

## Circular strings, wormholes, and minimum size

Luis J. Garay, Pedro F. González-Díaz, Guillermo A. Mena Marugán, and José M. Raya  
*Centro de Física Miguel A. Catalán, Instituto de Matemáticas y Física Fundamental, CSIC Serrano 121, 28006 Madrid, Spain*  
 (Received 17 January 1997)

The quantization of circular strings in an anti-de Sitter background spacetime is performed, obtaining a discrete spectrum for the string mass. A comparison with a four-dimensional homogeneous and isotropic spacetime coupled to a conformal scalar field shows that the string radius and the scale factor have the same classical solutions and that the quantum theories of these two models are formally equivalent. However, the physically relevant observables of these two systems have different spectra, although they are related to each other by a specific one-to-one transformation. We finally obtain a discrete spectrum for the spacetime size of both systems, which presents a nonvanishing lower bound. [S0556-2821(97)02412-0]

PACS number(s): 11.25.Db, 04.60.Ds, 04.60.Kz

### I. INTRODUCTION

Circular strings in curved backgrounds have been systematically studied [1–4], showing interesting deviations from their behavior in flat spacetimes. Exact classical solutions were found in Ref. [1], a canonical analysis was performed in Ref. [2], and a semiclassical quantization was carried out in Ref. [3].

One can realize that circular strings in an anti-de Sitter background classically expand and contract [1] in the same form as the scale factor of a four-dimensional homogeneous and isotropic spacetime conformally coupled to a massless scalar field in the presence of a negative cosmological constant [5]. In the Lorentzian regime, they both expand from zero to a maximum radius and then recollapse, following the same geometrical pattern. In the Euclidean regime, two asymptotically large regions of the string world sheet are connected by a throat just as happens in the wormhole minisuperspace model of Ref. [5]. One of the aims of this work is to find out to what extent this equivalence between circular strings and wormholes survives quantum mechanically.

We perform a proper quantization of circular strings in an anti-de Sitter background spacetime following the algebraic quantization program [6]. The same procedure was used for the study of quantum wormholes in anti-de Sitter spacetime in Ref. [5], so that the comparison between these two quantized models can be easily carried out. We will see that, although the quantum physics in these two systems is not identical, an equivalence can be nevertheless recovered by means of a unitary transformation.

In addition, as a result of the quantization process, we will obtain the mass spectrum of circular strings, whose semiclassical limit had already been found in Refs. [2,3].

Finally, we will discuss another related analogy between circular strings and wormholes: the existence of a minimum length. That a minimum length may appear in string theory and quantum gravity has been suggested from various points of view (see, e.g., Ref. [7]). Most of them involve semiclassical reasonings. In the models discussed here, the throat size for both strings and wormholes can actually be quantized. We will see that the corresponding spectra are discrete and possess a nonvanishing lower bound.

### II. CLASSICAL SOLUTIONS FOR CIRCULAR STRINGS

We will start with the string action (see, for instance, Ref. [8])

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu, \quad (2.1)$$

where  $\sigma^\alpha = (\tau, \sigma)$  are the coordinates,  $h_{\alpha\beta}$  is the two-metric in the world sheet,  $X^\mu$  ( $\mu = 1, \dots, 4$ ) are the coordinates on the background target spacetime,  $g_{\mu\nu}$  is its four-dimensional metric, and  $\alpha'$  is the inverse string tension.

