Chapter 6: Exploring Teacher's Epistemic Beliefs and Emotions in Inquiry-Based Teaching of Mathematics

Inés M. Gómez-Chacón and Constantino De la Fuente

Abstract In the present study, we describe the actions and decision-making of teachers in the teaching of modelling, based on their personal epistemology. The teacher's personal epistemology (epistemic reasoning, beliefs and emotions) acts as a component of the cognitive and emotional conditions of a task required of students. The strategy of analogy as a tool to foster the students' commitment and motivation in both real world and mathematical transitions, as well as the creation of multiple connections: both vertical (within the world of mathematics) and horizontal (within the real world, outside of mathematics) is prioritised. This paper proposes a conceptualisation of the term personal epistemology, from the current ontology and epistemology of mathematical knowledge.

Keywords Inquiry-based teaching · Modelling · Epistemology · Decision-making · Problem solving · Analogy

Introduction

Learning mathematics entails the development of an epistemological perspective about the content. It is recognised that the way in which mathematics is characterised in the classroom has much to do with the beliefs and epistemological views that the teacher holds. The subtle (explicit and implicit) messages communicated to students about mathematics and the nature of mathematical thinking affect, in turn, the way students grow in mathematical knowledge and the recognition they attribute to it.

When we want to promote modelling in the classroom through inquiry-based teaching we consider mathematics as "a process and an experience". Knowing maths

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is equated with doing maths. Research in mathematics education has focused on examining the characteristics of the context in which this "doing" is fostered. It is the "doing" – experimentation, abstraction, generalisation and specialisation – which constitutes mathematics, not a transmission through perfect communication by the teacher.

The 'Culture of inquiry' models the activity of research mathematicians who expand mathematical knowledge and horizons through inquiry processes (Ernest 1990). Today, one of the challenges we have is the design and implementation of these types of activities that can be motivating for the students within the cultural and institutional constraints in which they operate (Artigue and Blomhoj 2013; Jaworski 2004, 2006, 2014; Maass and Doorman 2013).

Designing research-based tasks requires deep analysis of the "mathematical experience" that is generated in both students and teachers. Thus, the creation of inquiry-based activity for their students is itself an inquiry process: teachers learn from the practices resulting from their teaching designs. This chapter proposes a reflection based both on a practical situation and on broader research (De la Fuente 2016; Gómez-Chacón and De la Fuente 2018).

We approach the "mathematical experience generated" in a modelling activity focusing on the personal epistemology of the teacher (epistemic reasoning, epistemic beliefs and emotions). Addressing the personal epistemology of the teacher and the interaction between it and mathematical modelling can provide points of interest for the design of teaching modelling. The mathematical experience is something complex which must be understood from the cognitive and affective point of view. Here we will distinguish two ways of approaching the affective dimension: (1) the threshold of the interplay between cognition and affect, which from the heuristic point of view is of great interest, (2) the distinction in the way of thinking about the emotional aspect, which has to do with the subject who feels (personal epistemology and decision making according to the identification with mathematical objects) (for further information see Gómez-Chacón (2018)).

The importance of an epistemological perspective in teaching and learning has been reviewed by different authors (Ernest 1991; Hersh 1986; Otte 1994). In this work the emphasis on epistemology is mainly presented as a meta-perspective, in how the teacher thinks about knowledge. Often a meta-perspective is expounded with a normative intent, driven by the assertion that a deeper reflection on epistemological issues would improve mathematical education. Research based on observational empirical studies of the epistemology of thinking in natural situations of interaction in the classroom (Schoenfeld 2010, 2016) is scarcer. Acquiring knowledge is not only a logical, sequential and standardised process, as the rationalists would say, but learning is considered 'intuitive'. Often the teacher is faced with questions about how to deal with modelling teaching, and how to make decisions in the face of modelling processes with students, most of which involve epistemic cognition. This study tries to identify emergent configurations; some more routinised and others more spontaneous and intuitive, which involve the decision making of the teacher and penetrate the teaching of modelling.

In what follows, sections "Personal Epistemology: Epistemic Beliefs and Emotions" and "Modelling and Epistemic Reasoning Processes" present antecedents on the topic and a theoretical frame of reference. These sections outline the concept of personal epistemology and the learning of modelling, in particular, the analogy as a heuristic tool for the transitions from real world modelling to the mathematical world. The methodology used (section "Objectives and Methodology") is briefly described, giving rise to the presentation of the results from the development of a Mathematical Research Project (MRP) in the classroom, which details the epistemic mediation of the teacher through his actions and decision making (section "Personal Epistemology of Mathematical Knowledge and Decision Making in the Classroom"), and finally some conclusions are presented.

Personal Epistemology: Epistemic Beliefs and Emotions

Personal epistemology is the study of people's thinking about knowledge and about knowing. Its study is born in the field of educational and cognitive psychology (Barzilai and Zohar 2014; Hofer and Bendixen 2012; Hofer and Pintrich 1997). Even though in the last decades the field of personal epistemology has developed in several different directions there is convergence in some central descriptive dimensions of personal epistemology: the nature of knowledge, the certainty of knowledge, the simplicity of knowledge, the source of knowledge, and the justification of knowledge (Hofer and Pintrich 1997).

Currently there is no single model guiding research on personal epistemology (Bendixen and Rule 2004). We briefly present some of the approaches to the concept of personal epistemology that the perspective of educational psychology poses: approaches to development, approaches to beliefs and approaches to resources.

Developmental Approaches

Developmental models of personal epistemology generally view students as holding integrated epistemic positions or perspectives. These models describe students' epistemic positions developing throughout the course of their life and studies, often following a typical trajectory (see Barzilai and Zohar 2014; Hofer and Pintrich 1997). Developmental approaches are concerned with identifying changes in students' thinking. Thinking skills and theories about knowledge and knowing are deeply and intricately linked, capturing the close link between people's views of knowledge and their reasoning processes by describing epistemic thinking as a "theory-in-action". The developmental perspective encompasses both the epistemic reasoning processes and the epistemic beliefs and theories that underlie them. Beliefs reflect assumptions, expectations and attitudes that may affect reasoning processes.

Leder et al. (2002), Maass and Schloeglmann (2009), and Schoenfeld (1985) have conducted extensive research in the field of beliefs in mathematical education.

In this approach, personal epistemology is a term that refers to the beliefs that people hold about knowledge, both as to its nature and its acquisition and justification (Hofer 2002). Although different models of competence have been proposed, there is a consensus that epistemological beliefs refer to "belief about the nature of knowledge and knowledge processes" (Hofer and Pintrich 1997: 112) and in some cases learning (Op't Eynde et al. 2006).

Epistemological beliefs have sometimes been explicitly described as a type of metacognitive knowledge or as schemas (Muis et al. 2015; Schoenfeld 1985). These models of epistemic beliefs and self-regulated learning are mainly concerned with understanding how and why epistemic beliefs impact learning and how they are conditions that serve as inputs to metalevel learning standards.

Resources Approach

A third important approach to the study of personal epistemology is the resources approach (Elby and Hammer 2010). This perspective emerges from the "knowledge in pieces" approach to the study and analysis of knowledge, and highlights the fragmented and contextual nature of students' epistemologies. Epistemological resources are specific cognitive resources highly linked to the context that people use to understand and reflect on their epistemic knowledge, activities and positions. Epistemological resources may gradually advance into beliefs as they become entirely articulated and more stable.

