UNIVERSIDAD COMPLUTENSE DE MADRID
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TESIS DOCTORAL

The Complemented Subspace Problem in Banach lattices
El Problema del Subespacio Complementado en Retículos de Banach

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Programa de Doctorado en Investigación Matemática

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Contents

Resumen

El *principal objetivo* de esta tesis es estudiar la cuestión de si todo subespacio complementado en un retículo de Banach debe ser isomorfo a un retículo de Banach. Esta pregunta se conoce como **Problema del subespacio complementado en retículos de Banach** (**CSP**). A pesar del notable interés que ha recibido dentro de la teoría de retículos de Banach, ha permanecido abierta durante décadas. En el **Capítulo 2** analizamos este problema y discutimos nuevas maneras de abordarlo, destacando su conexión con los *retículos de Banach libres*. En el **Capítulo 3** resolvemos el CSP, dando una *respuesta negativa* al mismo basándonos en una construcción de Plebanek y Salguero-Alarcón con la que previamente se había resuelto el problema del subespacio complementado para espacios $C(K)$ (también de manera negativa). En el **Capítulo 4**, se demuestra la existencia del *retículo de Banach libre complejo* y, de hecho, se da una descripción explícita del mismo apoyándonos en la representación funcional existente para el retículo de Banach libre (real) dada por Avilés, Rodríguez y Tradacete. Una de las razones que nos impulsan a considerar este objeto es que constituye un lugar canónico para estudiar el CSP para retículos complejos. El **Capítulo 5** está dedicado a estudiar los *homomorfismos reticulares que alcanzan la norma*. Probamos que en un AM-espacio todo homomorfismo reticular alcanza su norma. En el otro extremo, demostramos que cualquier retículo de Banach que posee un funcional estrictamente positivo admite un renormamiento reticular de manera que *ningún* homomorfismo alcance su norma. Finalmente, en el **Capítulo 6** demostramos que si K es un espacio compacto Hausdorff con $K^{(\alpha)} \neq \varnothing$, para $2 < \alpha < \omega$, entonces $C(K)$ contiene un subespacio linealmente isométrico a $C[1, \omega^\alpha]$ que hace *recuperación de fase estable* (SPR), dando una respuesta parcial afirmativa a una pregunta planteada por Freeman, Oikhberg, Pineau y Taylor.

Abstract

The *main objective* of this thesis is to study the question of whether every complemented subspace in a Banach lattice must be isomorphic to a Banach lattice. This question is known as the **Complemented Subspace Problem for Banach lattices** (**CSP**). Despite the considerable attention it has received within Banach lattice theory, it has remained open for decades. In **Chapter 2**, we analyze this problem and discuss new possible approaches to it, with a particular focus on its connection with *free Banach lattices*. In **Chapter 3**, we solve the **CSP**, providing a *negative answer* based on a construction by Plebanek and Salguero-Alarcón, with which a negative solution to the complemented subspace problem for $C(K)$-spaces had previously been given. In **Chapter 4**, the existence of the *complex free Banach lattice* is proven, giving an explicit description building on the existing functional representation for the (real) free Banach lattice due to Avilés, Rodríguez, and Tradacete. One of the reasons that lead us to consider this object is that it constitutes a canonical setting for studying the **CSP** for complex Banach lattices. **Chapter 5** is dedicated to studying *norm-attaining lattice homomorphisms*. We prove that every lattice homomorphism on an AM-space attains its norm. In contrast, we show that any Banach lattice which has a strictly positive functional admits a lattice renorming in such a way *no* homomorphism attains its norm. Finally, in **Chapter 6**, we prove that if K is a compact Hausdorff space with $K^{(\alpha)} \neq \varnothing$, for $2 < \alpha < \omega$, then $C(K)$ contains a subspace linearly isometric to $C[1, \omega^\alpha]$ that does *stable phase retrieval* (SPR), providing a partial affirmative answer to a question posed by Freeman, Oikhberg, Pineau, and Taylor.

List of Publications

The results obtained during my years as a predoctoral student supervised by Pedro Tradacete have led to the articles cited below:

1. E. Bilokopytov, E. García-Sánchez, D. **de Hevia**, G. Martínez-Cervantes, and P. Tradacete, *Norm-attaining lattice homomorphisms and renormings of Banach lattices*, Preprint available on arXiv (2025), 29 pp.

2. M. Camúñez, E. García-Sánchez, and D. **de Hevia**, *A characterization of complex stable phase retrieval in Banach lattices*, Preprint available on arXiv (2025), 14 pp.

3. E. García-Sánchez and D. **de Hevia**, *Subspaces of $C(K)$-spaces doing stable phase retrieval*, in preparation.

4. E. García-Sánchez, D. **de Hevia**, and P. Tradacete, *Free objects in Banach space theory*, Cutting-edge mathematics, RSME Springer Ser., vol. 13, Springer, Cham, [2024] ©2024, pp. 100–124. MR 4823784

5. D. **de Hevia**, G. Martínez-Cervantes, A. Salguero-Alarcón, and P. Tradacete, *A negative solution to the complemented subspace problem for Banach lattices*, Preprint available on arXiv (2025), 25 pp.

6. D. **de Hevia** and P. Tradacete, *Free complex Banach lattices*, J. Funct. Anal. **284** (2023), no. 10, Paper No. 109888, 26. MR 4552375

7. D. **de Hevia** and P. Tradacete, *Complemented subspaces of Banach lattices*, Banach J. Math. Anal. **19** (2025), no. 4, Paper No. 60. MR 4940175

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Introducción

El tema central de esta memoria es el estudio de los **subespacios complementados de retículos de Banach**. Si bien los dos últimos capítulos no tratan directamente esta cuestión, hay un concepto que une todas las partes de nuestro estudio: el de *retículo de Banach*. Recordemos que un **retículo de Banach** es un espacio de Banach real $(X, \|\cdot\|)$ dotado de un orden parcial $\leq$ que cumple las siguientes propiedades:

(i) el orden es *compatible con la estructura lineal* de X, esto es, dados $x, y \in X$, $x \leq y$, se tiene que $x + z \leq y + z$ para todo $z \in X$ y $ax \leq ay$ para todo real no negativo a;

(ii) el orden es *reticular*, esto es, para cualesquiera $x, y \in X$, el conjunto $\{x, y\}$ tiene una cota superior mínima (supremo) y una cota inferior máxima (ínfimo), los cuales denotaremos por $x \vee y$ y $x \wedge y$ respectivamente;

(iii) el orden es *compatible con la norma* de X, es decir, siempre que $|x| \leq |y|$ se tiene que $\|x\| \leq \|y\|$, donde $|x| := x \vee (-x)$.

En vista de esta definición, se puede comprobar que muchos de los espacios de funciones más habituales son precisamente retículos de Banach cuando los dotamos de los órdenes *más naturales posibles*, como por ejemplo:

- Los espacios $C(K)$ de funciones continuas en un espacio Hausdorff compacto K con la norma del supremo $\|\cdot\|_\infty$ y el orden puntual $\leq$, esto es, $f \leq g$ si $f(t) \leq g(t)$ para todo $t \in K$.
- El espacio c_0 y los espacios ℓ_p, para $1 \leq p \leq \infty$, con sus correspondientes normas $\|\cdot\|_p$ habituales y el orden coordenada a coordenada $\leq$, es decir, $(x_n)_{n=1}^\infty \leq (y_n)_{n=1}^\infty$ si $x_n \leq y_n$ para todo $n \in \mathbb{N}$.
- Más generalmente, los espacios $L_p(\mu)$, los espacios de Orlicz $L_M(\mu)$ y los espacios de Lorentz $L_{W,p}(\mu)$ con sus normas usuales y el orden en μ-casi todo punto.

La consideración de esta estructura de orden adicional en un espacio de Banach (cuando sea posible) *facilita el estudio de su geometría*. Esto se debe a las valiosas herramientas que proporciona la teoría de retículos de Banach para este fin, como los *teoremas de representación*. Referimos al lector a [109, Section 1.b] para una muestra de esta clase de resultados.

Por otra parte, recordemos que un subespacio (cerrado) F de un espacio de Banach E se dice que está **complementado** si coincide con *la imagen de alguna proyección en* E, es decir, si existe un operador (lineal y continuo) $P : E \to E$ tal que $P \circ P = P$ que cumpla $P(E) = F$. Notemos que si P es una proyección en E, entonces $I - P$ también lo es, y las imágenes de estos dos operadores determinan la *descomposición en suma directa*

$E = P(E) \oplus (I - P)(E)$. Recíprocamente, si tenemos una descomposición en suma directa $E = F \oplus G$, existe una proyección $P : E \to E$ tal que $F = P(E)$ y $G = (I - P)(E)$.

Entender de qué maneras es posible descomponer (en suma directa) un espacio de Banach es una operación básica dentro de la teoría de espacios de Banach que ha recibido una gran atención desde sus inicios. Citando a Plichko y Yost [132, Section 2], el primer ejemplo de subespacio *no complementado* aparece en la obra [19] de 1932 de Banach y se atribuye a Mazur. A lo largo de los años 30 llegaron nuevos ejemplos debidos a Banach y Mazur, Fichtenholz y Kantorovich o Murray, entre otros. En 1940, Phillips demuestra *esencialmente* que c_0 no está complementado en ℓ_∞ [127, Remark 7.5]; formalmente, esto fue observado por Sobczyk un año después [145].

En 1941, Sobczyk conjetura por primera vez que si para un espacio de Banach todos sus subespacios están complementados (por una constante uniforme), entonces dicho espacio debe ser isomorfo a un espacio de Hilbert [146, p. 79]. Esta conjetura también era conocida como el *problema de los subespacios complementados* y no debe confundirse con el *Problema del Subespacio Complementado* que enunciaremos próximamente. Conviene mencionar que en ese momento la situación en el *caso isométrico* ya era más clara: en 1940 Phillips había demostrado que todo espacio de Banach real cuyos subespacios de dimensión 2 están complementados por una proyección de norma 1 debe ser isométrico a un espacio de Hilbert [126] (este mismo resultado fue probado en el caso complejo poco después por Bohnenblust [33]). El citado *problema de los subespacios complementados* de Sobczyk fue resuelto afirmativamente por Lindenstrauss y Tzafriri en 1971 ([106], véase también [86]).

En la dirección opuesta a la pregunta anterior, Lindenstrauss planteó en 1971 lo siguiente en [100]: ¿existen espacios de Banach *indescomponibles*? Recordemos que un espacio de Banach E es indescomponible si no se puede expresar como una suma directa de dos espacios de dimensión infinita. Esto es, si no existe una proyección $P : E \to E$ tal que $\dim P(E) = \dim (I - P)(E) = \infty$. El primer ejemplo de espacio indescomponible fue construido por Gowers y Maurey [67]. De hecho, este espacio poseía la propiedad adicional de ser *hereditariamente indescomponible* (abreviadamente, H.I.), es decir, la propiedad de que todo subespacio suyo (cerrado e infinito dimensional) es indescomponible. Esta clase de espacios han sido intensamente estudiados desde entonces. Más recientemente, se han encontrado ejemplos de espacios $C(K)$ indescomponibles (veáse el artículo de Koszmider [91] de recopilación de resultados relativos a espacios $C(K)$ con pocos operadores). Cabe destacar que un espacio $C(K)$ nunca puede ser H.I., ya que siempre contiene una copia isomórfica de c_0.

Tras esta disgresión sobre retículos de Banach y complementación, vamos a centrarnos ahora en el *problema fundamental* de esta memoria:

Pregunta (CSP). *¿Todo subespacio complementado en un retículo de Banach es isomorfo a un retículo de Banach?*

Nos referiremos a esta pregunta como **Problema del Subespacio Complementado (para retículos de Banach)** y lo denotaremos de forma abreviada como **CSP** por sus siglas en inglés. En el Capítulo 3 resolveremos esta pregunta dando una respuesta negativa. El objetivo del Capítulo 2 es dar una perspectiva general sobre este problema y recopilar algunos resultados notables relativos al mismo, así como plantear nuevos enfoques para su estudio.

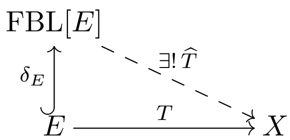
El CSP aparece mencionado de manera explícita en la primera frase del artículo [40] de 1987 de Casazza, Kalton y Tzafriri y se refieren a él como *uno de los problemas abiertos más importantes dentro de la teoría de retículos de Banach.* Esto sugiere que esta cuestión ya debía ser objeto de estudio y bien conocida desde años atrás por parte de los especialistas, si bien no hemos hallado una referencia previa. Una de las razones que pueden explicar esta *mención tardía* en la literatura es la *dificultad* para determinar si un espacio de Banach puede ser isomorfo a un retículo de Banach.

Los primeros ejemplos de espacios de Banach no isomorfos a retículos de Banach llegan en los años 70. Algunos de los criterios utilizados para conseguir esto son los siguientes:

- Un retículo de Banach X es *reflexivo* si solo si **no** contiene *subespacios isomorfos a* c_0 *o a* ℓ_1. En general, esto no es cierto para espacios de Banach: el espacio de James $\mathcal{J}$ [75] no contiene a c_0, ni ℓ_1, pero no es reflexivo.
- Todo retículo de Banach tiene *sucesiones básicas incondicionales*. Sin embargo, ya hemos comentado que existen espacios de Banach H.I. y estos no pueden contener tales sucesiones.
- Todo retículo de Banach posee *estructura local incondicional de Gordon-Lewis* (GL-lust). Entre los espacios de Banach que fallan esta propiedad encontramos $\mathcal{H}^\infty(\mathbb{D})$ (el espacio de las funciones holomorfas y acotadas en el disco abierto) [123], el espacio de los operadores compactos en ℓ_2, $c_p(\ell_2)$, con $1 \leq p \neq 2 \leq \infty$ con la p-norma Schatten [64, Theorem 5.1], o ciertos espacios de Sobolev [124, 125].

Una peculiaridad de todos estos criterios (los cuales se pueden encontrar en la primera sección del Capítulo 2) es que también son válidos para subespacios complementados en retículos de Banach (véase el Corolario 2.26). Por tanto, *no permiten distinguir* entre retículos de Banach y sus subespacios complementados. Una herramienta introducida recientemente que pensamos que puede ser de utilidad para aclarar este asunto es **el retículo de Banach libre generado por un espacio de Banach**.

Recordemos que *el retículo de Banach libre generado por un espacio de Banach* E es un par $(\mathrm{FBL}[E], \delta_E)$, donde $\mathrm{FBL}[E]$ es un retículo de Banach y $\delta_E : E \hookrightarrow \mathrm{FBL}[E]$ es una inclusión lineal isométrica, que posee la siguiente *propiedad universal*: para todo retículo de Banach X y todo operador $T : E \to X$, *existe un único* homomorfismo reticular $\widehat{T} : \mathrm{FBL}[E] \to X$ tal que $\widehat{T}\delta_E = T$ y, además, $\|\widehat{T}\| = \|T\|$. Es habitual expresar esta propiedad mediante el siguiente diagrama conmutativo:



En 2018, Avilés, Rodríguez y Tradacete demostraron no solo la existencia de este objeto, sino que dieron una representación funcional explícita del mismo [15]. Vamos a recordarla brevemente. Consideremos primero la siguiente expresión sobre el conjunto de funciones $f : E^* \to \mathbb{R}$:

$$\|f\|_{\mathrm{FBL}[E]} = \sup \left\{ \sum_{k=1}^{n} |f(x_k^*)| \; : \; n \in \mathbb{N},\, (x_k^*)_{k=1}^n \subseteq E^*,\, \sup_{x \in B_E} \sum_{k=1}^{n} |x_k^*(x)| \leq 1 \right\}.$$

Nótese que $H_1[E]$, el espacio de funciones positivamente homogéneas $f : E^* \to \mathbb{R}$ que satisfacen $\|f\|_{\mathrm{FBL}[E]} < \infty$, con el orden puntual y la norma anterior, es un retículo de Banach. Para cada $x \in E$, sea $\delta_x : E^* \to \mathbb{R}$ la función de evaluación $\delta_x(x^*) = x^*(x)$. Entonces $\mathrm{FBL}[E]$ puede identificarse con el *subretículo cerrado* de $H_1[E]$ generado por el conjunto $\{\delta_x : x \in E\}$, junto con el *embebimiento isométrico* $\delta_E : E \to \mathrm{FBL}[E]$ dado por $\delta_E(x) = \delta_x$ [15, Theorem 2.5].

Como hemos comentado, el retículo de Banach libre generado por un espacio de Banach fue introducido en [15] en 2018, *generalizando* el concepto de *retículo de Banach libre generado por un conjunto* que de Pagter y Wickstead habían definido y estudiado en su artículo [121] de 2015. La asignación $E \mapsto \mathrm{FBL}[E]$ puede considerarse como un *functor canónico* entre la categoría de espacios de Banach (con operadores lineales y continuos) y la categoría de retículos de Banach (con homomorfismos reticulares). Esto explica la utilidad que ha tenido hasta ahora para abordar preguntas relativas a la interacción entre ambas categorías, entre las que se incluyen: el $\mathrm{FBL}[E]$ se ha utilizado para proporcionar ejemplos de retículos de Banach que son *débil-compactamente generados como retículos pero no como espacios de Banach*, resolviendo una pregunta planteada por Diestel [15, Section 5]; se ha empleado para construir *push-outs* en la categoría de retículos de Banach [17], o para dar los primeros ejemplos de *homomorfismos reticulares que no alcanzan su norma* [42]; y se ha usado para mostrar la existencia de subespacios de retículos de Banach *sin sucesiones bibásicas* [119, Section 7], resolviendo una pregunta de [148]. Referimos al lector al trabajo de Oikhberg, Taylor, Tradacete y Troitsky [119], publicado en 2024, para un extenso estudio de estos objetos.

Nuestro interés por considerar los retículos de Banach libres para estudiar el CSP viene motivado por la siguiente observación:

Proposición (2.11)**.** *Sea E un espacio de Banach. Si E es C_1-isomorfo a un subespacio C_2-complementado de un retículo de Banach, entonces $\delta_E(E)$ está C_1C_2-complementado en $FBL[E]$.*

Es decir, los retículos de Banach libres nos proporcionan *un lugar canónico* donde estudiar este problema, ya que estar complementado en *algún* retículo, supone estar complementado en el correspondiente retículo de Banach libre (y de hecho, con *la mejor constante posible*). Asimismo, los retículos de Banach libres ofrecen también *una manera de distinguir* los retículos de Banach de sus subespacios complementados.

Proposición (2.13)**.** *Un espacio de Banach E es isomorfo a un retículo de Banach si y solo si existe un ideal I en $FBL[E]$ tal que $FBL[E] = \delta_E(E) \oplus I$.*

En cualquier caso, no parece sencillo obtener una descripción satisfactoria de los ideales en un retículo de Banach libre y esto dificulta el uso de este criterio en la práctica. Esto se discute en el apartado 2.2. El resto del Capítulo 2 está dedicado a discutir y plantear numerosas preguntas relativas al Problema del Subespacio Complementado. Una de las cuestiones abiertas que se analizan y que nos gustaría destacar *por su aparente sencillez* es la siguiente (Sección 2.5):

Pregunta (Problema del hiperplano)**.** *¿Todo hiperplano de un retículo de Banach es isomorfo a un retículo de Banach?*

El Capítulo 3 está destinado a dar **una solución negativa al Problema del Subespacio Complementado**. Para ello utilizamos un ejemplo ya existente: el espacio PS_2 construido en 2023 por Plebanek y Salguero-Alarcón en [131]. Dicho espacio fue construido para *resolver de forma negativa* el *Problema del Subespacio Complementado para espacios $C(K)$*:

Pregunta (CSP para espacios $C(K)$)**.** *¿Todo subespacio complementado en un espacio $C(K)$ es isomorfo a un espacio $C(K)$?*

Se trata de una pregunta ya mencionada por Pełczyński en su artículo de 1960 [122] y que ha sido intensamente estudiada, como ilustran los numerosos resultados parciales que se han obtenido al respecto (véase [138, Section 5] para una recopilación de los mismos).

La Sección 3.1 comienza con un interesante resultado de renormamiento para retículos que poseen una *estructura local semejante* a la de los espacios L_1:

Teorema (3.1)**.** *Sea X un retículo de Banach que es un espacio $\mathcal{L}_1$. Entonces X es reticularmente isomorfo a un espacio L_1.*

Aunque este resultado está enunciado en [2], su demostración no aparece explícitamente. Nuestra prueba no es trivial en absoluto: requiere de distintas herramientas de la teoría local de espacios de Banach (como la desigualdad de Grothendieck), así como de la teoría de retículos de Banach. Una consecuencia inmediata de este teorema es que un *retículo de Banach separable* (infinito-dimensional) que es $\mathcal{L}_1$ solo puede ser isomorfo a $L_1[0,1]$ o ℓ_1. Esto muestra lo *reducida* que es la clase de los retículos de Banach en este contexto, puesto que sabemos que existe un *continuo* de espacios $\mathcal{L}_1$ que son subespacios de ℓ_1 y no son mutuamente isomorfos [80]. A partir del teorema anterior, es fácil obtener la siguiente *versión dual*:

Corolario (3.2)**.** *Sea X un retículo de Banach que es un espacio $\mathcal{L}_\infty$. Entonces X es reticularmente isomorfo a un AM-espacio.*

Este corolario jugará un papel esencial en la prueba de que PS_2 también proporciona un contraejemplo al CSP para retículos de Banach. Puesto que el ejemplo que analizamos no es separable, el Problema del Subespacio Complementado *sigue abierto* en el caso **separable**. Hay que señalar que una respuesta afirmativa al CSP separable supondría que las dos siguientes célebres conjeturas son también ciertas (véase la Observación 3.5):

- Todo subespacio complementado de $L_1[0,1]$ es isomorfo a ℓ_1 o a $L_1[0,1]$.
- Todo subespacio complementado de $C[0,1]$ es isomorfo a un espacio $C(K)$.

La Sección 3.2 está dedicada a recordar la construcción del espacio PS_2, así como sus propiedades más significativas. Debido a su importancia en nuestra memoria, vamos a recordar someramente la forma del espacio PS_2. Sea $\mathrm{fin}(\mathbb{N})$ el conjunto de los subconjuntos finitos de $\mathbb{N}$. Recordemos que una familia $\mathcal{A}$ de *subconjuntos infinitos de* $\mathbb{N}$ se dice que es **casi disjunta** si $A \cap B \in \mathrm{fin}(\mathbb{N})$ para cualquier par $A, B \in \mathcal{A}$ distintos. Para una familia casi disjunta $\mathcal{A}$, definimos

$$\mathrm{JL}(\mathcal{A}) := \overline{\mathrm{span}}\big\{\mathbf{1}_A \,:\, A \in \mathrm{fin}(\mathbb{N}) \cup \mathcal{A} \cup \{\mathbb{N}\}\big\} \subseteq \ell_\infty,$$

y llamamos a este espacio el **espacio de Johnson-Lindenstrauss asociado a** $\mathcal{A}$, ya que fue introducido por primera vez en [79, Example 2]. Dado que $\mathrm{JL}(\mathcal{A})$ es un subretículo cerrado de ℓ_∞ que contiene la función constante $\mathbf{1}$, el *teorema de representación de Kakutani para AM-espacios* nos permite deducir que es reticularmente isométrico a un espacio $C(K)$ [109, Theorem 1.b.6]. No es difícil comprobar que $\mathrm{JL}(\mathcal{A})^*$ es linealmente isométrico a $\ell_1(\mathbb{N} \cup \mathcal{A} \cup \{\mathbb{N}\})$, de modo que K debe ser disperso. Este método particular de construcción de espacios $C(K)$ usando familias casi disjuntas ha producido varios ejemplos con propiedades exóticas [4, 68, 92, 131].

Sea $\mathcal{A} = \{A_\xi : \xi < \mathfrak{c}\}$ una fa milia casi disjunta de $\mathbb{N}$ de cardinalidad $\mathfrak{c}$. Escribimos $\widehat{\mathbb{N}} = \mathbb{N} \times \{0,1\}$ y para cada $\xi < \mathfrak{c}$ y $n \in \mathbb{N}$, denotamos $\widehat{A_\xi} = A_\xi \times \{0,1\}$ y $c_n = \{(n,0),(n,1)\}$. Para $\xi < \mathfrak{c}$, descomponemos $\widehat{A_\xi} = B^0_\xi \cup B^1_\xi$ de tal manera que para todo $n \in \mathbb{N}$, los conjuntos $B^0_\xi \cap c_n$ y $B^1_\xi \cap c_n$ tienen un único elemento. Nótese que $\mathcal{B} := \{B^0_\xi, B^1_\xi : \xi < \mathfrak{c}\}$ es una familia casi disjunta de $\widehat{\mathbb{N}}$ ahora. Con un ligero abuso de notación, denotaremos por $\mathrm{JL}(\mathcal{A})$ el subespacio cerrado de $\ell_\infty(\widehat{\mathbb{N}})$ generado por $\{\mathbf{1}_{c_n} : n \in \mathbb{N}\} \cup \big\{\mathbf{1}_{\widehat{A_\xi}} : \xi < \mathfrak{c}\big\} \cup \big\{\mathbf{1}_{\widehat{\mathbb{N}}}\big\}$. De manera similar, definimos

$$\mathrm{JL}(\mathcal{B}) := \overline{\mathrm{span}}\{\mathbf{1}_B \ :\ B \in \mathrm{fin}(\widehat{\mathbb{N}}) \cup \mathcal{B} \cup \{\widehat{\mathbb{N}}\}\} \subseteq \ell_\infty(\widehat{\mathbb{N}}).$$

Debe observarse que $\mathrm{JL}(\mathcal{A})$ es precisamente el subespacio de $\mathrm{JL}(\mathcal{B})$ que consiste en todas las funciones que son constantes en cada fibra c_n y la aplicación $P : \mathrm{JL}(\mathcal{B}) \to \mathrm{JL}(\mathcal{B})$ definida por

$$Pf(n,0) = Pf(n,1) = \frac{1}{2}\big(f(n,0) + f(n,1)\big), \quad n \in \mathbb{N},$$

es una proyección de norma uno cuya imagen es $\mathrm{JL}(\mathcal{A})$. Ahora, escribimos $X = \ker P$, obteniendo así $\mathrm{JL}(\mathcal{B}) = \mathrm{JL}(\mathcal{A}) \oplus X$. Nótese que X es también un subespacio 1-complementado de $\mathrm{JL}(\mathcal{B})$, ya que la proyección $Q = I - P$ (donde $I = \mathrm{id}_{\mathrm{JL}(\mathcal{B})}$) está dada por

$$Qf(n,0) = -Qf(n,1) = \frac{1}{2}\big(f(n,0) - f(n,1)\big), \quad n \in \mathbb{N}.$$

En [131], se construyen dos familias casi disjuntas, $\mathcal{A}$ y $\mathcal{B}$, de la forma que acabamos de describir, de tal manera que X no es isomorfo a un espacio $C(K)$. Dicho espacio será denotado por PS_2.

En el Capítulo 3 demostramos que PS_2 *ni siquiera puede ser isomorfo a un retículo de Banach*. Para ello, una de las claves es observar que ciertas peculiaridades de este espacio permiten simplificar considerablemente este problema. Concretamente, puesto que PS_2 es un *predual de* $\ell_1(\Gamma)$ (y, en particular, un espacio $\mathcal{L}_\infty$) que además tiene un *conjunto normante numerable* (por ser un subespacio de ℓ_∞), *ser isomorfo a un retículo equivale en este caso a ser isomorfo a un* ***subretículo de*** ℓ_∞ (Proposición 3.16). La herramienta determinante para lograr esta *reducción* de nuestro problema es el mencionado teorema de renormamiento descrito en la Sección 3.1.

Nos resultará conveniente reescribir la propiedad de que PS_2 *no pueda ser isomorfo a un subretículo de* ℓ_∞ a que PS_2 posea la **Propiedad Deseada (DP)**: Para cada *sucesión normante* $(e_n^*)_{n=1}^\infty \subseteq B_{\mathsf{PS}_2^*}$ existe $f \in \mathsf{PS}_2$ tal que no existe ninguna $g \in \mathsf{PS}_2$ para la cual $|e_n^*(f)| = e_n^*(g)$ para todo $n \in \mathbb{N}$. Un análisis minucioso de la construcción de PS_2 (explicado en detalle en el Teorema 3.19) revela que para cada sucesión normante en $B_{\mathsf{PS}_2^*}$ existe $\xi < \mathfrak{c}$ tal que $1_{B_\xi^0} - 1_{B_\xi^1} \in \mathsf{PS}_2$ *no tiene módulo* con respecto a esa sucesión, y por lo tanto PS_2 no puede ser isomorfo a un retículo de Banach.

En la última sección del mismo capítulo se muestra cómo modificando la construcción de PS_2 se puede obtener un contraejemplo para el *Problema del Subespacio Complementado para* ***retículos de Banach complejos*** (la definición de retículo complejo se recordará a continuación). De forma más precisa, existe una variación de dicho espacio, la cual denotamos por $\widetilde{\mathsf{PS}_2}$, de manera que $\widetilde{\mathsf{PS}_2} \oplus i\widetilde{\mathsf{PS}_2}$ está 1-complementado en un espacio $C(K)$ complejo, pero no puede ser isomorfo a un retículo de Banach complejo (Teorema 3.23). Esto es interesante, porque en la literatura podemos encontrar algunos resultados concernientes a subespacios 1-complementados en retículos complejos que difieren significativamente de aquellos del caso real [46, 87, 89]. Entre ellos destaca el siguiente resultado de Kalton y Wood [89] (véase también [56, 137]): todo subespacio 1-complementado de un espacio de Banach **complejo** con una base 1-incondicional, también tiene una base 1-incondicional. Esto no es cierto en el caso real, como muestran los ejemplos de [22, 97], si bien la pregunta más general de si todo subespacio complementado de un espacio con base incondicional también tiene base incondicional permanece abierta para escalares reales y complejos.

Estas diferencias existentes entre el caso real y complejo a la hora de estudiar el CSP nos motivaron a introducir el ***retículo de Banach libre complejo*** *generado por un espacio de Banach complejo*, el cual se describe y estudia en el Capítulo 4. Recordemos que un *retículo de Banach complejo* Z es la *complejificación de un retículo de Banach real* X (esto es, $Z = X \oplus iX$) donde la norma de Z está dada por $\|x+iy\|_Z := \||x+iy|\|_X$, $x+iy \in Z$, donde $|\cdot| : Z \to X_+$ es la aplicación *módulo*, que está dada por

$$|x+iy| := \sup_{\theta\in[0,2\pi]} \{x\cos\theta + y\sin\theta\}, \quad \text{para todo } x+iy \in Z.$$

Los espacios de tipo L_p o $C(K)$ complejos con sus normas usuales son ejemplos de retículos de Banach complejos. Por otra parte, un *homomorfismo reticular complejo* es la *complejificación* de un homomorfismo reticular real, es decir, $T : X \oplus iX \to Y \oplus iY$ es homomorfismo complejo si existe un homomorfismo real $S : X \to Y$ tal que $T(x+iy) = Sx + iSy$ para todo $x+iy \in X \oplus iX$. Es importante destacar que, en general, un espacio de Banach complejo no tiene por qué ser isomorfo a la complejificación de algún espacio real [35, 85]. Pese a ello, veremos que la definición de espacio de Banach complejo (sin hipótesis adicionales) y la de retículo complejo son *compatibles de la manera deseada*.

Dado un espacio de Banach complejo E, el ***retículo de Banach libre complejo*** *generado por* E es un retículo de Banach complejo $\mathrm{FBL}_{\mathbb{C}}[E]$ junto con una inclusión isométrica $\mathbb{C}$-lineal $\delta_E : E \to \mathrm{FBL}_{\mathbb{C}}[E]$ tal que para todo retículo de Banach complejo $X_{\mathbb{C}}$ y todo operador $\mathbb{C}$-lineal $T : E \to X_{\mathbb{C}}$, existe un único homomorfismo reticular $\widehat{T} : \mathrm{FBL}_{\mathbb{C}}[E] \to X_{\mathbb{C}}$ tal que $\widehat{T} \circ \delta_E = T$ y, además, $\|\widehat{T}\| = \|T\|$.

En la Sección 4.1 se demuestra *no solo la existencia de* $\mathrm{FBL}_{\mathbb{C}}[E]$, sino que también se da una *descripción explícita* de este objeto aprovechando la representación funcional existente del retículo de Banach libre real. Esbozamos a continuación la idea de la prueba. Dado un espacio de Banach complejo E, denotamos por $E_{\mathbb{R}}$ al espacio de Banach real resultante de *restringir* la multiplicación por escalares complejos de E a los reales. De este modo, podemos considerar el retículo de Banach libre (real) de $E_{\mathbb{R}}$. Se puede comprobar que esencialmente $\mathrm{FBL}_{\mathbb{C}}[E] = \mathrm{FBL}[E_{\mathbb{R}}] \oplus i\mathrm{FBL}[E_{\mathbb{R}}]$. Y decimos *esencialmente*, porque para garantizar que las extensiones a $\mathrm{FBL}_{\mathbb{C}}[E]$ preservan la norma de las operadores definidos en E, previamente necesitamos renormar $\mathrm{FBL}[E_{\mathbb{R}}]$ con lo siguiente:

$$\|f\|_{\mathrm{FBL}_{\mathbb{C}}[E]} = \sup\left\{ \sum_{j=1}^{m} |f(\mathfrak{Re}\, z_j^*)| \,:\, m\in\mathbb{N},\ (z_j^*)_{j=1}^m \subseteq E^*,\ \sup_{z\in B_E} \sum_{j=1}^{m} |z_j^*(z)| \le 1 \right\}.$$

Por otra parte, definimos $\delta_E : E \to \mathrm{FBL}_{\mathbb{C}}[E]$ por

$$\delta_E(z) = \delta_{E_{\mathbb{R}}}(z) - i\delta_{E_{\mathbb{R}}}(iz), \qquad z \in E,$$

que es una inclusión $\mathbb{C}$-lineal isométrica (Lema 4.2). En el Teorema 4.3 se prueba que el retículo complejo $\mathrm{FBL}_{\mathbb{C}}[E] = \mathrm{FBL}[E_{\mathbb{R}}] \oplus i\mathrm{FBL}[E_{\mathbb{R}}]$, dotado de la norma $\||\cdot|\|_{\mathrm{FBL}_{\mathbb{C}}[E]}$, junto con la aplicación δ_E, es el *retículo de Banach libre complejo generado por* E.

Una propiedad llamativa que se desprende de la construcción de este objeto es que, si E es un espacio complejo, y $\overline{E}$ denota su conjugado complejo, entonces $\mathrm{FBL}_{\mathbb{C}}[E]$ es $\mathbb{C}$-reticularmente isométrico a $\mathrm{FBL}_{\mathbb{C}}[\overline{E}]$ (Proposición 4.8, y véase también el recíproco *parcial* obtenido en la Proposición 4.15). Esto muestra que *un mismo retículo de Banach libre complejo* puede provenir de espacios de Banach *diferentes*: recuérdense los ejemplos de espacios complejos no isomorfos a sus conjugados construidos por Bourgain [35] o Kalton [85]. Esto *contrasta con el caso real*, donde no se conocen ejemplos de espacios de Banach no isomorfos cuyos correspondientes retículos de Banach libres sí sean reticularmente isomorfos [119, Remark 10.25]. Por otro lado, en la Sección 4.4 estudiamos el *espectro* de los homomorfismos complejos del tipo $\overline{T} : \mathrm{FBL}_{\mathbb{C}}[E] \to \mathrm{FBL}_{\mathbb{C}}[E]$, donde $\overline{T}\delta_E = \delta_E T$ para un operador $T : E \to E$, analizando la relación entre $\sigma(T)$ y $\sigma(\overline{T})$.

El Capítulo 5 está dedicado a investigar el fenómeno de *alcance de la norma* para funcionales que son *homomorfismos de retículos*. Decimos que un funcional (lineal) x^* :

$E \to \mathbb{R}$ *alcanza su norma* si existe un vector $x \in E$, $\|x\| \leq 1$, tal que $\|x^*\| = |x^*(x)|$. El estudio del *alcance de la norma* para funcionales ha recibido (y sigue recibiendo) una gran atención desde hace décadas y constituye una parte fundamental dentro de la teoría de espacios de Banach. Algunos de los resultados más destacados en este ámbito son la célebre *caracterización de James de la reflexividad* –un espacio de Banach E es reflexivo si solo si todo $x^* \in E^*$ alcanza su norma– [76], o el teorema de Bishop-Phelps –para todo espacio de Banach E el conjunto $\mathrm{NA}(E, \mathbb{R})$ es denso en E^*– [31].

Cuando estudiamos el alcance de la norma para funcionales en retículos de Banach podemos plantearnos además qué sucede con aquellos funcionales que guardan *cierta compatibilidad* con la estructura de orden del retículo. En este sentido, queremos señalar que ha habido un interés creciente por comprender esta propiedad para *funcionales positivos*, como reflejan, por ejemplo, [3], [112] y [120]. En lo que respecta a nuestro trabajo, la principal motivación se encuentra en el artículo de Dantas, Martínez-Cervantes, Rodríguez Abellán y Rueda Zoca [42]. En él, los autores dan *el primer ejemplo* de *homomorfismo reticular* que *no* alcanza su norma.

Hay que destacar que los homomorfismos reticulares constituyen una *clase muy restringida* dentro de los funcionales de un retículo de Banach. De hecho, no todo retículo de Banach posee homomorfismos de retículos (aparte del homomorfismo trivial 0): los espacios $L_p[0, 1]$, para $1 \leq p < \infty$, *no* poseen homomorfismos reticulares no triviales, es decir, $\mathrm{Hom}(L_p[0, 1], \mathbb{R}) = \{0\}$. Se puede comprobar también que para cualquier compacto Hausdorff K, $\mathrm{Hom}(C(K), \mathbb{R}) = \{\lambda\delta_t \ : \ t \in K, \lambda \geq 0\}$, donde δ_t denota el funcional evaluación en el punto t, o si X es ℓ_p, con $1 \leq p < \infty$, o c_0, se tiene que $\mathrm{Hom}(X, \mathbb{R}) = \{\lambda e_n^* \ : \ n \in \mathbb{N}, \lambda \geq 0\}$, siendo $(e_n^*)_{n=1}^{\infty}$ los funcionales biortogonales asociados a la base canónica. Por tanto, en estos espacios clásicos se cumple, en efecto, que todo funcional que preserva las operaciones reticulares alcanza su norma.

Con el objetivo de encontrar un contraejemplo a lo anterior, en [42] los autores se fijan en los *retículos de Banach libres*, puesto que son una clase de retículos de Banach con *muchos homomorfismos reticulares*. Para ser más precisos, dado un espacio de Banach E, tenemos que $\mathrm{Hom}(\mathrm{FBL}[E], \mathbb{R}) = \{\widehat{x^*} \ : \ x^* \in E^*\}$, donde $\widehat{x^*} : \mathrm{FBL}[E] \to \mathbb{R}$ denota el único homomorfismo reticular en $\mathrm{FBL}[E]$ que extiende a x^* (esto es, $\widehat{x^*} \circ \delta_E = x^*$). Notemos que si x^* alcanza la norma en $x \in B_E$, entonces $\widehat{x^*}$ alcanza su norma en $\delta_x \in B_{\mathrm{FBL}[E]}$. Es decir, $x^* \in \mathrm{NA}(E, \mathbb{R})$ implica que $\widehat{x^*} \in \mathrm{NA}(\mathrm{FBL}[E], \mathbb{R})$. En [42] los autores *conjeturan* que la implicación recíproca también es cierta, y son capaces de comprobar su validez para algunos espacios de Banach, entre los que se incluye, por ejemplo $E = \ell_1$. Puesto que ℓ_1 no es reflexivo, existen funcionales que no alcanzan su norma y, por el resultado que acabamos de comentar, sus correspondientes extensiones al $\mathrm{FBL}[\ell_1]$ tampoco lo hacen. Esto proporciona el contraejemplo deseado.

La investigación que desarrollamos en el Capítulo 5 busca entender de manera *más general*, sin enfocarse en los retículos de Banach libres, el *alcance* de la norma de los homomorfismos reticulares en retículos de Banach. Comenzamos analizando una clase muy especial de homomorfismos reticulares: los *funcionales coordenada*. Recordemos que dado un átomo x_0 en un retículo de Banach X, el subespacio generado por él mismo es una *banda de proyección* cuya proyección asociada viene dada por $P_{x_0}(x) = \lambda_{x_0}(x)x_0 = \sup_n (x \wedge nx_0)$, $x \in X_+$. El funcional λ_{x_0} es un homomorfismo reticular que denominamos *funcional coordenada del átomo* x_0. Es sencillo comprobar que para cualquier *renormamiento reticular* $|||\cdot|||$ de X (esto es, una norma equivalente de manera que $(X, |||\cdot|||)$ sea también un retículo de Banach sin modificar su orden) se tiene que λ_{x_0} alcanza su norma en $\frac{x_0}{|||x_0|||}$ (ver Proposición 5.1). Una observación interesante sobre esta clase de funcionales es que coinciden precisamente con los homomorfismos reticulares que son orden continuos (Proposición 5.2).

En vista de la mencionada Proposición 5.1 es natural preguntarse si los funcionales coordenada son *los únicos* homomorfismos reticulares que alcanzan su norma para cualquier renormamiento reticular. La respuesta es que, en muchas ocasiones, podemos conseguir renormar nuestro retículo para que *ningún homomorfismo* (salvo los funcionales coordenada) alcance su norma:

Teorema (5.7)**.** *Sea X un retículo de Banach que tiene un funcional estrictamente positivo μ. Si renormamos X mediante $\|\cdot\|_\mu := \|\cdot\| + \mu\big(|\cdot|\big)$, entonces los únicos homomorfismos reticulares que alcanzan su norma son funcionales coordenada de átomos.*

Entre los retículos que poseen un funcional estrictamente positivo encontramos todos aquellos que son *separables*. Pero también pertenecen a esta clase espacios como ℓ_∞ o $L_\infty[0,1]$, lo que nos permite dar ejemplos de *retículos Dedekind completos que tienen homomorfismos que no alcanzan su norma*, respondiendo de manera negativa una cuestión planteada en [42]. En general, sin embargo, la respuesta a la pregunta que planteamos es *negativa*: δ_{ω_1} es un homomorfismo reticular de $C[0,\omega_1]$ que no es funcional coordenado, pero alcanza su norma para cualquier renormamiento (Ejemplo 5.10).

En las secciones 5.3 y 5.4 analizamos el alcance de la norma de los homomorfismos definidos en AM-espacios. Se trata de una familia de retículos con *muchos homomorfismos reticulares*, ya que para cualquier AM-espacio X, el conjunto $\mathrm{Hom}(X,\mathbb{R})$ es *normante* y, de hecho, esta propiedad caracteriza a los AM-espacios (Proposición 5.22). Y aunque los AM-espacios constituyen una clase próxima a los espacios $C(K)$ (concretamente, se pueden identificar con subretículos de $C(K)$), estos no contienen necesariamente *una unidad*, ni tampoco tenemos a nuestra disposición un *lema de tipo Urysohn*. Por esta razón, el siguiente resultado no debe parecer trivial:

Teorema (5.15)**.** *Todo homomorfismo reticular en un AM-espacio alcanza su norma.*

La clave de la demostración del resultado previo es *una caracterización de los homomorfismos reticulares* interesante por sí sola; de hecho, será también determinante posteriormente para demostrar que todo homomorfismo del *retículo de Banach libre generado por un retículo* (una clase de retículos introducida por Avilés y Rodríguez-Abellán en [16]) alcanza su norma (Proposición 5.41). Enunciamos dicha caracterización:

Proposición (5.14)**.** *Sea X un retículo de Banach y $x^* \in X^*$ un homomorfismo reticular de norma 1. Entonces, x^* alcanza su norma si y solo si existe una sucesión creciente de elementos positivos $(x_n)_{n=1}^\infty$ en B_X de manera que $x^*(x_n) \to 1$.*

El Capítulo 6 trata sobre *recuperación de fase estable* (SPR, por sus siglas en inglés) en espacios $C(K)$. Recordemos que un subespacio E de un retículo de Banach X se dice que hace *recuperación de fase estable* con constante $C > 0$ si $\min_{|\lambda|=1} \|f - \lambda g\| \le C\big\||f| - |g|\big\|$, para cualesquiera $f, g \in E$.

Este trabajo tiene su origen en un proyecto de investigación supervisado por Mitchell A. Taylor titulado *Open problems in stable phase retrieval* que tuvo lugar en el ICMAT (Madrid) durante los días $26-30$ de junio de 2023, como parte de un curso de doctorado organizado por el ICMAT y el IMAG (ICMAT-IMAG Doc-Course in Functional Analysis). La investigación de aquellos días (con un esfuerzo importante adicional posterior) dio lugar a la publicación [39] conjunta con Camúñez y García-Sánchez, donde se da una caracterización de la SPR *compleja*. El caso complejo no se tratará en esta memoria y los resultados que presentaremos en ella pueden encontrarse en un trabajo siguiente al anterior en colaboración con García-Sánchez [60].

En [58], Freeman, Oikhberg, Pineau y Taylor prueban el siguiente resultado: un espacio $C(K)$ contiene un subespacio (isométrico a c_0) haciendo SPR si y solo si el conjunto de puntos de acumulación K' es infinito. A raíz de esto, los autores se preguntan si el hecho de que hubiera un *mayor número de puntos de acumulación* en el compacto K (en el sentido de que la derivada de Cantor-Bendixon $K^{(\alpha)}$ sea un conjunto infinito para $\alpha > 1$) garantizaría la existencia de *subespacios haciendo SPR más grandes que* c_0. Más precisamente, plantean lo siguiente [58, Question 6.4]: Si $K^{(\alpha)}$ es infinito, ¿entonces $C(K)$ contiene una copia de $C[1, \omega^\alpha]$ haciendo SPR? En el capítulo 6 analizamos esta pregunta y daremos una respuesta *completa y afirmativa* a la misma para el caso en que $\alpha \geq 2$ es un ordinal finito.

Para ello, una de las claves será observar que esta cuestión se puede *reducir* a estudiar *inclusiones que hacen SPR entre espacios de la forma* $C[1, \omega^\alpha]$ (Proposición 6.10). Posteriormente probaremos que, para $\alpha \geq 2$, existe un embebimiento isométrico SPR de $C[1, \omega^\alpha]$ en $C[1, \omega^2] \oplus_\infty C[1, \omega^\alpha]$ (Proposición 6.13), lo que nos permitirá deducir lo siguiente:

(i) Si $K^{(\alpha)} \neq 0$ para $3 \leq \alpha < \omega$, entonces existe un embebimiento isométrico SPR de $C[1, \omega^\alpha]$ en $C(K)$ (Corolario 6.14).

(ii) Si $|K''| \geq 2$, existe un embebimiento isométrico SPR de $C[1, \omega^2]$ en $C(K)$ (ver también el Corolario 6.14). Este resultado *no se puede mejorar*: $C[1, \omega^2]$ no se puede embeber isométricamente haciendo SPR en un espacio $C(K)$ con $|K''| = 1$ (Proposición 6.16).

Introduction

The main topic of this thesis is the study of **complemented subspaces of Banach lattices**. While the last two chapters do not directly address this issue, there is a unifying concept throughout our study: **Banach lattices**. Recall that a *Banach lattice* is a real Banach space $(X, \|\cdot\|)$ equipped with a partial order $\leq$ satisfying the following properties:

(i) the order is *compatible with the linear structure* of X; that is, for any $x, y \in X$ with $x \leq y$, we have $x + z \leq y + z$ for all $z \in X$ and $ax \leq ay$ for all non-negative real numbers a;

(ii) $\leq$ is a *lattice order*, that is, for any $x, y \in X$, the set $\{x, y\}$ has a least upper bound (supremum) and a greatest lower bound (infimum), which we denote by $x \vee y$ and $x \wedge y$, respectively;

(iii) the order is *compatible with the norm* of X; that is, whenever $|x| \leq |y|$, we have $\|x\| \leq \|y\|$, where $|x| := x \vee (-x)$.

In view of this definition, it should be observed that many of the most commonly used function spaces are precisely Banach lattices when these are endowed with *the most natural possible orders*, such as:

- $C(K)$-spaces, that is, spaces of continuous functions on a compact Hausdorff space K with the supremum norm $\|\cdot\|_\infty$ and the pointwise order $\leq$, meaning $f \leq g$ if $f(t) \leq g(t)$ for all $t \in K$.
- The space c_0 and the spaces ℓ_p, for $1 \leq p \leq \infty$, with their corresponding usual $\|\cdot\|_p$ norms and the coordinate-wise order $\leq$, meaning $(x_n)_{n=1}^\infty \leq (y_n)_{n=1}^\infty$ if $x_n \leq y_n$ for all $n \in \mathbb{N}$.
- More generally, $L_p(\mu)$ spaces, Orlicz spaces $L_M(\mu)$, and Lorentz spaces $L_{W,p}(\mu)$ with their usual norms and the order μ-almost everywhere.

The consideration of this additional order structure in a Banach space (when possible) *facilitates the study of its geometry*, thanks to the useful tools the theory of Banach lattices provides for this purpose, such as the *representation theorems*. We refer the reader to [109, Section 1.b] for a compilation of the most relevant results of this type.

On the other hand, recall that a (closed) subspace F of a Banach space E is said to be **complemented** if it is *the image of some projection on* E; that is, if there exists a (linear and continuous) operator $P : E \to E$ such that $P \circ P = P$ and $P(E) = F$. Note that if P is a projection on E, then $I - P$ is also one, and the images of these two operators determine the *direct sum decomposition* $E = P(E) \oplus (I - P)(E)$. Conversely, if we have a direct sum decomposition $E = F \oplus G$, there exists a projection $P : E \to E$ such that $F = P(E)$ and $G = (I - P)(E)$.

Understanding in what ways it is possible to decompose (as a direct sum) a Banach space is a basic operation within the theory of Banach spaces that has received significant attention since its beginnings. Quoting Plichko and Yost [132, Section 2], the first example of a *non-complemented* subspace appears in the 1932 work [19] of Banach and is attributed to Mazur. Throughout the 1930s, new examples arrived due to Banach and Mazur, Fichtenholz and Kantorovich or Murray, among others. In 1940, Phillips proved *essentially* that c_0 is not complemented in ℓ_∞ [127, Remark 7.5]; formally, this was observed by Sobczyk one year later [145].

In 1941, Sobczyk conjectures for the first time that if a Banach space has the property that all its subspaces are complemented (by a uniform constant), then that space must be isomorphic to a Hilbert space [146, p. 79]. This conjecture was also known as the *complemented subspaces problem* and should not be confused with the *Complemented Subspace Problem* that we will state shortly. It is worth noting that at that moment the situation in the *isometric case* was already clearer: in 1940 Phillips had shown that every real Banach space whose 2-dimensional subspaces are complemented by a projection of norm 1 must be isometric to a Hilbert space [126] (this same result was extended to the complex case shortly after by Bohnenblust [33]). The aforementioned *complemented subspaces problem* formulated by Sobczyk was solved affirmatively by Lindenstrauss and Tzafriri in 1971 ([106], see also [86]).

In the opposite direction to the previous question, Lindenstrauss posed the following in 1971 in [100]: do *indecomposable* Banach spaces exist? Recall that a Banach space E is indecomposable if it cannot be expressed as a direct sum of two infinite-dimensional spaces. That is, if there is no projection $P : E \to E$ such that $\dim P(E) = \dim (I - P)(E) = \infty$. The first example of an indecomposable space was constructed by Gowers and Maurey [67]. In fact, this space had the additional property of being *hereditarily indecomposable* (H.I., for short), that is, the property that every of its (closed and infinite-dimensional) subspaces is indecomposable. This class of spaces has been intensely studied since then. More recently, examples of indecomposable $C(K)$-spaces have been found (see Koszmider's survey article [91] concerning $C(K)$-spaces with few operators). It should be observed that $C(K)$-spaces can never be H.I. since they always contain an isomorphic copy of c_0.

After this digression on Banach lattices and complementation, we will now focus on the *fundamental problem* of this thesis:

Question (CSP)**.** *Is every complemented subspace of a Banach lattice isomorphic to a Banach lattice?*

We will refer to this question as the **Complemented Subspace Problem (for Banach lattices)** and we will denote by **CSP**, for short. In Chapter 3 we will solve this question in the negative. The aim of Chapter 2 is to give a general perspective on this problem and to gather some notable results related to it, as well as to propose new approaches for its study.

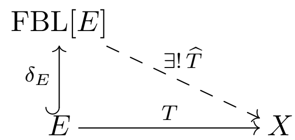
The CSP appears mentioned explicitly in the first sentence of the 1987 article [40] by Casazza, Kalton and Tzafriri and they refer to it as *one of the most important open problems within the theory of Banach lattices*. This suggests that this question must have already been an object of study and well-known among specialists for years, although we have not found a previous reference. One of the reasons that can explain this *late mention* in the literature is the *difficulty* in determining whether a Banach space can be isomorphic to a Banach lattice.

The first examples of Banach spaces not isomorphic to Banach lattices arrived in the 1970s. Some of the criteria used to achieve this are the following:

- A Banach lattice X is *reflexive* if and only if it **does not** contain *subspaces isomorphic to* c_0 *or to* ℓ_1. In general, this is not true for Banach spaces: James's space $\mathcal{J}$ [75] does not contain c_0, nor ℓ_1, but it is not reflexive.
- Every Banach lattice has *unconditional basic sequences*. However, we have already pointed out that there exist H.I. Banach spaces and these cannot contain such sequences.
- Every Banach lattice has *Gordon-Lewis local unconditional structure* (GL-LUST). Among the Banach spaces that fail this property we can find $\mathcal{H}^\infty(\mathbb{D})$ (the space of holomorphic and bounded functions on the open disk) [123], the space $c_p(\ell_2)$ of compact operators on ℓ_2, for $1 \leq p \neq 2 \leq \infty$, endowed with the Schatten p-norm [64, Theorem 5.1], or certain Sobolev spaces [124, 125].

A peculiarity of all these criteria (which can be found in the first section of Chapter 2) is that they are also valid for complemented subspaces in Banach lattices (see Corollary 2.26). Therefore, they *do not allow to distinguish* between Banach lattices and their complemented subspaces. A recently introduced tool that could be useful to clarify this matter is **the free Banach lattice generated by a Banach space**.

Recall that *the free Banach lattice generated by a Banach space* E is a pair $(\mathrm{FBL}[E], \delta_E)$, where $\mathrm{FBL}[E]$ is a Banach lattice and $\delta_E : E \hookrightarrow \mathrm{FBL}[E]$ is a linear isometric embedding, which has the following *universal property*: for every Banach lattice X and every operator $T : E \to X$, *there exists a unique* lattice homomorphism $\widehat{T} : \mathrm{FBL}[E] \to X$ such that $\widehat{T}\delta_E = T$ and, moreover, $\|\widehat{T}\| = \|T\|$. This property is usually expressed by means of the following commutative diagram:



In 2018, Avilés, Rodríguez, and Tradacete proved not only the existence of this object, but also they gave an explicit functional representation of it [15]. We will briefly recall it. Consider first the following expression over the set of functions $f : E^* \to \mathbb{R}$:

$$\|f\|_{\mathrm{FBL}[E]} = \sup\left\{ \sum_{k=1}^{n} |f(x_k^*)| \ : \ n \in \mathbb{N},\ (x_k^*)_{k=1}^n \subseteq E^*,\ \sup_{x \in B_E} \sum_{k=1}^{n} |x_k^*(x)| \leq 1 \right\}.$$

Note that $H_1[E]$, the space of positively homogeneous functions $f : E^* \to \mathbb{R}$ that satisfy $\|f\|_{\mathrm{FBL}[E]} < \infty$, with the pointwise order and the previous norm, is a Banach lattice. For each $x \in E$, let $\delta_x : E^* \to \mathbb{R}$ be the evaluation function $\delta_x(x^*) = x^*(x)$. Then $\mathrm{FBL}[E]$ can be identified with the *closed sublattice* of $H_1[E]$ generated by the set $\{\delta_x : x \in E\}$, along with the *isometric embedding* $\delta_E : E \to \mathrm{FBL}[E]$ given by $\delta_E(x) = \delta_x$ [15, Theorem 2.5].

As we have mentioned, the free Banach lattice generated by a Banach space was introduced in [15] in 2018, *generalizing* the concept of *free Banach lattice generated by a set* that de Pagter and Wickstead had defined and studied in their 2015 article [121]. The assignment $E \mapsto \mathrm{FBL}[E]$ can be considered as a *canonical functor* between the category of Banach spaces (with linear and continuous operators) and the category of Banach lattices (with lattice homomorphisms). This explains the usefulness it has had so far for addressing questions on the interplay between these categories, such as the following ones: $\mathrm{FBL}[E]$

has been used to provide examples of Banach lattices that are *weakly-compactly generated as lattices but not as Banach spaces*, solving a question posed by Diestel [15, Section 5]; it has been used to construct *push-outs* in the category of Banach lattices [17], or to give the first examples of *lattice homomorphisms that do not attain their norm* [42]; and it has been used to show the existence of subspaces of Banach lattices *without bibasic sequences* [119, Section 7], solving a question of [148]. We refer the reader to the work of Oikhberg, Taylor, Tradacete and Troitsky [119], published in 2024, for an extensive study of these objects.

Our interest in considering free Banach lattices to study the CSP stems from the following observation:

Proposition (2.11)**.** *Let E be a Banach space. If E is C_1-isomorphic to a C_2-complemented subspace of a Banach lattice, then $\delta_E(E)$ is C_1C_2-complemented in $FBL[E]$.*

That is, free Banach lattices provide *a canonical place* where to study this problem, since being complemented in *some* lattice, implies being complemented in the corresponding free Banach lattice (and in fact, with *the best possible constant*). Additionally, free Banach lattices also offer *a way of distinguishing* Banach lattices from their complemented subspaces.

