# MATRIX PRODUCT OPERATOR ALGEBRAS II: PHASES OF MATTER FOR 1D MIXED STATES

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Abstract. The classification of topological phases of matter is fundamental to understand and characterize the properties of quantum materials. In this paper we study phases of matter in one-dimensional open quantum systems. We define two mixed states to be in the same phase if both states can be transformed into the other by a shallow circuit of local quantum channels. We aim to understand the phase diagram of matrix product density operators that are renormalization fixed points. These states arise, for example, as boundaries of two-dimensional topologically ordered states. We first construct families of such states based on C*-weak Hopf algebras, the algebras whose representations form a fusion category. More concretely, we provide explicit local fine-graining and local coarse-graining quantum channels for the renormalization procedure of these states. Finally, we prove that those arising from C*-Hopf algebras are in the trivial phase.

## 1. Introduction

One of the main projects that quantum science is undertaking in the last decades is the understanding and classification of exotic topological phases of quantum matter. The approach to tackle this project is intrinsically connected to quantum information theory. On the one hand, topological phases of matter have been identified as valuable resources in quantum computing [21]. On the other, quantum information tools and ideas are playing a key role in the classification program.

Before going any further, it is important to define what it means that two systems belong to the same topological phase. Since topological properties have an inherent global nature, the key idea is that their ground states display similar global properties independently of their (possibly) different local features. For instance, a ferromagnetic state $|\uparrow\uparrow \cdots \uparrow\uparrow\rangle$ is topologically equivalent to an antiferromagnetic one $|\uparrow\downarrow \cdots \uparrow\downarrow\rangle$ since one can map one into the other with local operations, despite the fact that they have a very different magnetization behaviour.

A definition, motivated by quantum information, which tries to capture the global properties, is the existence of a short-depth (geometrically local) quantum circuit mapping one ground state into the other [11, 8]. Using Hastings-Wen's quasi-adiabatic evolution [23] and Lieb-Robinson bounds [31] one can prove that this property is implied by the more standard definition of phase based on the existence of a gapped path of Hamiltonians connecting both systems [2].

The main advantage of the definition based on quantum circuits is that it focuses on states rather than on Hamiltonians, which is crucial to extend it to more general setups, like the one we are addressing here: open quantum systems. However, this approach poses an additional problem: one has to identify the relevant class of states to classify. For closed quantum systems this relevant class is precisely the set of ground states of gapped short-range Hamiltonians. Again quantum information theory provided us with a characterization of this set: ground states of short-ranged gapped Hamiltonians fulfill an area law for the entanglement between neighbouring regions, which implies that they are well approximated by "tensor network states", in particular by matrix product states (MPS) and projected-entagled pair states (PEPS) [24, 1, 15].

A natural approach to classify phases is to first restrict the classification to "simple" states that nevertheless are representatives for each phase. Since topological properties are global, these representatives are taken to be insensitive to real space renormalization steps (being those a finite depth circuit), that is, they are renormalization fixed points (RFP). In 2D, for instance, the string-net models of Levin and Wen [29] are believed to provide a complete set of renormalization fixed points for non-chiral 2D topological phases.

The restriction to RFPs has two important benefits. On the one hand, RFPs in gapped phases have zero correlation length and thus they are *exactly* MPS and PEPS [15]; no approximation is needed. On the other hand, it is easier to identify the key global invariants and thus identify the different phases of RFP states.

These two points have been the crucial insights to successfully complete the classification of 1D phases with symmetries, the so-called symmetry protected topological (SPT) phases. Let us illustrate that this is the case by recalling the steps that led to the classification of 1D SPT phases. The first step was to prove that any MPS can be transformed into an RFP MPS in the same phase [40]. This restricts the classification problem to just RFP MPS. The second step was to identify the invariants of the phases using the set of RFP MPS. These invariants are a set of quantities which, on the one hand, are robust against short depth circuits and, on the other, are sufficient to identify each phase uniquely. For SPT phases with unique ground state, the invariants are the different equivalent classes of the second cohomology group of the symmetry group [12, 40]. For SPT phases with symmetry breaking and therefore degenerate ground states, the invariants are the different induced representations of the non-symmetry broken subgroup together with its second cohomology group [40]. The third step was to prove that any two RFP MPS that share the same invariants can be mapped into each other with a short depth quantum circuit. On top of that, a final and important step has been recently made: the breakthrough results of Ogata [37] show that one can even extend these arguments beyond the framework of MPS to cover all gapped short-range Hamiltonians.

All the previous results stand for closed quantum systems, where the object of interest is the ground state of a Hamiltonian. However, the question of classifying phases is far from being answered for open quantum systems, even in one dimension. Since isolation is never practically achieved, the characterization of those systems play a fundamental role in real applications.

In this manuscript, we take the first steps towards the classification of open quantum systems in 1D. A main difference between open and closed quantum systems is that evolutions in closed quantum systems (either Hamiltonian evolution or quantum circuits) are *reversible*, whereas this is no longer true in open quantum systems evolved under a Linblad master equation. For instance, if one starts in a topologically ordered state, like the toric code [27], one cannot find a short depth quantum circuit mapping it into a product state. Short depth quantum circuits cannot create *or destroy* global correlations. However, local depolarizing noise can convert the toric code (and indeed any topologically ordered state, no matter how complex) into a product state in a short amount of time [16]. Destroying global correlations is therefore *easy* in the open quantum systems regime. Constructing global correlations is, on the other hand, still hard. In fact, local fast dissipative evolutions cannot create global correlations [28]. This shows that in the open quantum setting, phases should not be thought of as classes of an equivalence relation, but rather as a partial order given by the existence of a local fast dissipative evolution mapping one state into another one. This partial order can also be understood as the complexity present in the different topological phases. This proposal, due to [16], is the one we are taking here. Concretely, we will say that a mixed state $\rho_1$ is more complex than another one $\rho_2$ if there is a short-depth (geometrically local) circuit of quantum channels, i.e. completely positive trace-preserving linear maps, mapping $\rho_1$ into $\rho_2$ .

There are several subtleties to make this definition formal. First of all, $\rho_1$ and $\rho_2$ should be well defined for all system size $n$. Second, one should ask only for getting sufficiently close to $\rho_2$, allowing for both polylog($n$) depth and polylog($n$) locality in the gates of the circuit. Finally, one could take either a discrete point of view, as here, or a continuous one, asking for a rapid mixing quasi-local Linbladian evolution that approximates $\rho_2$ starting from $\rho_1$. Since in this paper we are working only with RFP states, we will not need any of those subtleties here and we refer to [16] for a detailed analysis of those.

We notice that there are other definitions of phases on the open quantum system setting, like the works of Diehl et al. for Gaussian mixed states [17, 3, 4] and for quasi thermal states [22], where the authors generalize the notion of phases via gapped paths of Hamiltonians or via local unitary transformations respectively. We refer also to [16] for a detailed discussion about why the definition we are taking here seems more appropriate.

Encouraged by the successful classification of pure states sketched above, we will focus on RFP that are *gapped* mixed states, that is, mixed states which fulfill an area law for the mutual information. This is motivated by two facts. First, it is known that Gibbs states of short-range Hamiltonians fulfill an area law for the mutual information [42]. Second, fixed points of rapid mixing dissipative evolutions also fulfill an area law for the mutual information [9].

This naturally leads us to the set of RFP mixed states with a matrix product density operator (MPDO) representation. The structure of RFP MPDOs has been studied in detail in [14] where, up to minor technical conditions, the following is shown: (i) An MPDO is a RFP if there exist two quantum channels $\mathcal{T}$ and $\mathcal{S}$ that implement the local coarse graining and the local fine graining respectively, for which the given MPDO is a fixed point. (ii) The RFP condition for MPDOs is characterized operationally by the absence of length scales in the system; in particular by having zero correlation length and saturation of the area law. (iii) The existence of such $\mathcal{T}$ and $\mathcal{S}$ maps is equivalent to the fact that from the MPDO an MPO algebra can be constructed.

This result brings the classification of 1D mixed states into the realm of understanding and classification of MPO algebras. Notably, MPO algebras are precisely the mathematical objects behind the classification of RFP 2D topologically ordered pure states in terms of PEPS [39, 10]. This is not a lucky coincidence, but a consequence of the remarkable bulk-boundary correspondence originated in the work of Li and Haldane [30]. In PEPS the bulk-boundary mapping is very explicit [13] and allows one to establish a dictionary between bulk and boundary properties [41, 26, 38]. Indeed, RFP MPDOs are expected to contain the set of boundary states associated to RFP 2D non-chiral topologically ordered systems [14].

A throughout study of MPO algebras is done in the first paper of this series [33]. There, it is shown that MPO algebras are closely related to representations of semisimple finite-dimensional weak Hopf algebras, which are, in turn, the algebraic description of fusion categories. Here, for the sake of self-containedness, we first recall in Section 2 the fundamental notions and results on weak Hopf algebras with a compatible C*-structure, called C*-weak Hopf algebras. This allows us to develop in Section 3 the construction of a family of RFP MPDOs from any C*-weak Hopf algebra. In particular, we provide explicit definitions of the local coarse-graining and local fine-graining quantum channels $\mathcal{T}$ and $\mathcal{S}$ introduced before. In Section 4 we prove that the previous families of RFP MPDOs are in the trivial phase in the Hopf algebra case, in the sense that they can be obtained via a finite-depth and bounded-range circuit of quantum channels acting on the maximally mixed state. Moreover, we show that this result can be extended to the trivial sector of any (biconnected) C*-weak Hopf algebra.

## 2. C*-weak Hopf algebras and Tensor Networks

Throughout this section we recall elementary notions on C*-weak Hopf algebras and their representation theory in terms of matrix product operators, recently developed in [33]. From now on, we assume that all vector spaces are finite dimensional, over the field of complex numbers.

2.1. **Algebras and coalgebras.** We first recall that an *associative unital algebra* is a vector space $A$ with an associative linear map $A \otimes A \to A$, called multiplication, denoted by juxtaposition, and an element $1 \in A$, called unit, satisfying $1x = x1 = x$ for all $x \in A$. A *unital C*-algebra* is an associative unital algebra $A$ with an antilinear involutive algebra homomorphism $* : A \to A$, $x \mapsto x^*$, called $*$-operation, and a faithful $*$-representation, i.e. a couple consisting on a Hilbert space $V$ and an injective $*$-algebra homomorphism $A \to \mathrm{End}(V)$. Dually to the notion of algebra, a *coassociative counital coalgebra* is a vector space $C$ with a linear map $\Delta : C \to C \otimes C$, called comultiplication, which is coassociative, i.e.

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta, \tag{1}$$

and a linear functional $\varepsilon : C \to \mathbb{C}$, called counit, such that

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}. \tag{2}$$

Now, as usually done in the context of coassociative coalgebras, let

$$\Delta^{(1)} := \Delta \quad \text{and} \quad \Delta^{(n+1)} := (\Delta \otimes \mathrm{id}^{\otimes n}) \circ \Delta^{(n)} \tag{3}$$

for all $n \in \mathbb{N}$. Complementarily, we will use Sweedler's notation

$$x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(n+1)} := \Delta^{(n)}(x) \tag{4}$$

for all $x \in C$ and all $n \in \mathbb{N}$. More concretely, this means that for brevity we abuse notation and write $\Delta(x) = x_{(1)} \otimes x_{(2)}$ instead of the more appropriate expression $\Delta(x) = \sum_i x_{1,i} \otimes x_{2,i}$, and similarly for the triple coproduct, etc. Two relevant definitions are now in place: an element $x \in C$ is said to be *cocentral* if

$$x_{(1)} \otimes x_{(2)} = x_{(2)} \otimes x_{(1)} \tag{5}$$

i.e. its coproduct is invariant under shifts, and *non-degenerate* if for every element $y \in C$ there exist two linear functionals $\mu_{x,y}, \mu'_{x,y} : C \to \mathbb{C}$, called the left and right Radon-Nikodym derivatives of $y$ with respect to $x$, such that

$$\mu_{x,y}(x_{(1)})x_{(2)} = y = x_{(1)}\mu'_{x,y}(x_{(2)}). \tag{6}$$

Roughly speaking, one can recover every element from the coproduct of $x \in A$ by applying an appropiate linear functional. Notice that the previous equation stands for $(\mu_{x,y} \otimes \mathrm{id}) \circ \Delta(x) = y = (\mathrm{id} \otimes \mu'_{x,y}) \circ \Delta(x)$ in Sweedler's notation.

The structures of algebras and coalgebras combine to give the notion of a bialgebra. Incorporating antipodes, which are intuitively a generalization of the group inverse, we obtain the notion of a Hopf algebra. This provides an appropiate framework when studying quantum groups.

**Definition 1.** A C*-Hopf algebra is a vector space $A$ endowed with the structures of a unital C*-algebra and a coassociative counital coalgebra for which the comultiplication is an algebra homomorphism, i.e.

$$(xy)_{(1)} \otimes (xy)_{(2)} = x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)} \quad \text{and} \quad 1_{(1)} \otimes 1_{(2)} = 1 \otimes 1 \tag{7}$$

for all $x, y \in A$; the $*$-operation $* : A \to A$ is comultiplicative, i.e.

$$(x^*)_{(1)} \otimes (x^*)_{(2)} = (x_{(1)})^* \otimes (x_{(2)})^* \tag{8}$$

for all $x \in A$; the counit $\varepsilon : A \to \mathbb{C}$ is an algebra homomorphism, i.e.

$$\varepsilon(xy) = \varepsilon(x)\varepsilon(y) \quad \text{and} \quad \varepsilon(1) = 1 \tag{9}$$

for all $x, y \in A$, and there is a linear map $S : A \to A$, called antipode, such that

$$S(x_{(1)})x_{(2)} = \varepsilon(x)1 = x_{(1)}S(x_{(2)}) \tag{10}$$

for all $x \in A$.

**Example 2.** The group C*-algebra $\mathbb{C}G$ of a finite group $G$ is endowed with the structure of a C*-Hopf algebra by means of the linear extensions of the maps

$$\Delta(g) := g \otimes g, \;\; \varepsilon(g) := 1 \quad \text{and} \quad S(g) := g^* := g^{-1}$$

for all $g \in G$.

**Example 3.** Let $\mathbb{C}^G \cong (\mathbb{C}G)^*$ be the C*-algebra of complex-valued maps on a finite group $G$ with multiplication, unit and $*$-operation given by

$$(\varphi\psi)(g) := \varphi(g)\psi(g), \;\; 1_{\mathbb{C}^G} : g \mapsto 1 \quad \text{and} \quad \varphi^*(g) := \overline{\varphi(g)}$$

for all $\varphi, \psi : G \to \mathbb{C}$ and all $g \in G$. It becomes a C*-Hopf algebra with

$$\Delta(\varphi)(g \otimes h) := \varphi(gh), \;\; \varepsilon(\varphi) := \varphi(1_G) \quad \text{and} \quad S(\varphi)(g) := \varphi(g^{-1}),$$

for all $\varphi : G \to \mathbb{C}$ and all $g, h \in G$.

The following example is the smallest C*-Hopf algebra which is neither cocommutative, i.e. a group algebra, nor commutative, i.e. the dual of a group algebra.

