set X and Y^X is the class of all functions from X to Y . The x -projection of a set $A \subset \mathbb{R}^2$ is denoted by $\text{dom}(A)$. The symbols \mathcal{C} , Bor , and \mathcal{B} denote the class of all continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$, the class of all Borel functions and the σ -algebra of all Borel sets in \mathbb{R} , respectively.

The symbols \mathcal{N} and \mathcal{M} denote the ideals on \mathbb{R} consisting of all Lebesgue measure zero (null) sets and all meager sets, respectively. In what follows we will write $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$ to say that either $\mathcal{I} = \mathcal{M}$ or $\mathcal{I} = \mathcal{N}$. For $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$ the symbol $\text{cov}(\mathcal{I})$ denotes the *covering number for \mathcal{I}* , that is, the first cardinal number κ such that \mathbb{R} can be covered by κ -many sets from \mathcal{I} . It is well known that for $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$, $\text{cov}(\mathcal{I}) = \kappa$ implies that no $B \in \mathcal{B} \setminus \mathcal{I}$ can be covered by less than κ -many sets from \mathcal{I} . (For the case of $\mathcal{I} = \mathcal{M}$ this is a consequence of the fact that every Borel set has the Baire property; whereas if $\mathcal{I} = \mathcal{N}$ then it follows from [11, remark after theorem 4.12], compare also the proof of Lemma 2.2 below.) The coefficient *uniformity of measure*, $\text{non}(\mathcal{N})$, is the least cardinal κ such that there exists a set $A \subset \mathbb{R}$ of cardinality κ that is not null. For more information about the covering number and the uniformity, see [7]. In what follows we will extensively use the assumption that $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$, which is consistent with ZFC: it holds in the model of ZFC obtained by adding ω_2 random reals to the model for ZFC+CH, see e.g. [35] or [8]. (Compare also [26, theorem 3.19].) Notice that a theorem of Rothberger states that $\text{cov}(\mathcal{M}) \leq \text{non}(\mathcal{N})$ (see [11, theorem 7.3] or [26, theorem 2.2]), which implies that $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$ contradicts the property $\text{cov}(\mathcal{M}) = \mathfrak{c}$, the other assumption consistent with ZFC that we will extensively use in this work.

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is Sierpiński–Zygmund, $f \in \text{SZ}$, provided its restriction $f \upharpoonright M$ is discontinuous for any $M \subset \mathbb{R}$ of cardinality continuum. Often, it is slightly easier to construct a function $f: \mathbb{R} \rightarrow \mathbb{R}$, denoted as $f \in \text{SZ}(\text{Bor})$, with a seemingly stronger property that $f \upharpoonright M$ is not Borel for any $M \subset \mathbb{R}$ of cardinality continuum (i.e., such that $f \cap h$ has cardinality less than continuum for every $h \in \text{Bor}$), see [18, 24, 25], or [16]. It is well-known that Sierpiński–Zygmund functions can be constructed in ZFC, that is, without additional set-theoretical assumptions [32]. The situation becomes more complicated if we consider Sierpiński–Zygmund functions $f: \mathbb{R} \rightarrow \mathbb{R}$ that have additionally the Darboux (i.e., the intermediate value) property, $f \in \text{D}$. Such functions (i.e., in $\text{SZ} \cap \text{D}$) exist under additional set-theoretical assumptions. It seems that the first construction of such example can be found in a 1981 article [10] of Ceder, where it was proved that that under the assumption of the continuum hypothesis, CH, there exists a connectivity (hence Darboux) SZ function. Next, in a 1992 paper [23] Kellum noticed that Ceder’s function is in fact almost continuous. Finally, in a 1997 paper [4] Balcerzak, Ciesielski, and Natkaniec showed that such examples (i.e., in $\text{SZ} \cap \text{AC}$) exist also under the weaker assumption $\text{cov}(\mathcal{M}) = \mathfrak{c}$ but they (specifically, the maps in $\text{SZ} \cap \text{D}$) cannot be constructed in ZFC. Since then, the theory of Sierpiński–Zygmund functions having one of the Darboux properties was always developed with the assumption $\text{cov}(\mathcal{M}) = \mathfrak{c}$. For more on the history of the subject, see 2019 survey [18] of Ciesielski and Seoane-Sepúlveda. The following question is natural in this context.

Question 1.1 *Is the condition $\text{cov}(\mathcal{M}) = \mathfrak{c}$ equivalent to the statement “there exists a Darboux Sierpiński–Zygmund function”?*

Ciesielski and Seoane-Sepúlveda constructed in the survey [18] an example of Darboux Sierpiński–Zygmund function $f : \mathbb{R} \rightarrow \mathbb{R}$ under an additional set-theoretical $\text{cov}(\mathcal{N}) = \mathfrak{c}$. Since there are models of ZFC in which $\text{cov}(\mathcal{N}) = \mathfrak{c} > \text{cov}(\mathcal{M})$, this solves Question 1.1 in the negative.

As any almost continuous function is Darboux (i.e., $\text{AC} \subset \text{D}$, see e.g. [34]) it is natural to examine also the following variant of Question 1.1:

Question 1.2 *Is the condition $\text{cov}(\mathcal{M}) = \mathfrak{c}$ equivalent to the statement: “there exists an almost continuous Sierpiński–Zygmund function”?*

Of course, the sufficiency of the condition $\text{cov}(\mathcal{M}) = \mathfrak{c}$ was proved in the previously mentioned article [4]. One of the goals of this article is to answer this question, in negative, by showing that $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$ (under which $\text{cov}(\mathcal{M}) < \mathfrak{c}$) also implies that $\text{SZ}(\text{Bor}) \cap \text{AC} \neq \emptyset$. This is proved in Sect. 2.

Additionally, we show how other results concerning class $\text{SZ} \cap \text{AC}$, often known under assumption $\text{cov}(\mathcal{M}) = \mathfrak{c}$, can be deduced from the opposite assumption $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$ or its strengthening. Specifically, in Sect. 3 we study the lineability of $\text{SZ}(\text{Bor}) \cap \text{AC}$, in Sect. 4 its additivity coefficient, while in Sect. 5 the existence of functions in different subclasses of $\text{SZ}(\text{Bor}) \cap \text{AC}$.

2 Maps in $\text{SZ}(\text{Bor}) \cap \text{AC}$ when $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$

In what follows any closed set $E \subset \mathbb{R}^2$ which x -projection is a non-degenerate interval is called a *blocking set*. It is well-known that any $f : \mathbb{R} \rightarrow \mathbb{R}$ which meets all blocking sets is almost continuous, see e.g. [28]. (Compare also [17, lemma 5.1] and related history.)

We start with the following minor modification of [23, lemma 1].

Lemma 2.1 *For every blocking set $E \subset \mathbb{R}^2$ there is a Borel function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $\text{dom}(h \upharpoonright E)$ contains a non-trivial interval. In particular, $\text{dom}(h \upharpoonright E) \notin \mathcal{M} \cup \mathcal{N}$.*

Proof Let $E_n = E \cap ([-n, n] \times [-n, n])$ for $n \in \mathbb{N}$. Then each E_n is compact, hence $\text{dom}(E_n)$ is closed, and $\text{dom}(E) = \bigcup_{n \in \mathbb{N}} \text{dom}(E_n)$, so for some $n \in \mathbb{N}$ the set $\text{dom}(E_n)$ contains a non-degenerate interval J , hence $\text{dom}(E_n) \notin \mathcal{M} \cup \mathcal{N}$. Now, the function $\hat{h} : \text{dom}(E_n) \rightarrow \mathbb{R}$ defined by $\hat{h}(x) := \max\{y \in [-n, n] : \langle x, y \rangle \in E_n\}$ is Borel (in fact, it is upper semi-continuous) and it is contained in E . Then any Borel extension $h : \mathbb{R} \rightarrow \mathbb{R}$ of \hat{h} is as we need. □

To show that $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$ implies $\text{SZ}(\text{Bor}) \cap \text{AC} \neq \emptyset$ we will also use the following lemma. The property of the set S constructed there means that S is dense in the density topology on \mathbb{R} . Note that Tall (see [35, theorem 4.15]) was the first who noticed that in the model of ZFC obtained by adding ω_2 random reals to the model for ZFC+CH the density of the density topology is equal to $\omega_1 < \mathfrak{c}$.