Proposition (2.13)**.** *A Banach space E is isomorphic to a Banach lattice if and only if there exists an ideal I in $FBL[E]$ such that $FBL[E] = \delta_E(E) \oplus I$.*

In any case, it does not seem simple to obtain a satisfactory description of the ideals in a free Banach lattice and this makes the use of this criterion difficult in practice. This is discussed in Section 2.2. The rest of Chapter 2 is dedicated to discussing and posing numerous questions related to the Complemented Subspace Problem. One of the open questions that is analyzed and that we would like to highlight *for its apparent simplicity* is the following (Section 2.5):

Question (Hyperplane Problem)**.** *Is every hyperplane of a Banach lattice isomorphic to a Banach lattice?*

Chapter 3 is dedicated to providing **a negative solution to the Complemented Subspace Problem**. For this, we use an already existing example: the space PS_2 constructed in 2023 by Plebanek and Salguero-Alarcón in [131]. This space was constructed to *solve in the negative* the *Complemented Subspace Problem for $C(K)$ spaces*:

Question (CSP for $C(K)$ spaces)**.** *Is every complemented subspace of a $C(K)$-space isomorphic to a $C(K)$-space?*

This is a question already mentioned by Pełczyński in his 1960 article [122] and which has been intensely studied, as illustrated by the numerous partial results that have been obtained in this regard (see [138, Section 5] for a compilation of these). Section 3.1 begins with an interesting renorming result for lattices that has a *local structure similar* to that of an L_1-space:

Theorem (3.1)**.** *Let X be a Banach lattice that is an $\mathcal{L}_1$-space. Then X is lattice isomorphic to an L_1 space.*

Although this result is stated in [2], its proof does not appear explicitly. The proof we present here is not trivial at all: it requires different tools from the local theory of Banach spaces (such as Grothendieck's inequality), as well as from Banach lattice theory. An immediate consequence of this theorem is that a *separable Banach lattice* (of infinite dimension) that is $\mathcal{L}_1$ can only be isomorphic to $L_1[0,1]$ or ℓ_1. This shows *how reduced* the class of Banach lattices is in this context, since it is known that there exists a *continuum* of $\mathcal{L}_1$-spaces that are subspaces of ℓ_1 and are not mutually isomorphic [80]. From the previous theorem, it is easy to obtain the following *dual version*:

Corollary (3.2)**.** *Let X be a Banach lattice that is an $\mathcal{L}_\infty$-space. Then X is lattice isomorphic to an AM-space.*

This corollary will play an essential role in the proof that PS_2 also provides a counterexample to the CSP for Banach lattices. Since this space is not separable, the Complemented Subspace Problem *remains open* in the **separable** case. It must be noted that an affirmative answer to the separable CSP would imply that the following two famous conjectures are also true (see Observation 3.5):

- Every complemented subspace of $L_1[0,1]$ is isomorphic to ℓ_1 or to $L_1[0,1]$.
- Every complemented subspace of $C[0,1]$ is isomorphic to a $C(K)$ space.

Section 3.2 is dedicated to recalling the construction of the space PS_2, as well as its most significant properties. Due to its importance in our thesis, we will briefly recall the form of PS_2. Let $\mathrm{fin}(\mathbb{N})$ be the set of finite subsets of $\mathbb{N}$. Recall that a family $\mathcal{A}$ of *infinite subsets of* $\mathbb{N}$ is said to be **almost disjoint** if $A \cap B \in \mathrm{fin}(\mathbb{N})$ for every distinct $A, B \in \mathcal{A}$. For an almost disjoint family $\mathcal{A}$, we define

$$\mathrm{JL}(\mathcal{A}) := \overline{\mathrm{span}}\big\{\mathbf{1}_A \ :\ A \in \mathrm{fin}(\mathbb{N}) \cup \mathcal{A} \cup \{\mathbb{N}\}\big\} \subseteq \ell_\infty,$$

and we call this space the **Johnson-Lindenstrauss space associated to** $\mathcal{A}$, as it was first introduced in [79, Example 2]. Given that $\mathrm{JL}(\mathcal{A})$ is a closed sublattice of ℓ_∞ that contains the constant function $\mathbf{1}$, the *Kakutani representation theorem for AM-spaces* allows us to deduce that it is lattice isometric to a $C(K)$-space [109, Theorem 1.b.6]. It is not difficult to check that $\mathrm{JL}(\mathcal{A})^*$ is linearly isometric to $\ell_1(\mathbb{N} \cup \mathcal{A} \cup \{\mathbb{N}\})$, so K must be scattered. This particular method of construction of $C(K)$-spaces using almost disjoint families has produced several examples with exotic properties [4, 68, 92, 131].

Let $\mathcal{A} = \{A_\xi \ :\ \xi < \mathfrak{c}\}$ be an almost disjoint family of $\mathbb{N}$ of cardinality $\mathfrak{c}$. We write $\widehat{\mathbb{N}} = \mathbb{N} \times \{0,1\}$ and for each $\xi < \mathfrak{c}$ and $n \in \mathbb{N}$, we denote $\widehat{A_\xi} = A_\xi \times \{0,1\}$ and $c_n = \{(n,0),(n,1)\}$. For $\xi < \mathfrak{c}$, we decompose $\widehat{A_\xi} = B_\xi^0 \cup B_\xi^1$ in such a way that for all $n \in \mathbb{N}$, the sets $B_\xi^0 \cap c_n$ and $B_\xi^1 \cap c_n$ are singletons. Note that $\mathcal{B} := \{B_\xi^0, B_\xi^1 \ :\ \xi < \mathfrak{c}\}$ is an almost disjoint family of $\widehat{\mathbb{N}}$ now. With a slight abuse of notation, we will denote by $\mathrm{JL}(\mathcal{A})$ the closed subspace of $\ell_\infty(\widehat{\mathbb{N}})$ generated by $\{\mathbf{1}_{c_n} \ :\ n \in \mathbb{N}\} \cup \big\{\mathbf{1}_{\widehat{A_\xi}} \ :\ \xi < \mathfrak{c}\big\} \cup \big\{\mathbf{1}_{\widehat{\mathbb{N}}}\big\}$. Similarly, we define

$$\mathrm{JL}(\mathcal{B}) := \overline{\mathrm{span}}\big\{\mathbf{1}_B \ :\ B \in \mathrm{fin}(\widehat{\mathbb{N}}) \cup \mathcal{B} \cup \{\widehat{\mathbb{N}}\}\big\} \subseteq \ell_\infty(\widehat{\mathbb{N}}).$$

It must be observed that $\mathrm{JL}(\mathcal{A})$ is precisely the subspace of $\mathrm{JL}(\mathcal{B})$ that consists of all functions that are constant on each fiber c_n and the map $P : \mathrm{JL}(\mathcal{B}) \to \mathrm{JL}(\mathcal{B})$ defined by

$$Pf(n,0) = Pf(n,1) = \frac{1}{2}\big(f(n,0) + f(n,1)\big), \quad n \in \mathbb{N},$$

is a norm-one projection whose image is $\mathrm{JL}(\mathcal{A})$. Now, we write $X = \ker P$, obtaining thus $\mathrm{JL}(\mathcal{B}) = \mathrm{JL}(\mathcal{A}) \oplus X$. Note that X is also a 1-complemented subspace of $\mathrm{JL}(\mathcal{B})$, since the projection $Q = I - P$ (where $I = \mathrm{id}_{\mathrm{JL}(\mathcal{B})}$) is given by

$$Qf(n,0) = -Qf(n,1) = \frac{1}{2}\big(f(n,0) - f(n,1)\big), \quad n \in \mathbb{N}.$$

In [131], two almost disjoint families, $\mathcal{A}$ and $\mathcal{B}$, are constructed in the form we just described in such a way X is not isomorphic to a $C(K)$-space. This space will be denoted by PS_2.

In Chapter 3 we show that PS_2 *cannot even be isomorphic to a Banach lattice*. A key to this is observing that certain peculiarities of this space allow for a considerable simplification of the problem. More precisely, since PS_2 is a *predual of* $\ell_1(\Gamma)$ (and, in particular, an $\mathcal{L}_\infty$-space) that also has a *countable norming set* (for being a subspace of ℓ_∞), *being isomorphic to a lattice is equivalent in this case to being isomorphic to a* ***sublattice of*** ℓ_∞ (Proposition 3.16). The decisive tool for achieving this *reduction* of our problem is the aforementioned renorming theorem described in Section 3.1.

It will be convenient for us to rephrase the property that PS_2 *cannot be isomorphic to a sublattice of* ℓ_∞ as PS_2 has the **Desired Property (DP)**: For each *norming sequence* $(e_n^*)_{n=1}^\infty \subseteq B_{\mathsf{PS}_2^*}$ there exists $f \in \mathsf{PS}_2$ such that there is no $g \in \mathsf{PS}_2$ for which $|e_n^*(f)| = e_n^*(g)$ for all $n \in \mathbb{N}$. A careful analysis of the construction of PS_2 (which is explained in detail in Theorem 3.19) reveals that for each norming sequence in $B_{\mathsf{PS}_2^*}$ there exists $\xi < \mathfrak{c}$ such that $1_{B_\xi^0} - 1_{B_\xi^1} \in \mathsf{PS}_2$ *does not have a modulus* with respect to that sequence, and therefore PS_2 cannot be isomorphic to a Banach lattice.

In the last section of Chapter 3, we explain how to modify PS_2 construction to obtain a counterexample to the *Complemented Subspace Problem for* ***complex Banach lattices*** (the definition of *complex Banach lattice* will be recalled shortly). More precisely, it is possible to build a variation of this space, which we denote by $\widetilde{\mathsf{PS}}_2$, such that $\widetilde{\mathsf{PS}}_2 \oplus i\widetilde{\mathsf{PS}}_2$ is 1-complemented in a complex $C(K)$-space, but it cannot be isomorphic to a complex Banach lattice (Theorem 3.23). This is interesting, because in the literature we can find some results concerning 1-complemented subspaces in complex Banach lattices that differ significantly from those of the real case [46, 87, 89]. Among them, we would like to highlight the following result of Kalton and Wood [89] (see also [56, 137]): every 1-complemented subspace of a **complex** Banach space with a 1-unconditional basis, also has a 1-unconditional basis. This is not true in the real case, as shown by the examples of [22, 97], although the more general question of whether every complemented subspace of a space with an unconditional basis also has an unconditional basis remains open for both real and complex scalars.

These existing differences between the real and complex cases in the study of the CSP led us to introduce the ***complex free Banach lattice*** *generated by a complex Banach space*, which is described and studied in Chapter 4. Recall that a *complex Banach lattice* Z is the *complexification of a real Banach lattice* X (that is, $Z = X \oplus iX$) where the norm of Z is given by $\|x+iy\|_Z := \||x+iy|\|_X$, $x+iy \in Z$, where $|\cdot| : Z \to X_+$ is the *modulus* map, which is given by

$$|x+iy| := \sup_{\theta\in[0,2\pi]} \{x\cos\theta + y\sin\theta\}, \quad \text{for all } x+iy \in Z.$$

Complex L_p-spaces or $C(K)$-spaces with their usual norms are examples of complex Banach lattices. On the other hand, a *complex lattice homomorphism* is the *complexification* of a real lattice homomorphism, that is, $T : X \oplus iX \to Y \oplus iY$ is a complex lattice homomorphism if there exists a real homomorphism $S : X \to Y$ such that $T(x+iy) = Sx + iSy$ for all $x+iy \in X \oplus iX$. It is important to highlight that, in general, a complex Banach space does not have to be isomorphic to the complexification of some real space [35, 85]. Despite this, we will see that the definition of complex Banach space (without additional hypotheses) and that of complex lattice are *compatible in the desired way*.

Given a complex Banach space E, the ***complex free Banach lattice*** *generated by* E is a complex Banach lattice $\mathrm{FBL}_{\mathbb{C}}[E]$ together with an isometric $\mathbb{C}$-linear embedding $\delta_E : E \to \mathrm{FBL}_{\mathbb{C}}[E]$ such that for every complex Banach lattice $X_{\mathbb{C}}$ and every $\mathbb{C}$-linear operator $T : E \to X_{\mathbb{C}}$, there exists a unique complex lattice homomorphism $\widehat{T} : \mathrm{FBL}_{\mathbb{C}}[E] \to X_{\mathbb{C}}$ such that $\widehat{T} \circ \delta_E = T$ and, moreover, $\|\widehat{T}\| = \|T\|$.

In Section 4.1 *not only the existence of FBL$_\mathbb{C}[E]$ is proven*, but also an *explicit description* of this object is given taking advantage of the existing functional representation of the real free Banach lattice. We outline below the idea of the proof. Given a complex Banach space E, we denote by $E_\mathbb{R}$ the real Banach space resulting from *restricting* the multiplication by complex scalars of E to the reals. In this way, we can consider the free (real) Banach lattice of $E_\mathbb{R}$. It can be verified that essentially $\mathrm{FBL}_\mathbb{C}[E] = \mathrm{FBL}[E_\mathbb{R}] \oplus i\mathrm{FBL}[E_\mathbb{R}]$. And we say *essentially*, because to ensure that the extensions to $\mathrm{FBL}_\mathbb{C}[E]$ preserve the norm of the operators defined on E, previously we need to renorm $\mathrm{FBL}[E_\mathbb{R}]$ with the following:

$$\|f\|_{\mathrm{FBL}_\mathbb{C}[E]} = \sup \left\{ \sum_{j=1}^{m} |f(\mathfrak{Re}\, z_j^*)| \,:\, m \in \mathbb{N},\ (z_j^*)_{j=1}^m \subseteq E^*,\ \sup_{z\in B_E} \sum_{j=1}^{m} |z_j^*(z)| \leq 1 \right\}.$$

On the other hand, we define $\delta_E : E \to \mathrm{FBL}_\mathbb{C}[E]$ by

$$\delta_E(z) = \delta_{E_\mathbb{R}}(z) - i\delta_{E_\mathbb{R}}(iz), \qquad z \in E,$$

which is an isometric $\mathbb{C}$-linear embedding (Lemma 4.2). In Theorem 4.3 it is proven that the complex lattice $\mathrm{FBL}_\mathbb{C}[E] = \mathrm{FBL}[E_\mathbb{R}] \oplus i\mathrm{FBL}[E_\mathbb{R}]$, endowed with the norm $|||\cdot|||_{\mathrm{FBL}_\mathbb{C}[E]}$, together with the map δ_E, is the *complex free Banach lattice generated by E*.

A striking property that emerges from the construction of this object is that, if E is a complex space, and $\overline{E}$ denotes its complex conjugate, then $\mathrm{FBL}_\mathbb{C}[E]$ is (complex) lattice isometric to $\mathrm{FBL}_\mathbb{C}[\overline{E}]$ (see Proposition 4.8, and see also the *partial* converse obtained in Proposition 4.15). This shows that *the same complex free Banach lattice* can come from *different* Banach spaces: recall the examples of complex spaces not isomorphic to their conjugates constructed by Bourgain [35] or Kalton [85]. This *contrasts with the real case*, where no examples of non-isomorphic Banach spaces are known whose corresponding free Banach lattices are lattice isomorphic [119, Remark 10.25]. On the other hand, in Section 4.4 we study the *spectrum* of complex homomorphisms of the form $\overline{T} : \mathrm{FBL}_\mathbb{C}[E] \to \mathrm{FBL}_\mathbb{C}[E]$, where $\overline{T}\delta_E = \delta_E T$ for an operator $T : E \to E$, analyzing the relationship between $\sigma(T)$ and $\sigma(\overline{T})$.

Chapter 5 is dedicated to investigating the phenomenon of *norm attainment* for functionals that are *lattice homomorphisms*. We say that a (linear) functional $x^* : E \to \mathbb{R}$ *attains its norm* if there exists a vector $x \in E$, $\|x\| \leq 1$, such that $\|x^*\| = |x^*(x)|$. The study of *norm attainment* for functionals has received (and continues to receive) considerable attention for decades and constitutes a fundamental part within the theory of Banach spaces. Some of the most remarkable results in this field are the celebrated *James's characterization of reflexivity* –a Banach space E is reflexive if and only if every $x^* \in E^*$ attains its norm– [76], or the Bishop-Phelps theorem –for every Banach space E the set $\mathrm{NA}(E, \mathbb{R})$ is dense in E^*– [31].

When we study norm-attainment for functionals in Banach lattices we can also ask what happens with those functionals that has *certain compatibility* with the order structure of the lattice. In this sense, we want to point out that there has been a growing interest in understanding norm-attainment for *positive functionals*, as reflected, for example, in [3], [112] and [120]. With regard to our work, the main motivation is found in the article by Dantas, Martínez-Cervantes, Rodríguez Abellán and Rueda Zoca [42]. In this article, the authors give *the first example* of a *lattice homomorphism* that *does not* attain its norm.

It must be highlighted that lattice homomorphisms constitute a *very restricted class* within the functionals on a Banach lattice. In fact, not every Banach lattice has lattice homomorphisms (apart from the trivial homomorphism 0): the spaces $L_p[0,1]$, for $1 \leq p <$

∞, *do not* have non-trivial lattice homomorphisms, that is, $\mathrm{Hom}(L_p[0,1],\mathbb{R})=\{0\}$. It can also be verified that for any compact Hausdorff K, $\mathrm{Hom}(C(K),\mathbb{R})=\{\lambda\delta_t : t\in K,\ \lambda\geq 0\}$, where δ_t denotes the functional evaluation at point t, or if X is ℓ_p, with $1\leq p<\infty$, or c_0, one has that $\mathrm{Hom}(X,\mathbb{R})=\{\lambda e_n^* : n\in\mathbb{N},\ \lambda\geq 0\}$, being $(e_n^*)_{n=1}^\infty$ the biorthogonal functionals associated to the canonical basis. Therefore, in these classical spaces it is indeed true that every functional that preserves lattice operations attains its norm.

With the aim of finding a counterexample to the latter, in [42] the authors focus on the *free Banach lattices*, as they constitute a class of Banach lattices with *many lattice homomorphisms*. To be more precise, given a Banach space E, we have that $\mathrm{Hom}(\mathrm{FBL}[E],\mathbb{R})=\{\widehat{x^*} : x^*\in E^*\}$, where $\widehat{x^*}:\mathrm{FBL}[E]\to\mathbb{R}$ denotes the unique lattice homomorphism in $\mathrm{FBL}[E]$ that extends x^* (that is, $\widehat{x^*}\circ\delta_E=x^*$). Note that if x^* attains the norm in $x\in B_E$, then $\widehat{x^*}$ attains its norm in $\delta_x\in B_{\mathrm{FBL}[E]}$. That is, $x^*\in\mathrm{NA}(E,\mathbb{R})$ implies that $\widehat{x^*}\in\mathrm{NA}(\mathrm{FBL}[E],\mathbb{R})$. In [42] the authors *conjecture* that the reciprocal implication is also true, and they are able to verify its validity for some Banach spaces, among which is included, for example $E=\ell_1$. Since ℓ_1 is not reflexive, there exist functionals that do not attain their norm and, by the result we have just commented, their corresponding extensions to the $\mathrm{FBL}[\ell_1]$ also cannot be norm-attaining. This provides the desired counterexample.

The research that we develop in Chapter 5 seeks to understand in a *more general way*, without focusing on free Banach lattices, the *norm-attainment* of lattice homomorphisms on Banach lattices. We begin by analyzing a very special class of lattice homomorphisms: the *coordinate functionals*. Recall that given an atom x_0 in a Banach lattice X, the subspace generated by it is a *projection band* whose associated projection is given by $P_{x_0}(x)=\lambda_{x_0}(x)x_0=\sup_n\big(x\wedge nx_0\big)$, $x\in X_+$. The functional λ_{x_0} is a lattice homomorphism that we call *coordinate functional of the atom* x_0. It is not difficult to check that for any *lattice renorming* $|||\cdot|||$ of X (that is, an equivalent norm such that $(X,|||\cdot|||)$ is also a Banach lattice under the original order) one has that λ_{x_0} attains its norm in $\frac{x_0}{|||x_0|||}$ (see Proposition 5.1). An interesting observation about this class of functionals is that they coincide precisely with the lattice homomorphisms that are order continuous (Proposition 5.2).

In view of the mentioned Proposition 5.1 it is natural to ask if the coordinate functionals are *the only* lattice homomorphisms that attain their norm for any lattice renorming. The answer is that, in many occasions, we can manage to renorm our Banach lattice so that *no homomorphism* (except the coordinate functionals) attains its norm:

Theorem (5.7)**.** *Let X be a Banach lattice which has a strictly positive functional μ. If we renorm X with $\|\cdot\|_\mu:=\|\cdot\|+\mu\big(|\cdot|\big)$, then the only lattice homomorphisms attaining their norms are coordinate functionals of atoms.*

Among the Banach lattices that have a strictly positive functional we find all those that are *separable*. Moreover, spaces such as ℓ_∞ or $L_\infty[0,1]$ also belong to this class, which allows us to give examples of *Dedekind complete lattices that have homomorphisms that do not attain their norm*, answering in the negative a question posed in [42]. In general, however, the answer to the question that we pose is *negative*: δ_{ω_1} is a lattice homomorphism of $C[0,\omega_1]$ that is not a coordinate functional, but it attains its norm for any lattice renorming (Example 5.10).

In Sections 5.3 and 5.4 we analyze the norm-attainment of the lattice homomorphisms defined on AM-spaces. This is a family of Banach lattices with *many lattice homomorphisms*: for any AM-space X, the set $\mathrm{Hom}(X,\mathbb{R})$ is *norming* and, in fact, this property characterizes AM-spaces (Proposition 5.22). And although AM-spaces form a class close to $C(K)$-spaces (specifically, they can be identified with sublattices of $C(K)$), these do not necessarily have *a unit*, nor do we have at our disposal a *Urysohn-type lemma*. For this reason, the following result should not be considered trivial:

Theorem (5.15)**.** *Every lattice homomorphism on an AM-space attains its norm.*

The key to proving the previous result is *a characterization of lattice homomorphisms* that is interesting in itself; in fact, it will also be decisive later to show that every homomorphism on the *free Banach lattice generated by a lattice* (a class of lattices introduced by Avilés and Rodríguez-Abellán in [16]) attains its norm (Proposition 5.41). For completeness, we state the aforementioned characterization here:

Proposition (5.14)**.** *Let X be a Banach lattice and $x^* \in X^*$ a lattice homomorphism of norm 1. Then, x^* attains its norm if and only if there exists an increasing sequence of positive elements $(x_n)_{n=1}^{\infty}$ in B_X such that $x^*(x_n) \to 1$.*

Chapter 6 is dedicated to studying *stable phase retrieval* (SPR, for short) in $C(K)$-spaces. Recall that a subspace E of a Banach lattice X is said to do *stable phase retrieval* with constant $C > 0$ if $\min_{|\lambda|=1} \|f - \lambda g\| \leq C\big\||f| - |g|\big\|$, for any $f, g \in E$.

This work originated from a research project supervised by Mitchell A. Taylor titled *Open problems in stable phase retrieval*, which took place at ICMAT (Madrid) between June 26 − 30 2023, as part of the ICMAT-IMAG Doc-Course in Functional Analysis. The research conducted during those days (with significant additional effort afterward) led to the joint publication with Camúñez and García-Sánchez, [39], which characterizes *complex SPR*. The complex case will not be covered in this thesis. The results presented here can be found in a subsequent work, co-authored with García-Sánchez, [60].

In [58], Freeman, Oikhberg, Pineau, and Taylor prove the following result: a $C(K)$-space contains a subspace (isometric to c_0) doing **Stable Phase Retrieval (SPR)** if and only if the set of its accumulation points K' is infinite. Motivated by this, the authors ask whether a *large number of accumulation points* in the compact space K (in the sense that the Cantor-Bendixon derivative $K^{(\alpha)}$ is an infinite set for $\alpha > 1$) would ensure the existence of *bigger SPR subspaces than* c_0 in $C(K)$. More precisely, they pose the following question [58, Question 6.4]: if $K^{(\alpha)}$ is infinite, then does $C(K)$ contain a subspace isometric to $C[1, \omega^\alpha]$ doing SPR? In Chapter 6 we analyze this question and provide a *complete and affirmative* answer for the case where $\alpha \geq 2$ is a finite ordinal.

To achieve this, one of the keys will be to observe that this question can be *reduced* to studying *SPR embeddings between spaces of the form* $C[1, \omega^\alpha]$ (Proposition 6.10). Subsequently, we will prove that, for $\alpha \geq 2$, there exists an isometric SPR embedding of $C[1, \omega^\alpha]$ into $C[1, \omega^2] \oplus_\infty C[1, \omega^\alpha]$ (Proposition 6.13), which will allow us to deduce the following:

(i) If $K^{(\alpha)} \neq 0$ for $3 \leq \alpha < \omega$, then there exists an isometric SPR embedding of $C[1, \omega^\alpha]$ into $C(K)$ (Corollary 6.14).

(ii) If $|K''| \geq 2$, there exists an isometric **SPR** embedding of $C[1, \omega^2]$ into $C(K)$ (see also Corollary 6.14). This result *cannot be improved*: $C[1, \omega^2]$ cannot be isometrically embedded in an SPR way into a $C(K)$-space with $|K''| = 1$ (Proposition 6.16).

Chapter 1

Preliminary results

The purpose of this chapter is to fix the notation and terminology that will be used throughout this dissertation and to recall some well-known definitions and results that will be very relevant in the rest of the text.

We will begin by fixing the **basic notation** that will be used throughout the text. *Banach lattices* will preferably be denoted by X, Y, Z, while the letters E, F, or G will be reserved for *general Banach spaces*. A *subspace* of a Banach space will always be assumed to be closed (unless explicitly stated otherwise), and when we refer to $T : E \to F$ as an *operator*, we are assuming that T is a *linear and continuous* map. The symbols B_E, S_E and E^* stand for the *closed unit ball* of E, the *unit sphere* of E and the *dual* of E, respectively. The *adjoint* of an operator $T : E \to F$ will be denoted by $T^* : F^* \to E^*$. If A is a subset of a Banach space E, we define its *annihilator* by $A^\perp = \{x^* \in E^* : x^*(x) = 0 \text{ for every } x \in A\}$. Similarly, if A is a subset of E^*, we define its *pre-annihilator* $A_\perp = \{x \in E : x^*(x) = 0 \text{ for every } x^* \in A\}$.

Two Banach spaces are said to be *isomorphic* if there exists a bijective operator between them (an *isomorphism*); if this operator can additionally be taken to be norm-preserving, then it is said to be an *isometry* and, in this situation, those two Banach spaces are *isometric*. An operator $T : E \to F$ which is an isomorphism onto its image $T(E) \subseteq F$ is called an *embedding*; if in addition T is norm-preserving, then it is said to be an *isometric embedding*. A *projection* on E is an operator $P : E \to E$ such that $P \circ P = P$; a *complemented subspace* of E is the range of a projection on E. Given $1 \leq p < \infty$, a Banach lattice of the form $L_p(\Omega, \Sigma, \mu)$ (where (Ω, Σ, μ) is any measure space), equipped with its usual norm and the μ-almost everywhere order, will be called an *L_p-space*. A *$C(K)$-space* is any Banach lattice of the form $C(K) := \{f : K \to \mathbb{R} : f \text{ continuous on } K\}$, where K is a compact Hausdorff space, endowed with the supremum norm and the pointwise order.

1.1 Basic concepts in Banach lattice theory

Let us start by recalling some elementary notions of Banach lattice theory. There is extensive literature where these concepts can be found, and among the works we have most frequently consulted are [1, 7, 94, 109, 115, 141].

Definition 1.1. A (real) *Banach lattice* is a real Banach space $(X, \|\cdot\|)$ equipped with a partial order $\leq$ which satisfies the following properties:

i) For every $x, y \in X$ such that $x \leq y$, we have $x + z \leq y + z$ for every $z \in X$ and $\lambda x \leq \lambda y$ for every $\lambda \geq 0$.

ii) $\leq$ is a lattice order, that is, for every $x, y \in X$, the set $\{x, y\}$ has a supremum (least upper bound) and an infimum (greatest lower bound), which will be denoted by $x \vee y$ and $x \wedge y$, respectively.

iii) If $|x| \leq |y|$, then $\|x\| \leq \|y\|$, where $|x| := x \vee (-x)$.

On some occasions, we will use a more general concept than the previous one: the *vector lattice*. A *vector lattice* (or a *Riesz space* in some books [1, 7, 115]) is a real vector space X endowed with a partial order $\leq$ satisfying the above properties i) and ii); that is, $\leq$ is a lattice order such that $x+z \leq y+z$ for every $z \in X$ and $\lambda x \leq \lambda y$ for every scalar $\lambda \geq 0$ whenever $x \leq y$. Moreover, we will assume that *every vector lattice fulfills the following property (Archimedean Property):* if $x \in X$ and $u \in X_+$ satisfy $nx \leq u$ for every $n \in \mathbb{N}$, then $x \leq 0$. Note that every Banach lattice also satisfies this additional condition thanks to property iii).

Given a vector lattice X, we denote by $X_+ := \{x \in X \,:\, x \geq 0\}$ the *positive cone* of X. For $x \in X$, we define its *positive part* as $x^+ = x \vee 0$ and its *negative part* as $x^- = (-x) \vee 0$. The vector $|x| = x \vee (-x)$ is called the *absolute value* of x. Two vectors $x, y \in X$ are said to be *disjoint* if $|x| \wedge |y| = 0$ and we express this situation as $x \perp y$. Note that every element $x \in X$ can be expressed as $x = x^+ - x^-$ and this is the unique representation of x as the difference of positive disjoint elements. An *order interval* is a set of the form $[x, y] := \{z \in X \,:\, x \leq z \leq y\}$ and a subset $A \subseteq X$ is said to be *order bounded* if it is contained in an order interval (or, equivalently, if it is bounded above and below).

Among the most useful *formulas involving lattice and linear operations* for vector lattices, we can find the following (see, for instance, [7, Section 1] or [94, p. 2] for a more exhaustive list):

- $x \vee y + z = (x+z) \vee (y+z)$ and $x \wedge y + z = (x+z) \wedge (y+z)$.
- $|x+y| \leq |x| + |y|$ and $\big||x| - |y|\big| \leq |x-y|$.
- $x \vee y = -(-x) \wedge (-y)$ and $x \wedge y = -(-x) \vee (-y)$.
- $x + y = x \vee y + x \wedge y$
- $a(x \vee y) = (ax) \vee (ay)$ and $a(x \wedge y) = (ax) \wedge (ay)$ for all $a \geq 0$.
- $x = (x-y)^+ + x \wedge y$.
- $x \vee y = \frac{1}{2}\big(x + y + |x-y|\big)$ and $x \wedge y = \frac{1}{2}\big(x + y - |x-y|\big)$.

All the above expressions can be proven *by elementary means*. However, for Banach lattices, there is an alternative way to check them: we will briefly explain in the Subsection 1.2.3 that, thanks to *Krivine's functional calculus*, any *finite lattice-linear expression* which holds for the *reals* is *automatically valid for any Banach lattice*.

A vector lattice X is called *Dedekind complete* (or *order complete*) if every non-empty order bounded set has a supremum and an infimum in X. Sometimes it is useful to consider the *countable* version of this notion: X is called *σ-Dedekind complete* (or σ-order complete) if every order bounded *sequence* has a supremum and an infimum in X.

1.1.1. Distinguished subspaces of vector and Banach lattices. Some of the *special subspaces* we can define within the class of vector lattices are the following. A *sublattice* of X is a subspace such that if $x, y \in Y$, then so does $x \vee y$ (and $x \wedge y$). An *ideal* in X is a sublattice Y such that if $y \in Y$ and $|x| \leq |y|$, then $x \in Y$. A *band* in X is an ideal

Y such that if A is a non-empty set in Y such that $\sup(A)$ exists in X, then $\sup(A) \in Y$. A band Y is called a *projection band* if there is a projection P of X onto Y such that $0 \leq Px \leq x$ for every $x \in X_+$. Such a projection P is called a *band projection.* As usual, we will assume that *these subspaces* are *closed* when X is a Banach lattice unless we state otherwise.

Let I be a (closed) ideal in a Banach lattice X and consider the quotient map $Q : X \to X/I$, defined by $Qx := \overline{x}$, where $\overline{x}$ stands for the equivalence class of x in X/I. We have the following [115, Proposition 1.3.13 and Corollary 1.3.14].

Proposition 1.2. *Let X be a Banach lattice and I be an ideal in X. Then, the quotient space X/I ordered by the cone $Q(X_+)$ and equipped with the usual norm given by $\|\overline{x}\| = \inf\{\|x + z\| : z \in I\}$ is a Banach lattice and $Q : X \to X/I$ is a lattice homomorphism.*

Since an arbitrary intersection of sublattices (resp. ideals, bands) is again a sublattice (resp. ideal, band), given any non-empty subset A of a vector lattice X there is a smallest sublattice (resp. ideal, band) containing A and we call it *the sublattice* (resp. *ideal*, *band*) *generated by A*. Note that the ideal generated by A, denoted by I_A, can be easily described:

$$I_A = \left\{ x \in X \ : \ \exists\, x_1, \ldots, x_n \in A \text{ and } \lambda \in \mathbb{R}_+ \text{ with } |x| \leq \lambda \sum_{k=1}^{n} |x_k| \right\}.$$

In particular, the ideal generated by an element $x_0 \in X$ is

$$I_{x_0} = \{x \in X \ : \ |x| \leq \lambda x_0 \text{ for some } \lambda \geq 0\}$$

and will call it *the principal ideal generated by* x_0. These ideals are of great importance in vector lattice theory, as they can be *represented as $C(K)$-spaces* (we will explain this in more detail later).

Given a non-empty subset A of a vector lattice X, we define $A^\vee := \left\{\bigvee_{k=1}^{n} x_k \ : \ n \in \mathbb{N}, x_1, \ldots, x_n \in A\right\}$ and $A^\wedge := \left\{\bigwedge_{k=1}^{n} x_k : n \in \mathbb{N}, x_1, \ldots, x_n \in A\right\}$. If A is a linear subspace of X, then the sublattice generated by A is $\mathrm{lat}(A) = A^{\vee\wedge}$ and $\mathrm{lat}(A) = A^{\vee\wedge} = A^\vee - A^\vee = A^\wedge - A^\wedge$ (see [7, Exercise 8 of Section 4.1]). In particular, if E is a separable subspace of a Banach lattice X, then the closed sublattice generated by E is also separable. The situation is different for ideals: if $X = C(K)$, for some compact Hausdorff K, then the principal ideal generated by the constant function $\mathbf{1}_K$ is the whole space (so this ideal will not be separable unless $C(K)$ is).

As we have already noted, when we refer to sublattices, ideals, or bands in *Banach lattices*, these subspaces will be assumed to be *closed.* However, when we refer to sublattices (or ideals or bands) *generated by a set*, even in the Banach lattice setting, we will *not* assume that these subspaces *are necessarily closed*, unless specified otherwise.

The constant function $\mathbf{1}_K$ in a $C(K)$-space is the canonical example of a *strong unit.* Recall that a positive vector x_0 in a vector lattice X is said to be a *strong (order) unit* if for all $x \in X_+$, there is $\lambda \in \mathbb{R}_+$ such that $|x| \leq \lambda x_0$. In this situation, $I_{x_0} = X$.

A sublattice Y of a vector lattice X is said to be *order dense* in X if for every $x \in X$, $x > 0$, there exists $y \in X$ such that $0 < y \leq x$. For example, the space of sequences of finite support, c_{00}, is order dense in c_0 (and in ℓ_p for any $1 \leq p \leq \infty$).

1.1.2. Distinguished operators of vector and Banach lattices. Let X, Y be vector lattices and $T : X \to Y$ be a linear mapping. Then:

- T is *positive* if $Tx \geq 0$ whenever $x \geq 0$ (equivalently, if $T(X_+) \subseteq Y_+$).
- T is a *regular operator* if it can be expressed as the difference of two positive operators
- T is a *lattice homomorphism* if $T(x \vee y) = Tx \vee Ty$ (and $T(x \wedge y) = Tx \wedge Ty$ for every $x, y \in X$. Note that, in this case, T must be positive: given $x \geq 0$, we have $Tx = T(x \vee 0) = Tx \vee 0 \geq 0$. The *set of lattice homomorphisms* from X into Y will be denoted by $\mathrm{Hom}(X, Y)$.

It can be checked that a positive operator between two Banach lattices is automatically continuous [7, Theorem 4.3] and, therefore, the above mappings are always (linear and continuous) *operators* if X and Y are Banach lattices. It can also be shown that the dual of a Banach lattice is also a Banach lattice, whose positive cone consists of the positive functionals, that is, $X^*_+ = \{x^* \in X^* \ : \ x^*(x) \geq 0 \text{ for every } x \in X_+\}$ [115, Proposition 1.3.7 and Corollary 1.3.4].

Proposition 1.3. *Let X be a Banach lattice. Then, every (linear and continuous) functional $x^* \in X^*$ is regular. Moreover, X^* is a Dedekind complete Banach lattice with the order $x^* \leq y^*$ if $x^*(x) \leq y^*(x)$ for every $x \in X_+$ and its lattice operations are given by the* Riesz-Kantorovich formulas*:*

$$\begin{aligned}(x^* \vee y^*)(x) &= \sup\big\{x^*(x-y) + y^*(y) \ : \ y \in [0,x]\big\}\\ (x^* \wedge y^*)(x) &= \inf\big\{x^*(x-y) + y^*(y) \ : \ y \in [0,x]\big\},\\ |x^*|(x) &= \sup\big\{|x^*(y)| \ : \ |y| \leq x\big\},\end{aligned}$$

for every $x \in X_+$.

Another very important property concerning duals of Banach lattices is presented here [109, Proposition 1.a.2]:

Proposition 1.4. *Let X be a Banach lattice. The canonical isometric embedding $J : X \to X^{**}$, given by $Jx(x^*) = x^*(x)$, $x^* \in X^*$, preserves the lattice operations. That is, $J : X \to X^{**}$ is a lattice isometric embedding.*

1.1.3. Atoms. A non-zero positive element $x_0 \in X$ is said to be an *atom* if $0 \leq y \leq x$ implies that $y = ax$ for some real $a \geq 0$. Note that if x_0 and y_0 are atoms in X, then they are either disjoint or proportional. The elements of the canonical basis of ℓ_p, $1 \leq p < \infty$, or c_0, are examples of atoms in these spaces. But not always a Banach lattice contains an atom. Regarding *the number of atoms* we say that a Banach lattice X is:

- *non-atomic* (or *atomless*) if it does not contain atoms. For example, $C[0,1]$ or $L_p[0,1]$ for any $1 \leq p \leq \infty$;
- *purely atomic* if $X = \overline{\mathrm{span}}\{e_\alpha \ : \ \alpha \in \Gamma\}$, where $\{e_\alpha \ : \ \alpha \in \Gamma\}$ is the collection of all norm-one atoms of X. Recall also that every finite-dimensional Banach lattice is purely atomic [94, p. 9].
- *discrete* (or *atomic*) if for every $x \in X_+$, $x \neq 0$, there is an atom x_0 such that $x_0 \leq x$. For example, ℓ_∞.

The following proposition establishes a very useful connection between the atoms of the dual of a Banach lattice and lattice homomorphisms [94, Lemma 1, p. 59]:

Proposition 1.5. *Let X be a Banach lattice. Then, $x^* \in X^* \setminus \{0\}$ is a lattice homomorphism if and only if x^* is an atom in X^*.*

1.1.4. Order continuous Banach lattices. A Banach lattice X is said to have an *order continuous norm* (resp. *σ-order continuous norm*) or, briefly, to be *order continuous* (resp. *σ-order continuous*) if, for every decreasing net (resp. decreasing sequence) $(x_\alpha)_\alpha$ in X with $\inf_\alpha x_\alpha = 0$, we have that $\lim_\alpha \|x_\alpha\| = 0$.

Order continuous Banach lattices play a fundamental role within the theory of Banach lattices. There are multiple ways to characterize Banach lattices having this property. We collect some of these characterizations in the following proposition (for more details, see [109, Section 1.a] or [115, Section 2.4]):

Theorem 1.6. *Let X be a Banach lattice. Then the following assertions are equivalent:*

(i) X is order continuous.

(ii) X is σ-Dedekind complete and σ-order continuous.

(iii) Every monotone order bounded sequence in X converges in the norm topology of X.

(iv) Every disjoint order bounded sequence in X_+ is convergent to zero.

(v) If $J : X \to X^{**}$ stands for the canonical embedding of X into its bidual, then $J(X)$ is an ideal in X^{**}.

(vi) Every order interval of X is weakly compact.

(vii) Every ideal of X is the range of a positive projection.

A Banach lattice X is said to be a *KB-space* if every *norm-bounded* monotone sequence is convergent. Observe that by the fourth equivalence of the previous proposition, every KB-space must be order continuous. The converse is not true, as c_0 shows. KB-spaces can also be characterized by several useful ways (see [109, Theorem 1.b.4] and [115, Theorem 2.4.12]).

Theorem 1.7. *Let X be a Banach lattice. Then the following assertions are equivalent:*

(i) X is a KB-space.

(ii) X does not contain a subspace isomorphic to c_0.

(iii) X does not contain a sublattice lattice isomorphic to c_0.

(iv) X is weakly sequentially complete.

(v) X is a projection band in its bidual X^{**}.

1.1.5. Representation theorems. A Banach lattice X is said to be an *abstract L_p-space* for some $1 \le p < \infty$ whenever its norm is *p-additive* in the sense that $\|x+y\|^p = \|x\|^p + \|y\|^p$ for all $x, y \in X$, with $x \wedge y = 0$. For the particular case $p = 1$, we will refer to an *abstract L_1-space* as an *AL-space*. It is clear that every L_p-space, for some $1 \le p < \infty$, is an abstract L_p-space. The *converse* of this result is also true (see [109, Theorem 1.b.2] or [7, Theorem 4.27]):

Theorem 1.8. *An abstract L_p-space, $1 \le p < \infty$, is lattice isometric to $L_p(\Omega, \Sigma, \mu)$ for some measure space (Ω, Σ, μ).*

On the other hand, we say that X is an *abstract M-space* (*AM-space*, for short) whenever its norm is an *M-norm*, that is, if $x \wedge y = 0$, then $\|x+y\| = \max\{\|x\|, \|y\|\}$. A very simple example of an AM-space is any $C(K)$-space. It is also easy to check that any sublattice of a $C(K)$-space also belongs to this class. The converse of the latter is also true and is known as the *Kakutani representation theorem for AM-spaces* [84] (see also [109, Theorem 1.b.6] or [7, Theorem 4.29]).

Theorem 1.9. *Any AM-space X is lattice isometric to a sublattice of a $C(K)$-space. If moreover X has a strong unit x_0 with the property that $\|x\| \leq 1$ if and only if $|x| \leq x_0$, then there exists a (surjective) lattice isometry $T : X \to C(K)$ such that $Tx_0 = \mathbf{1}_K$.*

A highly useful consequence of Kakutani's representation theorem for AM-spaces is that Banach lattices are *locally representable* as $C(K)$-spaces.

Proposition 1.10. *Let X be a Banach lattice and let $x_0 \in X$. Then, the principal ideal I_{x_0} generated by x_0 in X under the norm*

$$\|x\|_\infty = \inf\{\lambda > 0 \,:\, |x| \leq \lambda |x_0|\}, \qquad x \in I_{x_0},$$

is an AM-space and x_0 is a strong unit. Thus, there exists a compact Hausdorff space K and a lattice isometry $T : C(K) \to (I_{x_0}, \|\cdot\|_\infty)$ such that $T\mathbf{1}_K = x_0$.

There is an important *duality* between AL-spaces and AM-spaces [7, Theorem 4.23]:

Theorem 1.11. *A Banach lattice X is an AL-space (resp. an AM-space) if and only if X^* is an AM-space (resp. an AL-space).*

We state now an important property concerning the extreme points of the unit ball of an AL-space. Since we have not found a reference for this (likely well-known) fact, we provide a proof below.

Proposition 1.12. *If X is an AL-space, then $\operatorname{ext} B_X = \big(\operatorname{ext} B_{X_+} \cup (-\operatorname{ext} B_{X_+})\big) \setminus \{0\}$.*

Proof. Take $x \in \operatorname{ext} B_X$. We define $\alpha = \|x^+\|$. Since X is an AL-space, if $\alpha \neq 0, 1$, we would have the decomposition

$$x = \alpha \frac{x^+}{\alpha} + (1-\alpha)\frac{-x^-}{1-\alpha},$$

contradicting the fact that x is an extreme point of B_X. Thus, $x \in \operatorname{ext} B_{X_+} \cup (-\operatorname{ext} B_{X_+})$ and $x \neq 0$ (in fact, $\|x\| = 1$).

Conversely, let us fix an element $x \in \big(\operatorname{ext} B_{X_+} \cup (-\operatorname{ext} B_{X_+})\big) \setminus \{0\}$. Without loss of generality, we may assume that $x \geq 0$. Take any $y, z \in B_X$ such that

$$x = \frac{1}{2}y + \frac{1}{2}z. \tag{1.1}$$

Since $\|x\| = 1$, this implies that $\|y\| = \|z\| = 1$. If y (or z) is not positive, then

$$2x = y + z < |y| + |z|,$$

and since every AL-norm is strictly monotone, we have that $2 = 2\|x\| < \|y\| + \|z\| = 2$, a contradiction. Thus, if we had such a decomposition (1.1) of x, as x is an extreme point of $\operatorname{ext} B_{X_+}$, this would imply that $x = y = z$. Therefore, $x \in \operatorname{ext} B_X$. □

1.2 The free Banach lattice generated by a Banach space

1.2.1. Basics in free objects. Before giving the definition of the free Banach lattice generated by a Banach space, we will provide a brief introduction to the notion of *free object*, as it is a concept that frequently appears in many branches of mathematics. For a more precise and rigorous treatment of Category theory, we refer the reader to [142, Chapter 3].

A *category* Cat is a pair $(\mathsf{Ob}, \mathsf{Mor})$, where Ob is a class of sets called *objects* and Mor is a class of mappings defined between those objects, which will be called *morphisms*. Below are some examples of categories, several of which will be discussed in this section:

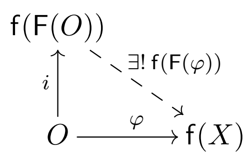
- Set: the *sets* with the *maps* as morphisms;
- Top: the *topological spaces*, with the *continuous* maps;
- Ban: the *Banach spaces*, with the *linear and continuous operators*;
- BL: the *Banach lattices*, with the *lattice homomorphisms*.

It is worth noting that the categories listed above are ordered so that each subsequent class is *more restrictive* than the one before it. For instance, any Banach lattice can be viewed as a Banach space if we *forget* its order structure or any Banach space can be seen as a topological space if we *forget* that its topology comes from a (complete) norm. These last two *assignments* are examples of **forgetful functors**: Given two categories Cat_1 and Cat_2, where Cat_2 is a *wider* category than Cat_1 a *forgetful functor* $\mathsf{f} : \mathsf{Cat}_1 \to \mathsf{Cat}_2$ is a map that *forgets* the additional structures that Cat_1 has over Cat_2.

We will avoid saying that BL is a subcategory of Ban, or that Ban is a subcategory of Top. The reason is that the usual definition of *subcategory* requires the morphisms of the subcategory to be a *subset* of the morphisms of the original category (see [142, Definitions 9.1.3 and Examples 9.2]).

For instance, ℓ_2 and $L_2[0,1]$ are *distinct Banach lattices*; they cannot even be lattice isomorphic given that the former has atoms and the latter is non-atomic. Nevertheless, as *Banach spaces*, they are indistinguishable. Similarly, the Banach spaces ℓ_p^n (for $1 \leq p \leq \infty$ and a fixed n) are not linearly isometric, but they are *topologically indistinguishable* because their norms are equivalent.

Definition 1.13. Let Cat_1, Cat_2 be categories such that there exists a forgetful functor $\mathsf{f} : \mathsf{Cat}_1 \to \mathsf{Cat}_2$. Given an object O in Cat_2, **the free object in Cat_1 generated by O** is an object $\mathsf{F}(O)$ in Cat_1 together with a morphism $i : O \to \mathsf{f}(\mathsf{F}(O))$ (in Cat_2) satisfying the following *universal property*: for any object X in Cat_1 and any morphism $\varphi : O \to \mathsf{f}(X)$ (in Cat_2), *there is a unique* morphism $\mathsf{F}(\varphi) : \mathsf{F}(O) \to X$ in Cat_1 such that $\mathsf{f}(\mathsf{F}(\varphi)) \circ i = \varphi$. It will be customary to represent this in the form of the following commutative diagram:



It should be noticed that if such an object exists, then it is *essentially unique* (up to an isomorphism in the category Cat_1). We will now present some well-known examples of free objects that arise in functional analysis.

Example 1.14 (The bidual of a Banach space). Consider the category of *dual Banach spaces* together with the *adjoint operators* (or equivalently, w^*-continuous linear maps). Given a Banach space E, let $J_E : E \to E^{**}$ denote the canonical embedding of E into its bidual. It is clear that every operator $T : E \to F^*$ can be uniquely extended to an adjoint operator $\overset{*}{T} : E^{**} \to F^*$, given by $\overset{*}{T} = (T^* \circ J_F)^*$, in such a way that the following diagram commutes:

$$\begin{array}{ccc} E^{**} & & \\ {\scriptstyle J_E}\Big\uparrow & \searrow^{\exists!\,\overset{*}{T}} & \\ E & \xrightarrow{\;T\;} & F^* \end{array}$$

Moreover, $\|\overset{*}{T}\| = \|T\|$. Thus, we can consider the pair $(E^{**},\ J_E)$ as the *free dual Banach space generated by* E.

Example 1.15 (The Stone-Čech compactification). The *Stone-Čech compactification* of a completely regular space (also called a Tychonoff space) T is a compact Hausdorff space βT together with a homeomorphism $i_T : T \to \beta T$ of T onto a dense set in βT, so that for every compact Hausdorff space K and every continuous map $f : T \to K$ there is a unique continuous map $\widehat{f} : \beta T \to K$ making the following diagram commute:

$$\begin{array}{ccc} \beta T & & \\ {\scriptstyle i_S}\Big\uparrow & \searrow^{\exists!\,\widehat{f}} & \\ T & \xrightarrow{\;f\;} & K \end{array}$$

Thus, we could consider βT as the *free compact Hausdorff space generated by* T. For more details on this construction, see [94, Section 7] or [155, Section 19].

Example 1.16 (The Lipschitz-free Banach space). Given a metric space M with a distinguished point $\mathbf{0}$, the *Lipschitz-free Banach space over* M is a Banach space $\mathcal{F}(M)$ together with a Lipschitz isometric embedding $\delta_M : M \to \mathcal{F}(M)$ with the property that for every Banach space X and every Lipschitz map $f : M \to X$ with $f(\mathbf{0}) = 0$, there is a unique linear operator $\widehat{f} : \mathcal{F}(M) \to X$ making the following diagram commute:

$$\begin{array}{ccc} \mathcal{F}(M) & & \\ {\scriptstyle \delta}\Big\uparrow & \searrow^{\exists!\,\widehat{f}} & \\ M & \xrightarrow{\;f\;} & X \end{array}$$

Moreover, the operator norm of the linear extension coincides with the Lipschitz norm of the function f: $\|\widehat{f}\| = \|f\|_{Lip}$. These objects have received considerable attention within Banach space theory over the past two decades, notably boosted by the publication of Godefroy and Kalton's article [63] which examined these spaces.

The last free object we present will be the one that appears most frequently in this thesis: the free Banach lattice over a Banach space. Following Definition 1.13, these are the free objects in the category BL generated by Banach spaces.

Definition 1.17. Given a real Banach space E, the **free Banach lattice generated by** E is a pair $(\mathrm{FBL}[E], \delta_E)$, where $\mathrm{FBL}[E]$ is a Banach lattice and $\delta_E : E \to \mathrm{FBL}[E]$ is an isometric embedding, which has the following *universal property*: for every Banach lattice X and every operator $T : E \to X$, there exists a unique lattice homomorphism

$\widehat{T} : \mathrm{FBL}[E] \to X$ such that $\widehat{T}\delta_E = T$ and, moreover, $\|\widehat{T}\| = \|T\|$. As usual, this property can be conveniently illustrated using the commutative diagram below:

$$\begin{array}{ccc} \mathrm{FBL}[E] & & \\ {\scriptstyle \delta_E}\uparrow & \searrow^{\exists!\,\widehat{T}} & \\ E & \xrightarrow{\quad T \quad} & X \end{array}$$

The existence of this free object was proven in the 2018 article [15] by Avilés, Rodríguez and Tradacete, where they indeed provide an *explicit functional representation* of this object, which will be described in the next subsection. Free Banach lattices have been the focus of recent intensive research and the reader may consult [119] for a thorough study of the properties of these objects.

The reader is also referred to the *survey* [59], a joint work with Tradacete and García-Sánchez, which studies and compares properties of some of the most well-studied free objects in Banach space theory, including *Lipschitz-free Banach spaces*, *free Banach lattices* and *holomorphic-free Banach spaces.*

1.2.2. An explicit description of FBL$[E]$. Let E be a Banach space and denote by $H[E]$ the linear subspace of $\mathbb{R}^{E^*}$ consisting of all *positively homogeneous functions* $f : E^* \to \mathbb{R}$, that is, $f(\lambda x^*) = \lambda f(x^*)$, for every $\lambda \geq 0$ and every $x^* \in E^*$. Given $f \in H[E]$, consider the expression

$$\|f\|_{\mathrm{FBL}[E]} = \sup\left\{ \sum_{k=1}^{n} |f(x_k^*)| \;:\; n \in \mathbb{N},\; (x_k^*)_{k=1}^n \subseteq E^*,\; \sup_{x \in B_E} \sum_{k=1}^{n} |x_k^*(x)| \leq 1 \right\}. \qquad (1.2)$$

Note that the vector space $H_1[E] := \{f \in H[E] \;:\; \|f\|_{\mathrm{FBL}[E]} < \infty\}$ endowed with the pointwise order and the above norm is a Banach lattice. Now, for each $x \in E$, let $\delta_x : E^* \to \mathbb{R}$ be defined by $\delta_x(x^*) := x^*(x)$, $x^* \in E^*$ and observe that $\delta_x \in H_1[E]$; in fact, $\|\delta_x\|_{\mathrm{FBL}[E]} = \|x\|$.

It was shown in [15, Theorem 2.5] that the *free Banach lattice generated by* E is the Banach lattice $\mathrm{FBL}[E] = \overline{\mathrm{lat}}\{\delta_x \;:\; x \in E\} \subseteq H_1[E]$ together with the linear isometric embedding $\delta_E : E \to \mathrm{FBL}[E]$ given by $\delta_E(x) := \delta_x$.

Given any Banach space E, it is not difficult to check using the universal property of $\mathrm{FBL}[E]$ that $\mathrm{Hom}(\mathrm{FBL}[E], \mathbb{R}) = \left\{\widehat{x^*} \;:\; x^* \in E^*\right\}$, where $\widehat{x^*} : \mathrm{FBL}[E] \to \mathbb{R}$ is the lattice homomorphism given by $\widehat{x^*}(f) = f(x^*)$, for $f \in \mathrm{FBL}[E]$ [15, Corollary 2.7].

In [77], this notion was extended to certain *subcategories* of the category BL, such as, for example, the category of *p-convex Banach lattices* and lattice homomorphisms. A reminder of these objects will be provided in the upcoming subsection.

1.2.3. Free p-convex Banach lattices. Before defining the concept of p-convexity in Banach lattices, we must recall that Banach lattices admit a *functional calculus for positively homogeneous functions* [109, Section 1.d].

Theorem 1.18 (Krivine's functional calculus). *Let X be a Banach lattice. For every $n \in \mathbb{N}$ and every $\mathbf{x} = (x_k)_{k=1}^n \subseteq X$, there exists a unique map $\Phi_{\mathbf{x}}$ from the vector lattice $\mathscr{H}_n$, of all the functions which are continuous and positively homogeneous on $\mathbb{R}^n$, into X such that:*

i) $\Phi_{\mathbf{x}}\pi_k = x_k$ *for* $1 \le k \le n$*, where* $\pi_k(x_1, \ldots, x_n) := x_k$*.*

ii) $\Phi_{\mathbf{x}}$ *is linear and preserves the lattice operations.*

An immediate consequence of this theorem is that any identity consisting of applying finitely many operations of addition, multiplication by scalars and finite suprema and infima to a finite sequence $(x_k)_{k=1}^n$ (a *lattice-linear expression of* $(x_k)_{k=1}^n$) that holds for $\mathbb{R}$ will also hold in any Banach lattice.

Remark 1.19. Let X, Y be Banach lattices and let $T : X \to Y$ be a lattice homomorphism. Then, given any finite sequence $\mathbf{x} = (x_k)_{k=1}^n$ in X, by the uniqueness of $\Phi_{\mathbf{x}}$, we have that $T \circ \Phi_{\mathbf{x}} = \Phi_{\mathbf{Tx}}$, where $\mathbf{Tx} = (Tx_k)_{k=1}^n$.

Given $1 \le p \le \infty$, a Banach lattice X is said to be *p-convex* if there is a constant $M \ge 1$ such that for every choice of vectors $(x_k)_{k=1}^n$ in X we have

$$\left\| \left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} \right\| \le M \left(\sum_{k=1}^n \|x_k\|^p \right)^{\frac{1}{p}}, \qquad \text{if } 1 \le p < \infty,$$

or

$$\left\| \bigvee_{k=1}^n |x_k| \right\| \le M \max_{1 \le k \le n} \|x_k\|, \qquad \text{if } p = \infty.$$

Note that such expressions $\left(\sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}}$ are well defined thanks to Krivine's functional calculus, as the functions $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(t_1, \ldots, t_n) = \left(\sum_{k=1}^n |t_k|^p\right)^{\frac{1}{p}}$ are continuous and positively homogeneous. The smallest possible value of M is called the *p-convexity constant* of X and will be denoted by $M^{(p)}(X)$. It should be noted that every Banach lattice X is 1-convex with $M^{(1)}(X) = 1$. On the other hand, it can be checked that if X is an ∞-convex Banach lattice, then it is lattice $M^{(\infty)}(X)$-isomorphic to an AM-space (see [115, Theorem 2.1.12]).

Given a Banach space E and $p \in [1, \infty]$, *the free p-convex Banach lattice over E* is a p-convex Banach lattice $\mathrm{FBL}^{(p)}[E]$ with p-convexity constant equal to 1 together with a linear isometric embedding $\delta_E : E \to \mathrm{FBL}^{(p)}[E]$ with the property that for every p-convex Banach lattice X and every operator $T : E \to X$, there exists a unique lattice homomorphism $\widehat{T} : \mathrm{FBL}^{(p)}[E] \to X$ such that $\widehat{T}\delta_E = T$ and, moreover, $\|\widehat{T}\| \le M^{(p)}(X)\|T\|$. Note that for $p = 1$ this definition coincides with the one of free Banach lattice generated by a Banach space introduced at the beginning of the section, that is, for any Banach space E, $\mathrm{FBL}[E]$ and $\mathrm{FBL}^{(1)}[E]$ coincide. Throughout the text, we will only use the notation $\mathrm{FBL}[E]$.