**Example 4** (Kac, Paljutkin [25])**.** Let $H_8$ be the C*-algebra generated by elements $x$, $y$ and $z$, subject to the relations

$$x^2 = 1, \;\; y^2 = 1, \;\; z^2 = \tfrac{1}{2}(1 + x + y - xy), \;\; xy = yx,$$
$$zx = yz, \;\; zy = xz, \;\; x^* = x, \;\; y^* = y \quad \text{and} \quad z^* = z^{-1}.$$

It becomes a C*-Hopf algebra by means of the maps defined by

$$\Delta(x) = x \otimes x,\ \ \Delta(y) = y \otimes y,\ \ \Delta(z) = \tfrac{1}{2}(z \otimes z + yz \otimes z + z \otimes xz - yz \otimes xz),$$
$$\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1,\ \ S(x) = x,\ \ S(y) = y \ \text{ and } \ S(z) = z.$$

In order to describe a sufficiently large family of renormalization fixed point mixed states, e.g. boundary states of 2D string-net models, it will be precise to introduce a generalization of C*-Hopf algebras. In this case, the relationship between the algebra and coalgebra structures is weakened, no longer forcing $\Delta(1) = 1 \otimes 1$. This is well-motivated from a representation theoretical point of view [36, 7] and may look odd from a purely algebraic one; thus the subsequent axioms can be skipped in a first reading. The following definition is due to G. Böhm and K. Szlachányi [5].

**Definition 5.** A C*-weak Hopf algebra is a vector space $A$ endowed with the structures of a unital C*-algebra and a coassociative counital coalgebra for which the comultiplication is multiplicative, i.e.

$$(xy)_{(1)} \otimes (xy)_{(2)} = x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)} \tag{11}$$

for all $x, y \in A$; the $*$-operation $* : A \to A$ is comultiplicative, i.e.

$$(x^*)_{(1)} \otimes (x^*)_{(2)} = (x_{(1)})^* \otimes (x_{(2)})^* \tag{12}$$

for all $x \in A$; the unit $1 \in A$ is weakly comultiplicative, i.e.

$$1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} = 1_{(1)} \otimes 1_{(2)}1_{(1')} \otimes 1_{(2')} = 1_{(1)} \otimes 1_{(1')}1_{(2)} \otimes 1_{(2')}, \tag{13}$$

where the prime symbol intends to distinguish different coproducts of the same element; the counit $\varepsilon : A \to \mathbb{C}$ is weakly multiplicative, i.e.

$$\varepsilon(xyz) = \varepsilon(xy_{(1)})\varepsilon(y_{(2)}z) = \varepsilon(xy_{(2)})\varepsilon(y_{(1)}z) \tag{14}$$

for all $x, y, z \in A$, and there is an antimultiplicative linear map $S : A \to A$, called antipode, satisfying

$$S(x_{(1)})x_{(2)} = \varepsilon(1_{(1)}x)1_{(2)} \quad \text{and} \quad x_{(1)}S(x_{(2)}) = 1_{(1)}\varepsilon(x1_{(2)}), \tag{15}$$

for all $x \in A$.

We refer the reader to [6, 7, 35, 18] for an account of the theory. Notice that this definition is equivalent to the ones given in the previous papers; see [36]. For completeness, we recall a few consequences of the axioms now. First of all, it can be proven that the antipode $S : A \to A$ is necessarily unique, invertible and both an algebra and a coalgebra antihomomorphism, i.e.

$$S(xy) = S(y)S(x) \quad \text{and} \quad S(x)_{(1)} \otimes S(x)_{(2)} = S(x_{(2)}) \otimes S(x_{(1)}) \tag{16}$$

for all $x, y \in A$, $S(1) = 1$ and $\varepsilon \circ S = \varepsilon$. Moreover, one can also derive that

$$1^* = 1,\ \ \varepsilon(x^*) = \overline{\varepsilon(x)} \quad \text{and} \quad S(x^*) = S^{-1}(x)^* \tag{17}$$

for all $x \in A$. Note that the notion of C*-weak Hopf algebra is self-dual, i.e. $A^* := \mathrm{Hom}(A, \mathbb{C})$ can be canonically endowed with the structure of a C*-weak Hopf algebra too by means of

$$(fg)(x) := f(x_{(1)})g(x_{(2)}), \quad 1_{A^*} := \varepsilon, \quad \Delta(f)(x \otimes y) := f(xy), \tag{18}$$

$$\varepsilon(f) := f(1), \quad S(f)(x) := f(S(x)) \quad \text{and} \quad f^*(x) := \overline{f(S(x)^*)} \tag{19}$$

for all $f, g \in A^*$ and $x, y \in A$. Finally, we introduce two essential $*$-subalgebras,

$$A^L := \{x \in A : x_{(1)} \otimes x_{(2)} = x1_{(1)} \otimes 1_{(2)} = 1_{(1)}x \otimes 1_{(2)}\}, \tag{20}$$

$$A^R := \{y \in A : y_{(1)} \otimes y_{(2)} = 1_{(1)} \otimes y1_{(2)} = 1_{(1)} \otimes 1_{(2)}y\}, \tag{21}$$

known as the target and source $*$-subalgebras of $A$, respectively.

The following example is the smallest proper C*-weak Hopf algebra (with non-involutive antipode). It is known as the Lee-Yang C*-weak Hopf algebra as it is reconstructed from one of the solutions of the pentagon equation arising from the Lee-Yang fusion rules:

**Example 6** (Böhm, Szlachányi [5])**.** Let $A_{\mathrm{LY}}$ be the direct sum $M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ of full-matrix C*-algebras. Let $\zeta$ be the unique positive solution to $z^4 + z^2 - 1 = 0$ and fix matrix units $e_1^{ij}$, $i, j = 1, 2$, in $M_2(\mathbb{C})$ and $e_2^{k\ell}$, $k, \ell = 1, 2, 3$, in $M_3(\mathbb{C})$. It can be equipped with the coassociative comultiplication defined by:

$$\begin{aligned}
\Delta(e_1^{11}) &:= e_1^{11} \otimes e_1^{11} + e_2^{11} \otimes e_2^{22},\\
\Delta(e_1^{12}) &:= e_1^{12} \otimes e_1^{12} + \zeta^2 e_2^{12} \otimes e_2^{21} + \zeta e_2^{13} \otimes e_2^{23},\\
\Delta(e_1^{22}) &:= e_1^{22} \otimes e_1^{22} + \zeta^4 e_2^{22} \otimes e_2^{11} +\\
&\qquad \zeta^3 e_2^{23} \otimes e_2^{13} + \zeta^3 e_2^{32} \otimes e_2^{31} + \zeta^2 e_2^{33} \otimes e_2^{33},\\
\Delta(e_2^{11}) &:= e_1^{11} \otimes e_2^{11} + e_2^{11} \otimes e_1^{22} + e_2^{11} \otimes e_2^{33},\\
\Delta(e_2^{12}) &:= e_1^{12} \otimes e_2^{12} + e_2^{12} \otimes e_1^{21} + e_2^{13} \otimes e_2^{32},\\
\Delta(e_2^{13}) &:= e_1^{12} \otimes e_2^{13} + e_2^{11} \otimes e_1^{22} + \zeta e_2^{12} \otimes e_2^{31} - \zeta^2 e_2^{13} \otimes e_2^{33},\\
\Delta(e_2^{22}) &:= e_0^{22} \otimes e_2^{22} + e_2^{22} \otimes e_0^{11} + e_2^{33} \otimes e_2^{22},\\
\Delta(e_2^{23}) &:= e_1^{22} \otimes e_2^{23} + e_2^{23} \otimes e_1^{21} + \zeta e_2^{32} \otimes e_2^{21} - \zeta^2 e_2^{33} \otimes e_2^{23},\\
\Delta(e_2^{33}) &:= e_1^{22} \otimes e_2^{33} + e_2^{33} \otimes e_1^{22} + \zeta^2 e_2^{22} \otimes e_2^{11} -\\
&\qquad \zeta^3 e_2^{23} \otimes e_2^{13} - \zeta^3 e_2^{32} \otimes e_2^{31} + \zeta^4 e_2^{33} \otimes e_2^{33}
\end{aligned}$$

as well as the counit and the antipode given by

$$\varepsilon(e_1^{ij}) = 1, \quad \varepsilon(e_2^{k\ell}) = 0, \quad S(e_1^{ij}) = e_1^{ji} \;\text{ and }\; S(e_2^{k\ell}) = \zeta^{\ell-k} e_2^{\sigma(\ell)\sigma(k)}$$

for all $i, j = 1, 2$ and $k, \ell = 1, 2, 3$, where $\sigma(1) := 2$, $\sigma(2) := 1$, $\sigma(3) := 3$, endowing it with the structure of a C*-weak Hopf algebra. This specification has been slightly adapted from the reference as we will propose a tensor network description in Example 13 consistent with its string-net model definition.

2.2. **Representations of C*-weak Hopf algebras.** A representation of a C*-weak Hopf algebra $A$ is a representation of the underlying C*-algebra, i.e. a couple $(V, \phi)$, where $V$ is a finite dimensional vector space and $\phi : A \to \mathrm{End}(V)$ is an algebra homomorphism, i.e. a linear map with $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$ and $\phi(1) = \mathbf{1}$. If, in addition, $V$ is a Hilbert space with inner product $(\cdot,\cdot)_V$ and $\phi(x^*) = \phi(x)^\dagger$ for all $x \in A$, it is said to be a $*$-representation of $A$. In any case, it is said that the representation is faithful if $\phi$ is injective.

Since $A$ is, in particular, a finite-dimensional C*-algebra, the set of equivalence classes of irreducible representations, also called *sectors*, is necessarily finite. Let $1, \ldots, r$ be labels for those equivalence classes. In what follows, we fix a complete set $(V_1, \pi_1), \ldots, (V_r, \pi_r)$ of pairwise non-equivalent irreducible $*$-representations of $A$. Let $\chi_1, \ldots, \chi_r$ stand for their corresponding characters. On account of self-duality, let $1, \ldots, s$ be labels for the sectors of the dual C*-weak Hopf algebra $A^*$, choose another complete set $(\hat{V}_1, \hat{\pi}_1), \dots, (\hat{V}_s, \hat{\pi}_s)$ of pairwise non-equivalent irreducible $*$-representations of $A^*$ and let $\hat{\chi}_1, \dots, \hat{\chi}_s$ stand for their characters.

By construction, the category of $*$-representations of a C*-weak Hopf algebra $A$ has a natural structure of a rigid monoidal category [36, 18]. The comultiplication $\Delta : A \to A \otimes A$ allows us to define the monoidal product

$$V \boxtimes W := \{z \in V \otimes W : \Delta(1)z = z\}, \quad \phi \boxtimes \psi := (\phi \otimes \psi) \circ \Delta,$$

of any two $*$-representations $(V, \phi)$ and $(W, \psi)$ of $A$. Additionally, the axioms of C*-weak Hopf algebras ensure the existence of a monoidal unit, called the "trivial" representation [6, 7]. It has the unusual feature that it can be reducible. This motivates the following definition.

**Definitions 7.** A C*-weak Hopf algebra $A$ is said to be *connected* if its "trivial" representation is irreducible, *coconnected* if its dual C*-weak Hopf algebra $A^*$ is connected, and *biconnected* if it is both connected and coconnected.

For simplicity, we assume that $(V_1, \pi_1)$ corresponds to the "trivial" representation of $A$ if $A$ is connected and, analogously, that $(\hat{V}_1, \hat{\pi}_1)$ stands for the "trivial" representation of $A^*$ if $A$ is coconnected.

2.3. **Dimensionality and the canonical regular element.** It turns out that, for every connected C*-weak Hopf algebra $A$ its Grothendieck ring, denoted $K_0(A)$ (i.e. the free $\mathbb{Z}$-module generated by the characters of representations of $A$ with addition and multiplication defined accordingly), is a fusion ring [18]. In particular, the characters $\chi_1, \dots, \chi_r$ form a basis of $K_0(A)$ satisfying

$$\chi_\alpha \chi_\beta = N^1_{\alpha\beta}\chi_1 + \cdots + N^r_{\alpha\beta}\chi_r$$

for some $N^\gamma_{\alpha\beta} \in \mathbb{N} \cup \{0\}$, for all sectors $\alpha, \beta, \gamma = 1, \dots, r$. Consider now any $*$-representation $(V, \phi)$ of $A$ and let $\chi_V = \nu_1\chi_1 + \cdots + \nu_r\chi_r$ be its character, where $\nu_1, \dots, \nu_r \in \mathbb{N} \cup \{0\}$ are the multiplicities of $(V_1, \pi_1), \dots, (V_r, \pi_r)$ within $(V, \phi)$. Define the $r \times r$ matrix $N_V$ with coefficients $(N_V)_{\beta\gamma} := \nu_1 N^\gamma_{1\beta} + \cdots + \nu_r N^\gamma_{r\beta}$, $\beta, \gamma = 1, \dots, r$. Since $K_0(A)$ is, in particular, a transitive ring [20], $N_V$ is a matrix with strictly positive entries. Hence, by virtue of the Frobenius-Perron Theorem,

$$d(V) := \text{spectral radius of } N_V$$

is an algebraically simple positive eigenvalue, known as the Frobenius-Perron dimension of $(V, \phi)$. Though it is not needed, we remark that this notion coincides with the one of quantum dimension from the category of $*$-representations; see Proposition 8.23 from [18] for a rigorous statement. For simplicity, we let $d_\alpha := d(V_\alpha)$ for all sectors $\alpha = 1, \dots, r$. Also, let

$$\mathcal{D}^2 := d_1^2 + \cdots + d_r^2$$

denote the Frobenius-Perron dimension of the algebra. Dually, if $A$ is coconnected, let $\hat{d}_1, \dots, \hat{d}_s$ denote the dual Frobenius-Perron dimensions of $A$. If $A$ is a biconnected C*-weak Hopf algebra, then $d_1^2 + \cdots + d_r^2 = \hat{d}_1^2 + \cdots + \hat{d}_s^2$.

Now we recall the following result, which proves the existence of a distinguished element in each coconnected C*-weak Hopf algebra satisfying a "pulling-through equation". These properties turn out to be enough to understand the properties of renormalization fixed points.

**Theorem 8** (Molnár et al. [33])**.** *Let $A$ be a coconnected C*-weak Hopf algebra. Then, the element*

$$\Omega := \tfrac{1}{\mathcal{D}^2}(\hat{d}_1\hat{\chi}_1 + \cdots + \hat{d}_s\hat{\chi}_s) \tag{22}$$

*is a cocentral, non-degenerate positive idempotent in $A$, called the canonical regular element of $A$. Moreover, there exists a linear map $T : A \to A$ satisfying*

$$T(x)\Omega_{(1)} \otimes \Omega_{(2)} = \Omega_{(1)} \otimes x\Omega_{(2)} \tag{23}$$

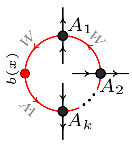
*for all $x \in A$. In particular, $T$ is an involutive algebra antihomomorphism.*

Here, the previous equation is referred to as a *pulling-through equation*, and resembles the definition of left integrals in the theory of C*-weak Hopf algebras. Since $T$ is an involutive algebra antihomomorphism, it is easy to see by applying it in the first term of the coproduct that it can be rewritten in the form

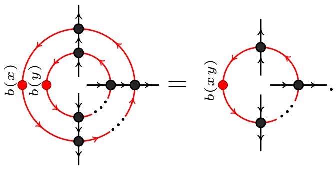
$$T(\Omega_{(1)})x \otimes \Omega_{(2)} = T(\Omega_{(1)}) \otimes x\Omega_{(2)} \tag{24}$$

for all $x \in A$. In particular, if $A$ is a C*-Hopf algebra then $\Omega$ coincides with the well-known Haar integral and $T$ with the antipode; see Appendix B.