Lemma 2.2 *There exists a set $S \subset \mathbb{R}$ of cardinality $\text{non}(\mathcal{N})$ such that $B \cap S \neq \emptyset$ for every $B \in \mathcal{B} \setminus \mathcal{N}$. Moreover, if $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N})$, then for every $\mathcal{E} \subset \mathcal{B}$ of cardinality less than $\text{cov}(\mathcal{N})$*

- (•) *there is a $D \subset \mathbb{R} \setminus \bigcup \mathcal{E}$ of cardinality at most $\text{non}(\mathcal{N})$ such that $B \cap D \neq \emptyset$ for every $B \in \mathcal{B} \setminus \mathcal{N}$ with the property that $B \cap E \in \mathcal{N}$ for every $E \in \mathcal{E}$.*

Proof To see that the main part of the lemma holds, fix any set $A \subset \mathbb{R}$ with $|A| = \text{non}(\mathcal{N})$ and $A \notin \mathcal{N}$. Let $E \subset \mathbb{R}$ be a measurable hull of A . Clearly $|E| = \mathfrak{c}$, so there exists a Borel isomorphism $\varphi: \mathbb{R} \rightarrow E$ which maps null sets in \mathbb{R} onto null subsets of E , see e.g. [11, remark after theorem 4.12]. One can easily verify that $S := \varphi^{-1}(A)$ is as needed: clearly $|S| = |A| = \text{non}(\mathcal{N})$ and if $B \in \mathcal{B} \setminus \mathcal{N}$, then $\varphi(B) \cap A \neq \emptyset$ (as $\varphi(B)$ is a subset of E of positive measure and E is a hull of A) so $B \cap S = \varphi^{-1}(\varphi(B) \cap A) \neq \emptyset$.

To see the additional part of the lemma, fix an \mathcal{E} as in the statement. First notice that (•) holds if we additionally assume that $\mathcal{E} \subset \mathcal{N}$. Indeed, under such assumption, there exists an $x \in \mathbb{R}$ such that $(x+S) \cap \bigcup \mathcal{E} = \emptyset$ since otherwise $\mathbb{R} = \bigcup_{E \in \mathcal{E}} (E - S) = \bigcup_{s \in S, E \in \mathcal{E}} (E - s)$ is the union of less than $\text{cov}(\mathcal{N})$ -many null sets, a contradiction. The set $D_0 := x + S \subset \mathbb{R} \setminus \bigcup \mathcal{E}$ satisfies (•).

Finally, if we do not assume that $\mathcal{E} \subset \mathcal{N}$, let $H \in \mathcal{B}$ be a measurable hull of $\mathbb{R} \setminus \bigcup \mathcal{E}$. If $H \in \mathcal{N}$, then $D := \emptyset$ satisfies (•). So, we can assume that $H \notin \mathcal{N}$. Notice that $\tilde{\mathcal{E}} := \{E \cap H : E \in \mathcal{E}\}$ is contained in \mathcal{N} and let D_0 be as above for the family $\tilde{\mathcal{E}}$. Then $D := H \cap D_0$ satisfies (•) since for every $B \in \mathcal{B} \setminus \mathcal{N}$ such that $B \cap E \in \mathcal{N}$ for every $E \in \mathcal{E}$ we have $B \cap H \notin \mathcal{N}$ so that $B \cap D = (B \cap H) \cap D_0 \neq \emptyset$. □

The following lemma is the main tool of this paper and shows that if $F \in \mathbb{R}^{\mathbb{R}}$ extends f from the lemma, then $F \in \text{AC}$. This result will be used several times in this paper, when the functions of interest will be constructed by transfinite induction containing the sequence $\langle \langle G^\alpha, f^\alpha, D^\alpha \rangle : \alpha < \mathfrak{c} \rangle$ satisfying the assumptions (M), (N), (D), and (F).

For the case when $\text{cov}(\mathcal{M}) = \mathfrak{c}$, Lemma 2.3 is a close variation of [12, lemma 2.5]. We include its proof also for this case to emphasize the similarities and differences with its proof in the case when $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$.

Lemma 2.3 *Assume that either $\text{cov}(\mathcal{M}) = \mathfrak{c}$ or $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$ and let $\text{Bor} = \{h_\alpha : \alpha < \mathfrak{c}\}$. Assume also that the sequence $\langle \langle G^\alpha, f^\alpha, D^\alpha \rangle : \alpha < \mathfrak{c} \rangle$ satisfies the following properties for every $\alpha < \mathfrak{c}$:*

- (M) *if $\text{cov}(\mathcal{M}) = \mathfrak{c}$, then $\mathbb{R} \setminus G^\alpha \in \mathcal{M}$ and $h_\alpha \upharpoonright G^\alpha$ is continuous;*
- (N) *if $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$, then $G^\alpha = \mathbb{R}$;*
- (D) *D^α is a dense subset of $Z_\alpha := G^\alpha \setminus \bigcup_{\beta < \alpha} (D^\beta \cup \text{dom}(h_\alpha \cap h_\beta))$ which in the case when $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$ has also the property that $B \cap D^\alpha \neq \emptyset$ for every $B \in \mathcal{B} \setminus \mathcal{N}$ such that $B \cap E \in \mathcal{N}$ for every E in the family $\mathcal{E}_\alpha := \{\text{dom}(h_\alpha \cap h_\beta) : \beta < \alpha\}$;*
- (F) *$f^\alpha = h_\alpha \upharpoonright D^\alpha$.*

If $f := \bigcup_{\alpha < \mathfrak{c}} f^\alpha$, then $K \cap f \neq \emptyset$ for every blocking set K .

Before we prove the lemma we like to show how it implies the following theorem, the main result of this section.

Theorem 2.4 *If either $\text{cov}(\mathcal{M}) = \mathfrak{c}$ or $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$, then there exists an $F \in \text{SZ}(\text{Bor}) \cap \text{AC}$.*

Proof First notice that the sets G^α as in (M), dense G_δ -sets, exist by a well known result, see e.g. [27, p. 306]. The sequence $\langle \langle G^\alpha, f^\alpha, D^\alpha \rangle : \alpha < \mathfrak{c} \rangle$ as in Lemma 2.3 can be constructed by an easy transfinite induction on $\alpha < \mathfrak{c}$, where

- if $\text{cov}(\mathcal{M}) = \mathfrak{c}$ then D^α is countable;

- if $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$ then $|D^\alpha| \leq \text{non}(\mathcal{N})$ and the additional property of the set D^α is ensured by Lemma 2.2 applied to the family $\mathcal{E} := \mathcal{E}_\alpha \cup \{\{x\} : x \in \bigcup_{\beta < \alpha} D^\beta\}$.

So, let $f : E \rightarrow \mathbb{R}$ be as in Lemma 2.3 and $g : \mathbb{R} \rightarrow \mathbb{R}$ be any map in $\text{SZ}(\text{Bor})$. Then $F := g \upharpoonright (\mathbb{R} \setminus E) \cup f$ is as needed. Indeed, $F \in \text{AC}$ is ensured by Lemma 2.3.

To see that $F \in \text{SZ}(\text{Bor})$ fix an $h \in \text{Bor}$ and choose a $\beta < \mathfrak{c}$ such that $h = h_\beta$. By (F) and (D), for any $\alpha > \beta$ and $x \in D^\alpha$ we have $f(x) = f^\alpha(x) \neq h_\beta(x) = h(x)$. So, $\text{dom}(f \cap h) \subset \bigcup_{\gamma \leq \beta} D^\gamma$ has cardinality less than \mathfrak{c} . Also, $|g \cap h| < \mathfrak{c}$, giving desired $|F \cap h| < \mathfrak{c}$. □

Proof of Lemma 2.3 Fix a blocking set $K \subset \mathbb{R}^2$. It is enough to show that there exists an $x \in \mathbb{R}$ such that $\langle x, f(x) \rangle \in K$.