The existence of the $\mathrm{FBL}^{(p)}[E]$ for any $1 \le p \le \infty$ was proven in [77], where the authors also give a functional representation of this object in the same spirit as that of $\mathrm{FBL}[E]$. In this case, they consider again the space $H[E] \subseteq \mathbb{R}^{E^*}$ of positively homogeneous functions and define

$$\|f\|_{\mathrm{FBL}^{(p)}[E]} = \sup \left\{ \left(\sum_{k=1}^n |f(x_k^*)|^p \right)^{\frac{1}{p}} : n \in \mathbb{N},\ (x_k^*)_{k=1}^n \subseteq E^*,\ \sup_{x \in B_E} \sum_{k=1}^n |x_k^*(x)|^p \le 1 \right\},$$

when $p < \infty$, and

$$\|f\|_{\mathrm{FBL}^{(\infty)}[E]} = \sup_{x^* \in B_{E^*}} |f(x^*)|,$$

when $p = \infty$. It can be shown that

$$\mathrm{FBL}^{(p)}[E] := \overline{\mathrm{lat}\{\delta_E(x) \,:\, x \in E\}}^{\|\cdot\|_{\mathrm{FBL}^{(p)}[E]}} \subseteq H_p[E] = \{f \in H[E] \,:\, \|f\|_{\mathrm{FBL}^{(p)}[E]} < \infty\},$$

together with the linear isometric embedding $\delta_E : E \to \mathrm{FBL}^{(p)}[E]$ given by $\delta_E(x)(x^*) := x^*(x)$ is the free p-convex Banach lattice generated by E. Moreover, in the case $p = \infty$, it is pointed out in [119, Proposition 2.2] that with this procedure we obtain that $\mathrm{FBL}^{(\infty)}[E]$ is precisely $C_{ph}(B_{E^*})$, the space of positively homogeneous w^*-continuous functions on B_{E^*}, equipped with the supremum norm and the pointwise order.

For details of these constructions, see [77] or [119] (the latter reference also provides a comprehensive study of these objects).

1.3 Complex Banach lattices

In this section we recall some relevant properties and definitions regarding *complex Banach lattices* and also general complex Banach spaces. We refer the reader to [141, Chapter II, Section 11] or [1, Section 3.2] for further information on this topic.

Definition 1.20. A **complex Banach lattice** is the complexification $X_{\mathbb{C}} = X \oplus iX$ of a (real) Banach lattice X, *equipped with the norm* $\|x + iy\|_{X_{\mathbb{C}}} := \||x + iy|\|_X$, where $|\cdot| : X_{\mathbb{C}} \to X_+$ is *the modulus* map given by

$$|x + iy| = \sup_{\theta \in [0,2\pi]} \{x \cos\theta + y \sin\theta\}, \qquad \text{for every } x + iy \in X_{\mathbb{C}}. \tag{1.3}$$

It can be shown that $\sup_{\theta\in[0,2\pi]}\{x\cos\theta + y\sin\theta\} = \left(|x|^2 + |y|^2\right)^{\frac{1}{2}}$, for every $x, y \in X$ [109, pp. 42–43], so the above infinite supremum is well defined in any Banach lattice thanks to Krivine's functional calculus. It should also be noticed that the *Banach lattice complexification* of the real $C(K)$ (resp., $L_p(\Omega, \Sigma, \mu)$) coincides with the complex $C(K; \mathbb{C})$ (resp., the complex $L_p(\Omega, \Sigma, \mu; \mathbb{C})$) (see [1, Section 3.2, Exercises 3 and 5]).

In general, given a real Banach space E, if $E_{\mathbb{C}}$ denotes the complexification of the real vector space E, $E \oplus iE$, we will assume that this complex vector space is equipped with the norm

$$\|x + iy\| = \sup_{\theta \in [0,2\pi]} \|x \cos\theta + y \sin\theta\|, \qquad \text{for every } x + iy \in E_{\mathbb{C}}. \tag{1.4}$$

It is not difficult to check that the norm induced by (1.3) and the one in (1.4) are equivalent in the class of complex Banach lattices.

A complex subspace Y of a complex Banach lattice $X_{\mathbb{C}}$ is said to be a *complex sublattice* if $|x + iy| \in Y$ whenever $x + iy \in Y$. Equivalently, we can define a complex sublattice Y of $X_{\mathbb{C}}$ as the complexification of a real sublattice Z of X [135, Lemma 1.3]. Similarly, a complex subspace I of a complex Banach lattice $X_{\mathbb{C}}$ is said to be a *complex ideal* if whenever $|z| \le |w|$ and $w \in I$ imply that $z \in I$. Equivalently, a complex ideal I of $X_{\mathbb{C}}$ can be defined as the complexification $I = I_0 \oplus iI_0$ of a real ideal I_0 of X [1, Section 3.2, Ex. 7].

On the other hand, recall that for every $\mathbb{C}$-linear operator $T : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ between two complex Banach lattices $X_{\mathbb{C}}$, $Y_{\mathbb{C}}$ there exists a unique pair of operators $T_1, T_2 : X \to Y$ such that

$$T(x+iy) = (T_1 + iT_2)(x+iy) = T_1x - T_2y + i(T_2x + T_1y),$$

for every $x+iy \in X_{\mathbb{C}}$ (see, for example, [1, Section 1.1]). If $T_2 = 0$, that is, if $T(x+iy) = Sx + iSy$ for some operator $S : X \to Y$, we say that T is a *real operator* and that T is the *complexification* of S, written $T = S_{\mathbb{C}}$.

A real operator $T_{\mathbb{C}} = T + iT : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ is said to be *positive* (resp. a *lattice homomorphism*) if $T : X \to Y$ is positive (resp. a *lattice homomorphism*). Complex lattice homomorphisms may be defined also as those $\mathbb{C}$-linear operators which preserve the modulus, that is, $T|z| = |Tz|$ for every $z \in X_{\mathbb{C}}$.

For a complex Banach space E, its dual space E^* consists of all bounded $\mathbb{C}$-linear maps between E and $\mathbb{C}$. Any complex Banach space E can be seen as a real Banach space $E_{\mathbb{R}}$ if we restrict the scalar multiplication (of E) to the reals. Moreover, given $z^* \in E^*$ we can consider its real part $\mathfrak{Re}\, z^*$, which is an element of $(E_{\mathbb{R}})^*$ and $\|z^*\|_{E^*} = \|\mathfrak{Re}\, z^*\|_{(E_{\mathbb{R}})^*}$. Conversely, given $x^* \in (E_{\mathbb{R}})^*$, we can define

$$z^*(z) = x^*(z) - ix^*(iz), \qquad \text{for every } z \in E,$$

and $\|z^*\|_{E^*} = \|x^*\|_{(E_{\mathbb{R}})^*}$. The previous comments show that $(E^*)_{\mathbb{R}}$ and $(E_{\mathbb{R}})^*$ are linearly isometric (cf. [1, Theorem 1.9]).

1.4 Unconditional bases

Throughout this and the following section, all presented concepts and results are valid for *real and complex* Banach spaces. Recall that a sequence of elements $(e_n)_{n=1}^{\infty}$ in a Banach space E is said to be a *basis* of E if for each $x \in E$ there is a *unique* sequence of scalars $(a_n)_{n=1}^{\infty}$ such that

$$x = \sum_{n=1}^{\infty} a_n e_n,$$

where the above identity means that the sequence of partial sums $\left(\sum_{k=1}^{n} a_k e_k\right)_{n=1}^{\infty}$ converges to x with respect to the norm of X. It can be checked that for every natural n the map $x \mapsto a_n$ is a linear functional on E, which will be denoted by e_n^*. The functionals $(e_n^*)_{n=1}^{\infty}$ are called the *biorthogonal functionals* associated to $(e_n)_{n=1}^{\infty}$.

A sequence $(e_n)_{n=1}^{\infty}$ in a Banach space E is called a *basic sequence* if it is a basis for $\overline{\operatorname{span}}\{e_n : n \in \mathbb{N}\}$. A very useful criterion to recognize that a sequence in a Banach space is basic is the following [5, Proposition 1.1.9]:

Proposition 1.21. *A sequence $(e_n)_{n=1}^{\infty}$ of non-zero elements of a Banach space E is basic if and only if there is a positive constant K such that*

$$\left\| \sum_{k=1}^{m} a_k e_k \right\| \leq K \left\| \sum_{k=1}^{n} a_k e_k \right\|$$

for every sequence of scalars $(a_k)_{k=1}^{\infty}$ and all integers m, n such that $m \leq n$.

A well-known theorem due to Mazur states that *every infinite-dimensional Banach space contains a basic sequence*, that is, an infinite-dimensional subspace with a basis [5, Theorem 1.4.5].

Given a sequence $(x_n)_{n=1}^{\infty}$ in a Banach space E, we say that the (formal) series $\sum_{n=1}^{\infty} x_n$ is *unconditionally convergent* (or that $(x_n)_{n=1}^{\infty}$ is unconditionally summable) if $\sum_{n=1}^{\infty} x_{\pi(n)}$

converges for every permutation π of $\mathbb{N}$. In the following proposition, we present multiple alternative ways to recognize the unconditional convergence of a series (see [108, Proposition 1.c.1] or [44, Theorem 1.9]):

Proposition 1.22. *Let $(x_n)_{n=1}^\infty$ be a sequence of vectors in a Banach space E. Then, the following conditions are equivalent:*

i) The series $\sum_{n=1}^\infty x_n$ is unconditionally convergent;

ii) The series $\sum_{k=1}^\infty x_{n_k}$ converges for every strictly increasing sequence of natural numbers $(n_k)_{k=1}^\infty$.

iii) The series $\sum_{n=1}^\infty \varepsilon_n x_n$ converges for every choice of signs $(\varepsilon_n)_{n=1}^\infty$ (that is, $\varepsilon_n = \pm 1$ for every $n \in \mathbb{N}$).

iv) For every $\varepsilon > 0$ there exists a natural number n so that $\left\|\sum_{k\in F} x_k\right\| \leq \varepsilon$ for every finite set F of natural numbers which satisfies $\min F > n$.

Definition 1.23. A basis $(u_n)_{n=1}^\infty$ of a Banach space E is said to be **unconditional** if for every $x \in E$ the series $x = \sum_{n=1}^\infty u_n^*(x)u_n$ converges unconditionally, where $(u_n^*)_{n=1}^\infty$ are the biorthogonal functionals associated to $(u_n)_{n=1}^\infty$.

Proposition 1.24. *A basis $(u_n)_{n=1}^\infty$ of a Banach space E is unconditional if and only if there is a constant $K \geq 1$ such that for all $n \in \mathbb{N}$,*

$$\left\|\sum_{k=1}^n a_k u_k\right\| \leq K \left\|\sum_{k=1}^n b_k u_k\right\| \tag{1.5}$$

whenever $(a_k)_{k=1}^n$, $(b_k)_{k=1}^n$ are finite sequences of scalars satisfying $|a_k| \leq |b_k|$ for $k = 1, \ldots, n$.

The **unconditional basis constant** K_u of $(u_n)_{n=1}^\infty$ is the least constant K such that equation (1.5) holds. We say that $(u_n)_{n=1}^\infty$ is K-unconditional whenever $K \geq \mathsf{K}_u$.

A consequence of the preceding proposition is that for every sequence of signs $(\varepsilon_n)_{n=1}^\infty$, the operator $T_{(\varepsilon_n)} : E \to E$ defined by $T_{(\varepsilon_n)}\big(\sum_{n=1}^\infty a_n u_n\big) := \sum_{n=1}^\infty \varepsilon_n a_n u_n$ is an isomorphism. It can be checked that

$$\mathsf{K}_u = \sup\big\{\|T_{(\varepsilon_n)}\| \,:\, (\varepsilon_n)_{n=1}^\infty \text{ sequence of signs}\big\}.$$

Thus, if E has an unconditional basis $(u_n)_{n=1}^\infty$, then we can consider the norm

$$|||x||| = \left|\left|\left|\sum_{n=1}^\infty u_n^*(x)u_n\right|\right|\right| = \sup\left\{\left\|\sum_{n=1}^\infty \varepsilon_n u_n^*(x)u_n\right\| \,:\, (\varepsilon_n)_{n=1}^\infty \text{ sequence of signs}\right\},$$

which is equivalent to the original one and the unconditional basis constant of $(u_n)_{n=1}^\infty$ in $(X, |||\cdot|||)$ is 1. That is, every Banach space space with an unconditional basis is isomorphic to a space with a 1-unconditional basis. One of the reasons why the latter is relevant is because spaces with a 1-unconditional basis are Banach lattices under the coordinate-wise under determined by its biorthogonal functionals.

Proposition 1.25. *Suppose that X is a Banach space which has a 1-unconditional basis $(u_n)_{n=1}^\infty$. Then X endowed with the coordinate-wise order determined by the biorthogonal functionals $(u_n^*)_{n=1}^\infty$ associated to $(u_n)_{n=1}^\infty$ is an order continuous Banach lattice.*

Proof. The fact that X is a Banach lattice under the order $x \leq y$ if $u_n^*(x) \leq u_n^*(y)$ for every $n \in \mathbb{N}$ is an immediate consequence of Proposition 1.24; note that the modulus is given by $|x| = \sum_{n=1}^{\infty} |u_n^*(x)| u_n$. It remains to show that X is order continuous. Let $(x_\alpha)_\alpha$ be a decreasing net such that $\inf_\alpha x_\alpha = 0$. Fix any element x_{α_0} of the net. By Proposition 1.22, given $\varepsilon > 0$, there exists a natural number n so that $\left\| \sum_{k \in F} u_k^*(x_{\alpha_0}) u_k \right\| \leq \frac{\varepsilon}{2}$ for every finite subset of $\{n+1, n+2, \ldots\}$. Note that since X is a Banach lattice and $(x_\alpha)_\alpha$ is decreasing, the last assertion also holds for every x_α with $\alpha \geq \alpha_0$.

On the other hand, given that $\inf_\alpha x_\alpha = 0$, for every $k = 1, \ldots, n$ there exists x_{α_k} such that $|u_k^*(x_{\alpha_k})| \leq \frac{\varepsilon}{2n}$. Let α_N be such that $\alpha_N \geq \alpha_0, \alpha_1, \ldots, \alpha_n$. It is clear that $\|x_\alpha\| \leq \varepsilon$ for every $\alpha \geq \alpha_N$ and, by the arbitrariness of $\varepsilon > 0$, this shows that the norm is order continuous. □

We say that a sequence of non-zero vectors $(x_n)_{n=1}^{\infty}$ in a Banach lattice X is *disjoint* if for every distinct $j, k \in \mathbb{N}$, $|x_j| \wedge |x_k| = 0$. It can be checked that if $(x_k)_{k=1}^{n}$ is disjoint, then

$$\left| \sum_{k=1}^{n} x_k \right| = \sum_{k=1}^{n} |x_k| = \left| \sum_{k=1}^{n} \varepsilon_k x_k \right|,$$

where $\varepsilon_k = \pm 1$ for every $k = 1, \ldots, n$. By Propositions 1.21 and 1.24, the latter implies that every disjoint sequence $(x_n)_{n=1}^{\infty} \subseteq X$ is a 1-unconditional basic sequence in X. Since every infinite-dimensional Banach lattice has an infinite disjoint sequence (see, for instance, [94, p. 9, Theorem 11]), *every Banach lattice has an unconditional basic sequence*. However, ***not*** *every Banach space has an unconditional basic sequence*: the first example showing this was given by Gowers and Maurey in 1993 [67].

1.5 The $\mathcal{L}_p$-spaces

The $\mathcal{L}_p$-spaces, for $1 \leq p \leq \infty$, are a class of Banach spaces that *locally resemble* L_p-spaces, in the sense that their finite-dimensional subspaces are close to those of an L_p-space. These spaces were introduced by Lindenstrauss and Pełczyński in 1968 [103] and have played a fundamental role in the study of summing operators (also called absolutely summing operators), as can be seen in [44]. Let us recall the definition of this notion:

Definition 1.26. Given $1 \leq p \leq \infty$ and $\lambda \geq 1$, a Banach space E is said to be an $\mathcal{L}_{p,\lambda}$-*space* if for every finite-dimensional subspace B of E there is a finite dimensional subspace C of E such that $B \subseteq C$ and C is λ-isomorphic to ℓ_p^n, where $n = \dim C$. We say that a Banach space E is an $\mathcal{L}_p$-*space* if it is an $\mathcal{L}_{p,\lambda}$-space for some λ. We will write $\mathcal{L}_{p,1^+}$ to denote that a Banach space is an $\mathcal{L}_{p,1+\varepsilon}$-space for every $\varepsilon > 0$.

It can be checked that any L_p-space, for $1 \leq p \leq \infty$, is an $\mathcal{L}_{p,1^+}$-space; similarly, every $C(K)$-space is an $\mathcal{L}_{\infty,1^+}$-space [44, Theorem 3.2]. We will now gather some of the most relevant properties of these spaces. All the results collected in the following theorem were proven in [103, Section 7].

Theorem 1.27 (Lindenstrauss-Pełczyński, 1968). *Let $1 \leq p \leq \infty$ and let E be an $\mathcal{L}_p$-space. Then:*

(i) E is isomorphic to a subspace of an L_p-space.

(ii) *E is isomorphic to a complemented subspace of an L_p-space if and only if E is complemented in E^{**}. Note that this always holds if $1 < p < \infty$, as in these cases E is reflexive by (i).*

(iii) *If $1 \leq p < \infty$, E has a complemented subspace isomorphic to ℓ_p.*

In view of the third condition, it is natural to ask whether every $\mathcal{L}_\infty$-space contains a complemented subspace isomorphic to c_0. However, Bourgain and Delbaen in [36, Section 5] built an example (actually, a *family* of examples) of an $\mathcal{L}_\infty$-space which does not contain isomorphic copies of c_0.

The following results from [105] complement those of the previous theorem and also provide very useful information about the structure of $\mathcal{L}_p$-spaces.

Theorem 1.28 (Lindenstrauss-Rosenthal, 1969).

(i) *A Banach space E is an $\mathcal{L}_p$-space ($1 \leq p \leq \infty$) if and only if E^* is an $\mathcal{L}_q$-space (where $\frac{1}{p} + \frac{1}{q} = 1$).*

(ii) *If $p \in \{1, 2, \infty\}$, every complemented subspace of an $\mathcal{L}_p$-space is also an $\mathcal{L}_p$-space. For $1 < p \neq 2 < \infty$ we have that every complemented subspace of $\mathcal{L}_p$-space which is not isomorphic to a Hilbert space is isomorphic to an $\mathcal{L}_p$-space.*

(iii) *Let E be an $\mathcal{L}_p$-space, $1 \leq p \leq \infty$. Then there is a constant ρ such that for every finite-dimensional subspace B of E there is a further finite-dimensional subspace C of E such that $C \supset B$, C is ρ-isomorphic to ℓ_p^n (where $n = \dim C$), and such that there is a projection of norm $\leq \rho$ from E onto C.*

Chapter 2
The Complemented Subspace Problem

We survey classical and recent developments on the structure of complemented subspaces of Banach lattices. Several natural questions and directions of future research are presented. We provide an approach to some of these problems using tools from the theory of free Banach lattices. This chapter is based on the article:

[72] D. de Hevia and P. Tradacete, *Complemented subspaces of Banach lattices*, Banach J. Math. Anal. **19** (2025), no. 4, Paper No. 60. MR 4940175

2.1 General overview of the Problem

Given a class $\mathcal{C}$ of Banach spaces, we refer to the following question as *the Complemented Subspace Problem for* $\mathcal{C}$ (CSP for $\mathcal{C}$, for short): is every complemented subspace of a member of $\mathcal{C}$ isomorphic to another member of $\mathcal{C}$? The question we are primarily interested in is when $\mathcal{C}$ is the class of Banach lattices, that is, *the CSP for Banach lattices.* Since we will mention this problem numerous times, we will abbreviate it further by simply calling it *the CSP* instead of the CSP for Banach lattices.

Question 2.1 (CSP). Is every complemented subspace of a Banach lattice isomorphic to a Banach lattice?

As we already pointed out in the introduction, this problem was first mentioned in a 1987 article by Casazza, Kalton, and Tzafriri, where they refer to it as *One of the most important problems in the theory of Banach lattices, which is still open.* This suggests that this problem was already well known and had been under study earlier. This would not be surprising, as understanding how one can decompose a space into a direct sum is a fundamental question within the theory of Banach spaces. An illustration of this latter point is the intense study *the complemented subspace problem* has undergone for the classes of L_1-spaces and $C(K)$-spaces. The following two problems were posed in 1960 by Pełczyński [122].

Question 2.2 (CSP for $C(K)$-spaces). Is every complemented subspace of a $C(K)$-spaces isomorphic to a $C(K)$-space?

Question 2.3 (CSP for L_1-spaces). Is every complemented subspace of an L_1-space isomorphic to an L_1-space?

While the CSP for L_1-spaces is still open, *the CSP for $C(K)$-spaces has recently been answered in the negative by Plebanek and Salguero-Alarcón* [131]: a 1-complemented *non-separable* subspace –denoted by PS_2– of a $C(K)$-space is exhibited which is not isomorphic to any $C(K)$-space. Shortly thereafter, Martínez-Cervantes, Salguero-Alarcón, Tradacete and the author of this memoir have proved in [70] that PS_2 is, in fact, not isomorphic to any Banach lattice, showing *that CSP (for Banach lattices) also has a* ***negative solution***. This example will be explained in detail in the next chapter.

Questions 2.2 and 2.3 are of particular relevance in the *separable case*. In this realm, it is conjectured that every complemented subspace of $C[0,1]$ is isomorphic to a $C(K)$-space and that every complemented subspace of $L_1[0,1]$ is isomorphic to ℓ_1 or to $L_1[0,1]$ (see the dicussion at the end of Chapter 5 of [5]).

The *CSP for Banach lattices* presents an *additional difficulty* compared to these particular cases, which may account for its late mention in the literature: before attempting to distinguish between Banach lattices and their complemented subspaces, we must first ensure that *there actually exist Banach spaces not isomorphic to lattices*; and this is by no means trivial. Indeed, the first examples of Banach spaces that cannot be isomorphic to Banach lattices appeared in the 1970s. The next section gathers some isomorphic properties characteristic of Banach lattices, setting them apart from general Banach spaces.

2.1.1. Banach space properties of Banach lattices. The following results were first proven by James for *spaces with an unconditional basis* in 1951 [74] and they were later extended by Bessaga and Pełczyński to *subspaces* of spaces with an unconditional basis in 1958 [26]. Subsequently, Tzafriri generalized these results to *subspaces of order continuous Banach lattices*, and this is the version we present below [153, Theorem 13 and Theorem 18]. Recall that a Banach space E is *weakly sequentially complete* if for every sequence $(x_n)_{n=1}^\infty \subseteq E$ such that $\big(x^*(x_n)\big)_{n=1}^\infty$ is convergent for every $x^* \in E^*$, then $(x_n)_{n=1}^\infty$ is weakly convergent in E (that is, there exists $x \in E$ such that $x^*(x_n) \to x^*(x)$ for every $x^* \in E^*$).

Theorem 2.4 (Tzafriri, 1972). *Let X be an order continuous Banach lattice and let E be a subspace of X. Then:*

(i) E is weakly sequentially complete if and only if no subspace of E is isomorphic to c_0.

(ii) E is reflexive if and only if no subspace of E is isomorphic to either c_0 or ℓ_1.

Example 2.5. The James space $\mathcal{J}$ introduced in [75] contains no subspaces isomorphic to c_0 or ℓ_1. However, $\mathcal{J}$ is neither weakly sequentially complete nor reflexive. In [36], Bourgain and Delbaen construct examples of $\mathcal{L}_\infty$-spaces that contain no isomorphic copies of c_0 or ℓ_1, and are not reflexive either.

Another well-known result concerning subspaces of spaces with an unconditional basis is that they are *saturated by unconditional basic sequences*. Indeed, let F be a subspace of a space E which has an unconditional basis $(u_n)_{n=1}^\infty$. By [108, Proposition 1.a.11], E contains a basic sequence which is equivalent to a block basis of $(u_n)_{n=1}^\infty$, so that basic sequence is actually unconditional. Similar to the preceding theorem, this result can be extended to subspaces of order continuous lattices [55, Theorem 4.1].

Theorem 2.6 (Figiel, Johnson, and Tzafriri, 1975). *If X is an order continuous Banach lattice, then every subspace of it has an unconditional basic sequence.*

Example 2.7. The first example of a Banach space without unconditional basic sequences was given by Gowers and Maurey in 1993 [67]. In fact, the example they constructed has the stronger property of being *hereditarily indecomposable* (H.I., for short).

In [101], Lindenstrauss asks whether every Banach space contains uniformly complemented copies of $(\ell_p^n)_{n=1}^\infty$ for some $p = 1, 2, \infty$. That is, for every infinite-dimensional Banach space X, do there exist a constant λ and $p \in \{1, 2, \infty\}$ such that for every natural number n, there is a finite-dimensional subspace B of X with $d(B, \ell_p^n) \leq \lambda$ and a projection of norm $\leq \lambda$ from X onto B? In 1975, Johnson and Tzafriri showed that this question has a positive response for Banach lattices [82, Corollary 1]. In fact, they proved the following:

Theorem 2.8 (Johnson and Tzafriri, 1975). *Let X be a Banach lattice which does not contain uniformly (complemented) $(\ell_\infty^n)_{n=1}^\infty$. If E is a subspace of X, then E contains uniformly complemented $(\ell_1^n)_{n=1}^\infty$ or $(\ell_2^n)_{n=1}^\infty$.*

Example 2.9. In [128], Pisier produces a separable Banach space E with the property that there is a constant $\delta > 0$ such that all finite rank projections $P : E \to E$ satisfy $\|P\| \geq \delta\,(\operatorname{rank} P)^{\frac{1}{2}}$, where rank stands for the dimension of $P(E)$. In particular, this space cannot uniformly complemented $(\ell_p^n)_{n=1}^\infty$ for any $1 \leq p \leq \infty$.

Remark 2.10. In Corollary 2.26, we will show that Theorems 2.4, 2.6 and 2.8 are **valid** for **complemented subspaces of Banach lattices**. Therefore, in particular, the examples 2.5, 2.7 and 2.9 are Banach spaces which cannot be isomorphic to complemented subspaces of Banach lattices. Thus, the criteria we have provided so far *do not allow us to distinguish Banach lattices from their complemented subspaces.*

On the other hand, two notions of *local unconditional structure*, which generalize that of $\mathcal{L}_p$-space, **DPR-lust** (introduced in [47]) and **GL-lust** (introduced in [64]), have proven useful for providing examples of Banach spaces not isomorphic to Banach lattices. For the sake of completeness, we recall the definition of these concepts below.

A Banach space E is said to have *local unconditional structure in the sense of Dubinsky, Pełczyński and Rosenthal with constant* $\lambda \geq 1$ (λ-DPR-lust, for short) if for every finite-dimensional subspace B of E there is a finite-dimensional subspace C of E such that $C \supseteq B$ and C has an unconditional basis with constant $\mathsf{K}_u \leq \lambda$. Equivalently, E has λ-DPR-lust if there are $\lambda \geq 1$ and an upwards directed family under set inclusion $(E_\gamma)_\gamma \in \Gamma$ of finite dimensional subspaces of E such that $E = \cup_{\gamma \in \Gamma} E_\gamma$ and each E_γ admits a λ-unconditional basis. We say that a Banach space has *DPR-lust* if it has λ-DPR-lust for some $\lambda \geq 1$.

On the other hand, a Banach space E has *local unconditional structure in the sense of Gordon-Lewis* with constant $\lambda \geq 1$ (λ-GL-lust, for short) if for every finite-dimensional subspace B of E there is a space C with an unconditional basis and operators $T : B \to C$ and $S : C \to E$ such that $ST = \iota_B$ (where $\iota_B : B \hookrightarrow E$ being the canonical inclusion of B as a subspace) and $\|S\|\|T\|\mathsf{K}_u \leq \lambda$, where K_u is the unconditional constant of a basis of C. As one might expect, a Banach space is said to have *GL-lust* if it has λ-GL-lust for some λ. It is clear that λ-DPR-lust implies λ-GL-lust.

It can be checked that *every Banach lattice has λ-DPR-lust for every $\lambda > 1$* (see, for instance, [38, Theorem 6.4]). However, the converse of this is not true: there exists a separable Banach space which is $\mathcal{L}_{\infty,1^+}$ (and hence, in particular, has λ-DPR-lust for every $\lambda > 1$), but is not isomorphic to a Banach lattice (see the remark after [25, Proposition 5.27]). This contrasts with the fact that for $1 \leq p < \infty$, being $\mathcal{L}_{p,1^+}$ implies being isometric to an L_p-space [152, Corollary 7].

Both notions of local unconditional structure are preserved by isomorphisms and, moreover, *GL-lust* is *inherited by complemented subspaces*; in fact, it can be shown that a Banach space E has GL-lust if and only if E^{**} is complemented in a Banach lattice (see [55, Remark 2.3] or [95, Theorem 2]). In contrast, it remains unknown whether every complemented subspace in a Banach lattice has DPR-lust (the *main conjecture* in [55] is that this is true), and it is not even known if these two properties could be equivalent. The most remarkable result in this direction is that X GL-lust if and only if $X \oplus c_0$ has DPR-lust [95, p. 49]. So, similar to the first results discussed in this section, we cannot use these properties to distinguish Banach lattices from their complemented subspaces either. For more information on local unconditional structures, we refer the reader to [44, Chapter 17], [149, Section 34] and [95].

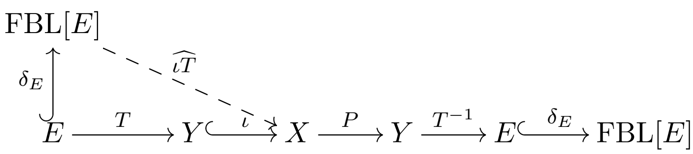
Among the examples that fail these properties (and thus cannot be isomorphic to Banach lattices), we can find the Schatten p-classes $c_p(\ell_2)$, for $1 \leq p \neq 2 \leq \infty$ [64, Theorem 5.1], the space of bounded holomorphic functions on the disk $\mathcal{H}^\infty(\mathbb{D})$ [123], the James space $\mathcal{J}$ again [95, Theorem 8], the Kalton-Peck space introduced in [88] (see [81]), or certain Sobolev spaces and spaces of smooth functions [124, 125, 151]. Several other isomorphic properties of Banach lattices (and, often, their complemented subspaces) can be found in [25, 38, 55, 61, 78, 95, 153].

2.2 Connections of the CSP with free Banach lattices

Free Banach lattices have a close connection with the Complemented Subspace Problem for Banach lattices. They serve as a canonical object to study this question in the sense that if a Banach space E is complemented in some Banach lattice, then it must be complemented in the corresponding $\mathrm{FBL}[E]$. The following proposition pinpoints this idea. An analogous result for $C(K)$-spaces can be found in [23, Lemma, p. 247].

Proposition 2.11. *Let E be a Banach space. If E is C_1-isomorphic to a C_2-complemented subspace of a Banach lattice, then $\delta(E)$ is C_1C_2-complemented in FBL$[E]$*

Proof. By hypothesis, there exist a subspace Y of a Banach lattice X, an isomorphism $T : E \to Y$ such that $\|T\|\|T^{-1}\| \leq C_1$ and a projection $P : X \to X$ onto Y such that $\|P\| \leq C_2$. By the universal property of $\mathrm{FBL}[E]$, there is a unique lattice homomorphism $\widehat{\iota T} : \mathrm{FBL}[E] \to X$ which extends ιT (where ι stands for the natural inclusion of Y into X as a subspace) and $\|\widehat{\iota T}\| = \|\iota T\| = \|T\|$. We have the following diagram:



Define $Q = \delta_E T^{-1} P \widehat{\iota T}$. Observe that $Q(\mathrm{FBL}[E]) \subseteq \delta_E(E)$ and $Q^2 = Q$. In addition,

$$Q\delta_E(x) = \delta_E T^{-1} P \widehat{\iota T} \delta_E(x) = \delta_E T^{-1} P \iota T(x) = \delta_E T^{-1} T(x) = \delta_E(x), \text{ for } x \in E,$$

so $Q(\mathrm{FBL}[E]) = \delta_E(E)$. Finally, note that

$$\|Q\| = \|\delta_E T^{-1} P \widehat{\iota T}\| = \|T^{-1} P \widehat{\iota T}\| \leq \|T^{-1}\|\|P\|\|\widehat{\iota T}\| = \|T^{-1}\|\|P\|\|T\| \leq C_1 C_2.$$

□

Remark 2.12. The preceding argument can be replicated to obtain analogous results for other free objects. In particular, recall Example 1.14, which shows that the bidual of a Banach space is the *free dual Banach space generated by it*. By mimicking the above proof, it can be proven that if a Banach space E is C_1-isomorphic to a C_2-complemented subspace of a dual Banach space, then $J(E)$ is C_1C_2-complemented in E^{**} (where J stands for the canonical embedding of E into its bidual). This is a well-known observation that can be found, for instance, in [98, Remark, p.16].

In fact, using free Banach lattices again, we can identify exactly when a complemented subspace of Banach lattice is actually isomorphic to a Banach lattice:

Proposition 2.13. *Given a Banach space E, then E is isomorphic to a Banach lattice if and only if there is an ideal I in $FBL[E]$ such that $FBL[E] = \delta_E(E) \oplus I$. More precisely, if $T : E \to X$ is an isomorphism, then $FBL[E] = \delta_E(E) \oplus ker(\widehat{T})$.*

Proof. Let X be a Banach lattice and suppose that there exists an isomorphism $T : E \to X$. By the universal property of $\mathrm{FBL}[E]$, there is a lattice homomorphism $\widehat{\mathrm{T}} : \mathrm{FBL}[E] \to X$ such that $\widehat{\mathrm{T}} \circ \delta_E = T$. Therefore, $P = \delta_E T^{-1}\widehat{T}$ defines a projection on $\mathrm{FBL}[E]$ with range $\delta_E(E)$ and $\ker(P) = \ker(\widehat{\mathrm{T}})$ is an ideal I in $\mathrm{FBL}[E]$. We have that $\mathrm{FBL}[E] = \delta_E(E) \oplus I$.

To prove the converse implication, recall that the quotient of a Banach lattice over a norm-closed ideal is a Banach lattice by [115, Corollary 1.3.14]. Thus, if we have a decomposition $\mathrm{FBL}[E] = \delta_E(E) \oplus I$, then $\mathrm{FBL}[E]/I$ is a Banach lattice isomorphic to $\delta_E(E)$ and therefore also to E. □

Remark 2.14. By just mimicking the above proof, one may show that a Banach space E is isomorphic to a p-convex Banach lattice if and only if there is an ideal in $\mathrm{FBL}^{(p)}[E]$ such that $\mathrm{FBL}^{(p)}[E] = \delta_E(E) \oplus I$ (for any $1 \le p \le \infty$). In particular, E is isomorphic to an AM-space if and only if $C_{ph}(B_{E^*}) = \mathrm{FBL}^{(\infty)}[E] = \delta_E(E) \oplus I$.

Consequently, at least at a theoretical level, free Banach lattices provide a possible criterion for distinguishing between Banach lattices and their complemented subspaces. In order to be able to use Proposition 2.13, a natural first step would be to understand how closed ideals in free Banach lattices look like. It is well known that ideals in $C(K)$-spaces are *sets of zeros*, that is, given an ideal $I \subseteq C(K)$, there is a closed subset $F \subseteq K$ such that $I = \{f \in C(K) : f(t) = 0 \text{ for all } t \in F\}$ ([115, Proposition 2.1.9]). We will show in the next proposition that *the same* holds for $\mathrm{FBL}^{(\infty)}[E]$.

Given a Banach space E, we will say that a set $K \subseteq B_{E^*}$ is *positively homogeneous* in B_{E^*} if $K = \mathbb{R}_+K \cap B_{E^*}$, that is, if $0 \in K$ and if $0 \neq x^* \in K$, then $\lambda x^* \in K$ for every $\lambda \in \big(0, \frac{1}{\|x^*\|}\big]$. Throughout this section, if K is a positively homogeneous set in B_{E^*}, then Z_K represents the set of functions of $\mathrm{FBL}^{\infty}[E]$ (or $\mathrm{FBL}[E]$) which vanish on K.

Let us start with a simple (and probably well-known) lemma for which we have not found a precise reference, so we include its proof below.

Lemma 2.15. *Let X be an AM-space and $I \subseteq X$ a (closed) ideal. Then X/I is an AM-space.*

Proof. Recall that by [115, Corollary 1.3.14] we already know that X/I is a Banach lattice, so we simply have to check that $\|\overline{x}\vee\overline{y}\| = \max\{\|\overline{x}\|, \|\overline{y}\|\}$ for every pair of positive elements $\overline{x}, \overline{y} \in X/I$. Let us fix $\varepsilon > 0$ and $\overline{x}, \overline{y} \in \big(X/I\big)_+$. Then, there exist $z, w \in I$ such that $\|x + z\| \le \|\overline{x}\| + \varepsilon$ and $\|y + w\| \le \|\overline{y}\| + \varepsilon$. Thus, we have that

$$\varepsilon+\max\{\|\overline{x}\|,\|\overline{y}\|\} \geq \max\{\|x+z\|,\|y+w\|\} = \big\||x+z|\vee|y+w|\big\| \geq \big\|\overline{|x+z|}\vee\overline{|y+w|}\big\|. \quad (2.1)$$

Now, observe that $\big||x+z|\vee|y+w|-|x|\vee|y|\big| \leq |z|+|w|$ and, since $|w|$ and $|z|$ are in I, we conclude that $|x+z|\vee|y+w|-|x|\vee|y| \in I$. Moreover, given that $\overline{x}$ and $\overline{y}$ are positive elements, we have that $|x+z|\vee|y+w|-x\vee y \in I$. Thus, from equation (2.1) we get that

$$\varepsilon+\max\{\|\overline{x}\|,\|\overline{y}\|\} \geq \big\|\overline{|x+z|}\vee\overline{|y+w|}\big\| = \|\overline{x}\vee\overline{y}\|,$$

and since $\varepsilon>0$, $\overline{x}$ and $\overline{y}$ were arbitrarily chosen, we conclude that X/I is an AM-space. □

Proposition 2.16. *Let E be a Banach space. Given an ideal I in $FBL^{(\infty)}(E) = C_{ph}(B_{E^*})$, there exists a w^*-closed positively homogeneous subset K of B_{E^*} such that $I = \{f \in C_{ph}(B_{E^*}) \ : \ f|_K = 0\}$. Moreover, the operator $T : C_{ph}(B_{E^*})/I \to C_{ph}(K)$ defined by $T\overline{f} := f|_K$ is a (surjective) lattice isometry.*

Proof. Define $K := \{x^* \in B_{E^*} \ : \ f(x^*) = 0 \text{ for all } f \in I\}$. Observe that by the w^*-continuity on B_{E^*} of the elements of $I \subseteq C_{ph}(B_{E^*})$, K is a w^*-closed subset of B_{E^*} and also, by the positive homogeneity of these functions, if $x^* \in K$, then $\lambda\cdot x^* \in K$ for every $\lambda \in \big[0, \frac{1}{\|x^*\|}\big]$. Let us write $Z_K := \{f \in C_{ph}(B_{E^*}) \ : \ f|_K = 0\}$.

It is clear that $I \subseteq I_K$, so let us see the reverse inclusion. Suppose that $f \notin I$, hence $\bar{f} \in C_{ph}(B_{E^*})/I$ is non-zero. By the previous lemma, we know that $C_{ph}(B_{E^*})/I$ is an AM-space, so there exists a norm-one lattice homomorphism $\overline{z^*}$ on $C_{ph}(B_{E^*})/I$ such that $\overline{z^*}(\bar{f}) \neq 0$ (see, for instance, [29, Proposition 5.4]). Now, denote by $Q : C_{ph}(B_{E^*}) \to C_{ph}(B_{E^*})/I$ the canonical quotient map $Qg := \bar{g}$, which is a lattice homomorphism [115, Proposition 1.3.13]. Therefore, $\overline{z^*}\circ Q \in B_{C_{ph}(B_{E^*})^*}$ is a lattice homomorphism and so there is $x^* \in B_{E^*}$ such that $\widehat{x^*} = \overline{z^*}\circ Q$. Note that $x^* \in K$, because for every $g \in I$ we have

$$g(x^*) = \widehat{x^*}(g) = \overline{z^*}(Qg) = \overline{z^*}(\bar{0}) = 0.$$

On the other hand, $f(x^*) = \overline{z^*}(\bar{f}) \neq 0$, so $f \notin Z_K$.

For the last statement of the proposition, let us consider the mapping $T : C_{ph}(B_{E^*})/I \to C_{ph}(K)$ defined by $T\bar{f} := f|_K$. It is clear that T is well defined (since $I = Z_K$) and that it is a lattice homomorphism. Let us check its surjectivity first. Recall that, by [29, Proposition 5.4], $C_{ph}(B_{E^*})/I$ can be identified with $C_{ph}\Big(K_{C_{ph}(B_{E^*})/I}\Big)$ in a lattice isometric way, where $K_{C_{ph}(B_{E^*})/I} = \mathrm{Hom}\big(C_{ph}(B_{E^*})/I\big) \cap B_{C_{ph}(B_{E^*})/I}$. Given $f \in C_{ph}(K)$, we define $\overline{f} : K_{C_{ph}(B_{E^*})/I} \to \mathbb{R}$ by $\overline{f}(\overline{z^*}) := f(Q^*\overline{z^*})$, for $\overline{z^*} \in K_{C_{ph}(B_{E^*})/I}$. Observe that $Q^*\overline{z^*} = \overline{z^*}\circ Q$ is a lattice homomorphism on $C_{ph}(B_{E^*})$ which is zero on I and hence there exists a unique $x^* \in K$ such that $\widehat{x^*} = Q\overline{z^*}$. Therefore, $\overline{f}$ is well defined and it is easy to see that it is w^*-continuous positively homogeneous on $K_{C_{ph}(B_{E^*})/I}$, so $\overline{f} \in C_{ph}(B_{E^*})/I$. Finally, note that by construction $T\overline{f} = f$.

It remains to show that T is norm-preserving. Fix $\bar{f} \in C_{ph}(B_{E^*})/I$ with $\|\bar{f}\| = 1$. Given that $C_{ph}(B_{E^*})/I$ is an AM-space, there exists a norm-one lattice homomorphism $\overline{z^*}$ on $C_{ph}(B_{E^*})/I$ such that $\overline{z^*}(\bar{f}) = 1$. We have shown above that there is $x^* \in K$ such that $\widehat{x^*} = Q^*\overline{z^*}$, so we have

$$1 = \|\bar{f}\| = \overline{z^*}(\bar{f}) = Q^*\overline{z^*}(f) = f(x^*) \leq \sup_{y^*\in K} |f(y^*)| = \|\, f|_K\,\|.$$

Now, suppose that $f(x^*) > 1$ for some $x^* \in K$. Since $\|\bar{f}\| = 1$, there exists $g \in I$ such that $f(x^*) > \|f+g\|$. Therefore, we have

$$f(x^*) > \|f+g\| = \sup_{y^*\in B_{E^*}} |f(y^*)+g(y^*)| \geq |f(x^*)|,$$

which is absurd. □

As a consequence of the previous proposition, we can obtain the following characterization of Banach spaces isomorphic to AM-spaces; this should be compared with a very similar and already known characterization of Banach spaces isomorphic to $C(K)$-spaces (see, for instance, [130, Lemma 2.2]).

Corollary 2.17. *A Banach space E is isomorphic to an AM-space if and only if there exists a w^*-closed subset K of B_{E^*} such that K is norming for E and for every $x \in E$, there exists $y \in E$ such that $x^*(y) = |x^*(x)|$ for every $x^* \in K$.*

Proof. Suppose that E is isomorphic to an AM-space X and let $T : E \to X$ be an isomorphism. By the universal property of $\mathrm{FBL}^{(\infty)}[E] = C_{ph}(B_{E^*})$, there exists a unique lattice homomorphism $\widehat{T} : C_{ph}(B_{E^*}) \to X$ such that $\widehat{T}\delta_E = T$. As $\ker(\widehat{T})$ is an ideal in $C_{ph}(B_{E^*})$, by the previous proposition there is w^*-closed positively homogeneous subset K of B_{E^*} such that $\bar{f} \in C_{ph}(B_{E^*})/\ker(\widehat{T}) \mapsto f|_K \in C_{ph}(K)$ defines a surjective lattice isometry. Therefore, the mapping $x \in E \mapsto \delta_x|_K \in C_{ph}(K)$ is an isomorphism, so K satisfies the desired conditions.

Conversely, let K be a w^*-closed subset of B_{E^*} which is norming for E and for every $x \in E$ we can find $y \in E$ such that $x^*(y) = |x^*(x)|$ for all $x^* \in K$. Let us define $T : E \to C(K)$ by $Tx := \delta_x$, where $\delta_x(x^*) := x^*(x)$ for $x^* \in K$. Since K is norming, we deduce that T is bounded below, so T is an isomorphism onto its range. Moreover, note that $T(X)$ is a sublattice of $C(K)$ since, by hypothesis, for every $x \in X$, there is $y \in X$ such that $Ty = |\delta_x|$. Therefore, $T(X)$ is an AM-space. □

However, the description of ideals in free Banach lattices seems more complicated than in $C(K)$-spaces. We will now see that ideals in free Banach lattices need not be *sets of zeros* (in the sense of Proposition 2.16); in fact, an ideal in $\mathrm{FBL}[E]$ can have no zeros. To illustrate this circumstance, we will use the following simple observation:

Proposition 2.18. *Let X be a Banach lattice. Then, $x^* \in X^*$ is a lattice homomorphism if and only if $ker(\beta) \subseteq ker(\widehat{x^*})$, where $\beta : FBL[X] \to X$ is the unique lattice homomorphism such that $\beta \circ \delta_X = id_X$.*

Proof. Suppose that $x^* : X \to \mathbb{R}$ is a lattice homomorphism. Then, $x^* \circ \beta$ and $\widehat{x^*}$ are lattice homomorphisms on $\mathrm{FBL}[X]$ which clearly agree on $\delta_X(X)$. By uniqueness of the extension, they actually coincide on $\mathrm{FBL}[X]$, so $\widehat{x^*} = x^* \circ \beta$. This shows that $\ker(\beta) \subseteq \ker(\widehat{x^*})$.

Now, assume that $\ker(\beta) \subseteq \ker(\widehat{x^*})$. Then the functional $\overline{x^*} : \mathrm{FBL}[X]/\ker(\beta) \to \mathbb{R}$, given by $\overline{x^*}(\bar{f}) := f(x^*)$, is a well-defined lattice homomorphism on $\mathrm{FBL}[X]/\ker(\beta)$. On the other hand, note that the mapping $\bar{\beta} : \mathrm{FBL}[X]/\ker(\beta) \to X$ defined by $\bar{\beta}(\bar{f}) = \beta f$ is a bijective lattice homomorphism and it is not difficult to determine its inverse explicitly: the composition $Q\delta_X : X \to \mathrm{FBL}[X]/\ker(\beta)$, where $Q : \mathrm{FBL}[X] \to \mathrm{FBL}[X]/\ker(\beta)$ stands for the canonical quotient defined by $Qf := \bar{f}$. Indeed, for every $x \in X$, we have

$$\bar{\beta}Q\delta_X(x) = \bar{\beta}(\overline{\delta_X(x)}) = \beta\delta_X(x) = x.$$

Also, for $f \in \mathrm{FBL}[X]$, we have

$$Q\delta_X\bar{\beta}(\bar{f}) = Q\delta_X\beta(f) = Q(f) = \bar{f},$$

where the second-to-last identity follows from the fact that $\beta\delta_X = id_X$ and $ker(Q) = ker(\beta)$. Thus, $Q\delta_X$ is also a lattice homomorphism, so $\overline{x^*} \circ Q\delta_X$ is a lattice homomorphism on X. But observe that

$$\overline{x^*} \circ Q\delta_X(x) = \overline{x^*}\big(\overline{\delta_X(x)}\big) = \delta_X(x)(x^*) = x^*(x), \quad \text{for } x \in X,$$

so $x^* = \overline{x^*} \circ Q\delta_X$ and this shows that x^* is a lattice homomorphism. □

Consequently, if we take a Banach lattice X which does not have lattice homomorphisms (for example, $L_p[0,1]$, for any $1 \leq p < \infty$), then $\ker(\beta)$ is an ideal in $\mathrm{FBL}[X]$ which does not have zeros (in the sense that for every non-zero $x^* \in B_{X^*}$ we can find $f \in \ker(\beta)$ such that $f(x^*) \neq 0$). Comparing the previous proposition with Proposition 2.16 it is natural to wonder whether having a decomposition $\mathrm{FBL}[X] = \delta_X(X) \oplus \ker(\beta)$, with $\ker(\beta)$ being a set of zeros, implies that X is an AM-space. This does not necessarily have to be the case, as the following shows.

Proposition 2.19. *Given a Banach lattice X, $Hom(X,\mathbb{R})$ separates the points of X if and only if there exists a w^*-closed positively homogeneous subset K of B_{X^*} such that $ker(\beta) = \{f \in FBL[X] \,:\, f|_K = 0\}$. In this situation $K = Hom(X,\mathbb{R}) \cap B_{X^*}$.*

Proof. Let us assume that $\mathrm{Hom}(X,\mathbb{R})$ separates the points of X and define $K := \mathrm{Hom}(X,\mathbb{R}) \cap B_{X^*}$, which is a w^*-closed positively homogeneous subset of B_{X^*}, and $Z_K := \{f \in \mathrm{FBL}[X] \,:\, f|_K = 0\}$. By the preceding proposition, $\ker(\beta) \subseteq Z_K$. Now, take $f \in Z_K$. Since $f \in \mathrm{FBL}[X]$, there exists a unique decomposition $f = \delta_X(x) + g$, for some $x \in X$ and $g \in \ker(\beta)$. Then, $\delta_X(x) \in Z_K$ and since K separates the points of X we deduce that $x = 0$. Hence $f = g \in \ker(\beta)$.

For the reverse implication, suppose that $\ker(\beta) = Z_K$, for some w^*-compact positively homogeneous subset K of B_{X^*}. Proposition 2.18 guarantees that each element $x^* \in K$ is a lattice homomorphism on X. Since $\delta_X(X) \cap Z_K = \{0\}$, given $x \neq 0$, there is $x^* \in K$ such that $x^*(x) = \delta_x(x^*) \neq 0$. Therefore, K separates the points of X and, in particular, so does $\mathrm{Hom}(X,\mathbb{R})$. □

The next result gathers *an isomorphic version* of Propositions 2.18 and 2.19. Before proving this, let us recall the definition of a prominent class of lattice homomorphisms between free Banach lattices. Given two Banach spaces E and F, the universal property of $\mathrm{FBL}[E]$ ensures that for every operator $T : E \to F$ there is a unique lattice homomorphism $\overline{T} : \mathrm{FBL}[E] \to \mathrm{FBL}[F]$ which makes the following diagram commutative

$$\begin{array}{ccc} \mathrm{FBL}[E] & \overset{\overline{T}}{\dashrightarrow} & \mathrm{FBL}[F] \\ \uparrow{\scriptstyle \delta_E} & & \uparrow{\scriptstyle \delta_F} \\ E & \xrightarrow{\quad T \quad} & F \end{array}$$

that is, $\overline{T}\delta_E = \delta_F T$. It is easy to check that $\overline{T}$ is a lattice isomorphism if and only T is an isomorphism and also that this extension is given by $\overline{T}f = f \circ T^*$, for $f \in \mathrm{FBL}[E]$ [119, Lemma 3.1]. We refer the reader to [119, Section 3] for more properties concerning these operators.

Proposition 2.20. *Let E be a Banach space and X a Banach lattice. If $T : E \to X$ is an isomorphism, then $\overline{T} : FBL[E] \to FBL[X]$ is a lattice isomorphism (the only one satisfying $\overline{T}\delta_E = \delta_X T$), which is given by $\overline{T}f = f \circ T^*$, for $f \in FBL[E]$. Let $\beta : FBL[X] \to X$ and $\widehat{T} : FBL[E] \to X$ be the unique lattice homomorphisms satisfying $\beta\delta_X = id_X$ and $\widehat{T}\delta_E = T$, respectively. The following assertions hold:*

(i) $\overline{T}\big(ker(\widehat{T})\big) = ker(\beta)$;

(ii) $\overline{T}\big(ker(\widehat{T^*x^*})\big) = ker(\widehat{x^*})$ *for any* $x^* \in X^*$*;*

(iii) $x^* \in Hom(X,\mathbb{R})$ *if and only if* $ker(\widehat{T}) \subseteq ker(\widehat{T^*x^*})$*;*

(iv) $Hom(X,\mathbb{R})$ *separates the points of* X *if and only if there exists a* w^**-closed positively homogeneous subset* K *of* B_{E^*} *such that* $ker(\widehat{T}) = Z_K = \{f \in FBL[X] : f|_K = 0\}$.

Proof. (i) The composition $\beta\overline{T} : \mathrm{FBL}[E] \to X$ is a lattice homomorphism such that for every $x \in E$ we have $\beta\overline{T}\big(\delta_E(x)\big) = \beta\delta_X(Tx) = Tx = \widehat{T}\big(\delta_E(x)\big)$. Since the sublattice generated by $\{\delta_E(x) : x \in E\}$ is dense in $\mathrm{FBL}[E]$, then $\beta\overline{T} = \widehat{T}$, so $\overline{T}\big(\ker(\widehat{T})\big) \subseteq \ker(\beta)$. For the other inclusion, observe that the latter implies that $\beta = \widehat{T}\overline{T}^{-1}$.

(ii) This can be checked in a similar way to the previous claim using on this occasion the identity of lattice homomorphisms $\widehat{x^*T} = \widehat{T^*x^*}$.

(iii) By Proposition 2.18 we know that $x^* \in X^*$ is a lattice homomorphism if and only if $\ker(\beta) \subseteq \ker(\widehat{x^*})$. And it can be easily derived from (i) and (ii) that the last-mentioned condition is equivalent to $\ker(\widehat{T}) \subseteq \ker(\widehat{T^*x^*})$.

(iv) Proposition 2.19 states that $\mathrm{Hom}(X,\mathbb{R})$ separates the points of X is equivalent to the fact that $\ker(\beta) = \{f \in \mathrm{FBL}[X] : f(x^*) = 0 \text{ for all } x^* \in \mathrm{Hom}(X,\mathbb{R})\}$. So if $\mathrm{Hom}(X,\mathbb{R})$ separates the points of X, then by (i) we have

$$\ker(\widehat{T}) = \overline{T}^{-1}\big(\ker(\beta)\big) = \Big\{\overline{T}^{-1}f : \overline{T}^{-1}f(T^*x^*) = 0 \text{ for all } x^* \in \mathrm{Hom}(X,\mathbb{R})\Big\},$$

and this shows that $\ker(\widehat{T}) = Z_K$ for $K = T^*\big(\mathrm{Hom}(X,\mathbb{R})\big) \cap B_{E^*}$. Conversely, suppose that there is a w^*-closed positively homogeneous subset K of B_{E^*} such that $\ker(\widehat{T}) = Z_K$. Therefore, for any $z^* \in K$, $\ker(\widehat{T}) \subseteq \ker(\widehat{z^*})$. Then, using (i) and (ii), we obtain $\ker(\beta) = \overline{T}\big(\ker(\widehat{T})\big) \subseteq \overline{T}\big(\ker(\widehat{z^*})\big) = \ker(\widehat{(T^{-1})^*z^*})$, so by Proposition 2.18 we conclude that $(T^{-1})^*(K) \subseteq \mathrm{Hom}(X,\mathbb{R})$. Moreover, $(T^{-1})^*(K)$ separates the points of X. Indeed, given a non-zero $x \in X$, as $\delta_E(E) \cap Z_K = \{0\}$, there exists $z^* \in K$ such that

$$0 \neq \delta_E(T^{-1}x)(z^*) = \overline{T^{-1}}\delta_X(x)(z^*) = \delta_X(x)\big((T^{-1})^*z^*\big) = (T^{-1})^*z^*(x).$$

Therefore, $\mathrm{Hom}(X,\mathbb{R})$ separates the points of X. □

We have seen in the fourth assertion of the preceding result that being isomorphic to a Banach lattice whose set of lattice homomorphisms separates its points implies the existence of a w^*-compact subset $K \subseteq B_{E^*}$ such that $\mathrm{FBL}[E] = \delta_E(E) \oplus Z_K$. We will demonstrate now that the converse also holds.

Proposition 2.21. *Let* E *be a Banach space and suppose that there is a* w^**-closed subset* K *of* B_{E^*} *such that* $FBL[E] = \delta_E(E) \oplus Z_K$*. Then* E *is isomorphic to a Banach lattice* X *such that* $Hom(X,\mathbb{R})$ *separates the points of* X*.*

Proof. First, note that $Z_K = \{f \in \mathrm{FBL}[E] : f(x^*) = 0 \text{ for all } x^* \in K\}$ is a closed ideal in $\mathrm{FBL}[E]$. Thus, by [115, Corollary 1.3.14], $X := \mathrm{FBL}[E]/Z_K$ is a Banach lattice. For every $x^* \in K$, we define $\overline{x^*} : X \to \mathbb{R}$ by $\overline{x^*}(\bar{f}) := f(x^*)$, which is a well-defined lattice homomorphism, so $\{\overline{x^*} : x^* \in K\} \subseteq \mathrm{Hom}(X,\mathbb{R})$. We will show that $\{\overline{x^*} : x^* \in K\}$ separates the points of X. Take any non-zero $\bar{f} \in \mathrm{FBL}[E]/Z_K$. As $\mathrm{FBL}[E] = \delta_E(E) \oplus Z_K$, there exists $0 \neq x \in E$ such that $\bar{f} = \overline{\delta_E(x)}$. In particular, given that $\delta_E(E) \cap Z_K = \{0\}$ there must be $x^* \in K$ such that $x^*(x) \neq 0$. As a result, we have

$$\overline{x^*}(\bar{f}) = \overline{x^*}\big(\overline{\delta_E(x)}\big) = \delta_E(x)(x^*) = x^*(x) \neq 0,$$

and this shows that $\{\overline{x^*} : x^* \in K\} \subseteq \mathrm{Hom}(X,\mathbb{R})$ separates the points of X. □

Given an isomorphism $T : E \to X$ from a Banach space E onto a Banach lattice X, we know by Proposition 2.13 that we have a decomposition $\mathrm{FBL}[E] = \delta_E(E) \oplus \ker(\widehat{T})$. The ideal $\ker(\widehat{T})$ is not a set of zeros unless $\mathrm{Hom}(X, \mathbb{R})$ separates the points of X (Proposition 2.20). However, we could still wonder the following question: Is there always a w^*-compact $K \subseteq B_{E^*}$ such that $\mathrm{FBL}[E] = \delta_E(E) \oplus Z_K$? If the latter were true, that would imply by the preceding proposition that every Banach lattice is isomorphic to a Banach lattice whose set of lattice homomorphisms separates its points. The next example will show that this is not always the case.