We now reduce our model by imposing anti-de Sitter symmetry in target spacetime and considering circular strings on the equatorial plane. More explicitly, we will consider background spacetimes of the form

$$ds^2 = s \xi(r) dt^2 + \frac{1}{\xi(r)} dr^2 + r^2 d\Omega_2^2, \quad (2.2)$$

where the radial coordinate  $r$  must be positive,  $d\Omega_2^2$  is the unit metric on the two-sphere, the function  $\xi(r)$  has the form

$$\xi(r) = 1 + \lambda r^2, \quad (2.3)$$

$s$  takes the values  $\pm 1$  depending on whether the Euclidean ( $s = +1$ ) or Lorentzian ( $s = -1$ ) regime is being considered, the cosmological constant is  $-3\lambda$ , and

$$X^1 = t(\tau), \quad X^2 = r(\tau), \quad X^3 = \theta = \frac{\pi}{2}, \quad X^4 = \sigma. \quad (2.4)$$

We will work in the gauge

$$h_{\alpha\beta} = \begin{pmatrix} sN^2 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.5)$$

The conformal gauge is achieved when the lapse function  $N$  is set equal to 1.

Then, denoting the derivative with respect to  $\tau$  by an overdot, the action takes the canonical form

$$S = \int d\tau (p_t \dot{t} + p_r \dot{r} - NH), \quad (2.6)$$

where the Hamiltonian constraint  $H$ , which is the only surviving constraint (coming from the fact that the world sheet metric  $h_{\alpha\beta}$  is nondynamical), is

$$H = \frac{1}{2} \left( \frac{\alpha'}{\xi(r)} p_t^2 + s \alpha' \xi(r) p_r^2 - \frac{r^2}{\alpha'} \right), \quad (2.7)$$

$p_t$  and  $p_r$  being the momenta canonically conjugate to the variables  $t$  and  $r$ , respectively.

Let us briefly summarize the classical propagation of this system [1]. The variable  $t$  is cyclic and therefore

$$\dot{p}_t = 0, \quad (2.8)$$

so that  $p_t$  is a constant of motion. Introducing this result in the Hamiltonian constraint, together with the expression of  $p_r$  in terms of  $\dot{r}$ , we obtain the equation (in the conformal gauge  $N=1$ )

$$\dot{r}^2 + s[\alpha'^2 p_t^2 - r^2 \xi(r)] = 0, \quad (2.9)$$

which can be solved in terms of Jacobian elliptic functions [9]

$$r(\tau) = r_m \left| \operatorname{cn} \left( \frac{\sqrt{-s} \tau}{\sqrt{1-2m}}, m \right) \right|. \quad (2.10)$$

The parameters  $m$  and  $r_m$  are functions of the constant of motion  $p_t$ :

$$m = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{1+4\alpha'^2 \lambda p_t^2}} \right), \quad r_m = \sqrt{\frac{1}{\lambda} \frac{m}{1-2m}}, \quad (2.11)$$

$r_m$  being the turning point of the potential in Eq. (2.9). The absolute value has been taken in order to account for the physical restriction  $r > 0$ . Notice then that Eq. (2.10) is just a formal solution at the singularity  $r=0$ .

The equation of motion for the variable  $t$  (again in the conformal gauge  $N=1$ ) is

$$\dot{t} = \alpha' \frac{p_t}{\xi(r)} \quad (2.12)$$

and its solution is

$$\begin{aligned} t(\tau) &= \alpha' p_t \int_0^\tau \frac{d\eta}{1 + \lambda r(\eta)^2} \\ &= \frac{2}{\sqrt{\lambda}} \sqrt{\frac{m(1-2m)}{1-m}} \Pi \left( \arccos \frac{r}{r_m}, \frac{m}{1-m}, m \right), \end{aligned} \quad (2.13)$$

where  $\Pi(\phi, n, m)$  is the elliptic integral of the third kind [9] and it is understood that the function  $\arccos$  is defined modulo  $\pi$  because of the absolute value in Eq. (2.10).