In the development of research, these approaches have often acted disjointedly, without taking into account, in an integrated way, that which each of them considers key components that underpin the concept of personal epistemology. One of the strongest criticisms of empirical studies on personal epistemology under these approaches to educational psychology is that they have little regard for the context and specific domains of knowledge (Bromme 2005).

In the field of Mathematical Education we must highlight authors who have adopted a different perspective and have implemented this epistemic integration, although not under the coined denomination of personal epistemology (Schoenfeld 2010, 2016). For example, Schoenfeld has worked with metacognition as a central aspect of cognition and has related it to belief systems (Schoenfeld 1987). The author has developed a theory of decision making, centred around teachers (Schoenfeld 2010). This work is indicative of the fact that in the field of Mathematics Education there is a fundamental and productive dialectic between theory and practice; and contextual and knowledge components, allowing for the development of observational tools for reliable naturalistic interventions in which classrooms can serve as laboratories.

The central idea of the Schoenfeld model is based on the claim that it is possible to describe, explain and predict teachers' performance, decision making and actions during teaching based on their knowledge, beliefs and goals. In the last 20 years, the literature on teachers has identified and broadly described the knowledge, beliefs and goals of the teacher. Schoenfeld proposes to go some steps further, describing the ways in which these elements interact and result in teachers' in-the-moment decision-making. In summary the Schoenfeld model is articulated in: resources (especially knowledge); goals; orientations (an abstraction of beliefs, including values, preferences, etc.); and decision-making (which can be modelled as a form of subjective cost-benefit analysis).

Based on the Schoenfeld model, a deeper exploration of the construct that this author poses with *Orientations* would be interesting. We find it crucial to pinpoint epistemic beliefs and emotions -as an operative way of unpacking this category- as well as the interaction between epistemic reasoning and epistemic beliefs/emotions reflected in decision-making (Gómez-Chacón 2017). This interaction is understood as the application of heuristics and strategies in specific contexts and specific content in order to make judgments about *right or wrong* or *true or false* in mathematical knowledge. Thus, this study proposes the term *personal epistemology* as a multi-faceted concept that operates at cognitive, metacognitive and affective levels. For an operational level of the term *personal epistemology* we have distinguished between epistemic reasoning, epistemic beliefs, epistemic emotions and decision-making.

The term *epistemic beliefs* shall be used to refer to a person's beliefs about the nature of human knowledge; its certainty and how it is conceptualised, and a person's beliefs about the criteria for, and the process of, knowing. Likewise, the epistemic emotions are defined as emotions that arise when the object is the knowledge and the processes that involve the knowing are caused by cognitive qualities of task information and the processing of that information (Gómez-Chacón 2017; Pekrun and Linnenbink-Garcia 2012).

Modelling and Epistemic Reasoning Processes

As mentioned in the preceding section, the analysis of personal epistemology will address the teacher's resources, particularly their knowledge on modelling and heuristics in problem solving.

Recognising Modelling Skills

Results from research on the learning of mathematical modelling from different perspectives (Blum and Leiß 2005; Haines and Crouch 2005), show that student learning in the transition from the real world to the mathematical model is hampered by the lack of knowledge and experience related to abstraction. This behaviour is less obvious when one moves from the mathematical model to the real world, in fact in this case the higher process level is more likely to be used.

Several research studies on modelling have considered students' mathematical modelling skills in terms of the expert-novice continuum (Crouch and Haines 2004). Students have difficulty keeping the real-world demands and the model in mind all at once. Novices tend to spend less time analysing the problem (Schoenfeld 1987), they have difficulty distinguishing between relevant and irrelevant aspects, and believe to have understood the problem sufficiently when they have not. Beginners immediately tend to start generating equations without recognising particular underlying abstract problem types or being able to access relevant concepts and procedures (Glaser and Chi 1988).

In order to tackle these challenges, the experts attempt to find an answer to the following question: How could these students further increase their level of mathematical expertise? A certain level can be achieved by extended relevant motivated practice (with feedback) on all aspects of building models for a variety of problem types (Ericsson et al. 1993) or scaffolding of technical aspects to help beginning modellers through stages what can initially appear to be demanding and unfamiliar approach to problem solving and though metacognitive modelling competences (Stillman and Galbraith 1998; Stillman 2011).

Such practice needs to be aimed towards developing recognition of underlying problem categories and formation of sufficiently and relevantly detailed problem representations that mediate between the abstract model and the real world problem context. We consider that students would also need to have extended practice to improve speed and accuracy in accessing and deploying appropriate mathematical procedures for particular model categories and with good heuristic-based strategic help supports.

As it shows by Stender (2017), up to now, there is no empirical evidence of how good heuristic-based strategic help supports the modelling process and how the support must be adapted to groups of students of different ages, knowledge or culture. In this study is indicated that this "facilitators toolkit" might be a strong instrument to support students that are working on complex modelling problems and is exemplified by the use of analogy as heuristic tool.

Real-World: Mathematical World Transitions: Analogy as a Heuristic Tool

Some aspects of analogy are specified in this section to support the understanding of the scope of the empirical data (section "Personal Epistemology of Mathematical Knowledge and Decision Making in the Classroom"). We will refer to the role of analogy in the search for patterns that allow the abstraction of contexts and generalisation of ideas, facilitating in the modelling processes the transitions from the real world to the mathematical world.

In the problem-solving mathematical understanding, analogy may be considered the prior use of solution procedures to solve the problem. Solving problems by analogy involves the recognition of high-level relationships existing between two domains, although the two domains share very few similarities in their superficial characteristics (Gentner 1983, 1989).

With respect to analogical transfer, the psychology of mathematics education distinguishes between two components of analogical thinking: access and use. Access relates to remembering the appropriate solution procedure (memory), while use refers to the correct implementation of the solution procedure. Researchers have noted that much of the inability of trainees to transfer the procedure of solving a new problem to an old one lies in access. People fail to remember the right memory solution.

Once the student is told which solution procedure to use, the solution rates increase significantly. In the field of educational psychology a theoretical access-use framework for the analogical understanding of problem-solving has been developed by Novick and Holyoak (1991) and in the mathematical framework a key model is Polya (1945, 1954).

Authors in the field of psychology (Novick and Holyoak 1991) highlight two findings that are particularly relevant to mathematical learning: first, they find that students who best solve problems are those who abstract the structural characteristics of the problem; this is what cognitive psychologists call the "induction scheme". Those who induce an appropriate scheme develop a better conceptual understanding of the type of problems represented in the experiment.

Secondly, positive correlations are found between the transfer of solutions recognised and the qualification obtained in mathematics. This type of research suggests that the so-called "conditions of applicability" are critical to success in solving problems. In other words, being able to solve the problem is contingent on being able to recognise which solution is appropriate. For this purpose, two processes are important: understanding principles and executing procedures.

According to Polya's approach, "analogy" is not a method of solution; "looking for something analogous" or other variations of "looking for a related problem". Polya presents it as a heuristic suggestion involving the articulation of processes of generalisation and specialisation. In Polya's second book dedicated to solving problems (Polya 1954) -the first volume is entitled *Induction and analogy in mathematics*- the first thing he notes is that "Yet as we start discussing analogy we tread on a less solid ground" (p. 13) and he suggests that the only way to deal with the matter in a useful way is to conceptually specify analogy.