Example 2.22. Let us consider $L_1[0, 1]$ and suppose that there exists a Banach lattice X which is isomorphic to $L_1[0, 1]$ and such that $\mathrm{Hom}(X, \mathbb{R})$ separates its points. Thanks to a result we will show later (Theorem 3.1), we can deduce that X is lattice isomorphic to an L_1-space. Thus, we can directly assume that X is an L_1-space. By [94, Corollary to Theorem 9, Section 14], X must be lattice isometric to ℓ_1, $L_1[0, 1]$ or $L_1[0, 1] \oplus_1 \ell_1(\Gamma)$, where $|\Gamma| \leq \aleph_0$. Since we are assuming that the lattice homomorphisms of X separates its points, then $X = \ell_1$. But $L_1[0, 1]$ is not isomorphic to ℓ_1, so we have arrive at a contradiction. It should be noted that this argument cannot be extended to $L_p[0, 1]$ for $1 < p < \infty$, since it is well known that for these spaces the Haar basis is unconditional [5, Theorem 6.1.7].

We conclude this section by pointing out that all the results shown here can be easily adapted to the complex setting, using the notion of free complex Banach lattice introduced in Chapter 4.

2.3 The relevance of projection constants

Proposition 2.11 brings an additional peculiarity that we have not discussed so far: if E is a C-complemented subspace of some Banach lattice (i.e., E is the range of a projection of norm C), then $\delta_E(E)$ is complemented in $\mathrm{FBL}[E]$ *with constant less than or equal to* C. The reason for taking into account the projection constant is that there are significant differences between what happens for the *contractive case* ($C = 1$) and the general case in some of the most relevant classes of Banach lattices. Let us recall some well-known results in this direction:

- *For any* $1 \leq p < \infty$*, every* 1*-complemented subspace of an* L_p*-space is isometric to an* L_p*-space.* This was proven by Tzafriri in 1969 in [152] (see also [24]), extending to general measure spaces previous results for L_p-spaces over probability spaces due to Douglas [45] (case $p = 1$) and Ando [11] (cases $1 < p \neq 2 < \infty$). This result was generalized to Hilbert-valued L_p-spaces by Raynaud [134]. We refer the reader to the extensive survey by Randrianantoanina [133] for more information on 1-complementation in Köthe function spaces and sequence spaces.

  If we do not assume that the projections have norm 1, the situation changes significantly. In this regard, we should mention that Bourgain, Rosenthal and Schechtman showed the existence of uncountably many mutually non-isomorphic complemented subspaces of $L_p[0, 1]$, for any $1 < p \neq 2 < \infty$ [37]. In contrast, for $p = 1$, as we already mentioned, it is conjectured that the only possible complemented subspaces (up to isomorphism) are ℓ_1 and $L_1[0, 1]$ [5, Conjecture 5.7.7].

- *Every separable* 1*-complemented subspace of a* $C(K)$*-space is isomorphic to a* $C(K)$*-space.* This is a consequence of the following two facts: in [110, Theorem 3 (i)] it was shown that every 1-complemented subspace of a $C(K)$-space is linearly isometric to some $C_\sigma(K)$-space; shortly after, Samuel [140] proved that separable $C_\sigma(K)$-spaces are isomorphic to $C(K)$-spaces (see also [20, Lemma 5]). In 1973, Benyamini generalized the preceding result: every separable G-space is isomorphic to a $C(K)$-space [20]. Recall that a G-space is exactly a 1-complemented subspace in an AM-space (up to a linear isometry) [110, Theorem 3 (ii)]. In the non-separable setting, this result is no longer true: PS_2 is a 1-complemented subspace of a $C(K)$-space which is not even isomorphic to a Banach lattice [70, 131].

For not necessarily contractive projections, the following *conjecture* deserves to be mentioned again: Is every complemented subspace of $C[0,1]$ linearly isomorphic to $C(K)$? [5, Conjecture 5.7.8].

- *For complex scalars, it was proven by Kalton and Wood in 1976 [89] that every* 1*-complemented subspace of a Banach space with a* 1*-unconditional basis must have a* 1*-unconditional basis* (see also [56, 137]). This theorem does not hold in the real case (see [22] or the last example of [97]). However, the more general question of whether every complemented subspace of a space with an unconditional basis has an unconditional basis is still open in both real and complex cases [108, Problem 1.d.5].

Since all norms on a finite-dimensional vector space are equivalent, every finite-dimensional Banach space is trivially isomorphic to a Banach lattice. In particular, finite-dimensional Banach spaces are complemented subspaces of Banach lattices. A naive question would be whether these spaces are complemented by some uniform constant (not depending on the dimension). Well-known examples show this is not the case: n-dimensional Schatten p-class operators S_p^n for $p \neq 2$ [64, Theorem 5.1], appropriate finite-dimensional subspaces of James' space [83] (see also [149, Theorem 34.3]). Next result provides a similar argument making use of free Banach lattices.

Proposition 2.23. *For every $C \geq 1$, there exists a finite-dimensional Banach space which is not C-complemented in any Banach lattice.*

Proof. Let us consider the James space $\mathcal{J}$ (see [75] or [5, Section 3.4]). The reason why we are interested in taking this particular Banach space is that it has the following three properties:

(i) $\mathcal{J}$ has a (monotone) basis.

(ii) $\mathcal{J}$ is isometric to its bidual $\mathcal{J}^{**}$.

(iii) $\mathcal{J}$ cannot be isomorphic to any complemented subspace of a Banach lattice.

We will make use of $\mathcal{J}^{**}$ to construct a sequence of finite-dimensional spaces which cannot be uniformly complemented in their corresponding free Banach lattices. Note that $\mathcal{J}^{**}$ also satisfies the properties $(i)-(iii)$. Let $(e_n)_{n=1}^\infty$ be a monotone basis of $\mathcal{J}^{**}$ with associated basis projections $(P_n)_{n=1}^\infty$ and let $E_n = \operatorname{span}\{e_k : 1 \leq k \leq n\}$ denote the range of P_n.

Suppose that there exists a constant $C \geq 1$ such that for every natural n there is a projection $Q_n : \mathrm{FBL}[E_n] \to \mathrm{FBL}[E_n]$ with range $\delta_{E_n}(E_n)$ such that $\|Q_n\| \leq C$. For every $n \in \mathbb{N}$, $\overline{P_n} : \mathrm{FBL}[\mathcal{J}^{**}] \to \mathrm{FBL}[\mathcal{J}^{**}]$ defines a projection whose range is $\overline{\iota_n}(\mathrm{FBL}[E_n])$ (where ι_n stands for the canonical inclusion of E_n into $\mathcal{J}^{**}$ as a subspace). Observe that $\overline{\iota_n} : \mathrm{FBL}[E_n] \to \mathrm{FBL}[\mathcal{J}^{**}]$ is a lattice isometric embedding given that E_n is 1-complemented in $\mathcal{J}^{**}$ [119, Theorem 3.7]. We will denote by $\widetilde{Q_n}$ the projection on $\overline{\iota_n}(\mathrm{FBL}[E_n])$ defined by $\widetilde{Q_n}(\overline{\iota_n} f) = \overline{\iota_n}(Q_n f)$, for every $n \in \mathbb{N}$ and every $f \in \mathrm{FBL}[E_n]$.

For every natural n, let $R_n := \widetilde{Q_n\overline{P_n}}$, which is a projection on $\mathrm{FBL}[\mathcal{J}^{**}]$ onto $\overline{\iota_n}\big(\delta_{E_n}(E_n)\big) = \delta_{\mathcal{J}^{**}}(E_n) \subseteq \delta_{\mathcal{J}^{**}}(\mathcal{J}^{**})$. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and define

$$Rf := \delta_{\mathcal{J}^{**}}\left(w^* - \lim_{\mathcal{U}} \delta_{\mathcal{J}^{**}}^{-1}(R_n f)\right), \qquad f \in \mathrm{FBL}[\mathcal{J}^{**}].$$

Observe that R is a projection (with $\|R\| \leq C$) from $\mathrm{FBL}[\mathcal{J}^{**}]$ onto $\delta_{\mathcal{J}^{**}}(\mathcal{J}^{**})$, which is a contradiction with the property *(iii)* mentioned above. Therefore, by Proposition 2.11, this argument shows that for every $C \geq 1$ there exists $n \in \mathbb{N}$ such that E_n cannot be C-complemented in any Banach lattice. □

We say that a Banach space E is *almost contractively complemented in a Banach lattice* if for every $\varepsilon > 0$, there is a Banach lattice X_ε such that E is $(1+\varepsilon)$-isomorphic to a $(1+\varepsilon)$-complemented subspace of X_ε.

Proposition 2.24. *Suppose that E is almost contractively complemented in a Banach lattice and there is a contractive projection $Q : E^{**} \to E$. Then E is contractively complemented in a Banach lattice.*

Proof. By Proposition 2.11, for every $n \in \mathbb{N}$ there is a projection P_n on $\mathrm{FBL}[E]$ with range $\delta(E)$ and $\|P_n\| \leq 1 + \frac{1}{n}$. Thus, P_n^{**} are projections on $\mathrm{FBL}[E]^{**}$ such that $\|P_n^{**}\| \leq 1 + \frac{1}{n}$ and with ranges isometric to E^{**} (namely, all projections have the same range $\delta_E^{**}(E^{**})$). Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and define

$$Pf = w^* - \lim_{\mathcal{U}} P_n^{**} f, \qquad \text{for every } f \in \mathrm{FBL}[E]^{**}.$$

It is not difficult to check that P is a contractive projection whose range is $\delta_E^{**}(E^{**})$. This shows that E^{**} is 1-complemented in $\mathrm{FBL}[E]^{**}$ and since E is 1-complemented in E^{**}, then E is 1-complemented in $\mathrm{FBL}[E]^{**}$. □

We do not know whether the hypothesis that E is contractively complemented in E^{**} in the above proposition is actually necessary. This should be compared with the fact that if a Banach space E is *almost contractively complemented in a $C(K)$-space*, then it is actually isometric to a $C_\sigma(K)$-space [9, Theorem 0.2].

2.4 Complementation in Banach lattices with extra properties

If a Banach space which is complemented in a Banach lattice has a certain property, it is sometimes possible to construct a Banach lattice with this extra property in which it also embedds as a complemented subspace. For instance, if a separable Banach space E is complemented in a Banach lattice, then it is complemented in a separable Banach lattice; specifically, in its free Banach lattice $\mathrm{FBL}[E]$ (this follows from the fact that the sublattice generated by a subset of a Banach lattice preserves the density character). In a similar direction, recall [109, Proposition 1.c.6] (see also [55, Proposition 2.6 (i)]):

Proposition 2.25. *Let E be a Banach space which does not contain isomorphic copies of c_0 (respectively, E does not contain $(\ell_\infty^n)_{n=1}^\infty$ uniformly) and is a complemented subspace of a Banach lattice X. Then, there is a Banach lattice Y which does not contain isomorphic copies of c_0 (resp., $(\ell_\infty^n)_{n=1}^\infty$ uniformly) and contains E as a complemented subspace.*

The above proposition allows us to deduce some isomorphic properties of complemented subspaces of Banach lattices from the theorems stated in Section 2.1.

Corollary 2.26. *If E is a complemented subspace of a Banach lattice, then it has the following properties:*

(i) E is weakly sequentially complete if and only if no subspace of E is isomorphic to c_0.

(ii) E is reflexive if and only if E contains no subspaces isomorphic to c_0 or ℓ_1.

(iii) E contains an unconditional basic sequence.

(iv) E contains uniformly complemented $(\ell_p^n)_{n=1}^{\infty}$ for some $p = 1, 2, \infty$.

Proof. (i) and (ii). If E does not contain an isomorphic copy of c_0, then by the previous theorem there exists a Banach lattice Y which does not contain isomorphic copies of c_0 and contains E as a complemented subspace. Thus, Y is order continuous. In particular, E is a subspace of an order continuous Banach lattice and the results follow from Theorem 2.4.
(iii). If E contains a subspace isomorphic to c_0, then the image of the canonical basis of c_0 through this isomorphism is an unconditional basic sequence in E. If E does not contain c_0, then it is complemented in a Banach lattice Y which does not contain c_0. Since Y is order continuous, by Theorem 2.6 E contains an unconditional basic sequence.
(iv) can be deduced from Theorem 2.8 using an argument analogous to the preceding ones. □

We can also get an analogous result for *reflexive complemented subspaces of Banach lattices*. While this fact is likely well-known, we have not found an explicit reference for it, so we include its proof below.

Proposition 2.27. *If E is a reflexive Banach space which is complemented in a Banach lattice X, then E is a complemented subspace of some reflexive Banach lattice.*

Proof. Let $P : X \to X$ be a projection onto E. It should be noted that P is a weakly compact operator since its range $P(X) = E$ is a reflexive Banach space. As a consequence of [6, Corollary 2.7], we know that $P^2 = P$ factors through a reflexive Banach lattice Y. That is, there exist operators $T : X \to Y$ and $S : Y \to X$ such that $P = ST$. Now, define $Q := TPS$, which is an operator on Y. Observe that $T|_E$ is an isomorphism into its image and Q is a projection on Y (this can be deduced straightforwardly from the identity $P = ST$) with range $T(E)$. To check the last fact, note that for every $x \in E = P(X)$ we have

$$Q(Tx) = TPS(Tx) = TP(ST)(x) = TPP(x) = Tx.$$

□

The following *dual version* of Proposition 2.25 was established in [54, Theorem 1.2].

Proposition 2.28. *Let E be a complemented subspace of a Banach lattice X and assume that c_0 does not embed into E^*. Then E is complemented in a Banach lattice Y such that c_0 does not embed into Y^*.*

Recall that by Proposition 2.11 we know that if a Banach space E is a complemented subspace of some Banach lattice X, then E must be complemented in $\mathrm{FBL}[E]$. So it is natural to wonder whether we can take $Y = \mathrm{FBL}[E]$ in the three previous propositions. Let us analyze this:

- FBL$[E]$ contains an isomorphic of c_0 whenever E is a Banach space such that $\dim E \geq 2$. Indeed, let E be a Banach space of dimension ≥ 2 and let F be a 2-dimensional subspace of E. Then FBL$[F]$ is a complemented sublattice of FBL$[E]$ and, moreover, FBL$[F]$ is 2-lattice isomorphic to $C(S_{F^*}) \approx C[0,1]$ (see [118, Remark 3.1 (i)]). Thus, c_0 embeds isomorphically into FBL$[F]$, and hence into FBL$[E]$. Therefore, we cannot deduce Propositions 2.25 and 2.27 using free Banach lattices (at least, not in a trivial way).
- However, in [119, Theorem 9.20] it is shown that ℓ_1 is a complemented subspace of E if and only if ℓ_1 is a complemented subspace of FBL$[E]$. Moreover, recall that by Bessaga-Pełczyński's theorem [108, Proposition 2.e.8], for any Banach space F, ℓ_1 embeds complementably into F if and only if c_0 embeds isomorphically into F^*. Consequently, we can take $Y =$ FBL$[E]$ in Proposition 2.28.

As a result of Proposition 2.11 and [119, Corollary 9.25 and Lemma 9.26], we can also state a *local version* of the previous proposition:

Proposition 2.29. *Let E be a complemented subspace of a Banach lattice X and assume that E^* does not contain $(\ell_\infty^n)_{n=1}^\infty$ uniformly. Then E is complemented in a Banach lattice Y such that Y^* does not contain uniformly subspaces isomorphic to ℓ_∞^n. Namely, we can take $Y = FBL[E]$.*

We now turn to analyze the case when E is an $\mathcal{L}_\infty$-space. When E is an $\mathcal{L}_1$-space, we have the following result:

Proposition 2.30. *If E is an $\mathcal{L}_1$-space which is complemented in a Banach lattice, then it is complemented in some $L_1(\mu)$-space.*

Proof. As E is an $\mathcal{L}_1$-space, then thanks to [103, Proposition 7.1] we know that it is isomorphic to a subspace of an $L_1(\mu)$-space. Since every $L_1(\mu)$-space is weakly sequentially complete and this property passes to subspaces, E cannot contain isomorphic copies of c_0. By [109, Proposition 1.c.6] (see [55, Proposition 2.6]) E is a complemented subspace of a certain Banach lattice X which does not contain isomorphic copies of c_0. By [109, Theorem 1.c.4], the canonical image of X in X^{**} is a projection band of X^{**}. Thus, E is complemented in its bidual (recall Remark 2.12), and by [103, Corollary 1 of Theorem 7.1] we conclude that E is a complemented subspace of an $L_1(\mu)$-space. □

Remark 2.31. The above proof actually shows that if E is an $\mathcal{L}_1$-space the following assertions are equivalent:

(i) E is complemented in a Banach lattice;

(ii) E is complemented in its bidual;

(iii) E is complemented in an L_1-space.

Not every $\mathcal{L}_1$-space satisfies the above conditions (see the examples D_k constructed in [107, p. 211]).

Can we establish an analogous version of the preceding result for $\mathcal{L}_\infty$-spaces? Since an $\mathcal{L}_\infty$-space which is isomorphic to a Banach lattice must be isomorphic to an AM-space (this will be shown in Corollary 3.2), so it is natural to pose the following:

Question 2.32. If E is an $\mathcal{L}_\infty$-space which is complemented in a Banach lattice, is it then complemented in some AM-space?

Note that in the separable setting, *AM-space* can be replaced by *C(K)-space* [20] (while in the non-separable case this cannot be done, see [21]). One of the motivations behind this question is the construction due to Benyamini and Lindenstrauss of an isometric predual of ℓ_1 which cannot be complemented in any $C(K)$-space [23]. Since this space is separable, it cannot be isomorphic to a Banach lattice (given that this is equivalent to being isomorphic to a $C(K)$-space for separable spaces). But could this space be complemented in a Banach lattice? If Question 2.32 had an affirmative answer, then the answer to the latter would be negative.

Question 2.33 (Separable CSP). Let E be a *separable* Banach space which is complemented in a Banach lattice X. Is E isomorphic to a Banach lattice?

But a priori, Benyamini-Lindenstrauss' example could provide a negative answer to the CSP for separable Banach lattices. The separable version of CSP is closely related to understanding the complemented subspaces of $L_1[0,1]$ and $C[0,1]$, as noted in [70, Remark 2.5].

By [119, Theorem 9.21], if E is an $\mathcal{L}_\infty$-space, then FBL$[E]$ satisfies *an upper* 2*-estimate.* In fact, we will see next that, in this case, FBL$[E]$ is actually 2-convex.

Lemma 2.34. *Let X be a 2-convex Banach lattice. Then, $FBL[X]$ is 2-convex with $M^{(2)}(FBL[X]) \le K_G M^{(2)}(X)$.*

Proof. By [119, Proposition 9.38], the following two assertions are equivalent for any $C \ge 1$:

(1) FBL$^{(2)}[X]$ is lattice C-isomorphic to FBL$[X]$.

(2) Every contraction $T : X \to L_1(\mu)$ is 2-convex with constant C.

Let us check (2). Let $T : X \to L_1(\mu)$ be an operator such that $\|T\| \le 1$ and let $(x_k)_{k=1}^n$ be an arbitrary finite sequence in X. Then, by [109, Theorem 1.f.14], we have

$$\left\| \left(\sum_{k=1}^n |Tx_k|^2 \right)^{\frac{1}{2}} \right\| \le K_G \left\| \left(\sum_{k=1}^n |x_k|^2 \right)^{\frac{1}{2}} \right\| \le K_G M^{(2)}(X) \left(\sum_{k=1}^n \|x_k\|^2 \right)^{\frac{1}{2}},$$

which shows that T is 2-convex with constant $\le K_G M^{(2)}(X)$. □

Remark 2.35. Conversely, it should also be noticed that if X is a Banach lattice such that FBL$[X]$ is 2-convex, then X is also 2-convex with $M^{(2)}(X) \le M^{(2)}(\text{FBL}[X])$. Indeed, given an arbitrary finite sequence $(x_k)_{k=1}^n$ in X, we have

$$\begin{aligned}
\left\| \left(\sum_{k=1}^n |x_k|^2 \right)^{\frac{1}{2}} \right\| &= \left\| \left(\sum_{k=1}^n |\beta \delta_X(x_k)|^2 \right)^{\frac{1}{2}} \right\| \overset{(*)}{=} \left\| \beta \left(\sum_{k=1}^n |\delta_X(x_k)|^2 \right)^{\frac{1}{2}} \right\| \\
&\le \left\| \left(\sum_{k=1}^n |\delta_X(x_k)|^2 \right)^{\frac{1}{2}} \right\| \le M^{(2)}(\text{FBL}[X]) \left(\sum_{k=1}^n \|\delta_X(x_k)\|^2 \right)^{\frac{1}{2}} \\
&= M^{(2)}(\text{FBL}[X]) \left(\sum_{k=1}^n \|x_k\|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

where $\beta : \mathrm{FBL}[X] \to X$ is the unique lattice homomorphism such that $\beta\delta_X = \mathrm{id}_X$. The equality $(*)$ is a consequence of Krivine's functional calculus (see, for instance, [77, Lemma 2.1]).

Proposition 2.36. *If E is an $\mathcal{L}_p$-space for some $2 \le p \le \infty$, then FBL$[E]$ is 2-convex.*

Proof. First, recall that as E is an $\mathcal{L}_p$-space, [105, Theorem III (c)] ensures the existence of a constant $\rho \ge 1$ such that for every finite-dimensional subspace G of E there is a finite-dimensional subspace F of E such that $F \supseteq G$, $d(F, \ell_p^{\dim \mathrm{F}}) \le \rho$, and such that there is a projection of norm $\le \rho$ from E onto F. We will show that for every $(f_k)_{k=1}^n \subseteq \mathrm{FBL}[E]$,

$$\left\| \left(\sum_{k=1}^n |f_k|^2 \right)^{\frac{1}{2}} \right\|_{\mathrm{FBL}[E]} \le K_G\, \rho^2 \left(\sum_{k=1}^n \|f_k\|_{\mathrm{FBL}[E]}^2 \right)^{\frac{1}{2}}.$$

Since FVL$[E]$ is norm-dense in FBL$[E]$, we can find $x_1, \dots, x_m \in E$ with the property that for every $k = 1, \dots, n$ there exists $g_k \in \mathrm{lat}\{\delta_{x_1}, \dots, \delta_{x_m}\}$ such that $\|f_k - g_k\|_{\mathrm{FBL}[E]} \le \frac{\varepsilon}{2K_G\rho^2 n}$. Define $G = \mathrm{span}\{x_j \ : \ j = 1, \dots, m\} \subseteq E$ and let F be such that $F \supseteq G$, $d(F, \ell_p^{\dim \mathrm{F}}) \le \rho$ and there is a projection $P : E \to F$ ($P\iota = \mathrm{id}_F$, where $\iota : F \hookrightarrow E$) such that $\|P\| \le \rho$. Let us make some observations:

(1) For every $(a_k)_{k=1}^n$, $(b_k)_{k=1}^n \subseteq \mathbb{R}^n$ we have
$$\left| \left\|(a_k)_{k=1}^n\right\|_2 - \left\|(b_k)_{k=1}^n\right\|_2 \right| \le \left\|(a_k - b_k)_{k=1}^n\right\|_2 \le \left\|(a_k - b_k)_{k=1}^n\right\|_1,$$
so by Krivine's functional calculus [109, Theorem 1.d.1], we get:
$$\left\| \left(\sum_{k=1}^n |f_k|^2 \right)^{\frac{1}{2}} - \left(\sum_{k=1}^n |g_k|^2 \right)^{\frac{1}{2}} \right\|_{\mathrm{FBL}[E]} \le \left\| \sum_{k=1}^n |f_k - g_k| \right\|_{\mathrm{FBL}[E]} \le \frac{\varepsilon}{2}$$

(2) Given $g \in \bar{\iota}(\mathrm{FBL}[F])$, we have $\bar{\iota}\overline{P}g = g$, so $\|\bar{\iota}\overline{P}g\|_{\mathrm{FBL}[E]} \le \|\overline{P}g\|_{\mathrm{FBL}[F]}$. Also, by [77, Lemma 2.1], we have $\overline{P}\left(\sum_{k=1}^n |g_k|^2\right)^{\frac{1}{2}} = \left(\sum_{k=1}^n |\overline{P}g_k|^2\right)^{\frac{1}{2}}$.

(3) Since F is ρ-isomorphic to $\ell_p^{\dim F}$ (and $2 \le p \le \infty$) then by the previous lemma we deduce that FBL$[F]$ is 2-convex with constant $M^{(2)}(F) \le K_G\rho$.

With the above comments in mind, we deduce that

$$\begin{aligned}
\left\| \left(\sum_{k=1}^n |f_k|^2 \right)^{\frac{1}{2}} \right\|_{\mathrm{FBL}[E]} &\overset{(1)}{\le} \left\| \left(\sum_{k=1}^n |g_k|^2 \right)^{\frac{1}{2}} \right\|_{\mathrm{FBL}[E]} + \frac{\varepsilon}{2} \overset{(2)}{\le} \left\| \left(\sum_{k=1}^n |\overline{P}g_k|^2 \right)^{\frac{1}{2}} \right\|_{\mathrm{FBL}[F]} + \frac{\varepsilon}{2} \\
&\overset{(3)}{\le} K_G\rho \left(\sum_{k=1}^n \|\overline{P}g_k\|_{\mathrm{FBL}[F]}^2 \right)^{\frac{1}{2}} + \frac{\varepsilon}{2} \le K_G\rho^2 \left(\sum_{k=1}^n \|g_k\|_{\mathrm{FBL}[E]}^2 \right)^{\frac{1}{2}} + \frac{\varepsilon}{2} \\
&\le K_G\rho^2 \left(\sum_{k=1}^n \|f_k\|_{\mathrm{FBL}[E]}^2 \right)^{\frac{1}{2}} + \varepsilon,
\end{aligned}$$

and since $\varepsilon > 0$ is arbitrary, we obtain the desired inequality. □

In particular, it follows that if an $\mathcal{L}_\infty$-space is complemented in a Banach lattice, then it is complemented in a 2-convex Banach lattice. However, note that by [119, Proposition 9.30], FBL$[E]$ is at most 2-convex, so the convexity in Proposition 2.36 cannot be improved.

2.5 Hyperplanes in Banach lattices

A noteworthy particular case of the CSP is *the Hyperplane Problem for Banach lattices*:

Question 2.37. Let X be a Banach lattice. Is every hyperplane of X linearly isomorphic to a Banach lattice?

Note that since all hyperplanes of a Banach space are mutually isomorphic, in order to answer the above question it will be enough to find a certain hyperplane which is isomorphic (or not) to a Banach lattice. Also note that if the above question has a positive answer, then every finite codimensional subspace of a Banach lattice would be linearly isomorphic to a Banach lattice.

In the particular case when a Banach lattice X has a non-trivial lattice homomorphism $x^* \in X^*$, then $\ker(x^*)$ is a hyperplane which is also an ideal in X. Thus $\ker(x^*)$ is a Banach lattice with the Banach lattice structure inherited from X and this shows that, in this case, the hyperplanes of X are isomorphic to Banach lattices. Let us look closer at some general instances of this situation:

(i) If X has a 1-unconditional basis $(u_n)_{n=1}^\infty$, with biorthogonal functionals $(u_n^*)_{n=1}^\infty$, then $\mathrm{Hom}(X,\mathbb{R}) = \{\lambda u_n^* \,:\, \lambda \geq 0,\ n \in \mathbb{N}\}$. Note that $\ker(u_{n_0}^*)$ also has 1-unconditional basis (for any $n_0 \in \mathbb{N}$), so hyperplanes do have an unconditional basis in this case. We would like to emphasize that whether hyperplanes are isomorphic or not to the original space is not the matter here; we simply want to know whether they can carry a Banach lattice structure. So our problem is different from the classical one considered in [66] and, in fact, Gowers' counterexample has an unconditional basis, so its hyperplanes are isomorphic to Banach lattices.

(ii) Recall that AM-spaces can be characterized as those Banach lattices whose set of lattice homomorphisms is 1-norming (for instance, see Proposition 5.22). In particular, given a non-trivial AM-space X, we can find a non-zero $x^* \in \mathrm{Hom}(X,\mathbb{R})$, and so $\ker(x^*)$ is also an AM-space.

(iii) From the preceding comment we know that hyperplanes in a $C(K)$-space are isomorphic to AM-spaces. But in this case, more can be said: every hyperplane of a $C(K)$-space is isomorphic to a $C(K)$-space. Indeed, take $t_1, t_2 \in K$, $t_1 \neq t_2$, and consider

$$X = \big\{f \in C(K) \,:\, f(t_1) = f(t_2)\big\} = \big\{f \in C(K) \,:\, (\delta_{t_1} - \delta_{t_2})(f) = 0\big\}.$$

Therefore, X is the kernel of $\delta_{t_1} - \delta_{t_2} \in C(K)^*$, so it is a hyperplane of $C(K)$. Moreover, it is clear that X is a closed sublattice of $C(K)$ such that $\mathbf{1}_K \in X$, so by Kakutani's representation theorem for AM-spaces X is lattice isometric to a $C(K)$-space. On the other hand, recall that there exist $C(K)$-spaces whose hyperplanes are not isomorphic to the whole space (see, for instance, the first examples due to Koszmider [90] –assuming **CH**– and Plebanek [129] –accomplished in ZFC), but we will not look at this question here.

At the moment we have *essentially* one example of a complemented subspace of a Banach lattice which is not isomorphic to a Banach lattice, namely PS_2 [70]. Can this space be isomorphic to a hyperplane of some Banach lattice? We will see that this is not the case.

Remark 2.38. Suppose that $\mathsf{PS}_2 \oplus_\infty \mathbb{R}$ were isomorphic to a Banach lattice. Since PS_2 is a complemented subspace of a $C(K)$-space, it is an $\mathcal{L}_\infty$-space. Therefore, $\mathsf{PS}_2 \oplus_\infty \mathbb{R}$ is also an $\mathcal{L}_\infty$-space, so by Corollary 3.2 we may assume that it is isomorphic to an AM-space. But, we have seen in (ii) that hyperplanes of an AM-space are Banach lattices, so PS_2 would be isomorphic to a Banach lattice, and this is a contradiction.

Remark 2.39. Although on many occasions it is not hard to check that the hyperplanes of a certain Banach lattice are isomorphic to Banach lattices, the Hyperplane Problem for Banach lattices is still an open question in both the separable and non-separable cases. It should be noted that in the *separable* setting it is sufficient to study this problem *only for reflexive Banach lattices*. Indeed, suppose that X is a separable Banach lattice. We distinguish two cases:

- If X contains a complemented copy of c_0 or ℓ_1, then, since $c_0 \approx c_0 \oplus \mathbb{R}$ and $\ell_1 \approx \ell_1 \oplus \mathbb{R}$, we deduce that X is isomorphic to its hyperplanes. In particular, its hyperplanes are isomorphic to Banach lattices.
- If X has no complemented copies of c_0 and no complemented copies of ℓ_1, we will see below that this implies that X must be reflexive:
  - Since X is separable, not having complemented copies of c_0 implies not containing isomorphic copies of c_0. Hence, by [115, Theorem 2.4.12], X is a KB-space.
  - Moreover, as X has no complemented copies of ℓ_1, then by a Bessaga-Pełczyński's result [108, Proposition 2.e.8], X^* does not contain isomorphic copies of c_0. Thus, using again [115, Theorem 2.4.12], we infer that X^* is a KB-space.
  - Now, given that X and X^* are KB-spaces, [115, Theorem 2.4.15] ensures that X is reflexive.

It is well known (see [1, Theorem 5.59]) that given a Banach lattice X and a positive projection $P : X \to X$, then its range $E = P(X)$ endowed with the inherited order of X becomes a Banach lattice with the following:

- its lattice operations are given by $x \vee_E y = P(x \vee y)$, $x \wedge_E y = P(x \wedge y)$ and $|x|_E = P|x|$;
- the norm $|||\cdot|||$ defined by $|||x||| = \||x|_E\| = \|P|x|\|$.

In particular, the above shows that the range of a positive projection is isomorphic to a Banach lattice. This prompts the following question: given a Banach lattice X, is there always a positive projection $P : X \to X$ onto one of its hyperplanes? An affirmative answer to the latter would imply a positive solution to the hyperplane problem. However, this will not be the case in general, as we will see below. To prove this we will rely on the following simple observation:

Lemma 2.40. *Let X be a Banach lattice and let x_0 be a non-zero positive element of X. If x_0 is not an atom, then there exists $0 \le y_0 \le x_0$ such that neither y_0 nor $x_0 - y_0$ can dominate a positive multiple of x_0.*

Proof. Since x_0 is not an atom in X, there must exist a non-proportional vector $x \in X$ to x_0 such that $0 \le x \le x_0$. Now, define

$$\lambda_0 := \sup\{\lambda > 0 \,:\, \lambda x \leq x_0\}.$$

Observe that this supremum exists given that X is a Banach lattice and so its norm is monotone on X_+. Moreover, this supremum is actually a maximum, as the set X_+ is closed. Consider $y := x_0 - \lambda_0 x$. This new vector satisfies that $0 \leq y \leq x_0$ and is not proportional to x_0 either, but it has an additional feature: y cannot dominate a positive multiple of x_0. Indeed, suppose that there exists $\lambda > 0$ such that

$$x_0 - \lambda_0 x = y \geq \lambda x_0.$$

Since $x_0 \geq y$ and y is not proportional to x_0, then $\lambda < 1$. Therefore, the above inequality is equivalent to $x_0 \geq \frac{\lambda_0}{1-\lambda} x$ and this contradicts the definition of λ_0. Now, define $\mu_0 := \sup\{\lambda > 0 \,:\, \lambda y \leq x_0\}$ and $y_0 := \mu_0 y = \mu_0(x_0 - \lambda_0 x)$. Then we have the following: both $y_0, x_0 - y_0 \in [0, x_0]$ cannot dominate a positive multiple of x_0. □

Proposition 2.41. *Let X be a Banach lattice. If there exists a hyperplane in X which is complemented by a positive projection, then X has an atom.*

Proof. Let $P : X \to X$ be a positive projection whose range $P(X)$ is a hyperplane of X. Then, there exist $x_0 \in X$, $x_0^* \in X^*$ such that $x_0^*(x_0) = 1$ and

$$Px = x - x_0^*(x)x_0 \geq 0, \qquad \text{for every } x \in X_+. \tag{2.2}$$

By decomposing x_0 into its corresponding positive and negative parts, we get

$$1 = x_0^*(x_0) = x_0^*(x_0^+) - x_0^*(x_0^-),$$

so $x_0^*(x_0^+) \geq \frac{1}{2}$ or $x_0^*(x_0^-) \leq -\frac{1}{2}$. Without lost of generality, we may assume that $x_0^*(x_0^+) \geq \frac{1}{2}$; indeed, if not, we can replace x_0^* and x_0 with $-x_0^*$ and $-x_0$ respectively and then we are in the desired situation. If x_0^+ is not an atom, then by the previous lemma we can find $0 \leq y_0 \leq x_0^+$ such that neither y_0 nor $x_0^+ - y_0$ can dominate a positive multiple of x_0^+. As we have $x_0^+ = y_0 + (x_0^+ - y_0)$, then $x_0^*(y_0) \geq \frac{1}{4}$ or $x_0^*(x_0^+ - y_0) \geq \frac{1}{4}$. We may assume that $x_0^*(y_0) \geq \frac{1}{4}$, but if we are in the other case we can argue similarly. If we evaluate expression 2.2 at y_0, we obtain

$$y_0 \geq x_0^*(y_0)x_0 \quad \Longleftrightarrow \quad y_0 + x_0^*(y_0)x_0^- \geq x_0^*(y_0)x_0^+ \geq \frac{1}{4}x_0^+.$$

Taking infima on both sides of the last inequality with respect to x_0^+ and using [115, Theorem 1.1.1 (ix)], we get that $y_0 \geq \frac{1}{4}x_0^+$, and this is a contradiction with the fact that y_0 does not dominate any positive multiple of x_0^+. □

We close this section with a characterization of those Banach spaces whose hyperplanes are isomorphic to Banach lattices using free Banach lattices.

Proposition 2.42. *Let E be a Banach space. The following assertions are equivalent:*

(i) E is linearly isomorphic to a Banach lattice whose hyperplanes are isomorphic to Banach lattices.

(ii) E is linearly isomorphic to a Banach lattice X such that $Hom(X, \mathbb{R}) \neq \{0\}$.

(iii) There is $x^* \in E^* \setminus \{0\}$ and there is an ideal I in $FBL[E]$ such that $FBL[E] = \delta_E(E) \oplus I$ with $I \subseteq ker(\widehat{x^*})$.

(iv) For every $x^* \in E^*$ there is an ideal I in $FBL[E]$ such that $FBL[E] = \delta_E(E) \oplus I$ with $I \subseteq ker(\widehat{x^*})$.

Proof. $(i) \Rightarrow (ii)$ By hypothesis, E is isomorphic to $Y \oplus \mathbb{R}$, where Y is a Banach lattice. Then, the Banach space $X := Y \oplus_1 \mathbb{R}$ equipped with the coordinate-wise order is a Banach lattice and the functional $x^* : X \to \mathbb{R}$ given by $x^*(y,t) := t$ defines a non-trivial lattice homomorphism on it.

$(ii) \Rightarrow (iii)$ Let $T : E \to X$ be a lattice isomorphism onto a Banach lattice X such that $\mathrm{Hom}(X,\mathbb{R}) \neq \{0\}$. By Proposition 2.13, $\mathrm{FBL}[E] = \delta_E(E) \oplus \ker(\widehat{T})$. Given a non-zero lattice homomorphism x^* on X, we deduce from Proposition 2.20 that $\ker(\widehat{T}) \subseteq \ker(\widehat{T^*x^*})$.

$(iii) \Rightarrow (iv)$ Suppose that there exists $x_1^* \in S_{E^*}$ and an ideal I in $\mathrm{FBL}[E] = \delta_E(E) \oplus I$ such that $I \subseteq \ker(\widehat{x_1^*})$. Fix any $x_2^* \in S_{E^*}$, $x_2^* \neq x_1^*$. As $\ker(x_1^*)$, $\ker(x_2^*)$ are both closed hyperplanes of E, there is an isomorphism $T : \ker(x_1^*) \to \ker(x_2^*)$ [51, Exercise 2.7] (in fact, all closed hyperplanes of a Banach space are mutually isomorphic with a uniform constant ≤ 25 [1, Lemma 9.5.4]). Now, take $x_1, x_2 \in E$ such that $x_1^*(x_1) = x_2^*(x_2) = 1$, and consider the operator $S : E \to E$ defined by

$$Sx := T\big(x - x_1^*(x)x_1\big) + x_1^*(x)x_2, \qquad x \in E.$$

It is clear that S is bounded. Moreover, it is bijective with inverse given by

$$S^{-1}y := T^{-1}\big(y - x_2^*(y)x_2\big) + x_2^*(y)x_1, \qquad y \in E.$$

Thus, $\overline{S} : \mathrm{FBL}[E] \to \mathrm{FBL}[E]$, the unique lattice homomorphism satisfying $\overline{S}\delta_E = \delta_E S$, is a (surjective) lattice isomorphism. From the latter we deduce that $\overline{S}(I)$ is also an ideal in $\mathrm{FBL}[E]$ and it is immediate to check that $\overline{S}(\delta_E(E)) = \delta_E(E)$, so we obtain the decomposition $\mathrm{FBL}[E] = \delta_E(E) \oplus \overline{S}(I)$.

On the other hand, observe that $S^*x_2^* = x_1^*$, given that for every $x \in E$ we have

$$S^*x_2^*(x) = x_2^*(Sx) = x_2^*\Big(T\big(x - x_1^*(x)x_1\big)\Big) + x_1^*(x) = x_1^*(x),$$

where for the last equality one must remember that $T : \ker(x_1^*) \to \ker(x_2^*)$. Now, note that $\widehat{x_2^*}\overline{S} = \widehat{S^*x_2^*} = \widehat{x_1^*}$ and this implies that $\overline{S}\big(\ker(\widehat{x_1^*})\big) \subseteq \ker(\widehat{x_2^*})$. Consequently, $\overline{S}(I) \subseteq \ker(\widehat{x_2^*})$.

$(iii) \Rightarrow (i)$ (It is obvious that $(iv) \Rightarrow (iii)$). Suppose that there are $x^* \in E^*$, $x^* \neq 0$, and an ideal I in $\mathrm{FBL}[E]$ such that $\mathrm{FBL}[E] = \delta_E(E) \oplus I$ with $I \subseteq \ker(\widehat{x^*})$. In the same way as in Proposition 2.18 (or also in Proposition 2.21), we can consider the lattice homomorphism $\overline{x^*} : \mathrm{FBL}[E]/I \to \mathbb{R}$ defined by $\overline{x^*}(\overline{f}) = f(x^*)$. Note that if $x^*(x) \neq 0$ for some $x \in E$, then $\overline{x^*}(\overline{\delta_x}) = \delta_x(x^*) = x^*(x) \neq 0$ so, in particular, $\overline{x^*} \neq 0$. Therefore, $\ker(\overline{x^*})$ is a closed hyperplane of the Banach lattice $X = \mathrm{FBL}[E]/I$ which is also a Banach lattice. Since X is isomorphic to E, this concludes the proof. □

2.6 More open questions

2.6.1. Primariness of the class of Banach lattices. Recall that a Banach space E is said to be *primary* if whenever $E = F \oplus G$, then either $E \approx F$ or $E \approx G$. Let us recall some Banach spaces which do have this property:

- In [122], Pełczyński proved that c_0 and ℓ_p, for $1 \leq p < \infty$, are *prime* (every –infinite-dimensional– complemented subspace is isomorphic to the whole space); afterwards, Lindenstrauss showed that ℓ_∞ is also a prime Banach space [99].
- Every separable $C(K)$-space is primary: for K being compact metric uncountable, this is due to Lindenstrauss and Pełczyński [104, Corollary 1 to Theorem 2.1]; for a countable K this was solved in [8] by Alspach and Benyamini. With regard to the uncountable case, a stronger result due to Rosenthal [136] should be mentioned: any complemented subspace of $C[0,1]$ with non-separable dual must be isomorphic to $C[0,1]$.
- The first proof of the primariness of L_p, for any $1 \leq p < \infty$, can be found in the seminar notes by Maurey from 1974 [114], although the author attributes this result to Enflo. In [10] and [50] it is mentioned that Maurey's proof is based on *unpublished techniques* due to Enflo presented at a conference in 1973. Alternative proofs are given in [10, Theorem 1.3], for the case $1 < p < \infty$, and in [50, Corollary 5.4], for the case $p = 1$).

Partly inspired by these results, a related more general open question is the following:

Question 2.43 (Primariness of the class of Banach lattices). Let X be a Banach lattice and suppose that we have a decomposition $X = Y \oplus Z$ into two infinite-dimensional Banach spaces Y and Z. Must then at least one of the factors be isomorphic to a Banach lattice?

This question could also be formulated for the class of $C(K)$-spaces and the class of L_1-spaces, that is, if a $C(K)$-space (resp. an L_1-space) is isomorphic to $Y \oplus Z$, must Y or Z be isomorphic to a $C(K)$-space (resp. an L_1-space)? As mentioned at the beginning of this section, both separable $C(K)$-spaces and separable L_1-spaces are in fact *primary*. Nevertheless, the current question seems to be open in the non-separable setting.

2.6.2. Projections in AM-spaces. We have already mentioned in Question 2.32, whether every $\mathcal{L}_\infty$-space which is complemented in a Banach lattice must be complemented in some AM-space.

We now know that the space PS_2 is not isomorphic to a Banach lattice [70] so, in particular, it is not isomorphic to an AM-space. On the other hand, in [21] Benyamini gave an example of an AM-space which is not complemented in any $C(K)$-space. However, the following question remains open:

Question 2.44. Suppose that X is an AM-space isomorphic to a complemented subspace of a $C(K)$-space. Must X be isomorphic to a $C(K)$-space?

2.6.3. Complemented subspaces of spaces with unconditional basis. Among the oldest questions concerning the structure of complemented subspaces is whether every complemented subspace of a Banach space with an unconditional basis must have an unconditional basis. The most relevant positive result in this direction is given in [89]: every 1-complemented subspace of a complex Banach space with a 1-unconditional basis also has a 1-unconditional basis. Other relevant partial results can be found in [49, 156] where under extra assumptions if $X \oplus Y$ is a decomposition of a Banach space with unconditional basis, then one can partition the basis to build bases both for X and Y.

However, a formally weaker version of this problem (motivated by Corollaries 2.3 and 2.4 of [70]) is also open:

Question 2.45. If a *Banach lattice* X is isomorphic to a complemented subspace of a space with an unconditional basis, must X have an unconditional basis?

In connection with this, let us recall that every Banach space with an unconditional basis is isomorphic to a complemented subspace of a space with a *symmetric basis* [102]. Motivated by this, it might be natural to wonder whether every separable Banach lattice must be isomorphic to a complemented subspace of a *rearrangement invariant space*. This is not always the case as the following argument suggested by W. B. Johnson shows: Take a separable Banach lattice without the approximation property, such as the one constructed by Szankowski in [147] (in fact this can be even taken as an appropriate sublattice of $\ell_r(L_p(0,1))$, with $1 \leq r < p < \infty$). If this space were complemented in a rearrangement invariant space X, it would have the bounded approximation property (BAP), leading to a contradiction (observe that rearrangement invariant spaces do in fact have the MAP as a consequence of the density of simple functions). However, we have the following:

Question 2.46. Is every Banach lattice with BAP isomorphic to a complemented subspace of a rearrangement invariant space?

It is also worth mentioning here that it is not known whether every separable super-reflexive Banach lattice embeds into a separable super-reflexive rearrangement invariant space.

2.6.4. Existence of complemented disjoint sequences. Motivated by the study of disjointly homogeneous Banach lattices (where every pair of disjoint sequences share an equivalent subsequence), the following question was posed in [57]:

Question 2.47. Does every *separable* Banach lattice contain a disjoint sequence whose span is complemented?

Note that if the Banach lattice were not assumed to be separable in the previous question, the answer would clearly be no: in ℓ_∞, disjoint sequences span (lattice-isometric) copies of c_0, so they cannot be complemented subspaces. This question has positive answer for non-reflexive spaces (as these contain either a complemented sublattice isomorphic to c_0 or ℓ_1 [115, Proposition 2.3.11 and Theorem 2.4.15]) and for rearrangement invariant spaces (because the subspace generated by characteristic functions of pairwise disjoint sets is always complemented by means of a conditional expectation operator [109, Theorem 2.a.4]).

A closely related question, also motivated by the theory of indecomposable Banach spaces, would be the following:

Question 2.48. Does every Banach lattice (of infinite dimension) contain an infinite-dimensional complemented sublattice?

Note that in the previous question we did not require such a sublattice to also have infinite codimension. In that case, the answer would be negative, as there exist indecomposable $C(K)$-spaces (various examples of which are mentioned in [91]). However, what is asked in Question 2.48 is trivially true for $C(K)$-spaces: $\ker(\delta_t)$, where $\delta_t : C(K) \to \mathbb{R}$ is the evaluation at $t \in K$, is an infinite-dimensional complemented ideal (and thus a sublattice). A useful description of finite-codimensional sublattices can be found in [30, Section 5].

On the other hand, observe that Question 2.48 has an affirmative answer in the separable case. Indeed, if X contains a lattice isomorphic copy of c_0, then this sublattice must be complemented; if X does not contain c_0, then X is order continuous so it has *many projection bands* [109, Theorem 1.a.13].

Chapter 3

A negative solution to the Complemented Subspace Problem for Banach lattices

Building on a recent construction of Plebanek and Salguero-Alarcón, it is shown that a complemented subspace of a Banach lattice need not be linearly isomorphic to a Banach lattice. This solves a long-standing open question in Banach lattice theory. This chapter is based on:

> [70] D. de Hevia, G. Martínez-Cervantes, A. Salguero-Alarcón, and P. Tradacete, *A negative solution to the complemented subspace problem for Banach lattices*, Preprint available on arXiv (2025), 25 pp.

3.1 A useful renorming of Banach lattices

As we have already mentioned, one of the main goals of this chapter is to prove that the space PS_2, built by Plebanek and Salguero-Alarcón in [131] (we will recall its construction in Section 3.2), cannot be isomorphic to a Banach lattice, *solving in the negative the CSP for Banach lattices* (Section 3.3). Moreover, in Section 3.4, we will show how to modify PS_2 to get a counterexample for the Complemented Subspace Problem *for complex Banach lattices.*

The task of proving that PS_2 cannot be isomorphic to a Banach lattice will be *significantly simplified* thanks to some particular properties of this space: it will be enough to show that it cannot be isomorphic to a sublattice of ℓ_∞ (Proposition 3.16). The key to achieving this reduction of our problem will be the following interesting renorming theorem for Banach lattices. We will dedicate this section to proving this result and exploring some of its relevant consequences. The theorem is based on the following result stated in [2]; since the proof is not given explicitly there, we include one below for the convenience of the reader.

Theorem 3.1. *Let X be a Banach lattice which is an $\mathcal{L}_1$-space. Then X is lattice isomorphic to an L_1-space.*

Proof. Fix $\lambda > 1$ such that X is an $\mathcal{L}_{1,\lambda}$-space. Since X is an $\mathcal{L}_1$-space, then it is isomorphic to a subspace of a certain $L_1(\mu)$ [103, Proposition 7.1]. Hence, X cannot contain isomorphic copies of c_0 and, thus, X is an order continuous Banach lattice [7, Theorem 4.60].

Let us consider the following norm in X:

$$|||x||| = \sup \left\{ \sum_{i=0}^{m} \|x_i\| \ : \ m \in \mathbb{N},\ x_0, \dots, x_m \in X \text{ pairwise disjoint s.t. } \sum_{i=0}^{m} x_i = x \right\}. \tag{3.1}$$

We claim that $|||\cdot|||$ defines an equivalent AL-norm for X. This fact implies, by Kakutani's representation theorem, that X endowed with this new norm is lattice isometric to an L_1-space [7, Theorem 4.27] and thus we obtain that X is lattice isomorphic to an L_1-space.

Let us detail the proof of the previous claim. We begin by showing the next inequalities:

$$\|x\| \leq |||x||| \leq (K_G\lambda)^2 \|x\|, \qquad \text{for every } x \in X, \tag{3.2}$$

where K_G stands for Grothendieck's constant for real scalars. The first inequality is trivial, so we shall focus on the second one. Fix a natural number m and $x_0, \dots, x_m$ pairwise disjoint vectors in X. For each $i \in \{0, \dots, m\}$, let B_i be the band generated by x_i in X. Since X is order continuous, in particular, X is σ-complete and thus each B_i is a projection band [115, Proposition 1.2.11]. Let us denote by P_i its corresponding band projection from X onto B_i for $0 \leq i \leq m$. For each $0 \leq i \leq m$, take $x_i^* \in S_{X^*}$ such that $x_i^*(x_i) = \|x_i\|$.

Consider the bounded linear operator

$$\begin{aligned} S : X &\longrightarrow \ell_2^{m+1} \\ x &\mapsto (x_i^* \circ P_i(x))_{i=0}^m \end{aligned}$$

Since X is an $\mathcal{L}_{1,\lambda}$-space, we have that S is a 1-summing operator with $\pi_1(S) \leq K_G\lambda\|S\|$ [44, Theorem 3.1]. Now, we are going to show that $\|S\| \leq K_G\lambda$. To this end, we introduce the following family of operators, which are defined for each $x \in X$ by the linear extension of

$$\begin{aligned} T_x : \ell_\infty^{m+1} &\longrightarrow X \\ e_i &\longmapsto P_i(x) \end{aligned}$$

Using again the fact that X is an $\mathcal{L}_{1,\lambda}$-space, by [44, Theorem 3.7] we have that T_x is 2-summing with $\pi_2(T_x) \leq K_G\lambda\|T_x\|$ for every $x \in X$. Taking into account that the elements $(P_i(x))_{i=0}^m$ are pairwise disjoint and that each P_i is a band projection, we obtain that

$$\left| T_x\big((a_i)_{i=0}^m\big) \right| = \left| \sum_{i=0}^{m} a_i P_i(x) \right| = \sum_{i=0}^{m} |a_i||P_i(x)| \leq |x| \big\|(a_i)_{i=0}^m\big\|_\infty$$

and, thus, $\|T_x\| \leq \|x\|$.

Now, observe that

$$\sup \left\{ \left(\sum_{i=0}^{m} |y^*(e_i)|^2 \right)^{\frac{1}{2}} \ : \ y^* \in B_{(\ell_\infty^{m+1})^*} \right\} = \sup \left\{ \left(\sum_{i=0}^{m} |a_i|^2 \right)^{\frac{1}{2}} \ : \ (a_j)_{j=0}^m \in B_{\ell_1^{m+1}} \right\} = 1,$$

and since T_x is 2-summing with $\pi_2(T_x) \leq K_G\lambda\|x\|$, for every $x \in X$, the next inequality holds:

$$\left(\sum_{i=0}^{m} \|P_i(x)\|^2 \right)^{\frac{1}{2}} = \left(\sum_{i=0}^{m} \|T_x(e_i)\|^2 \right)^{\frac{1}{2}} \leq K_G\lambda\|x\|, \qquad \text{for every } x \in X.$$

From the above identity, it follows that

$$\|Sx\| = \left(\sum_{i=0}^{m} |x_i^*(P_i(x))|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{i=0}^{m} \|P_i(x)\|^2 \right)^{\frac{1}{2}} \leq K_G \lambda \|x\|, \qquad \text{for every } x \in X,$$

so we get that $\pi_1(S) \leq (K_G\lambda)^2$. Given that the vectors $(x_i)_{i=0}^m$ are pairwise disjoint and $\sum_{i=0}^m x_i = x$, for every $x^* \in B_{X^*}$ we have that

$$\sum_{i=0}^{m} |x^*(x_i)| \leq \sum_{i=0}^{m} |x^*|(|x_i|) = |x^*| \left(\sum_{i=0}^{m} |x_i| \right) = |x^*|(|x|).$$

Therefore, $\sup \left\{ \sum_{i=0}^m |x^*(x_i)| \,:\, x^* \in B_{X^*} \right\} \leq \|x\|$ and we finally obtain

$$\sum_{i=0}^{m} \|x_i\| = \sum_{i=0}^{m} \|Sx_i\| \leq (K_G\lambda)^2 \|x\|.$$

Since the preceding inequality does not depend on the choice of $m \in \mathbb{N}$ and $x_0, \ldots, x_m \in X$, this proves the identity (3.2).

It is straightforward to check that the map $|||\cdot|||$ defined in (3.1) is a norm on X and is complete because it is equivalent to the complete norm $\|\cdot\|$, as we have already exhibited in (3.2). It remains to show that it is indeed a lattice norm. To this end, take $x, y \in X$ such that $|x| \leq |y|$ and fix a natural number m and a finite sequence $(y_i)_{i=0}^m$ of pairwise disjoint vectors in X such that $y = \sum_{i=0}^m y_i$. By the Riesz Decomposition Property (see [7, Theorem 1.13]), there exist $x_0, \ldots, x_m \in X$ satisfying $x = \sum_{i=0}^m x_i$ and $|x_i| \leq |y_i|$ for each $i = 0, \ldots, m$. Thus, the vectors $(x_i)_{i=0}^m$ are also pairwise disjoint, and since $\|\cdot\|$ is a lattice norm, we have $\|x_i\| \leq \|y_i\|$ for each $i = 0, \ldots, m$. This implies that $|||x||| \leq |||y|||$. Finally, it is easy to check that $\|x+y\| = \|x\| + \|y\|$ for every disjoint pair $x, y \in X$, so we conclude that $|||\cdot|||$ is an AL-norm. By Kakutani's Theorem this finishes the proof. □

Corollary 3.2. *Let X be a Banach lattice which is an $\mathcal{L}_\infty$-space. Then X is lattice isomorphic to an AM-space.*

Proof. Since X is an $\mathcal{L}_\infty$-space, its dual X^* is an $\mathcal{L}_1$-space [105, Theorem III (a)]. By the previous proposition, X^* is lattice isomorphic to an L_1-space and, thus, X^{**} is lattice isomorphic to a $C(K)$-space. Since X is a sublattice of X^{**} [115, Proposition 1.4.5 (ii)], X is lattice isomorphic to a sublattice of a $C(K)$-space, which is an AM-space. □

Corollary 3.3. *Let X be a complemented subspace in an L_1-space. If X is isomorphic to a Banach lattice, then it is isomorphic to an L_1-space.*

Proof. Since X is complemented in an L_1-space, by [105, Theorem III (b)] we get that X is an $\mathcal{L}_1$-space. If X is also isomorphic to a Banach lattice, the preceding proposition ensures that X is isomorphic to an L_1-space. □

Corollary 3.4. *Let X be a separable complemented subspace in a $C(K)$-space. If X is isomorphic to a Banach lattice, then it is isomorphic to a $C(K)$-space.*

Proof. By [105, Theorem 3.2] complemented subspaces of $C(K)$-spaces are $\mathcal{L}_\infty$-spaces. If X is also isomorphic to a Banach lattice, Corollary 3.2 shows that X is isomorphic to an AM-space. The conclusion follows from the fact that separable AM-spaces are isomorphic to $C(K)$-spaces [20]. □

Remark 3.5. Corollaries 3.3 and 3.4 imply that a positive answer to the CSP for Banach lattices in the separable setting would also yield a positive answer to the CSP for L_1-spaces and for $C(K)$-spaces in the separable setting.

