PREFERENCES, CLASSIFICATION AND INTUITIONISTIC FUZZY SETS

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Abstract

In this paper we analyze some models being used for fuzzy preference representation and fuzzy classification, putting them in relation to intuitionistic fuzzy sets. In particular, preference modeling is viewed as a classification problem, which is deeply related to intuitionistic fuzzy sets. In particular, the interest of a more sophisticated model based upon intuitionistic fuzzy sets, viewed as a general classification problem, is suggested. In this way some limitation of intuitionistic fuzzy sets might be overcome.

Keywords: Classification structures, Preference structures.

1 Introduction.

Intuitionistic fuzzy sets were initially introduced by Atanassov [6] (see also [11]), as a generalization of Fuzzy Sets [27] (see also [16]). For an intuitionistic fuzzy set defined on a family X of objects under consideration, each object x is being simultaneously described by the degrees of membership and non-membership to a certain property,

$$\{(x, \mu_A(x), \nu_A(x)), x \in X\}$$

the first coordinate $\mu_A(x)$ representing the degree to which "$x \in A$" holds, and the second coordinate $\nu_A(x)$ representing the degree to which "$x \notin A$" holds.

Hence, an ordinary intuitionistic fuzzy set can be modeled as a mapping

$$\chi^A : X \to [0,1]^2$$

and by definition the restriction

$$\mu_A(x) + \nu_A(x) \leq 1 \qquad \forall x \in X$$

was imposed in [6], in such a way that

$$\pi_A(x) = 1 - \mu_A(x) + \nu_A(x) \qquad \forall x \in X$$

can be understood as a degree of *non-determinacy* of the object x with respect to the intuitionistic fuzzy set A (a recent and complete overview on intuitionistic fuzzy sets can be found in Atanassov [7]).

Although from a pure mathematical point of view, Atanassov's intuitionistic fuzzy sets can not be distinguished from interval valued fuzzy sets (see, e.g., [8, 26]), both approaches have totally different starting points. In this paper we suggest that the main reason for such a confusion is the fact that Atanassov proposed a Ruspini classification system restricted to three classes, which represents a very particular classification system.

2 Fuzzy partitions and classification systems.

It is a fact, at least in classical set theory, that each assertion goes with its negation. In particular, once we know that "$x \in A$" is true, then "$x \notin A$" is false, and the other way round too, so within the crisp context, making explicit only one of both indexes clarifies the whole situation.

But this is not the way it works under fuzziness: there is no a unique and univoque negation (see

[25]), and the knowledge of the degree to which "$x \in A$" holds does not imply in general the knowledge of the degree to which "$x \notin A$" holds, and these two values may not even sum up to 1. Hence, an intuitionistic fuzzy set can be introduced as an assignation problem of every object into two particular fuzzy clases: the degree to which "$x \in A$" holds and the degree to which "$x \notin A$" holds, still allowing a third class for *non-determinacy* (the complement of their aggregated class). As a particular case, fuzzy sets appear whenever

$$\nu_A(x) = 1 - \mu_A(x) \qquad \forall x \in X$$

In Atanassov [6] we can find an example of an intuitionistic fuzzy set not being a fuzzy set (see [7] for more details).

Anyway, the above basic definition of an intuitionistic fuzzy set can be explained as a fuzzy partition in the sense of Ruspini [24], allowing three fuzzy categories, i.e., a mapping

$$\chi^A : X \to [0,1]^3$$

where

$$\chi^A(x) = (\mu_A(x), \nu_A(x), \pi_A(x))$$

in such a way that

$$\mu_A(x) + \nu_A(x) + \pi_A(x) = 1 \qquad \forall x \in X$$

But as soon as we assimilate each ordinary intuitionistic fuzzy set with a particular classification problem, we can bring here most of the arguments contained in Amo *et al.* [3], as an alternative support for a sophisticated generalization of the above ordinary intuitionistic fuzzy sets:

1. The number of possible classes needs not to be restricted to two (fuzzy sets) or three (intuitionistic fuzzy sets) but to any family of classes allowing a classification.

2. There is no need to impose a particular structure to elements within such a family of classes (a finite linear order with five or seven classes is quite common, but any other lattice may appear).