As shown in Ref. [1], inserting this classical solution in the metric element (2.2) for anti-de Sitter spacetime, we obtain the world sheet metric

$$ds^2 = r(\tau)^2 (s d\tau^2 + d\sigma^2). \quad (2.14)$$

In the Lorentzian regime ( $s = -1$ ),  $r(\tau)$  is described by an oscillating Jacobian elliptic function whose period is

$$T = 2\sqrt{1-2m} K(m), \quad (2.15)$$

where  $K(m)$  is the complete elliptic integral of the first kind [9]. The metric (2.14) describes a circular string that starts expanding from zero radius (at  $\tau = -T/2$ ) until it reaches a maximum radius (at  $\tau = 0$ ) and then recollapses again (at  $\tau = +T/2$ ). The same dynamics has been found in Ref. [5] for the scale factor of a homogeneous and isotropic spacetime conformally coupled to a massless scalar field in the presence of a negative cosmological constant. In the Euclidean regime, the classical solution describes a world sheet that is just the two-dimensional analogue of the anti-de Sitter wormhole solutions also obtained in Ref. [5].

For vanishing cosmological constant, the classical equations of motion have the solution

$$r(\tau) = r_m |\cos \sqrt{-s} \tau|, \quad t(\tau) = \alpha' p_t \tau. \quad (2.16)$$

Here,  $r_m^2 = \alpha'^2 p_t^2$ . Again, we see that this classical behavior is the same as that of the scale factor of the minisuperspace model mentioned above (for  $\lambda=0$ ) [5].

### III. QUANTIZATION OF CIRCULAR STRINGS

Starting from the anti-de Sitter case, we now proceed to quantize circular strings. The formalism for vanishing cosmological constant can be attained by taking the limit  $\lambda \rightarrow 0$ .

In the Lorentzian regime, the classical variable  $r(\tau)$  is periodic with period  $T$  given in Eq. (2.15), and it then follows from Eq. (2.13) that the variable  $t(\tau)$  satisfies

$$t(\tau + T) = t(\tau) + t(T), \quad (3.1)$$

where

$$t(T) = \sqrt{\frac{4m}{\lambda} \frac{1-2m}{1-m}} \Pi \left( \frac{m}{1-m}, m \right) \equiv \mathcal{T}(\alpha' p_t), \quad (3.2)$$

the function  $\Pi(n, m)$  being the complete elliptic integral of the third kind [9]. From this property, we can easily check that the classical solution  $r(t)$  is periodic in  $t$  with period  $\mathcal{T}$ . Consequently,  $2\pi t/\mathcal{T}$  can be regarded as an angular variable defined on  $S^1$ . Besides, the classical solution does not depend on the sign of it. Then, time reversal symmetry allows us to restrict  $p_t$  to be positive. We can perform a canonical change of variables from  $(t, p_t)$  to  $(\beta, j)$  defined by

$$\beta = \frac{2\pi t}{\mathcal{T}(p_t)}, \quad j = \frac{1}{2\pi\alpha'} \int_0^{\alpha' p_t} dz \mathcal{T}(z) \equiv G(p_t^2), \quad (3.3)$$

where, in terms of the modulus  $m$  defined in Eq. (2.11), the function  $G$  has the form<sup>1</sup>

$$G(p_t^2) = \frac{1}{\pi\alpha'\lambda} \frac{1}{\sqrt{1-2m}} \left\{ m\Pi\left(\frac{m}{1-m}, m\right) + [(1-m)K(m) - E(m)] \right\}, \quad (3.4)$$

$E(m)$  being the complete elliptic integral of the second kind [9]. One can check that

$$G(x) \sim \frac{1}{4}\alpha'x \quad \text{when } \lambda x \rightarrow 0, \quad (3.5)$$

$$G(x) \sim \frac{1}{2\sqrt{\lambda}}\sqrt{x} \quad \text{when } \lambda x \rightarrow +\infty. \quad (3.6)$$

Moreover, the function  $G(x)$  is continuous, strictly increasing, and its image is the whole positive real axis, i.e.,  $G(\mathbb{R}^+) = \mathbb{R}^+$ . Therefore, the function  $G(x)$  is invertible in  $\mathbb{R}^+$ , and so is the change of variables from  $(t, p_t)$  to  $(\beta, j)$ . Note that  $\beta \in S^1$ ,  $j \in \mathbb{R}^+$ , and their Poisson brackets are  $\{\beta, j\} = 1$ , as should happen for action-angle variables.