Remark 3.6. Proposition 3.1 can also be used to show that if a Banach lattice X is linearly isomorphic to ℓ_1, then it must be lattice isomorphic to ℓ_1 (see also [2]). Note that this does not extend to isometries, in the following sense: A Banach lattice linearly isometric to ℓ_1 need not be lattice isometric. In fact, the proof of Proposition 3.1 tells us that if a Banach lattice X is linearly isometric to ℓ_1, then it is K_G^2-lattice isomorphic to ℓ_1 (where K_G is Grothendieck's constant). Moreover, in the 2-dimensional setting it can be checked that the isomorphism constant is sharp: take ℓ_∞^2 which is linearly isometric to ℓ_1^2, and one can check that any lattice isomorphism will give constant at least 2 (which coincides with the square of Grothendieck's constant for dimension 2 [93]).

Remark 3.7. It would be natural to wonder whether Proposition 3.1 and Corollary 3.2 can be extended to $\mathcal{L}_p$-spaces for $p \in [1, +\infty]$. This is however not the case, since for $p \in (1, +\infty) \setminus \{2\}$, $\ell_p \oplus_p \ell_2$ is a Banach lattice which is also an $\mathcal{L}_p$-space, but it is not isomorphic (even as a Banach space) to any L_p-space [103, Example 8.2].

3.2 The Plebanek-Salguero space

Let us denote by $\omega = \{0, 1, 2, ...\}$ the set of non-negative integers, and write $\text{fin}(\omega)$ for the family of all finite subsets of ω. Given $\mathcal{F} \subseteq \mathcal{P}(\omega)$, $[\mathcal{F}]$ denotes the smallest Boolean subalgebra of $\mathcal{P}(\omega)$ containing $\mathcal{F}$.

A family $\mathcal{A}$ of infinite subsets of ω is *almost disjoint* whenever $A \cap B$ is finite for every distinct $A, B \in \mathcal{A}$. Every almost disjoint family $\mathcal{A}$ gives rise to a *Johnson-Lindenstrauss* space $\text{JL}(\mathcal{A})$, which is the closed linear span inside ℓ_∞ of the set of characteristic functions $\{1_n : n \in \omega\} \cup \{1_A : A \in \mathcal{A}\} \cup \{1_\omega\}$, where 1_n represents $1_{\{n\}}$. Alternatively, let us write $\mathfrak{A} = [\text{fin}(\omega) \cup \mathcal{A}]$. Then $\text{JL}(\mathcal{A})$ is precisely the closure in ℓ_∞ of the subspace $s(\mathfrak{A})$ consisting of all *simple* $\mathfrak{A}$-measurable functions; that is, functions of the form $f = \sum_{i=1}^n a_i \cdot 1_{B_i}$, where $n \in \omega$, $a_i \in \mathbb{R}$ and $B_i \in \mathfrak{A}$.

It is easy to check that $\text{JL}(\mathcal{A})$ is isometrically isomorphic to a $C(K)$-space (see [109, Theorem 1.b.6]). The underlying compact space can be realized as the Stone space consisting of all ultrafilters of $\mathfrak{A}$. More explicitly, we can define

$$K_\mathcal{A} = \omega \cup \{p_A : A \in \mathcal{A}\} \cup \{\infty\}$$

and specify a topology on $K_\mathcal{A}$ as follows:

- points in ω are isolated;
- given $A \in \mathcal{A}$, a basic neighbourhood of p_A is of the form $\{p_A\} \cup A \setminus F$, where $F \in \text{fin}(\omega)$;
- $K_\mathcal{A}$ is the one-point compactification of the locally compact space $\omega \cup \{p_A : A \in \mathcal{A}\}$.

The compact space $K_\mathcal{A}$ is often referred to as the *Alexandrov-Urysohn compact space* associated with $\mathcal{A}$. It is a separable, scattered compact space with empty third derivative. Please observe that $\text{JL}(\mathcal{A})$ coincides with the subspace $\{f|_\omega : f \in C(K_\mathcal{A})\}$ of ℓ_∞.

On the other hand, the dual of $\text{JL}(\mathcal{A})$ is isometrically isomorphic to the space $M(\mathfrak{A})$ of real-valued *finitely* additive measures on $\mathfrak{A}$. Indeed, every $\mu \in M(\mathfrak{A})$ defines a functional on $s(\mathfrak{A})$ by means of integration [48, III.2], and every functional on $\mathrm{JL}(\mathcal{A})$ arises in this way. Let us recall that the norm of any measure $\nu \in M(\mathfrak{A})$ is given by $\|\nu\| = |\nu|(\omega)$, where the *variation* $|\nu|$ is defined as

$$|\nu|(A) = \sup\{|\nu(B)| + |\nu(A \setminus B)| : B \in \mathfrak{A}, B \subseteq A\}.$$

In particular, since $\mathrm{JL}(\mathcal{A})$ is isometrically isomorphic to $C(K_{\mathcal{A}})$, and $K_{\mathcal{A}}$ is scattered, $M(\mathfrak{A})$ is isometrically isomorphic to $\ell_1(K_{\mathcal{A}}) = \overline{\operatorname{span}}\{\delta_n, \delta_{P_A}, \delta_\infty \ : \ n \in \omega,\ A \in \mathcal{A}\} \equiv \ell_1(\omega) \oplus_1 \ell_1(\mathcal{A}) \oplus_1 \mathbb{R}$. Therefore, every $\nu \in M(\mathfrak{A})$ can be decomposed as $\nu = \mu + \bar{\nu}$, where μ is supported on ω and $\bar{\nu}$ is an element of $M(\mathfrak{A})$ which vanishes on finite sets of ω.

The spaces $\mathrm{JL}(\mathcal{A})$, originally introduced in [79], have recently found use in Banach space theory as counterexamples or as a tool to produce them, including the following:

- The first example of two Banach spaces which are Lipschitz isomorphic but not linearly isomorphic [4].
- A Banach space X that cannot be linearly isometric to any Banach lattice, but such that $X^{\mathcal{U}}$ is linearly isometric to $c_0^{\mathcal{U}}$ for some ultrafilter $\mathcal{U}$ [68].
- A $C_0(K)$-space admitting only few operators and decompositions [92].
- Further applications can be found in [13, 113, 130].

They were in particular used in [131] to obtain a negative solution for the complemented subspace problem for $C(K)$-spaces.

3.2.1. General facts. The approach in [131] is to construct two almost disjoint families $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\omega \times 2)$ such that the corresponding Johnson-Lindenstrauss spaces enjoy the following properties, as stated in [131, Theorem 1.3]:

- $\mathrm{JL}(\mathcal{B})$ is isomorphic to $\mathrm{JL}(\mathcal{A}) \oplus \mathsf{PS}_2$, where both $\mathrm{JL}(\mathcal{A})$ and PS_2 are 1-complemented subspaces of $\mathrm{JL}(\mathcal{B})$.
- PS_2 is not isomorphic to any $C(K)$-space.

Let us describe how such families $\mathcal{A}$ and $\mathcal{B}$ are defined and how they interact with each other. For this, we will work in the countable set $\omega \times 2$ rather than in ω. Let us say that a subset $C \subseteq \omega \times 2$ is a *cylinder* if it is of the form $C = C_0 \times 2$ for some $C_0 \subseteq \omega$. Given $n \in \omega$, let us write $c_n = \{n\} \times 2$. A partition B^0, B^1 of a cylinder $C = C_0 \times 2$ *splits* C (or is a *splitting* of C) if for every $n \in C_0$, the sets $B^0 \cap c_n$ and $B^1 \cap c_n$ are singletons.

Consider two almost disjoint families $\mathcal{A}$ and $\mathcal{B}$ such that:

- $\mathcal{A} = \{A_\xi : \xi < \mathfrak{c}\}$ is a family of cylinders in $\omega \times 2$.
- $\mathcal{B} = \{B^0_\xi, B^1_\xi : \xi < \mathfrak{c}\}$ satisfies that the pair B^0_ξ, B^1_ξ is a splitting of A_ξ for $\xi < \mathfrak{c}$.

In this context, we will adopt the slight abuse of notation of [131, Section 3] by declaring $\mathrm{JL}(\mathcal{A})$ to be the closed subspace of $\ell_\infty(\omega \times 2)$ spanned by $\{c_n : n \in \omega\} \cup \{1_A : A \in \mathcal{A}\} \cup \{1_{\omega\times 2}\}$; that is to say, in the definition of $\mathrm{JL}(\mathcal{A})$ we consider only finite *cylinders* instead of all finite subsets of $\omega \times 2$. With these considerations, it is straightforward to see that $\mathrm{JL}(\mathcal{A})$ sits inside $\mathrm{JL}(\mathcal{B})$ as the subspace formed by all functions of $\mathrm{JL}(\mathcal{B})$ which are constant on cylinders, and the map $P : \mathrm{JL}(\mathcal{B}) \to \mathrm{JL}(\mathcal{B})$ defined as

$$Pf(n,0) = Pf(n,1) = \frac{1}{2}\big(f(n,0) + f(n,1)\big)$$

is a norm-one projection whose image is $\mathrm{JL}(\mathcal{A})$ [131, Proposition 3.1]. Let us write $X = \ker P$, so that we have $\mathrm{JL}(\mathcal{B}) = \mathrm{JL}(\mathcal{A}) \oplus X$. Then the map $Q = \mathrm{Id}_{\mathrm{JL}(\mathcal{B})} - P$ acts as

$$Qf(n,0) = -Qf(n,1) = \frac{1}{2}\big(f(n,0) - f(n,1)\big),$$

and so it is a norm-one projection onto X. Therefore both X and $\mathrm{JL}(\mathcal{A})$ are isometric to 1-complemented subspaces of $\mathrm{JL}(\mathcal{B})$, and the space X can be defined as follows:

$$X = \{f \in \mathrm{JL}(\mathcal{B}) : f(n,0) = -f(n,1) \text{ for all } n \in \omega\}. \tag{3.3}$$

In order to ensure that X is not isomorphic to a $C(K)$-space, the families $\mathcal{A}$ and $\mathcal{B}$ will be chosen to satisfy certain delicate combinatorial properties. Actually, the space which we denote by PS_2 is such a space X for a particular choice of $\mathcal{A}$ and $\mathcal{B}$ such that X is not a $C(K)$-space.

3.2.2. Norming and free subsets. We now focus on how to produce the almost disjoint families $\mathcal{A}$ and $\mathcal{B}$, intertwined as in Section 3.2.1, so that the resulting space X is not isomorphic to a $C(K)$-space. These techniques will be essential for our counterexample below. We start with the basic idea.

Definition 3.8. Given a Banach space X and a weak* closed subset K of B_{X^*}, we say that

- K is *norming* for X if there is $0 < c \le 1$ such that $\sup_{x^* \in K} |x^*(x)| \ge c\,\|x\|$ for every $x \in X$.
- K is *free* if for every $f \in C(K)$ there exists $x \in X$ such that $f(x^*) = x^*(x)$ for every $x^* \in K$.

We will speak of a *c-norming* set for X whenever we need explicit mention of the constant c in the definition of a norming set. Observe that K is a norming and free subset of B_{X^*} precisely when the natural operator $J : X \to C(K)$ defined by $J(x)(x^*) = \delta_x|_K(x^*) = x^*(x)$ is an (onto) isomorphism [130, Lemma 2.2].

The use of norming free sets makes it possible to construct certain Banach spaces which are not $C(K)$-spaces. Applied to our particular case, the idea is to prevent every candidate for a norming set in the dual ball of X to be free. We now indicate how to proceed. First, set $\mathfrak{B} := [\mathrm{fin}(\omega) \cup \mathcal{B}]$ and observe that, for any choice of the families $\mathcal{A}$ and $\mathcal{B}$ as in Section 3.2.1, X^* can be isometrically identified *in a canonical way* with the subspace $\mathrm{JL}(\mathcal{A})^\perp = \{\nu \in JL(\mathcal{B})^* = M(\mathfrak{B}) : \nu|_{\mathrm{JL}(\mathcal{A})} = 0\}$ of $M(\mathfrak{B})$, that is, $\mathrm{JL}(\mathcal{A})^\perp$ is formed by all measures of $M(\mathfrak{B})$ which vanish on every cylinder. Indeed, let us consider the operator $T : \mathrm{JL}(\mathcal{A})^\perp \to X^*$ defined by $T\nu := \nu|_X$. Given $x^* \in X^*$, we define $\nu = Q^* x^* \in \mathrm{JL}(\mathcal{B})^*$, which satisfies that for every $f \in \mathrm{JL}(\mathcal{A})$, $\nu(f) = Q^*x^*(f) = x^*(Qf) = x^*(0) = 0$ (and so $\nu \in \mathrm{JL}(\mathcal{A})^\perp$) and $\nu|_X = x^*$ given that $Qf = f$ for all $f \in X$). The latter shows that T is surjective and it remains to show that T is norm-preserving. Given any $\nu \in \mathrm{JL}(\mathcal{A})^\perp$, we have

$$\begin{aligned}\|T\nu\| &= \sup_{f \in B_X} \|\nu(f)\| \le \sup_{f \in B_{\mathrm{JL}(\mathcal{B})}} \|\nu(f)\| = \sup_{f \in B_{\mathrm{JL}(\mathcal{B})}} \|\nu(Pf) + \nu(Qf)\| \\ &= \sup_{f \in B_{\mathrm{JL}(\mathcal{B})}} \|\nu(Qf)\| \le \sup_{f \in B_X} \|\nu(f)\| = \|T\nu\|,\end{aligned}$$

so $\|T\nu\| = \sup_{f \in B_{\mathrm{JL}(\mathcal{B})}} \|\nu(f)\| = \|\nu\|$.

Therefore, every functional $\nu \in X^* \equiv \mathrm{JL}(\mathcal{A})^\perp \subseteq \mathrm{JL}(\mathcal{B})^* \equiv \ell_1(\omega \times 2) \oplus_1 \ell_1(\mathcal{B}) \oplus_1 \mathbb{R}$ can be seen as a pair of measures $(\mu, \bar{\nu}) \in \ell_1(\omega \times 2) \oplus_1 \ell_1(\mathfrak{c} \times 2)$ where $\mu(n,0) = -\mu(n,1)$ for every $n \in \omega$ and $\bar{\nu}(\xi, 0) = -\bar{\nu}(\xi, 1)$ for every $\xi < \mathfrak{c}$, where here we are using the notation $\mu(n,i) := \mu\big(\{(n,i)\}\big)$ and $\bar{\nu}(\xi, i) := \bar{\nu}(B^i_\xi)$, for $i \in \{0,1\}$.

On the other hand, for every $n \in \omega$, the function

$$f_n = 1_{(n,0)} - 1_{(n,1)} \tag{3.4}$$

is always a norm-one element of X. Hence, every norming set for X must contain a sequence of functionals $(\nu_n)_{n\in\omega}$ such that $\inf_{n\in\omega} |\nu_n(n,0)| = \frac{1}{2} \inf_{n\in\omega} |\nu_n(f_n)| > 0$. This motivates the following definition:

Definition 3.9. A bounded sequence $(\mu_n)_{n\in\omega}$ in $\ell_1(\omega \times 2)$ is *admissible* if

- $\inf_{n\in\omega} |\mu_n(n,0)| > 0$.
- $\mu_k(n,0) = -\mu_k(n,1)$ for every $k, n \in \omega$.

Remark 3.10. Consequently, every norming set for X must contain a sequence $(\nu_n)_{n\in\omega} = (\mu_n, \bar{\nu}_n)_{n\in\omega} \in \ell_1(\omega \times 2) \oplus_1 \ell_1(\mathfrak{c} \times 2)$ such that:

i) $(\mu_n)_{n\in\omega}$ is admissible;

ii) $\bar{\nu}_n(\alpha, 0) + \bar{\nu}_n(\alpha, 1) = 0$ for every $n \in \omega$ and $\alpha < \mathfrak{c}$;

iii) Since for every natural n, $|\bar{\nu_n}|(B) > 0$ for at most countably many $B \in \mathcal{B}$, then there exists $\xi < \mathfrak{c}$ such that $\bar{\nu}_n(\alpha, i) = 0$ for all $\alpha \geq \xi$, $n \in \omega$ and $i \in \{0,1\}$.

The main idea of [131] is to prevent every sequence $(\nu_n)_{n\in\omega}$ contained in $B_{\mathrm{JL}(\mathcal{A})^\perp}$ of the form described in the preceding remark from lying inside a free subset in the dual unit ball of PS_2. In this way, there are no norming free subsets for PS_2, and therefore it cannot be isomorphic to a $C(K)$-space.

Our subsequent argument also makes use of admissible sequences to prove that PS_2 is, in fact, not isomorphic to a Banach lattice. This proof relies on the following simple observation:

Lemma 3.11. *[131, Lemma 3.3] Assume $\mathfrak{B}$ is a Boolean subalgebra of $\mathcal{P}(\omega)$. If $M \subseteq M_1(\mathfrak{B}) = B_{\mathrm{JL}(\mathcal{B})^*}$ lies inside a free subset, then for every $B \in \mathfrak{B}$ and $\varepsilon > 0$ there is $g \in s(\mathfrak{B})$ such that $\big|\langle \mu, g\rangle - |\mu(B)|\big| < \varepsilon$ for every $\mu \in M$.*

3.2.3. Separation of measures. Let $M_1(\mathfrak{B})$ denote the unit ball of the space $M(\mathfrak{B})$.

Definition 3.12. Given a Boolean subalgebra $\mathfrak{B}$ of $\mathcal{P}(\omega \times 2)$, two subsets of measures $M, M' \subseteq M_1(\mathfrak{B})$ are *$\mathfrak{B}$-separated* if there is $\varepsilon > 0$ and a finite collection $B_1, ..., B_n \in \mathfrak{B}$ such that for every pair $(\mu, \mu') \in M \times M'$, there is $k \in \{1, ..., n\}$ such that $|\mu(B_k) - \mu'(B_k)| \geq \varepsilon$.

The notion of $\mathfrak{B}$-separation is essential in the construction of PS_2. In particular, Definition 3.12 in tandem with Lemma 3.11 is what prevents a certain sequence of measures from lying inside a free set. We will also need the following fact about $\mathfrak{B}$-separation to show that PS_2 is not isomorphic to a Banach lattice:

Lemma 3.13. *[131, Lemma 4.2] Let M, $M' \subseteq M_1(\mathfrak{B})$ be two sets of measures. If there exist $\varepsilon > 0$ and a simple $\mathfrak{B}$-measurable function g such that for every $(\mu, \mu') \in M \times M'$ we have $|\langle \mu, g\rangle - \langle \mu', g\rangle| \geq \varepsilon$, then M and M' are $\mathfrak{B}$-separated.*

3.2.4. The heart of the construction of PS_2. The almost disjoint families $\mathcal{A}$ and $\mathcal{B}$ are constructed through an inductive process of length $\mathfrak{c}$ which is explained in [131, Section 6]. Let us now describe this process paying special attention to the properties that will be needed later to show that PS_2 is not isomorphic to a Banach lattice.

Recall that the idea is to construct a family $\mathcal{A} = \{A_\xi : \xi < \mathfrak{c}\}$ of cylinders in $\omega \times 2$ and define suitable splittings B^0_ξ, B^1_ξ of A_ξ for every $\xi < \mathfrak{c}$. Given any $\Lambda \subseteq \mathfrak{c}$, we will denote $\mathfrak{B}(\Lambda) = [\mathrm{fin}(\omega \times 2) \cup \{B^0_\alpha, B^1_\alpha \colon \alpha \in \Lambda\}]$. In particular, $\mathfrak{B}(\xi)$ stands for $\mathfrak{B}(\{\alpha : \alpha < \xi\})$, and the final algebra is denoted $\mathfrak{B} = [\mathrm{fin}(\omega \times 2) \cup \{B^0_\alpha, B^1_\alpha \colon \alpha < \mathfrak{c}\}]$.

Let us explain how the sets B^0_ξ, B^1_ξ are obtained for any given $\xi < \mathfrak{c}$. First, observe that we can "code" *all* sequences in $M_1(\mathfrak{B})$ of the form detailed in Remark 3.10, $(\nu^\xi_n)_{n\in\omega} = (\mu^\xi_n, \bar{\nu}^\xi_n)_{n\in\omega}$ for $\xi < \mathfrak{c}$, in such a way that $\bar{\nu}^\xi_n(\alpha, i) = 0$ for all $\alpha \geq \xi$, $i = 0, 1$ and every $n \in \omega$. Moreover, every sequence $(\nu_n)_{n\in\omega}$ in the unit ball of $\ell_1(\omega \times 2) \oplus_1 \ell_1(\mathfrak{c} \times 2)$ which satisfies properties *i)–iii)* of the aforementioned remark is of the form $(\nu^\xi_n)_{n\in\omega}$ for exactly one $\xi < \mathfrak{c}$.

The sets B^0_ξ and B^1_ξ are defined, together with three infinite auxiliary subsets $J^\xi_2 \subseteq J^\xi_1 \subseteq J^\xi_0 \subseteq \omega$, so that the sequence $(\nu^\xi_n)_{n\in\omega}$ cannot eventually lie in a free set of $M_1(\mathfrak{B})$. This is done as follows. First, let us define $c = \inf_{n\in\omega} |\mu^\xi_n(n, 0)|$, which is a strictly positive number, and let p^ξ_n be the one element subset of $c_n = \{(n, 0), (n, 1)\}$ for which $\mu^\xi_n(p^\xi_n) > 0$. Now, consider three infinite subsets $J^\xi_2 \subseteq J^\xi_1 \subseteq J^\xi_0 \subseteq \omega$ such that the differences $\omega \setminus J^\xi_0$, $J^\xi_0 \setminus J^\xi_1$ and $J^\xi_1 \setminus J^\xi_2$ are also infinite, and in such a way that the following assertions are verified for some fixed $\delta \in (0, \frac{c}{16})$:

- (P1) For every $n \in J^\xi_0$, $|\mu^\xi_n|\big((J^\xi_0 \times 2) \setminus c_n\big) < \delta$ (this is exactly (5.a) in the proof of [131, Lemma 5.3]).
- (P2) There is $a \geq c$ such that $|\mu^\xi_n(p^\xi_n) - a| < \delta$ for every $n \in J^\xi_1$ (this is (5.b) in the proof of [131, Lemma 5.3]).
- (P3) For any $\alpha < \xi$, the pairs
  - $\{\nu^\alpha_n : n \in J^\alpha_2\}$ and $\{\nu^\alpha_n : n \in J^\alpha_1 \setminus J^\alpha_2\}$,
  - $\{\nu^\alpha_n : n \in J^\alpha_1\}$ and $\{\nu^\alpha_n : n \in J^\alpha_0 \setminus J^\alpha_1\}$

  are not $\mathfrak{B}(\xi \setminus \{\alpha\})$-separated (this is exactly the Key Property in [131, p. 16]).

The justification of the existence of such a trio of sets can be found in [131, p. 16]. We also remark the fact that although the computations in [131, Lemma 5.3] require that $0 < \delta < c/16$, in the proof of our main Theorem 3.19 we will only need to assume that $\delta < c/11$. In any case, with the sets J^ξ_0, J^ξ_1 and J^ξ_2 in our power, we finally declare:

$$B^0_\xi = \left(\bigcup_{n\in J^\xi_2} p^\xi_n\right) \cup \left(\bigcup_{n\in J^\xi_1\setminus J^\xi_2} c_n \setminus p^\xi_n\right), \quad B^1_\xi = (J^\xi_1 \times 2) \setminus B^0_\xi.$$

To conclude with the construction, one needs to ensure that, as a consequence of (P1)–(P3), $(\nu^\xi_n)_{n\in\omega}$ cannot lie inside a free subset of $M(\mathfrak{B})$. This is taken care of in [131, Lemmata 5.3 and 5.5]. Our proof that PS_2 is not isomorphic to a Banach lattice is also substantially based on Properties (P1)–(P3). We now record two observations which will pave the way in the next Sections.

Remark 3.14. The final algebra $\mathfrak{B}$ is such that (P3) is satisfied for every $\xi < \mathfrak{c}$. This immediately implies that, given any $\xi < \mathfrak{c}$, the pairs

- $\{\nu_n^\xi : n \in J_2^\xi\}$ and $\{\nu_n^\xi : n \in J_1^\xi \setminus J_2^\xi\}$,
- $\{\nu_n^\xi : n \in J_1^\xi\}$ and $\{\nu_n^\xi : n \in J_0^\xi \setminus J_1^\xi\}$

are not $\mathfrak{B}(\mathfrak{c} \setminus \{\xi\})$-separated.

In the sequel, the following remark will be applied to functions of the form $f = 1_{B_0^\xi} - 1_{B_1^\xi}$ for a given $\xi < \mathfrak{c}$.

Remark 3.15. Fix any $\xi < \mathfrak{c}$ and consider the sequence $(\nu_n^\xi)_{n\in\omega}$ in $M_1(\mathfrak{B})$. Since $\bar{\nu}_n^\xi(\alpha, i) = 0$ for every $\alpha \geq \xi$ and $i \in \{0,1\}$, we have $|\bar{\nu}_n^\xi|(B_\alpha^i) = 0$ whenever $\alpha \geq \xi$ and $i \in \{0,1\}$. Hence, for any $n \in \omega$, ν_n^ξ agrees with μ_n^ξ inside any set B_α^i for $\alpha \geq \xi$ and $i \in \{0,1\}$. In particular, if $f \in \mathsf{PS}_2$ has its support contained in the set $B_\xi^0 \cup B_\xi^1$, then $\langle \nu_n^\xi, f\rangle = \langle \mu_n^\xi, f\rangle$ for every $n \in \omega$.

3.3 The Plebanek-Salguero space is not a Banach lattice

We now combine the results in Section 3.1 with the fundamental properties of the space PS_2 to show that it is not linearly isomorphic to any Banach lattice. First of all, notice that PS_2^* is a 1-complemented subspace of $\mathrm{JL}(\mathcal{B})^* \equiv C(K_\mathcal{B})^* \equiv \ell_1(K_\mathcal{B})$ and therefore linearly isometric to $\ell_1(\Gamma)$ for some Γ. Furthermore, since the set $\{\delta_{(n,0)}, \delta_{(n,1)} : n \in \omega\}$ is 1-norming for $\mathrm{JL}(\mathcal{B})$, just taking the restrictions to PS_2 we deduce that PS_2 also has a countable 1-norming set. These two characteristics of PS_2 will make easier to prove that this space cannot be isomorphic to a Banach lattice, as the next proposition shows.

Proposition 3.16. *Let X be an isomorphic predual of $\ell_1(\Gamma)$ which has a countable norming set. If X is isomorphic to a Banach lattice, then it is isomorphic to a sublattice of ℓ_∞.*

Proof. Let Y be a Banach lattice which is isomorphic to X. Since X is a predual of $\ell_1(\Gamma)$, Y is an $\mathcal{L}_\infty$-space. Hence, by Corollary 3.2, we may assume that Y is an AM-space. Then, Y^* is an AL-space isomorphic to $\ell_1(\Gamma)$, so Y^* is, in fact, lattice isometric to $\ell_1(\Gamma)$ (see [94, Corollary to Theorem 3 of Section 15 and Theorem 4 of Section 18]). Let us denote by $(e_\gamma^*)_{\gamma\in\Gamma}$ the canonical basis of $\ell_1(\Gamma)$ and let $(y_n^*)_{n\in\omega}$ be a countable c-norming set in B_{Y^*} for some $c > 0$. We can write each y_n^* as

$$y_n^* = \sum_{\gamma\in\Gamma} \lambda_\gamma^n e_\gamma^*,$$

where $\sum_{\gamma\in\Gamma} |\lambda_\gamma^n| \leq 1$. Thus, for every $n \in \omega$, the set $S_n = \{\gamma \in \Gamma : \lambda_\gamma^n \neq 0\}$ is countable and, consequently, $S = \bigcup_{n\in\omega} S_n$ is also a countable set. We claim that the set $\{e_\gamma^* : \gamma \in S\}$ is c-norming for Y. Indeed, let us first note that for every $y \in Y$ we have

$$|y_n^*(y)| = \left|\sum_{\gamma\in S_n} \lambda_\gamma^n e_\gamma^*(y)\right| = \sum_{\gamma\in S_n} |\lambda_\gamma^n||e_\gamma^*(y)| \leq \left(\sum_{\gamma\in S_n} |\lambda_\gamma^n|\right) \sup_{\gamma\in S_n} |e_\gamma^*(y)| \leq \sup_{\gamma\in S_n} |e_\gamma^*(y)|.$$

From the latter we deduce that

$$c\,\|y\| \leq \sup_{n\in\omega} |y_n^*(y)| \leq \sup_{n\in\omega}\sup_{\gamma\in S_n} |e_\gamma^*(y)| \leq \sup_{\gamma\in S} |e_\gamma^*(y)| \quad \text{for all } y \in Y.$$

Finally, observe that Y is lattice embeddable into $\ell_\infty(S)$ through the lattice embedding given by $y \mapsto \big(e_\gamma^*(y)\big)_{\gamma\in S}$. □

The preceding proposition motivates the following definition:

Definition 3.17. We say that a Banach space X has the **Desired Property (DP)** if for every norming sequence $(e_n^*)_{n\in\omega}$ in X^* there exists an element $f \in X$ such that no element $g \in X$ satisfies

$$e_n^*(g) = |e_n^*(f)| \text{ for every } n \in \omega.$$

We will see next that a Banach space has the (DP) if and only if it is not isomorphic to a sublattice of ℓ_∞. Therefore, by Proposition 3.16, in order to prove that PS_2 is not isomorphic to a Banach lattice it will be sufficient to check that this space has the (DP).

Corollary 3.18. *Given an isomorphic predual X of $\ell_1(\Gamma)$ which has a countable norming set, the following statements are equivalent:*

(i) X is isomorphic to a Banach lattice.

(ii) X is isomorphic to a sublattice of ℓ_∞.

(iii) X does not have the (DP). That is, there exists a norming sequence $(x_n^*)_{n\in\omega}$ in B_{X^*} such that for every $f \in X$ there is an element $g \in X$ such that

$$x_n^*(g) = |x_n^*(f)|, \text{ for every } n \in \omega.$$

Proof. $(i) \Leftrightarrow (ii)$ is just Proposition 3.16.

$(ii) \Rightarrow (iii)$. Let $T\colon X \to Y$ be an invertible operator onto a sublattice Y of ℓ_∞ and let $C = \|T\|\|T^{-1}\|$. If we denote by $(e_n^*)_{n\in\omega} \subseteq \ell_\infty^*$ the canonical basis of ℓ_1, the natural order in ℓ_∞ is given by

$$f \leq g \quad \text{if and only if} \quad e_n^*(f) \leq e_n^*(g), \quad \text{for every } n \in \omega.$$

It is clear that $(e_n^*)_{n\in\omega}$ is 1-norming in ℓ_∞ and, hence, the sequence of restrictions

$$y_n^* := e_n^*\big|_Y, \quad n \in \omega$$

is 1-norming in Y. Now, define

$$x_n^* := \frac{1}{\|T\|} T^* y_n^*, \quad n \in \omega.$$

It is straightforward to check that $(x_n^*)_{n\in\omega} \subseteq B_{X^*}$ is $1/C$-norming in X and, given $f \in X$, if we take $g = T^{-1}|Tf|$, then for every $n \in \omega$ we have

$$x_n^*(g) = \frac{1}{\|T\|} T^* y_n^*\big(T^{-1}|Tf|\big) = \frac{1}{\|T\|} y_n^*\big(|Tf|\big) = |x_n^*(f)|.$$

$(iii) \Rightarrow (ii)$. Suppose that X fails the (DP). That is, there exists a norming sequence $(x_n^*)_{n\in\omega} \subseteq B_{X^*}$ such that for every $f \in X$ there is an element $g \in X$ such that $x_n^*(g) = \big|x_n^*(f)\big|$ for every $n \in \omega$. We define the following mapping

$$\begin{aligned} T : X &\longrightarrow \quad \ell_\infty \\ f &\longmapsto \big(x_n^*(f)\big)_{n\in\omega}, \end{aligned}$$

which is clearly linear and bounded below (since $(x_n^*)_{n\in\omega}$ is norming). Hence, $Y = T(X)$ is a closed subspace of ℓ_∞. Moreover, it is a sublattice. Indeed, by hypothesis, for any $\big(x_n^*(f)\big)_{n\in\omega} \in Y$ there exists $g \in X$ such that $\big(x_n^*(g)\big)_{n\in\omega} = \big(|x_n^*(f)|\big)_{n\in\omega} = \big|\big(x_n^*(f)\big)_{n\in\omega}\big|$; that is, the absolute value of $\big(x_n^*(f)\big)_{n\in\omega}$ also belongs to Y. □

Theorem 3.19. PS_2 *is not isomorphic to a Banach lattice.*

Proof. We will prove this fact by showing that PS_2 has the (DP). Fix a norming sequence $(e_n^*)_{n\in\omega}$ in $B_{\mathsf{PS}_2^*}$. Our aim is to find an $f \in \mathsf{PS}_2$ such that no $g \in \mathsf{PS}_2$ satisfies

$$\langle e_n^*, g\rangle = |\langle e_n^*, f\rangle|, \qquad \text{for every } n \in \omega. \tag{3.5}$$

The very definition of PS_2 allows us to write, for every $n \in \omega$, $e_n^* = \mu_n + \bar{\nu}_n$, where $\mu_n \in \ell_1(\omega \times 2)$ and $\bar{\nu}_n \in \ell_1(\mathfrak{c} \times 2)$ such that $\|\mu_n\|_1 + \|\bar{\nu}_n\|_1 \le 1$ and $\mu_n(k,0) = -\mu_n(k,1)$ for every $k \in \omega$ and $\bar{\nu}_n(\alpha,0) = -\bar{\nu}_n(\alpha,1)$ for every $\alpha < \mathfrak{c}$, since we are identifying PS_2^* with $\mathrm{JL}(\mathcal{A})^\perp$ (see Section 3.2.2 for more details). Given that every element of $(\bar{\nu}_n)_{n\in\omega}$ vanishes on finite subsets of $\omega \times 2$, we have

$$e_n^*(f_k) = e_n^*\left(1_{(k,0)} - 1_{(k,1)}\right) = \mu_n(k,0) - \mu_n(k,1) = 2\,\mu_n(k,0), \quad \text{for all } k, n \in \omega,$$

where f_k had already been defined in equation (3.4).

In addition, as $(e_n^*)_{n\in\omega}$ is a norming set, there exists $\tilde{c} > 0$ such that

$$2\sup_n |\mu_n(k,0)| = \sup_n |e_n^*(f_k)| \ge \tilde{c}, \qquad \text{for every } k \in \omega.$$

Since $\sup_n |\mu_n(k,0)| > \frac{\tilde{c}}{4}$ for every $k \in \omega$, there exists a function $\pi : \omega \to \omega$ such that $\left|\mu_{\pi(k)}(k,0)\right| > \frac{\tilde{c}}{4}$ for every $k \in \omega$. Moreover, as $\|\mu_n\|_1 \le 1$ for all $n \in \omega$, it follows that the set $\pi^{-1}(n)$ must be finite for every $n \in \omega$. Therefore, we can find an infinite subset $\omega_0 \subseteq \omega$ such that $\pi|_{\omega_0}$ is injective. Consequently, $(e_{\pi(n)}^*)_{n\in\omega_0} = (\mu_{\pi(n)}, \bar{\nu}_{\pi(n)})_{n\in\omega_0}$ is a sequence of the form described in Remark 3.10. Thus, there exists $\xi < \mathfrak{c}$ such that $(e_{\pi(n)}^*)_{n\in\omega_0} = (\nu_n^\xi)_{n\in\omega}$, with the notation of Section 3.2.4. Recall that, by the way the enumeration has been carried out, we have $\bar{\nu}_n^\xi(\alpha, i) = 0$ whenever $\alpha \ge \xi$, $i \in \{0,1\}$ and $n \in \omega$. Moreover, by virtue of Remark 3.14, the pairs of measures

- $\{\nu_n^\xi : n \in J_2^\xi\}$ and $\{\nu_n^\xi : n \in J_1^\xi \setminus J_2^\xi\}$,
- $\{\nu_n^\xi : n \in J_1^\xi\}$ and $\{\nu_n^\xi : n \in J_0^\xi \setminus J_1^\xi\}$

are not $\mathfrak{B}(\mathfrak{c} \setminus \{\xi\})$-separated.

For the rest of the proof, we will drop the superindex ξ, and simply write ν_n, μ_n and $\bar{\nu}_n$ for the measures ν_n^ξ, μ_n^ξ and $\bar{\nu}_n^\xi$, respectively (we will also denote p_n instead of p_n^ξ). Now, consider the function

$$f = 1_{B_\xi^0} - 1_{B_\xi^1} \in \mathsf{PS}_2. \tag{3.6}$$

Since f is supported in $B_\xi^0 \cup B_\xi^1$, Remark 3.15 asserts that $\langle \nu_n, f\rangle = \langle \mu_n, f\rangle$ for every $n \in \omega$. Let us suppose that there exists an element $g \in \mathsf{PS}_2$ such that

$$\langle \nu_n, g\rangle = |\langle \nu_n, f\rangle| = |\langle \mu_n, f\rangle| \text{ for every } n \in \omega.$$

We will arrive at a contradiction with the separation of the above pairs of sets. Note that the latter means that the function f defined in (3.6) cannot have an absolute value in PS_2 with respect to the sequence $(\nu_n)_{n\in\omega} = (e^*_{\pi(n)})_{n\in\omega_0}$; in particular, f cannot have an absolute value with respect to $(e^*_n)_{n\in\omega}$ (which is what we were looking for in (3.5)). Our argument closely follows that of [131, Lemma 5.3]. First, we introduce some notation: given $a, b \in \mathbb{R}$ and $\delta > 0$, we write $a \approx_\delta b$ to mean $|a-b| < \delta$.

Let us pick $\delta > 0$ satisfying property (P1). Since the subspace of PS_2 consisting of all simple $\mathfrak{B}$-measurable functions in PS_2 is dense, there is such a function $h \in \mathsf{PS}_2$ such that $\|g-h\| < \delta$. Therefore,

$$|\langle \mu_n, f\rangle| = |\langle \nu_n, f\rangle| \approx_\delta \langle \nu_n, h\rangle, \text{ for every } n \in \omega.$$

Without loss of generality, we assume that $h = rf + s$, where $r \in \mathbb{R}$ and s is a simple $\mathfrak{B}(\mathfrak{c}\setminus\{\xi\})$-measurable function lying in PS_2. Let us further suppose that $r \geq 0$. Otherwise, we may apply our argument to the function $-g$ instead of g; that is, if we show that $-|f|$ cannot exist, then neither can $|f|$. Hence, we have:

$$|\langle \mu_n, f\rangle| \approx_\delta r\langle \mu_n, f\rangle + \langle \nu_n, s\rangle, \text{ for every } n \in \omega. \tag{3.7}$$

Now, observe that properties (P1) and (P2), together with the definition of B^0_ξ and B^1_ξ, yield, for every $n \in J^\xi_0$,

$$\langle \mu_n, f\rangle \approx_\delta \int_{c_n} f d\mu_n = f(p_n)\mu_n(p_n) + f(c_n\setminus p_n)\mu_n(c_n\setminus p_n) \approx_{2\delta} \begin{cases} 2a & \text{if } n \in J^\xi_2, \\ -2a & \text{if } n \in J^\xi_1 \setminus J^\xi_2, \\ 0 & \text{if } n \in J^\xi_0 \setminus J^\xi_1. \end{cases}$$

Hence,

$$\langle \mu_n, f\rangle \approx_{3\delta} \begin{cases} 2a & \text{if } n \in J^\xi_2, \\ -2a & \text{if } n \in J^\xi_1 \setminus J^\xi_2, \\ 0 & \text{if } n \in J^\xi_0 \setminus J^\xi_1. \end{cases} \tag{3.8}$$

We deduce from the previous equation that

$$|\langle \mu_n, f\rangle| \approx_{3\delta} \begin{cases} 2a & \text{if } n \in J^\xi_1, \\ 0 & \text{if } n \in J^\xi_0 \setminus J^\xi_1. \end{cases} \tag{3.9}$$

Finally, using (3.8) and (3.9), we infer from (3.7) the following relations:

$$\begin{cases} 2a \approx_{\delta(4+3r)} 2ra + \langle \nu_n, s\rangle & \text{if } n \in J^\xi_2, \\ 2a \approx_{\delta(4+3r)} -2ra + \langle \nu_n, s\rangle & \text{if } n \in J^\xi_1 \setminus J^\xi_2, \\ 0 \approx_{\delta(4+3r)} \langle \nu_n, s\rangle & \text{if } n \in J^\xi_0 \setminus J^\xi_1. \end{cases} \tag{3.10}$$

First, suppose that $0 \leq r \leq 1/2$. The first two relations of (3.10) give, for every $n \in J^\xi_1$,

$$\langle \nu_n, s\rangle \geq 2(1-r)a - \delta(4+3r) \geq a - \frac{11}{2}\delta,$$

while the third one gives, for every $k \in J^\xi_0 \setminus J^\xi_1$,

$$\langle \nu_k, s\rangle \leq \delta(4+3r) \leq \frac{11}{2}\delta.$$

Thus, for any $n \in J_1^\xi$ and any $k \in J_0^\xi \setminus J_1^\xi$, we have, using $\delta < c/11$ and $a \geq c$,

$$\langle \nu_n, s\rangle - \langle \nu_k, s\rangle \geq a - 11\delta > 0.$$

This already implies (see Lemma 3.13) that the sets $\{\nu_n : n \in J_1^\xi\}$ and $\{\nu_n : n \in J_0^\xi \setminus J_1^\xi\}$ are $\mathfrak{B}(\mathfrak{c} \setminus \{\xi\})$-separated. On the other hand, if $r \geq 1/2$, then using relations (3.10) again, we infer that for every $n \in J_1^\xi \setminus J_2^\xi$ and every $k \in J_2^\xi$

$$\begin{aligned}\langle \nu_n, s\rangle - \langle \nu_k, s\rangle &\geq 2a(1+r) - \delta(4+3r) - \big(2a(1-r) + \delta(4+3r)\big)\\ &= 2r(2a - 3\delta) - 8\delta \geq 2a - 11\delta > 0.\end{aligned}$$

Hence, the sets $\{\nu_n : n \in J_2^\xi\}$ and $\{\nu_n : n \in J_1^\xi \setminus J_2^\xi\}$ are $\mathfrak{B}(\mathfrak{c} \setminus \{\xi\})$-separated, again by Lemma 3.13. Thus, in both cases we arrive at a contradiction. □

3.3.1. Relation to other classes of $\mathcal{L}_\infty$-spaces. Apart from the class of AM-spaces, other well-known classes of $\mathcal{L}_\infty$-spaces are those of G-spaces and $C_\sigma(K)$-spaces. *G-spaces* can be characterized as the closed subspaces X of some $C(K)$ for which there exists a certain set of triples $A = \{(t_\alpha, t'_\alpha, \lambda_\alpha) : \alpha \in \Gamma\} \subseteq K \times K \times \mathbb{R}$ so that $X = \{f \in C(K) : f(t_\alpha) = \lambda_\alpha f(t'_\alpha) \text{ for all } \alpha \in \Gamma\}$. On the other hand, $C_\sigma(K)$-*spaces* are the closed subspaces X of some $C(K)$ which are of the form

$$X = \{f \in C(K) : f(\sigma t) = -f(t) \text{ for all } t \in K\},$$

where $\sigma : K \to K$ is a homeomorphism with $\sigma^2 = \mathrm{Id}$.

These classes are also rather natural in the context of 1-complemented subspaces. Indeed, G-spaces are precisely those Banach spaces which are 1-complemented in some AM-space, while a Banach space is 1-complemented in a $C(K)$-space if and only if it is a $C_\sigma(K)$-space [110, Theorem 3].

It is clear that $C_\sigma(K)$-spaces are G-spaces. On the other hand, Kakutani's representation theorem asserts that every AM-space X is of the form $X = \{f \in C(K) : f(t_\alpha) = \lambda_\alpha f(t'_\alpha) \text{ for all } \alpha \in \Gamma\}$ for some compact space K and a certain set of triples $A = \{(t_\alpha, t'_\alpha, \lambda_\alpha) : \alpha \in \Gamma\} \subseteq K \times K \times [0, \infty)$. Therefore, AM-spaces are in particular G-spaces. The properties of PS_2 and Theorem 3.19 show that the latter are a strictly larger class:

Corollary 3.20. *There is a $C_\sigma(K)$-space which is not isomorphic to an AM-space. In particular, there is a G-space which is not isomorphic to an AM-space.*

In fact, equation (3.3) witnesses PS_2 is a $C_\sigma(K)$-space for $K = K_{\mathcal{B}}$: we can write

$$\mathsf{PS}_2 = \{f \in C(K_{\mathcal{B}}) : f(\sigma p) = -f(p) \text{ for all } p \in K_{\mathcal{B}}\},$$

where $\sigma : K_{\mathcal{B}} \to K_{\mathcal{B}}$ is defined as

$$\sigma(n,i) = (n, 1-i), \qquad \sigma(p_{B_\xi^i}) = p_{B_\xi^{i-1}}, \qquad \sigma(\infty) = \infty,$$

for $n \in \omega$, $i \in \{0,1\}$ and $\xi < \mathfrak{c}$. This yields two interesting consequences. First, let us observe that such map σ has ∞ as its only fixed point. It is shown in [94, Corollary of Theorem 10, Section 10] that every $C_\sigma(K)$-space, where K is a scattered compact space and σ has no fixed points, is isometrically isomorphic to a $C(K)$-space. This result is no longer true if σ has only one fixed point, as the existence of PS_2 shows.

On the other hand, in [69, Theorem 7] it is shown that a Banach space X has an ultrapower isometric to an ultrapower of c_0 if and only if X is isometric to a $C_\sigma(K)$-space for K a totally disconnected compact space with a dense subset of isolated points and such that σ has a unique fixed point which is not isolated in K. We therefore conclude that $(\mathsf{PS}_2)^{\mathcal{U}}$ is isometric to $(c_0)^{\mathcal{U}}$ for some ultrafilter $\mathcal{U}$. This should be compared with [68, Theorem 4.1], in which the authors construct a Banach space X such that $X^{\mathcal{U}}$ is isometric to $(c_0)^{\mathcal{U}}$ for some ultrafilter $\mathcal{U}$, but X is not *isometric* to any Banach lattice.

3.4 The CSP for complex Banach lattices

In this section, we will show how PS_2 can be *modified* to provide a negative solution to the Complemented Subspace Problem for *complex Banach lattices*. We will explain how to construct a variation $\widetilde{\mathsf{PS}_2}$ of Plebanek-Salguero's space in such a way $\widetilde{\mathsf{PS}_2} \oplus i\widetilde{\mathsf{PS}_2}$ is a 1-complemented subspace of a complex $C(K)$-space which cannot be isomorphic to any complex Banach lattice. We recall that a *complex Banach lattice* is the complexification $X_\mathbb{C} = X \oplus iX$ of a real Banach lattice X, equipped with the norm $\|x+iy\|_{X_\mathbb{C}} = \||x+iy|\|_X$, where $|\cdot| : X_\mathbb{C} \to X_+$ is *the modulus* map given by

$$|x+iy| = \sup_{\theta\in[0,2\pi]} \{x\cos\theta + y\sin\theta\}, \qquad \text{for every } x+iy \in X_\mathbb{C}. \tag{3.11}$$

We refer the reader to Section 1.3 for more terminology concerning complex Banach lattices.

As we mentioned in the introduction, one of the main motivations that led us to consider the complex version of the CSP is the following result of Kalton and Wood [89] (see also [56, 137]): every 1-complemented subspace of a complex space with a 1-unconditional basis has a 1-unconditional basis. This result is not true in the real case (see, for instance, [22],[97]), even though it is still unknown whether every complemented subspace of a space with an unconditional basis also has an unconditional basis. Additionally, there are other results for complex Banach lattices which fail in the real setting. Let us recall the following facts:

- An M-projection P on $X_\mathbb{C}$ (respectively, an L-projection), i.e. a projection satisfying $\|x\| = \max\{\|Px\|, \|x-Px\|\}$ for every $x \in X_\mathbb{C}$ (resp., $\|x\| = \|Px\| + \|x-Px\|$), is always a band projection [46].
- If a complex Banach lattice can be written as $X_\mathbb{C} = E \oplus F$ such that $\|x+y\| = \||x| \vee |y|\|$ for all $x \in E$ and $y \in F$, then $|x| \wedge |y| = 0$ for $x \in E$, $y \in F$; in other words, $E \oplus F$ is a band decomposition of $X_\mathbb{C}$ [87].

For the purpose of providing a counterexample to the CSP for complex Banach lattices, a natural approach would be to take $(\mathsf{PS}_2)_\mathbb{C} = \mathsf{PS}_2 \oplus i\mathsf{PS}_2$, the complexification of PS_2, which is one complemented in $\mathrm{JL}(\mathcal{B})_\mathbb{C} \equiv C_\mathbb{C}(K_\mathcal{B})$ (which is a complex Banach lattice). Nevertheless, we do not know whether $(\mathsf{PS}_2)_\mathbb{C}$ is isomorphic to a complex Banach lattice. Instead, we will show how the construction of PS_2 can be slightly modified in order to give a negative solution to the CSP in the complex setting as well. This variation of PS_2, which we denote by $\widetilde{\mathsf{PS}_2}$, will have the same form as the space X described in Subsection 3.2.1, so it will also be 1-complemented in a $C(K)$-space; but $\widetilde{\mathsf{PS}_2}$ will have the additional feature that its complexification cannot be isomorphic to a complex Banach lattice. In particular, it will not be isomorphic to a real Banach lattice either (see Corollary 3.24 below).

We start with a complex version of the notion of admissible sequence (cf. Definition 3.9).

Definition 3.21. We say that a sequence $(\mu_n)_{n\in\omega}$ in the unit ball of $\ell_1(\omega\times 2)\oplus i\ell_1(\omega\times 2)$ is $\mathbb{C}$-*admissible* if $\mu_n(k,0) = -\mu_n(k,1)$ for every $n,k\in\omega$ and $\inf_{n\in\omega}|\mu_n(n,0)| > 0$.

Note that if $\mu_n(c_k) = 0$ for every $n,k\in\omega$, then it follows that $\mathfrak{Re}\,\mu_n(c_k) = \mathfrak{Im}\,\mu_n(c_k) = 0$ for $n,k\in\omega$. Nevertheless, this does not imply that $\inf_{n\in\omega}|\mathfrak{Re}\,\mu_n(n,0)| > 0$ or $\inf_{n\in\omega}|\mathfrak{Im}\,\mu_n(n,0)| > 0$, so $(\mathfrak{Re}\,\mu_n)_{n\in\omega}$ or $(\mathfrak{Im}\,\mu_n)_{n\in\omega}$ are not necessarily *real* admissible sequences. This is an obvious obstruction to directly show that the complexification of PS_2 cannot be isomorphic to a complex Banach lattice. Instead, working directly with the complex version of the notion of *admissibility* (Definition 3.21) we will show that with small modifications in the construction of PS_2 one can produce a space $\widetilde{\mathsf{PS}_2}$ with the following *desired property*.

Definition 3.22. We say that a Banach space X has the **Complex Desired Property ($\mathbb{C}$-DP)** if for every norming sequence $(e_n^*)_{n\in\omega}$ in X^* there exists an element $f\in X$ such that no element $g\in X$ satisfies

$$e_n^*(g) = |e_n^*(f)| \text{ for every } n\in\omega.$$

As we have already mentioned, the new space $\widetilde{\mathsf{PS}_2}$ does have the same shape as the space X explained in Subsection 3.2.1. Following the same notation, $\widetilde{\mathsf{PS}_2}$ is therefore the range of the contractive projection $Q = \mathrm{Id}_{\mathrm{JL}(\mathcal{B})} - P$, whereas $\mathrm{JL}(\mathcal{A})$ is 1-complemented in $\mathrm{JL}(\mathcal{B})$ by P. Note that the operator $Q_{\mathbb{C}} : \mathrm{JL}(\mathcal{B})_{\mathbb{C}} \to \mathrm{JL}(\mathcal{B})_{\mathbb{C}}$ defined by $Q_{\mathbb{C}}(f_1+if_2) := Qf_1 + iQf_2$ is a norm-one $\mathbb{C}$-linear projection with range $(\widetilde{\mathsf{PS}_2})_{\mathbb{C}}$ (see [1, Lemma 1.7]); similarly, $\mathrm{JL}(\mathcal{A})_{\mathbb{C}}$ is the range of the contractive projection $P_{\mathbb{C}}$. It should also be noted that now have $(\widetilde{\mathsf{PS}_2})^*_{\mathbb{C}} \equiv \mathrm{JL}(\mathcal{A})^{\perp}_{\mathbb{C}}$.

Following similar steps as in [131], it is possible to construct two almost disjoint families $\mathcal{A} = \{A_\xi : \xi < \mathfrak{c}\}$ and $\mathcal{B} = \{B_\xi^0, B_\xi^1 : \xi < \mathfrak{c}\}$ in $\mathcal{P}(\omega\times 2)$ and a suitable enumeration of sequences $(\nu_n^\xi)_{n\in\omega} = (\mu_n^\xi, \bar{\nu}_n^\xi)_{n\in\omega}$, for $\xi < \mathfrak{c}$, in the unit ball of $\big(\ell_1(\omega\times 2)\oplus_1 \ell_1(\mathfrak{c}\times 2)\big)_{\mathbb{C}}$ satisfying the properties below:

i) $(\mu_n^\xi)_{n\in\omega}$ is $\mathbb{C}$-admissible,

ii) $\bar{\nu}_n^\xi(\alpha,0) + \bar{\nu}_n^\xi(\alpha,1) = 0$ for every $\alpha < \mathfrak{c}$,

iii) $\bar{\nu}_n^\xi(\alpha,j) = 0$ whenever $\alpha \geq \xi$, $n\in\omega$ and $j\in\{0,1\}$;

in such a way that if $c := \inf_{n\in\omega}|\mu_n^\xi(n,0)|$ and δ represents a fixed number in the interval $(0, c/22)$, there are three infinite sets $J_2^\xi \subseteq J_1^\xi \subseteq J_0^\xi \subseteq \omega$ such that $\omega\setminus J_0^\xi$, $J_0^\xi\setminus J_1^\xi$ and $J_1^\xi\setminus J_2^\xi$ are also infinite, with the following properties:

(Q1) For every $n\in J_0^\xi$, $|\mu_n^\xi|\big((J_0^\xi\times 2)\setminus c_n\big) < \delta$.

(Q2) There exist $a\in\mathbb{C}$ with $|a|\geq c$ and $p_n^\xi \in \{\{(n,0)\},\{(n,1)\}\}$, such that $|\mu_n^\xi(p_n^\xi) - a| < \delta$ for every $n\in J_1^\xi$ and $\mathfrak{Re}\,a \geq \frac{c}{\sqrt{2}}$ or $\mathfrak{Im}\,a \geq \frac{c}{\sqrt{2}}$.

(Q3) For every $\alpha < \xi$, the pairs

- $\{\nu_n^\alpha : n \in J_2^\alpha\}$ and $\{\nu_n^\alpha : n \in J_1^\alpha \setminus J_2^\xi\}$,
- $\{\nu_n^\alpha : n \in J_1^\alpha\}$ and $\{\nu_n^\alpha : n \in J_0^\alpha \setminus J_1^\alpha\}$

are not $\mathfrak{B}(\xi \setminus \{\alpha\})$-separated.

Properties (Q1)–(Q3) can be obtained by adjusting lemmata 5.3 and 5.5 from [131] to the definition of $\mathbb{C}$-admissibility. To avoid cumbersome repetitions, we will not give an explicit proof of these as similar computations will be detailed in the proof of Theorem 3.23. Let us however sketch the idea of how (Q2) could be verified: Since, for every $\xi < \mathfrak{c}$, $(\nu_n^\xi)_{n\in\omega} = (\mu_n^\xi, \bar{\nu}_n^\xi)_{n\in\omega} \subseteq B_{(\widetilde{\mathsf{PS}_2})^*_{\mathbb{C}}}$ and $(\mu_n^\xi)_{n\in\omega}$ is $\mathbb{C}$-admissible, we have

$$1 \geq \left|\nu_n^\xi\left(1_{(n,0)} - 1_{(n,1)}\right)\right| = 2\left|\mu_n^\xi(n,0)\right| \geq 2c > 0, \qquad \text{for every } n \in \omega.$$

Hence, passing to a subsequence we may assume that $\left(\mu_n^\xi(n,0)\right)_{n\in\omega}$ converges to some $b \in \mathbb{C}$. As $\inf_{n\in\omega} |\mu_n^\xi(n,0)| = c > 0$, then $|b| \geq c$. Thus, $|\mathfrak{Re}\, b| \geq \frac{c}{\sqrt{2}}$ or $|\mathfrak{Im}\, b| \geq \frac{c}{\sqrt{2}}$. Let us suppose for instance that $|\mathfrak{Re}\, b| \geq \frac{c}{\sqrt{2}}$. Since $\mu_n^\xi(c_k) = 0$ for every $n, k \in \omega$, in particular,

$$\mathfrak{Re}\, \mu_n^\xi(n,0) = -\mathfrak{Re}\, \mu_n^\xi(n,1) \qquad \text{for every } n \in \omega.$$

Consequently, for each $n \in \omega$, we can choose $p_n^\xi = \{(n,0)\}$ or $p_n^\xi = \{(n,1)\}$ such that $\mathfrak{Re}\, \mu_n^\xi(p_n^\xi) = |\mathfrak{Re}\, \mu_n^\xi(n,0)| \geq 0$, so $\mathfrak{Re}\, \mu_n^\xi(p_n^\xi) \to |\mathfrak{Re}\, b|$. Finally, passing again to a subsequence if necessary, we obtain $\mu_n^\xi(p_n^\xi) \to a$ with $|a| \geq c$ and $\mathfrak{Re}\, a = |\mathfrak{Re}\, b| \geq \frac{c}{\sqrt{2}}$.

Additionally, let us remark that property (Q3) also implies an analogue of Remark 3.14, and therefore, for any $\xi < \mathfrak{c}$, the pairs

- $\{\nu_n^\xi : n \in J_2^\xi\}$ and $\{\nu_n^\xi : n \in J_1^\xi \setminus J_2^\xi\}$,
- $\{\nu_n^\xi : n \in J_1^\xi\}$ and $\{\nu_n^\xi : n \in J_0^\xi \setminus J_1^\xi\}$

are not $\mathfrak{B}(\mathfrak{c} \setminus \{\xi\})$-separated.

We now proceed to prove our main result in this section.