To see this let $\mathcal{I} = \mathcal{M}$ when $\text{cov}(\mathcal{M}) = \mathfrak{c}$ and $\mathcal{I} = \mathcal{N}$ otherwise. By Lemma 2.1, $\text{dom}(h_\xi \cap K) \in \mathcal{B} \setminus \mathcal{I}$ for some $\xi < \mathfrak{c}$.¹ Let $\alpha < \mathfrak{c}$ be the first ordinal with this property. Then

- $\text{dom}(h_\alpha \cap K) \in \mathcal{B} \setminus \mathcal{I}$ and $\text{dom}(h_\beta \cap K) \in \mathcal{I}$ for each $\beta < \alpha$.

The second part of • implies that

$$\text{dom}(h_\alpha \cap K) \cap \text{dom}(h_\alpha \cap h_\beta) \subset \text{dom}(h_\beta \cap K) \in \mathcal{I} \text{ for every } \beta < \alpha. \tag{1}$$

Consider two cases.

$\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$: By (1) and part of (D) applied to the set $B := \text{dom}(h_\alpha \cap K)$ there is an $x \in \text{dom}(h_\alpha \cap K) \cap D^\alpha$ for which $\langle x, f(x) \rangle = \langle x, h_\alpha(x) \rangle \in K$, as needed.

$\text{cov}(\mathcal{M}) = \mathfrak{c}$: The first part of • implies that there is a non-empty open interval J_0 such that $\text{dom}(h_\alpha \cap K) \cap J_0$ is residual in J_0 , that is, the set $J_0 \setminus \text{dom}(h_\alpha \cap K)$ is in \mathcal{M} . So, by (1) and $\text{cov}(\mathcal{M}) = \mathfrak{c}$, for every non-empty open interval $J \subset J_0$ we have

$$\emptyset \neq \text{dom}(h_\alpha \cap K) \cap J \cap G^\alpha \setminus \bigcup_{\beta < \alpha} (D^\beta \cup \text{dom}(h_\alpha \cap h_\beta)) \subset J \cap Z_\alpha \tag{2}$$

so, by (D), there is an $x \in D^\alpha \cap J \cap Z_\alpha$. In particular, by (F), $\langle x, f(x) \rangle = \langle x, h_\alpha(x) \rangle$. So, to finish the proof it is enough to show that $\langle x, h_\alpha(x) \rangle \in K$.

This would be obvious if we could ensure that x belongs also to $\text{dom}(h_\alpha \cap K)$; however, it is possible that $\text{dom}(h_\alpha \cap K) \cap D^\alpha = \emptyset$, what prevents such choice of x . So, we will use another argument to show that $\langle x, h_\alpha(x) \rangle \in K$.

To see this, notice that, by the property (2), for every $n < \omega$ there exists an $x_n \in \text{dom}(h_\alpha \cap K) \cap (x - 2^{-n}, x + 2^{-n}) \cap G^\alpha$. Such choice and the continuity of $h_\alpha \upharpoonright G^\alpha$ implies that the sequence $\langle x_n, h_\alpha(x_n) \rangle_n$ converges to $\langle x, h_\alpha(x) \rangle$. At the same time each $\langle x_n, h_\alpha(x_n) \rangle$ belongs to the closed set K so $\langle x, h_\alpha(x) \rangle = \lim_{n \rightarrow \infty} \langle x_n, h_\alpha(x_n) \rangle$ also belongs to K , as we needed. □

In the case $\text{cov}(\mathcal{M}) = \mathfrak{c}$, the key property in the proof of Lemma 2.3 is that the first part of • implies existence of a non-empty open interval J_0 for which $J_0 \setminus \text{dom}(h_\alpha \cap K) \in \mathcal{I}$. Of course, there is no such property for $\mathcal{I} = \mathcal{N}$ and this is the main reason why a similar proof does not work in this case.

3 Lineability of the family $\text{SZ}(\text{Bor}) \cap \text{AC}$

Recall that a family $\mathcal{F} \subset \mathbb{R}^\mathbb{R}$ is κ -lineable, where κ stands for any finite or infinite cardinality, if $\mathcal{F} \cup \{0\}$ contains a linear subspace of the space $\mathbb{R}^\mathbb{R}$ (over \mathbb{R}) of dimension κ . The study

¹ Recall that one-to-one projection of a Borel set is Borel, see e.g. [33, theorem 4.5.4].

in lineability, introduced by Gurariĭ, has been a rapidly developing trend in both recent real and functional analysis, see e.g. [2, 3], or [9].

In 2015, Płotka, assuming CH, proved that the family $AC \cap SZ$ is \mathfrak{c}^+ -lineable [31]. (Clearly, such a result cannot be proved in ZFC, as it is consistent that $AC \cap SZ = \emptyset$.) It was noticed, in a 2017 paper [13] of Ciesielski, Gámez-Merino, Mazza, and Seoane-Sepúlveda (and repeated in a survey [18]), that the argument from [31] actually works under weaker assumption $\text{cov}(\mathcal{M}) = \mathfrak{c}$. This, however, was not precise, since (as we will see below) the argument requires also the assumption that \mathfrak{c} is regular, which does not follow from $\text{cov}(\mathcal{M}) = \mathfrak{c}$.

The goal of this section is to prove the following theorem, which clarifies the situation under the assumption of $\text{cov}(\mathcal{M}) = \mathfrak{c}$ and shows that the same result holds also under assumption $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$.

Theorem 3.1 *Assume that \mathfrak{c} is a regular cardinal and either $\text{cov}(\mathcal{M}) = \mathfrak{c}$ or $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$. Then the family $AC \cap SZ(\text{Bor})$ is \mathfrak{c}^+ -lineable.*

It is worth to mention that the class AC (as well as any of its subclass discussed in Sect. 5) is $2^{\mathfrak{c}}$ -lineable, see [1] and [9]. However, $2^{\mathfrak{c}}$ -lineability for the class $SZ(\text{Bor})$ is undecidable in ZFC. More precisely, it is not difficult to see that $SZ(\text{Bor})$ is \mathfrak{c}^+ -lineable. Thus, if $2^{\mathfrak{c}} = \mathfrak{c}^+$ (e.g., under GCH), then $SZ(\text{Bor})$ is $2^{\mathfrak{c}}$ -lineable. On the other hand, there are models of ZFC in which SZ is not $2^{\mathfrak{c}}$ -lineable see [20] or [18, section 3]. Even more, the assumptions of Theorem 3.1 also do not decide $2^{\mathfrak{c}}$ -lineability of $AC \cap SZ(\text{Bor})$. Specifically, either of the set-theoretical assumptions of the theorem is consistent with $2^{\mathfrak{c}} = \mathfrak{c}^+$: in the model of ZFC obtained by adding ω_2 Cohen reals to a model of ZFC+GCH (where we have $\text{cov}(\mathcal{M}) = \mathfrak{c}$) and, respectively, in the extension of a model of ZFC+GCH by adding ω_2 random reals (where we have $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$, see next section). At the same time, if one starts with the model M of ZFC+GCH, extends it by adding ω_4 subsets of ω_1 using countable supported functions, then we obtain a model M_1 of ZFC such that in any ω_2 -cc generic extension of M_1 the family SZ is not \mathfrak{c}^{++} -lineable, see [18, remark (κ) in the proof of theorem 3.3]. Hence, if we add to M_1 either ω_2 Cohen reals (to get $\text{cov}(\mathcal{M}) = \mathfrak{c}$) or ω_2 random reals (to get $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$), then in such extensions SZ is not $2^{\mathfrak{c}}$ -lineable.

The proof of Theorem 3.1 will easily follow from the next lemma, a variant and consequence of Lemma 2.3. Notice that the assumption $\mathcal{G} \subset SZ(\text{Bor}) \cup \{0\}$ in its statement is crucial, as we prove in Theorem 4.1.

Lemma 3.2 *If \mathfrak{c} is regular and either $\text{cov}(\mathcal{M}) = \mathfrak{c}$ or $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$, then for every additive group $\mathcal{G} \subset SZ(\text{Bor}) \cup \{0\}$ of cardinality $\leq \mathfrak{c}$ there exists an f from $E \subset \mathbb{R}$ into \mathbb{R} such that for every $g \in \mathcal{G}$:*

- (i) $|(f + g \upharpoonright E) \cap h| < \mathfrak{c}$ for every $h \in \text{Bor}$;
- (ii) $(f + g \upharpoonright E) \cap K \neq \emptyset$ for every blocking set K .