Then, the first term of the Hamiltonian (2.7), which contains all the dependence on  $t$  and  $p_t$ , can be written as

$$\frac{\alpha'}{2\xi(r)}p_t^2 = \frac{\alpha'}{2\xi(r)}G^{-1}(j), \quad (3.7)$$

with  $G^{-1}$  being the inverse of  $G$ .

Let us now concentrate on the remaining part of the Hamiltonian constraint. We can employ the following canonical variables to describe the radial phase space:

$$\tilde{h} = \frac{1}{2}\xi(r)\left(\alpha'\xi(r)p_r^2 + \frac{r^2}{\alpha'}\right) \quad (3.8)$$

and

$$\begin{aligned} \tilde{\theta} &= \int_{r_m}^r \frac{dz}{\xi(z)\sqrt{2\alpha'\tilde{h} - z^2\xi(z)}} \\ &= \frac{(1-2m)^{3/2}}{1-m} \Pi\left(\arccos \frac{r}{r_m}, \frac{m}{1-m}, m\right), \end{aligned} \quad (3.9)$$

where the parameter  $m$  is given in Eq. (2.11) with  $p_t^2$  replaced by  $2\tilde{h}/\alpha'$  and, as before, it is understood that the function  $\arccos$  is defined modulo  $\pi$ . Note that  $\tilde{h}$  is always positive. Also,  $r(\tilde{\theta}, \tilde{h})$  can be shown to be periodic in  $\tilde{\theta}$  with period  $\mathcal{T}(\sqrt{2\alpha'\tilde{h}})/\sqrt{2\alpha'\tilde{h}}$  and one can see that this period lies within the interval  $(0, \pi)$  for positive  $\tilde{h}$ .

Another canonical transformation provides us with the angle-action variables for the radial part

$$\theta = \frac{2\pi\sqrt{2\alpha'\tilde{h}}}{\mathcal{T}(\sqrt{2\alpha'\tilde{h}})}\tilde{\theta}, \quad h = \frac{1}{2\pi}\int_0^{\tilde{h}} dz \frac{\mathcal{T}(\sqrt{2\alpha'z})}{\sqrt{2\alpha'z}} = G\left(\frac{2\tilde{h}}{\alpha'}\right), \quad (3.10)$$

where the function  $G(x)$  is given in Eq. (3.4). As corresponds to action-angle variables,  $\theta \in S^1$  and  $h \in \mathbb{R}^+$ . In terms of the angle-action variables  $(\beta, j)$  and  $(\theta, h)$ , the Lorentzian Hamiltonian acquires the expression

$$H = \frac{\alpha'}{2\xi(r)}[G^{-1}(j) - G^{-1}(h)], \quad (3.11)$$

where  $r$  is to be understood as a function of  $\theta$  and  $h$ .

We now introduce annihilation and creation variables for both pairs of angle-action variables

$$\begin{aligned} A_r &= \sqrt{h}e^{-i\theta}, & A_r^\dagger &= \sqrt{h}e^{i\theta}, \\ A_t &= \sqrt{j}e^{-i\beta}, & A_t^\dagger &= \sqrt{j}e^{i\beta}, \end{aligned} \quad (3.12)$$

which, as usual, verify  $\{A_z, A_z^\dagger\} = -i$  and  $\bar{A}_z = A_z^\dagger$  for  $z = r, t$ ; i.e., the set  $\{A_r, A_r^\dagger, A_t, A_t^\dagger\}$  is closed under Poisson brackets and complex conjugation. With them, we can construct the step variables

$$J_+ = \frac{1}{\sqrt{2}}A_r^\dagger A_t^\dagger, \quad J_- = \frac{1}{\sqrt{2}}A_r A_t, \quad (3.13)$$