Theorem 3.23. $\widetilde{\mathsf{PS}_2} \oplus i\widetilde{\mathsf{PS}_2}$ *is not isomorphic to a complex Banach lattice.*

Proof. We will prove this statement by showing that $(\widetilde{\mathsf{PS}_2})_{\mathbb{C}}$ does have the ($\mathbb{C}$-DP), which is equivalent to the fact that $(\widetilde{\mathsf{PS}_2})_{\mathbb{C}}$ is not isomorphic to a complex Banach lattice (and this can be checked with a straightforward adaptation of the proof of Corollary 3.18).

Fix a norming sequence $(e_n^*)_{n\in\omega}$ in $B_{(\widetilde{\mathsf{PS}_2})^*_{\mathbb{C}}}$. Our aim is to find an $f \in (\widetilde{\mathsf{PS}_2})_{\mathbb{C}}$ such that no $g \in (\widetilde{\mathsf{PS}_2})_{\mathbb{C}}$ satisfies

$$\langle e_n^*, g\rangle = |\langle e_n^*, f\rangle|, \text{ for every } n \in \omega.$$

We shall denote $e_n^* = e_{n,0}^* + ie_{n,1}^*$, where $e_{n,j}^* \in \widetilde{\mathsf{PS}_2}^*$ for $j = 0,1$. The identification $\widetilde{\mathsf{PS}_2} \equiv \mathrm{JL}(\mathcal{A})^\perp$ allows us to write, for every $n \in \omega$ and $j \in \{0,1\}$, $e_{n,j}^* = \mu_{n,j} + \bar{\nu}_{n,j}$, where $\mu_{n,j} \in \ell_1(\omega \times 2)$ (which is determined by its values on finite sets of $\omega \times 2$) and $\bar{\nu}_{n,j} \in \ell_1(\mathfrak{c} \times 2)$ (which vanishes on finite subsets of $\omega \times 2$-) fulfill $\mu_{n,j}(k,0) = -\mu_{n,j}(k,1)$ for every $k \in \omega$ and $\bar{\nu}_{n,j}(\alpha,0) = -\bar{\nu}_{n,j}(\alpha,1)$ for every $\alpha < \mathfrak{c}$ (for details, see Section 3.2.2). Moreover, as $(e_n^*)_{n\in\omega}$ is a norming set, there exists $\tilde{c} > 0$ such that

$$2\sup_n |\mu_n(k,0)| = \sup_n |\mu_n(f_k)| = \sup_n |e_n^*(f_k)| \geq \tilde{c}, \quad \text{for every } k \in \omega,$$

where $f_k = 1_{(k,0)} - 1_{(k,1)}$. Arguing in the same way as we did in the proof of Theorem 3.19, we can find an injective map $\pi : \omega_0 \to \omega$ for some infinite $\omega_0 \subseteq \omega$ in such a way that $(\mu_{\pi(n)})_{n\in\omega_0}$ is a $\mathbb{C}$-admissible sequence. Therefore, the sequence $(e^*_{\pi(n)})_{n\in\omega_0}$ is coded by some $(\nu_n^\xi)_{n\in\omega} = (\mu_n^\xi, \bar{\nu}_n^\xi)_{n\in\omega}$ (where $\xi < \mathfrak{c}$) satisfying properties *i)–iii)* mentioned in the previous comments to this theorem. Recall that the enumeration was chosen so as to satisfy $\bar{\nu}_n^\xi(\alpha, j) = 0$ for all $\alpha \geq \xi$, $n \in \omega$ and $j \in \{0,1\}$.

Again, for the sake of simplicity, for the remainder of the proof of the theorem we will omit the superscript ξ in ν_n^ξ, μ_n^ξ, $\bar{\nu}_n^\xi$ and p_n^ξ. By (Q3), the pairs of measures

- $\{\nu_n : n \in J_2^\xi\}$ and $\{\nu_n : n \in J_1^\xi \setminus J_2^\xi\}$,
- $\{\nu_n : n \in J_1^\xi\}$ and $\{\nu_n : n \in J_0^\xi \setminus J_1^\xi\}$

are not $\mathfrak{B}(\mathfrak{c} \setminus \{\xi\})$-separated. Let $c = \inf_{n\in\omega} |\mu_n(n,0)|$ and let a be the complex number appearing in property (Q2). We will only consider the case when $\mathfrak{Re}\, a \geq \frac{c}{\sqrt{2}}$, but the proof can be easily adapted to the case when $\mathfrak{Im}\, a \geq \frac{c}{\sqrt{2}}$

Let us consider the function

$$f = 1_{B_\xi^0} - 1_{B_\xi^1} \in (\widetilde{\mathsf{PS}_2})_{\mathbb{C}}. \tag{3.12}$$

We will check that there cannot exist a function in $(\widetilde{\mathsf{PS}_2})_{\mathbb{C}}$ giving the modulus of f with respect to the sequence $(\nu_n)_{n\in\omega} = (e^*_{\pi(n)})_{n\in\omega_0}$; in particular, this would imply that f cannot have a modulus with respect to $(e_n^*)_{n\in\omega}$. Let us suppose that there exists an element $g \in (\widetilde{\mathsf{PS}_2})_{\mathbb{C}}$ such that

$$\langle \nu_n, g\rangle = |\langle \nu_n, f\rangle| = |\langle \mu_n, f\rangle|, \text{ for every } n \in \omega,$$

where in the second equality we are using that $\bar{\nu}_n(\xi, j) = 0$ for $n \in \omega$ and $j \in \{0,1\}$ (see Remark 3.15). We will arrive at a contradiction with the separation of the two pairs of sets of measures defined above. The computations will be very similar to those performed in the proof of Theorem 3.19 (following again very closely the argument of [131, Lemma 5.3]). We will keep this notation: given $a, b \in \mathbb{R}$ and $\delta > 0$, we write $a \approx_\delta b$ to mean $|a - b| < \delta$.

Let us fix $\delta > 0$ as in (Q1)–(Q3). Since the subspace of $(\widetilde{\mathsf{PS}_2})_{\mathbb{C}}$ consisting of simple $\mathfrak{B}$-measurable functions is dense in $(\widetilde{\mathsf{PS}_2})_{\mathbb{C}}$, there is such a function $h \in (\widetilde{\mathsf{PS}_2})_{\mathbb{C}}$ such that $\|g - h\| < \delta$. Therefore,

$$|\langle \mu_n, f\rangle| = \langle \nu_n, g\rangle \approx_\delta \langle \nu_n, h\rangle, \text{ for every } n \in \omega.$$

Without loss of generality, we assume that $h = rf + s$, where $r \in \mathbb{C}$ and s is a simple $\mathfrak{B}(\mathfrak{c} \setminus \{\xi\})$-measurable function lying in $(\widetilde{\mathsf{PS}_2})_{\mathbb{C}}$. Let us further suppose that $r \geq 0$; if $r = |r|e^{i\theta}$ we may apply our argument to the function $e^{i\theta}g$ instead of g. That is, if we prove that $e^{-i\theta}|f|$ cannot exist, then $|f|$ does not exist either. Hence, we have:

$$|\langle \mu_n, f\rangle| \approx_\delta r\langle \mu_n, f\rangle + \langle \nu_n, s\rangle, \text{ for every } n \in \omega. \tag{3.13}$$

Now, observe that properties (Q1) and (Q2), together with the definition of B_ξ^0 and B_ξ^1, yield, for every $n \in J_0^\xi$

$$\langle \mu_n, f\rangle \approx_\delta \int_{c_n} f d\mu_n = f(p_n)\mu_n(p_n) + f(c_n\setminus p_n)\mu_n(c_n\setminus p_n) \approx_{3\delta} \begin{cases} 2a & \text{if } n\in J_2^\xi, \\ -2a & \text{if } n\in J_1^\xi\setminus J_2^\xi, \\ 0 & \text{if } n\in J_0^\xi\setminus J_1^\xi. \end{cases}$$

Hence,

$$\langle \mu_n, f\rangle \approx_{3\delta} \begin{cases} 2a & \text{if } n\in J_2^\xi, \\ -2a & \text{if } n\in J_1^\xi\setminus J_2^\xi, \\ 0 & \text{if } n\in J_0^\xi\setminus J_1^\xi. \end{cases} \tag{3.14}$$

We deduce from the previous equation that

$$|\langle \mu_n, f\rangle| \approx_{3\delta} \begin{cases} 2|a| & \text{if } n\in J_1^\xi, \\ 0 & \text{if } n\in J_0^\xi\setminus J_1^\xi. \end{cases} \tag{3.15}$$

Now, using (3.14) and (3.15), we infer from (3.13) the following relations:

$$\begin{cases} 2|a| \approx_{\delta(4+3r)} & 2ra + \langle \nu_n, s\rangle & \text{if } n\in J_2^\xi, \\ 2|a| \approx_{\delta(4+3r)} & -2ra + \langle \nu_n, s\rangle & \text{if } n\in J_1^\xi\setminus J_2^\xi, \\ 0 \approx_{\delta(4+3r)} & \langle \nu_n, s\rangle & \text{if } n\in J_0^\xi\setminus J_1^\xi. \end{cases} \tag{3.16}$$

First, suppose that $0\le r\le 1/2$. The first two relations of (3.16) give for every $n\in J_1^\xi$

$$|\langle \nu_n, s\rangle| \ge 2|1\pm re^{i\alpha}||a| - \delta(4+3r) \ge |a| - \frac{11}{2}\delta,$$

where $a=|a|e^{i\alpha}$, while the third one gives for every $k\in J_0^\xi\setminus J_1^\xi$,

$$|\langle \nu_k, s\rangle| \le \delta(4+3r) = \frac{11}{2}\delta.$$

Thus, for any $n\in J_1^\xi$ and any $k\in J_0^\xi\setminus J_1^\xi$, we have, using that $\delta<c/22$ and $|a|\ge c$,

$$|\langle \nu_n, s\rangle| - |\langle \nu_k, s\rangle| \ge |a| - 11\delta > 0.$$

This already implies (by adjusting Lemma 3.13 to the complex setting) that the sets $\{\nu_n : n\in J_1^\xi\}$ and $\{\nu_n : n\in J_0^\xi\setminus J_1^\xi\}$ are $\mathfrak{B}(\mathfrak{c}\setminus\{\xi\})$-separated. On the other hand, if $r\ge 1/2$, then using relations (3.16) again, we infer that for every $n\in J_1^\xi\setminus J_2^\xi$ and every $k\in J_2^\xi$

$$\begin{aligned}\mathfrak{Re}\,\langle \nu_n, s\rangle - \mathfrak{Re}\,\langle \nu_k, s\rangle &\ge 2r\mathfrak{Re}\,a + 2|a| - \delta(4+3r) - \big(\delta(4+3r) + 2|a| - 2r\mathfrak{Re}\,a\big) = \\ &= 2r(2\mathfrak{Re}\,a - 3\delta) - 8\delta \ge 2\mathfrak{Re}\,a - 11\delta > 0,\end{aligned}$$

since we have supposed that $\mathfrak{Re}\,a\ge \frac{c}{\sqrt{2}}$. It is clear that if $\mathfrak{Im}\,a\ge\frac{c}{\sqrt{2}}$, we may obtain, using the same procedure as shown above, that $\mathfrak{Im}\,\langle \nu_n, s\rangle - \mathfrak{Im}\,\langle \nu_k, s\rangle \ge 2\mathfrak{Re}\,a - 11\delta > 0$ for every $n\in J_1^\xi\setminus J_2^\xi$ and every $k\in J_2^\xi$. Hence, the sets $\{\nu_n : n\in J_2^\xi\}$ and $\{\nu_n : n\in J_1^\xi\setminus J_2^\xi\}$ are $\mathfrak{B}(\mathfrak{c}\setminus\{\xi\})$-separated. This is a contradiction. □

We have already remarked that $\widetilde{\mathsf{PS}_2}$ is 1-complemented in a $C(K)$-space (see the paragraph that comes after Definition 3.22). We will see below that it is easy to deduce from our last result that $\widetilde{\mathsf{PS}_2}$ cannot be isomorphic to a Banach lattice. Therefore this modification of PS_2 is also a counterexample to the CSP for (real) Banach lattices.

Corollary 3.24. $\widetilde{\mathsf{PS}_2}$ *is not isomorphic to a Banach lattice.*

Proof. Suppose that there exist a Banach lattice X and an isomorphism $T:\widetilde{\mathsf{PS}_2}\to X$. Recall that since $(\widetilde{\mathsf{PS}_2})_{\mathbb{C}}$ is a subspace of $\mathrm{JL}(\mathcal{B})_{\mathbb{C}}\equiv C(K_{\mathcal{B}})_{\mathbb{C}}$ and in this space the complex Banach lattice norm induced by (3.11) coincides with the one defined in (1.4), then the norm of $(\widetilde{\mathsf{PS}_2})_{\mathbb{C}}$ is also given by $\|f_1+if_2\|_{(\widetilde{\mathsf{PS}_2})_{\mathbb{C}}}=\sup_{\theta\in[0,2\pi]}\|f_1\cos\theta+f_2\sin\theta\|$.

The operator $T_{\mathbb{C}}:(\widetilde{\mathsf{PS}_2})_{\mathbb{C}}\to X_{\mathbb{C}}$ given by $T_{\mathbb{C}}(f_1+if_2):=Tf_1+iTf_2$ is clearly $\mathbb{C}$-linear and bijective. Let us now check that $T_{\mathbb{C}}$ is continuous. First note that by definition of the modulus map (3.11), for every $f_1,f_2\in\widetilde{\mathsf{PS}_2}$ we have $|Tf_1+iTf_2|\leq|Tf_1|+|Tf_2|$. Therefore, for every $f_1+if_2\in(\widetilde{\mathsf{PS}_2})_{\mathbb{C}}$ we have

$$\begin{aligned}\|T_{\mathbb{C}}(f_1+if_2)\|_{X_{\mathbb{C}}}&=\||Tf_1+iTf_2|\|_X\leq\|Tf_1\|_X+\|Tf_2\|_X\leq\|T\|\big(\|f_1\|_{\widetilde{\mathsf{PS}_2}}+\|f_2\|_{\widetilde{\mathsf{PS}_2}}\big)\\&\leq 2\|T\|\|f_1+if_2\|_{(\widetilde{\mathsf{PS}_2})_{\mathbb{C}}},\end{aligned}$$

and by the bounded inverse theorem it follows that $T_{\mathbb{C}}$ is an isomorphism. This is a contradiction with the previous theorem, so $\widetilde{\mathsf{PS}_2}$ cannot be isomorphic to a Banach lattice. □

Remark 3.25. Regarding the CSP for complex Banach lattices, one question that remains open is whether every complemented subspace of a complex Banach lattice is linearly isomorphic to the complexification of some real Banach space. Note that if a complex Banach space is the complexification of a real Banach space then, in particular, it is isomorphic to its complex conjugate. As far as we are concerned, all the known examples of complex Banach spaces non-isomorphic to their corresponding complex conjugates (see, for instance, [12, 35, 52, 85]) fail GL-lust, so they cannot be complemented subspaces of complex Banach lattices.

Chapter 4

Free Complex Banach lattices

The construction of the free Banach lattice generated by a real Banach space is extended to the complex setting. It is shown that for every complex Banach space E there is a complex Banach lattice $\mathrm{FBL}_{\mathbb{C}}[E]$ containing a linear isometric copy of E and satisfying the following universal property: for every complex Banach lattice $X_{\mathbb{C}}$, every operator $T : E \to X_{\mathbb{C}}$ admits a unique lattice homomorphic extension $\widehat{T} : \mathrm{FBL}_{\mathbb{C}}[E] \to X_{\mathbb{C}}$ with $\|\widehat{T}\| = \|T\|$. The free complex Banach lattice $\mathrm{FBL}_{\mathbb{C}}[E]$ is shown to have analogous properties to those of its real counterpart. However, examples of non-isomorphic complex Banach spaces E and F can be given so that $\mathrm{FBL}_{\mathbb{C}}[E]$ and $\mathrm{FBL}_{\mathbb{C}}[F]$ are lattice isometric. The spectral theory of induced lattice homomorphisms on $\mathrm{FBL}_{\mathbb{C}}[E]$ is also explored. This chapter is based on the following article:

[71] D. de Hevia and P. Tradacete, *Free complex Banach lattices*, J. Funct. Anal. **284** (2023), no. 10, Paper No. 109888, 26. MR 4552375

4.1 Construction of $\mathrm{FBL}_{\mathbb{C}}[E]$

Definition 4.1. Given a complex Banach space E, the **free complex Banach lattice generated by** E is a *complex* Banach lattice $\mathrm{FBL}_{\mathbb{C}}[E]$ together with a $\mathbb{C}$-linear isometric embedding $\delta_E : E \to \mathrm{FBL}_{\mathbb{C}}[E]$ such that for every complex Banach lattice $X_{\mathbb{C}}$ and every $\mathbb{C}$-linear operator $T : E \to X_{\mathbb{C}}$, there is a unique lattice homomorphism $\widehat{T} : \mathrm{FBL}_{\mathbb{C}}[E] \to X_{\mathbb{C}}$ such that $\widehat{T} \circ \delta_E = T$. Moreover, $\|\widehat{T}\| = \|T\|$.

Observe that if this object exists for a complex Banach space E, then it is essentially unique in the sense that if there exists any other complex Banach lattice $L_{\mathbb{C}}$ with the previous property, we have that $L_{\mathbb{C}}$ is lattice isometric to $\mathrm{FBL}_{\mathbb{C}}[E]$.

As we indicated in the Introduction, one of the main reasons that lead us to prove the existence of this object and investigate it is *its connection to the CSP for* ***complex Banach lattices***. In an analogous manner to Proposition 2.11, it is easy to check that if E is a complex Banach space that is C_1-isomorphic to a C_2-complemented subspace of a complex Banach lattice, then $\delta_E(E)$ is C_1C_2-complemented in $\mathrm{FBL}_{\mathbb{C}}[E]$. This could be especially useful for finding affirmative answers to the CSP in particular situations, such as for *separable spaces that are* 1*-complemented in complex Banach lattices*. We have already highlighted some peculiar results at the beginning of Section 3.4 that invite greater optimism regarding the complex case compared to the real one. Among these, we wish to again underscore Kalton and Wood's result concerning unconditional bases [89]: every 1-complemented subspace of a complex space with a 1-unconditional basis has a 1-

unconditional basis. In any case, free Banach lattices are interesting objects in their own that have been intensely studied in recent years (see [119] for an extensive survey on the topic), and our construction seeks to enrich this theory.

In this section we shall prove *the existence of the free complex Banach lattice generated by a complex Banach space*, providing an *explicit description* of this object based on the one given by Avilés, Rodríguez, and Tradacete in [15] for the free Banach lattice generated by a real Banach space. Given a complex Banach space E, we can consider the real Banach space $E_{\mathbb{R}}$. We can equip the vector lattice $\mathrm{FBL}[E_{\mathbb{R}}]$ with the following norm:

$$\|f\|_{\mathrm{FBL}_{\mathbb{C}}[E]} = \sup\left\{\sum_{j=1}^{m} |f(\mathfrak{Re}\, z_j^*)| \,:\, m\in\mathbb{N}, (z_j^*)_{j=1}^m \subseteq E^*, \sup_{z\in B_E}\sum_{j=1}^{m}|z_j^*(z)|\leq 1\right\}.$$

It should be noted that $\|\cdot\|_{\mathrm{FBL}_{\mathbb{C}}[E]}$ is a lattice norm on $\mathrm{FBL}[E_{\mathbb{R}}]$ (with the pointwise ordering) and is equivalent to the (real) free Banach lattice norm recalled in (1.2):

$$\frac{1}{2}\|f\|_{\mathrm{FBL}[E_{\mathbb{R}}]} \leq \|f\|_{\mathrm{FBL}_{\mathbb{C}}[E]} \leq \|f\|_{\mathrm{FBL}[E_{\mathbb{R}}]}, \qquad f\in \mathrm{FBL}[E_{\mathbb{R}}]. \tag{4.1}$$

We define $\mathrm{FBL}_{\mathbb{C}}[E]$ as the complexification of the real Banach lattice $\mathrm{FBL}[E_{\mathbb{R}}]$ endowed with the complex Banach lattice norm $\||\cdot|\|_{\mathrm{FBL}_{\mathbb{C}}[E]}$. Observe that if $f = f_1 + if_2 \in \mathrm{FBL}_{\mathbb{C}}[E] = \mathrm{FBL}[E_{\mathbb{R}}] \oplus i\mathrm{FBL}[E_{\mathbb{R}}]$, the modulus of f is given by

$$|f|(x^*) = \sqrt{f_1(x^*)^2 + f_2(x^*)^2}, \qquad \text{for every } x^*\in (E_{\mathbb{R}})^*,$$

and, thus

$$\||f|\|_{\mathrm{FBL}_{\mathbb{C}}[E]} = \sup\left\{\sum_{j=1}^{m} |f(\mathfrak{Re}\, z_j^*)| \,:\, (z_j^*)_{j=1}^m \subseteq E^*, \sup_{z\in B_E}\sum_{j=1}^{m}|z_j^*(z)|\leq 1\right\}. \tag{4.2}$$

For simplicity, henceforth, we shall take the above expression as the definition of $\|\cdot\|_{\mathrm{FBL}_{\mathbb{C}}[E]}$, that is, the norm $\||\cdot|\|_{\mathrm{FBL}_{\mathbb{C}}[E]}$ will be represented by $\|\cdot\|_{\mathrm{FBL}_{\mathbb{C}}[E]}$.

Let $\delta_E : E \to \mathrm{FBL}_{\mathbb{C}}[E]$ be given by

$$\delta_E(z) = \delta_{E_{\mathbb{R}}}(z) - i\delta_{E_{\mathbb{R}}}(iz), \qquad z\in E. \tag{4.3}$$

Observe that δ_E is a $\mathbb{C}$-linear map. Indeed, δ_E is $\mathbb{R}$-linear, as $\delta_{E_{\mathbb{R}}}$ has this property, and $\delta_E(iz) = \delta_{E_{\mathbb{R}}}(iz) + i\delta_{E_{\mathbb{R}}}(z) = i\delta_E(z)$. Moreover, this mapping is norm-preserving.

Lemma 4.2. *The map δ_E is a $\mathbb{C}$-linear isometric embedding.*

Proof. Let us note that for every $z\in E$ and for every $z^*\in E^*$ we have that

$$\begin{aligned}\delta_E(z)(\mathfrak{Re}\, z^*) &= \delta_{E_{\mathbb{R}}}(z)(\mathfrak{Re}\, z^*) - i\delta_{E_{\mathbb{R}}}(iz)(\mathfrak{Re}\, z^*) = \mathfrak{Re}\, z^*(z) - i\mathfrak{Re}\, z^*(iz)\\ &= \mathfrak{Re}\, z^*(z) + i\mathfrak{Im}\, z^*(z) = z^*(z).\end{aligned}$$

Using the above identity, it is straightforward to check that $\|\delta_E(z)\|_{\mathrm{FBL}_{\mathbb{C}}[E]} = \|z\|$ for all $z\in E$ in view of the definition (4.2) of $\|\cdot\|_{\mathrm{FBL}[E]_{\mathbb{C}}}$. □

Theorem 4.3. *The complex Banach lattice $FBL_{\mathbb{C}}[E] = FBL[E_{\mathbb{R}}] \oplus iFBL[E_{\mathbb{R}}]$ with the norm $\|\cdot\|_{FBL_{\mathbb{C}}[E]}$, together with the map δ_E given above, form the free complex Banach lattice generated by E.*

Proof. By Lemma 4.2, δ_E is a $\mathbb{C}$-linear isometric embedding.

Given a complex Banach lattice $X_{\mathbb{C}} = X \oplus iX$, where X is a (real) Banach lattice, we can consider the projection onto the real part $\mathfrak{Re} : X_{\mathbb{C}} \to X$ given by $\mathfrak{Re}\,(x+iy) = x$ for $x, y \in X$. This defines a $\mathbb{R}$-linear projection. For a $\mathbb{C}$-linear operator $T : E \to X_{\mathbb{C}}$, let $S : \mathrm{FBL}[E_{\mathbb{R}}] \to X$ denote the unique lattice homomorphism such that $S \circ \delta_{E_{\mathbb{R}}} = \mathfrak{Re} \circ T$. Let $\widehat{T} : \mathrm{FBL}[E_{\mathbb{R}}]_{\mathbb{C}} \to X_{\mathbb{C}}$ be the complexification of the operator S, that is, $\widehat{T}(f+ig) = Sf + iSg$. Thus, $\widehat{T}$ is a complex lattice homomorphism.

Moreover, for every $z \in E$, using the fact that T is $\mathbb{C}$-linear, we have that

$$\begin{aligned}\widehat{T}\delta_E(z) &= \widehat{T}(\delta_{E_{\mathbb{R}}}(z) - i\delta_{E_{\mathbb{R}}}(iz)) = S\delta_{E_{\mathbb{R}}}(z) - iS\delta_{E_{\mathbb{R}}}(iz)\\ &= \mathfrak{Re}\,T(z) - i\mathfrak{Re}\,T(iz) = \mathfrak{Re}\,T(z) - i\mathfrak{Re}\,iT(z) = T(z),\end{aligned}$$

so $\widehat{T}\delta_E = T$.

Now, let us see the uniqueness of the extension $\widehat{T}$. Let $\widehat{T}_1 : \mathrm{FBL}[E_{\mathbb{R}}]_{\mathbb{C}} \to X_{\mathbb{C}}$ be another complex lattice homomorphism such that $\widehat{T}_1\delta_E = T$. As a complex lattice homomorphism, $\widehat{T}_1$ satisfies $\widehat{T}_1(f+ig) = S_1 f + iS_1 g$ for every $f, g \in \mathrm{FBL}[E_{\mathbb{R}}]$ for some lattice homomorphism S_1. By the definition of δ_E, it follows that $S_1\delta_{E_{\mathbb{R}}} = \mathfrak{Re}\,T = S\delta_{E_{\mathbb{R}}}$. Since $\mathrm{FBL}[E_{\mathbb{R}}] = \overline{\mathrm{lat}}\{\delta_{E_{\mathbb{R}}}(x) : x \in E\}$, we conclude that S and S_1 agree on $\mathrm{FBL}[E_{\mathbb{R}}]$ and, consequently, $\widehat{T} = \widehat{T}_1$.

We claim that $\|\widehat{T}\| = \|T\|$. First, it should be noted that since $\widehat{T}\delta_E = T$ and δ_E is an isometric embedding, we have that $\|T\| \leq \|\widehat{T}\|$. It remains to show that

$$\||\widehat{T}(f+ig)|\|_X \leq \|T\|\|f+ig\|_{\mathrm{FBL}_{\mathbb{C}}[E]},$$

for all $f, g \in \mathrm{FBL}[E_{\mathbb{R}}]$. Since $\widehat{T}$ is a lattice homomorphism, the preceding inequality is equivalent to

$$\|Sf\|_X \leq \|T\|\|f\|_{\mathrm{FBL}_{\mathbb{C}}[E]}, \qquad \text{for every } f \in (\mathrm{FBL}[E_{\mathbb{R}}])^+. \tag{4.4}$$

Moreover, by density of $\mathrm{lat}\{\delta_{E_{\mathbb{R}}}(x) : x \in E\}$ in $\mathrm{FBL}[E_{\mathbb{R}}]$, it suffices to check the above identity when f has the form (see, for instance [7, Section 4.1, Ex. 8])

$$f = \bigvee_{i=1}^{p} \delta_{E_{\mathbb{R}}}(x_i) - \bigvee_{j=1}^{q} \delta_{E_{\mathbb{R}}}(y_j).$$

Fix a positive element

$$f = \bigvee_{i=1}^{p} \delta_{E_{\mathbb{R}}}(x_i) - \bigvee_{j=1}^{q} \delta_{E_{\mathbb{R}}}(y_j).$$

Since $Sf \geq 0$, equation (4.4) is equivalent to $y^*(Sf) \leq \|T\|\|f\|_{\mathrm{FBL}_{\mathbb{C}}[E]}$ for every $y^* \in (B_{X^*})^+$ (see [115, Proposition 1.3.5]). Take an arbitrary decomposition $y^* = \sum_{k=1}^{p} y_k^*$, where $y_1^*, \ldots, y_p^* \in (X^*)^+$. For every $k \in \{1, \ldots, p\}$, define

$$x_k^* = \|T\|^{-1}(\mathfrak{Re}\,T)^*(y_k^*) \in (E_{\mathbb{R}})^*$$

(note that $\mathfrak{Re}\,T$ is a $\mathbb{R}$-linear operator from $E_{\mathbb{R}}$ to X). Hence, if we put

$$z_k^*(z) = x_k^*(z) - ix_k^*(iz), \qquad \text{for } z \in E,$$

then z_k^* defines a $\mathbb{C}$-linear functional on E with real part $\mathfrak{Re}\,z_k^* = x_k^*$ for $k = 1, \ldots, p$.

For each $z \in B_E$ and for each $k \in \{1, \ldots, p\}$ let $\theta_{z,k}$ be a real number such that

$$|z_k^*(z)|e^{i\theta_{z,k}} = z_k^*(z).$$

Using in step $(*)$ the complex homogeneity of T, we have that

$$\begin{aligned}
\sup_{z\in B_E}\sum_{k=1}^{p}|z_k^*(z)| &= \sup_{z\in B_E}\sum_{k=1}^{p}e^{-i\theta_{z,k}}z_k^*(z) = \sup_{z\in B_E}\sum_{k=1}^{p}z_k^*\left(e^{-i\theta_{z,k}}z\right)\\
&= \sup_{z\in B_E}\sum_{k=1}^{p}x_k^*\left(e^{-i\theta_{z,k}}z\right) = \sup_{z\in B_E}\sum_{k=1}^{p}\frac{1}{\|T\|}y_k^*\left(\mathfrak{Re}\,T\left(e^{-i\theta_{z,k}}z\right)\right)\\
&\leq \sup_{z\in B_E}\sum_{k=1}^{p}\frac{1}{\|T\|}y_k^*\left(\left|T\left(e^{-i\theta_{z,k}}z\right)\right|\right) \overset{(*)}{=} \sup_{z\in B_E}\sum_{k=1}^{p}y_k^*\left(\frac{1}{\|T\|}|T(z)|\right)\\
&= \sup_{z\in B_E}y^*\left(\frac{1}{\|T\|}|T(z)|\right) \leq \|y^*\| \leq 1.
\end{aligned}$$

By the definition of the norm $\|\cdot\|_{\mathrm{FBL}_{\mathbb{C}}[E]}$ it follows that

$$\begin{aligned}
\|f\|_{\mathrm{FBL}_{\mathbb{C}}[E]} &\geq \sum_{k=1}^{p}f(\mathfrak{Re}\,z_k^*) = \sum_{k=1}^{p}f(x_k^*) = \sum_{k=1}^{p}\left(\bigvee_{i=1}^{p}\delta_{E_{\mathbb{R}}}(x_i)(x_k^*) - \bigvee_{j=1}^{q}\delta_{E_{\mathbb{R}}}(y_j)(x_k^*)\right)\\
&= \frac{1}{\|T\|}\sum_{k=1}^{p}\left(\bigvee_{i=1}^{p}y_k^*(\mathfrak{Re}\,Tx_i) - \bigvee_{j=1}^{q}y_k^*(\mathfrak{Re}\,Ty_j)\right)\\
&\geq \frac{1}{\|T\|}\sum_{k=1}^{p}\left(y_k^*(\mathfrak{Re}\,Tx_k) - y_k^*\left(\bigvee_{j=1}^{q}\mathfrak{Re}\,Ty_j\right)\right)\\
&= \frac{1}{\|T\|}\left(\sum_{k=1}^{p}y_k^*(\mathfrak{Re}\,Tx_k) - y^*\left(\bigvee_{j=1}^{q}\mathfrak{Re}\,Ty_j\right)\right).
\end{aligned}$$

If we take the supremum over all decompositions of y^* into p positive elements of X^*, bearing in mind the Riesz-Kantorovich formulas [7, Theorem 1.21], we obtain from the previous expression that

$$\|T\|\|f\|_{\mathrm{FBL}_{\mathbb{C}}[E]} \geq y^*\left(\bigvee_{k=1}^{p}\mathfrak{Re}\,Tx_k - \bigvee_{j=1}^{q}\mathfrak{Re}\,Ty_j\right) = y^*(Sf).$$

Remember that S is a lattice homomorphism such that $S\delta_{E_{\mathbb{R}}} = \mathfrak{Re}\,T$. This concludes the proof. □

It should be recalled that the lattice homomorphisms of $\mathrm{FBL}[E]^*$ are precisely the evaluations functionals $\varphi_{x^*}(f) = f(x^*)$, $f \in \mathrm{FBL}[E]$, with $x^* \in E^*$ [15, Corollary 2.7]. Now, we establish an analogous result in the complex case.

Corollary 4.4. *$\varphi \in FBL_{\mathbb{C}}[E]^*$ is a lattice homomorphism if and only if there is $z^* \in E^*$ such that $\varphi(f+ig) = f(\mathfrak{Re}\,z^*) + ig(\mathfrak{Re}\,z^*)$ for all $f+ig \in FBL_{\mathbb{C}}[E]$.*

Proof. It is clear that the evaluation functional

$$\varphi_{\mathfrak{Re}\,z^*}(f+ig) = f(\mathfrak{Re}\,z^*) + ig(\mathfrak{Re}\,z^*)$$

is a lattice homomorphism for any $z^* \in E^*$.

Conversely, let φ be a lattice homomorphism in $\mathrm{FBL}_{\mathbb{C}}[E]^*$. We define $z^* = \varphi \circ \delta_E \in E^*$. Then, $\varphi_{\mathfrak{Re}\, z^*}$ is a lattice homomorphism in $\mathrm{FBL}_{\mathbb{C}}[E]^*$ such that

$$\varphi_{\mathfrak{Re}\, z^*} \circ \delta_E = z^* = \varphi \circ \delta_E.$$

By the uniqueness of the universal property of $\mathrm{FBL}_{\mathbb{C}}[E]$ we conclude that $\varphi = \varphi_{\mathfrak{Re}\, z^*}$. □

Remark 4.5. In a similar spirit as in [77], for $p \geq 1$ one can define $\mathrm{FBL}^p_{\mathbb{C}}[E]$, the free p-convex complex Banach lattice generated by a complex Banach space E. This can be done replacing the above norm with

$$\|f\|_{\mathrm{FBL}^p_{\mathbb{C}}[E]} = \sup \left\{ \left(\sum_{j=1}^m |f(\mathfrak{Re}\, z_j^*)|^p \right)^{\frac{1}{p}} \, : \, (z_j^*)_{j=1}^m \subseteq E^*, \ \sup_{z \in B_E} \sum_{j=1}^m |z_j^*(z)|^p \leq 1 \right\}.$$

For $p = \infty$, similarly to the real setting, it can be shown that $\mathrm{FBL}^{(\infty)}_{\mathbb{C}}[E]$ coincides with the sublattice generated by $\{\delta_E(z)\}_{z \in E}$ in $\mathcal{C}(B_{E^*})_{\mathbb{C}}$, which is precisely $\mathcal{C}_{ph}(B_{E^*})_{\mathbb{C}}$ (the space of all positively homogeneous w^*-continuous functions from B_{E^*} to $\mathbb{C}$).

Remark 4.6. Let E be a real Banach space. Since any function in $\mathrm{FBL}[E]$ is w^*-continuous on B_{E^*} (see [15, Lemma 4.10]), it follows that $\delta_E(E)$ is precisely the subset of functions of $\mathrm{FBL}[E]$ which are linear. The last statement remains true if we replace the word *linear* by *additive*, given that functions of $\mathrm{FBL}[E]$ are positively homogeneous. We may formulate a similar result to the previous one in the complex case. Let E be a complex Banach space. A function $f + ig \in \mathrm{FBL}_{\mathbb{C}}[E]$ belongs to $\delta_E(E)$ if, and only if, the map $z^* \in E^* \mapsto f(\mathfrak{Re}\, z^*) + ig(\mathfrak{Re}\, z^*)$ is $\mathbb{C}$-linear. To be a $\delta_E(z)$, for some $z \in E$, is also equivalent to the fact of $f + ig$ being additive and satisfying

$$(f + ig)(L_i x^*) = i(f + ig)(x^*), \qquad \text{for every } x^* \in (E_{\mathbb{R}})^*,$$

where $L_i : (E_{\mathbb{R}})^* \to (E_{\mathbb{R}})^*$ is defined by $L_i x^*(x) = x^*(ix)$ for all $x^* \in (E_{\mathbb{R}})^*$ and all $x \in E$.

Remark 4.7. It should be noted that the operations of taking complexification and taking free Banach lattices do not commute, in the sense that for a real Banach space E, $\mathrm{FBL}_{\mathbb{C}}[E_{\mathbb{C}}]$ need not coincide with $\mathrm{FBL}[E]_{\mathbb{C}}$. Indeed, taking $E = \mathbb{R}$, one can easily check that

$$\mathrm{FBL}[\mathbb{R}]_{\mathbb{C}} = \mathbb{C}^2 \neq \mathrm{FBL}_{\mathbb{C}}[\mathbb{C}],$$

as the latter is even infinite dimensional.

4.2 Complex conjugates and $\mathrm{FBL}_{\mathbb{C}}[E]$

As was mentioned in the Introduction, in the real setting it is an *open question* whether there can exist non-isomorphic Banach spaces such that their corresponding free Banach lattices are lattice isomorphic (see [119, Remark 10.25]). In this section we will analyze this problem in the complex case.

If E is a complex Banach space, its *complex conjugate* $\overline{E}$ is defined as the space E with the scalar multiplication $\alpha \odot x = \overline{\alpha} x$. If $E = F_{\mathbb{C}}$ for some real Banach space F, the map $T : E \to \overline{E}$ given by $T(x+iy) = y+ix$ is easily seen to be a $\mathbb{C}$-linear isomorphism. The first example of a complex Banach space which is not $\mathbb{C}$-isomorphic to its complex conjugate is due to Bourgain [35]. An elementary explicit example was later given by Kalton [85].

Proposition 4.8. *For every complex Banach space E, $FBL_{\mathbb{C}}[E]$ is lattice isometric to $FBL_{\mathbb{C}}[\overline{E}]$.*

Proof. First, notice that $E_{\mathbb{R}} = \overline{E}_{\mathbb{R}}$. Hence, with the construction of the free complex Banach lattice detailed in the preceding section, we obtain that $\mathrm{FBL}_{\mathbb{C}}[E]$ and $\mathrm{FBL}_{\mathbb{C}}[\overline{E}]$ are the same set. We are going to show that the identity $f \in \mathrm{FBL}_{\mathbb{C}}[E] \mapsto f \in \mathrm{FBL}_{\mathbb{C}}[\overline{E}]$ is norm-preserving.

Let z^* be an element of E^*. If we define

$$\psi_{z^*}(z) = \mathfrak{Re}\, z^*(z) - i\mathfrak{Im}\, z^*(z),$$

for every $z \in \overline{E}$, we obtain a bounded functional on $\overline{E}$. It is clear that ψ_{z^*} is $\mathbb{R}$-linear, so it suffices to check that $\psi_{z^*}(i \odot z) = i\psi_{z^*}(z)$. Indeed, for every $z \in \overline{E}$ we have

$$\begin{aligned}\psi_{z^*}(i \odot z) &= -\psi_{z^*}(iz) = -\mathfrak{Re}\, z^*(iz) + i\mathfrak{Im}\, z^*(iz) = \mathfrak{Im}\, z^*(z) + i\mathfrak{Re}\, z^*(z)\\ &= i(\mathfrak{Re}\, z^*(z) - i\mathfrak{Im}\, z^*(z)) = i\psi_{z^*}(z).\end{aligned}$$

In addition, note that the real parts and moduli of z^* and ψ_{z^*} agree on the set E. Conversely, if we begin with an element $w^* \in \overline{E}^*$, we can construct similarly a bounded functional on E^* which has the same real part and modulus as w^* at each point of E. It follows that

$$\begin{aligned}\|f\|_{\mathrm{FBL}_{\mathbb{C}}[E]} &= \sup\left\{\sum_{j=1}^m |f(\mathfrak{Re}\, z_j^*)| \,:\, (z_j^*)_{j=1}^m \subseteq E^*,\ \sup_{z\in B_E}\sum_{j=1}^m |z_j^*(z)| \le 1\right\}\\ &= \sup\left\{\sum_{j=1}^m |f(\mathfrak{Re}\, \psi_{z_j^*})| \,:\, (z_j^*)_{j=1}^m \subseteq E^*,\ \sup_{z\in B_{\overline{E}}}\sum_{j=1}^m |\psi_{z_j^*}(z)| \le 1\right\}\\ &= \sup\left\{\sum_{j=1}^m |f(\mathfrak{Re}\, w_j^*)| \,:\, (w_j^*)_{j=1}^m \subseteq \overline{E}^*,\ \sup_{z\in B_{\overline{E}}}\sum_{j=1}^m |w_j^*(z)| \le 1\right\}\\ &= \|f\|_{\mathrm{FBL}_{\mathbb{C}}[\overline{E}]}.\end{aligned}$$

□

Corollary 4.9. *There exist non-isomorphic complex Banach spaces E and F such that $FBL_{\mathbb{C}}[E]$ and $FBL_{\mathbb{C}}[F]$ are lattice isometric.*

Proof. Take any complex Banach space E non-isomorphic to its complex conjugate, then apply Proposition 4.8 with $F = \overline{E}$. □

The next proposition exhibits that the lattice homomorphisms between complex free Banach lattices are *composition operators*. We shall omit the proof of this fact since it can be readily adjusted from its real version [119, Proposition 10.11].

Proposition 4.10. *Given two complex Banach spaces E and F and a (complex) lattice homomorphism $T : FBL_{\mathbb{C}}[F] \to FBL_{\mathbb{C}}[E]$, we define a map $\Phi_T : E^* \to F^*$ given by*

$$\Phi_T z^*(w) = (T\delta_F(w))(\mathfrak{Re}\, z^*), \quad \text{for every} \;\; z^* \in E^*,\;\; w \in F.$$

The above map satisfies the following properties:

(i) For every $f + ig \in FBL_{\mathbb{C}}[E]$ we have that $T(f+ig) = (f+ig)\circ \Phi_T^{\mathfrak{Re}}$, where $\Phi_T^{\mathfrak{Re}} : (E_{\mathbb{R}})^* \to (F_{\mathbb{R}})^*$ is defined by $\Phi_T^{\mathfrak{Re}}(\mathfrak{Re}\, z^*) = \mathfrak{Re}\,(\Phi_T z^*)$.

(ii) Φ_T is positively homogeneous.

(iii) Φ_T is w^*-w^*-continuous on bounded sets.

(iv) For every $(z_j^*)_{j=1}^m \subseteq E^*$ we have

$$\sup_{w\in B_F}\sum_{j=1}^m |\Phi_T z_j^*(w)| \le \|T\| \sup_{z\in B_E}\sum_{j=1}^m |z_j^*(z)|. \tag{4.5}$$

Remark 4.11. Occasionally, it may be helpful to keep in mind the following identity for the $(1, weak)$-norm of a finite sequence $(z_j^*)_{j=1}^m \subseteq E^*$:

$$\sup_{z\in B_E}\sum_{j=1}^m |z_j^*(z)| = \sup\left\{ \Big\|\sum_{j=1}^m \varepsilon_j z_j^*\Big\| \ : \ |\varepsilon_j| = 1 \text{ for } j=1,\dots,m \right\}.$$

Remark 4.12. If T is a lattice isomorphism, then Φ_T is bijective and $\Phi_T^{-1} = \Phi_{T^{-1}}$. Therefore, if T is also an isometry, then we deduce from the inequality (4.5) that Φ_T (and, for the same reason, Φ_T^{-1}) preserves $(1, weak)$-norms of finite sequences.

Recall that given a Banach space Z, a supporting functional at a point $z \in Z$ is an element $f_z \in Z^*$ such that $\|f_z\| = 1$ and $f_z(z) = \|z\|$. Recall that Z is said to be *smooth* if for every element $z \in Z$, with $z \neq 0$, there exists a unique supporting functional f_z at z.

It was shown in [119, Theorem 10.18] that if E and F are real Banach spaces with smooth duals, then every lattice isometry $T : \mathrm{FBL}[E] \to \mathrm{FBL}[F]$ is necessarily induced by a linear isometry between E and F. Now, we are going to establish a version for complex scalars of this result in Proposition 4.15. It should be noted that this proposition provides a partial converse to Proposition 4.8.

The following elementary observation will be crucial in the proof of the next lemma.

Remark 4.13. For each pair x, y of vectors in a Banach space Z, the function

$$t \in (0, +\infty) \mapsto \frac{\|x+ty\| - \|x\|}{t}$$

is increasing. Indeed, given two positive real numbers $s < t$, by the convexity of the norm function $f(z) = \|z\|$, we have

$$\begin{aligned}\frac{f(x+sy) - f(x)}{s} &= \frac{f\big(\frac{s}{t}(x+ty) + \big(1-\frac{s}{t}\big)x\big) - f(x)}{s} \\ &\le \frac{\frac{s}{t} f(x+ty) + \big(1-\frac{s}{t}\big) f(x) - f(x)}{s} = \frac{f(x+ty) - f(x)}{t}.\end{aligned}$$

It is well-known that a Banach space Z is smooth at $z \in Z\backslash\{0\}$ if and only if the norm $\|\cdot\|$ (of Z) is *Gâteaux differentiable* at z, that is, if there exists $F(z) \in (Z_{\mathbb{R}})^*$ such that

$$F(z)(w) = \lim_{t\to 0}\frac{\|z+tw\| - \|z\|}{t}, \qquad \text{for every } w \in Z,$$

where the above limit is assumed to be taken in $\mathbb{R}$ [43, Corollary 1.5]. Moreover, the *Gâteaux derivative* $F(z)$ is a supporting functional at z (see, for instance, [43, p. 2]). Thus, in the case that $\|\cdot\|$ is smooth at $z \neq 0$, we have

$$\lim_{t\to 0} \frac{\|z+tw\|-\|z\|}{t} = \mathfrak{Re} f_z(w),$$

for every $w \in Z$.

Lemma 4.14. *Let z, w be given in a smooth complex Banach space Z with $z \neq 0$. Then,*

$$\lim_{t\to 0^+} \frac{\sup_{|\varepsilon|=1} \|z+\varepsilon tw\|-\|z\|}{t} = |f_z(w)|,$$

where $f_z \in Z^*$ stands for the unique supporting functional at z.

Proof. Since Z is smooth, we have that

$$\lim_{t\to 0} \frac{\|z+tw\|-\|z\|}{t} = \mathfrak{Re} f_z(w).$$

Thus, for every $\theta \in [0, 2\pi]$ we have that

$$\lim_{t\to 0} \frac{\|z+te^{i\theta}w\|-\|z\|}{t} = \mathfrak{Re} f_z(e^{i\theta}w) = \mathfrak{Re} f_z(w)\cos\theta - \mathfrak{Im} f_z(w)\sin\theta,$$

so that there exists $\theta_0 \in [0, 2\pi]$ such that $\lim_{t\to 0} \frac{\|z+te^{i\theta_0}w\|-\|z\|}{t} = |f_z(w)|$.

If $t > 0$, then by Remark 4.13 we have

$$\frac{\sup_{|\varepsilon|=1} \|z+\varepsilon tw\|-\|z\|}{t} = \sup_{|\varepsilon|=1} \frac{\|z+\varepsilon tw\|-\|z\|}{t} \geq \frac{\|z+te^{i\theta_0}w\|-\|z\|}{t} \geq |f_z(w)|,$$

and thus,

$$\liminf_{t\to 0^+} \frac{\sup_{|\varepsilon|=1} \|z+\varepsilon tw\|-\|z\|}{t} \geq |f_z(w)|.$$

On the other hand, fix $\delta > 0$. Given $|\varepsilon| = 1$, there is $t_\varepsilon > 0$ such that for all $t \in (0, t_\varepsilon)$ we have

$$\frac{\|z+\varepsilon tw\|-\|z\|}{t} < \lim_{t\to 0^+} \frac{\|z+\varepsilon tw\|-\|z\|}{t} + \delta \leq |f_z(w)| + \delta.$$

Thus, the family of sets

$$U_t = \left\{\varepsilon \in \partial\mathbb{D} \,:\, \frac{\|z+\varepsilon tw\|-\|z\|}{t} < |f_z(w)| + \delta\right\}, \qquad \text{for } t \in (0, +\infty),$$

is an open cover of $\partial\mathbb{D} = \{\varepsilon \in \mathbb{C} \,:\, |\varepsilon| = 1\}$. By the compactness of $\partial\mathbb{D}$, there exist $t_1, \ldots, t_n$ such that $\partial\mathbb{D} \subseteq \cup_{i=1}^n U_{t_i}$. By Remark 4.13, we have that $\cup_{i=1}^n U_{t_i} = U_{t_0}$, where $t_0 = \min_{1\leq i\leq n} t_i$. Therefore, we get that

$$\frac{\|z+\varepsilon tw\|-\|z\|}{t} < |f_z(w)| + \delta, \qquad \text{for every } t \in (0, t_0) \text{ and every } |\varepsilon| = 1.$$

Taking supremum over all $|\varepsilon| = 1$ in the above equation, and limit superior for $t \to 0^+$, we conclude that

$$\limsup_{t\to 0^+} \frac{\sup_{|\varepsilon|=1} \|z+\varepsilon tw\|-\|z\|}{t} \leq |f_z(w)|.$$

We have proven that

$$\lim_{t\to 0^+} \frac{\sup_{|\varepsilon|=1} \|z+\varepsilon tw\|-\|z\|}{t} = |f_z(w)|.$$

□

Proposition 4.15. *Let E, F be complex Banach spaces whose corresponding duals E^*, F^* are assumed to be smooth. If $FBL_{\mathbb{C}}[E]$ is lattice isometric to $FBL_{\mathbb{C}}[F]$, then E is isometric to F or $\overline{F}$.*

Proof. Let $T : \mathrm{FBL}_{\mathbb{C}}[E] \to \mathrm{FBL}_{\mathbb{C}}[F]$ be a surjective lattice isometry. We define the following semi-inner product on E^* (resp. on F^*): for $z^*, w^* \in E^*$ (resp. on F^*),

$$[w^*, z^*] = \begin{cases} 0 & \text{if } z^* = 0, \\ f_{z^*}(w^*) & \text{if } z^* \neq 0, \end{cases}$$

where $f_{z^*} \in E^{**}$ (resp. F^{**}) is the unique supporting functional at z^*.

By Proposition 4.10, T is the composition operator associated to a certain surjective positively homogeneous map $\Phi_T : F^* \to E^*$, w^*-w^*-continuous on bounded sets, which preserves $(1, weak)$-norms of tuples; Φ_T^{-1} also has all these properties. Then, by the previous lemma we have

$$\begin{aligned} \big|[w^*, z^*]\big| = |f_{z^*}(w^*)| &= \lim_{t\to 0^+} \frac{\sup_{|\varepsilon|=1} \|z^* + \varepsilon t w^*\| - \|z^*\|}{t} \\ &= \lim_{t\to 0^+} \frac{\sup_{|\varepsilon|=1} \|\Phi_T z^* + \varepsilon t \Phi_T w^*\| - \|\Phi_T z^*\|}{t} \\ &= |f_{\Phi_T z^*}(\Phi_T w^*)| = \big|[\Phi_T w^*, \Phi_T z^*]\big|, \end{aligned}$$

for every $z^*, w^* \in E^*$, with $z^* \neq 0$. By [73], there exist a map $\sigma : F^* \to \mathbb{C}$ with $|\sigma(z^*)| = 1$ for all $z^* \in F^*$ and a linear or conjugate linear surjective isometry $V : F^* \to E^*$ such that

$$\Phi_T z^* = \sigma(z^*) V z^*.$$

Therefore, we have $V z^* = \overline{\sigma(z^*)} \Phi_T z^*$ for all $z^* \in F^*$.

Note that V is w^*-w^*-continuous on B_{F^*}. Indeed, since V is linear, it suffices to check that $V z^*_\alpha \xrightarrow{w^*} 0$ for any $\{z^*_\alpha\} \subseteq B_{F^*}$ w^*-convergent to zero. This can easily be deduced from the fact that Φ_T is w^*-w^*-continuous on bounded sets and $|\sigma(z^*)| = 1$ for every $z^* \in F^*$.

Let us denote by $S : E^* \to \overline{E}^*$ the map defined by

$$S z^*(z) = \overline{z^*(z)}$$

for all $z^* \in E^*$, $z \in \overline{E}$. It should be noted that S is a conjugate-linear surjective isometry which is also w^*-w^*-continuous.

If V is linear, given that it is w^*-w^*-continuous on B_{F^*}, then V is the adjoint operator of a surjective isometry between E and F. If V is conjugate-linear, we may take the composition SV which must be the adjoint operator of a surjective isometry between $\overline{E}$ and F. □

Remark 4.16. The preceding proposition cannot be extended to the lattice isomorphic case. In [12], Anisca built a family of cardinality continuum of uniformly convex Banach spaces (and thus, with uniformly smooth duals) which are mutually non-isomorphic as complex Banach spaces even though they are real isometric. Therefore, the members of this collection have essentially (up to lattice isomorphism) the same free complex Banach lattice.

More generally, one might consider the notion of *complex structures*. Recall that a real Banach space E is said *to admit a complex structure* if there exists an operator $U : E \to E$

such that $U^2 =$ -Id (see [144, pp. 4-5]). In this situation, we can put the following scalar multiplication on E:

$$(a+ib)\cdot x = ax + b\,Ux, \qquad \text{for every } x\in E \text{ and every } a,b\in\mathbb{R}.$$

Thus, E becomes a complex Banach space, denoted by E^U, when renormed with

$$|||x||| = \sup_{\theta\in[0,2\pi]} \|\cos\theta x + \sin\theta\, Ux\|, \qquad \text{for every } x\in E. \tag{4.6}$$

Observe that from the construction of the free complex Banach lattice described in the preceding section we can infer that if E and F are two complex Banach spaces which are $\mathbb{R}$-linearly isomorphic, then their corresponding $\mathrm{FBL}_{\mathbb{C}}[E]$ and $\mathrm{FBL}_{\mathbb{C}}[F]$ are complex lattice isomorphic. Indeed, since $\mathrm{FBL}[E_{\mathbb{R}}]$ and $\mathrm{FBL}[F_{\mathbb{R}}]$ are lattice isomorphic, then $(\mathrm{FBL}[E_{\mathbb{R}}])_{\mathbb{C}}$ and $(\mathrm{FBL}[F_{\mathbb{R}}])_{\mathbb{C}}$ are complex lattice isomorphic. In addition, equation (4.1) shows that $\mathrm{FBL}_{\mathbb{C}}[E]$ (resp. $\mathrm{FBL}_{\mathbb{C}}[F]$) is complex lattice isomorphic to $(\mathrm{FBL}[E_{\mathbb{R}}])_{\mathbb{C}}$ (resp. $(\mathrm{FBL}[F_{\mathbb{R}}])_{\mathbb{C}}$).

It should be noted that if U, V are complex structures on E, then their associated complex Banach spaces E^U, E^V are isomorphic as real Banach spaces, given that their norms are equivalent to the original one defined on E (recall expression (4.6)). Nevertheless, E^U, E^V do not have to be $\mathbb{C}$-linearly isomorphic. In fact, with this terminology, the spaces constructed in [35, 85] have more than one complex structure, whereas spaces with exactly n non-equivalent complex structures were given in [52] (see also [12, 41] for the cases of continuum many and infinite countably many non-equivalent complex structures respectively). As a result, these provide examples of non-isomorphic complex Banach spaces whose corresponding free complex Banach lattices are lattice isomorphic.

4.3 Free complex vector lattices

The purpose of this section is to provide an alternative proof of the existence of $\mathrm{FBL}_{\mathbb{C}}[E]$, similar to the one given in [150]. Although this argument is conceptually simpler it has the drawback that it does not provide the explicit description of $\mathrm{FBL}_{\mathbb{C}}[E]$ given in Section 4.1. To this end, we will need first to consider the concept of a free complex vector lattice.

Let us begin by recalling some definitions about complex vector lattices. A *complex vector lattice* $X_{\mathbb{C}} = X\oplus iX$ is the complexification of a real vector lattice X such that for every $x+iy\in X_{\mathbb{C}}$ the *modulus* function $|\cdot| : X_{\mathbb{C}}\to X$, given by

$$|x+iy| = \sup_{\theta\in[0,2\pi]} \big\{x\,\cos\theta + y\sin\theta\big\},$$

is well-defined. Typically, the real vector lattice X is assumed to be *uniformly complete* (for instance, in [141, Section 2.11] or [115, Section 2.2]) to ensure the existence of the modulus. This contrasts with the case of Banach lattices, where the modulus map is always well-defined (cf. Section 1.3); in fact, every Banach lattice is uniformly complete. Another equivalent way to define complex vector lattices may be found in [117, 154], where an axiomatic definition of the modulus map on a vector lattice is given. We thank Timur Oikhberg for bringing the latter reference to our attention.

By a *complex vector sublattice* Y of $X_{\mathbb{C}}$ we mean a conjugation invariant (that is, $x-iy\in Y$ whenever $x+iy\in Y$) complex vector subspace such that $|z|\in Y$ whenever $z\in Y$. A $\mathbb{C}$-linear operator $T : X_{\mathbb{C}}\to Y_{\mathbb{C}}$ is said to be a *lattice homomorphism* if it is the complexification of a lattice homomorphism $S : X\to Y$ or, equivalently, if T preserves the modulus of the elements in $X_{\mathbb{C}}$.

Let $X_{\mathbb{C}}$ be a complex vector lattice and let A be a non-empty subset of $X_{\mathbb{C}}$. The *complex vector sublattice generated by* A in $X_{\mathbb{C}}$, which is represented by $\mathrm{lat}_{\mathbb{C}}(A)$, is the smallest complex vector sublattice of $X_{\mathbb{C}}$ which contains A. In the real case, we have a useful description of the elements of the sublattice generated by an arbitrary subset: every member of $\mathrm{lat}(A)$ is a lattice-linear combination of a finite number of elements of A [1, Lemma 5.63]. The upcoming remark provides a description of the elements of $lat_{\mathbb{C}}(A)$.