Moreover there is an extension $F \in \mathbb{R}^{\mathbb{R} \setminus \mathcal{G}}$ of f such that $F + \mathcal{G} \subset AC \cap SZ(\text{Bor})$.

Proof Let $\mathcal{G} = \{g_\beta : \beta < \mathfrak{c}\}$ and fix a sequence $\langle h_\alpha \in \text{Bor} : \alpha < \mathfrak{c} \rangle$ in which every $h \in \text{Bor}$ appears \mathfrak{c} many times. For every $\alpha < \mathfrak{c}$ choose $G^\alpha \subset \mathbb{R}$ and κ such that

- (m) if $\text{cov}(\mathcal{M}) = \mathfrak{c}$, then $\kappa = \omega$ and G^α is residual such that $h_\alpha \upharpoonright G^\alpha$ is continuous;
- (n) if $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$, then $\kappa = \text{non}(\mathcal{N})$ and $G^\alpha = \mathbb{R}$.

By double induction on $\beta \leq \alpha < \mathfrak{c}$ we define the sequences $\langle D_\beta^\alpha : \beta \leq \alpha < \mathfrak{c} \rangle$ of pairwise disjoint subsets of \mathbb{R} each of cardinality at most κ and $\langle f_\beta^\alpha : \beta \leq \alpha < \mathfrak{c} \rangle$ of partial maps, each with domain D_β^α , aiming for $f := \bigcup_{\beta \leq \alpha < \mathfrak{c}} f_\beta^\alpha$ with domain $E := \bigcup_{\beta \leq \alpha < \mathfrak{c}} D_\beta^\alpha$.

For every $\alpha < \mathfrak{c}$ let

$$T_\alpha := \bigcup \{ \text{dom}((h_\alpha - h_\delta) \cap (g_\gamma - g_\beta)) : \beta, \gamma, \delta \leq \alpha \ \& \ g_\beta \neq g_\gamma \}$$

and notice that, by our assumption on \mathcal{G} and regularity of \mathfrak{c} , T_α has a cardinality less than \mathfrak{c} . To ensure that such f is as needed, we make sure that the following inductive conditions are satisfied for every $\alpha < \mathfrak{c}$ and $\beta \leq \alpha$:

(d) D_β^α is a dense subset of $Z_\beta^\alpha := \mathbb{R} \setminus M_\beta^\alpha$, where

$$M_\beta^\alpha := (\mathbb{R} \setminus G^\alpha) \cup \bigcup_{\delta \leq \gamma < \alpha} D_\delta^\gamma \cup \bigcup_{\delta < \beta} D_\delta^\alpha \cup T_\alpha \cup \bigcup_{\delta < \alpha} \text{dom}(h_\alpha \cap h_\delta),$$

which in the case when $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$ has also the property that $B \cap D_\beta^\alpha \neq \emptyset$ for every $B \in \mathcal{B} \setminus \mathcal{N}$ such that $B \cap E \in \mathcal{N}$ for every E in the family $\mathcal{E}_\alpha := \{ \text{dom}(h_\alpha \cap h_\delta) : \delta < \alpha \}$;

(f) $f_\beta^\alpha(x) = h_\alpha(x) - g_\beta(x)$ for every $x \in D_\beta^\alpha$.

The possibility of such a construction is obvious unless $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$, in which case the additional property of the set D_β^α can be ensured by Lemma 2.2 applied to the family

$$\mathcal{E} := \mathcal{E}_\alpha \cup \left\{ \{x\} : x \in \bigcup_{\delta \leq \gamma < \alpha} D_\delta^\gamma \cup \bigcup_{\delta < \beta} D_\delta^\alpha \cup T_\alpha \right\}.$$

To see (ii) choose $g \in \mathcal{G}$, fix $\beta < \mathfrak{c}$ with $g = g_\beta$, and notice that the sequence

$$\langle (G^\alpha, f_\beta^\alpha + (g_\beta \upharpoonright D_\beta^\alpha), D_\beta^\alpha) : \beta \leq \alpha < \mathfrak{c} \rangle$$

satisfies the assumptions of Lemma 2.3, where the properties (M), (N), (D), and (F) are ensured, respectively, by (m), (n), (d), and (f).² Thus $\bigcup_{\beta \leq \alpha < \mathfrak{c}} f_\beta^\alpha + (g_\beta \upharpoonright D_\beta^\alpha)$, as well as its superset $f + g \upharpoonright E$ indeed intersects every blocking set.

To see (i) choose $g = g_\beta \in \mathcal{G}$ and $h = h_\delta \in \text{Bor}$. It is enough to show that $\text{dom}((f + g \upharpoonright E) \cap h) \subset \bigcup_{\beta \leq \gamma \leq \max\{\delta, \beta\}} D_\gamma^\alpha$ as this last set has cardinality less than \mathfrak{c} . To see this inclusion, fix $\alpha > \max\{\delta, \beta\}$, $\gamma \leq \alpha$, and $x \in D_\gamma^\alpha$. We need to show that $(f + g)(x) = f_\gamma^\alpha(x) + g_\beta(x) = h_\alpha(x) - g_\gamma(x) + g_\beta(x)$ is not equal to $h_\delta(x) = h(x)$.

But for $g_\beta = g_\gamma$ this means simply that $h_\alpha(x) \neq h_\delta(x)$, what is ensured by the fact that D_γ^α is disjoint with $\text{dom}(h_\alpha \cap h_\delta)$. So, assume that $g_\beta \neq g_\gamma$. Then, by (d), $x \notin T_\alpha$, that is, $h_\alpha(x) - h_\delta(x) \neq g_\gamma(x) - g_\beta(x)$. This clearly implies that $(f + g)(x) = h_\alpha(x) - g_\gamma(x) + g_\beta(x) \neq h_\delta(x) = h(x)$, as required.

Finally, to find indicated extension F , notice that there exists a $\psi \in \mathbb{R}^\mathbb{R}$ such that $\psi + \mathcal{G} \subset \text{SZ}(\text{Bor})$, see [15, theorem 2.1].³ Then $F := f \cup (\psi \upharpoonright (\mathbb{R} \setminus E))$ is a needed extension. Indeed, the properties (i), (ii), and the definition of ψ ensures that $F + \mathcal{G} \subset \text{AC} \cap \text{SZ}(\text{Bor})$. Such F cannot belong to \mathcal{G} , since otherwise also $-F \in \mathcal{G}$ and $F + (-F) \notin \text{SZ}(\text{Bor})$. \square

Proof of Theorem 3.1 By induction on $\xi \leq \mathfrak{c}^+$ construct a sequence $\langle V_\xi : \xi \leq \mathfrak{c}^+ \rangle$ of linear subspaces of $\text{AC} \cap \text{SZ}(\text{Bor}) \cup \{0\}$ such that $|V_\xi| \leq \mathfrak{c}$ for every $\xi < \mathfrak{c}^+$, $V_\lambda = \bigcup_{\eta < \lambda} V_\eta$ for every limit ordinal number $\lambda \leq \mathfrak{c}^+$, and $V_{\xi+1} := \bigcup_{r \in \mathbb{R}} (rf_\xi + V_\xi)$ for every $\xi < \mathfrak{c}^+$, where f_ξ is the function F from Lemma 3.2 used with $\mathcal{G} = V_\xi$. Then $V_{\mathfrak{c}^+}$ justifies \mathfrak{c}^+ -lineability of $\text{AC} \cap \text{SZ}(\text{Bor})$, as needed. \square

² Notice that M_β^α is dense in $Z_\alpha := G^\alpha \setminus \bigcup_{\beta \leq \delta < \alpha} (D_\beta^\delta \cup \text{dom}(h_\alpha \cap h_\beta))$ and that Bor equals to $\{h_\alpha \in \text{Bor} : \beta \leq \alpha < \mathfrak{c}\}$.

³ Actually, in [15, theorem 2.1] this fact is proved only for the SZ functions, however the same idea works also in our more general setting.