which, together with  $j$  (or, equivalently,  $h$ ), form an overcomplete set of classical observables. Indeed, if we take into account that the only solution to the Hamiltonian constraint (3.11) is  $j = h$ , it is easy to check that they weakly commute with the Hamiltonian constraint. Again, the set  $\{j, J_+, J_-\}$  is closed under complex conjugation,

$$\bar{j} = j \in \mathbb{R}^+, \quad \bar{J}_+ = J_-, \quad (3.14)$$

as well as under Poisson brackets, since, on the constraint surface, they generate the Lie algebra of  $\text{SO}(2,1)$ :

$$\{J_+, J_-\} = ij, \quad \{J_+, j\} = iJ_+, \quad \{J_-, j\} = -iJ_-. \quad (3.15)$$

In order to quantize the system, we need to promote the classical observables to quantum operators that act on a vector space. If we introduce the variables

$$\tilde{t} = \frac{1}{\sqrt{2}}(A_t + A_t^\dagger), \quad \tilde{r} = \frac{1}{\sqrt{2}}(A_r + A_r^\dagger) \quad (3.16)$$

and their canonically conjugate momenta, we can choose as representation space the vector space of complex functions on  $\mathbb{R}^2$  spanned by the basis

$$\psi_{nm}(\tilde{r}, \tilde{t}) = \varphi_n(\tilde{r})\varphi_m(\tilde{t}), \quad \tilde{r}, \tilde{t} \in \mathbb{R}, \quad (3.17)$$

where  $n, m$  are non-negative integers and  $\varphi_n(x)$  are the normalized harmonic oscillator wave functions. The classical annihilation and creation variables  $A_z$  and  $A_z^\dagger$  are now promoted to linear operators on this space, with action on the basis  $\psi_{nm}$  given by

<sup>1</sup>This function coincides with the function  $W(k)$  found in Ref. [3] up to a factor  $4\pi$ .

$$\begin{aligned}\hat{A}_t \psi_{nm} &= \sqrt{m} \psi_{n(m-1)}, & \hat{A}_t^\dagger \psi_{nm} &= \sqrt{m+1} \psi_{n(m+1)}, \\ \hat{A}_r \psi_{nm} &= \sqrt{n} \psi_{(n-1)m}, & \hat{A}_r^\dagger \psi_{nm} &= \sqrt{n+1} \psi_{(n+1)m}.\end{aligned}\quad (3.18)$$

We also promote the classical observables  $j$ ,  $h$ , and  $J_\pm$  to quantum operators  $\hat{j}$ ,  $\hat{h}$ , and  $\hat{J}_\pm$  defined as

$$\begin{aligned}\hat{j} &= \frac{1}{2}(\hat{A}_t^\dagger \hat{A}_t + \hat{A}_r \hat{A}_r^\dagger), & \hat{h} &= \frac{1}{2}(\hat{A}_r^\dagger \hat{A}_r + \hat{A}_t \hat{A}_t^\dagger), \\ \hat{J}_+ &= \frac{1}{\sqrt{2}} \hat{A}_t^\dagger \hat{A}_r^\dagger, & \hat{J}_- &= \frac{1}{\sqrt{2}} \hat{A}_r \hat{A}_t,\end{aligned}\quad (3.19)$$

and whose action on the basis  $\psi_{nm}$  is

$$\begin{aligned}\hat{h} \psi_{nm} &= (n+1/2) \psi_{nm}, \\ \hat{j} \psi_{nm} &= (m+1/2) \psi_{nm}, \\ \hat{J}_+ \psi_{nm} &= \frac{1}{\sqrt{2}} \sqrt{(n+1)(m+1)} \psi_{(n+1)(m+1)}, \\ \hat{J}_- \psi_{nm} &= \frac{1}{\sqrt{2}} \sqrt{nm} \psi_{(n-1)(m-1)}.\end{aligned}\quad (3.20)$$