This is what he calls "clarified analogy", whereby: "two systems are analogous if they agree in clearly definable relations of their respective parts". Thus a triangle in a plane can be said to be analogous to a tetrahedron in space, and the analogy is clarified in this case by specifying which are the relations in which they agree: two lines in a plane cannot enclose a part of it, whereas three can; likewise, three planes in space cannot enclose a part thereof, while four can: "The relation of the triangle to the plane is the same as that of the tetrahedron to space in so far as both the triangle and the tetrahedron are bounded by the minimum number of simple bounding elements" (p.14).

Table 1 Uses of analogy

Analogy. Uses in Maths class		
Use	1. Proceed by analogy (from a topic)	2. Establishment of analogies (between two topics)
Step 1	Analysis of the topic	Comparing the two topics
Step 2	Construction of a topic analogous to the initial one	Initial establishment of analogies between topics

This second book (Polya 1954) is devoted to the study of the formal structure of the reasoning made in the course of problem solving and that cannot be described with the classical deductive patterns of logic. In the Brief Dictionary of heuristics, Polya began by calling this "heuristic reasoning", whereas in the second book he called it "plausible reasoning" (Polya 1945). This clarified analogy may already be more than a suggestion, inasmuch that not only does it imply that it would not be wrong to do something, but it also specifies the type of transformation to be performed.[1]

Finally, in his third book, Polya (1962–1965) attempts to advance towards what he calls a "general method" -which, although announced in the first volume, fails to appear in the second volume- He addresses the definition of general models that could encompass many ways of elaborating problem-solving plans. It reveals the two kinds of reasoning: demonstrative reasoning, being precise, final and "automatic"; and plausible reasoning, being vague, provisional and specifically "human".[2]

In the process of conjecturing and justifying, complex chains of plausible reasoning are often elaborated, which may contain new nuances that enrich the patterns already known. A thorough analysis of these processes may make it easier to make them explicit and to model them, as this is usually done with known patterns.

The use of analogy, presented in section "Personal Epistemology of Mathematical Knowledge and Decision Making in the Classroom", which analyses a class session in the development of a MRP, will show us the epistemic type of mediations performed by the teacher.

Two uses of analogy have been identified (De la Fuente 2016) (Table 1):

Use 1: Proceed by analogy. This use is based on a topic or mathematical entity in a given context (a conceptual or procedural idea, a result, a model, a structure, . . .), analysing its qualities and proceeding by analogy building (designing or elaborating); in another context (more general or more concrete, generalising or particularising) another topic or analogous entity, which maintains the characteristics of the former, adapted to the new context. This process has two steps: (1) analysis of the mathematical topic; (2) construction of the analogue topic.

[1]Clarified analogy is a heuristic tool, the word "tool" meaning instrument of transformation. "To make a table" it is defined as a "skill" since it lacks the quality of "transformation" of the problem.

[2]The emphasis was added by the author.

Use 2: Establish analogies. This second use is based on two topics, entities or mathematical ideas (concepts, formulas, strategies or other procedures, demonstrations, results, models, structures, . . .) that are compared with each other. This is carried out through a comparative analysis using criteria to detect and establish possible similarities and common characteristics. The analogy is established when the common structure underlying both, independent of the contexts, is revealed; it is then said that, for those criteria, the two entities are analogous. In this case the process also has two steps: (1) comparing topics; (2) establishment of analogies.

In a MRP these two uses can be sequenced: first use 2 and then use 1; That is to say, once the analogy between the two mathematical topics (use 2) is established by analogy (use 1), in order to construct another entity of the same type, situated in a more general or more concrete context, by generalising or particularising, depending on the case, the initial entities.

Objectives and Methodology

Objectives

The aim of the study is to explore the interrelationships between teacher's personal epistemology and the knowledge of mathematical practice in the classroom. We show how the teacher's decision-making in the classroom and the self-regulation of learning in modelling activities is supported by the teacher's personal epistemology. In particular, the objectives of the study are:

1. To identify how the personal epistemology of the teacher (epistemic beliefs and emotions) act as a component of the cognitive and emotional conditions of a task required of students;
2. To explore what determines what that teacher does, on a moment-by-moment basis and what shaped the teachers' decision-making in the teaching of modelling.

Methods

The qualitative methodology used is based on methods of observation and case study (Bassey 1999). The criterion determining this case is that, on the one hand, this case is within a convenience sample, a selected sample category (Gliner et al. 2009) and on the other hand, the case (*Teacher-FC*) is a key informant, a secondary mathematics teacher considered of excellence, which allows to illustrate the interface between personal epistemology and inquiry-based teaching for modelling processes with a wealth of data.

We focus on one modelling activity, where the inquiry based-learning is the mathematical activity involved in the process of transforming a problem-solving task (PST) in a Mathematical Research Project (MRP). The Mathematical research project (MRP) is the real investigation, based on the initial problem statement, the steps followed being completely analogous to those of scientific research, with the teacher acting as project advisor. The process of generating a Mathematics Research Project (MRP) from a particular problem will not only require the students' creativity, but also the teacher's mediation for the establishment of a suitable creative mathematical working space.

The MRP: *Functional models for modifying exam marks* (see section "Personal Epistemology of Mathematical Knowledge and Decision Making in the Classroom"), was carried out with High School Students (17 years old), 25 students (15 boys and 10 girls) by the secondary mathematics teacher, denoted by Teacher-FC. Section "Personal Epistemology of Mathematical Knowledge and Decision Making in the Classroom" presents the activity developed during four class sessions, each lasting 1 h.

The research team includes two researchers and a secondary mathematics teacher (*Teacher-FC*). The method used for data collection has been the observation of classroom sessions, materials generated by the teacher and students in the development of the class and semi-structured interviews with the teacher. Although we focus here on a MRP, the study carried out with this teacher has been going on for more than 4 years with a diversity of projects and activities (De la Fuente 2016).The lesson was analysed by the research team. The analysis proceeded in stages. We decomposed the lesson into smaller and smaller "episodes," noting for each episode which goals were present, and observing how transitions corresponded to changes in the goals and personal epistemology of the teacher. In this way, we decomposed the entire lesson – starting with the lesson as a whole, and ultimately characterising what happened on a line-by-line basis.

The next step was to codify the material, taking as codes the indicators of actions and decisions made in the classroom by the teacher related to teach the analogy as a heuristic tool, together with a number of epistemological aspects (epistemic reasoning, epistemic beliefs and epistemic emotions) that had been registered in the teacher interviews. Encoding and analysing in detail each significant piece of every session permitted the understanding of the process of inquiry-based teaching and personal epistemology development and to report on the factors responsible for it: goals, modelling and epistemic reasoning, beliefs or strategic heuristic.

Section "Epistemic Reasoning in Act: Actions and Decisions in the Classroom" shows the whole lesson and breaks it into major episodes (lesson segments), each of which has its own internal structure according to a main category: the analogy process as a heuristic tool for facilitating the processes of modelling (the transitions from the real world to the mathematical world). The analogy topic is selected because it is noted to be frequently used after a 2-year observation period of this teacher. It is a key tool in the actions and decisions taken in the classroom.