3. Each particular structure of our family of classes will suggest certain natural links between some classes (for example, the aggregation of *consecutive* classes if they define a linear order as in [5, 10]).

4. Links between classes may not be unique in nature (several types of links may coexist for each family of classes, for example a *disjunctive* link and a *conjunctive* as in [1], see also [4] and [21, 22]).

In particular, we propose to view each intuitionistic fuzzy set as a classification system in the sense of [3], i.e. as a family of three classes, being this family of classes structured by means of a particular binary relation and where a logical triple allows the disjunction and conjunction of connected classes, plus negation (some classification system using intuitionistic fuzzy sets can be found in [17], where operators allow some learning too). The number of classes, its structure and the logical combining rules can be of course modified, leading to a more general model that may deserve more investigation.

In fact, we can find in the literature many alternative truth values structures. For example, as pointed out by Kerre [18], it should be noticed the relevance of the evaluation family of classes associated to the *four epistemic states*, truth "T", falsehood "F", ignorance "I" and contradiction "C" (see also [15]). This evaluation system can be represented as the lattice depicted below.

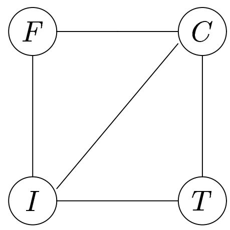

Figure 1: Original epistemic evaluation system

But such a 4-states evaluation representation is not the unique alternative for an *epistemic structure*. Since the particular structure we choose will play a key role when information is being managed, we should be extremely careful about the representation we associate to those four classes. For example, perhaps the structure below better fits our intuition in some specific problems.

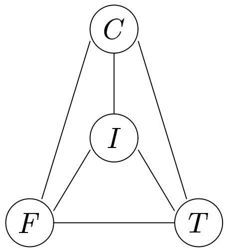

Figure 2: Alternative epistemic evaluation system

Hence, if we denote by

$$|\mathcal{C}| = card\{\mathcal{C}\}$$

the number of elements in $\mathcal{C}$, we propose that a rich enough model for many classification problems, including intuitionistic fuzzy models, is a L-fuzzy set

$$\mu : X \to [0,1]^{|\mathcal{C}|}$$

where a graph is being associated to a finite family of evaluation classes $\mathcal{C}$, together with some logical tools giving full meaning to possible combination of classes. In this way, some additional properties can be imposed, in order to assure certain behavior relative to allowed aggregation of classes (both in disjunctive and conjunctive sense) or relative to negation of classes ([25]).

3 Application to preference modeling.

A similar four-states classification system appears in preference modeling (see [13, 14] and [23] but also [9]), where preference intensity for each pair of alternatives x, y is being classified into four states too: x is worse than y ("$x < y$"), indifference ("$x \sim y$"), x is better than y ("$x > y$") and incomparability ("$x||y$").

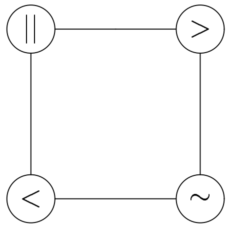

Figure 3: Preference evaluation system

In these case, states can be represented according to a *circular* structure (each vertex of the unit square is only connected with its two adjacent vertices), suggesting as *natural* certain aggregations, to be obtained by means of an appropriate disjunction operator (we could assume a fixed t-conorm [13], alternatively justified in [23], or even allow disjunction *evolve in time* as in [5, 10]). Aggregation of *non connected* classes should not be considered.

Moreover, as pointed out in [3], a conjunction operator will be also needed in order to evaluate *quality* of the classification system itself, and the whole logical structure should give us hints on how our classification system can be *improved* for future classifications (see [1]).

The need of a learning process for classification may also suggest that perhaps a nice preference structure should include, apart from the above four states $x < y$, $x \sim y$, $x > y$ and $x||y$, a *central* state meaning *undecisiveness* or *ignorance* "I", being this extra state connected with each other four states, as in the above *alternative* epistemic evaluation system: with no information, the whole preference intensity should be associated to such a state, and as far as we learn about our preferences, intensities *transfer* between connected states till they are fixed, hopefully assigning no intensity at all to such a *undecisiveness* state.