We can choose as the inner product in our representation space that of  $L^2(\mathbb{R}^2)$ , which guarantees that the complex conjugation relations can be directly translated to adjointness relations between our operators. Since the functions  $\psi_{nm}$  are orthonormal with this inner product, they provide a spectral resolution of the identity in terms of eigenfunctions of  $\hat{j}$  and  $\hat{h}$ . Recalling that the function  $G$ , and hence  $G^{-1}$ , is continuous, the spectral theorem [10] allows us then to define the operators  $G^{-1}(\hat{j})$  and  $G^{-1}(\hat{h})$  in the form

$$\begin{aligned}G^{-1}(\hat{h}) \psi_{nm} &= G^{-1}(n+1/2) \psi_{nm}, \\ G^{-1}(\hat{j}) \psi_{nm} &= G^{-1}(m+1/2) \psi_{nm}.\end{aligned}\quad (3.21)$$

Hence, the quantum solutions to the Hamiltonian constraint form the subspace  $V_p$  spanned by the wave functions  $\psi_{nm}$ . The Hilbert space of physical states is then just the Hilbert completion  $H_p$  of  $V_p$  with respect to the inner product of  $L^2(\mathbb{R}^2)$ .

Finally, from Eq. (3.20), it is easy to see that  $\hat{j}$  and  $\hat{h}$  coincide on  $H_p$  and  $\hat{J}_\pm$  leave the space  $H_p$  invariant. Therefore, we conclude that  $\hat{j}$  and  $\hat{J}_\pm$  are quantum observables. Furthermore, under commutators, they generate the Lie algebra of  $\text{SO}(2,1)$  on the physical space  $H_p$ . Thus,  $H_p$  carries a linear representation of the algebra of physical observables, which, in addition, can be seen to be irreducible.

#### IV. MASS SPECTRUM

The momentum  $p_t$  is a conserved quantity that is associated with an isometry of anti-de Sitter spacetime. It generates translations along the timelike Killing direction. Thus,

we can identify  $p_t$  with the string mass  $M$ . Indeed, a detailed calculation, using the embedding of the four-dimensional anti-de Sitter spacetime in a flat five-dimensional one, shows that the Casimir of the anti-de Sitter group is given by  $C = \alpha' p_t^2$ . Since we are considering homogeneous and isotropic configurations, the Casimir will directly give the dimensionless string mass, i.e.,  $C = \alpha' M^2$ , and, consequently,

$$M = p_t. \quad (4.1)$$

This result coincides with the semiclassical expression [3]

$$M = \frac{dS_{\text{class}}}{dT}, \quad (4.2)$$

where the action of the classical solution is

$$S_{\text{class}} = \frac{2}{\alpha' \lambda} \frac{1}{\sqrt{1-2m}} [E(m) - (1-m)K(m)] \quad (4.3)$$

and  $T$  is given in Eq. (3.2). From Eq. (4.1), we can obtain the circular string mass spectrum

$$M_n^2 = G^{-1}(n+1/2). \quad (4.4)$$

As we have seen, in the limit of vanishing cosmological constant,  $G(x) \sim \alpha' x/4$ , so that the mass spectrum in this limit turns out be

$$\alpha' M_n^2 \sim 4(n+1/2) + O(\lambda^{3/2} \ln \lambda) \quad \text{when } \lambda \rightarrow 0, \quad (4.5)$$

which corresponds to the well-known result for flat spacetime, except for the ground state, as the contribution from nonhomogeneous modes have not been considered here [8].

We can also expand  $G^{-1}(n+1/2)$  for large values of  $n$  and fixed nonvanishing  $\lambda$ , obtaining the following asymptotic behavior of the mass spectrum:

$$\begin{aligned}\alpha' M_n^2 &\sim 4\alpha' \lambda n^2 + \frac{32\sqrt{\pi\alpha'\lambda}}{\Gamma(\frac{1}{4})^2} n^{3/2} + \left(4\alpha'\lambda + \frac{128\pi}{\Gamma(\frac{1}{4})^4}\right) n \\ &+ O(\sqrt{n}),\end{aligned}\quad (4.6)$$

whose leading term coincides with that obtained in Ref. [3].