Personal Epistemology of Mathematical Knowledge and Decision Making in the Classroom

This section presents the results of the study. Before the study described, it was analysed based on observations that Teacher-FC used analogy quite often (emerges as a pattern of behaviour). The main result explained here is that, the teacher uses analogy in the teaching of Mathematical research project (MRP) as a heuristic tool to foster student engagement several times and that this frequency of use comes and is linked to the personal epistemology of the teacher.

Firstly, we present the results that come from the lesson analysis, in natural situations of interaction in the classroom, where the teacher takes in the situation and adapts accordingly (section "Epistemic Reasoning in Act: Actions and Decisions in the Classroom"). Certain pieces of information and knowledge become salient and are activated and they show up individual's resources, goals, and epistemic reasoning and beliefs, allowing us to model their behaviour. The sections "Epistemic Beliefs and Emotions" and "Epistemology and Ontology of Mathematical Knowledge", based on the data from lesson analysis and the interviews with the teacher, show up the dimensions that support the teaching practice of Teacher-FC: the dimension of interconnectivity between epistemic beliefs and emotions and decision making in the classroom; the dimensions of process and of the coherence of the mathematical experience offered to students based on the epistemology and ontology of mathematical knowledge.

Epistemic Reasoning in Act: Actions and Decisions in the Classroom

The mathematical modelling activity based on the study of functional models, which relates everyday life to the school environment was proposed to work in the class by Teacher-FC. The statement is as follows:

> *Mathematical Research Project (MRP): Functional models for modifying exam marks*[3]
>
> A high school student returned home saying that his math teacher was dissatisfied with his students' marks in a written test that they had done about functions, attributing it to perhaps the proposed questions had been rather difficult. The teacher decided to "fit" those marks using a correction factor: if the original mark was x (on a scale of 0 to 100), it would be $10\sqrt{x}$. That is, if the initial mark was 81, the corrected grade would be 90. Apparently, this factor is commonly used among teachers in Israel

The class sessions to carry out this project were analysed to attempt to answer the questions: What determines what that teacher does, on a moment-by-moment basis

[3]The statement was adapted from Arcavi, A. (2007). El desarrollo y el uso del sentido de los símbolos. UNO. Revista de Didáctica de las Matemáticas, nº 44, 59–75.

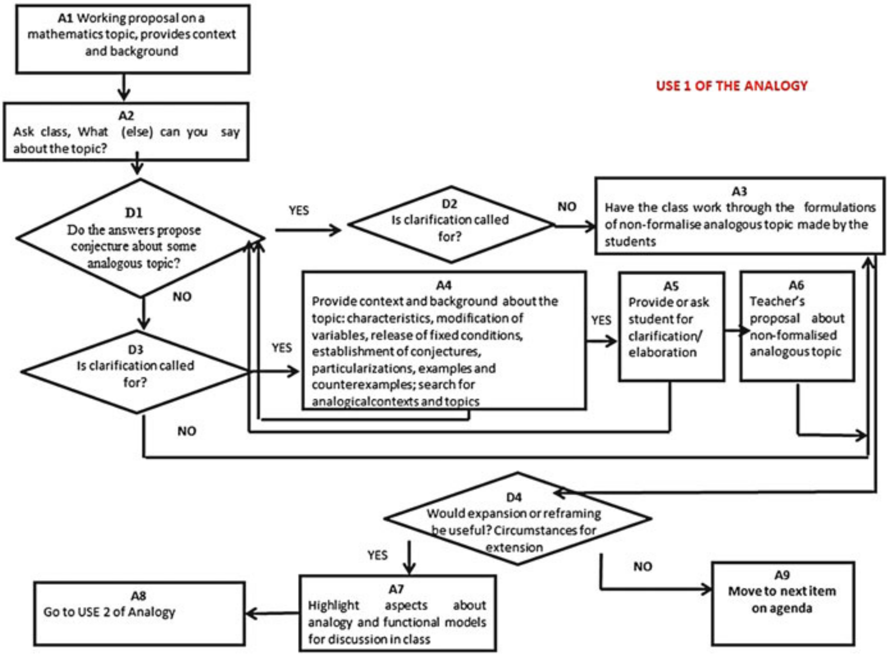

Fig. 1 Diagram of actions and decisions for discussing the MRP: "Functional models for modifying exam marks" (A1, A2, A3 . . . and D1, D2, D3 . . . coding refers to actions and decisions made in the classroom)

and what shaped teachers' decision-making in the teaching of modelling? Figure 1 synthesises the process followed with respect to the actions and decisions carried out in the classroom development by the Teacher-FC. An in-depth analysis showed that the actions and decisions had an articulating axis: the use of the heuristic tool analogy. The figure represents the pattern (routine) that the teacher uses in the access and use 1 of the analogy (section "Modelling and Epistemic Reasoning Processes").

For the *Teacher-FC* an epistemic belief is that:

> The most important mathematical knowledge is the ways of doing and the specific methods of working in mathematics. Among this knowledge I highlight the search for patterns, regularities and mathematical laws, in changing processes, from particularisations and generic examples. (Teacher-FC Interview 2016)

Although *Teacher-FC* did not explicit in interviews, the repeated observation of his mathematical practice in class (example is the MRP presented here, see lesson episodes related to D1 and A3, A4, A6, A7, A8 in Fig. 1) has led us to claim that: the patterns to be developed further are the inductive and analogical patterns. The dominance of analogy is shown as an epistemic reasoning in act (tacit), moment by moment in the class.

Following we illustrate this diagram describing the lesson episodes and pointing out the actions and decisions that are produced according to his goals and personal epistemology.

[A1[4]]: As the factor presented is common in Israel but it is totally unknown in Spain the teacher decided to present the previous paragraph in two phases: the first of the two points, asking the students to propose some appropriate correction factors if the exam was set by them.

What correction factors can we propose to the teacher in order to modify the marks? Express them in algebraic form and represent them graphically. Analyse the advantages and disadvantages of each.

[A2]: *Teacher-FC* asks the class: What can be said about the subject? After a short discussion, several correction factors are presented by the students. No student agrees with the Israeli teacher' proposed factor. Once formalized, we could describe them as follows: if x is a mark belonging to the interval [0, 10] and the mark obtained when correcting x, we can:

Increase all of the marks by the same fixed amount c, $y = x + c$

Increase each mark by a percentage, r, $y = x + \frac{rx}{100} = \left(1 + \frac{r}{100}\right)x$

Round the mark to the nearest whole number, which is greater than or equal to the mark, $y = Ent[x] + 1$

If the highest mark is the value a, transform this mark into 10 and transform the rest proportionally, $y = \frac{10}{a}x$

As we can see they are unformalized factors: raise all of the marks by a quantity, raise it by a percentage, round, etc. Expressed like this. Then the teacher asks them to express them in a more rigorous, formalised form and to represent them graphically, with the aim, if they had not seen it yet, that they see that they are not analogous to the Israeli model.

[D1-D3]: Once the advantages or disadvantages of each of them were discussed[5] and facing up to the fact that the students' answers do not formulate conjecture on some topic analogous to the Israeli' model the second part is formulated by the teacher.