4 Additivity of the family $SZ(\text{Bor}) \cap AC$

For $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$, the *additivity coefficient* of \mathcal{F} is defined as

$$\text{add}(\mathcal{F}) = \min \left(\{ |F| : F \subset \mathbb{R}^{\mathbb{R}} \ \& \ \forall g \in \mathbb{R}^{\mathbb{R}} \ g + F \notin \mathcal{F} \} \cup \{ (2^c)^+ \} \right).$$

(For more information on additivity see, for instance, [22].) The goal of this section is to study this coefficient for the class $SZ(\text{Bor}) \cap AC$.

There is an interesting relation between $\text{add}(\mathcal{F})$ and κ -lineability of \mathcal{F} , see e.g. [19, theorem 2.4] or [14, proposition 2.2]:

if $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ is closed under non-zero scalar multiplications and $\text{add}(\mathcal{F}) > \kappa \geq c$, then \mathcal{F} is κ^+ -lineable.

Thus, one may be tempted to provide an alternative proof of Theorem 3.1, that $SZ(\text{Bor}) \cap AC$ is c^+ -lineable, by showing that under the same set theoretical assumption we have $\text{add}(SZ(\text{Bor}) \cap AC) > c$. However, this is impossible, as shown by the following result.

Theorem 4.1 *If c is regular, then $\text{add}(SZ(\text{Bor}) \cap AC) \leq \text{add}(SZ \cap D) \leq c$.*

Proof The inequality $\text{add}(SZ(\text{Bor}) \cap AC) \leq \text{add}(SZ \cap D)$ follows immediately from the inclusion $SZ(\text{Bor}) \cap AC \subset SZ \cap D$ and an obvious remark that the operator add is monotone.

The proof of the second inequality is a small variation of the proof of [13, theorem 2.10]. To see this, let $\{r_\xi : \xi < c\}$ be an enumeration of \mathbb{R} and, for every $\xi < c$, define $A_\xi = \{r_\zeta : \zeta < \xi\}$. Let $F = \{r \cdot \chi_{A_\xi} : r \in \mathbb{R} \ \& \ \xi < c\}$, where χ_A is the characteristic function of A . Then $|F| = c$. Fix a $g : \mathbb{R} \rightarrow \mathbb{R}$. It is enough to prove that $g + F \notin SZ \cap D$.

To see this notice that $g = g + \chi_{A_0} \in g + F$. If $g \notin SZ \cap D$ we are done. So assume that $g \in SZ \cap D$. Then $g[\mathbb{R}]$ contains a non-trivial interval (c, d) . Take a $y \in (c, d)$ and notice that $A = g^{-1}(y)$ has cardinality smaller than c (as $g \in SZ$). Since c is regular, there is a $\xi < c$ with $A \subset A_\xi$. Choose an $r \in (0, \infty) \setminus (y - g[A_\xi])$. It is enough to prove that $G := g + r \cdot \chi_{A_\xi} \in g + F$ is not Darboux.

To see this, first notice that $y \notin G[\mathbb{R}]$. Indeed, $y \notin g[\mathbb{R} \setminus A_\xi] = G[\mathbb{R} \setminus A_\xi]$, as $g^{-1}(y) = A \subset A_\xi$. Also, $y \notin G[A_\xi]$ since for every $x \in A_\xi$ we have $G(x) = g(x) + r \neq y$, as guaranteed by the choice of r .

On the other hand, there are $p \in (c, y)$ and $q \in (y, d)$ with $g^{-1}(\{p, q\}) \cap A_\xi = \emptyset$. Thus, $p, q \in g[\mathbb{R} \setminus A_\xi] = G[\mathbb{R} \setminus A_\xi] \subset G[\mathbb{R}]$.

This means that $G[\mathbb{R}]$ is not connected, so that indeed $G \notin D$. □

The main goal of this section is to prove Theorem 4.6, which gives, consistently, a lower bound for $\text{add}(SZ(\text{Bor}) \cap AC)$ as ω . For this, we will need some preliminaries.

Let $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$. We use the symbol $\mathcal{I}_{<c}$ to denote the ideal of all subsets of \mathbb{R} that are the unions of less than c -many sets from \mathcal{I} . We say that $S \subset \mathbb{R}$ is *everywhere $\mathcal{I}_{<c}$ -positive* provided $B \cap S \notin \mathcal{I}_{<c}$ for every $B \in \mathcal{B} \setminus \mathcal{I}$. We will be interested in this notion only under the assumption that $\text{cov}(\mathcal{I}) = c$, in which case no $B \in \mathcal{B} \setminus \mathcal{I}$ belongs to $\mathcal{I}_{<c}$. Notice also that if c is a regular cardinal, then the ideal $\mathcal{I}_{<c}$ is *c-additive*, that is, a union of less than c -many sets from $\mathcal{I}_{<c}$ still belongs to $\mathcal{I}_{<c}$.

Lemma 4.2 *For every set $T \subset \mathbb{R}$ which is everywhere $\mathcal{I}_{<c}$ -positive and for every $g \in \mathbb{R}^{\mathbb{R}}$ there exist everywhere $\mathcal{I}_{<c}$ -positive $S \subset T$, $\gamma \in \text{Bor}$, and $G \in \mathcal{B}$ (possibly empty) such that*

- (a) $g = \gamma$ on $S \cap G$; and

(b) $\text{dom}(g \cap h) \cap (S \setminus G) \in \mathcal{I}_{<c}$ for every $h \in \text{Bor}$.

Proof Let \mathcal{U} be the maximal family of pairwise disjoint sets $B \in \mathcal{B} \setminus \mathcal{I}$ for which there exists $h_B \in \text{Bor}$ such that

- $\text{dom}(g \cap h_B) \cap T \cap E \notin \mathcal{I}_{<c}$ for every $E \in \mathcal{B} \setminus \mathcal{I}$ contained in B .

Notice that \mathcal{U} is at most countable, since $\mathcal{B} \setminus \mathcal{I}$ is ccc. So, $G = \bigcup \mathcal{U}$ is Borel. Let γ be any Borel extension of the partial function $\bigcup_{B \in \mathcal{U}} h_B \upharpoonright B$. Define

$$S := (T \setminus G) \cup \bigcup_{B \in \mathcal{U}} (\text{dom}(g \cap \gamma) \cap T \cap B)$$

and observe that S is everywhere $\mathcal{I}_{<c}$ -positive. Indeed, if $E \in \mathcal{B} \setminus \mathcal{I}$, then either $E \setminus G \notin \mathcal{I}$, so $E \cap S \supset (E \setminus G) \cap T \notin \mathcal{I}_{<c}$, or $E \cap B \notin \mathcal{I}$ for some $B \in \mathcal{U}$ and then $E \cap S \supset \text{dom}(g \cap \gamma) \cap T \cap (E \cap B) \notin \mathcal{I}_{<c}$.

One can easily verify that S , γ , and G satisfy the statement (a). To prove statement (b), suppose that $A := \text{dom}(g \cap h) \cap (S \setminus G) \notin \mathcal{I}_{<c}$ for some $h \in \text{Bor}$ and observe that then there is a $B \in \mathcal{B} \setminus \mathcal{I}$ such that $E \cap A \notin \mathcal{I}_{<c}$ for every $E \in \mathcal{B} \setminus \mathcal{I}$, $E \subset B$. Indeed, let \mathcal{C} be the maximal family of pairwise disjoint Borel sets $C \notin \mathcal{I}$ such that $C \cap A \in \mathcal{I}_{<c}$, and let $C_0 = \bigcup \mathcal{C}$. Then $C_0 \in \mathcal{B}$, $C_0 \cap A \in \mathcal{I}_{<c}$, and $B := \mathbb{R} \setminus C_0$ is as we need. (It is easy to see that $B \in \mathcal{B}$ but also $B \notin \mathcal{I}$ by the maximality of \mathcal{C} .) But this contradicts the maximality of the family \mathcal{U} . □

Lemma 4.3 *Assume that $\text{cov}(\mathcal{I}) = \mathfrak{c}$. For every finite family $\Gamma \subset \mathbb{R}^{\mathbb{R}}$ there exist everywhere $\mathcal{I}_{<c}$ -positive $S \subset \mathbb{R}$, $\{\gamma_g : g \in \Gamma\} \subset \text{Bor}$, and $\{B_g : g \in \Gamma\} \subset \mathcal{B}$ such that for every $g \in \Gamma$*

- (a) $g = \gamma_g$ on $S \cap B_g$; and
- (b) $\text{dom}(g \cap h) \cap (S \setminus B_g) \in \mathcal{I}_{<c}$ for every $h \in \text{Bor}$.