#### V. MINIMUM SIZE IN STRINGS AND GRAVITY

The classical solution for the radius  $r(\tau)$  of a circular string is entirely equivalent to the classical scale factor of a homogeneous and isotropic four-dimensional spacetime conformally coupled to a massless scalar field, as we have already pointed out. In view of this geometrical coincidence, we now address the question of whether an analogous relation between both systems still holds quantum mechanically.

Actually, there exists an obvious isomorphism between the Hilbert spaces and the algebras of observables of both models that preserve the adjointness relations among the observables (see Ref. [5]). In this sense, we can say that these two systems are quantum mechanically equivalent. However, we will see that observables with direct physical meaning,

such as the maximum radius of the string and that of the universe, have different spectra. Thus, the physical features reduce, in principle, the above equivalence to a formal level. Despite the physical differences between both models, they can be nonetheless related to each other by a well-defined transformation mediated by the function  $G$ , given in Eq. (3.4). This transformation preserves besides some features of these systems, as happens to be the case for the existence of a minimum invariant size [7].

Let us see in some detail how this minimum size appears. We will first consider the case of circular strings. We have seen that the observable  $\hat{p}_i^2$  has a discrete spectrum, its eigenvalues being  $G^{-1}(n+1/2)$ . From this observable  $\hat{p}_i^2$  (and using the spectral theorem), we can construct another observable

$$\hat{R} = \frac{1}{\sqrt{2\lambda}} \sqrt{(1 + \alpha'^2 \lambda \hat{p}_i^2)^{1/2} - 1}, \quad (5.1)$$

which also has a discrete spectrum

$$R_n = \frac{1}{\sqrt{2\lambda}} \sqrt{[1 + \alpha'^2 \lambda G^{-1}(n+1/2)]^{1/2} - 1}, \quad (5.2)$$

which, for  $\lambda=0$ , reduces to  $R_n = \sqrt{\alpha'(n+1/2)}$ .

We see that the spectrum of the observable  $\hat{R}$  is bounded from below, its smallest eigenvalue being

$$R_0 = \frac{1}{\sqrt{2\lambda}} \sqrt{[1 + \alpha'^2 \lambda G^{-1}(1/2)]^{1/2} - 1}, \quad (5.3)$$

so that the mean value of  $\hat{R}$  in any state  $|\psi\rangle$  of the string must be larger than or equal to  $R_0$ , i.e.,  $\langle \psi | \hat{R} | \psi \rangle \geq R_0$ . From Eqs. (2.11) and (5.1), we can interpret  $\hat{R}$  as the observable corresponding to the radius of a circular string at the time of maximum expansion, which coincides with the turning point of the potential in Eq. (2.9). Since it is bounded from below, we reach the conclusion that this quantity has a minimum value. In other words, there are no quantum circular strings with a spacetime size smaller than  $R_0$ . It could be argued that, rather than using the observable  $\hat{R}$ , we could have employed the string coordinate radius  $\hat{r}$ , which may acquire smaller values. However, this quantity is not invariant under time reparametrizations, as can be easily checked. Thus the radius  $\hat{r}$  depends on the choice of the time parameter. On the other hand,  $\hat{R}$  is indeed time reparametrization invariant; i.e., it is a true gauge-independent observable.

It is also worth noting that this situation is qualitatively different from what we find in standard quantum mechanics. Indeed, the analogue of the minimum length that we have derived in the case of strings cannot be obtained in standard quantum mechanics because, in the latter, the position operator is an observable. This is due to the fact that quantum mechanics, although can be formulated in a time-reparametrization-invariant way, has a preferred time parameter given *a priori* and, consequently, we cannot require that quantum-mechanical observables commute with the generator of time reparametrizations.