[A4]: The teacher deliberately requests a factor analogous to the Israeli for the Spanish context.

[A5]: The key is that the marks are between 0 and 10. The teacher asks to the students how to elaborate and obtain them.

[4][A1], [A2], [A3] ... and [D1], [D2], [D3] ... coding refers to actions and decisions made in the classroom, see Fig. 1.

[5]It is omitted in order to not to lengthen the document, but they have a lot of didactic interest; for example, some images of interval values [0, 10] do not always remain in the interval, so new marks may be impossible values: for example 12, etc.

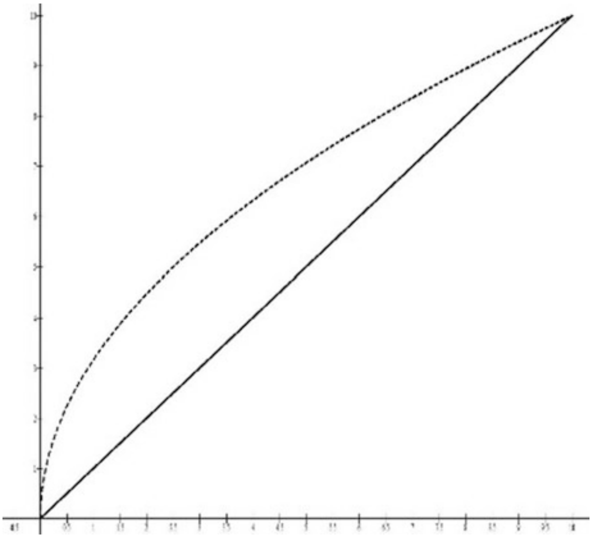

Fig. 2 Factor representation $y = x$; $y = \sqrt{10x}$

Adapt the correction factor of the teacher to our country, where the marks are between 0 and 10. Express it algebraically and make its graphical representation. Analyse its advantages and disadvantages with respect to the previous ones.

After a while, the students proposed to the teacher the desired factor (underlying [D1]) (Fig. 2).

[D2]: As no further clarification is required, the teacher made the class work through the formulations made by students of the non-formalized analogue topic. Students used the Graph program for implementing the teacher's proposal about comparison.

[A3]: The teacher proposed the comparison to clarify the essence of the factor sought, indicating that the two functions can be compared, $y = x$ is the factor that does not modify the initially obtained grades, and the factor $y = \sqrt{10x}$, which is analogous to that of the Israeli teacher, but adapted to the Spanish context (underlying [D2]).

[A3]: As we can notice in each of the two functions, the students transformed the interval [0, 10] into itself, so that the new marks remain values of the interval. This completes the analysis of the adequacy of the new factor to the Spanish context.

[D4]: The teacher made the decision to go deeper into the subject, doing so after studying the new factor, he continued with the following question:

Could we vary this factor to obtain similar ones? Test introducing some change in its algebraic expression: root index, exponent of 10 or exponent of x. Analyse the characteristics of each one and its suitability.

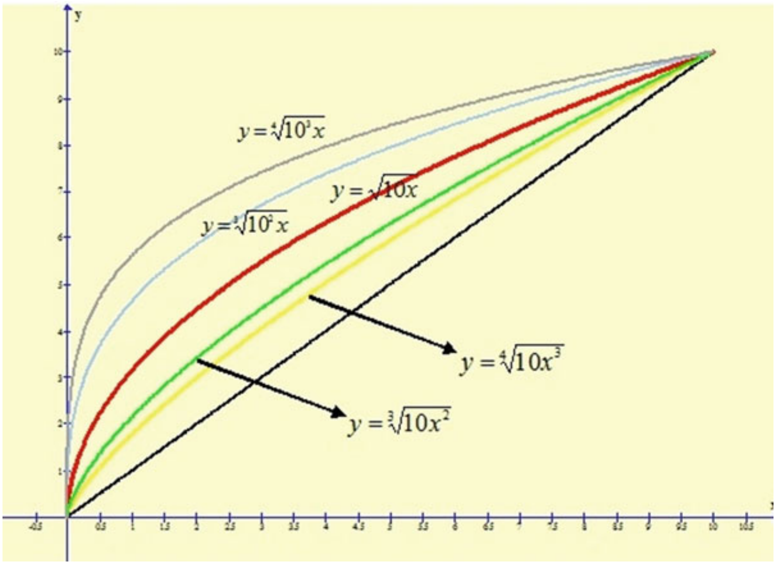

Fig. 3 Synthesis of factor family graphs

[A2]: With this question the teacher led the class-group back to ask what can be said about the topic and he poses new questions that could provide the student with the model to follow in order to achieve it.

Both teacher and students have passed through [D1] and [A3].

Then, although not described by the limits of the extension required in this article, students have proposed new correction factors.

[A3]: As an expression of the work done by the students the teacher summarised the results as follows, indicating: the result of which we present in the following graph, in which some of the functions analogous to $y = \sqrt{10x}$ appear (Fig. 3):

[A6]: In the teacher's proposal of the non-formalized analogue topic it is explained that:

The expressions of all of them can be grouped into two, representing the two families of factors:

$$F_n(x) = \sqrt[n]{10^{n-1}x} \qquad G_n(x) = \sqrt[n]{10x^{n-1}} \qquad x \in [0, 10], \quad n \in \mathrm{N}$$

And if instead of having marks between 0 and 10, they were of the interval [0, N], it would have the families of factors:

$$F_n(x) = \sqrt[n]{N^{n-1}x} \qquad G_n(x) = \sqrt[n]{N.x^{n-1}} \qquad x \in [0, N], \quad n \in \mathrm{N}$$

After the study of the previous models, we can return to the idea of continuing to generalize and we can raise another question in the classroom. Here the teacher can decide whether or not to raise a new question, a new twist to the topic; that is, we can return from [D4] to [A2].

The teacher returned from [D4] to [A2]: Cyclically returning to [A2] the teacher did so by the following question:

> Reflecting on the expressions of the factors F_n and G_n, find an expression that encompasses the two, with the condition that all of them become part of a single family of correction factors.

[D4] and [A7]: In the teacher's decision lies the idea that students become aware of how they can group them into a new generalisation of the model, as follows [D4] and [A7]:

$$H_n^i(x) = \sqrt[n]{N^i x^{n-i}} \qquad n \in \mathrm{N}, \quad i \in \{0, 1, 2, \ldots, n-1\}$$

In this new context, the teacher and students returned to the known factors so far, verifying that:

- For i=1, we obtain the factors G_n.
- For i=n-1, we obtain the factors F_n.
- For i=n/2, we obtain the factors $F_2 = G_2$.
- For i=0, we obtain the factors Identity $y=x$.

But the question can be further expressed if it is considered as the search for functional models that adapt to the situation and are of the trigonometric, logarithmic type.

Here there would be successive cycles [D4], [A2], [A3], with different types of functional models. For each one of them the cycle is repeated in the diagram of actions and decisions.

We present, by way of example, some of the factors proposed by the students (Figs. 4 and 5).[6]

Later, in other solutions given by students we can find other types of trigonometric models (Figs. 6 and 7).