Proof Fix an enumeration $\{g_1, \dots, g_n\}$ of Γ .

Notice that $S_0 = \mathbb{R}$ is everywhere $\mathcal{I}_{<c}$ -positive by our assumption that $\text{cov}(\mathcal{I}) = \mathfrak{c}$. By applying Lemma 4.2 iteratively we can find functions γ_{g_i} , sets B_{g_i} , and a sequence $S_0 \supset S_1 \supset \dots \supset S_n$ so that the conditions (a) and (b) are satisfied with S_i , B_{g_i} , and γ_{g_i} in place of S , B , and γ , respectively. Then $S := S_n$ is as needed. □

We will also need the following fact.

Fact 4.4 *In the model of ZFC obtained by adding ω_2 random reals to the model for ZFC+CH we have $\omega_1 = \text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \omega_2 = \mathfrak{c}$ (so \mathfrak{c} is a regular) and*

- (d) every $Z \notin \mathcal{N}$ includes a subset of power ω_1 with the same outer measure.

Proof This fact is stated in [35, paragraph above theorem 4.15] and is attributed to Professor Kenneth Kunen. However, [35] contains neither proof of this fact nor a reference to a printed source where a proof can be found. Therefore, for reader’s convenience, we provide here a sketch of its proof.⁴

Let M be a model of ZFC+CH and for an ordinal α let $\mathbb{B}(\alpha)$ be a random real forcing on 2^α , see [7, page 99]. We will show that (d) holds in the model $M_{\omega_2} := M[h]$ which is an extension of M via forcing $\mathbb{B}(\omega_2)$, where $h \in 2^{\omega_2}$ is M -generic over $\mathbb{B}(\omega_2)$. Recall that in

⁴ We thank dr Tomek Bartoszyński for helping us with sketching this argument.

this setting for every $\alpha < \omega_2$ there is an associated generic extension $M_\alpha := M[h \upharpoonright \alpha]$ of M over $\mathbb{B}(\alpha)$ and that

$$M_{\omega_2} \text{ is a generic extension of } M_\alpha \text{ over the random real forcing } \mathbb{B}(\omega_2 \setminus \alpha). \tag{3}$$

Also $M_\alpha \subset M_\beta$ for every $\alpha < \beta < \omega_2$ and we have CH in M_α .

First we will show that, in M_{ω_2} ,

(*) for every $A \subset \mathbb{R}$ with $A \notin \mathcal{N}$ there is a $B \subset A$ so that $|B| \leq \omega_1$ and $B \notin \mathcal{N}$.

Indeed, the assumption on A means that, in M_{ω_2} , there is a map ψ , a subset of $2^\omega \times \mathbb{R}$, such that if $c \in 2^\omega$ is a code for a Borel set⁵ $\hat{c} \in \mathcal{B} \cap \mathcal{N}$, then $\psi(c)$ is a real number in $A \setminus \mathcal{N}$. Since random real forcing is ccc, every pair in $2^\omega \times \mathbb{R}$ belongs to M_α for some $\alpha < \omega_2$. Therefore, there is a $\lambda < \omega_2$ of cofinality ω_1 such that if $C_\lambda \subset 2^\omega$ is the set of all Borel codes that are in M_λ , then $\psi \upharpoonright C_\lambda$ belongs to M_λ .

But this means that, in M_λ , the set $B := A \cap M_\lambda$ is not in \mathcal{N} . Now, (3) and the following property that can be found in [7, lemmas 6.3.11 and 6.3.12]

- any $B \subset \mathbb{R}$ not in \mathcal{N} is also not in \mathcal{N} in any random real extension

imply that $B \notin \mathcal{N}$ also in M_{ω_2} . Since CH holds in M_λ , we have also $|B| \leq \omega_1$, that is, B is the set that satisfies (*).

Now, to finish the proof (d), choose a non-null set $Z \subset \mathbb{R}$. Without loss of generality we can assume that it is bounded so that its outer measure $m^*(Z)$ is finite. Let $d := \sup\{m^*(C) : C \subset Z \text{ \& } |C| = \omega_1\}$ and observe that there is a $D \subset Z$ with $|D| = \omega_1$ and $m^*(D) = d$. We claim that $d = m^*(Z)$, that is, that D satisfies (d). Indeed, if $d < m^*(Z)$, then there is an $E \in \mathcal{B}$ disjoint with D and such that $A := E \cap Z$ is not in \mathcal{N} . Then, by (*), there is a $B \subset A$ so that $|B| \leq \omega_1$ and $B \notin \mathcal{N}$. But then $C := D \cup B$ contradicts the maximality of d . □

Lemma 4.5 *Assume that \mathfrak{c} is a regular cardinal and that one of the following two assumptions holds:*

- (μ) $\text{cov}(\mathcal{M}) = \mathfrak{c}$;
- (ν) $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$ and (d) from Fact 4.4 holds.

Then for every finite family $\mathcal{G} = \{g_i : i < n\} \subset \mathbb{R}^{\mathbb{R}}$ there exists an f from $E \subset \mathbb{R}$ to \mathbb{R} such that for every $i < n$

- (i) $|(f + g_i \upharpoonright E) \cap h| < \mathfrak{c}$ for every $h \in \text{Bor}$;
- (ii) $(f + g_i \upharpoonright E) \cap K \neq \emptyset$ for every blocking set K .

Proof Let $\text{Bor} = \{h_\alpha : \alpha < \mathfrak{c}\}$. For every $\alpha < \mathfrak{c}$ choose $G^\alpha \subset \mathbb{R}$ and κ such that

- (m) if (μ) holds, then $\kappa = \omega$ and G^α is residual such that $h_\alpha \upharpoonright G^\alpha$ is continuous;
- (n) if (ν) holds, then $\kappa = \text{non}(\mathcal{N})$ and $G^\alpha = \mathbb{R}$.

Put $\Gamma := \mathcal{G} - \mathcal{G}$ and let $S \subset \mathbb{R}$, $\{\gamma_g : g \in \Gamma\} \subset \text{Bor}$, and $\{B_g : g \in \Gamma\} \subset \mathcal{B}$ be as in Lemma 4.3.

By induction on $\alpha < \mathfrak{c}$ define the sequences $\langle D_i^\alpha : i < n \text{ \& } \alpha < \mathfrak{c} \rangle$ of pairwise disjoint subsets of S each of cardinality at most κ and $\langle f^\alpha : \alpha < \mathfrak{c} \rangle$ of partial maps, each from $D^\alpha := \bigcup_{i < n} D_i^\alpha$ to \mathbb{R} , aiming for $f := \bigcup_{\alpha < \mathfrak{c}} f^\alpha$. To ensure that such f is as needed, we make sure that the following inductive conditions are satisfied for every $\alpha < \mathfrak{c}$ and $i < n$:

⁵ For what such codes are, see e.g. [7, sec 1.2.D].

(d) D_i^α is a dense subset of $Z_i^\alpha := S \setminus M_i^\alpha$, where

$$M_i^\alpha := (\mathbb{R} \setminus G^\alpha) \cup \bigcup_{j < i} D_j^\alpha \\ \cup \bigcup_{\beta < \alpha} (D^\beta \cup \text{dom}(h_\alpha \cap h_\beta)) \cup \bigcup_{g \in \mathcal{G}} (\text{dom}((g - g_i) \cap (h_\alpha - h_\beta)) \setminus B_{g-g_i})$$

which in the case when $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$ has also the property that $B \cap D_i^\alpha \neq \emptyset$ for every $B \in \mathcal{B} \setminus \mathcal{N}$ such that $B \cap E \in \mathcal{N}$ for every E in the family $\mathcal{E}_\alpha := \{\text{dom}(h_\alpha \cap h_\beta) : \beta < \alpha\}$;

(f) $f^\alpha(x) = h_\alpha(x) - g_i(x)$ for every $x \in D_i^\alpha$.

The construction of such sequences is obvious, where the additional property of the set D_i^α in the case when $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$ can be ensured by Fact 4.4 applied to the set Z_i^α , compare proof of Lemma 2.2.