2.4. **A tensor network description of C*-weak Hopf algebras.** Now, we interpret representations of C*-weak Hopf algebras in terms of tensor networks; see [33] for an exhaustive discussion. As usually done in the tensor network literature, we employ a slightly enriched graphical notation, briefly described in Appendix A. Let us consider any sequence $\{(V_i, \phi_i) : i \in \mathbb{N}\}$ of representations of a C*-weak Hopf algebra $A$. It turns out that the endomorphisms $(\phi_1 \otimes \cdots \otimes \phi_k) \circ \Delta^{(k-1)}(x)$ can be described in terms of matrix product operators, for all $x \in A$. More concretely, there exist a Hilbert space $W$ and tensors $A_i \in \mathrm{End}(W) \otimes \mathrm{End}(V_i)$, $i \in \mathbb{N}$, independent of $x \in A$, such that one can write

(25) $(\phi_1 \otimes \cdots \otimes \phi_k) \circ \Delta^{(k-1)}(x) =$ 


for some linear map $b : A \to W \otimes W^*$, for all $k \in \mathbb{N}$. We will usually restrict to the translation-invariant case, for which $\phi_1 = \phi_2 = \cdots =: \phi$ and $A_1 = A_2 = \cdots$. Notice that the physical indices, associated to Hilbert spaces $V$ and $V^*$, are depicted by black lines, while the virtual indices, associated to Hilbert spaces $W$ and $W^*$, are depicted by red lines. Thus, from now on, we will drop the labels, since no misunderstanding can arise. For instance, we can express the multiplicativity of the coproduct (see Equation 11) with this simplified graphical notation as

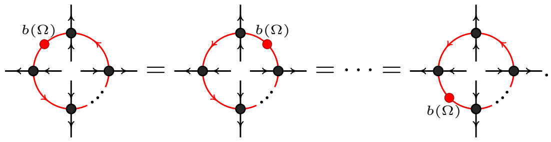
(26) 


for all $x, y \in A$.

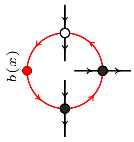
We finish this section by reinterpreting the different properties of the canonical regular element in graphical notation. First, it is easy to check by induction on $N \in \mathbb{N}$ that the fact that $\Omega \in A$ is a cocentral element implies

$$\Omega_{(1)} \otimes \Omega_{(2)} \otimes \cdots \otimes \Omega_{(N)} = \Omega_{(\tau(1))} \otimes \Omega_{(\tau(2))} \otimes \cdots \otimes \Omega_{(\tau(N))} \tag{27}$$

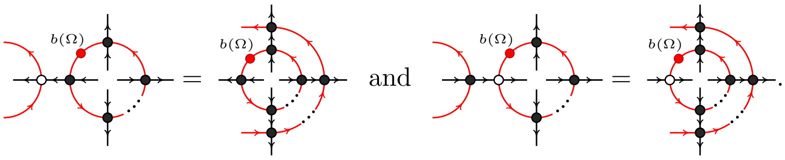
for any shift permutation $\tau$ of $\{1, \dots, N\}$. In turn, this can be rephrased as the translation-invariance of the associated MPOs:

(28) 


In order to interpret the action of the linear map $T : A \to A$, first note that the linear map $A \to \mathrm{End}(V^*)$, $x \mapsto (\phi \circ T(x))^{\mathrm{t}}$ defines a representation of $A$, where $(\,\cdot\,)^{\mathrm{t}}$ stands for the transpose operation. As discussed below, it is not necessarily a $*$-representation. By Equation 25, one can depict, e.g.

(29) $$\phi^{\otimes 3} \circ (T \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta^{(2)}(x) =$$ 

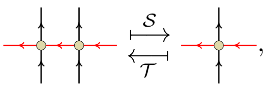

for all $x \in A$, for some white rank-four tensor, where all physical spaces in the picture are $V$. For the sake of clarity, we have reversed the direction of the physical arrows corresponding to the new tensor, since $T$ is an antimultiplicative map. With this notation, Equations 23 and 24 can be interpreted respectively as follows:

(30) 


Note that, since $x \in A$ is arbitrary, we have dropped the tensor $b(x)$ and reexpressed the identities using open boundary MPOs instead; see [33].

## 3. RFP MPDOs from C*-weak Hopf algebras

In this section we define a distinguished family of MPOs starting from a biconnected C*-weak Hopf algebra and show that they are RFP MPDOs, as defined in [14]. More concretely, we provide explicit expressions of both local coarse-graining and local fine-graining quantum channels $\mathcal{T}$ and $\mathcal{S}$ for which the generating rank-four tensor is a fixed point under the corresponding induced flows, i.e. such that

(31) 


very much in the spirit of standard renormalization group. All proofs in this section will be postponed to Appendix C.

### 3.1. Construction of the tensor.

The generating tensor of the RFP MPDOs is obtained here by appropiately weighting the tensor from the original MPO algebra, described in the previous section. This weighting is done by means of the canonical regular element of the dual C*-weak Hopf algebra. To this end, let us examine first the properties of this linear functional.

**Lemma 9.** *Let $A$ be a connected C*-weak Hopf algebra. Then, the canonical regular element $\omega \in A^*$ of the dual C*-weak Hopf algebra $A^*$ is the unique trace-like, faithful, positive linear functional on $A$ that is idempotent, i.e.*

(32) $$(\omega \otimes \omega) \circ \Delta = \omega.$$

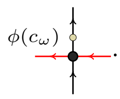
Note that this linear functional formally plays the role in C*-weak Hopf algebra of the character of the usual left-regular representation. Also, the previous equation simply rephrases, in terms of the original structure maps, the relation $\omega^2 = \omega$ in the dual C*-weak Hopf algebra; see Equation 18. Now, given a faithful representation of the C*-weak Hopf algebra, we define the appropiate weight extending the action of the previous linear functional to the representation space.

**Remark 10.** Let $A$ be a connected C*-weak Hopf algebra and let $(V, \phi)$ be any faithful $*$-representation of $A$. Then,

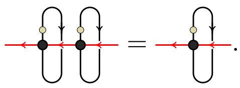
$$\mathrm{tr}(\phi(c_\omega)\phi(x)) = \omega(x) \tag{33}$$

for all $x \in A$, for some central, invertible and positive $c_\omega \in A$.

Then, an ansatz tensor is the one obtained by multiplying the MPO tensor in Equation 25 by $\phi(c_\omega)$ on the physical space:




Remarkably, this leads to a convenient reinterpretation of idempotence of $\omega \in A^*$, i.e. Equation 32, as an indicator of zero correlation length; see [14]:



It is clear that computations of correlation functions using the MPDOs generated by the previous tensor will be length-independent. In particular, it induces the following family of mixed states:

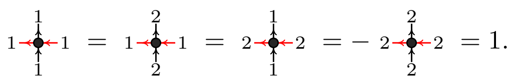
**Proposition 11.** *Let $A$ be a connected C*-weak Hopf algebra and let $(V, \phi)$ be any faithful $*$-representation of $A$. Then, the endomorphisms*

$$\rho_N(x) := \tfrac{1}{\omega(x)} \phi^{\otimes N}(c_\omega^{\otimes N} \Delta^{(N-1)}(x))$$

*are MPDO, for all positive non-zero $x \in A$ and all $N \in \mathbb{N}$.*

Let us illustrate the construction with an extremely modest example.

**Example 12.** Let $A := \mathbb{C}\mathbb{Z}_2$ be the C*-Hopf algebra arising from the cyclic group $G := \mathbb{Z}_2$ generated by $g \in G$; see Example 2. It posseses only two sectors, namely the equivalence classes of the trivial representation and the sign representation, each one-dimensional. Consider the faithful $*$-representation of $A$ given by $V := W := \mathbb{C}^2$, with basis elements $|1\rangle, |2\rangle$, and let $\phi : A \to \mathrm{End}(\mathbb{C}^2)$ be defined by $\phi(g) := \sigma_z$, the Pauli-Z matrix. It is easy to see that both Frobenius Perron dimensions are 1 and hence the canonical regular elements of $A$ and $A^*$ are given by $\Omega = 2^{-1}(e+g)$ and $\omega(x) = (x,e)_V$, for all $x \in A$, respectively. A tensor generating the corresponding MPOs is specified by the non-zero coefficients

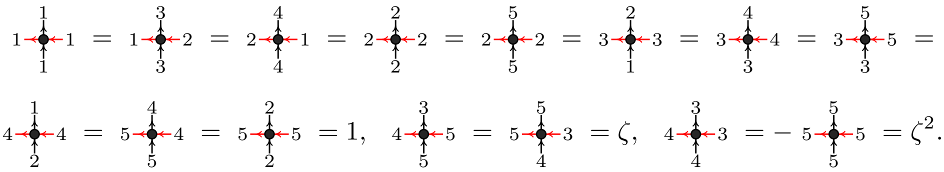



Moreover, in this case the weight is trivially given by $c_\omega = 2^{-1}e$ and thus

$$\rho_N(x) = \tfrac{1}{2^N}(\mathbf{1}^{\otimes N} + \tfrac{(x,g)_V}{(x,e)_V}\sigma_z^{\otimes N}),$$

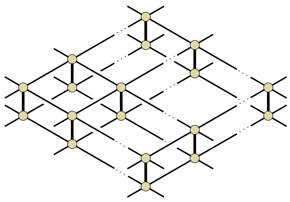
are the induced RFP MPDOs, for all positive non-zero $x \in A$. In particular, $\rho_N(\Omega) = 2^{-N}(\mathbf{1}^{\otimes N} + \sigma_z^{\otimes N})$ is the boundary state of the toric code; see [14].

**Example 13.** Let $A_{\text{LY}}$ be the Lee-Yang C*-weak Hopf algebra from Example 6. It possesses only two sectors, denoted 1 and $\tau$, for which it is easy to check that $d_1 = 1$ and $d_\tau = \zeta^{-2} = 2^{-1}(1+\sqrt{5})$, respectively. Let $V := W := \mathbb{C}^5$ and let $\phi : A_{\text{LY}} \to \text{End}(\mathbb{C}^5)$ be the faithful $*$-representation arising from the string-net specification; see [33, 10] for a derivation. A tensor generating the corresponding MPOs is then specified by the non-zero coefficients

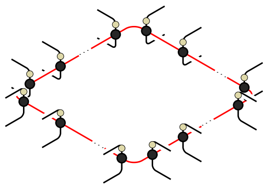


Finally, it is straightforward to check that $\phi(c_\omega) = 2(5+5^{1/2})^{-1}\mathbf{1}_2 \oplus 5^{-1/2}\mathbf{1}_3$.

3.2. **RFP structure.** These MPDOs arise as boundary states of RFP topologically ordered PEPS; see [41] for algebras based on groups. The boundary state of a PEPS is defined as



For topological RFP PEPS this corresponds precisely to

 ,

which is an MPO representation of the C*-weak Hopf algebra associated to the gauge symmetries of the PEPS tensor and, therefore, to the topological properties of the PEPS [33, 15]. This motivates the following proposition, which is the main result of this section.

**Proposition 14.** *Let $A$ be a biconnected C*-weak Hopf algebra and let $(V,\phi)$ be any faithful $*$-representation of $A$. Then, there exist two quantum channels*

$$\mathcal{T} : \text{End}(V) \to \text{End}(V \otimes V) \ \textit{and} \ \mathcal{S} : \text{End}(V \otimes V) \to \text{End}(V),$$

*called local coarse-graining and local fine-graning channels, respectively, such that*

$$\mathcal{T}(\rho_1(x)) = \rho_2(x) \ \textit{and} \ \mathcal{S}(\rho_2(x)) = \rho_1(x) \tag{34}$$

*for all positive non-zero $x \in A$.*

For the purpose of giving explicit definitions of both quantum channels, we introduce the following auxiliary result.

**Lemma 15.** *For any biconnected C*-weak Hopf algebra $A$ there is $\xi \in A$ such that*

$$\omega(\xi T(\Omega_{(1)}))\Omega_{(2)} = 1. \tag{35}$$

*In particular, it is strictly positive and its inverse is given by*

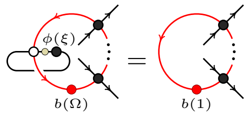
$$\xi^{-1} = \omega(T(\Omega_{(1)}))\Omega_{(2)} = \omega(\Omega_{(1)})\Omega_{(2)}, \tag{36}$$

*Furthermore, it satisfies*

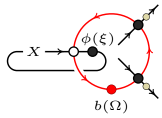
(37) $$T(\xi) = \xi \quad \textit{and} \quad T(x^*) = \xi^{-1} T(x)^* \xi$$

*for all* $x \in A$, *and it can be decomposed in the form* $\xi = \xi_L \xi_R$, *for two positive elements* $\xi_L \in A^L$ *and* $\xi_R \in A^R$. *If* $A$ *is a* C*-*Hopf algebra, then* $\xi = \mathcal{D}^2 1$.

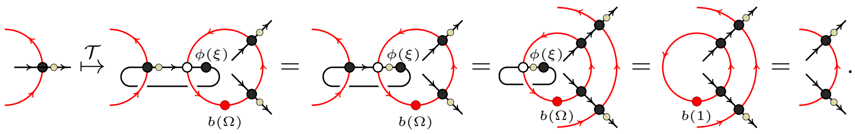
In graphical notation, Equation 35 is represented as follows:

(38)



With that, one can define the coarse-graining map by the expression

$\mathcal{T} : \mathrm{End}(V) \to \mathrm{End}(V \otimes V), \quad X \mapsto$



and prove that it has the property of duplicating the tensor defining the MPDO:




In the first step we have used that the weight $\phi(c_\omega) \in \mathrm{End}(V)$ can be freely moved along the physical indices since $c_\omega \in A$ is a central element. The second equality follows from the pulling-through identity in Equation 30, while the third is due to the defining property of $\xi \in A$ in Equation 38. Finally, the last equality easily follows from Equation 26.

## 4. Phase classification of RFP MPDOs

In this section we prove that RFP MPDOs arising from C*-Hopf algebras belong to the trivial phase. Namely, we provide explicit definitions of depth-two circuits of finite-range quantum channels that map the maximally mixed state to the RFP MPDOs. Finally, we show that our construction cannot be extended to arbitrary biconnected C*-weak Hopf algebras, which lead us to conjecture that there are non-trivial phases in this context.

### 4.1. An illustrative example.

In order to deepen the intuition towards the general case of C*-Hopf algebras, let first examine the simplest non-trivial example.

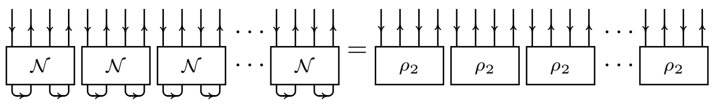
**Example 16.** RFP MPDOs arising from the group C*-Hopf algebra $A := \mathbb{C}\mathbb{Z}_2$, introduced in Examples 2 and 12, are in the trivial phase. Specifically, we build

$$\rho_N := \rho_N(\Omega) = \tfrac{1}{2^N}(\mathbf{1}^{\otimes N} + \sigma_z^{\otimes N})$$

via a depth-two circuit of range-two quantum channels from the maximally mixed state $\mathrm{tr}(\mathbf{1})^{-N}\mathbf{1}^{\otimes N}$. We assume without loss of generality that $N \in \mathbb{N}$ is even and propose the following procedure:

**Step 1** ("initialization"). We first construct $N/2$ copies $(\rho_2)^{\otimes N/2}$ of the mixed state $\rho_2$ between pairs of nearest neighbors by replacing the product states separately.