Remark 4.17. Let A be a non-empty subset of a complex vector lattice $X_{\mathbb{C}} = X \oplus iX$. First, we put

$$E_1 = \mathrm{lat}(\mathfrak{Re}\,(A) \cup \mathfrak{Im}\,(A)) \subseteq X \quad \text{and} \quad F_1 = \{|(x_1, x_2)| \,:\, x_1, x_2 \in E_1\}.$$

Given $n \in \mathbb{N}$, such that $n \geq 2$, we define

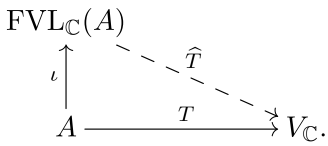
$$E_n = \mathrm{lat}(E_{n-1} \cup F_{n-1}) \quad \text{and} \quad F_n = \{|(x_1, x_2)| \,:\, x_1, x_2 \in E_n\}.$$

In this way, we obtain an increasing sequence $\{E_n\}_{n=1}^{\infty}$ of sublattices of X. It is straightforward to check that $E = \cup_{n=1}^{\infty} E_n$ is a sublattice of X and

$$\mathrm{lat}_{\mathbb{C}}(A) = E \oplus iE.$$

We can define an analogous concept of *free vector lattice over a set* (see [121, Definition 3.1]) in the complex setting.

Definition 4.18. If A is any non-empty set, *the free complex vector lattice* over A is a pair $(\mathrm{FVL}_{\mathbb{C}}(A), \iota)$, where $\mathrm{FVL}_{\mathbb{C}}(A)$ is a complex vector lattice and $\iota : A \to \mathrm{FVL}_{\mathbb{C}}(A)$ is a map, with the following universal property: for any complex vector lattice $V_{\mathbb{C}}$ and any map $T : A \to V_{\mathbb{C}}$, there exists a unique complex vector lattice homomorphism $\widehat{T} : \mathrm{FVL}_{\mathbb{C}}(A) \to V_{\mathbb{C}}$ such that $\widehat{T} \circ \iota = T$, i.e the following diagram commutes



It is not difficult to see that if such an object exists, then it is esentially unique up to complex vector lattice isomorphism and this justifies why we refer to it as "the" free complex vector lattice over A. The following proposition ensures the existence of this object. The proof makes use of the free (real) vector lattice over a set S, which is denoted $\mathrm{FVL}(S)$ (cf. [18, 32]). We are grateful to Enrique García-Sánchez for pointing out a gap in the initial version of this proof.

Proposition 4.19. *For any non-empty set A, $FVL_{\mathbb{C}}(A)$ exists.*

Proof. Let us write $\tilde{A} = \big(A \times \{0\}\big) \cup \big(A \times \{1\}\big)$. Given $(x, y) \in \tilde{A} \times \tilde{A}$, we define $\eta_{(x,y)}(f) = f(x, y)$ for every $f \in \mathbb{R}^{\tilde{A} \times \tilde{A}}$. We recall that $\mathrm{FVL}(\tilde{A} \times \tilde{A})$ is the sublattice generated by $\{\eta_{(x,y)} \,:\, (x, y) \in \tilde{A} \times \tilde{A}\}$ in $\mathbb{R}^{\tilde{A} \times \tilde{A}}$ (see [121, Theorem 3.6]).

Let $V_{\mathbb{C}} = V \oplus iV$ be a complex vector lattice and $T : A \to V_{\mathbb{C}}$ a map. We define a mapping $\tilde{T} : \tilde{A} \to V_{\mathbb{C}}$ by $\tilde{T}(x, 0) = Tx$ and $\tilde{T}(x, 1) = -Tx$, for $x \in A$. Let us consider the following map:

$$\begin{aligned} S : \tilde{A} \times \tilde{A} &\longrightarrow V \\ (x, y) &\longmapsto \tfrac{1}{2}\big(\mathfrak{Re}\tilde{T}x + \mathfrak{Re}\tilde{T}y - \mathfrak{Im}\tilde{T}x + \mathfrak{Im}\tilde{T}y\big). \end{aligned}$$

By the universal property of $\mathrm{FVL}(\tilde{A}\times\tilde{A})$, there is a unique lattice homomorphism $\widehat{S}:\mathrm{FVL}(\tilde{A}\times\tilde{A})\to V$ such that $\widehat{S}\circ\eta=S$.

Now, we define a function $\iota : A\to\mathrm{FVL}(\tilde{A}\times\tilde{A})_{\mathbb{C}}$ by

$$\iota(x)=\eta_{\big((x,0),(x,0)\big)}+i\eta_{\big((x,1),(x,0)\big)}$$

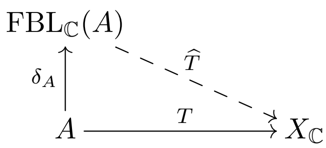
for every $x\in A$. The complex lattice homomorphism $\widehat{S}_{\mathbb{C}}:\mathrm{FVL}(\tilde{A}\times\tilde{A})_{\mathbb{C}}\to V_{\mathbb{C}}$ given by $\widehat{S}_{\mathbb{C}}(f+ig)=\widehat{S}f+i\widehat{S}g$, for every $f,g\in\mathrm{FVL}(\tilde{A}\times\tilde{A})$, extends the map T. Indeed, given $x\in A$, we have

$$\begin{aligned}\widehat{S}_{\mathbb{C}}\circ\iota(x)&=\widehat{S}\eta_{\big((x,0),(x,0)\big)}+i\widehat{S}\eta_{\big((x,1),(x,0)\big)}=S\big((x,0),(x,0)\big)+iS\big((x,1),(x,0)\big)\\&=\mathfrak{Re}Tx+i\mathfrak{Im}Tx=Tx.\end{aligned}$$

Therefore, the complex vector sublattice generated by $\{\iota(x)\ :\ x\in A\}$ in $\mathrm{FVL}(\tilde{A}\times\tilde{A})_{\mathbb{C}}$ is $\mathrm{FVL}_{\mathbb{C}}(A)$. Observe that the later condition guarantees the uniqueness of the complex lattice homomorphism which extends T. □

We can also provide a complex version of the notion of *free Banach lattice over a set* introduced by de Pagter and Wickstead [121, Definition 4.1].

Definition 4.20. The *free complex Banach lattice* over a non-empty *set* A is a pair $(\mathrm{FBL}_{\mathbb{C}}(A),\delta_A)$, where $\mathrm{FBL}_{\mathbb{C}}(A)$ is a complex Banach lattice and $\delta_A : A\to\mathrm{FBL}_{\mathbb{C}}(A)$ is a bounded map, with the property that for any complex Banach lattice $X_{\mathbb{C}}$ and any bounded map $T : A\to X_{\mathbb{C}}$ there is a unique complex vector lattice homomorphism $\widehat{T}:\mathrm{FBL}_{\mathbb{C}}(A)\to X_{\mathbb{C}}$ such that $\widehat{T}\circ\delta_A=T$ and $\|\widehat{T}\|=\sup\{\|T(a)\| : a\in A\}$, i.e. the following diagram commutes



As usual, if such an object exists, then it is essentially unique up to lattice isometric isomorphism.

The existence of the free Banach lattice over a set was proved in [121, Theorem 4.7] (the real version of the concept defined above). In [150], Troitsky found a simpler proof of this fact, which is not difficult to adjust to the complex case. We omit the proof of the following proposition on account of it can be proved in a very similar fashion to [150, Theorem 2.1].

Proposition 4.21. *Let A be a non-empty set and let $FVL_{\mathbb{C}}(A)=F\oplus iF$ be the free complex vector lattice over A. There exists a maximal lattice seminorm ν on F with $\nu(|\iota(a)|)\leq 1$ for all $a\in A$. It is a lattice norm and the completion of $FVL_{\mathbb{C}}(A)$ respect to $\nu(|\cdot|)$ is $FBL_{\mathbb{C}}(A)$.*

The existence of $\mathrm{FBL}_{\mathbb{C}}(A)$ can also be deduced readily from the existence of the free complex Banach lattice generated by a complex Banach space because $\mathrm{FBL}_{\mathbb{C}}[\ell_1(A)_{\mathbb{C}}]$ turns out to be $\mathrm{FBL}_{\mathbb{C}}(A)$ (compare with [15, Corollary 2.9]).

In [150], Troitsky also provides a description of the free Banach lattice over a real Banach space. The next proposition is an adaptation of Troitsky's construction [150, Theorem 3.1] to the complex case and it can be proved in a very similar fashion.

Proposition 4.22. *Let E be a complex Banach space. Let $L_{\mathbb{C}} = L \oplus iL$ the complex vector sublattice of $\mathbb{R}^{(E_{\mathbb{R}})^*} \oplus i\mathbb{R}^{(E_{\mathbb{R}})^*}$ generated by $\{\delta_E(x) : x \in E\}$. There exists a maximal lattice seminorm ν on L such that $\nu(|\delta_E(x)|) \leq \|x\|$ for every $x \in L$. The function ν is a lattice norm and the norm completion of $L_{\mathbb{C}}$ respect to $\nu(|\cdot|)$ is $FBL_{\mathbb{C}}[E]$.*

4.4 Spectra

Given an endomorphism on a complex Banach space $T : E \to E$, let us denote $\overline{T} : \mathrm{FBL}_{\mathbb{C}}[E] \to \mathrm{FBL}_{\mathbb{C}}[E]$ the unique lattice homomorphism given by the universal property which makes the following diagram commutative

$$\begin{array}{ccc} \mathrm{FBL}_{\mathbb{C}}[E] & \overset{\overline{T}}{\dashrightarrow} & \mathrm{FBL}_{\mathbb{C}}[E] \\ \uparrow{\scriptstyle \delta_E} & & \uparrow{\scriptstyle \delta_E} \\ E & \xrightarrow{\quad T \quad} & E \end{array}$$

and also satisfies $\|\overline{T}\| = \|T\|$. Note that in this way we can associate a lattice homomorphism to each bounded linear operator. Our aim in this section is to collect some observations concerning spectral theory via this correspondence.

As usual for an operator $T : X \to X$ on a complex Banach space we denote its spectrum $\sigma(T)$ as the (non-empty) compact set consisting of those $\lambda \in \mathbb{C}$ such that $\lambda - T$ is not invertible (in other words, $\lambda - T$ is not a surjective isomorphism). Let $r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$ denote the corresponding spectral radius. As usual $\sigma_p(T)$ denotes the set of eigenvalues (or point spectrum) of T and

$$\begin{aligned} \sigma_a(T) &= \{\lambda \in \sigma(T) : \lambda - T \text{ is not bounded below}\}, \\ \sigma_r(T) &= \{\lambda \in \sigma(T) \backslash \sigma_p(T) : \overline{(\lambda - T)(X)} \neq X\}, \\ \sigma_c(T) &= \{\lambda \in \sigma(T) \backslash \sigma_p(T) : \overline{(\lambda - T)(X)} = X\} \end{aligned}$$

denote respectively the approximate point spectrum, the residual spectrum and the continuous spectrum. Recall that

$$\sigma_p(T) \subseteq \sigma_a(T) \subseteq \sigma(T) \quad \text{and} \quad \sigma(T) = \sigma_p(T) \cup \sigma_r(T) \cup \sigma_c(T),$$

the latter being a disjoint union.

Proposition 4.23. *Let $T : E \to E$ be a bounded linear operator and $\overline{T} : FBL_{\mathbb{C}}[E] \to FBL_{\mathbb{C}}[E]$ the associated lattice homomorphism.*

(i) $\sigma_a(T) \subseteq \sigma_a(\overline{T})$ *and* $\sigma_p(T) \subseteq \sigma_p(\overline{T})$.

(ii) $0 \in \sigma(T)$ *if and only if* $0 \in \sigma(\overline{T})$.

(iii) The spectral radii satisfy $r(\overline{T}) = r(T)$.

Proof. (i) Let $T : E \to E$, and suppose $\lambda \notin \sigma_a(\overline{T})$. Thus, there is $\alpha > 0$ such that $\|(\lambda - \overline{T})f\| \geq \alpha\|f\|$ for every $f \in \mathrm{FBL}_{\mathbb{C}}[E]$. Hence, we have that for $z \in E$

$$\|(\lambda - T)z\| = \|(\lambda - \overline{T})\delta_E(z)\| \geq \alpha\|\delta_E(z)\| = \alpha\|z\|.$$

Thus, $\lambda \notin \sigma_a(T)$.

Similarly, the statement about point spectra follows from observing that if $Tz = \lambda z$, then $\overline{T}\delta_E(z) = \delta_E(\lambda z) = \lambda\delta_E(z)$.

(ii) Suppose $0 \notin \sigma(T)$, that is $T : E \to E$ is an isomorphism. Then there is $T^{-1} : E \to E$ such that $TT^{-1} = T^{-1}T = \mathrm{id}_E$. Since $\overline{\mathrm{id}_E} = \mathrm{id}_{\mathrm{FBL}_{\mathbb{C}}[E]}$, it is clear that in this case, $\overline{T^{-1}} = \overline{T}^{-1}$, so that $\overline{T}$ is an isomorphism and $0 \notin \sigma(\overline{T})$.

Conversely, suppose $\overline{T}$ is an isomorphism. Note that $\overline{T}$ maps the range of δ_E onto itself. Indeed, we have that $\overline{T}(\delta_E(E)) = \delta_E T(E) \subseteq \delta_E(E)$ and so $\overline{T}(\delta_E(E))$ is a closed subspace of $\mathrm{FBL}_{\mathbb{C}}[E]$ because $\overline{T}$ is an isomorphism. If $F = \overline{T}(\delta_E(E)) \subsetneq \delta_E(E)$, then we can find a nonzero $z^* \in E^*$ which is identically zero on $\delta_E^{-1}(F)$. Since $\overline{T}$ is a lattice homomorphism, its range would be contained in the closed sublattice generated by F. But $\overline{\mathrm{lat}}\{F\} \subseteq \{f \in \mathrm{FBL}_{\mathbb{C}}[E] : f(\mathfrak{Re}\, z^*) = 0\}$ and hence $\overline{T}$ would not be onto, which is a contradiction.

Now, we claim that the inverse

$$\overline{T}^{-1} : \mathrm{FBL}_{\mathbb{C}}[E] \to \mathrm{FBL}_{\mathbb{C}}[E]$$

also maps $\delta_E(E)$ to $\delta_E(E)$. Indeed, suppose that for some $w \in E$ we have $\overline{T}^{-1}\delta_E(w) = f \in \mathrm{FBL}_{\mathbb{C}}[E]\backslash\delta_E(E)$. Since $\overline{T}(\delta_E(E)) = \delta_E(E)$, there is $z \in E$ such that $\overline{T}\delta_E(z) = \delta_E(w)$. Hence, we would have that $\overline{T}f = \delta_E(w) = \overline{T}\delta_E(z)$, which is a contradiction with the injectivity of $\overline{T}$. It follows then that

$$\delta_E^{-1}\overline{T}^{-1}\delta_E : E \to E$$

is the inverse of T, thus showing that $0 \notin \sigma(T)$.

(iii) For every $n \in \mathbb{N}$, we have that $\overline{T}^n$ is a lattice homomorphism on $\mathrm{FBL}_{\mathbb{C}}[E]$ which extends T^n. Hence, $\overline{T^n} = \overline{T}^n$, and in particular

$$\|\overline{T}^n\| = \|T^n\|.$$

It follows from Gelfand's formula for the spectral radius [1, Theorem 6.12] that

$$r(\overline{T}) = \lim_n \|\overline{T}^n\|^{\frac{1}{n}} = \lim_n \|T^n\|^{\frac{1}{n}} = r(T).$$

□

Remark 4.24. The following facts, which are known in the real case (see [119, Section 3]), can be easily extended to the complex setting:

(i) T is injective if and only if $\overline{T}$ is injective.

(ii) T has dense range if and only if $\overline{T}$ has dense range.

Taking this into account, the second part of the former proposition can be refined as follows: $0 \in \sigma_p(T)$ if and only if $0 \in \sigma_p(\overline{T})$; $0 \in \sigma_c(T)$ if and only if $0 \in \sigma_c(\overline{T})$; $0 \in \sigma_r(T)$ if and only if $0 \in \sigma_r(\overline{T})$.

Remark 4.25. In general, we can have $\sigma_p(T) \neq \sigma_p(\overline{T})$, $\sigma_a(T) \neq \sigma_a(\overline{T})$ or $\sigma(T) \neq \sigma(\overline{T})$. Consider the operator $T : \mathbb{C} \to \mathbb{C}$ defined by $Tz = -z$, which has $\sigma(T) = \sigma_a(T) = \sigma_p(T) = \{-1\}$. Next, let us take the function $f : \mathbb{R}^2 \to \mathbb{C}$ given by $f(x, y) = \sqrt{x^2 + y^2} + i \cdot 0$, $(x, y) \in \mathbb{R}^2$. Observe that $f \in \mathrm{FBL}_{\mathbb{C}}[\mathbb{C}]$ and

$$\overline{T}f(x, y) = f(-x, -y) = \sqrt{x^2 + y^2} = f(x, y), \qquad \text{for every } (x, y) \in \mathbb{R}^2,$$

which shows that f is a non-trivial fixed point of $\overline{T}$, and hence $1 \in \sigma_p(\overline{T})$.

In view of the results exposed in Proposition 4.23 one may ask whether it is always true that $\sigma(T)$ is contained in $\sigma(\overline{T})$. The remainder of this section will be dedicated to exploring this question. Let us begin by showing that the positive elements of $\sigma(T)$ belong also to $\sigma(\overline{T})$.

Proposition 4.26. *If $\lambda \in \sigma(T) \cap [0, +\infty)$, then $\lambda \in \sigma(\overline{T})$.*

Proof. Since $\sigma(T) = \sigma_a(T) \cup \sigma_r(T)$, by the first part of Proposition 4.23 it is enough to show that $\sigma_r(T) \cap [0, +\infty) \subseteq \sigma(\overline{T})$. Fix any $\lambda \in \sigma_r(T) \cap [0, +\infty)$. Then $\lambda \in \sigma_p(T^*)$ (see [1, Theorem 6.19]). Take $z^* \in E^*$, $z^* \neq 0$, such that $T^* z^* = \lambda z^*$. For every $f \in \mathrm{FBL}_{\mathbb{C}}[E]$, by the positive homogeneity of f, we have that

$$(\lambda - \overline{T}) f(\mathfrak{Re}\, z^*) = \lambda f(\mathfrak{Re}\, z^*) - f(\mathfrak{Re}\, T^* z^*) = \lambda f(\mathfrak{Re}\, z^*) - f(\lambda \mathfrak{Re}\, z^*) = 0,$$

so that $\lambda - \overline{T}$ cannot have dense range and, in particular, $\lambda \in \sigma(\overline{T})$. □

Given $\theta \in \mathbb{R}$, let us denote by M_θ the operator defined on E by $M_\theta z = e^{i\theta} z$, for every $z \in E$. These operators will play an important role in the proof of Proposition 4.29, which is a generalization of the preceding proposition. Before this, we are going to describe the spectrum of the operators $\{M_\theta\}_{\theta \in \mathbb{R}}$.

Proposition 4.27. *Suppose that $e^{i\theta}$ is a primitive nth root of unity. Then $\sigma(\overline{M_\theta}) = \{e^{i\frac{2k\pi}{n}} : k = 0, 1, \ldots, n-1\}$.*

Proof. First, note that since $e^{i\theta} \in \sigma_p(M_\theta)$, then $e^{i\theta} \in \sigma_p(\overline{M_\theta}) \subseteq \sigma(\overline{M_\theta})$ (statement (1) of Proposition 4.23). Given that the spectrum of a lattice homomorphism is cyclic (see [1, Theorem 7.23]), we have that

$$\{e^{i\frac{2k\pi}{n}} : k = 0, 1, \ldots, n-1\} \subseteq \sigma(\overline{M_\theta}).$$

Now, let us see that $\sigma(\overline{M_\theta}) \subseteq \{e^{i\frac{2k\pi}{n}} : k = 0, 1, \ldots, n-1\}$. Observe that $\overline{M_\theta}$ is a lattice isometry, since M_θ is an isometry, and hence, $\sigma(\overline{M_\theta}) \subseteq \partial\mathbb{D}$. Now, recall that boundary points of the spectrum are in the approximate point spectrum, that is, $\partial\sigma(\overline{M_\theta}) \subseteq \sigma_a(\overline{M_\theta})$ (see [1, Theorem 6.18]). As $\sigma(\overline{M_\theta}) \subseteq \partial\mathbb{D}$, we have that $\partial\sigma(\overline{M_\theta}) = \sigma(\overline{M_\theta})$. Therefore, $\sigma(\overline{M_\theta}) = \sigma_a(\overline{M_\theta})$, so it is enough to prove that $\sigma_a(\overline{M_\theta}) \subseteq \{e^{i\frac{2k\pi}{n}} : k = 0, 1, \ldots, n-1\}$ to conclude.

Take $\lambda \in \sigma_a(\overline{M_\theta})$. Then, there is a sequence $(f_k)_{k=1}^\infty \subseteq \mathrm{FBL}_{\mathbb{C}}[E]$ with $\|f_k\| = 1$ for every $k \in \mathbb{N}$, such that $\|\lambda f_k - \overline{M_\theta} f_k\| \to 0$. It is straightforward to check by induction that

$$\|\lambda^n f_k - \overline{M_\theta}^n f_k\| \to 0$$

as $k \to \infty$. Indeed, let us suppose that $\|\lambda^m f_k - \overline{M_\theta}^m f_k\| \to 0$ for some $m \in \mathbb{N}$. For every $k \in \mathbb{N}$ we have that

$$\begin{aligned}\|\lambda^{m+1} f_k - \overline{M_\theta}^{m+1} f_k\| &= \|\lambda^{m+1} f_k - \lambda^m \overline{M_\theta} f_k + \lambda^m \overline{M_\theta} f_k - \overline{M_\theta}^{m+1} f_k\| \\ &\leq |\lambda^m| \|\lambda f_k - \overline{M_\theta} f_k\| + \|\overline{M_\theta}\| \|\lambda^m f_k - \overline{M_\theta}^m f_k\|,\end{aligned}$$

and, by hypothesis, the last term converges to zero. Since $\overline{M_\theta}^n = \mathrm{id}$, we deduce that $\|\lambda^n f_k - \overline{M_\theta}^n f_k\| = |\lambda^n - 1| \to 0$ as $k \to \infty$, that is $\lambda^n = 1$. Consequently, $\lambda \in \{e^{i\frac{2k\pi}{n}} : k = 0, 1, \ldots, n-1\}$. □

Remark 4.28. If $\theta = 2\pi t$, for some irrational number t, then $\sigma(\overline{M_\theta}) = \partial\mathbb{D}$. Indeed, observe that $\sigma(\overline{M_\theta}) \subseteq \partial\mathbb{D}$ since $\overline{M_\theta}$ is an isometry. On the other hand, since $\sigma(\overline{M_\theta})$ is closed and cyclic, we have that $\partial\mathbb{D} = \overline{\{e^{in\theta} \,:\, n \in \mathbb{N}\}} \subseteq \sigma(\overline{M_\theta})$.

Proposition 4.29. *Let $T : E \to E$ be a complex operator. If $\lambda \in \sigma(T)$, then $|\lambda| \in \sigma(\overline{T})$.*

Proof. Let us write $\lambda = |\lambda|e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Therefore, $|\lambda| \in \sigma(M_{-\theta}T)$, and by Proposition 4.26 we get that

$$|\lambda| \in \sigma(\overline{M_{-\theta}T}) = \sigma(\overline{M_{-\theta}}\,\overline{T}) \subseteq \sigma(\overline{M_{-\theta}})\sigma(\overline{T}),$$

the latter inclusion following from commutativity of $\overline{M_{-\theta}}$ and $\overline{T}$ [139, Theorem 11.23].

Since $\sigma(\overline{M_{-\theta}}) \subseteq \partial\mathbb{D}$ and $\sigma(\overline{T})$ is cyclic by [1, Theorem 7.23], it follows that $|\lambda| \in \sigma(\overline{T})$. □

We will see next that in the case when E is a Banach lattice and T is a lattice homomorphism, then we always have $\sigma(T) \subseteq \sigma(\overline{T})$. This fact will cover many instance of classical operators arising in the literature:

Example 4.30. The following operators are remarkable examples of complex lattice homomorphisms.

1. Let K be a compact Hausdorff space and let $h : K \to K$ a continuous mapping. The *composition operator* $C_h : \mathcal{C}(K)_\mathbb{C} \to \mathcal{C}(K)_\mathbb{C}$ defined by $C_h(f) = f \circ h$.
2. If $g \in \mathcal{C}(K)_\mathbb{C}$ is a positive function, the *multiplication operator* $M_g(f) = gf$, for every $f \in \mathcal{C}(K)_\mathbb{C}$
3. For $1 \le p \le \infty$, the *backward shift operator* $B : \ell_p \to \ell_p$ defined by
$$B(z_1, z_2, \dots) = (z_2, z_3, \dots),$$
and the *forward shift operator* $F : \ell_p \to \ell_p$ defined by
$$F(z_1, z_2, \dots) = (0, z_1, z_2, \dots).$$

Lemma 4.31. *If $X_\mathbb{C}$ is a complex Banach lattice, then there exists an ideal $I_\mathbb{C}$ in $FBL_\mathbb{C}[X_\mathbb{C}]$ such that $FBL_\mathbb{C}[X_\mathbb{C}] = \delta_{X_\mathbb{C}}(X_\mathbb{C}) \oplus I_\mathbb{C}$.*

Proof. By the universal property of $\mathrm{FBL}_\mathbb{C}[X_\mathbb{C}]$, there is a complex lattice homomorphism $\beta : \mathrm{FBL}_\mathbb{C}[X_\mathbb{C}] \to X_\mathbb{C}$ such that $\beta\delta_{X_\mathbb{C}} = \mathrm{id}$. It is easy to check that the composition $P = \delta_{X_\mathbb{C}}\beta : \mathrm{FBL}_\mathbb{C}[X_\mathbb{C}] \to \mathrm{FBL}_\mathbb{C}[X_\mathbb{C}]$ defines a projection onto $\delta_{X_\mathbb{C}}(X_\mathbb{C})$. Since $\delta_{X_\mathbb{C}}$ is an isometric embedding, we deduce that $\ker(P) = \ker(\beta)$ and note that $\ker(\beta)$ is an ideal in $\mathrm{FBL}_\mathbb{C}[X_\mathbb{C}]$ as β is a complex lattice homomorphism. □

Proposition 4.32. *If $T : X_\mathbb{C} \to X_\mathbb{C}$ is a lattice homomorphism, then $\sigma(T) \subseteq \sigma(\overline{T})$.*

Proof. For simplicity, throughout this proof we shall write $Z = X_\mathbb{C}$. By the previous lemma we know that $\mathrm{FBL}_\mathbb{C}[Z]$ can be decomposed into a direct sum $\mathrm{FBL}_\mathbb{C}[Z] = \delta_Z(Z) \oplus I_\mathbb{C}$, where $I_\mathbb{C} = \ker(\beta)$. Since $\beta\delta_Z = \mathrm{id}$, by the definition of δ_Z, we have that

$$(x_1, x_2) = \beta\delta_Z(x_1, x_2) = (\beta\delta_{Z_\mathbb{R}}(x_1, x_2), \beta\delta_{Z_\mathbb{R}}(x_2, -x_1)),$$

for every $(x_1, x_2) \in Z$, so $\beta\delta_{Z_\mathbb{R}}(x_1, x_2) = x_1$.

Let us see that $\overline{T}(f+ig) \in \ker(\beta) = I_{\mathbb{C}}$ whenever $f+ig \in \ker(\beta) = I_{\mathbb{C}}$. This is equivalent to the fact that $\overline{T}f \in I$ whenever $f \in I$. Fix any $f \in I$. Since $f \in \mathrm{FBL}[Z_{\mathbb{R}}]$, there is a sequence $(f_n)_{n=1}^{\infty}$ in $\mathrm{FVL}[Z_{\mathbb{R}}]$ such that $\|f_n - f\|_{\mathrm{FBL}_{\mathbb{C}}[Z_{\mathbb{R}}]} \to 0$. For each $n \in \mathbb{N}$, we shall write

$$f_n = \bigvee_{k=1}^{m(n)} \delta_{Z_{\mathbb{R}}}\left(x_{1,k}^{(n)}, x_{2,k}^{(n)}\right) - \bigvee_{j=1}^{m(n)} \delta_{Z_{\mathbb{R}}}\left(y_{1,j}^{(n)}, y_{2,j}^{(n)}\right).$$

By the continuity of β and $\overline{T}$, we have that the sequence $\beta\overline{T}f_n$ converges to $\beta\overline{T}f$. The next identity will enable us to show that the limit $\beta\overline{T}f$ is equal to zero:

$$\begin{aligned}\beta\overline{T}f_n &= \beta\overline{T}\left(\bigvee_{k=1}^{m(n)} \delta_{Z_{\mathbb{R}}}\left(x_{1,k}^{(n)}, x_{2,k}^{(n)}\right) - \bigvee_{j=1}^{m(n)} \delta_{Z_{\mathbb{R}}}\left(y_{1,j}^{(n)}, y_{2,j}^{(n)}\right)\right)\\ &= \beta\left(\bigvee_{k=1}^{m(n)} \delta_{Z_{\mathbb{R}}}\left(Tx_{1,k}^{(n)}, Tx_{2,k}^{(n)}\right) - \bigvee_{j=1}^{m(n)} \delta_{Z_{\mathbb{R}}}\left(Ty_{1,j}^{(n)}, Ty_{2,j}^{(n)}\right)\right)\\ &= \bigvee_{k=1}^{m(n)} Tx_{1,k}^{(n)} - \bigvee_{j=1}^{m(n)} Ty_{1,j}^{(n)} = T\left(\bigvee_{k=1}^{m(n)} x_{1,k}^{(n)} - \bigvee_{j=1}^{m(n)} y_{1,j}^{(n)}\right) = T(\beta f_n).\end{aligned}$$

Thus, by the continuity of T we obtain that $(\beta\overline{T}f_n)_{n=1}^{\infty}$ converges to $T\beta f$, which is zero, since we are assuming that $f \in I = \ker(\beta)$. By the uniqueness of the limit, $\beta\overline{T}f = 0$, so $\overline{T}f \in I$. Finally, given that $I_{\mathbb{C}}$ and $\delta_Z(Z)$ are $\overline{T}$-invariant subspaces, it can be easily verified that (see [1, Section 6.1, Ex. 17])

$$\sigma(\overline{T}) = \sigma(T) \cup \sigma\big(\overline{T}\big|_{I_{\mathbb{C}}}\big).$$

□

Remark 4.33. The previous proposition remains true for operators which are *similar* to lattice homomorphisms: an operator T on a complex Banach space X is said to be similar to a lattice homomorphism if there exist a complex Banach lattice $Y_{\mathbb{C}}$ and an invertible operator $S : X \to Y_{\mathbb{C}}$ such that STS^{-1} is a lattice homomorphism.

Remark 4.34. Let E be an uncomplemented subspace of ℓ_1 which is isomorphic to ℓ_1 [34]. Let $T : \ell_1 \to \ell_1$ be an isomorphism onto E. We have that $0 \in \sigma_a(\overline{T})$ even though $0 \in \sigma(T)\backslash\sigma_a(T)$. Indeed, suppose otherwise that $\overline{T}$ is bounded below. Let us denote by $S : \ell_1 \to E$ the operator defined by $Sx = Tx$ for every $x \in \ell_1$, so that $\iota S = T$, where ι stands for the canonical inclusion of E into ℓ_1. Thus, $\overline{T} = \overline{\iota}\,\overline{S}$ and since $\overline{S}$ is a lattice isomorphism of $\mathrm{FBL}_{\mathbb{C}}[\ell_1]$ onto $\mathrm{FBL}_{\mathbb{C}}[E]$ and we are assuming that $\overline{T}$ is bounded below, it follows that $\overline{\iota}$ is a lattice embedding $\mathrm{FBL}_{\mathbb{C}}[E]$ into $\mathrm{FBL}_{\mathbb{C}}[\ell_1]$. This implies that any $R : E \to L_1(\mu)$ extends to $\widehat{R} : \ell_1 \to L_1(\mu)$ with $\|\widehat{R}\| = \|R\|$ (see [119, Section 4]). In particular, $S^{-1} : E \to \ell_1$ extends to $\widehat{S^{-1}} : \ell_1 \to \ell_1$ and then we could define a projection $P = \iota S\widehat{S^{-1}}$ of ℓ_1 onto E. This is a contradiction.

Although the last remark suggests that Proposition 4.32 might not hold for T being an arbitrary operator, Jochen Glück and Phillip Krokor have recently shown that $\sigma(T) \subseteq \sigma(\overline{T})$ without any assumptions on the operator T or the space E where it is defined [62].

Chapter 5

Norm-attaining lattice homomorphisms and renormings of Banach lattices

This chapter focuses on studying norm-attainment for lattice homomorphisms defined on Banach lattices and its preservation through lattice renormings. Our main motivation stems from article [42], where the authors present the first example of a lattice homomorphism that does not attain its norm. We show that a Banach lattice with a strictly positive functional can be lattice renormed in such a way that no lattice homomorphism (except for the order continuous ones) attains its norm. As a consequence of the latter, one can exhibit examples of Dedekind complete Banach lattices admitting a renorming with a non-norm-attaining lattice homomorphism, answering negatively a question posed in [42] by Dantas, Rodríguez Abellán, Rueda Zoca and Martínez-Cervantes. In addition, we prove that every lattice homomorphism on an AM-space attains its norm and study the preservation of this property by lattice renormings. The chapter is based on:

[29] E. Bilokopytov, E. García-Sánchez, D. de Hevia, G. Martínez-Cervantes, and P. Tradacete, *Norm-attaining lattice homomorphisms and renormings of Banach lattices*, Preprint available on arXiv (2025), 29 pp.

5.1 Coordinate functionals on a Banach lattice

Following the notation of the rest of the thesis, the set of all functionals in X^* that are lattice homomorphisms is denoted by $\mathrm{Hom}(X,\mathbb{R})$. Recall that if E is a Banach space, we say that a functional $x^* \in E^*$ *attains its norm* if there is $x \in B_E$ such that $|x^*(x)| = \|x^*\|$. The set of all functionals in E^* that attain their norm is denoted by $\mathrm{NA}(E,\mathbb{R})$.

As we mentioned in the Introduction, the central topic of the present work *are lattice homomorphisms that attain their norm.* Later in the chapter some specific classes of Banach lattices will be studied (Sections 5.3–5.6), but first we will begin by focusing our scope on general facts about the stability of the norm attainment of lattice homomorphisms under *lattice renormings* of a Banach lattice (Sections 5.1 and 5.2). By *lattice renorming* of a Banach lattice X we mean an equivalent norm $|||\cdot|||$ such that $(X, |||\cdot|||)$ is also a Banach lattice with respect to the original lattice order. It turns out that *coordinate functionals of an atom* are lattice homomorphisms that always attain the norm, independently of the lattice renorming, as will be shown in the next proposition.

Recall that the ideal generated by an atom $x_0 \in X_+$ is always a projection band and its corresponding band projection $P_{x_0} : X \to X$ is given by $P_{x_0}(x) = \lambda_{x_0}(x)x_0 = \sup_n(x \wedge nx_0)$, for $x \in X_+$ [94, Theorem 9, p. 8]. In this case, the functional λ_{x_0} is a lattice homomorphism which is called the *coordinate functional of* x_0 and satisfies $\ker \lambda_{x_0} = \{x_0\}^d$, and $\lambda_{x_0}(x_0) = 1$. We recall that for a subset A of a vector lattice X we define the *disjoint* *complement* of A as $A^d = \{x \in X : x \perp y \text{ for every } y \in A\}$. Note that a positive multiple of a coordinate functional of an atom is also a coordinate functional of an atom (as for $r > 0$, we would have $P_{x_0}x = r\lambda_{x_0}(x)\frac{1}{r}x_0$ for $x \in X$, so $r\lambda_{x_0} = \lambda_{\frac{1}{r}x_0}$).

Proposition 5.1. *Let x^* be a non-zero lattice homomorphism on a Banach lattice X, and let $x_0 \in X_+$ be an atom. The following conditions are equivalent:*

(i) x^* is a (positive) multiple of the coordinate functional of x_0;

(ii) For any lattice norm on X, x^* attains its norm at a multiple of x_0;

(iii) There is a lattice norm on X such that x^* attains its norm at x_0;

(iv) $x^*(x_0) > 0$;

(v) $\{x_0\}^d \subseteq \ker x^*$.

In this case, $y^* \in X^*$ is disjoint with x^* if and only if $y^*(x_0) = 0$.

Proof. $(i)\Rightarrow(ii)$: We may assume that $x^* = \lambda_{x_0}$ (the coordinate functional of x_0) and let P_{x_0} denote its corresponding band projection. For any norm $\|\cdot\|$ on X and any $x \in X$ we have $\|x\| \geq \|P_{x_0}x\| = \|\lambda_{x_0}(x)\,x_0\| = |\lambda_{x_0}(x)|\,\|x_0\|$. It follows that

$$\|\lambda_{x_0}\| \leq \frac{1}{\|x_0\|} = \frac{\lambda_{x_0}(x_0)}{\|x_0\|} = \lambda_{x_0}\left(\frac{x_0}{\|x_0\|}\right) \leq \|\lambda_{x_0}\|,$$

and so λ_{x_0} attains its norm at $\frac{x_0}{\|x_0\|}$.

$(ii) \Rightarrow (iii)$ is trivial in light of the fact that X is a Banach lattice, so it admits at least one lattice norm.

$(iii) \Rightarrow (iv)$ is trivial.

$(iv)\Rightarrow (v)$: For any $x \perp x_0$ we have

$$|x^*(x)| \wedge x^*(x_0) = x^*\big(|x| \wedge x_0\big) = x^*(0) = 0,$$

and since $x^*(x_0) > 0$ we conclude that $x \in \ker x^*$.

$(v)\Rightarrow(i)$: Since $\{x_0\}^d$ is of codimension 1 and $x^* \neq 0$, it follows that $\{x_0\}^d = \ker x^*$. As x^* has the same kernel as the coordinate functional of x_0, we conclude that the two are multiples of each other.

Now we assume that $0 \leq y^* \perp x^*$. By the Riesz-Kantorovich formula, for every $n \in \mathbb{N}$ there is $y_n \in [0, x_0]$ such that $x^*(y_n) \leq \frac{1}{n}$ and $y^*(x_0 - y_n) \leq \frac{1}{n}$. Since x_0 is an atom there is $r_n \geq 0$ such that $y_n = r_n x_0$. We then have $r_n x^*(x_0) = x^*(y_n) \leq \frac{1}{n}$, hence $r_n \to 0$, and $(1 - r_n)y^*(x_0) = y^*(x_0 - y_n) \leq \frac{1}{n}$. The last inequality yields $y^*(x_0) \leq \frac{1}{n(1-r_n)} \to 0$. Thus, $y^*(x_0) = 0$ as desired. Without the assumption that $y^* \geq 0$ we still have that $|y^*| \perp x^*$, hence $0 = -|y^*|(x_0) \leq y^*(x_0) \leq |y^*|(x_0) = 0$.

In order to prove the converse, observe that, by the Riesz-Kantorovich formula, $|y^*|(x_0) = 0$, and then using it again we have that $\big(|y^*| \wedge x^*\big)(x_0) = 0 = \big(|y^*| \wedge x^*\big)(x)$, for any $x \perp x_0$. It follows that $|y^*| \wedge x^* = 0$. □

We now present some criteria for a lattice homomorphism on a vector lattice to be a coordinate functional of an atom. It turns out that coordinate functionals are precisely *those lattice homomorphisms that are order continuous*. Recall that a lattice homomorphism x^* is *order continuous* if for every decreasing net $(x_\alpha)_\alpha$ with $\inf_\alpha \|x_\alpha\| = 0$, $x^*(x_\alpha) \to 0$.

Proposition 5.2. *Let $x^* \neq 0$ be a lattice homomorphism on a vector lattice X. The following conditions are equivalent:*

(i) There exists an atom $x_0 \in X$ such that $x^*(x_0) > 0$.

(ii) x^* is a coordinate functional of an atom in X;

(iii) $\ker x^*$ is a projection band;

(iv) $\ker x^*$ is not order dense;

(v) x^* is order continuous.

Proof. $(i) \Rightarrow (ii)$ follows from Proposition 5.1 (the relevant implications $(iv) \Rightarrow (v) \Rightarrow (i)$ hold for any vector lattice).

$(ii) \Rightarrow (iii) \Rightarrow (iv)$ are straightforward.

$(iv) \Rightarrow (i)$: $\ker x^*$ is an ideal of codimension 1. If it is not order dense, then $(\ker x^*)^d$ is a non-trivial ideal. It then must have dimension 1, and so it is the span of an atom, say x_0. In particular, $x^*(x_0) \neq 0$.

$(ii) \Rightarrow (v)$: There is an atom $x_0 \in X$ such that $Px = x^*(x)\, x_0$, for every $x \in X$. Since P is order continuous, it follows that so is x^*.

$(v) \Rightarrow (iv)$: It follows that $\ker x^*$ is a proper band, and so cannot be order dense. □

We can add some additional conditions to the characterization above when the domain is a Banach lattice instead of a vector lattice.

Proposition 5.3. *Let $x^* \neq 0$ be a lattice homomorphism on a Banach lattice X. The following conditions are equivalent:*

(i) x^* is a coordinate functional of an atom in X;

(ii) $\{x^*\}^d$ is w^*-closed in X^*;

(iii) $\{x^*\}^d \cap B_{X^*}$ is w^*-compact;

(iv) $x^* \notin \overline{\{x^*\}^d}^{w^*}$.

Proof. $(i) \Rightarrow (ii)$ follows from Proposition 5.1.

$(ii) \Rightarrow (iii)$ and $(ii) \Rightarrow (iv)$ are straightforward.

$(iii) \Rightarrow (ii)$ follows from Krein-Smulian theorem.

$(iv) \Rightarrow (i)$: Since $\{x^*\}^d$ is a subspace of X^* it follows that $\overline{\{x^*\}^d}^{w^*} = (\{x^*\}^d{}_\perp)^\perp$, where $\{x^*\}^d{}_\perp \subseteq X$ is the set of vectors that vanish on every element of $\{x^*\}^d$. Hence, there is $x_0 \in \{x^*\}^d{}_\perp$ such that $x^*(x_0) > 0$. Since $\{x^*\}^d$ is an ideal of X^* of codimension 1, it follows from the dual Riesz-Kantorovich formula that $\{x^*\}^d{}_\perp$ is an ideal in X of dimension 1. Hence, x_0 is an atom, and so according to Proposition 5.2 we conclude that x^* is a coordinate functional of an atom. □

Using the last item in Proposition 5.2 and Proposition 5.1 we immediately get the following fact.

Corollary 5.4. *Every lattice homomorphism on an order continuous Banach lattice is a coordinate functional of an atom, and in particular, attains its norm.*

The fact that every lattice homomorphism on an order continuous Banach lattice attains its norm was previously obtained in [42, Theorem 4.1]. But now, the additional information provided by our latest corollary allows us to extend this result to the case of finite-rank operators.

Corollary 5.5. *Every finite-rank order continuous lattice homomorphism attains its norm. In particular, every finite-rank lattice homomorphism from an order continuous Banach lattice attains its norm.*

Proof. Let $T : X \to Y$ be a finite-rank order continuous lattice homomorphism. Replacing Y with TX does not affect order continuity of T, and so we may assume that $Y = TX$, which is lattice isomorphic to $\mathbb{R}^n$, for some $n \in \mathbb{N}$ and with the appropriate lattice norm. Let $e_1^*, \dots, e_n^*$ be the coordinate functionals on $\mathbb{R}^n$ (which we identify with TX). Let $x_k^* := e_k^* \circ T$, $k = 1, \dots, n$, which is an order continuous lattice homomorphism on X. It follows that $Tx = \big(x_1^*(x), ..., x_n^*(x)\big)$, for every $x \in X$. Due to surjectivity of T, we have $x_k^* \neq 0$, and so by Corollary 5.4 there is an atom $x_k \in X$ such that x_k^* is the coordinate functional of x_k, for $k = 1, \dots, n$. Let P_k be the band projection onto the span of x_k, so that $P_k x = x_k^*(x) x_k$, for every $x \in X$. Let P be the band projection onto $Z := \operatorname{span}\{x_1, \dots, x_n\}$. We have that $P_k P = P_k$, hence $x_k^*(x) x_k = P_k x = P_k P x$, and so $x_k^*(x) = x_k^*(Px)$, for every $x \in X$ and $k = 1, \dots, n$. It follows that $T = TP$, $TB_X = TB_Z$ and since Z is finitely dimensional, T attains its norm on B_Z. □

Let us consider now the case when X is close to being order continuous, in the sense that X^a is of finite codimension in X. By the *order continuous part* X^a of a Banach lattice X we mean the largest (closed) ideal of X on which the norm is order continuous [115, Proposition 2.4.10]. Note that even when X^a has codimension 1 there can be a lattice homomorphism that does not attain its norm. This is the case for example for $X = c$ (the space of convergent sequences of reals) with a renorming as in item (2) after Theorem 5.7 below.

Proposition 5.6. *Let X be a Banach lattice. If X^a has codimension n in X, then X has exactly n distinct norm-one lattice homomorphisms which are not coordinate functionals of atoms.*

Proof. Since X^a is a closed ideal of codimension n in X, the quotient X/X^a is an n-dimensional Banach lattice, hence lattice isomorphic to $\mathbb{R}^n$. Let $Q : X \to X/X^a \equiv \mathbb{R}^n$ be the quotient map, and let $x_1^*, \dots, x_n^*$ be as in the proof of Corollary 5.5. Let x^* be a lattice homomorphism on X. We will show that x^* is not a coordinate functional of an atom if and only if it is a multiple of some x_j^*.

First, assume that x^* vanishes on X^a. Then x^* must be a linear combination of $x_1^*, \dots, x_n^*$. Since lattice homomorphisms are either disjoint or proportional, we conclude that x^* is a positive multiple of some x_j^*. Also, every atom in X is contained in X^a, hence by Proposition 5.1, we deduce that x^* cannot be a coordinate functional on X.

Now we assume that $x^*|_{X^a} \neq 0$, so that $x^*|_{X^a}$ is a non-trivial lattice homomorphism on the order continuous Banach lattice X^a. By Corollary 5.4, there exists an atom $x_0 \in X^a$ such that $x^*(x_0) > 0$. As X^a is an ideal in X, it follows that x_0 is an atom in X as well. Now, by Proposition 5.1, we conclude that x^* must be a coordinate functional on X. □

5.2 Stability of norm attainment of lattice homomorphism under renormings

In view of the Proposition 5.1, one might wonder what happens to a lattice homomorphism which is not a coordinate functional. Can we always find a lattice norm so that such a lattice homomorphism does not attain its norm? The answer will be affirmative in many occasions:

Theorem 5.7. *Let X be a Banach lattice which has a strictly positive functional μ. If we renorm X with $\|\cdot\|_\mu := \|\cdot\| + \mu(|\cdot|)$, then the only lattice homomorphisms attaining their norms are coordinate functionals of atoms.*

Recall that a linear functional $\mu : X \to \mathbb{R}$ is said to be *strictly positive* if $\mu(x) > 0$ whenever $x > 0$. Before proving this result, let us see some consequences:

(1) If we equip $C[0,1]$ or $L_\infty[0,1]$ with the norm $|||f||| = \|f\|_\infty + \int_0^1 |f(t)|dt$, then no (non-trivial) lattice homomorphism is norm-attaining.

(2) If we consider $\ell_\infty = C(\beta\mathbb{N})$ with the renorming $|||(x_n)_n||| = \|(x_n)_n\|_\infty + \sum_{n=1}^\infty \frac{1}{2^n}|x_n|$, then no evaluation δ_t for $t \in \beta\mathbb{N}\backslash\mathbb{N}$ attains its norm.

(3) Since every separable Banach lattice has a strictly positive functional (see [109, Proposition 1.b.15]), then any separable Banach lattice can always be renormed in such a way that the only norm-attaining lattice homomorphism are the coordinate functionals.

It was shown in [42, Theorem 4.1] (see also Corollary 5.4) that every lattice homomorphism on an order continuous Banach lattice attains its norm. The authors asked whether the same result can be extended to σ-Dedekind complete Banach lattices. It should be noted that the above examples answer this question *in the negative*: recall that ℓ_∞ and $L_\infty[0,1]$ are both Dedekind complete Banach lattices and, as this property depends only on the order of X (and not on the norm), renormings (1) and (2) provide *examples of Dedekind complete Banach lattices with lattice homomorphisms that do not attain their norm.*

Back to the proof of Theorem 5.7, let us fix the notation. Let x^* be a lattice homomorphism on a Banach lattice X. Then, x^* is an atom in X^*, and so we denote the corresponding coordinate functional by $\lambda_{x^*} \in X^{**}$. That is, if $P_{x^*} : X^* \to X^*$ represents the band projection onto the span of x^*, given $\mu \in X^*$ we have $P_{x^*}(\mu) = \lambda_{x^*}(\mu)x^*$.

Lemma 5.8. *Let x^* be a lattice homomorphism on a Banach lattice X and let $\mu \in X_+^*$. Then:*

(i) The norm $\|x\|_\mu := \|x\| + \mu(|x|)$, $x \in X$, is an equivalent lattice norm on X, such that $\|x^*\|_\mu = \frac{1}{1+\lambda_{x^*}(\mu)}\|x^*\|$.

(ii) If x^* attains its $\|\cdot\|_\mu$-norm at x_0, then $\mu(x_0) = \lambda_{x^*}(\mu)x^*(x_0)$.

(iii) If also $\mu \perp x^*$, then $\|x^*\|_\mu = \|x^*\|$, and if x^* attains its $\|\cdot\|_\mu$-norm at x_0, then $\mu(x_0) = 0$.

Proof. Without loss of generality, we may assume that $\|x^*\| = 1$. Let $r := \lambda_{x^*}(\mu)$. We first show that $\|x^*\|_\mu \geq \frac{1}{1+r}$.

Fix $\varepsilon > 0$ and let $x \in B_{X_+}$ such that $x^*(x) \geq 1 - \varepsilon$. As x^* and $\mu - rx^*$ are disjoint, by the Riesz-Kantorovich formula there exists $y \in [0,x]$ such that $x^*(x-y) \leq \varepsilon$ and $(\mu - rx^*)(y) \leq \varepsilon$. From the first inequality, we deduce

$$\varepsilon \geq x^*(x-y) = x^*(x) - x^*(y) \geq 1-\varepsilon - x^*(y)$$

which means that $x^*(y) \geq 1-2\varepsilon$. Now, from the second one (keeping in mind that $\|y\| \leq 1$ because $x \in B_{X_+}$), we get

$$\varepsilon \geq \mu(y) - rx^*(y) \geq \mu(y) - r$$

or equivalently, $\mu(y) \leq \varepsilon + r$. Hence, we have

$$\|x^*\|_\mu \geq \frac{x^*(y)}{\|y\|_\mu} = \frac{x^*(y)}{\|y\| + \mu(y)} \geq \frac{1-2\varepsilon}{1+\varepsilon+r},$$

and since $\varepsilon > 0$ is arbitrary we conclude that $\|x^*\|_\mu \geq \frac{1}{1+r}$.

Now suppose that $|x^*(x)| = 1$ for some $x \in X$. Note that $x^*(|x|) = |x^*(x)| = 1$, and $\mu \geq rx^*$, hence

$$\|x\|_\mu = \|x\| + \big(\mu - rx^*\big)(|x|) + rx^*(|x|) \geq |x^*(x)| + rx^*(|x|) = 1+r.$$

This shows that $\|x^*\|_\mu \leq \frac{1}{1+r}$. Moreover, if $x_0 \in X_+$ is such that $|x^*(x_0)| = 1$ and $\|x_0\|_\mu = 1+r$, then $\big(\mu - rx^*\big)(x_0) = 0$, hence $\mu(x_0) = r = \lambda_{x^*}(\mu)$. This justifies the second claim in *(ii)*. *(iii)* is a special case of *(i)* and *(ii)*, since $\mu \perp x^*$ is equivalent to $r = \lambda_{x^*}(\mu) = 0$. □

Proof of Theorem 5.7. Suppose μ is a strictly positive functional and consider the renorming

$$\|x\|_\mu := \|x\| + \mu\big(|x|\big), \qquad x \in X.$$

Let x^* be a lattice homomorphism on X which is not a coordinate functional of an atom and assume that $x_0 \in X_+$ is such that $\|x_0\|_\mu = 1$ and $x^*(x_0) = \|x^*\|_\mu$. Then, according to Lemma 5.8 we have that $\mu(x_0) = rx^*(x_0)$, where $r \geq 0$ is such that $\mu - rx^* \geq 0$ and $\mu - rx^* \perp x^*$. It follows from Proposition 5.1 that x_0 is not an atom. Hence, there are disjoint non-zero $u, v \in [0, x_0]$. We have that $x^*(u) \wedge x^*(v) = x^*(u \wedge v) = 0$, and so we may assume that $x^*(u) = 0$. From strict positivity we have

$$\mu(x_0) > \mu(x_0 - u) \geq rx^*(x_0 - u) = rx^*(x_0) - rx^*(u) = rx^*(x_0),$$

which is a contradiction. □

Corollary 5.9. *Let X be a Banach lattice and let $x^* \in Hom(X,\mathbb{R})$, $x^* \neq 0$. Suppose that there exists $x_0 \in X$ such that for every lattice renorming $|||\cdot|||$ of X, x^* attains its norm at $\frac{x_0}{|||x_0|||}$. Then x_0 is an atom in X.*

Proof. Suppose that x_0 is not an atom in X. Arguing as in the proof of Theorem 5.7 we can find a non-zero $y_0 \in [0, x_0]$ such that $x^*(y_0) = 0$. Let $\mu \in X_+^*$ be such that $\mu(y_0) > 0$ and define $\|\cdot\|_\mu$ as in Lemma 5.8. Let $r \geq 0$ be such that $\mu - rx^* \geq 0$ and $\mu - rx^* \perp x^*$. By part *(ii)* of Lemma 5.8 we have $\mu(x_0) = rx^*(x_0)$, hence

$$\mu\left(y_0\right) = \left(\mu - rx^*\right)\left(y_0\right) + rx^*(y_0) \leq \left(\mu - rx^*\right)\left(x_0\right) = 0,$$

which is a contradiction. □

In general, there can be lattice homomorphisms that are not coordinate functionals but attain their norm for any lattice renorming, as shown in the next example.

Example 5.10. Let ω_1 denote the first uncountable ordinal, and let $[0, \omega_1]$ denote the corresponding interval of ordinals which is compact for the order topology (also called *interval topology*), which is the topology generated by the subbase of all sets of the form $[0, \alpha)$ and $(\beta, \omega_1]$, for $\alpha, \beta \in [0, \omega_1]$ [155, Problem 6D]. It is clear that the evaluation functional $\delta_{\omega_1} \in C[0, \omega_1]^*$ is a lattice homomorphism which is not a coordinate functional of an atom. Let us show that if $|||\cdot|||$ is a lattice norm on $C[0, \omega_1]$, then δ_{ω_1} attains its norm. Assume that $|||\delta_{\omega_1}||| = 1$. Then, for every $n \in \mathbb{N}$ there is $f_n \in C[0, \omega_1]_+$ such that $f_n(\omega_1) = 1$ and $1 \leq |||f_n||| \leq 1 + \frac{1}{n}$. Since continuous functions on $[0, \omega_1]$ are eventually constant, for every n there is $\alpha_n < \omega_1$ such that f_n is identically 1 on $[\alpha_n, \omega_1]$. Now, define $\alpha := \sup_{n \in \mathbb{N}} \alpha_n$, which is again a countable ordinal. Since $(\alpha, \omega_1]$ is a clopen set, $f = \chi_{(\alpha, \omega_1]} \in C[0, \omega_1]$. Note that $0 \leq f \leq f_n$ for every $n \in \mathbb{N}$, therefore $|||f||| \leq 1$ and $f(\omega_1) = 1$.

We have already pointed out just after Theorem 5.7 that being Dedekind complete does not ensure that every lattice homomorphism attains its norm. But what happens if we assume that the Banach lattice is σ-order continuous? In the following example we show that this condition does not guarantee that $\mathrm{Hom}(X, \mathbb{R}) \subseteq \mathrm{NA}(X, \mathbb{R})$ either.

Example 5.11. Let Γ be an uncountable set endowed with the discrete topology. If $K = \alpha\Gamma = \Gamma \cup \{\infty\}$ is the one-point compactification of Γ, then $C(K)$ is σ-order continuous but not order continuous. Indeed, suppose that $\{f_n\}_{n=1}^\infty$ is a decreasing sequence in $C(K)$ such that $\bigwedge_{n=1}^\infty f_n = 0$. We claim that $f_n(\infty) \to 0$. If not, there is $\varepsilon_0 > 0$ such that $f_n(\infty) \geq \varepsilon_0$. For every natural n, $S_n = \mathrm{supp}\big(f_n - f_n(\infty)\big)$ is countable, so $S = \cup_{n=1}^\infty S_n$ is also countable. Now, fix any $\gamma_0 \in \Gamma \setminus S$ (this set is non-empty as Γ is uncountable) and note that $f_n(\gamma_0) = f_n(\infty) \geq \varepsilon_0$ for all $n \in \mathbb{N}$. Thus, we arrive at the following contradiction

$$0 = \bigwedge_{n=1}^{\infty} f_n \geq \varepsilon_0 \, \mathbf{1}_{\{\gamma_0\}} > 0,$$

and this implies that $f_n(\infty) \to 0$. With the latter in mind, it is easy to conclude that $\|f_n\|_\infty \to 0$. Let $\varepsilon > 0$ and N be such that $f_N(\infty) < \varepsilon$. Then, $\{\gamma \in \Gamma : |f_N(\gamma)| \geq \varepsilon\} = \{\gamma_1, \ldots, \gamma_k\}$, and since $f_n(\gamma_i) \to 0$ for every $i = 1, \ldots, k$, we find $N' \geq N$ such that $f_{N'}(\gamma_i) < \varepsilon$ for all i. Then $\|f_n\|_\infty \leq \varepsilon$ whenever $n \geq N'$.

Now, take an infinite sequence of points $(t_n)_{n=1}^\infty$ in $\Gamma \subseteq K$ and define

$$|||f||| = \|f\|_\infty + \sum_{n=1}^{\infty} \frac{1}{2^n} |f(t_n)|, \qquad f \in C(K),$$

which is a lattice renorming of $C(K)$. Therefore, $(C(K), |||\cdot|||)$ is a σ-order continuous Banach lattice which has a lattice homomorphism, specifically δ_∞, which does not attain its norm.

The following result will come handy in applications later on. It is somewhat inspired by the examples in this section (see also Proposition 5.19).