It remains to show that the function $f := \bigcup_{\alpha < \mathfrak{c}} f^\alpha$ defined on the set $E := \bigcup_{\alpha < \mathfrak{c}} \bigcup_{j < n} D_j^\alpha$ indeed has the properties (i)–(ii).

To see (i), for every $j < n$ let $E_j := \bigcup_{\alpha < \mathfrak{c}} D_j^\alpha$. Notice that $E = \bigcup_{j < n} E_j$. Fix an $h \in \text{Bor}$. To see that $|(f + g_i \upharpoonright E) \cap h| < \mathfrak{c}$ it is enough to show that $|(f + g_i \upharpoonright E_j) \cap h| < \mathfrak{c}$ for every $j < n$. So, fix a $j < n$.

First choose $\beta < \mathfrak{c}$ such that $h = h_\beta$ and notice that, by (f), for every $\alpha > \beta$ and $x \in D_j^\alpha \setminus B_{g_j-g_i}$ we have $(f + g_i)(x) = (h_\alpha - g_j + g_i)(x) \neq h_\beta(x) = h(x)$, since the definition of M_i^α ensures that $(g_j - g_i)(x) \neq (h_\alpha - h_\beta)(x)$. Therefore, $|(f + g_i \upharpoonright (E_j \setminus B_{g_j-g_i})) \cap h| < \mathfrak{c}$ as its x -axis projection is contained in $\bigcup_{\alpha \leq \beta} D_j^\alpha$.

To see that $|(f + g_i \upharpoonright (E_j \cap B_{g_j-g_i})) \cap h| < \mathfrak{c}$ choose $\beta < \mathfrak{c}$ such that $h = h_\beta - \gamma_{g_j-g_i}$. Then, for every $\alpha > \beta$ and $x \in D_j^\alpha \cap B_{g_j-g_i}$ we have $h_\alpha(x) \neq h_\beta(x)$ and therefore,

$$(f + g_i)(x) = (h_\alpha - g_j + g_i)(x) = (h_\alpha - \gamma_{g_j-g_i})(x) \neq (h_\beta - \gamma_{g_j-g_i})(x) = h(x).$$

Hence $|(f + g_i \upharpoonright (E_j \cap B_{g_j-g_i})) \cap h| < \mathfrak{c}$ as its x -axis projection is contained in $\bigcup_{\alpha \leq \beta} D_j^\alpha$.

The property (ii) follows from Lemma 2.3 when we notice that, for every $i < n, \kappa$ and the sequence $\langle (G^\alpha, (f^\alpha \upharpoonright D_i^\alpha) + (g_i \upharpoonright D_i^\alpha), D_i^\alpha) : \alpha < \mathfrak{c} \rangle$ satisfies its assumption, so the map $\bigcup_{\alpha < \mathfrak{c}} ((f^\alpha \upharpoonright D_i^\alpha) + (g_i \upharpoonright D_i^\alpha))$ contained in f intersects every blocking set K . \square

Theorem 4.6 *Assume that \mathfrak{c} is a regular cardinal and that one of the following two assumptions holds:*

- (μ) $\text{cov}(\mathcal{M}) = \mathfrak{c}$;
- (ν) $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$ and (d) from Fact 4.4 holds.

Then for every finite $\mathcal{G} \subset \mathbb{R}^\mathbb{R}$ there is an $F \in \mathbb{R}^\mathbb{R}$ such that $F + \mathcal{G} \subset \text{SZ}(\text{Bor}) \cap \text{AC}$. In particular, $\text{add}(\text{SZ}(\text{Bor}) \cap \text{AC}) \geq \omega$, hence every function $f : \mathbb{R} \rightarrow \mathbb{R}$ can be represented as a sum of two functions from the class $\text{SZ}(\text{Bor}) \cap \text{AC}$.

Proof Recall that for every family $\mathcal{F} \subset \mathbb{R}^\mathbb{R}$ with $|\mathcal{F}| \leq \mathfrak{c}$ there exists a function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g + \mathcal{F} \subset \text{SZ}(\text{Bor})$, see the proof of Lemma 3.2. In particular, there exists a $g \in \mathbb{R}^\mathbb{R}$ be so that $g + \mathcal{G} \subset \text{SZ}(\text{Bor})$. Let $f : E \rightarrow \mathbb{R}$ be as in Lemma 4.5. Then map $F := f \cup (g \upharpoonright (\mathbb{R} \setminus E))$ is as needed. \square

Problem 4.7 *Can either Theorem 4.1 or Theorem 4.6 be proved without the assumption that \mathfrak{c} is a regular cardinal?*

Problem 4.8 *Under the assumptions of Theorem 4.6 we have the inequalities $\omega \leq \text{add}(\text{SZ}(\text{Bor}) \cap \text{AC}) \leq \mathfrak{c}$. Can the lower bound be replaced by ω_1 ? Can the upper bound be replaced by ω_1 if $\omega_1 < \mathfrak{c}$?*

5 Subclasses of $SZ(\text{Bor}) \cap AC$ when $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$

Recently (see e.g. [6, 12, 18]) there has been a considerable interest in the classes $SZ \cap \mathcal{F}$, including subclasses of $SZ \cap AC$, where \mathcal{F} is one of the classes in the algebra generated by Darboux-like families of functions. Recall, that the Darboux-like families of functions usually include eight classes, of which we are interested here in AC and the following two classes defined below, see e.g. [21] of [17].⁶

PR of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with *perfect road*, that is, such that for every $x \in \mathbb{R}$ there exists a perfect $P \subset \mathbb{R}$ containing x such that x is a bilateral limit point of P (i.e., with x being a limit point of $(-\infty, x) \cap P$ and of $(x, \infty) \cap P$) and that $f \upharpoonright P$ is continuous at x .

CIVP of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with *Cantor Intermediate Value Property*, that is, such that for all $p, q \in \mathbb{R}$ with $f(p) \neq f(q)$ and for every perfect set K between $f(p)$ and $f(q)$, there exists a perfect set P between p and q such that $f[P] \subset K$.

Clearly $\text{CIVP} \subset \text{PR}$. It is known that $D \cap B_1 = AC \cap B_1 = \text{PR} \cap B_1 = \text{CIVP} \cap B_1$, see e.g. [21].

The main result of this section is as follows.

Theorem 5.1 *If either $\text{cov}(\mathcal{M}) = \mathfrak{c}$ or $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$, then the following classes are non-empty: $SZ(\text{Bor}) \cap AC \setminus \text{PR}$, $SZ(\text{Bor}) \cap AC \cap \text{PR} \setminus \text{CIVP}$, and $SZ(\text{Bor}) \cap AC \cap \text{CIVP}$.*

Under the assumption that $\text{cov}(\mathcal{M}) = \mathfrak{c}$ and for the class SZ in place of $SZ(\text{Bor})$ this result was previously known, see [12] and the citations included there. However, nothing was previously known about these classes when $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$ and this is where the novelty of Theorem 5.1 lies.

It is also clear that among all subclasses in the algebra of Darboux-like classes of maps the only subclasses of AC that can have non-empty intersection with SZ are those listed in Theorem 5.1.

Our proof of Theorem 5.1 relies on the following simple consequence of Lemma 2.3.

Lemma 5.2 *Let $M \in \mathcal{M} \cap \mathcal{N}$. If either $\text{cov}(\mathcal{M}) = \mathfrak{c}$ or $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$, then there exists an f from $E \subset \mathbb{R} \setminus M$ into \mathbb{R} such that:*

- (A) $K \cap f \neq \emptyset$ for every blocking set K ;
- (B) $|f \cap h| < \mathfrak{c}$ for every $h \in \text{Bor}$;
- (C) f is unbounded on $P \cap E$ for every perfect subset P of $\mathbb{R} \setminus M$.