This is easily done by means of the quantum channel $\mathcal{N} : X \otimes Y \mapsto \mathrm{tr}(X \otimes Y)\rho_2$. In the Choi-Jamiołkowski picture, this process can be depicted as follows:




When the system size is an odd natural number simply replace three of them with the mixed state $\rho_3$, for example.

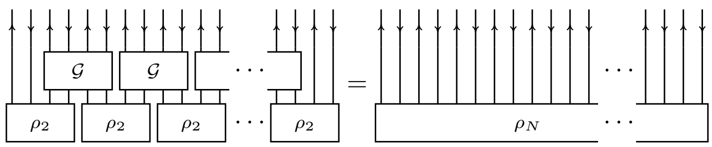
**Step 2** ("gluing")**.** Now, we "glue" together all these copies of $\rho_2$ in order to obtain the target mixed state $\rho_N$. This is done inductively by means of the following quantum channel, called from now on a *gluing* map:

$$\mathcal{G} : X \otimes Y \mapsto \frac{1}{2^2}(\mathrm{tr}(X \otimes Y)\mathbf{1} \otimes \mathbf{1} + \mathrm{tr}(X\sigma_z \otimes Y\sigma_z)\sigma_z \otimes \sigma_z) \tag{39}$$

for all $X, Y \in \mathrm{End}(\mathbb{C}^2)$. It is easy to check that it is a quantum channel and that

$$\mathrm{id} \otimes \mathcal{G} \otimes \mathrm{id}(\tfrac{1}{2^2}(\mathbf{1}^{\otimes 2} + \sigma_z^{\otimes 2}) \otimes \tfrac{1}{2^2}(\mathbf{1}^{\otimes 2} + \sigma_z^{\otimes 2})) = \tfrac{1}{2^4}(\mathbf{1}^{\otimes 4} + \sigma_z^{\otimes 4}).$$

By induction, it is clear that a simultaneous application of these quantum channels leads to the mixed state $\rho_N$. Again, in the Choi-Jamiołkowski picture this procedure can be depicted as follows:




4.2. **Phase classification of C*-Hopf algebras.** The previous construction can be generalized to arbitrary C*-Hopf algebras as follows. In the first place, the role of the previous element is replaced by the RFP MPDO associated to the canonical regular element. In addition, we introduce a family of quantum channels that "glue" together two RFP MPDOs associated to the canonical regular element $\Omega \in A$ into a larger one, associated to any arbitrary positive non-zero element of $A$.

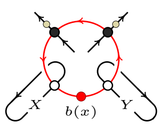
**Lemma 17.** *Let $A$ be a C*-Hopf algebra and let $(V, \phi)$ be any faithful $*$-representation of $A$. Then, for all positive non-zero $x \in A$ there exists a quantum channel $\mathcal{G}_x$ on* $\mathrm{End}(V \otimes V)$*, called "gluing" map, such that*

$$\mathrm{id}^{\otimes M-1} \otimes \mathcal{G}_x \otimes \mathrm{id}^{\otimes N-1}(\rho_M(\Omega) \otimes \rho_N(\Omega)) = \rho_{M+N}(x) \tag{40}$$

*for all $M, N \in \mathbb{N}$.*

See Appendix D for a proof. However, let us propose now an explicit expression for the gluing map and check with graphical notation that Equation 40 holds. To this end, fix any positive non-zero $x \in A$ and assume without loss of generality that $M = N = 2$. Define the linear map

$$\mathcal{G}_x : \mathrm{End}(V \otimes V) \to \mathrm{End}(V \otimes V), \quad X \otimes Y \mapsto \frac{1}{\omega(x)}$$

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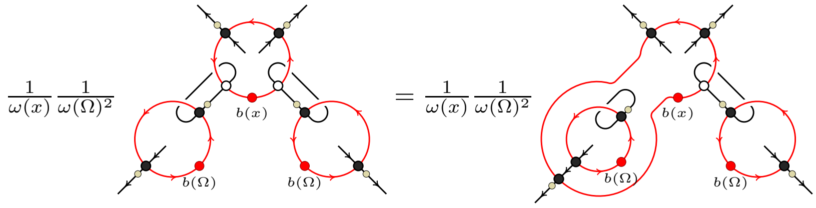
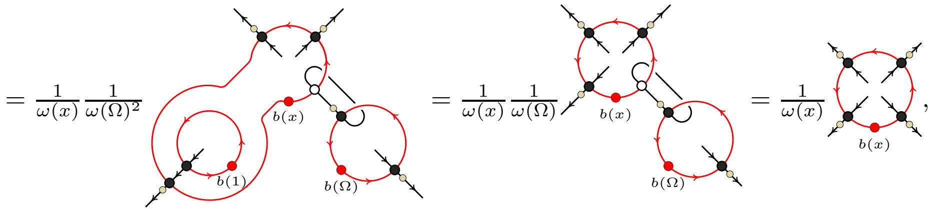
for all $X, Y \in \mathrm{End}(V)$. First, recall that $\phi(c_\omega)$ can be moved freely along the physical vector spaces and apply the pulling-through identity in Equation 23. Then,




now, since $\omega(\Omega_{(1)})\Omega_{(2)} = \omega(\Omega)1$ by virtue of Lemma 15,




where the last step follows by repeating the previous steps on the right side of the figure.

Similar to the construction described in details for the boundary state of the toric code, the existence of such a quantum channel immediately induces a finite-depth circuit of quantum channels manifesting the triviality of these states.

**Theorem 18.** *Let $A$ be a $C^*$-Hopf algebra and let $(V, \phi)$ be any faithful $*$-representation of $A$. Then, for all positive non-zero $x \in A$ and all $N \in \mathbb{N}$ there exists a depth-two circuit of bounded-range quantum channels that maps $\mathrm{tr}(\mathbf{1})^{-N}\mathbf{1}^{\otimes N}$ into $\rho_N(x)$. That is, the sequence $(\rho_N(x))_{N=1}^{\infty}$ is in the trivial phase.*

*Proof.* Assume without loss of generality that $N \in \mathbb{N}$ is even. The circuit consists of two layers, as presented above in Example 16. In the first layer, we replace the maximally mixed state $\mathrm{tr}(\mathbf{1})^{-N}\mathbf{1}^{\otimes n}$ with the sequence of $N/2$ tensor products $\rho_2(\Omega) \otimes \cdots \otimes \rho_2(\Omega)$ as previously done. Now, by virtue of Lemma 17, let $\mathrm{id} \otimes \mathcal{G}_\Omega \otimes \cdots \otimes \mathcal{G}_\Omega \otimes \mathcal{G}_x \otimes \mathrm{id}$ be the second layer of quantum channels, where all subindices are $\Omega \in A$ except for one, which is $x \in A$. This second layer of channels then glues together all local MPDOs into the single MPDO $\rho_N(x)$. □

4.3. **Phase classification of C*-weak Hopf algebras.** For general RFP MPDOs constructed from biconnected C*-weak Hopf algebras a straightforward generalization of the previous procedure is not possible anymore, since the comultiplication is no longer unit-preserving.

**Remark 19.** There are no trace-preserving gluing maps for general biconnected C*-weak Hopf algebras mimicking Lemma 17.

See Appendix E for a proof. Unfortunately, the description of the phases in this general case is still an open problem. Nevertheless, some evidence indicate the existence of non-trivial phases, as we conjecture here.

**Conjecture 20.** *RFP MPDOs arising from the Lee-Yang C*-weak Hopf algebra of Example 6 do not belong to the trivial phase.*

However, these obstructions can be circumvented if one restricts to the trivial sector. The following result establishes the existence of an special gluing map, motivated by the characterization of simple RFP MPDO tensors in [14].

**Lemma 21.** *Let $A$ be a biconnected C*-weak Hopf algebra and let $(V, \phi)$ be any faithful $*$-representation of $A$. There is a quantum channel $\mathcal{G}_1$ on $\operatorname{End}(V \otimes V)$ called "gluing" map, such that*

$$\mathrm{id}^{\otimes M-1} \otimes \mathcal{G}_1 \otimes \mathrm{id}^{\otimes N-1}(\rho_M(1) \otimes \rho_N(1)) = \rho_{M+N}(1) \tag{41}$$

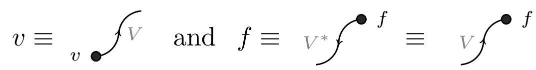
*for all $M, N \in \mathbb{N}$.*

A proof is given in Appendix E. As an immediate corollary, similar to the case of C*-Hopf algebras, we obtain the following result.

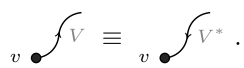
**Theorem 22.** *Let $A$ be a biconnected C*-weak Hopf algebra and let $(V, \phi)$ be any faithful $*$-representation of $A$. Then, for all $N \in \mathbb{N}$ there exist two depth-two circuits of bounded-range quantum channels that map $\operatorname{tr}(\mathbf{1})^{-N}\mathbf{1}^{\otimes N}$ into $\rho_N(1)$ and $\rho_N(\hat{\chi}_1)$. That is, the sequences $(\rho_N(1))_{N=1}^{\infty}$ and $(\rho_N(\hat{\chi}_1))_{N=1}^{\infty}$ are in the trivial phase.*

## Appendix A. A Graphical Notation for Tensor Networks

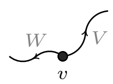
As usually done in the literature of tensor networks, it is useful to employ a graphical notation, briefly described here. Firstly, if $V$ is a vector space, we represent vectors $v \in V$ and linear functionals $f \in V^*$ by a shape, e.g. a circle, with a line sticking out of it:



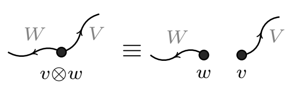

respectively. Notice that for the vector in $V$ we represent the arrow outgoing from the shape, as opposed to the representation of the linear functional on $V$. We identify each vector space $V$ and its double dual $V^{**}$, since they are canonically isomorphic. Thus, one can write



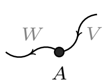

For the tensor product space of two vector spaces $V$ and $W$, vectors $v \in V \otimes W$ can be depicted by a shape with two lines sticking out of it, one for each index:



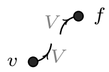

The labels $V$ and $W$ allow to identify which is the corresponding index, even if $V = W$. Other option, used here, consists on prescribing colors to distinguish them. Also, notice that we have implicitly identified $V \otimes W$ and $W \otimes V$, but these are again canonically isomorphic. The tensor product $v \otimes w \in V \otimes W$ of two vectors $v \in V$ and $w \in W$ can be depicted by placing the factors next to each other in the same picture:

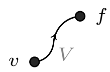



Besides, since $\operatorname{Hom}(V, W)$ and $V^* \otimes W$ are canonically isomorphic, one can depict linear maps $A : V \to W$ in the following form:



Now, it is intuitive to interpret the evaluation map $\mathrm{ev}_V : V^* \otimes V \to \mathbb{C}$ by joining the corresponding lines, i.e.

$f \otimes v =$ 

$\mapsto$ 

$= f(v)$

For the sake of clarity, open indices will be usually matched by their location and direction, and labels naming tensors and vector spaces will be omitted.

## Appendix B. A Characterization of the Canonical Regular Element

Here, we recall additional results about C*-weak Hopf algebras not introduced in the main text. As a matter of fact, we are interested in describing the canonical regular element in terms of these results.

Every C*-weak Hopf algebra $A$ possesses a unique non-degenerate, idempotent, positive element $h \in A$, which is a two-sided integral, i.e.

$$hx = h1_{(1)}\varepsilon(x1_{(2)}) \quad \text{and} \quad xh = \varepsilon(1_{(1)}x)1_{(2)}h \tag{42}$$

for all $x \in A$, called the *Haar integral* of $A$. In particular, this element is invariant under the action of the antipode, i.e. $S(h) = h$. In the sequel, let $\hat{h} \in A^*$ stand for the Haar integral of the dual C*-weak Hopf algebra $A^*$. Furthermore, there exists a unique element $t \in A$ such that

$$\hat{h}(t_{(1)})t_{(2)} = 1, \;\; tx = t1_{(1)}\varepsilon(x1_{(2)}) \quad \text{and} \quad t_{(1)} \otimes t_{(2)} = S^2(t_{(2)}) \otimes t_{(1)} \tag{43}$$

for all $x \in A$, called the dual left integral of $\hat{h} \in A^*$.

Every C*-weak Hopf algebra $A$ possesses a unique invertible, positive element $g \in A$, implementing $S^2 : A \to A$ as an inner automorphism, i.e.

$$S^2(x) = gxg^{-1} \tag{44}$$

for all $x \in A$, and normalized in the sense that $\chi_a(g^{-1}) = \chi_a(g) \neq 0$ for all sectors $a = 1, \ldots, r$, known as the *canonical group-like element* of $A$, cf. [6]. As its name implies, it is a group-like element, i.e.

$$g_{(1)} \otimes g_{(2)} = g1_{(1)} \otimes g1_{(2)} = 1_{(1)}g \otimes 1_{(2)}g \tag{45}$$

Moreover, it is a trivial group-like element, i.e. $g \in A_{\min}$ and, more specifically,

$$g = g_L g_R^{-1} \quad \text{for} \quad g_L := (h_{(1)}\hat{h}(h_{(2)}))^{1/2} \in A^L, \;\; g_R := S(g_L), \;\; g_L, g_R > 0. \tag{46}$$

We let $\hat{g} \in A^*$ denote the canonical grouplike element of $A^*$ and $\hat{g}_L \in (A^*)^L$ and $\hat{g}_R \in (A^*)^R$ the dual versions of the previous elements.

**Lemma 23.** *Let $A$ be a C*-weak Hopf algebra. Then,*

$$\hat{g}(x_{(1)})x_{(2)} = g_L x g_L^{-1} \quad \textit{and} \quad x_{(1)}\hat{g}(x_{(2)}) = g_R x g_R^{-1} \tag{47}$$

*for all $x \in A$. In particular,*

$$\hat{g}(1_{(1)})1_{(2)} = 1 = 1_{(1)}\hat{g}(1_{(2)}) \tag{48}$$

*Proof.* All are simple consequences of Scholium 2.7 and Lemma 4.13 from [6]. □

**Proposition 24.** *If $A$ is a coconnected C*-weak Hopf algebra, then*

$$\Omega = \tfrac{1}{\mathcal{D}^2\varepsilon(1)}\hat{g}(t_{(1)})t_{(2)} = \tfrac{1}{\mathcal{D}^2\varepsilon(1)}t_{(1)}\hat{g}(t_{(2)}). \tag{49}$$

*If $A$ is a connected C*-weak Hopf algebra, then*

$$\omega(x) = \tfrac{1}{\mathcal{D}^2\varepsilon(1)}\hat{h}(g_L^{-1}g_R^{-1}x) = \tfrac{1}{\mathcal{D}^2\varepsilon(1)}\hat{h}(xg_L^{-1}g_R^{-1}) \tag{50}$$

*for all $x \in A$.*

*Proof.* Assume e.g. that $A$ is connected. The following formula is valid in this case:

$$\chi_\alpha(g) = \varepsilon(1) d_\alpha \tag{51}$$

for all sectors $\alpha = 1, \dots, r$. Additionally, in every C*-weak Hopf algebra there is a linear functional $\tau : A \to \mathbb{C}$ given by $\tau = \chi_1(g)\chi_1 + \cdots + \chi_r(g)\chi_s$; see [6]. Then,

$$\omega = \frac{1}{\mathcal{D}^2}(d_1\chi_1 + \cdots + d_s\chi_s) = \frac{1}{\mathcal{D}^2\varepsilon(1)}(\chi_1(g)\chi_1 + \chi_s(g)\chi_s) = \frac{1}{\mathcal{D}^2\varepsilon(1)}\tau.$$

Both Equations 49 and 50 are then given by Proposition 4.14(iii) from [6]. ☐

**Proposition 25** (Molnár et al. [33]). *The map $T : A \to A$ in Theorem 8 is*

$$T(x) = S(x_{(1)})\hat{g}(x_{(2)}) = \hat{g}(x_{(1)})S^{-1}(x_{(2)}) \tag{52}$$

*for all $x \in A$.*

Finally, let us particularize the previous notions and results in the context of C*-Hopf algebras. We refer the reader to [34] for more details.