We have also seen that homogeneous and isotropic spacetimes filled with conformal matter are described by the same dynamics as circular strings. Moreover, from the quantization carried out in Ref. [5], we also obtain a discrete spectrum for the observable  $\hat{A}$ , the analogue of the former  $\hat{R}$  but with  $2\hat{p}_i^2$  replaced with the energy operator of the conformal field and the string parameter  $\alpha'$  with Newton's constant  $G_N$ . This spectrum has the same form as in Eq. (5.2) with the function  $G^{-1}$  replaced with the identity function (times a factor  $8/G_N$ ):

$$A_n = \frac{1}{\sqrt{2\lambda}} \sqrt{[1 + 8G_N \lambda (n+1/2)]^{1/2} - 1}. \quad (5.4)$$

Again, it is bounded from below and discrete. The observable  $\hat{A}$  represents the radius of the universe at the time of maximum Lorentzian expansion, i.e., what we can regard as the spacetime size of the universe. The fact that this observable  $\hat{A}$  has a minimum value  $A_0$  means that the smallest size that the universe can have is not zero, but  $A_0$ . As happened in the string case, the observable  $\hat{A}$ , rather than the scale factor  $\hat{a}$ , gives the time-reparametrization-invariant size of the universe. From the Euclidean point of view,  $\hat{A}$  represents the wormhole throat radius, and the fact that  $\langle \psi | \hat{A} | \psi \rangle \geq A_0$  implies that tunneling effects in large spacetimes mediated by wormholes will also have this minimum size.  $A_0$  can thus be regarded as the smallest wormhole throat radius.

Furthermore, this may also have consequences for the low energy effective physics in a background (flat or anti-de Sitter) spacetime. In this case, it can be argued that there exists a minimum uncertainty in the position. Indeed, the quantum fluctuations of spacetime would have a minimum spacetime volume of Planck's order. This would amount to have an uncertainty in the metric of the same order and, consequently, in the determination of any spacetime distance. On the other hand, these spacetime fluctuations would have associated with them fluctuations of the matter fields that would give rise to a bare vacuum energy. It has been proposed that wormholes can effectively drive the cosmological constant to zero as seen from the low energy point of view [11] even though, at the more fundamental level, it will be present at least in the form discussed above.

This scenario is in complete agreement with many other analyses suggesting the existence of a minimum length [7]. Here, we have presented a simple example where, starting from a proper quantum theory, this minimum length appears naturally.

In this discussion, we have employed a conformal scalar field as representative of the matter content. Nevertheless, one could expect that also other fields with a discrete energy spectrum, e.g., radiation fields, would give rise to a minimum length. Moreover, even in the absence of matter, the fluctuations of the gravitational field might also induce a minimum scale.

## VI. CONCLUSION

In this work, we have constructed a canonical quantum theory for circular strings in an anti-de Sitter background.

As a result, we have obtained its mass spectrum which, in the limit of large quantum numbers, agrees with the result of Ref. [3].

We have also seen that the classical solutions for the string radius are the same as the classical solutions for the scale factor of a four-dimensional homogeneous and isotropic spacetime conformally coupled to a massless scalar field in the presence of a negative cosmological constant and that their quantum theories are equivalent via a suitable isomorphism. Despite the fact that corresponding quantities with direct physical meaning (such as the string mass and the conformal matter energy) have different spectra, the underlying isomorphism allows us to introduce a specific one-to-one transformation between these two systems.

Among the physical features that they share, it is worth

noting the existence of a minimum spacetime size which, in both cases, is a direct consequence of the discrete spectrum of the mass-energy operator.

#### ACKNOWLEDGMENTS

The authors want to thank C. Barceló for helpful discussions. L.J.G. was supported by funds provided by DGICYT and MEC (Spain) under Contract Adjunct to the Project No. PB94-0107. P.G.-D. acknowledges DGICYT for financial support under Research Projects Nos. PB94-0107 and PB93-0139, and MEC Spanish German Joint Action No. 161.B. G.A.M.M. has been partially supported by funds provided by MEC and DGICYT under Research Project No. PB93-0139.

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