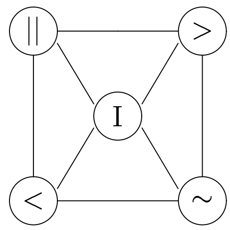

Figure 4: Alternative preference evaluation system

4 Final comments.

The examples above show that we often find out that the family of valuation classes is being defined with a binary relation associated to it, together with a logical structure defined on it. The family of valuation classes can vary, its associated binary relation can be modified, and future changes associated to some *learning process* can be supported by an arbitrary logical structure (not necessarily a standard De Morgan's triple, see [5, 10]). These arguments underlie in [1, 2, 4], where a fuzzy model was considered for the classification of land cover from remotely sensed data: each pixel was classified by means of the whole family of degrees of membership to every class under consideration. In fact, a concept should not be understood as one property, neither a property together with its negation property, but as a structured family of properties (and of course such a structure depends on the context, see [21]). An aggregation procedure can be then incorporated to our model, in order to allow the aggregated evaluation of adjacent classes (aggregation can not be properly defined for non-adjacent classes).

As pointed out above, the model we propose here to be analyzed in detail is a particular L-fuzzy set [16], where

$$L = [0,1]^{|\mathcal{C}|}$$

and $\mathcal{C}$ is a structured family of classes (a graph is being defined on it). Once a particular structure has been fixed, each object will be described by means of a vector

$$\mu(x) \in [0,1]^{|\mathcal{C}|}$$

but understanding its meaning needs the associated rules for disjunction, conjunction and negation, to be applied within the particular binary relation defined on $\mathcal{C}$ (see [1, 3], where some measures for relevance, overlapping and redundancy were considered). In this context, the recursive approach proposed in [5, 10] represents an interesting possibility for those connectives, allowing a sequential aggregation of adjacent classes.

An important specific case will be Ruspini's partition [24], where $\mathcal{C}$ is a finite set and

$$\sum_{j \in \mathcal{C}} \mu_j(x) = 1$$

Ruspini's definition did not assume any particular structure on the family of classes $\mathcal{C}$, but we should remind that the standard 5-valued scale *None, Poor, Average, Very and Complete*,

$$\mathcal{C} = \{\text{ N, P, A, V, C}\}$$

assumes a linear order.

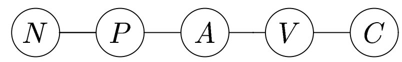

Figure 5: Standard 5-valued evaluation system

Moreover, as pointed out in [3], partitions will appear in practice only after a long learning process about the family of classes to be considered (and not always defining such a partition is an objective). The difficulties of defining a fuzzy partition can explain the relative small impact in practice of such a key concept for classification. The fact that our family of classes does not define a partition means that classes can be redefined (in order to avoid overlapping, for example), some classes are superfluous or an extra class is needed [1]). In no way should be imposed by definition that the sum or any aggregation of all degrees of membership $\{\mu_j(x), j \in \mathcal{C}\}$ must be exactly 1 for all $x \in X$ [3].

Notice also that Atanassov's intuitionistic fuzzy sets do not assume a linear valuation family of classes, as the above standard 5-valued scale assumes or the Lukasiewicz-based three-valued family of classes implies: the middle class *possibility* "P" is in between *truth* "T" and *falsehood* "F" (see e.g., [20]).

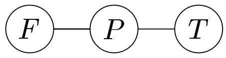

Figure 6: Lukasiewicz evaluation system

Although three classes are being also considered in Atanassov's intuitionistic fuzzy sets [6], *non-determinacy* "N" should not be considered a middle class between *truth* "T" and *falsehood* "F". Intuitionistic fuzzy sets are not Lukasiewicz-based fuzzy classification systems.

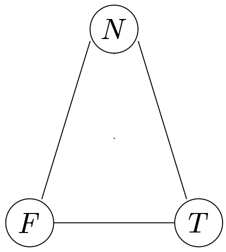

Figure 7: Intuitionistic evaluation system

The model we propose to be developed contains intuitionistic fuzzy sets, and makes extremely clear its essential classification nature, still not becoming too complex. Much more sophisticated models may find analogous criticisms as, for example, second-order fuzzy sets [12, 28]: even if a model shows great expressive power and it is conceptually appealing, if computational demands are excessive, it will almost never be utilized in any application (see [19]).

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