**Proposition 26.** *Let $A$ be a C*-Hopf algebra. Then:*

(i) $S^2 = \mathrm{id}$ *and the canonical grouplike element is $g = 1$;*
(ii) $d_\alpha = \dim_{\mathbb{C}}(\hat{V}_\alpha)$ *for all sectors $\alpha = 1, \dots, r$;*
(iii) *the dual left integral of the Haar measure $\hat{h}$ is $t = \mathcal{D}^2\Omega$;*
(iv) *the canonical regular element and the Haar integral coincide, i.e. $\Omega = h$;*
(v) *the map $T : A \to A$ from Theorem 8 coincides with $S : A \to A$;*
(vi) $g_L = g_R = \mathcal{D}^{-1}1$.

*Proof.* (i) It is well-known that in a C*-Hopf algebra $S^2 = \mathrm{id}$. Since $1 \in A$ also satisfies the defining properties of the canonical group-like element, which is unique, we can conclude that $g = 1$. (ii) Consider that $\varepsilon(1) = 1$ by Equation 7 and hence Equation 51 reads $\dim_{\mathbb{C}}(V_\alpha) = \chi_\alpha(1) = \chi_\alpha(g) = \varepsilon(1)d_\alpha = d_\alpha$, for all sectors $\alpha = 1, \dots, r$. (iii) Since the axioms of C*-Hopf algebras are self-dual, this implies that $\hat{g} = \varepsilon$ and hence $\Omega = (\mathcal{D}^2\varepsilon(1))^{-1}t = \mathcal{D}^{-2}t$, where the first expression follows from Equation 49. (iv) Every C*-Hopf algebra is *unimodular* [34], i.e. every left integral is a two-sided integral, and the subspace of two-sided integrals is one-dimensional. Hence $t = \eta h$ for some $\eta \in \mathbb{C}$. Since $\Omega^2 = \Omega$ and $h^2 = h$, the only possibility left is $\eta = \mathcal{D}^2$. (v) This follows trivially as a consequence of Proposition 25 since $\hat{g} = \varepsilon$. (vi) Recall the definition of $g_L$ and $g_R$ in Equation 46 and consider both steps (iii) and (iv). ☐

## Appendix C. Proofs of Section 3

In this appendix we provide proofs of all results presented in Section 3. Let us first prove the following result, restated as follows.

**Lemma 9.** *Let $A$ be a connected C*-weak Hopf algebra. Then, the canonical regular element $\omega \in A^*$ of the dual C*-weak Hopf algebra $A^*$ is the unique trace-like, faithful, positive linear functional on $A$ that is idempotent, i.e.*

$$(\omega \otimes \omega) \circ \Delta = \omega. \tag{32}$$

*Proof.* In first place, $\omega$ is obviously a trace-like, faithful, positive linear functional by construction. Additionally, recall the fact that it fulfills

$$\chi_\alpha \omega = \omega \chi_\alpha = d_\alpha \omega \tag{53}$$

for all $\alpha = 1, \dots, r$; see e.g. [20]. This implies, in particular, that $\omega$ is idempotent. Assume now that $f : A \to \mathbb{C}$ is any linear functional satisfying the previous properties. In the first place, since it is a trace-like linear functional it can be expanded in the form $f = f_1\chi_1 + \cdots + f_r\chi_r$, for some $f_1, \dots, f_r \in \mathbb{C}$. Secondly, since it is a positive faithful linear functional, it is easy to check that $f_\alpha > 0$ for all $\alpha = 1, \dots, r$, by simply evaluating the corresponding minimal central idempotents. Define the $r \times r$ matrix $N_f$ with real coefficients

$$(N_f)_{\beta\gamma} := f_1 N_{1\beta}^{\gamma} + \cdots + f_r N_{r\beta}^{\gamma}, \quad \beta, \gamma = 1, \dots, r.$$

As commented above, since $A$ is a connected C*-weak Hopf algebra, the entries of $N_f$ are strictly positive real numbers. On the other hand, in the basis $\{\chi_1, \dots, \chi_r\}$,

$$\begin{aligned} &N_f f = f^2 = f, \\ &N_f\omega = f\omega = f_1\chi_1\omega + \cdots + f_r\chi_r\omega = f_1 d_1\omega + \cdots + f_r d_r\omega \propto \omega, \end{aligned}$$

since $f^2 = f$ by hypothesis and $\omega$ satisfies Equation 53. However, by the classical Frobenius-Perron theorem there is a unique eigenvector with strictly positive entries, up to normalization. Thus, $f = \omega$, as we wanted to prove. □

**Remark 10.** Let $A$ be a connected C*-weak Hopf algebra and let $(V, \phi)$ be any faithful $*$-representation of $A$. Then,

$$\operatorname{tr}(\phi(c_\omega)\phi(x)) = \omega(x) \tag{33}$$

for all $x \in A$, for some central, invertible and positive $c_\omega \in A$.

*Proof.* Let $e_1, \dots, e_r \in A$ be the minimal central idempotents of $A$ associated to the characters $\chi_1, \dots, \chi_r$ of the irreducible $*$-representations, respectively. Also, let $\nu_1, \dots, \nu_r \in \mathbb{N}$ denote the corresponding multiplicities within $(V, \phi)$. Let

$$c_\omega := \tfrac{1}{\mathcal{D}^2\varepsilon(1)}\left(\tfrac{d_1}{\nu_1}e_1 + \cdots + \tfrac{d_r}{\nu_r}e_r\right) \in A.$$

It is clearly a central, invertible and positive element of $A$. Moreover,

$$\begin{aligned} \operatorname{tr}(\phi(c_\omega)\phi(x)) &= \tfrac{1}{\mathcal{D}^2\varepsilon(1)}\operatorname{tr}(\phi(\tfrac{d_1}{\nu_1}xe_1 + \cdots + \tfrac{d_r}{\nu_r}xe_r)) \\ &= \tfrac{1}{\mathcal{D}^2\varepsilon(1)}(\tfrac{d_1}{\nu_1}\operatorname{tr}(\phi(xe_1)) + \cdots + \tfrac{d_r}{\nu_r}\operatorname{tr}(\phi(xe_r))) \\ &= \tfrac{1}{\mathcal{D}^2\varepsilon(1)}(\nu_1\tfrac{d_1}{\nu_1}\chi_1(x) + \cdots + \nu_r\tfrac{d_r}{\nu_r}\chi_r(x)) \\ &= \tfrac{1}{\mathcal{D}^2\varepsilon(1)}(d_1\chi_1(x) + \cdots + d_r\chi_r(x)) = \omega(x) \end{aligned}$$

for all $x \in A$, as we wanted to prove. □

**Proposition 11.** *Let $A$ be a connected C*-weak Hopf algebra and let $(V, \phi)$ be any faithful $*$-representation of $A$. Then, the endomorphisms*

$$\rho_N(x) := \tfrac{1}{\omega(x)}\phi^{\otimes N}(c_\omega^{\otimes N}\Delta^{(N-1)}(x))$$

*are MPDO, for all positive non-zero $x \in A$ and all $N \in \mathbb{N}$.*

*Proof.* Fix an arbitrary positive non-zero $x \in A$ and any $N \in \mathbb{N}$. Using Sweedler's notation one can rewrite the previous operators in the form:

$$\rho_N(x) = \tfrac{1}{\omega(x)}\phi(c_\omega x_{(1)}) \otimes \cdots \otimes \phi(c_\omega x_{(N)}).$$

Let us check that $\rho_N(x)$ is positive semidefinite. On the one hand, $\omega(x) > 0$ since $\omega$ is a positive linear functional. On the other, let $x = y^*y$ for some $y \in A$. Then,

$$\rho_N(x) = \tfrac{1}{\omega(x)}\phi(c_\omega(y^*y)_{(1)}) \otimes \cdots \otimes \phi(c_\omega(y^*y)_{(N)}) \qquad \text{by Eq. 11}$$

$$= \tfrac{1}{\omega(x)}\phi(c_\omega(y^*)_{(1)}(y)_{(1)}) \otimes \cdots \otimes \phi(c_\omega(y^*)_{(N')}y_{(N)}) \qquad \text{by Eq. 11}$$
$$= \tfrac{1}{\omega(x)}\phi(c_\omega y^*_{(1')}y_{(1)}) \otimes \cdots \otimes \phi(c_\omega y^*_{(N')}y_{(N)}) \qquad \text{by Eq. 12}$$
$$= \tfrac{1}{\omega(x)}\phi((y_{(1')}c_\omega^{\frac{1}{2}})^* y_{(1)}c_\omega^{\frac{1}{2}}) \otimes \cdots \otimes \phi((y_{(N')}c_\omega^{\frac{1}{2}})^* y_{(N)}c_\omega^{\frac{1}{2}})$$
$$= \tfrac{1}{\omega(x)}\phi((y_{(1')}c_\omega^{\frac{1}{2}})^*)\phi(y_{(1)}c_\omega^{\frac{1}{2}}) \otimes \cdots \otimes \phi((y_{(N')}c_\omega^{\frac{1}{2}})^*)\phi(y_{(N)}c_\omega^{\frac{1}{2}})$$
$$= \tfrac{1}{\omega(x)}\phi(y_{(1)}c_\omega^{\frac{1}{2}})^\dagger\phi(y_{(1')}c_\omega^{\frac{1}{2}}) \otimes \cdots \otimes \phi(y_{(N')}c_\omega^{\frac{1}{2}})^\dagger\phi(y_{(N)}c_\omega^{\frac{1}{2}}) = \sigma^\dagger\sigma,$$

where we have defined $\sigma := \omega(x)^{-1/2}\phi(y_{(1)}c_\omega^{1/2}) \otimes \cdots \otimes \phi(y_N c_\omega^{1/2})$, which is clearly positive semidefinite. Finally, it is easy to check that, in addition,

$$\mathrm{tr}(\rho_N(x)) = \tfrac{1}{\omega(x)}\mathrm{tr}(\phi(c_\omega x_{(1)}))\cdots\mathrm{tr}(\phi(c_\omega x_{(N)})) = \tfrac{1}{\omega(x)}\omega(x_{(1)})\cdots\omega(x_{(N)}) = 1,$$

where we have used that $\omega(x_{(1)})\cdots\omega(x_{(N)}) = \omega^{\otimes n} \circ \Delta^{(N-1)}(x) = \omega(x)$, which easily follows by induction on $N \in \mathbb{N}$ as a consequence of Equations 3 and 32. □

**Lemma 27.** *Let $A$ be a coconnected $C^*$-weak Hopf algebra. Then,*

$$\omega \circ S = \omega = \omega \circ T. \tag{54}$$

*Proof.* On the one hand, $\omega \circ S = \omega$ is a well-known fact by Equation 50; see [6]. On the other, by virtue of Proposition 25 and Lemma 23,

$$\omega(T(x)) = \omega(S(x_{(1)}))\hat{g}(x_{(2)}) = \omega(x_{(1)})\hat{g}(x_{(2)}) = \omega(g_R x g_R^{-1}) = \omega(x),$$

where in the last step we have used that $\omega$ is a trace-like linear functional. □

Let us first restate and prove the following lemma:

**Lemma 15.** *For any biconnected $C^*$-weak Hopf algebra $A$ there is $\xi \in A$ such that*

$$\omega(\xi T(\Omega_{(1)}))\Omega_{(2)} = 1. \tag{35}$$

*In particular, it is strictly positive and its inverse is given by*

$$\xi^{-1} = \omega(T(\Omega_{(1)}))\Omega_{(2)} = \omega(\Omega_{(1)})\Omega_{(2)}, \tag{36}$$

*Furthermore, it satisfies*

$$T(\xi) = \xi \quad \text{and} \quad T(x^*) = \xi^{-1}T(x)^*\xi \tag{37}$$

*for all $x \in A$, and it can be decomposed in the form $\xi = \xi_L\xi_R$, for two positive elements $\xi_L \in A^L$ and $\xi_R \in A^R$. If $A$ is a $C^*$-Hopf algebra, then $\xi = \mathcal{D}^2 1$.*

*Proof.* Since $\Omega \in A$ is non-degenerate, there exists a linear functional $\mu \in A^*$ such that $\mu(\Omega_{(1)})\Omega_{(2)} = 1$. On the other hand, since $\omega \in A^*$ is non-degenerate, there exists an element $\xi \in A$ such that $\omega(\xi x) = \mu(T(x))$ for all $x \in A$. Therefore,

$$\omega(\xi T(\Omega_{(1)}))\Omega_{(2)} = \mu(T(T(\Omega_{(1)})))\Omega_{(2)} = \mu(\Omega_{(1)})\Omega_{(2)} = 1,$$

where we have used that $T : A \to A$ is involutive, i.e. $T \circ T = \mathrm{id}$. Besides, since $\omega : A \to \mathbb{C}$ is a trace-like linear functional, it follows by Equation 24 that

$$1 = \omega(\xi T(\Omega_{(1)}))\Omega_{(2)} = \omega(T(\Omega_{(1)})\xi)\Omega_{(2)} = \omega(T(\Omega_{(1)}))\xi\Omega_{(2)},$$

and hence $\xi \in A$ is invertible. Its inverse is trivially given by

$$\xi^{-1} = \omega(T(\Omega_{(1)}))\Omega_{(2)} = \omega(\Omega_{(1)})\Omega_{(2)}.$$

where the last equality follows from Lemma 27. By virtue of Equations 49 and 50, it is easy to conclude from Equation 35 that $\xi$ is necessarily given by the expression

$$\xi = \mathcal{D}^4\varepsilon(1)^2 g_L g_R. \tag{55}$$

Consequently, a natural choice of positive elements $\xi_L \in A^L$ and $\xi_R \in A^R$ is

$$\xi_L := \mathcal{D}^2\varepsilon(1)g_L, \quad \xi_R := \mathcal{D}^2\varepsilon(1)g_R. \tag{56}$$