Proof Fix an enumeration $\{P_\alpha : \alpha < \mathfrak{c}\}$ of all perfect subsets of \mathbb{R} disjoint with M . Let $\text{Bor} = \{h_\alpha : \alpha < \mathfrak{c}\}$ and let κ and $\langle \langle G^\alpha, f^\alpha, D_i^\alpha \rangle : i < 2 \ \& \ \alpha < \mathfrak{c} \rangle$ be defined such that for every $\alpha < \mathfrak{c}$: $D^\alpha := \bigcup_{i < 2} D_i^\alpha, f^\alpha : D^\alpha \rightarrow \mathbb{R}$,

- (M) if $\text{cov}(\mathcal{M}) = \mathfrak{c}$, then $\kappa = \omega, \mathbb{R} \setminus G^\alpha \in \mathcal{M}$, and $h_\alpha \upharpoonright G^\alpha$ is continuous;
- (N) if $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$, then $\kappa = \text{non}(\mathcal{N})$ and $G^\alpha = \mathbb{R}$;
- (D₀) D_0^α is a dense subset of $Z_\alpha := G^\alpha \setminus \bigcup_{\beta < \alpha} (D^\beta \cup \text{dom}(h_\alpha \cap h_\beta))$ of cardinality $\leq \kappa$ which in the case when $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$ has also the property that $B \cap D^\alpha \neq \emptyset$ for every $B \in \mathcal{B} \setminus \mathcal{N}$ such that $B \cap E \in \mathcal{N}$ for every E in the family $\mathcal{E}_\alpha := \{\text{dom}(h_\alpha \cap h_\beta) : \beta < \alpha\}$;

⁶ Other classes include: D , earlier mentioned class Conn of connectivity functions, SCIVP which is obviously disjoint with SZ , Ext contained in SCIVP so also disjoint with SZ , and the largest class, PC , of peripherally continuous functions. It is known that $\text{Ext} \subsetneq AC \subsetneq \text{Conn} \subsetneq D \subsetneq \text{PC}$ and $\text{Ext} \subsetneq \text{SCIVP} \subsetneq \text{CIVP} \subsetneq \text{PR} \subsetneq \text{PC}$.

- (F_0) $f^\alpha(x) = h_\alpha(x)$ for every $x \in D_0^\alpha$;
- (D_1) D_1^α is a countable dense subset of $P_\alpha \setminus (D_0^\alpha \cup \bigcup_{\beta < \alpha} D^\beta)$;
- (F_1) there is an enumeration $\{x_n : n < \omega\}$ of D_1^α such that for every $n < \omega$ we have $f^\alpha(x_n) \in (n, \infty) \setminus \{h_\beta(x_n) : \beta < \alpha\}$.

Such a sequence can be constructed by an easy transfinite induction on $\alpha < \mathfrak{c}$, where the additional property of the set D_0^α in the case when $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$ can be ensured by Lemma 2.2 applied to the family $\mathcal{E} := \mathcal{E}_\alpha \cup \{\{x\} : x \in \bigcup_{\beta < \alpha} D^\beta\}$. We claim that $f := \bigcup_{\alpha < \mathfrak{c}} f^\alpha$ is as needed.

Indeed, the sequence $\langle \langle G^\alpha, f^\alpha \upharpoonright D_0^\alpha, D_0^\alpha \rangle : \alpha < \mathfrak{c} \rangle$ satisfies the assumptions of Lemma 2.3, so $f^0 := \bigcup_{\alpha < \mathfrak{c}} f^\alpha \upharpoonright D_0^\alpha$ and its extension f satisfy (A).

To see (B) fix an $h \in \text{Bor}$, choose $\beta < \mathfrak{c}$ such that $h = h_\beta$, and notice that $\text{dom}(f \cap h) \subset \bigcup_{\gamma \leq \beta} D^\gamma$. The property (C) is ensured by (F_1). □

Proof of Theorem 5.1 Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a function in the class $\text{SZ}(\text{Bor})$ and for an $M \in \mathcal{M} \cap \mathcal{N}$ let $f : E \rightarrow \mathbb{R}$ be as in Lemma 5.2. Define $f_M : \mathbb{R} \setminus M \rightarrow \mathbb{R}$ as $f_M := g \upharpoonright (\mathbb{R} \setminus (E \cup M)) \cup f$ and notice that f_M satisfies (A)–(C) from Lemma 5.2.

Then $f_\emptyset \in \text{SZ}(\text{Bor}) \cap \text{AC} \setminus \text{PR}$, where $f_\emptyset \notin \text{PR}$ is justified by (C).

In the remainder of this proof, fix an $M \in \mathcal{M} \cap \mathcal{N}$ defined as $M := \bigcup \mathcal{G}$, where $\mathcal{G} := \{P_{p,q} \subset (p, q) : p < q \ \& \ p, q \in \mathbb{Q}\}$ is a family of pairwise disjoint compact perfect sets from \mathcal{N} . This is the set \hat{M} defined at the beginning of section 4 in [12].

In [12, lemma 4.1] it is constructed⁷ a $g_0 : M \rightarrow \mathbb{R}$ such that

- (R) $|g_0 \cap h| < \mathfrak{c}$ for every $h \in \text{Bor}$;
- (S) for every perfect $K \subset \mathbb{R}$ and $a < b$ there exists a perfect $P \subset M \cap (a, b)$ such that $g_0[P] \subset K$.

Then $F_0 := g_0 \cup f_M$ belongs to $\text{SZ}(\text{Bor}) \cap \text{AC} \cap \text{CIVP}$.

Similarly, for \mathfrak{C} denoting the classic Cantor ternary set in $[0, 1]$, in [12, lemma 4.3] it is constructed⁸ a $g_1 : M \rightarrow \mathbb{R}$ such that

- (α) $|g_1 \cap h| < \mathfrak{c}$ for every $h \in \text{Bor}$;
- (β) $g_1[\hat{M}] \cap \mathfrak{C} = \emptyset$;
- (γ) for any $\langle s, t \rangle \in \mathbb{R}^2$ there is a perfect set $P \subset M \cup \{s\}$ having s as a bilateral limit point and such that $\lim_{x \rightarrow s, x \in P} g_1(x) = t$.

Then $F_1 := g_1 \cup f_M$ belongs to $\text{SZ}(\text{Bor}) \cap \text{AC} \cap \text{PR} \setminus \text{CIVP}$, compare the proof of [12, theorem 4.4]. □

In [12] it is also shown that, besides classes listed in Theorem 5.1, also the following classes are nonempty under the assumption that $\text{cov}(\mathcal{M}) = \mathfrak{c}$:

- $\text{SZ} \cap (\text{D} \setminus \text{Conn}) \setminus \text{PR}$,
- $\text{SZ} \cap (\text{D} \setminus \text{Conn}) \cap \text{PR} \setminus \text{CIVP}$,
- $\text{SZ} \cap (\text{D} \setminus \text{Conn}) \cap \text{CIVP}$,
- $\text{SZ} \cap (\text{Conn} \setminus \text{AC}) \setminus \text{PR}$,
- $\text{SZ} \cap (\text{Conn} \setminus \text{AC}) \cap \text{PR} \setminus \text{CIVP}$,
- $\text{SZ} \cap (\text{Conn} \setminus \text{AC}) \cap \text{CIVP}$.

In this context the following is a natural question.

⁷ Actually, in [12, lemma 4.1] the property (A) is proved only for the partial continuous functions h . However, the same proof gives also our version of (A).

⁸ The comment on the construction of g_0 applies here as well.

Problem 5.3 Does $\text{non}(\mathcal{N}) < \text{cov}(\mathcal{N}) = \mathfrak{c}$ imply that the above six classes are non-empty? What about when the class SZ is replaced with SZ(Bor)?

It is our believe that the answer to this question, in its both versions, is positive. The key of proving this result should be to first prove that, under our assumption, the classes $\text{SZ}(\text{Bor}) \cap (\mathbb{D} \setminus \text{Conn})$ and $\text{SZ}(\text{Bor}) \cap (\text{Conn} \setminus \text{AC})$ are nonempty. Then technic developed in [12] and used in the proof of Theorem 5.1 should allow to refine such results to the aforementioned six classes.

It might be also interesting to consider the same problem under a weaker set theoretical assumption of just $\text{cov}(\mathcal{N}) = \mathfrak{c}$. A positive answer to this version, for at least the first three classes, is suggested by the fact that already $\text{cov}(\mathcal{N}) = \mathfrak{c}$ implies $\text{SZ}(\text{Bor}) \cap \mathbb{D} \neq \emptyset$, as shown in [18, theorem 4.4].

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