[6]In the first of the graphs a logarithmic function and the reciprocal exponential, which would be analogous to the logarithmic one, but for the reverse case, in which the marks had to be lowered. This last question can be worked simultaneously to the one of raising them. The second graph is the same approach with trigonometric functional models. The norm that is mentioned in the graph refers to that the functions transforming the interval [0, 10] into itself, a condition that they all must fulfill.

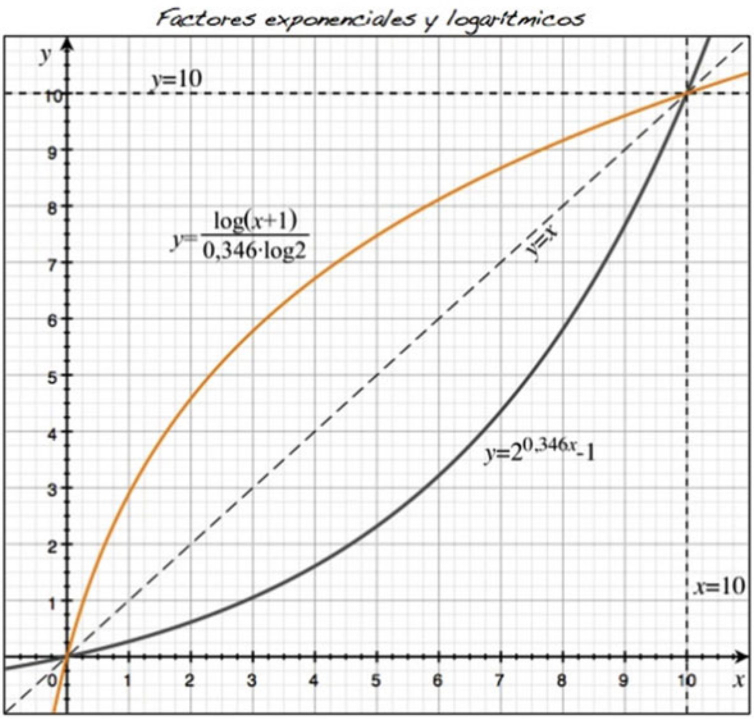

Fig. 4 Exponential and logarithmic factors

[A7]: The teacher continued to encourage students to extend the research project. Let us note that the teacher's types of orientations and commentaries, in the analysis of the student's solutions, focused on suggestions of improvements for the attainment of the project objectives. The teacher and the students assessed the results, proposed essay improvements and suggestions for the elimination of errors and modification of results, suggested new paths, posed new questions, etc.

For instance, in some of the models presented, the function was not strictly increasing as it had been demanded until then, this opens up new possibilities, analysing the meaning and interpreting the consequences that occur in the affected marks.

One of the most interesting questions that one student posed was the following: what is the greatest value of p so that the function $y = x + p.\left(1 - \cos\frac{\pi x}{5}\right)$ continues to transform the interval [0, 10] into itself. The same can be said for other functions that have maximum or minimum.

This led the teacher and the students to consider the following model (Fig. 8).

The teacher with the students verified this result. This solution seems incredible, but it is true: it leaves unchanged marks that are integers, raises the marks of the

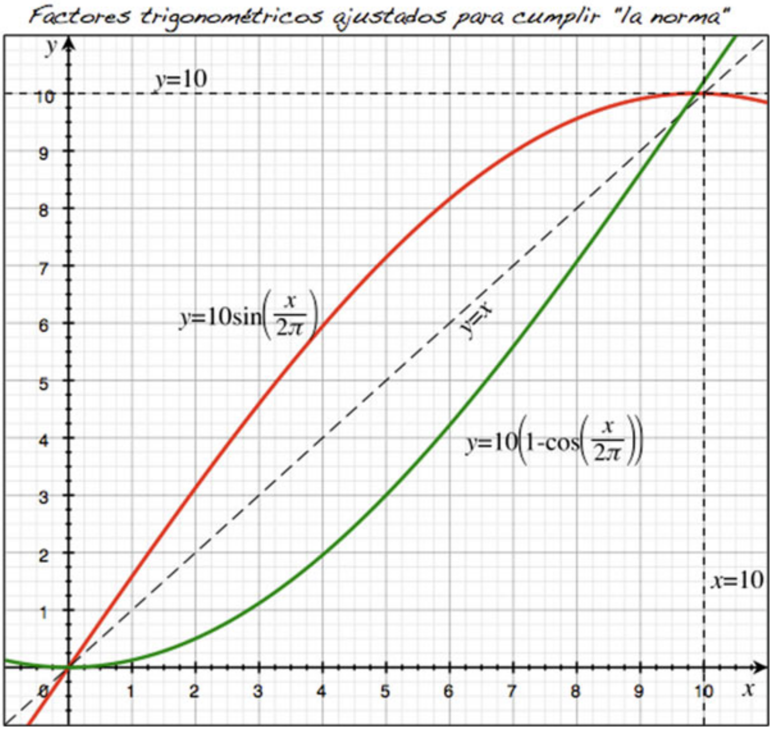

Fig. 5 Trigonometric factors adjusted to meet the norm

intervals (0, 1); (2. 3); . . . and lowers those belonging to the intervals (1, 2); (3. 4), . . .

In summary, through these episodes it can be verified that the actions and decision contexts, in which the teacher's epistemic reasoning emerge, mark the epistemological norms that serve as an input for the self-regulated learning of the learner. The analogy is considered as a heuristic tool to foster student engagement and to analyse the complexity of a modelling problem. In the sections "Epistemic Beliefs and Emotions" and "Epistemology and Ontology of Mathematical Knowledge", we will specify how this epistemic reasoning of the teacher is affected by his epistemic beliefs and emotions.

Epistemic Beliefs and Emotions

Teacher-FC uses Polya-style heuristics as a decision-making mechanism for mediations with the students as well as for the establishment of knowledge domains ([D1], [D2], [D3] and [D4] in Fig. 1). In this teacher there is evidence of the

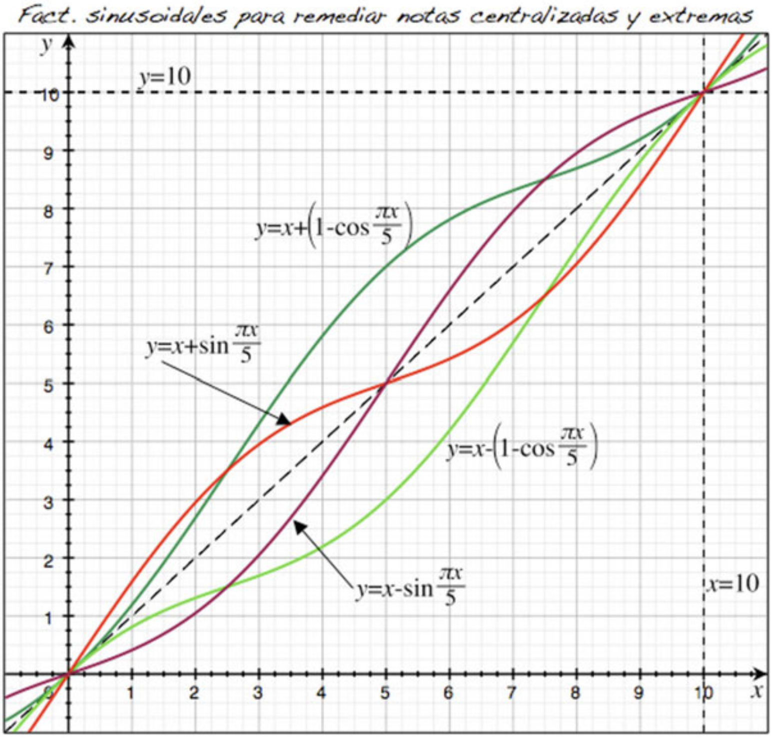

Fig. 6 Sinusoidal factors to fix centralized and extreme marks

epistemic belief that "plausible patterns describe the logical form of mathematical reasoning" (e.g. [A4] in Fig. 1). As researchers, the repeated observation of this teacher's action would lead us to affirm that "there is a model of competence in problem solving" based on the good management of interference that can occur in the coexistence of reasoning that responds to plausible patterns with reasoning that responds to deductive patterns. As in the way that plausible patterns are made explicit to students they are being given a tool for their reasoning in problem solving.