In order to prove that $T(\xi) = \xi$, note by the previous expressions that it turns out to be enough to check that $T(g_L) = g_R$ and $T(g_R) = g_L$. We refer to Equations 70 and 71 below for elementary proofs of these facts. Let us now move to the proof of Equation 37. For simplicity, we prove the equivalent formula $\xi T(x)\xi^{-1} = T(x^*)^*$ for all $x \in A$. To this end, we first recall that

$$\xi y \xi^{-1} = g_L g_R y g_L^{-1} g_R^{-1} = \hat{g}(y_{(1)})y_{(2)}\hat{g}(y_{(3)}) \tag{57}$$

for all $y \in A$; see Equations 2.20a-2.21b and 4.36-4.38 from [6]. On the other hand, by virtue of the the fact that $S^{-1}(\hat{g}) = \hat{g}^{-1}$ and the positivity of $\hat{g} \in A^*$,

$$\hat{g}^{-1}(y) = \hat{g}(S^{-1}(y)) = \hat{g}^*(S^{-1}(y)) = \overline{\hat{g}(S(S^{-1}(y))^*)} = \overline{\hat{g}(y^*)} \tag{58}$$

for all $y \in A$. Thus,

$$\begin{aligned}
\xi T(x)\xi^{-1} &= \xi S(x_{(1)})\xi^{-1}\hat{g}(x_{(2)}) && \text{by Eq. 52}\\
&= \hat{g}(S(x_{(1)})_{(1)})S(x_{(1)})_{(2)}\hat{g}(S(x_{(1)})_{(3)})\hat{g}(x_{(2)}) && \text{by Eq. 57}\\
&= \hat{g}(S(x_{(3)}))S(x_{(2)})\hat{g}(S(x_{(1)}))\hat{g}(x_{(4)}) && \text{by Eq. 16}\\
&= \hat{g}^{-1}(x_{(1)})S(x_{(2)})\hat{g}^{-1}(x_{(3)})\hat{g}(x_{(4)}) && \text{by Eq. 46}\\
&= \hat{g}^{-1}(x_{(1)})S(x_{(2)}) && \text{by Eq. 18}\\
&= \hat{g}^{-1}(x_{(1)})S^{-1}(S^2(x_{(2)})) && \text{by Eq. 46}\\
&= \hat{g}^{-1}(x_{(1)})\hat{g}(x_{(2)})S^{-1}(x_{(3)})\hat{g}^{-1}(x_{(4)}) && \text{by Eq. 44}\\
&= S^{-1}(x_{(1)})\hat{g}^{-1}(x_{(2)}) && \text{by Eq. 18}\\
&= S^{-1}(x_{(1)})\overline{\hat{g}(x^*_{(2)})} && \text{by Eq. 58}\\
&= S(x^*_{(1)})^*\overline{\hat{g}(x^*_{(2)})} && \text{by Eq. 17}\\
&= S((x^*)_{(1)})^*\overline{\hat{g}((x^*)_{(2)})} && \text{by Eq. 12}\\
&= T(x^*)^*, && \text{by Eq. 52}
\end{aligned}$$

for all $x \in A$, as we wanted to prove. Finally, if $A$ is a C*-Hopf algebra, it is already known by Proposition 26 that $g_L = g_R = \mathcal{D}^{-1}1$. This, together with the definition of $\xi$ in Equation 55 and that $\varepsilon(1) = 1$, leads to the expression $\xi = \mathcal{D}^2 1$. □

As commented, we are able now to give explicit expressions for both local coarse-graining and fine-graining quantum channels. We restate and prove the following proposition now:

**Proposition 14.** *Let $A$ be a biconnected C*-weak Hopf algebra and let $(V, \phi)$ be any faithful $*$-representation of $A$. Then, there exist two quantum channels*

$$\mathcal{T} : \mathrm{End}(V) \to \mathrm{End}(V \otimes V) \ \textit{and}\ \mathcal{S} : \mathrm{End}(V \otimes V) \to \mathrm{End}(V),$$

*called local coarse-graining and local fine-graning channels, respectively, such that*

$$\mathcal{T}(\rho_1(x)) = \rho_2(x) \ \textit{and}\ \mathcal{S}(\rho_2(x)) = \rho_1(x) \tag{34}$$

*for all positive non-zero $x \in A$.*

*Proof.* As previously done, let us define the local coarse-graining quantum channel

$$\mathcal{T}(X) := \mathrm{tr}(\phi(\xi T(\Omega_{(1)}))X)\phi(c_\omega\Omega_{(2)}) \otimes \phi(c_\omega\Omega_{(3)}) \tag{59}$$

for all $X \in \mathrm{End}(V)$. First, let us check that $\mathcal{T}(\rho_1(x)) = \rho_2(x)$ for all positive non-zero $x \in A$. Indeed,

$$\begin{aligned}
\mathcal{T}(\rho_1(x)) &= \tfrac{1}{\omega(x)}\mathrm{tr}(\phi(\xi T(\Omega_{(1)})c_\omega x))\phi(c_\omega\Omega_{(2)}) \otimes \phi(c_\omega\Omega_{(3)}) \\
&= \tfrac{1}{\omega(x)}\omega(\xi T(\Omega_{(1)})x)\phi(c_\omega\Omega_{(2)}) \otimes \phi(c_\omega\Omega_{(3)}) && \text{by Eq. 33} \\
&= \tfrac{1}{\omega(x)}\omega(\xi T(\Omega_{(1)}))\phi(c_\omega x_{(1)}\Omega_{(2)}) \otimes \phi(c_\omega x_{(2)}\Omega_{(3)}) && \text{by Eq. 24} \\
&= \tfrac{1}{\omega(x)}\phi(c_\omega x_{(1)}1_{(1)}) \otimes \phi(c_\omega x_{(2)}1_{(2)}) && \text{by Eq. 35} \\
&= \tfrac{1}{\omega(x)}\phi(c_\omega x_{(1)}) \otimes \phi(c_\omega x_{(2)}) = \rho_2(x) && \text{by Eq. 11}
\end{aligned}$$

Second, this map is trace-preserving:

$$\begin{aligned}
\mathrm{tr}(\mathcal{T}(X)) &= \mathrm{tr}(\phi(\xi T(\Omega_{(1)}))X)\mathrm{tr}(\phi(c_\omega\Omega_{(2)}))\mathrm{tr}(\phi(c_\omega\Omega_{(3)})) \\
&= \mathrm{tr}(\phi(\xi T(\Omega_{(1)}))X)\omega(\Omega_{(2)})\omega(\Omega_{(3)}) && \text{by Eq. 33} \\
&= \mathrm{tr}(\phi(\xi T(\Omega_{(1)}))X)\omega(\Omega_{(2)}) && \text{by Eq. 32} \\
&= \mathrm{tr}(\phi(\xi T(\xi^{-1}))X) && \text{by Eq. 36} \\
&= \mathrm{tr}(\phi(\xi\xi^{-1})X)\mathrm{tr}(X) && \text{by Eq. 37}
\end{aligned}$$

Finally, since $\Omega = \Omega^2 = \Omega\Omega^*$ (in fact, only positivity of $\Omega$ is needed), we can rewrite the map in the following form:

$$\begin{aligned}
\mathcal{T}(X) &= \mathrm{tr}(\phi(\xi T(\Omega_{(1)}(\Omega^*)_{(1')})X)\phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\Omega_{(2)}(\Omega^*)_{(2')})) \\
&= \mathrm{tr}(\phi(\xi T(\Omega_{(1)}(\Omega^*)_{(1')})X)\phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\Omega_{(2)})\Delta((\Omega^*)_{(2')})) \\
&= \mathrm{tr}(\phi(\xi T(\Omega_{(1)}\Omega^*_{(1')})X)\phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\Omega_{(2)})\Delta(\Omega_{(2')})^*) \\
&= \mathrm{tr}(\phi(\xi T(\Omega^*_{(1')})T(\Omega_{(1)}))X)\phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\Omega_{(2)})\Delta(\Omega_{(2')})^*) \\
&= \mathrm{tr}(\phi(T(\Omega_{(1')})^*\xi T(\Omega_{(1)}))X)\phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\Omega_{(2)})\Delta(\Omega_{(2')})^*) \\
&= \mathrm{tr}(\phi(T(\Omega_{(1')})^*\xi^{\frac{1}{2}}\xi^{\frac{1}{2}}T(\Omega_{(1)}))X)\phi^{\otimes 2}((c_\omega^{\frac{1}{2}})^{\otimes 2}\Delta(\Omega_{(2)})\Delta(\Omega_{(2')})^*(c_\omega^{\frac{1}{2}})^{\otimes 2}) \\
&= \mathrm{tr}(\phi(T(\Omega_{(1')})^*\xi^{\frac{1}{2}})\phi(\xi^{\frac{1}{2}}T(\Omega_{(1)}))X)\phi^{\otimes 2}((c_\omega^{\frac{1}{2}})^{\otimes 2}\Delta(\Omega_{(2)}))\phi^{\otimes 2}(\Delta(\Omega_{(2')})^*(c_\omega^{\frac{1}{2}})^{\otimes 2}) \\
&= \mathrm{tr}(\phi(\xi^{\frac{1}{2}}T(\Omega_{(1)}))X\phi(\xi^{\frac{1}{2}}T(\Omega_{(1')}))^\dagger)\phi^{\otimes 2}((c_\omega^{\frac{1}{2}})^{\otimes 2}\Delta(\Omega_{(2)}))\phi^{\otimes 2}((c_\omega^{\frac{1}{2}})^{\otimes 2}\Delta(\Omega_{(2')}))^\dagger \\
&= \mathrm{tr}_1(Q(X \otimes \mathbf{1} \otimes \mathbf{1})Q^\dagger)
\end{aligned}$$

where $\mathrm{tr}_1 := \mathrm{tr} \otimes \mathrm{id} \otimes \mathrm{id}$ is the partial trace on the first subsystem and

$$Q := \phi^{\otimes 3}(\xi^{\frac{1}{2}}T(\Omega_{(1)}) \otimes c_\omega^{\frac{1}{2}}\Omega_{(2)} \otimes c_\omega^{\frac{1}{2}}\Omega_{(3)})$$

Thus, $\mathcal{T}$ is completely positive. Now, let us define a local fine-graining quantum channel $\mathcal{S}$. Consider first the following hermitian projectors

$$P := \phi^{\otimes 2}(\Delta(1)), \quad P^\perp := \phi^{\otimes 2}(1 \otimes 1 - \Delta(1)), \quad P + P^\perp = \mathbf{1} \otimes \mathbf{1} \tag{60}$$

and let $\rho_0 \in \mathrm{End}(V)$ be any mixed state. Define

$$\mathcal{S}(X) := \mathrm{tr}(\phi(\Delta(\xi T(\Omega_{(1)})))X)\phi(c_\omega\Omega_{(2)}) + \mathrm{tr}(P^\perp X)\rho_0 \tag{61}$$

for all $X \in \mathrm{End}(V \otimes V)$. We first check that it satisfies $\mathcal{S}(\rho_2(x)) = \rho_1(x)$ for all positive non-zero $x \in A$. Notice that the second summand in the right-hand side of Equation 61 simply vanishes, i.e. $P^{\perp}\rho_2(x) = 0$, since $\rho_2(x)$ is supported on the orthogonal subspace $P \cdot \mathrm{End}(V^{\otimes 2})$. Thus,

$$
\begin{aligned}
\mathcal{S}(\rho_2(x)) &= \tfrac{1}{\omega(x)}\mathrm{tr}(\phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\xi T(\Omega_{(1)})x)))\phi(c_\omega\Omega_{(2)}) && \text{by Eq. 11}\\
&= \tfrac{1}{\omega(x)}(\omega\otimes\omega)(\Delta(\xi T(\Omega_{(1)})x))\phi(c_\omega\Omega_{(2)}) && \text{by Eq. 33}\\
&= \tfrac{1}{\omega(x)}\omega(\xi T(\Omega_{(1)})x)\phi(c_\omega\Omega_{(2)}) && \text{by Eq. 32}\\
&= \tfrac{1}{\omega(x)}\omega(\xi T(\Omega_{(1)}))\phi(c_\omega x\Omega_{(2)}) && \text{by Eq. 23}\\
&= \tfrac{1}{\omega(x)}\phi(c_\omega x) = \rho_1(x) && \text{by Eq. 35}
\end{aligned}
$$

for all positive non-zero $x \in A$, as we wanted to prove. Secondly, let us check that it is trace-preserving:

$$
\begin{aligned}
\mathrm{tr}(\mathcal{S}(X)) &= \mathrm{tr}(\phi^{\otimes 2}(\Delta(\xi T(\Omega_{(1)})))X)\mathrm{tr}(\phi(c_\omega\Omega_{(2)})) + \mathrm{tr}(P^{\perp}X)\\
&= \mathrm{tr}(\phi^{\otimes 2}(\Delta(\xi T(\Omega_{(1)})))X)\omega(\Omega_{(2)}) + \mathrm{tr}(P^{\perp}X) && \text{by Eq. 33}\\
&= \mathrm{tr}(\phi^{\otimes 2}(\Delta(\xi T(\xi^{-1})))X) + \mathrm{tr}(P^{\perp}X) && \text{by Eq. 36}\\
&= \mathrm{tr}(\phi^{\otimes 2}(\Delta(\xi\xi^{-1}))X) + \mathrm{tr}(P^{\perp}X) && \text{by Eq. 37}\\
&= \mathrm{tr}(PX) + \mathrm{tr}(P^{\perp}X) = \mathrm{tr}((P+P^{\perp})X) = \mathrm{tr}(X) && \text{by Eq. 60}
\end{aligned}
$$

for all $X \in \mathrm{End}(V \otimes V)$. That $\mathcal{S}$ is completely positive can be proved analogously and we do not include it here: simply notice that the second summand in Equation 61 is clearly a completely positive map, and a similar argument to that for $\mathcal{T}$ applies to the first summand. □

## Appendix D. Proofs of Section 4 (C*-Hopf algebras)

In this appendix we derive a proof of Lemma 17. We first provide the following auxiliary result, related to the trace-preserving condition of the gluing map.

**Lemma 28.** *Let $A$ be a C*-Hopf algebra. Then,*

$$x_{(1)} \otimes \omega(x_{(2)})x_{(3)} = \omega(x)1 \otimes 1 \tag{62}$$

*for all $x \in A$.*

*Proof.* Since $\Omega \in A$ is non-degenerate, the right Radon-Nikodym derivative $\mu_x \in A^*$ of an arbitrary element $x \in A$ with respect to $\Omega$ exists and satisfies

$$x = \Omega_{(1)}\mu_x(\Omega_{(2)}). \tag{63}$$

As an immediate consequence,

$$\omega(x) = \omega(\Omega_{(1)})\mu_x(\Omega_{(2)}) = \mathcal{D}^{-2}\mu_x(1). \tag{64}$$

where the last equality follows from Lemma 15. Then, it is easy to conclude that

$$
\begin{aligned}
x_{(1)}\omega(x_{(2)}) \otimes x_{(3)} &= \Omega_{(1)} \otimes \omega(\Omega_{(2)})\Omega_{(3)}\mu_x(\Omega_{(4)}) && \text{by Eq. 63}\\
&= \Omega_{(4)} \otimes \omega(\Omega_{(1)})\Omega_{(2)}\mu_x(\Omega_{(3)}) && \text{by Eq. 27}\\
&= \mathcal{D}^{-2}1_{(3)} \otimes 1_{(1)}\mu_x(1_{(2)}) && \text{by Eq. 35}\\
&= \mathcal{D}^{-2}\mu_x(1)1 \otimes 1 && \text{by Eq. 7}\\
&= \omega(x)1 \otimes 1, && \text{by Eq. 64}
\end{aligned}
$$

as we wanted to prove. □

**Lemma 17.** *Let $A$ be a $C^*$-Hopf algebra and let $(V,\phi)$ be any faithful $*$-representation of $A$. Then, for all positive non-zero $x \in A$ there exists a quantum channel $\mathcal{G}_x$ on $\mathrm{End}(V\otimes V)$, called "gluing" map, such that*

$$\mathrm{id}^{\otimes M-1}\otimes \mathcal{G}_x \otimes \mathrm{id}^{\otimes N-1}(\rho_M(\Omega)\otimes\rho_N(\Omega)) = \rho_{M+N}(x) \tag{40}$$

*for all $M, N \in \mathbb{N}$.*

*Proof.* Fix any positive non-zero $x \in A$. We first recall the definition of the gluing map previously given in Section 4. For simplicity, let $\mathcal{G}_x := \mathcal{T}\circ\mathcal{G}$ for the linear map $\mathcal{G} : \mathrm{End}(V\otimes V) \to \mathrm{End}(V)$ defined by the expression

$$\mathcal{G}(X\otimes Y) := \tfrac{1}{\omega(x)}\mathrm{tr}(\phi(S(x_{(1)}))X)\phi(c_\omega x_{(2)})\mathrm{tr}(\phi(S(x_{(3)}))Y) \tag{65}$$

for all $X, Y \in \mathrm{End}(V)$. Then, it is enough to check that $\mathcal{G}(\rho_2(\Omega)\otimes\rho_2(\Omega)) = \rho_3(\Omega)$. To this end, let us recall that, in the case of C*-Hopf algebras,

$$\omega(\Omega_{(1)})\Omega_{(2)} = \tfrac{1}{D^2}1 = \omega(\Omega)1, \tag{66}$$

where the first equality is stated in Lemma 15 and the second equality follows by applying the counit in the first one, since $\varepsilon(1) = 1$. Then,

$$\begin{aligned}
&(\mathrm{id}\otimes\mathcal{G}\otimes\mathrm{id})(\rho_2(\Omega)\otimes\rho_2(\Omega)) =\\
&= \tfrac{1}{\omega(x)}\tfrac{1}{\omega(\Omega)^2}\phi(c_\omega\Omega_{(1)})\otimes\omega(S(x_{(1)})\Omega_{(2)})\phi(c_\omega x_{(2)})\omega(S(x_{(3)})\Omega_{(1')})\otimes\phi(c_\omega\Omega_{(2')})\\
&= \tfrac{1}{\omega(x)}\tfrac{1}{\omega(\Omega)^2}\phi(c_\omega x_{(1)}\Omega_{(1)})\otimes\omega(\Omega_{(2)})\phi(c_\omega x_{(2)})\omega(S(x_{(3)})\Omega_{(1')})\otimes\phi(c_\omega\Omega_{(2')})\\
&= \tfrac{1}{\omega(x)}\tfrac{1}{\omega(\Omega)^2}\phi(c_\omega x_{(1)}\Omega_{(1)})\otimes\omega(\Omega_{(2)})\phi(c_\omega x_{(2)})\omega(\Omega_{(1')})\otimes\phi(c_\omega x_{(3)}\Omega_{(2')})\\
&= \tfrac{1}{\omega(x)}\phi(c_\omega x_{(1)}1)\otimes\phi(c_\omega x_{(2)})\otimes\phi(c_\omega x_{(3)}1) = \rho_3(x).
\end{aligned}$$

This calculation can be explained as follows. In the first place, we have replaced the trace with the canonical regular element $\omega \in A^*$ since by Remark 10 the weight $c_\omega \in A$, which is central, defines a linear extension of $\omega$ to the representation space. In the second and third steps we have applied the pulling-through identity; see Equation 23. Finally, we apply twice Equation 66 to get rid of $\Omega$ and the coefficients $\omega(\Omega)^{-1}$. As an aside, note that $\omega(\Omega_{(1)})\Omega_{(2)} = \Omega_{(1)}\omega(\Omega_{(2)})$ since $\Omega$ is cocentral; see Equation 27. Since $\mathcal{T}$ is a quantum channel it only remains to prove that $\mathcal{G}$ is also a quantum channel. On the one hand, that $\mathcal{G}$ is trace-preserving is a straightforward consequence of Lemma 28:

$$\begin{aligned}
\mathrm{tr}(\mathcal{G}(X\otimes Y)) &= \tfrac{1}{\omega(x)}\mathrm{tr}(\phi(S(x_{(1)}))X)\omega(x_{(2)})\mathrm{tr}(\phi(S(x_{(3)}))Y) && \text{by Eq. 33}\\
&= \tfrac{1}{\omega(x)}\omega(x)\mathrm{tr}(\phi(S(1))X)\mathrm{tr}(\phi(S(1))Y) && \text{by Eq. 62}\\
&= \mathrm{tr}(X)\mathrm{tr}(Y) = \mathrm{tr}(X\otimes Y) && \text{by Eq. 16}
\end{aligned}$$

for all $X, Y \in \mathrm{End}(V)$. On the other hand, in order to prove that $\mathcal{G}$ is completely positive, let $x = yy^*$ for some element $y \in A$. Then, we can rewrite it as follows

$$\begin{aligned}
&\mathcal{G}(X\otimes Y) =\\
&= \tfrac{1}{\omega(x)}\mathrm{tr}(\phi(S((yy^*)_{(1)}))X)\phi(c_\omega(yy^*)_{(2)})\mathrm{tr}(\phi(S(yy^*)_{(3)})Y)\\
&= \tfrac{1}{\omega(x)}\mathrm{tr}(\phi(S(y_{(1)}y^*_{(1')}))X)\phi(c_\omega y_{(2)}y^*_{(2')})\mathrm{tr}(\phi(S(y_{(3)}y^*_{(3')}))Y)\\
&= \tfrac{1}{\omega(x)}\mathrm{tr}(\phi(S(y^*_{(1')})S(y_{(1)}))X)\phi(c_\omega y_{(2)}y^*_{(2')})\mathrm{tr}(\phi(S(y^*_{(3')})S(y_{(3)}))Y)\\
&= \tfrac{1}{\omega(x)}\mathrm{tr}(\phi(S(y_{(1')})^*S(y_{(1)}))X)\phi(c_\omega y_{(2)}y^*_{(2')})\mathrm{tr}(\phi(S(y_{(3')})^*S(y_{(3)}))Y)\\
&= \tfrac{1}{\omega(x)}\mathrm{tr}(\phi(S(y_{(1)}))X\phi(S(y_{(1')})^*))\phi(c_\omega y_{(2)}y^*_{(2')})\mathrm{tr}(\phi(S(y_{(3)}))Y\phi(S(y_{(3')})^*))\\
&= \mathrm{tr}_{1,3}(Q(X\otimes\mathbf{1}\otimes Y)Q^\dagger)
\end{aligned}$$

where we have defined

$$Q := \frac{1}{\omega(x)^{1/2}}\phi^{\otimes 4}(S(y_{(1)}) \otimes c_\omega^{\frac{1}{2}} y_{(2)} \otimes \otimes S(y_{(3)})). \tag{67}$$

Therefore, $\mathcal{G}$ is completely positive. Indeed, in the first step we have applied that the comultiplication is multiplicative and the $*$-operation is a coalgebra homomorphism; see Equations 11 and 12. In the second step we have used that $S$ is an; see Equation 16. The third equality follows from the relation between the antipode and the $*$-operation; see Equation 17. Note that, for C*-Hopf algebras, $S = S^{-1}$; see Proposition 26. The fourth step is a simple consequence of the fact that $\phi$ is a $*$-representation and the cyclic property of the trace. Finally, the middle term can be rewritten in the form $\phi(c_\omega y_{(2)} y^*_{(2')}) = \phi(c_\omega^{1/2} y_{(2)})\phi(c_\omega^{1/2} y_{(2')})^\dagger$ since $c_\omega \in A$ is positive central element and $\phi$ is a $*$-representation. □

## Appendix E. Proofs of Section 4 (C*-weak Hopf algebras)

In this appendix we prove Lemma 21. In order to perform an analogous construction of this gluing map to the one given in the C*-Hopf algebra case, we first derive an appropiate version of the usual pulling-through identity in Equation 23 to the trivial sector:

**Lemma 29.** *Let $A$ be a biconnected C*-weak Hopf algebra. Then,*

$$x_L S(1_{(1)}) \otimes 1_{(2)} \otimes S(1_{(3)}) y_R = S(1_{(1)}) \otimes y_R 1_{(2)} x_L \otimes S(1_{(3)}) \tag{68}$$

*for all $x_L \in A^L$ and $y_R \in A^R$.*

*Proof.* First, recall Equations 2.31a and 2.31b from [6]:

$$x_L S(1_{(1)}) \otimes 1_{(2)} = S(1_{(1)}) \otimes 1_{(2)} x_L,$$
$$y_R 1_{(1)} \otimes S(1_{(2)}) = 1_{(1)} \otimes S(1_{(2)}) y_R.$$

for all $x_L \in A^L$ and $y_R \in A^R$. This, together with Equations 20 and 21, leads by taking coproducts accordingly to the following identities:

$$x_L S(1_{(1)}) \otimes 1_{(2)} \otimes 1_{(3)} = S(1_{(1)}) \otimes 1_{(2)} x_L \otimes 1_{(3)},$$
$$1_{(1)} \otimes y_R 1_{(2)} \otimes S(1_{(3)}) = 1_{(1)} \otimes 1_{(2)} \otimes S(1_{(3)}) y_R,$$

respectively, for all $x_L \in A^L$ and $y_R \in A^R$. Finally, since $A^L$ and $A^R$ commute, we conclude the result by combining both identities. □

In addition, we adapt slightly Lemma 15 to the trivial sector, which is a key property concerning complete positivity of the gluing map in Lemma 21. The following result solves this problem.

**Lemma 30.** *Let $A$ be a biconnected C*-weak Hopf algebra. Then,*

$$\xi_R S(x_L^*) = S(x_L)^* \xi_R \quad \text{and} \quad S(y_R)\xi_L = \xi_L S(y_R^*)^* \tag{69}$$

*for all $x_L \in A^L$ and $y_R \in A^R$.*

*Proof.* In the first place, note that $T$ coincides with $S$ and $S^{-1}$ restricted to $A^L$ and $A^R$, respectively. Indeed, by virtue of Proposition 25, Lemma 23 and Equation 17,

$$T(x_L) = S(x_L 1_{(1)})\hat{g}(1_{(2)}) = S(x_L), \tag{70}$$

$$T(y_R) = \hat{g}(1_{(1)}) S^{-1}(1_{(2)} y_R) = S^{-1}(y_R) = S(y_R^*)^*, \tag{71}$$

for all $x_L \in A^L$ and $y_R \in A^R$. Then, recall Lemma 15 to conclude that

$$S(x_L^*) = T(x_L^*) = \xi_L^{-1}\xi_R^{-1} T(x_L)^* \xi_L \xi_R = \xi_L^{-1}\xi_R^{-1} S(x_L)^* \xi_L \xi_R = \xi_R^{-1} S(x_L)^* \xi_R,$$

where in the last step we have used that $S(x_L) \in A^R$ and $A^L$ and $A^R$ commute. The remaining identity is proved similarly. ☐

The following auxiliary results arise naturally in the course of the derivation of the properties of the gluing map.

**Lemma 31.** *Let $A$ be a bicoconnected C*-weak Hopf algebra. Then,*

$$\hat{h}(\Omega_{(1)})\Omega_{(2)} = \tfrac{1}{\mathcal{D}^2\varepsilon(1)}1. \tag{72}$$

*Proof.* It is simple to check that

$$\hat{h}(\Omega_{(1)})\Omega_{(2)} = \tfrac{1}{\mathcal{D}^2\varepsilon(1)}\hat{h}(t_{(1)})t_{(2)}\hat{g}(t_{(3)}) = \tfrac{1}{\mathcal{D}^2\varepsilon(1)}1_{(1)}\hat{g}(1_{(2)}) = \tfrac{1}{\mathcal{D}^2\varepsilon(1)}1,$$

where the first step is a consequence of the characterization of $\Omega$ in Proposition 24, the second follows from the definition of dual left integral in Equation 43 and the third equality is due to Lemma 23. ☐

**Lemma 32.** *Let $A$ be a biconnected C*-weak Hopf algebra. Then,*

$$1_{(1)}\hat{h}(1_{(2)}) \otimes 1_{(3)} = \tfrac{1}{\varepsilon(1)}1 \otimes 1. \tag{73}$$

*Proof.* Equivalently, we will check that

$$(\varphi\hat{h}\psi)(1) = \tfrac{1}{\varepsilon(1)}\varphi(1)\psi(1)$$

for all $\varphi, \psi \in A^*$. Recall that $\hat{h} \in A^*$ is a one-dimensional projector supported on the trivial sector [6, Lemma 4.8]. Hence,

$$(\varphi\hat{h}\psi\hat{h})(\hat{\chi}_1) = (\varphi\hat{h})(\hat{\chi}_1)(\psi\hat{h})(\hat{\chi}_1) \quad \text{and} \quad (\varphi\hat{h})(\hat{\chi}_a) = \delta_{a1} \tag{74}$$

for all $\varphi, \psi \in A^*$ and all sectors $a = 1, \dots, s$. In particular

$$(f\hat{h})(\hat{\chi}_1) = (f\hat{h})(\hat{d}_1\hat{\chi}_1 + \cdots + \hat{d}_s\hat{\chi}_s) = \mathcal{D}^2(f\hat{h})(\Omega) = \tfrac{1}{\varepsilon(1)}f(1) \tag{75}$$

for all $f \in A^*$. Thus, we conclude that:

$$\tfrac{1}{\varepsilon(1)}(\varphi\hat{h}\psi)(1) = (\varphi\hat{h}\psi\hat{h})(\hat{\chi}_1) = (\varphi\hat{h})(\hat{\chi}_1)(\psi\hat{h})(\hat{\chi}_1) = \tfrac{1}{\varepsilon(1)^2}\varphi(1)\psi(1),$$

where the first equality follows from Equation 75 using $f := \varphi\hat{h}\psi$, the second equality is simply Equation 74 and the third equality follows from Equation 75 considering $f := \varphi, \psi$. ☐

**Lemma 33.** *Let $A$ be a biconnected C*-weak Hopf algebra. Then,*

$$1_{(1)} \otimes \omega(1_{(2)})1_{(3)} = \mathcal{D}^2\xi_R^{-1} \otimes \xi_L^{-1}. \tag{76}$$

*Proof.* Note by the definition of $A^L$ and $A^R$ in Equations 20 and 21, respectively, and the decomposition $\xi^{-1} = \xi_L^{-1}\xi_R^{-1}$ in Lemma 15, that

$$(\xi^{-1})_{(1)} \otimes (\xi^{-1})_{(2)} \otimes (\xi^{-1})_{(3)} = \xi_L^{-1}1_{(1)} \otimes 1_{(2)} \otimes \xi_R^{-1}1_{(3)} \tag{77}$$