Finally, note that the epistemic emotions that *Teacher-FC* emphasizes are intellectual courage, will, self-confidence and doubt. In his words:

> I think learning is really a process of mental change that normally requires someone to treat the content more than once, in a cyclical way (because the mathematical contents are very complex, polyhedral, so we cannot apprehend all their faces with only one contact with them). This implies that learning requires, to the student, intellectual effort and intention to do so; otherwise you can repeat it memorably, but it is not true learning. (Teacher-FC's Interview 2016)

These epistemic beliefs and emotions not only constitute the framework for decision-making in their practice but also implicitly define which model of competence is related to "cognitive and affective aspects" in students. In most of the

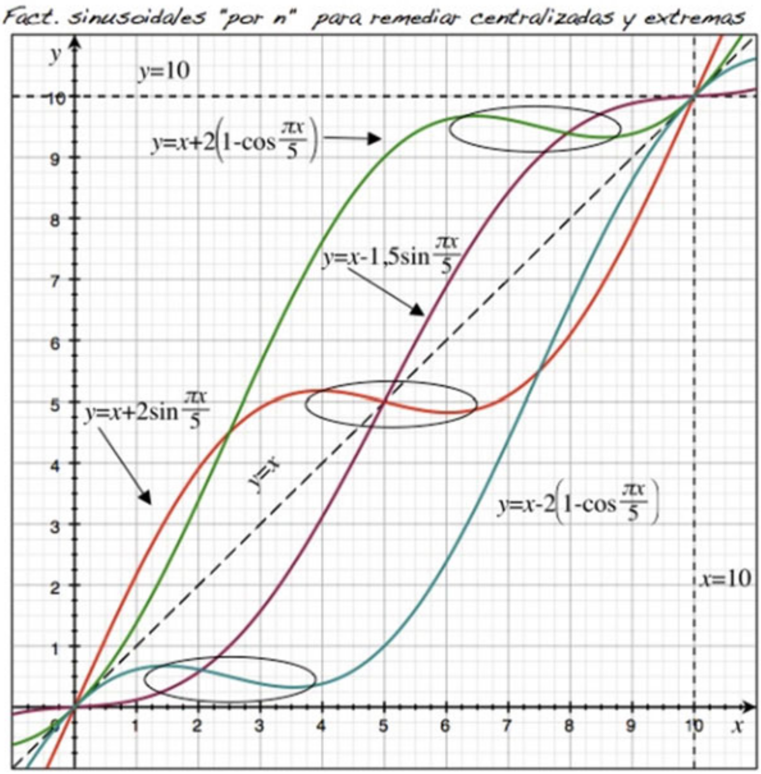

Fig. 7 Sinusoidal factors "by n" to fix centralized and extreme marks

development of the activity (section "Epistemic Reasoning in Act: Actions and Decisions in the Classroom"), *Teacher-FC* kept his students in three "meta-level" questions: What role does analogy play in modelling processes from the real world to the mathematical world? What do they learn by using analogy (reasoning, beliefs, epistemic emotions)? To what extent does mathematical trust depend on intellectual courage using these tools?

Epistemology and Ontology of Mathematical Knowledge

As it was indicated at the beginning of the diagnosis described here, the personal epistemology of *Teacher-FC* has as initial point his behaviour in action, moment to moment (section "Epistemic Reasoning in Act: Actions and Decisions in the Classroom"). This analysis shows a unity between subjective and objective knowledge of mathematics when using inquiry-based teaching. In the creation of "mathematical

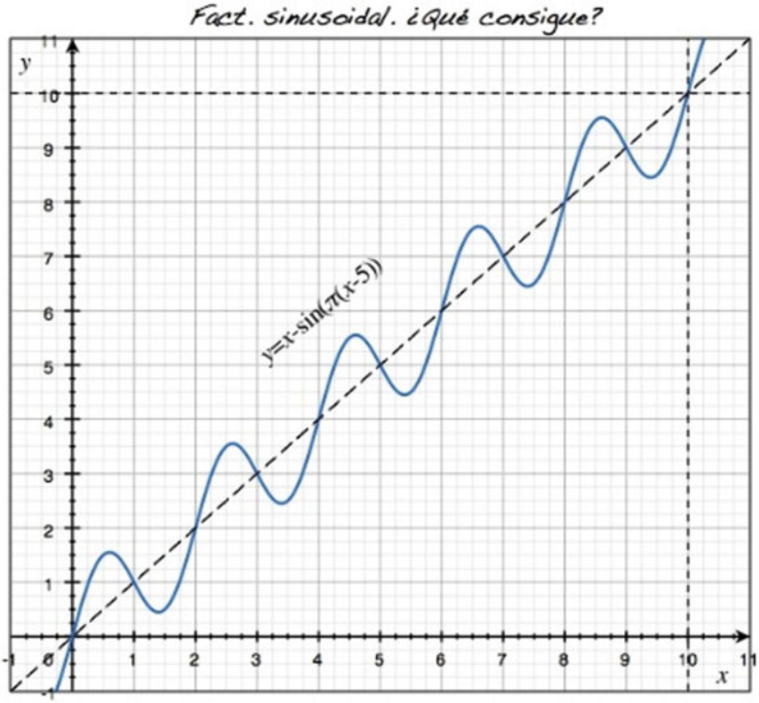

Fig. 8 Sinusoidal factors, What does it get?

experience" in the classroom, the personal epistemology of the teacher is manifested in the ontological dimension of mathematics, in epistemic beliefs regarding knowledge structure, knowledge stability and sources of knowledge and justification.

Teacher-FC, as a mathematician, has certain beliefs about the ontology of mathematics that influence his approach to teaching and learning. His position could be denoted as constructivist in the sense that both truths and mathematical objects are established by constructive methods. As Ernest (1991) points out the point of view of constructivists (intuitionists) is that "human mathematical activity is fundamental in the creation of new knowledge and that both mathematical truths and the existence of mathematical objects must be established by constructive methods" (Ernest 1991, p.29).