Then, the statement follows from the following calculation:

$$\begin{aligned}
1_{(1)} \otimes \omega(1_{(2)})1_{(3)} &= \mathcal{D}^2\varepsilon(1)\hat{h}(\Omega_{(1)})\Omega_{(2)} \otimes \omega(\Omega_{(3)})\Omega_{(4)} && \text{by Eq. 72}\\
&= \mathcal{D}^2\varepsilon(1)\hat{h}(\Omega_{(3)})\Omega_{(4)} \otimes \omega(\Omega_{(1)})\Omega_{(2)} && \text{by Eq. 27}\\
&= \mathcal{D}^2\varepsilon(1)\hat{h}((\xi^{-1})_{(2)})(\xi^{-1})_{(3)} \otimes (\xi^{-1})_{(1)} && \text{by Eq. 36}\\
&= \mathcal{D}^2\varepsilon(1)\hat{h}(1_{(2)})\xi_R^{-1}1_{(3)} \otimes \xi_L^{-1}1_{(1)} && \text{by Eq. 77}\\
&= \mathcal{D}^2\xi_R^{-1} \otimes \xi_L^{-1} && \text{by Eq. 73}
\end{aligned}$$

as we wanted to prove. ☐

**Lemma 34.** *Let $A$ be a biconnected C*-weak Hopf algebra. Then,*

$$\omega(1_{(1)})1_{(2)}\omega(1_{(3)}) = \mathcal{D}^2\omega(1)\xi^{-1}. \tag{78}$$

*Proof.* First, it will be useful to compute the constant $\omega(1)$ in a more operative way. The following calculation is a direct consequence of Proposition 24 and Equation 56:

$$\omega(1) = \tfrac{1}{\mathcal{D}^2\varepsilon(1)}\hat{h}(g_L^{-1}g_R^{-1}) = \tfrac{1}{\mathcal{D}^2\varepsilon(1)}\mathcal{D}^4\varepsilon(1)^2\hat{h}(\xi_L^{-1}\xi_R^{-1}) = \mathcal{D}^2\varepsilon(1)\hat{h}(\xi_L^{-1}\xi_R^{-1}). \tag{79}$$

Now, by an analogous reasoning as in the previous proof:

$$\begin{aligned}
\omega(1_{(1)})1_{(2)}\omega(1_{(3)}) &= \mathcal{D}^2\varepsilon(1)\hat{h}(\Omega_{(1)})\omega(\Omega_{(2)})\Omega_{(3)}\omega(\Omega_{(4)}) && \text{by Eq. 72}\\
&= \mathcal{D}^2\varepsilon(1)\hat{h}(\Omega_{(4)})\omega(\Omega_{(1)})\Omega_{(2)}\omega(\Omega_{(3)}) && \text{by Eq. 27}\\
&= \mathcal{D}^2\varepsilon(1)\hat{h}(\xi_R^{-1}1_{(3)})\xi_L^{-1}1_{(1)}\omega(1_{(2)}) && \text{by Eq. 36}\\
&= \mathcal{D}^4\varepsilon(1)\hat{h}(\xi_R^{-1}\xi_L^{-1})\xi_L^{-1}\xi_R^{-1} && \text{by Eq. 76}\\
&= \mathcal{D}^2\omega(1)\xi_L^{-1}\xi_R^{-1} = \mathcal{D}^2\omega(1)\xi^{-1} && \text{by Eq. 79}
\end{aligned}$$

as we wanted to prove. □

**Remark 19.** There are no trace-preserving gluing maps for general biconnected C*-weak Hopf algebras mimicking Lemma 17.

*Proof.* Suppose by contradiction that there exists a trace-preserving linear map $\mathcal{G} : \mathrm{End}(V \otimes V) \to \mathrm{End}(V \otimes V)$ that is a gluing map. In particular,

$$(\mathrm{id} \otimes \mathcal{G} \otimes \mathrm{id})(\rho_2(\Omega) \otimes \rho_2(\Omega)) = \rho_4(\Omega).$$

On the one hand, after performing a partial trace on the second and third subsystems, the left-hand side would be trivially given by the product state

$$\begin{aligned}
\mathrm{tr}_{2,3}(\rho_2(\Omega) \otimes \rho_2(\Omega)) &= \tfrac{1}{\omega(\Omega)^2}\phi(c_\omega\Omega_{(1)})\omega(\Omega_{(2)}) \otimes \omega(\Omega_{(1')})\phi(c_\omega\Omega_{(2')}) && \text{by Eq. 33}\\
&= \tfrac{1}{\omega(\Omega)^2}\phi(c_\omega\xi^{-1}) \otimes \phi(c_\omega\xi^{-1}) && \text{by Eq. 36}
\end{aligned}$$

However, the right-hand side would take the following form:

$$\begin{aligned}
\mathrm{tr}_{3,4}(\rho_4(\Omega)) &= \tfrac{1}{\omega(\Omega)^2}\phi(c_\omega\Omega_{(1)}) \otimes \omega(\Omega_{(2)})\omega(\Omega_{(3)})\phi(c_\omega\Omega_{(4)}) && \text{by Eq. 33}\\
&= \tfrac{1}{\omega(\Omega)^2}\phi(c_\omega\Omega_{(1)}) \otimes \omega(\Omega_{(2)})\phi(c_\omega\Omega_{(3)}) && \text{by Eq. 32}\\
&= \tfrac{1}{\omega(\Omega)^2}\phi(c_\omega\Omega_{(2)}) \otimes \omega(\Omega_{(3)})\phi(c_\omega\Omega_{(1)}) && \text{by Eq. 27}\\
&= \tfrac{1}{\omega(\Omega)^2}\phi(c_\omega\xi_R^{-1}1_{(2)}) \otimes \phi(c_\omega\xi_L^{-1}1_{(1)}) && \text{by Eq. 36}
\end{aligned}$$

which is not necessarily a product state. This contradicts the previous equation. □

**Lemma 21.** *Let $A$ be a biconnected C*-weak Hopf algebra and let $(V, \phi)$ be any faithful $*$-representation of $A$. There is a quantum channel $\mathcal{G}_1$ on $\mathrm{End}(V \otimes V)$ called "gluing" map, such that*

$$\mathrm{id}^{\otimes M-1} \otimes \mathcal{G}_1 \otimes \mathrm{id}^{\otimes N-1}(\rho_M(1) \otimes \rho_N(1)) = \rho_{M+N}(1) \tag{41}$$

*for all $M, N \in \mathbb{N}$.*

*Proof.* For simplicity, let $\mathcal{G}_1 := \mathcal{T} \circ \mathcal{G}$, where $\mathcal{T}$ stands for the local coarse-graining quantum channel from Section 3 and $\mathcal{G} : \mathrm{End}(V \otimes V) \to \mathrm{End}(V)$ is given by

$$\mathcal{G}(X \otimes Y) := \tfrac{1}{\mathcal{D}^2}\mathrm{tr}(\phi(S(1_{(1)})\xi_L)X)\phi(c_\omega 1_{(2)})\mathrm{tr}(\phi(\xi_R S(1_{(3)}))Y)$$

for all $X, Y \in \mathrm{End}(V)$. First, assume that $M = N = 2$ without loss of generality and let us check that it fulfills $\mathcal{G}(\rho_2(1) \otimes \rho_2(1)) = \rho_3(1)$. To this end, it turns out to be enough to prove:

$$\mathcal{G}(\phi(c_\omega x_L) \otimes \phi(c_\omega x_R)) = \omega(1)\phi(c_\omega x_L x_R) \tag{80}$$

for all $x_L \in A^L$ and $x_R \in A^R$. Indeed, in that case,

$$\begin{aligned}(\mathrm{id} \otimes \mathcal{G} \otimes \mathrm{id})(\rho_2(1)^{\otimes 2}) &= \tfrac{1}{\omega(1)^2}\phi(c_\omega 1_{(1)}) \otimes \mathcal{G}(\phi(c_\omega 1_{(2)}) \otimes \phi(c_\omega 1_{(1')})) \otimes \phi(c_\omega 1_{(2')}) \\ &= \tfrac{1}{\omega(1)}\phi(c_\omega 1_{(1)}) \otimes \phi(c_\omega 1_{(2)} 1_{(1')}) \otimes \phi(c_\omega 1_{(2')}) \\ &= \tfrac{1}{\omega(1)}\phi(c_\omega 1_{(1)}) \otimes \phi(c_\omega 1_{(2)}) \otimes \phi(c_\omega 1_{(3)}) = \rho_3(1).\end{aligned}$$

by the weak comultiplicativity of the counit and the fact that $1_{(1)} \otimes 1_{(2)} \in A^R \otimes A^L$; see Equation 13 and [6]. Thus, let us move to the proof of Equation 80:

$$\begin{aligned}\mathcal{G}(\phi(c_\omega x_L) \otimes \phi(c_\omega x_R)) &= \tfrac{1}{D^2}\omega(S(1_{(1)})\xi_L x_L)\phi(c_\omega 1_{(2)})\omega(\xi_R S(1_{(3)})x_R) && \text{by Eq. 33} \\ &= \tfrac{1}{D^2}\omega(\xi_L x_L S(1_{(1)}))\phi(c_\omega 1_{(2)})\omega(S(1_{(3)})x_R \xi_R) \\ &= \tfrac{1}{D^2}\omega(S(1_{(1)}))\phi(c_\omega x_R \xi_R 1_{(2)} \xi_L x_L)\omega(S(1_{(3)})) && \text{by Eq. 68} \\ &= \tfrac{1}{D^2}\omega(1_{(1)})\phi(c_\omega x_R \xi_R 1_{(2)} \xi_L x_L)\omega(1_{(3)}) && \text{by Eq. 54} \\ &= \omega(1)\phi(c_\omega x_R \xi_R \xi_R^{-1} \xi_L^{-1} \xi_L x_L) && \text{by Eq. 78} \\ &= \omega(1)\phi(c_\omega x_R x_L)\end{aligned}$$

as we wanted to prove. Additionally, $\mathcal{G}$ is trace-preserving as an immediate consequence of Lemma 33:

$$\begin{aligned}\mathrm{tr}(\mathcal{G}(X \otimes Y)) &= \tfrac{1}{D^2}\mathrm{tr}(\phi(S(1_{(1)})\xi_L)X)\omega(1_{(2)})\mathrm{tr}(\phi(\xi_R S(1_{(3)}))Y) && \text{by Eq. 33} \\ &= \mathrm{tr}(\phi(S(\xi_R^{-1})\xi_L)X)\mathrm{tr}(\phi(\xi_R S(\xi_L^{-1}))Y) && \text{by Eq. 76} \\ &= \mathrm{tr}(\phi(\xi_L^{-1}\xi_L)X)\mathrm{tr}(\phi(\xi_R \xi_R^{-1})Y) && \text{by Eq. 56} \\ &= \mathrm{tr}(X)\mathrm{tr}(Y) = \mathrm{tr}(X \otimes Y).\end{aligned}$$

Finally, in order to check that $\mathcal{G}$ is a completely positive linear map, let us first consider the following two calculations:

$$\begin{aligned}\mathrm{tr}(\phi(S(x_R y_R^*)\xi_L)X) &= \mathrm{tr}(\phi(S(y_R^*)S(x_R)\xi_L)X) && \text{by Eq. 16} \\ &= \mathrm{tr}(\phi(S(y_R^*)\xi_L S(x_R^*)^*)X) && \text{by Eq. 69} \\ &= \mathrm{tr}(\phi(S(y_R^*)\xi_L^{\frac{1}{2}}\xi_L^{\frac{1}{2}}S(x_R^*)^*)X) && \text{by Eq. 56} \\ &= \mathrm{tr}(\phi(S(y_R^*)\xi_L^{\frac{1}{2}})\phi(\xi_L^{\frac{1}{2}}S(x_R^*)^*)X) \\ &= \mathrm{tr}(\phi(\xi_L^{\frac{1}{2}}S(x_R^*)^*)X\phi(S(y_R^*)\xi_L^{\frac{1}{2}})) \\ &= \mathrm{tr}(\phi(\xi_L^{\frac{1}{2}}S(x_R^*)^*)X\phi((\xi_L^{\frac{1}{2}}S(y_R^*))^*)) && \text{by Eq. 56} \\ &= \mathrm{tr}(\phi(\xi_L^{\frac{1}{2}}S(x_R^*)^*)X\phi(\xi_L^{\frac{1}{2}}S(y_R^*)^*)^\dagger)\end{aligned}$$

for all $x_R, y_R \in A^R$ and, analogously,

$$\begin{aligned}\mathrm{tr}(\phi(\xi_R S(x_L y_L^*))Y) &= \mathrm{tr}(\phi(\xi_R S(y_L^*)S(x_L))Y) && \text{by Eq. 16} \\ &= \mathrm{tr}(\phi(S(y_L)^*\xi_R S(x_L))Y) && \text{by Eq. 69} \\ &= \mathrm{tr}(\phi(S(y_L)^*\xi_R^{\frac{1}{2}}\xi_R^{\frac{1}{2}}S(x_L))Y) && \text{by Eq. 56}\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{tr}(\phi(S(y_L)^*\xi_R^{\frac{1}{2}})\phi(\xi_R^{\frac{1}{2}}S(x_L))Y) \\
&= \operatorname{tr}(\phi(\xi_R^{\frac{1}{2}}S(x_L))Y\phi(S(y_L)^*\xi_R^{\frac{1}{2}})) \\
&= \operatorname{tr}(\phi(\xi_R^{\frac{1}{2}}S(x_L))Y\phi((\xi_R^{\frac{1}{2}}S(y_L))^*)) \qquad \text{by Eq. 56} \\
&= \operatorname{tr}(\phi(\xi_R^{\frac{1}{2}}S(x_L))Y\phi(\xi_R^{\frac{1}{2}}S(y_L))^\dagger)
\end{aligned}
$$

for all $x_L, y_L \in A^L$. Now, recall that $1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} \in A^R \otimes A \otimes A^L$; see [6]. This allows us to rewrite $\mathcal{G}$ in the following form:

$$
\begin{aligned}
\mathcal{G}(X \otimes Y) &= \tfrac{1}{\mathcal{D}^2}\operatorname{tr}(\phi(S((1{\cdot}1^*)_{(1)})\xi_L)X)\phi(c_\omega(1{\cdot}1^*)_{(2)})\operatorname{tr}(\phi(\xi_R S((1{\cdot}1^*)_{(3)}))Y) \\
&= \tfrac{1}{\mathcal{D}^2}\operatorname{tr}(\phi(S(1_{(1)}(1^*)_{(1')})\xi_L)X)\phi(c_\omega 1_{(2)}(1^*)_{(2')})\operatorname{tr}(\phi(\xi_R S(1_{(3)}1_{(3')}))Y) \\
&= \tfrac{1}{\mathcal{D}^2}\operatorname{tr}(\phi(S(1_{(1)}1^*_{(1')})\xi_L)X)\phi(c_\omega 1_{(2)}1^*_{(2')})\operatorname{tr}(\phi(\xi_R S(1_{(3)}1^*_{(3')}))Y) \\
&= \operatorname{tr}_{1,3}(Q(X \otimes \mathbf{1} \otimes Y)Q^\dagger)
\end{aligned}
$$

where the last step follows from the previous calculations, and we have defined

$$
Q := \tfrac{1}{\mathcal{D}}\phi^{\otimes 3}(\xi_L^{\frac{1}{2}}S(1^*_{(1)}) \otimes c_\omega^{\frac{1}{2}}1_{(2)} \otimes \xi_R^{\frac{1}{2}}S(1_{(3)})). \tag{81}
$$

This concludes the proof. □

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