Following the categories of epistemological beliefs indicated in section "Personal Epistemology: Epistemic Beliefs and Emotions": structure, stability and source of knowledge in interviews this teacher points out:

In relation to the *knowledge structure*, *Teacher-FC* indicates for the High School students that in the mathematical knowledge the most important thing is the knowledge of the mathematical practice and he indicates as specific knowledge of the teacher: search of patterns, regularities and mathematical laws, in changing process, from particularisations and generic examples; to mathematization of situations, contextualisation. In his own words:

> I started thinking that concepts were the gods of mathematical knowledge. After almost 35 years of professional experience as a math teacher, I believe that:
>
> – I have dethroned concepts like the kings of knowledge in my mind. This does not mean that I do not believe that concepts have no importance, they do, but in a teaching-learning context with teenagers, concepts are not the most important.
> – I have discovered that the most important mathematical knowledge is the ways of doing things and the specific methods of working in mathematics, as professional mathematicians do. Among this knowledge I think the following deserve to be highlighted: a) the search for patterns, regularities and mathematical laws, in changing processes, based on particularisations and generic examples; b) the mathematization of situations (from formal or academic mathematic to the real or everyday situations), through modelling (use of models, its construction and analysis of its adequacy and limitations); c) contextualization (from the academic, formal or mathematical to the real or everyday) as a reverse process to mathematization, through the search for new contexts, register of representation of ideas and the establishment of connections between contexts and different ways of representing or contextualizing ideas. (Teacher-FC's Interview 2016)

In relation to the stability of knowledge, he indicates his evolution in the vision of mathematical knowledge: the passage of knowledge as something static to something that is constantly changing. In his expression:

> I also thought that what I knew, or thought I knew, was fully settled, but the passage of time has made me change my mind:
>
> – All knowledge, both emanating from established theories and that emanating from what we discover day by day, is moldable, modifiable, improvable. And this occurs at all levels, from elementary school to the areas of university research. And this should be experienced by our students, otherwise they will not know what mathematics really is.
> – It is better for the teacher to convey that knowledge is static, because that allows him to better control the classroom environment, there are fewer questions, less questioning and easier to finish a topic, start another, manage everything without leaving our field of security or our comfort space. The opposite involves living with the uncertainty of the changes, with the insecurity of uncomfortable questions, with the lack of total control of the class; and that is very complicated for a teacher to manage. (Teacher-FC's Interview 2016)

Regarding to the source of mathematical knowledge, *Teacher-FC* also mentions its evolution. Going from the management of the same from authorities, whether moral or scientific, to the idea that this derives from empirical evidence and reasoning. In his words:

> I was always very obedient to my teachers and I believed in them and, in general, in their professional qualification, until in 3rd year of my degree. In that year I bought Proofs and Refutations, by I. Lakatos. With this book I realised that there were different conceptions of what mathematical knowledge is, its structure, its teaching, etc.
>
> Now I think that mathematical knowledge, in entailing discovery and creation, has a double aspect of "practical" experimentation (in the sense of Miguel de Guzman's words) in the generation and elaboration phase, and theoretical idealism (in the sense of formalism and maximum rigor in reasoning) in the final presentation phase for fellow scientists. As for the sources of knowledge, I believe that knowledge, from authorities, is necessary, but to make it meaningful in oneself one has to "experience" it on a personal level: unravel it, break it

down, analyse it, so that, in this process, we can internalise it, truly learn it, apprehend it and make it our own.

I always tell students that they do some MRP, those that were worried that they would not discover something new: do not think that the goal is that you invent something new; if you take some existing mathematical knowledge, you study it thoroughly, you crumble it. If you analyse it and you become an expert in it, that for you already means discovering something new, because those ideas, new to you, that have rooted in your mind and have acquired life, form a living universe that will amaze and impel you to look for more, to know better . . . And that is part of the process of discovery and creation. (Teacher-FC's Interview 2016).

Discussion and Conclusions

This concluding section is based on two core elements: (1) the objectives of the study and the teaching of modelling and (2) open questions about the conceptualisation of personal epistemology.

The present study takes a step forward to describe teacher action (decision making and actions) based on their personal epistemology (1 and 2 research objective). It was shown that the *Teacher-FC*, based on their personal epistemology, prioritises the strategy of analogy as a heuristic tool to foster student engagement and motivation in the real-world and mathematical world transitions, while also creating multiple connections, vertical (in mathematics) and horizontal (with the real world outside of mathematics).

The use of the subjective valuations implicitly expressed by the teacher in the lesson, through the heuristic tool of the analogy, capture aspects of this teacher's deep concern in teaching. Concern that has been explicitly expressed in the epistemic beliefs regarding the structure of knowledge, the stability of knowledge and the sources of knowledge alongside its justification, and the epistemic emotion of intellectual courage (as an emotional rudder to guide judgment and action).

These epistemic beliefs and emotions are something that involve the nature of their rule of action, i.e., routines of action (in Schoenfeld's terms). As noted, a detailed analysis reveals that the *Teacher-FC*, highly values students who do math and believe that these have the ability to point out questions as a result of student feedback. He has a style of teaching based on "routine of interrogation" that consists in asking questions and giving answers which integrate those given by the students. This routine is shown in Fig. 1. The routine seems to lead to the conscious level, since it acts as a component of the cognitive and emotional requirements of the task required of the students. It is also evident that the actions and decision contexts -in which the epistemic reasoning and the epistemic beliefs mark the epistemological norms in the classroom- are intended to achieve the self-regulated learning on the part of the student. In this regard, the process of epistemic cognition (reasoning and belief) fosters the establishment of a habit in the teacher's teaching style: the use of analogy as a tool for modelling.

Regarding the teaching of modelling, this study demonstrates that the use of heuristic strategies can be reconstructed within the modelling process. An example using an analogy presented here shows how strategic interventions can be created, based on heuristic strategies. The heuristic strategies can act as a conceptual toolkit for facilitators to analyse the complexity of a modelling problem, identify the important steps in the modelling process, and pre-formulate possible strategic support.

Finally this study raises two open questions; one of them is how to verify the interaction between personal epistemology, reasoning and decision-making in teaching, "acting in the moment". It was shown that it was not possible to perceive its presence as something static, yet rather something incardinated between the past and the future. The terms "reasoning" and "decision" often imply that the decision-maker has knowledge: (a) about the situation that requires a decision, (b) about the different options for action (answers), and (c) about the consequences of each of these options (results), immediately and the in future. The second question raised relates to the conceptualisation of *personal epistemology*, as a multifaceted and integrated perspective in which it is necessary to consider cognitive, metacognitive and affective aspects in contextualised and specific knowledge domains. Studies such as that by Schoenfeld (2010) or the one now presented involving teachers, make use of theoretical constructs as operational tools to analyse the decision-making and actions of teachers. Knowledge, goals and beliefs become resources for practice. These categories extend the traditional dimensions of the concept of personal epistemology (certainty of knowledge, simplicity of knowledge, source of knowledge, and justification of knowledge Hofer and Pintrich 1997).

We consider that the concept of personal epistemology allows us to relate the creation of subjective and objective knowledge in mathematics to the way in which the individual acts to develop his structures of thought, while the teachers adapt the manner in which they act in order to favour the mathematical thinking structures of their students. This suggests that the contexts of "discovery" (creation) and justification cannot be completely separated, since justifications, like proofs, are the product of human creativity as concepts, conjectures and theories. Inquiry-based teaching identifies all mathematics students as mathematicians, and here we reflect on a spectrum of potential reasons behind the teacher's decisions and actions aimed at developing this creation.

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