Lemma 5.12. *Let z^* be a non-zero lattice homomorphism on a Banach lattice X. Assume that there is a sequence $(y_n^*)_{n=1}^\infty \subseteq X_+^*$ such that $z^* \perp y_n^*$, for every $n \in \mathbb{N}$, but $z^* \in \overline{\mathrm{span}(y_n^*)_{n=1}^\infty}^{w^*}$. Then, there is a lattice renorming of X in which z^* does not attain its norm.*

Proof. Without loss of generality we may assume that $\|y_n^*\| \leq 1$, for every $n \in \mathbb{N}$. Let $\mu := \sum_{n=1}^\infty \frac{1}{2^n} y_n^* \in X^*$, which is disjoint with z^* (this may be checked using [115, Theorem

1.1.1 (ix)]). If $z_0 \in X_+$ is such that $\|z_0\|_\mu = 1$ and $z^*(z_0) = \|z^*\|_\mu$, then by Lemma 5.8 we have $\mu(z_0) = 0$. For every $n \in \mathbb{N}$, since $z_0 \geq 0$ and $y_n^* \geq 0$, it follows that $0 = \mu(z_0) \geq y_n^*(z_0) \geq 0$. Thus $y_n^*(z_0) = 0$, for every $n \in \mathbb{N}$, and as $z^* \in \overline{\mathrm{span}(y_n^*)_{n=1}^\infty}^{w^*}$, we conclude that $0 = z^*(z_0) = \|z^*\|_\mu$. This is absurd, since $z^* \neq 0$. □

5.3 Norm attainment of lattice homomorphisms on AM-spaces

Although the set of positive functionals over a Banach lattice is always large since any element of the dual is a difference of positive functionals, the set of lattice homomorphisms might be trivial for some Banach lattices. For instance, $L_p[0,1]$ does not have non-zero lattice homomorphisms whenever $1 \leq p < \infty$ (see [1, Lemma 2.31 (1)]). A natural class of Banach lattices with a *large set* of lattice homomorphisms is the class of AM-spaces (to get a more accurate idea, see Proposition 5.22). Recall that an *AM-space* is a Banach lattice X such that the norm satisfies

$$\|x \vee y\| = \max\{\|x\|, \|y\|\} \quad \text{for all } x, y \in X_+.$$

AM-spaces and $C(K)$-spaces are closely related. Specifically, a well-known theorem due to Kakutani asserts that a Banach lattice X is an AM-space if and only if it embeds as a closed sublattice of some $C(K)$-space [84]. It is also well known that on $C(K)$ spaces, lattice homomorphisms correspond to (positive multiples) of point evaluations. Therefore, on $C(K)$, every lattice homomorphism attains its norm; in fact, all of them attain their norm at the strong unit $\mathbf{1}_K$. However, an AM-space in general does not necessarily have a strong unit.

Similarly, every positive functional on a $C(K)$-space attains its norm at the constant function $\mathbf{1}_K$. It turns out that in the separable setting, this property characterizes $C(K)$-spaces among AM-spaces [120, Proposition 19.26]: given a separable AM-space X, every positive functional on X attains its norm if and only if X is lattice isometric to a $C(K)$-space. The following simple example shows that this cannot be generalized to the non-separable case.

Example 5.13. Let Γ be an uncountable set and consider the closed sublattice of $\ell_\infty(\Gamma)$

$$X = \{f \in \ell_\infty(\Gamma) \, : \, \mathrm{supp}(f) \text{ is countable}\}.$$

Note that X is a non-separable AM-space, but it does not admit a strong unit, so it cannot be lattice isometric to a $C(K)$-space. Let $x^* \in X^*$ be a norm-one positive functional. Then, for every $n \in \mathbb{N}$, there is $f_n \in B_{X_+}$ such that $x^*(f_n) \geq 1 - \frac{1}{n}$. By definition of X, for every natural n, the set $S_n := \mathrm{supp}(f_n)$ is countable, so the union $S = \cup_{n=1}^\infty S_n$ is also countable. Therefore, $\mathbf{1}_S \in X$ and since $\mathbf{1}_S \geq f_n$ for every $n \in \mathbb{N}$, this implies that $x^*(\mathbf{1}_S) = 1$.

Lattice homomorphisms, as opposed to positive functionals, have a similar behavior on $C(K)$-spaces and general AM-spaces in terms of norm attainment: every lattice homomorphism on an AM-space attains its norm (Theorem 5.15). This fact will be easily deduced from an *interesting characterization of lattice homomorphisms* that we present below. In fact, this characterization will also be decisive to prove later that every lattice homomorphism on a free Banach lattice over a lattice attains its norm (Proposition 5.41).

Proposition 5.14. *Let X be a Banach lattice and $x^* \in X^*$ a lattice homomorphism with $\|x^*\| = 1$. Then $x^* \in NA(X, \mathbb{R})$ if and only if there exists an increasing sequence $(x_n)_{n=1}^\infty$ in B_{X_+} such that $x^*(x_n) \to 1$ as $n \to \infty$.*

Proof. Necessity is trivial, by considering the constant sequence defined by $x_n := x \in B_{X_+}$ where $x^*(x) = 1$. Let us prove sufficiency. Let $(x_n)_{n=1}^\infty \subseteq B_{X_+}$ be an increasing sequence such that $0 < x^*(x_n) \to 1$. It follows that $x^*(x_n)$ is increasing. Define

$$y_n := \bigwedge_{k=1}^{n} \frac{x_k}{x^*(x_k)}, \qquad \text{for } n \in \mathbb{N}.$$

The above sequence $(y_n)_{n=1}^\infty$ is clearly decreasing and we claim that it is a Cauchy sequence in X. Given $m, n \in \mathbb{N}$, $m > n$, we have

$$\begin{aligned} 0 \le y_n - y_m = y_n - y_n \wedge \left(\bigwedge_{k=n+1}^{m} \frac{x_k}{x^*(x_k)} \right) &= \left(\bigwedge_{k=1}^{n} \frac{x_k}{x^*(x_k)} - \bigwedge_{k=n+1}^{m} \frac{x_k}{x^*(x_k)} \right)^+ \\ \le \left(\frac{x_n}{x^*(x_n)} - \bigwedge_{k=n+1}^{m} x_k \right)^+ \le \left(\frac{x_{n+1}}{x^*(x_n)} - x_{n+1} \right)^+ &= \left(\frac{1}{x^*(x_n)} - 1 \right) x_{n+1}. \end{aligned}$$

By the monotonicity of the norm of X we obtain $\|y_n - y_m\| \le \frac{1}{x^*(x_n)} - 1$, and this shows that $(y_n)_{n=1}^\infty$ is Cauchy. Let us denote by y the limit of this sequence in X and note that for every $n \in \mathbb{N}$,

$$x^*(y_n) = \bigwedge_{k=1}^{n} \frac{x^*(x_k)}{x^*(x_k)} = 1, \qquad \text{and} \qquad \|y_n\| \le \frac{\|x_n\|}{x^*(x_n)} \le \frac{1}{x^*(x_n)},$$

so $x^*(y) = 1 = \|y\|$. □

As a consequence, we get:

Theorem 5.15. *Every lattice homomorphism on an AM-space attains its norm.*

Proof. Let x^* be a norm-one lattice homomorphism on an AM-space X. Then, for every $n \in \mathbb{N}$, there is $y_n \in B_{X_+}$ such that $x^*(y_n) \ge 1 - \frac{1}{n}$. Now, define for every natural $n \ge 1$, $x_n := \bigvee_{k=1}^n y_n$. Clearly, $(x_n)_{n=1}^\infty$ is an increasing sequence in X. Moreover, since X is an AM-space, $x_n \in B_{X_+}$, and by the positivity of x^*, we also have $x^*(x_n) \ge 1 - \frac{1}{n}$ for $n \in \mathbb{N}$. Consequently, by Proposition 5.14, x^* attains its norm. □

As in the order continuous case, the previous theorem can be generalized for lattice homomorphisms of finite rank.

Corollary 5.16. *Let X be an AM-space and Y an arbitrary Banach lattice. Then, every finite-rank lattice homomorphism $T : X \to Y$ attains its norm.*

Proof. We may assume that $Y = T(X)$. Hence, Y is a finite-dimensional Banach lattice, so its order is determined by a 1-unconditional basis $\{y_k, y_k^*\}_{k=1}^n$. Thus, for every $x \in X$, we can write

$$Tx = \sum_{k=1}^{n} y_k^*(Tx) y_k = \sum_{k=1}^{n} x_k^*(x) y_k,$$

where $x_k^* := y_k^* \circ T$ for $1 \le k \le n$. Note that for every $x \in B_{X_+}$,

$$Tx = \sum_{k=1}^{n} x_k^*(x) y_k \le \sum_{k=1}^{n} \|x_k^*\| y_k =: y_0,$$

so $\|T\| \leq \|y_0\|$. On the other hand, as each x_k^* is a lattice homomorphism on the AM-space X, by Theorem 5.15, we can find $x_k \in B_{X_+}$ such that $x_k^*(x_k) = \|x_k^*\|$ for all k. Now, we define $x_0 := \vee_{k=1}^n x_k$, which belongs to B_{X_+} as X is an AM-space. Therefore, $Tx_0 = y_0$ and, in particular, this shows that T attains its norm at x_0. □

The following is an example of a Banach lattice X where every lattice homomorphism in X^* attains its norm, but finite-rank lattice homomorphisms need not attain their norm.

Example 5.17. Let us consider the following norm in $c \oplus c$:

$$\|(x_n, y_n)\| = \max\left\{ \|(x_n)\|_\infty + \sum_{n=1}^{\infty} \frac{1}{2^n}|y_n|, \;\; \|(y_n)\|_\infty \right\}.$$

It is clear that $\|(x_n, y_n)\|_\infty \leq \|(x_n, y_n)\| \leq 2\|(x_n, y_n)\|_\infty$. Note that every lattice homomorphism on this Banach lattice attains its norm: for example, $e_n^* \oplus 0$ attains its norm at $(e_n, \mathbf{0})$ and $e_\infty^* \oplus 0$ at $(\mathbf{1}, \mathbf{0})$ (where $\mathbf{0}$ and $\mathbf{1}$ represent the sequences which are constantly 0 and 1, respectively). The operator $T : (c \oplus c, \|\cdot\|) \to \ell_1^2$ defined by

$$T(x_n, y_n) := \big(\lim_n x_n, \lim_n y_n\big) = (e_\infty^* \oplus 0)(x_n, y_n)\, e_1 + (0 \oplus e_\infty^*)(x_n, y_n)\, e_2$$

is clearly a lattice homomorphism of norm at most 2. To see that $\|T\| = 2$ denote $z^n = \mathbf{1} - \sum_{k=1}^n e_k \in c$, and consider the sequence $(\mathbf{1}, z^n) \in c \oplus c$. We have that $\|T(\mathbf{1}, z^n)\|_1 = 2$ for every n, whereas $\|(\mathbf{1}, z^n)\| \to 1$ as $n \to \infty$. On the other hand, suppose that there exists a positive element $(x, y) \in c \oplus c$ with $\|(x, y)\| = 1$ such that $\|T(x, y)\|_1 = 2$. This implies that $\lim_n x_n = \lim_n y_n = 1$. In particular, (y_n) cannot be the constant sequence $\mathbf{0}$, so $\|(x_n)\|_\infty + \sum_{n=1}^\infty \frac{1}{2^n}|y_n| > 1$ and $\|(x, y)\| > 1$.

Another easy consequence of Theorem 5.15 is the following version of *Urysohn's lemma for AM-spaces.*

Corollary 5.18. *Let X be an AM-space and let $x_1^*, \dots, x_n^*$ be distinct lattice homomorphisms on X of norm 1. For every $a_1, \dots, a_n \in [0, 1]$, there exists $x \in B_X$ such that $x_k^*(x) = a_k$ for every $k = 1, \dots, n$.*

Proof. It suffices to show that there is an element $x \in B_X$ such that $x_1^*(x) = 1$ and $x_k^*(x) = 0$ for $k = 2, \dots, n$. By Theorem 5.15, there exists $y \in B_{X_+}$ such that $x_1^*(y) = 1$. By Kakutani's representation theorem, there is a lattice homomorphism $T : C(K) \to X$, for some compact Hausdorff K, such that $T\mathbf{1}_K = y$. Therefore, $T^*x_1^*, \dots, T^*x_n^*$ are lattice homomorphisms on $C(K)$. Since $x_1^*, \dots, x_n^*$ are linearly independent, they are disjoint; since lattice operations on X^* commute with taking the restriction to an ideal, it follows that $T^*x_1^*, \dots, T^*x_n^*$ are disjoint.

There exist $r_1, \dots, r_n \geq 0$ and $t_1, \dots, t_n \in K$ such that $T^*x_k^* = r_k\delta_{t_k}$. Note that $1 = x_1^*(y) = (T^*x_1^*)(\mathbf{1}_K) = r_1\mathbf{1}_K(t_1) = r_1$. For those $k \neq 1$ for which $r_k \neq 0$, since $r_k\delta_{t_k} \perp \delta_{t_1}$ we have $t_k \neq t_1$. By Urysohn's lemma there is a continuous function $f : K \to [0, 1]$ such that $f(t_1) = 1$ and $f(t_k) = 0$ for $k = 2, \dots, n$ with $r_k > 0$. Then, putting $x = Tf$ yields $x_1^*(x) = f(t_1) = 1$ and $x_k^*(x) = r_k f(t_k) = 0$, for all $k = 2, \dots, n$. □

5.4 Lattice homomorphisms on AM-spaces and renormings

Next, we continue with the study initiated in Section 5.2 of norm attainment of lattice homomorphisms after renormings, this time in the class of AM-spaces. The next proposition illustrates that it is a highly non-isomorphic property, at least for $C(K)$-spaces.

Proposition 5.19. *Let K be an infinite compact space. Then $C(K)$ has an equivalent lattice norm with a non-norm-attaining lattice homomorphism.*

Proof. Take a sequence $(t_n)_n$ in K with an accumulation point $t \in \overline{\{t_n : n \in \mathbb{N}\}} \setminus \{t_n : n \in \mathbb{N}\}$. Consider the equivalent lattice norm

$$|||f||| = \|f\|_\infty + \sum_{n=1}^{\infty} \frac{1}{2^n} |f(t_n)|.$$

Then the evaluation functional $\delta_t \in C(K)^*$ satisfies $|||\delta_t||| = 1$. However, by definition of t and the sequence $(t_n)_n$ we have that if $f(t) = 1$ then $|||f||| > 1$, so δ_t is not norm-attaining. □

Note that Example 5.10 illustrates that the previous argument does not imply that for every accumulation point t of the compact space K we can find a renorming such that the evaluation functional δ_t does not attain its norm.

Recall the following version of Stone-Weierstrass theorem (see e.g. [30, Theorem 2.1]).

Proposition 5.20. *Let K be compact and Hausdorff, and let X be a (not necessarily closed) sublattice of $C(K)$. Then $\overline{X} = \Big(\big\{\delta_s - \gamma\delta_t \mid s,t \in K,\ \gamma \geq 0\big\} \cap X^\perp\Big)_\perp$. In particular, X is a dense sublattice of a closed sublattice $Y \subseteq C(K)$ if and only if whenever for $t, s \in K$ the functionals $\delta_s|_Y$ and $\delta_t|_Y$ are linearly independent, $\delta_s|_X$ and $\delta_t|_X$ are also linearly independent.*

We now characterize lattice homomorphisms on AM-spaces.

Proposition 5.21. *Let K be a compact Hausdorff space and let X be a closed sublattice of $C(K)$. If x^* is a norm-one lattice homomorphism on X, then there is $t \in K$ such that $x^* = \delta_t|_X$. Thus, for every $x^* \in \mathrm{Hom}(X, \mathbb{R})$ there is $t \in K$ such that $x^* = \|x^*\|\, \delta_t|_X$.*

Proof. First, we consider the set $\mathcal{E}_{x^*}$ of norm preserving positive extensions of x^* to $\mathcal{C}(K)$, that is,

$$\mathcal{E}_{x^*} = \{y^* \in \mathcal{C}(K)^* \ :\ \|y^*\| \leq 1, \quad y^* \geq 0, \quad y^*\big|_X = x^*\},$$

which is non-empty by [111, Corollary 1.3]. Moreover, since $\mathcal{E}_{x^*}$ is w^*-closed and convex, we deduce from Krein-Milman theorem that the set of its extreme points is non-empty.

Fix an extreme point y_0^* of $\mathcal{E}_{x^*}$. We will prove that y_0^* is also an extreme point of the set $\{y^* \in B_{\mathcal{C}(K)^*} \ :\ y^* \geq 0\}$. Let $y_1^*, y_2^* \in B_{\mathcal{C}(K)^*_+}$ be such that $y_0^* = \frac{1}{2}y_1^* + \frac{1}{2}y_2^*$. If we restrict these functionals to X, we obtain

$$x^* = y_0^*\big|_X = \frac{1}{2}\, y_1^*\big|_X + \frac{1}{2}\, y_2^*\big|_X\,.$$

Since x^* is an atom in X^*, we deduce from the above identity that $y_1^*\big|_X = y_2^*\big|_X = x^*$. This shows that $y_1^*, y_2^* \in \mathcal{E}_{x^*}$ and given that y_0^* is an extreme point of $\mathcal{E}_{x^*}$, we have $y_0^* = y_1^* = y_2^*$. Therefore, y_0^* is a non-trivial extreme point of $\{y^* \in B_{\mathcal{C}(K)^*} \ :\ y^* \geq 0\}$, so there exists $t \in K$ such that $y_0^* = \delta_t$. □

Let us present an alternative proof of a weaker version of the preceding proposition, which is of independent interest. Namely, we will show that every homomorphism on X is a restriction of a positive multiple of a point evaluation.

Alternative proof of the claim. Let $0 \neq x^* \in \mathrm{Hom}(X, \mathbb{R})$, and let $N := \ker x^*$, which is a proper closed ideal in X. Let H be the (not necessarily closed) ideal generated by N in $C(K)$, that is,

$$H := \big\{ h \in C(K) \,:\, \exists x \in N \text{ such that } |h| \leq x \big\}$$

We claim that $\overline{H} \cap X = N$. Indeed, if $x \in X_+$ and $(h_n)_{n=1}^\infty \subseteq H$ are such that $h_n \to x$, then $x \wedge h_n^+ \to x$. For every $n \in \mathbb{N}$ there is $x_n \in N$ such that $x_n \geq h_n$, hence we have $|x - x \wedge x_n^+| \leq |x - x \wedge h_n^+| \to 0$, and so $x \wedge x_n^+ \to x$, as $n \to \infty$. As $x \wedge x_n^+ \in N$ for every $n \in \mathbb{N}$, we conclude that $x \in N$.

As $\overline{H}$ is a proper closed ideal in $C(K)$, there is a closed $A \subseteq K$ such that $h \in \overline{H}$ if and only if h vanishes on A (see e.g. [115, Proposition 2.1.9]). Since $X \not\subseteq \overline{H}$, there is $a \in A$ and $x \in X$ such that $x(a) \neq 0$. It follows that $\ker \delta_a \cap X$ is a closed subspace in X of codimension 1. Since we also have $N \subseteq H \subseteq \ker \delta_a$, we conclude that $\ker x^* = N = \ker \delta_a \cap X$, and so x^* is a multiple of $\delta_a|_X$. □

Let E be a Banach space and A be a *positively homogeneous set in* B_{E^*}, that is, a subset of B_{E^*} with the property that $A = \mathbb{R}_+ A \cap B_{E^*}$ or, equivalently, if $0 \in A$ and if $0 \neq x^* \in A$, then $\lambda x^* \in A$ for every $\lambda \in \big(0, \frac{1}{\|x^*\|}\big]$. Let $C_{ph}(A)$ be the space of all w^*-continuous functions f on A which are *positively homogeneous*, i.e. $f(rx^*) = rf(x^*)$, for all $r \in [0,1]$ and $x^* \in A$. If the subset A is w^*-compact, $C_{ph}(A)$ endowed with the supremum norm is an AM-space, but in general it does not have a strong unit.

Recall also that a subset $A \subseteq E^*$ is λ-norming, for some $0 < \lambda \leq 1$, if $\sup\{|x^*(x)| \,:\, x^* \in A \cap B_{E^*}\} \geq \lambda \|x\|$ for every $x \in E$. For a Banach lattice X let $K_X := \mathrm{Hom}(X, \mathbb{R}) \cap B_{X^*}$, endowed with the weak* topology, making it a compact space. The following proposition formalizes the idea that AM-spaces are distinguished among Banach lattices as those having *many lattice homomorphisms*. This justifies the interest in studying the preservation of their norm-attainment through lattice renormings. Moreover, the proposition provides an explicit representation for an AM-space X as $C_{ph}(K_X)$ that will be useful for our purposes. The result is mentioned in [65, Corollaire 1.31], and despite its similarities with Kakutani's Theorem [84], it does not seem to be very well known.

Proposition 5.22. *A Banach lattice X is λ-lattice isomorphic to an AM-space if, and only if, the set K_X is $\frac{1}{\lambda}$-norming. In this case it is λ-lattice isomorphic to $C_{ph}(K_X)$.*

Proof. First, note that if X is a closed sublattice of $C(K)$, for some K, then

$$\|x\| = \sup_{t \in K} |x(t)| = \sup_{t \in K} \big| \delta_t|_X (x) \big| \leq \sup_{x^* \in K_X} |x^*(x)| \leq \|x\|,$$

and so $K_X = \mathrm{Hom}(X, \mathbb{R}) \cap B_{X^*}$ is 1-norming.

Suppose that X is λ-lattice isomorphic to an AM-space Y, for some $\lambda \geq 1$. Let $T\colon X \to Y$ be a lattice isomorphism such that $\|T\|\|T^{-1}\| \leq \lambda$. As $K_Y = \mathrm{Hom}(Y, \mathbb{R}) \cap B_{Y^*}$ is 1-norming, for every $x \in X$ we have that

$$\begin{aligned}\sup\{|x^*(x)| \,:\, x^* \in K_X\} &= \sup\big\{|(T^{-1})^* x^*(Tx)| \,:\, x^* \in K_X\big\} \\ &\geq \frac{1}{\|T\|} \sup\{|y^*(Tx)| \,:\, y^* \in K_Y\} \\ &\geq \frac{1}{\|T\|}\|Tx\| \geq \frac{1}{\|T\|\|T^{-1}\|}\|x\| \geq \frac{1}{\lambda}\|x\|.\end{aligned}$$

Conversely, let us suppose that $\mathrm{Hom}(X,\mathbb{R})$ is $\frac{1}{\lambda}$-norming. Define $J : X \to C_{ph}(K_X)$ by $[Jx](x^*) := x^*(x)$. It is easy to see that Jx is in fact an element of $C_{ph}(K_X)$. Clearly, J is a lattice homomorphism, and it follows from our assumption that $\frac{1}{\lambda}\|x\| \leq \|Jx\| \leq \|x\|$. Hence, JX is a closed sublattice of $C_{ph}(K_X)$. We finish proving that JX is dense in $C_{ph}(K_X)$. According to part (i) of Proposition 5.21 it is enough to show that if x^* and y^* in K_X are not proportional, then there is $x \in X$ such that $(Jx)(x^*) = 0 \neq (Jx)(y^*)$, which is trivial. We conclude that X is λ-lattice isomorphic to $C_{ph}(K_X)$. □

Remark 5.23. Note that if a Banach lattice X is merely isomorphic (as a Banach space) to an AM-space, then it is already *lattice* isomorphic to some AM-space (this is a consequence of Corollary 3.2). In general, it is possible for two Banach lattices to be isometric as Banach spaces, while their collections of lattice homomorphisms *differ completely*: ℓ_2 and $L_2[0,1]$ are linearly isometric and we know that $\mathrm{Hom}(\ell_2,\mathbb{R}) = \{\lambda e_n^* : \lambda \geq 0,\ n \in \mathbb{N}\}$, whereas $\mathrm{Hom}(L_2[0,1],\mathbb{R}) = \{0\}$. Also, the proposition fails if $\mathrm{Hom}(X,\mathbb{R})$ is merely total (i.e., $x^*(x) = 0$ for every $x^* \in \mathrm{Hom}(X,\mathbb{R})$ if and only if $x = 0$), and not norming: consider $X = \ell_p$, for $p \in [1,+\infty)$.

Although it deviates somewhat from the topic we are currently discussing, this representation of AM-spaces readily yields a *Banach-Stone type theorem* for this class of Banach lattices, which was first observed in [94, Corollary 2, p. 188]:

Proposition 5.24 (Banach-Stone for AM-spaces). *Let X_1, X_2 be two AM-spaces. If X_1 is linearly isometric to X_2, then X_1 is lattice isometric to X_2.*

Proof. Let $\alpha : X_1 \to X_2$ be a surjective linear isometry and write $K_i = \mathrm{Hom}(X_i,\mathbb{R})$ for $i = 1,2$. Since $\alpha^* : X_2^* \to X_1^*$ is also a surjective linear isometry, it maps extreme points of $B_{X_2^*}$ to extreme points of $B_{X_1^*}$. Hence, by Proposition 1.12 (also keep in mind that the dual of an AM-space is an AL-space), for every $x^* \in K_1$ there exists a unique $y^* \in K_2 \cup (-K_2)$ such that $\alpha^* y^* = x^*$. Thus, we may define a function $\beta : K_1 \to K_2$ by $\beta x^* = y^*$ if $y^* \in K_2$ and $\beta x^* = -y^*$ if $y^* \in (-K_2)$. This mapping is clearly bijective and positively homogeneous. As K_1 and K_2 are w^*-compact, to see that β is an homeomorphism is enough to prove that β is w^*-continuous. Indeed, let $x_\alpha^* \xrightarrow{w^*} x^*$ in K_1 and let $y_\alpha^*, y^* \in \pm K_2$ such that $\alpha^* y_\alpha^* = x_\alpha^*$ and $\alpha^* y^* = x^*$. Since $(\alpha^*)^{-1}$ is w^*-continuous, $y_\alpha^* \xrightarrow{w^*} y^*$ in $B_{X_2^*}$. If $x^* = 0$, then $y^* = 0$ and it is easy to see that $\beta x_\alpha^* \xrightarrow{w^*} 0$ in K_2. Now, suppose that $x^* \neq 0$ and assume that $y^* > 0$. If there were a subnet $(y_{\alpha_\beta}^*)$ of (y_α^*) such that $y_{\alpha_\beta}^* \leq 0$ for every β, then $y^* \leq 0$. Hence, there must exist α_0 such that $y_\alpha > 0$ for every $\alpha \geq \alpha_0$. Thus, $\beta x_\alpha^* \xrightarrow{w^*} y^* = \beta x^*$ in K_2. The operator $T : \mathcal{C}_{ph}(K_2) = X_2 \to \mathcal{C}_{ph}(K_1) = X_1$ defined by $Tf = f \circ \beta$ is a surjective lattice isometry. □

We can now extend Proposition 5.19 to general AM-spaces, and in the process recover a classical result ([109, Lemma 1.b.10]).

Theorem 5.25. *Given an AM-space X, the following conditions are equivalent:*

(i) X is lattice isometric to $c_0(\Gamma)$ for some cardinal Γ;

(ii) X is order continuous;

(iii) Every lattice homomorphism on X is a multiple of the coordinate functional of some atom;

(iv) Every lattice homomorphism on X attains its norm for every lattice renorming of X;

(v) For every $x \in X$ and $\varepsilon > 0$ there are only finitely many lattice homomorphisms x^* of norm 1 with $|x^*(x)| \geq \varepsilon$.

Proof. $(i)\Rightarrow(ii)$ is clear, $(ii)\Rightarrow(iii)$ follows from Corollary 5.4, $(iii)\Rightarrow(iv)$ follows from Proposition 5.1.

$(iv)\Rightarrow(v)$: Assume that there is $\varepsilon_0 > 0$, $x_0 \in X_+$ and an infinite sequence $(y_n^*)_{n=1}^\infty$ of distinct homomorphisms in S_{X^*} such that $y_n^*(x_0) \geq \varepsilon_0$, for every $n \in \mathbb{N}$. As $\mathrm{Hom}(X,\mathbb{R}) \cap B_{X^*}$ is w^*-compact, we can find a lattice homomorphism $z^* \in B_{X^*}$ in $\overline{(y_n^*)_{n=1}^\infty}^{w^*}$ such that $z^* \neq y_n^*$ for every $n \in \mathbb{N}$. Note that $z^* \neq 0$ since it must fulfill $z^*(x_0) \geq \varepsilon_0 > 0$. Observe also that each y_n^* is non-proportional, and hence disjoint, with z^*. It follows from Lemma 5.12 that there is a lattice renorming of X such that z^* does not attain its norm. Contradiction.

$(v)\Rightarrow(i)$: Let us enumerate the set $\mathrm{Hom}(X,\mathbb{R}) \cap S_{X^*} = \{x_\alpha^* : \alpha \in \Gamma\}$. It is immediate that $x \in X \mapsto (x_\alpha^*(x))_{\alpha\in\Gamma} \in c_0(\Gamma)$ defines a lattice embedding of X into $c_0(\Gamma)$. This embedding is an isometry due to Proposition 5.22. In order to check the surjectivity of the mapping, by Proposition 5.20 it is enough to show that for every distinct $\beta, \gamma \in \Gamma$ there is $x \in X$ such that $x_\beta^*(x) = 0 \neq x_\gamma^*(x)$, which follows immediately from Corollary 5.18. □

Remark 5.26. Thanks to the Banach-Stone result for AM-spaces previously mentioned (Proposition 5.24), we could add a sixth equivalent statement to the previous theorem: X is linearly isometric to $c_0(\Gamma)$.

One might wonder whether the above result, specifically the equivalence $(iii)\Leftrightarrow(iv)$, could be generalized to arbitrary Banach lattices. This is not possible, as the following example, inspired by [96], shows. We are going to construct a non-atomic Banach lattice X which has exactly one lattice homomorphism (up to scaling), that is norm-attaining for any lattice renorming of X.

Example 5.27. Let ω_1 denote the first uncountable ordinal, and let $C([0,\omega_1], L_2[0,1])$ be the space of continuous functions on the ordinal interval $[0,\omega_1]$ with values on $L_2[0,1]$. Consider

$$X = \big\{F \in C\big([0,\omega_1], L_2[0,1]\big) \;:\; F(\omega_1) = a_F \mathbf{1}_{[0,1]} \text{ for some } a_F \in \mathbb{R}\big\}.$$

It is clear that $x^*(F) = a_F$ is a lattice-homomorphism. For any lattice norm in X, it attains its norm by a similar argument as in Example 5.10. Now, assume that $y^* \in X^*$ is a lattice homomorphism. For every countable limit ordinal α the space $C\big([0,\alpha], L_2[0,1]\big)$ embeds as a projection band in X, and so according to [42, Corollary 3.5], y^* vanishes on $C\big([0,\alpha], L_2[0,1]\big)$. Hence, y^* vanishes on the kernel of x^*, and so the two functionals are proportional.

Note that X is atomless. Indeed, given $F \in X$, $F > 0$, there exists an isolated point $\alpha \in [0,\omega_1)$ such that $F(\alpha) > 0$ and keeping in mind that $L_2[0,1]$ is non-atomic, there is $g \in L_2[0,1] \cap [0, F(\alpha)]$ which is non-proportional to $F(\alpha)$; then $\mathbf{1}_\alpha g \in [0,F]$ is non-proportional to F implying that the latter is not an atom.

We conclude this section with some results in the spirit of Section 5.1. The following is a characterization of coordinate functionals of atoms available exclusively for AM-spaces which generalizes the (probably) well-known fact that the norm-one lattice homomorphisms on $C(K)$ of this type are precisely the evaluations δ_t at isolated points of K. Note that the conditions in the following result are not equivalent in general to the conditions in Proposition 5.3, as Example 5.27 shows.

Proposition 5.28. *Let x^* be a lattice homomorphism on an AM-space X with $\|x^*\| = 1$. The following conditions are equivalent:*

(i) *x^* is a coordinate functional of an atom in X.*

(ii) *$K_X \backslash \{\lambda x^* : \lambda > 0\}$ is w^*-closed.*

(iii) *No nonzero multiple of x^* is a w^*-accumulation point of $K_X \cap S_{X^*}$.*

(iv) *There is a compact Hausdorff space K such that X embeds as a closed sublattice of $C(K)$ in a way that $x^* = \delta_t|_X$, and $1_{\{t\}} \in X$, where t is an isolated point in K.*

In this case, x^* is an isolated point in $K_X \cap S_{X^*}$ with respect to the weak topology.*

Proof. (i)⇒(ii) follows from Proposition 5.3. (ii)⇒ (iii) is straightforward. (iv)⇒(i) is obvious.

(iii)⇒(ii): Assume that there is $\theta \in (0, 1]$ and a net $(x^*_\alpha)_\alpha \subseteq K_X \backslash \{\lambda x^* : \lambda > 0\}$ such that x^*_α w^*-converges to θx^*. Without loss of generality, we may assume that the net $(\|x^*_\alpha\|)_\alpha$ converges to some scalar $\mu \in [\lambda, 1]$. Therefore, the normalized net $(x^*_\alpha/\|x^*_\alpha\|)_\alpha \subseteq K_X \cap S_{X^*} \backslash \{\lambda x^* : \lambda > 0\}$ w^*-converges to $\frac{\theta}{\mu} x^*$, which contradicts (iii).

(ii)⇒ (iv): By Proposition 5.22 we may assume that $X = C_{ph}(K_X)$. Let $L := K_X \backslash \{\lambda x^* : \lambda > 0\}$, which is closed by assumption, and let $K := L \cup \{x^*\} \subseteq K_X$, which is therefore a compact Hausdorff space. Let $Y = \{f : K \to \mathbb{R} : f|_L \in C_{ph}(L)\} \subseteq C(K)$, and let $R : X \to Y$ be the restriction operator, which is easily seen to be a lattice isometric embedding. To show its surjectivity, for $g \in Y$ consider its positively homogeneous extension f defined by

$$f(\lambda x^*) = \lambda, \qquad \text{for } \lambda \in (0, 1).$$

Observe that L and $M := \{\lambda x^* : \lambda \in [0, 1]\}$ are w^*-closed subsets of K_X whose union is K_X and f is w^*-continuous on each of them and is well defined in $L \cap M = \{0\}$, so f is w^*-continuous on K_X. Clearly, $t := x^*$ fulfills the requirements.

The last assertion follows immediately from (iii). □

The last condition is not equivalent to the rest, as the following example demonstrates.

Example 5.29. Let $X = C[0, 1]$ endowed with the norm $\|f\| := \|f\|_\infty \vee 2|f(0)|$, which is isometrically isomorphic to the sublattice of $C(K)$, where $K = \{-1\} \cup [0, 1]$, of functions f satisfying $f(-1) = 2f(0)$. We then have that $2\delta_0 \in K_X \cap S_{X^*}$ is an isolated point, despite not being a coordinate functional of an atom.

We also have the following supplement to Proposition 5.6. Again, Example 5.27 shows that the implications (ii)⇒(i) and (iii)⇒(i) are not valid for general Banach lattices.

Proposition 5.30. *For an AM-space X the following conditions are equivalent:*

(i) *X^a has codimension n in X.*

(ii) *X has exactly n distinct lattice norm-one homomorphisms which are not coordinate functionals of atoms.*

(iii) *The set S' of w^*-accumulation points of $S = K_X \cap S_{X^*}$ contains exactly n linearly independent elements.*

Proof. (i)⇒(ii) follows from Proposition 5.6.

$(ii)\Rightarrow(i)$: Assume that $x_1^*, \dots, x_n^*$ are the distinct norm-one lattice homomorphisms which are not coordinate functionals of atoms. Since X^a is an order continuous AM-space, by Theorem 5.25 it is lattice isometric to $c_0(\Gamma)$, so in particular, it is a closed span of its atoms (atoms in X^a, hence in X). As $x_1^*, \dots, x_n^*$ vanish on atoms, they vanish on X^a. This shows that X^a is of co-dimension at least n. Let $Y := X/X^a$, and let $Q : X \to Y$ be the quotient map. It is enough to show that $\dim Y \leq n$. Since Y is an AM-space, this amounts to showing that there are at most n linearly independent homomorphisms on Y. Let y^* be a non-zero homomorphism on Y. Then, $Q^* y^* = y^* \circ Q$ vanishes on all atoms, and so is a positive multiple of x_k^*, for some $k = 1, \dots, n$. As Q^* is an injection, the claim follows.

$(ii)\Leftrightarrow(iii)$: First note that $S' \subseteq K_X$, as K_X is w^*-closed. According to Proposition 5.28, the non-zero elements of S' are precisely the homomorphisms which are not coordinate functionals of atoms (up to scaling). □

Remark 5.31. Note that in the particular case of $X = C(K)$ the previous proposition shows that $C(K)^a$ has codimension n in $C(K)$ if and only if K has exactly n non-isolated points.

5.5 Free Banach lattices over Banach spaces

When it comes to the study of norm-attaining lattice homomorphisms, the class of free Banach lattices is certainly a relevant one. Indeed, the first explicit examples of Banach lattices with lattice homomorphisms not attaining their norm were found within this class [42, Corollary 5.2]. In this section, we will focus on *free (p-convex) Banach lattices generated by Banach spaces*. Specifically, we will continue the line of research started in [42], where it was conjectured that the norm-attaining lattice homomorphisms in a free Banach lattice generated by a Banach space E were in correspondence with the norm-attaining functionals in E.

We now provide an analogue of Proposition 5.19 for free Banach lattices over Banach spaces. Recall that the lattice homomorphisms in $\mathrm{FBL}^{(p)}[E]$ are precisely the extensions to $\mathrm{FBL}^{(p)}[E]$, as lattice homomorphisms, of the functionals $x^* \in E^*$, which will be denoted by $\widehat{x^*}$ (i.e. $\widehat{x^*} \circ \delta_E = x^*$). However, these can also be seen as appropriate multiples of the evaluations on points $x^* \in B_{E^*}$, when viewing $\mathrm{FBL}^{(p)}[E]$ as a sublattice of $C_{ph}(B_{E^*})$.

Corollary 5.32. *Let E be a Banach space of $\dim(E) \geq 2$. Given a norm-attaining lattice homomorphism $\widehat{x^*} \in FBL^{(p)}[E]$ (for any $1 \leq p \leq \infty$), there exists a lattice renorming $|||\cdot|||$ of $FBL^{(p)}[E]$ in such a way that $\widehat{x^*}$ does not attain its norm.*

Proof. We may assume that $\|\widehat{x^*}\| = 1$. Let $(x_n^*)_{n=1}^\infty \subset S_{E^*}$ be a sequence which converges to x^* in E^* such that x_n^* is not proportional to x^* for any $n \in \mathbb{N}$, and define $\mu := \sum_{n=1}^\infty \frac{1}{2^n} \widehat{x_n^*}$. Note that since $\widehat{x_n^*} \perp \widehat{x^*}$ for all n, it follows that μ and x^* are disjoint. On the other hand, $(x_n^*)_{n=1}^\infty$ converges to x^* in the weak* topology of B_{E^*}, and since elements of $\mathrm{FBL}^{(p)}[E]$ are w^*-continuous on B_{E^*}, it follows that $(\widehat{x_n^*})_{n=1}^\infty$ converges to $\widehat{x^*}$ in the weak* topology of $\mathrm{FBL}^{(p)}[E]^*$. Referring to Lemma 5.12 completes the proof. □

It was conjectured in [42, Conjecture 5.5] that a functional $x^* \in E^*$ attains its norm if and only if the lattice homomorphism $\widehat{x^*} \in \mathrm{FBL}[E]^*$ attains its norm. We have shown in Theorem 5.15 that every lattice homomorphism in $\mathrm{FBL}^{(\infty)}[E]^*$ attains its norm. Thus, the analogous conjecture turns out to be false in the setting of free ∞-convex Banach lattices generated by a Banach space. Nevertheless, we do not know the answer for free p-convex Banach lattices with $p \in [1, \infty)$:

Question 5.33. Let E be a Banach space and $p \in [1, \infty)$. Does a functional $x^* \in E^*$ attain its norm if and only if the lattice homomorphism $\widehat{x^*} \in \mathrm{FBL}^{(p)}[E]^*$ attains its norm?

In [42, Definition 5.7], the authors introduced the following property: A Banach space E has *property (P)* if for every $x^* \notin \mathrm{NA}(E, \mathbb{R})$, the set

$$C := \{y^* \in E^* \,:\, |x^*(x)| + |y^*(x)| \le \|x^*\| \text{ for every } x \in B_E\}$$

satisfies that x^* is in the w^*-closure of $\mathbb{R}_+ C := \{\lambda y^* \,:\, \lambda > 0,\, y^* \in C\}$. It is claimed in [42, Lemma 5.8] that Banach spaces with property (P) satisfy the above conjecture. However, there is a gap in the proof of this fact: at some point, the authors use the w^*-continuity on bounded sets of the functions of FBL$[E]$ and assume that this is sufficient to ensure that the elements of FBL$[E]$ are w^*-continuous on $\overline{\mathbb{R}_+ C}^{w^*}$. Since this set is not bounded in general, the argument is not valid.

For this reason, we propose the following slight modification and generalization of property (P): For $1 \le p < \infty$, a Banach space has *property* (P_p) if for every $x^* \notin \mathrm{NA}(E, \mathbb{R})$, the set

$$C_p := \{y^* \in E^* \,:\, |x^*(x)|^p + |y^*(x)|^p \le \|x^*\|^p \text{ for every } x \in B_E\}$$

satisfies that x^* is in $\bigcup_{n\in\mathbb{N}} \overline{(\mathbb{R}_+ C) \cap n B_{E^*}}^{w^*}$. It is clear that if $p \le q$ then a Banach space with property (P_p), also has property (P_q).

Proposition 5.34. *Given $1 \le p < \infty$, let E be a Banach space with property (P_p). Then, $x^* \in NA(E, \mathbb{R})$ if and only if $\widehat{x^*} \in NA(FBL^{(p)}[E], \mathbb{R})$.*

Proof. The proof follows the same argument as in [42, Lemma 5.8], despite the fact that there was a gap using the original definition of property (P). ☐

Fortunately, the classes of Banach spaces provided in [42] satisfying property (P) also satisfy (P_p) for every $1 \le p < \infty$. More specifically, we have:

- $E = c_0$ has property (P_p) with $x^* \in \overline{(\mathbb{R}_+ C_p) \cap 2B_{E^*}}^{w^*}$ for every $x^* \notin \mathrm{NA}(E, \mathbb{R})$.
- $E = \ell_1(\Gamma)$ has property (P_p) with $x^* \in \overline{(\mathbb{R}_+ C_p) \cap B_{E^*}}^{w^*}$ for every $x^* \notin \mathrm{NA}(E, \mathbb{R})$.

For the first claim, see the proof of [42, Theorem 5.10]; the second claim is a particular case of the following result.

Proposition 5.35. *If μ is a localizable measure (i.e. such that $L_1(\mu)^* = L_\infty(\mu)$), then $L_1(\mu)$ has property (P_p).*

Proof. Let x^* be a norm-one linear functional on $L_1(\mu)$ which does not attain its norm. Let us denote by g the function in $B_{L_\infty(\mu)}$ such that $x^*(f) = \int f g d\mu$ for every $f \in L_1(\mu)$: we will identify $x^* = g$. First, note that $A = \{|g| = 1\}$ is a μ-null set: if not, take $B \subseteq A$ of finite measure and define $f = \frac{\mathrm{sgn}(g)}{\mu(B)} \chi_B$, which is a norm-one function in $L_1(\mu)$ such that $x^*(f) = 1$ and this would contradict the fact that $x^* \notin \mathrm{NA}(L_1(\mu), \mathbb{R})$. Hence the decreasing sequence of measurable sets defined by $A_n = \{1 \ge |g| > 1 - \frac{1}{n}\}$ has μ-null intersection.

For every $n \in \mathbb{N}$, we define $y_n^* := \frac{1}{n} g \cdot \chi_{A_n^c}$; note that $0 \le |g| \cdot \chi_{A_n^c} \le |g| \in B_{L_\infty(\mu)}$, and so $n y_n^* \in B_{L_\infty(\mu)}$. We have

$$|x^*| + |y_n^*| = |g|\chi_{A_n} + |g|\chi_{A_n^c} + \frac{1}{n}|g|\chi_{A_n^c} = |g|\chi_{A_n} + \left(1 + \frac{1}{n}\right)|g|\chi_{A_n^c} \le 1, \ \mu\text{-a.e.}$$

Thus, $(y_n^*)_{n=1}^\infty$ is a sequence in the set

$$C_1 = \left\{ y^* \in L_1(\mu)^* \ : \ |x^*(f)| + |y^*(f)| \le 1 \text{ for every } f \in B_{L_1(\mu)} \right\}.$$

Let us show that $\mathbb{R}_+ C_1 \cap B_{L_1(\mu)^*} \ni n y_n^* \xrightarrow{w^*} x^*$. Indeed, for every $f \in L_1(\mu)$ we have that $fg \in L_1(\mu)$, and so

$$(x^* - n y_n^*)(f) = \int (g - g \cdot \chi_{A_n^c}) f \, d\mu = \int_{A_n} g f \, d\mu \to 0, \ n \to \infty.$$

This shows that $L_1(\mu)$ has property (P_1) and consequently (P_p) for every $1 \le p < \infty$. □

We now present an example of a Banach lattice X such that $\mathrm{Hom}(X, \mathbb{R}) \subseteq \mathrm{NA}(X, \mathbb{R})$, but it is neither σ-Dedekind complete nor an AM-space.

Example 5.36. Consider the free Banach lattice of a reflexive Banach space E with $\dim E = \infty$. This space is not σ-order complete (see [119, Proposition 2.11]). Moreover, $\mathrm{FBL}[E]$ is at most 2-convex (see [119, Proposition 9.30]), therefore cannot be ∞-convex, and so it is not lattice isomorphic to an AM-space. On the other hand, since E is reflexive, given $x^* \in E^*$, there exists $x \in B_E$ such that $|x^*(x)| = \|x^*\|$. Thus $\widehat{x^*}$ attains its norm at δ_x.

5.6 Free Banach lattices over lattices

In this section we focus on the setting of free Banach lattices generated by lattices. Here, a *lattice* is a non-empty set $\mathbb{L}$ with a partial order such that for every $x, y \in \mathbb{L}$, the set $\{x, y\}$ has both a supremum $x \vee y$ and an infimum $x \wedge y$. Throughout this section, by *lattice homomorphism* we refer to any map $T : \mathbb{L} \to \mathbb{M}$ between two lattices $\mathbb{L}$ and $\mathbb{M}$ that preserves lattice operations, i.e., $T(x \vee y) = Tx \vee Ty$ and $T(x \wedge y) = Tx \wedge Ty$ for every $x, y \in \mathbb{L}$. To avoid confusion, we will use the term *linear lattice homomorphism* when referring to linear and bounded operators between Banach lattices that preserve lattice operations.

The free Banach lattice generated by a lattice $\mathbb{L}$, denoted $\mathrm{FBL}\langle \mathbb{L} \rangle$, was introduced in [16], following a similar approach to [15]. Later, it was deduced in [14, Theorem 3.9] that $\mathrm{FBL}\langle \mathbb{L} \rangle$ is always 2-isomorphic to an AM-space. In this section, we provide a direct proof of this fact, and furthermore we show that every linear lattice homomorphism on the free Banach lattice generated by a lattice attains its norm. But first, we extend the definition of the free Banach lattice generated by a lattice to the setting of p-convex Banach lattices for $1 \le p \le \infty$, in a analogous way to [77].

Definition 5.37. Given $1 \le p \le \infty$, the *free p-convex Banach lattice* over a lattice $\mathbb{L}$ is a p-convex Banach lattice $\mathrm{FBL}^{(p)}\langle \mathbb{L} \rangle$ (with p-convexity constant 1) together with a norm-bounded lattice homomorphism $\phi : \mathbb{L} \to \mathrm{FBL}^{(p)}\langle \mathbb{L} \rangle$ with the property that for every p-convex Banach lattice X (with p-convexity constant 1) and every norm-bounded lattice homomorphism $T : \mathbb{L} \to X$ there is a unique linear lattice homomorphism $\widehat{T} : \mathrm{FBL}^{(p)}\langle \mathbb{L} \rangle \to X$ such that $T = \widehat{T} \circ \phi$ and $\|\widehat{T}\| = \|T\|$. Here *norm-bounded* means $\|T\| := \sup\{\|T(x)\| \ : \ x \in \mathbb{L}\} < \infty$.

As in the case of free Banach lattices generated by Banach spaces, when $p = 1$ we recover the free Banach lattice generated by a lattice, $\mathrm{FBL}\langle\mathbb{L}\rangle$, while $\mathrm{FBL}^{(\infty)}\langle\mathbb{L}\rangle$ can also be called the *free AM-space over* $\mathbb{L}$, since the classes of AM-spaces and ∞-convex Banach lattices with constant 1 coincide.

Remark 5.38. In the definition above, the assumption of norm-boundedness of T followed by the requirement $\|\widehat{T}\| = \|T\|$ can be replaced with the assumption that $\|T\| \leq 1$ and the requirement that $\|\widehat{T}\| \leq 1$.

The existence of $\mathrm{FBL}\langle\mathbb{L}\rangle$ was established in [16, Section 2]. More specifically, it is shown that $\mathrm{FBL}\langle\mathbb{L}\rangle$ can be identified with the quotient of $\mathrm{FBL}[\ell_1(\mathbb{L})]$ with respect to the closed ideal I generated by the set

$$\{u(x) \vee u(y) - u(x \vee y),\, u(x) \wedge u(y) - u(x \wedge y) : x, y \in \mathbb{L}\},$$

where $u : \mathbb{L} \to \mathrm{FBL}[\ell_1(\mathbb{L})]$ denotes the canonical embedding (recall that for any set A, $\mathrm{FBL}[\ell_1(A)]$ coincides with $\mathrm{FBL}(A)$, the free Banach lattice generated by the set A, introduced in [121]). Arguing analogously, it can be shown that $\mathrm{FBL}^{(p)}\langle\mathbb{L}\rangle$ exists for every $1 \leq p \leq \infty$ and coincides with the quotient of $\mathrm{FBL}^{(p)}[\ell_1(\mathbb{L})]$ with respect to the closed ideal I generated by the same set above.

In order to investigate these objects more thoroughly, let us start by recalling and setting the notation we will use. We will write:

$$\mathbb{L}^* = \{x^* : \mathbb{L} \to [-1, 1] \;:\; x^* \text{ is a lattice homomorphism}\}.$$

By Tychonoff's theorem, $[-1, 1]^{\mathbb{L}}$ is a compact space with respect to the product topology and it is not difficult to check that $\mathbb{L}^*$ is a positively homogeneous closed subset of it, so $\mathbb{L}^*$ is a compact space with the product topology. We will always assume that $\mathbb{L}^*$ is equipped with this topology. Let $C_{ph}(\mathbb{L}^*)$ stand for the space of continuous positively homogeneous functions on $\mathbb{L}^*$ endowed with the supremum norm $\|\cdot\|_\infty$. For every $x \in \mathbb{L}$, we consider the evaluation function $\delta_x : \mathbb{L}^* \to \mathbb{R}$ given by $\delta_x(x^*) = x^*(x)$. It is easy to see that $\delta_x \in C_{ph}(\mathbb{L}^*)$, for every $x \in \mathbb{L}$.

Proposition 5.39. $C_{ph}(\mathbb{L}^*) = \overline{\mathrm{lat}}^{\|\cdot\|_\infty}\{\delta_x : x \in \mathbb{L}\}$ *together with the map* $\phi_\infty(x) = \delta_x$ *is the free AM-space over* $\mathbb{L}$.

Proof. Let us first prove the equality. According to Proposition 5.20 it is enough to show that if x^* and y^* in $\mathbb{L}^*\backslash\{0\}$ are not positive multiples, there is $f \in \mathrm{lat}\{\delta_x,\ x \in \mathbb{L}\}$ such that $f(x^*) = 0 \neq f(y^*)$. First, suppose that $x^* = \lambda y^*$ for some $\lambda < 0$. There exists $x \in \mathbb{L}$ such that $y^*(x) \neq 0$. Then either $y^*(x) > 0 > x^*(x)$, or $y^*(x) < 0 < x^*(x)$. In the former case we define $f := \delta_x^+ = \delta_x \vee 0$, then $f(x^*) = 0 < f(y^*)$. The latter case is analogous. Now let us assume that x^* and y^* are not proportional, so that there are $u, v \in \mathbb{L}$ such that $(x^*(u), x^*(v))$ and $(y^*(u), y^*(v))$ are not proportional. Without loss of generality we suppose $x^*(v) \neq 0$. Take $f := \delta_u - \frac{x^*(u)}{x^*(v)}\delta_v$. It is clear that $f(x^*) = 0$. On the other hand, $f(y^*) = y^*(u) - \frac{x^*(u)}{x^*(v)}y^*(v) \neq 0$, as $(x^*(u), x^*(v))$ and $(y^*(u), y^*(v))$ are not proportional.

Now let X be an AM-space and $T : \mathbb{L} \to X$ a lattice homomorphism with range contained in B_X. By Proposition 5.22 we may assume that $X = C_{ph}(K_X)$. Every $z^* \in K_X$ maps B_X into $[-1, 1]$, and so $\psi(z^*) := z^* \circ T$ is a lattice homomorphism from $\mathbb{L}$ into $[-1, 1]$, i.e. $\psi(z^*) \in \mathbb{L}^*$. It is easy to see that $\psi : K_X \to \mathbb{L}^*$ is positively homogeneous and continuous with respect to the weak* topology on K_X and the product topology on $\mathbb{L}^*$. Hence, the composition operator $\widehat{T} : C_{ph}(\mathbb{L}^*) \to C_{ph}(K_X)$ given by $\widehat{T}f(z^*) = f(\psi(z^*))$ is a well-defined linear lattice homomorphism of norm at most 1. Finally, for every $x \in \mathbb{L}$ and $z^* \in K_X$ we have

$$\widehat{T}\delta_x(z^*) = \delta_x(\psi(z^*)) = \psi(z^*)(x) = z^*(Tx),$$

hence $\widehat{T}\delta_x = Tx$.

For the uniqueness of extension assume that $S : C_{ph}(\mathbb{L}^*) \to X$ is a linear lattice homomorphism which agrees with $\widehat{T}$ on δ_x, for every $x \in \mathbb{L}$, then they have to agree on the closed sublattice generated by these vectors. As $\overline{\mathrm{lat}}^{\|\cdot\|_\infty}\{\delta_x : x \in \mathbb{L}\} = C_{ph}(\mathbb{L}^*)$, we conclude that $\widehat{T} = S$. □

Let us now switch to the case $1 \leq p < \infty$. For $f \in \mathbb{R}^{\mathbb{L}^*}$, define

$$\|f\|_p = \sup\left\{\left(\sum_{i=1}^n |f(x_i^*)|^p\right)^{\frac{1}{p}} : n \in \mathbb{N},\ x_1^*, \ldots, x_n^* \in \mathbb{L}^*,\ \sup_{x \in \mathbb{L}} \sum_{i=1}^n |x_i^*(x)|^p \leq 1\right\}. \quad (5.1)$$

It was proven in [16, Theorem 1.2] that

$$\mathrm{FBL}\langle\mathbb{L}\rangle = \overline{\mathrm{lat}}^{\|\cdot\|_1}\{\delta_x : x \in \mathbb{L}\},$$

where the closure is taken in $\{f \in \mathbb{R}^{\mathbb{L}^*} : \|f\|_1 < \infty\}$, and $\phi(x) = \delta_x$, for every $x \in \mathbb{L}$. It is not difficult to check that the same proof can be adapted verbatim to show that

$$\mathrm{FBL}^{(p)}\langle\mathbb{L}\rangle = \overline{\mathrm{lat}}^{\|\cdot\|_p}\{\delta_x : x \in \mathbb{L}\},$$

where this time the closure is taken in $\{f \in \mathbb{R}^{\mathbb{L}^*} : \|f\|_p < \infty\}$.

Clearly, $\|\cdot\|_\infty \leq \|\cdot\|_p$ for every $1 \leq p < \infty$. Since $\mathrm{lat}\{\delta_x : x \in \mathbb{L}\}$ is contained in $C_{ph}(\mathbb{L}^*)$, it follows that the identity operator $id_p : \mathrm{FBL}^{(p)}\langle\mathbb{L}\rangle \to C_{ph}(\mathbb{L}^*)$ is bounded. On the other hand, the universal property of $\mathrm{FBL}\langle\mathbb{L}\rangle$ implies that there exists a contractive linear lattice homomorphism $\widehat{\phi_p} : \mathrm{FBL}\langle\mathbb{L}\rangle \to \mathrm{FBL}^{(p)}\langle\mathbb{L}\rangle$ that extends the canonical map $\phi_p : \mathbb{L} \to \mathrm{FBL}^{(p)}\langle\mathbb{L}\rangle$, and hence, coincides with the identity map on $\mathrm{lat}\{\delta_x : x \in \mathbb{L}\}$. It is clear that $id_p \circ \widehat{\phi_p} = id_1$, so actually $\widehat{\phi_p}$ must be the identity from $\mathrm{FBL}\langle\mathbb{L}\rangle$ into $\mathrm{FBL}^{(p)}\langle\mathbb{L}\rangle$, and we can properly say that $\|f\|_p \leq \|f\|_1$ for every $f \in \mathrm{FBL}\langle\mathbb{L}\rangle$. Additionally, it can be deduced from [14, Theorem 3.6] that $\|\cdot\|_1 \leq 2\|\cdot\|_\infty$ (where the authors show that $\mathrm{FBL}\langle\mathbb{L}\rangle$ is 2-linearly lattice isomorphic to an AM-space for any $\mathbb{L}$). We include here a direct proof of this fact.

Proposition 5.40. *Let $1 \leq p < \infty$. For every $f \in C_{ph}(\mathbb{L}^*)$ we have $\|f\|_p \leq 2^{\frac{1}{p}}\|f\|_\infty$. Moreover, $\left(C_{ph}(\mathbb{L}^*), \|\cdot\|_p\right)$ together with the map $\phi_p(x) = \delta_x$ is the free p-convex Banach lattice over $\mathbb{L}$.*

Proof. In order to prove the first claim, it is enough to show that if $x_1^*, \ldots, x_n^* \in \mathbb{L}^*$ are such that $\sup_{x \in \mathbb{L}} \sum_{i=1}^n |x_i^*(x)|^p \leq 1$, then $\left(\sum_{i=1}^n |f(x_i^*)|^p\right)^{\frac{1}{p}} \leq 2^{\frac{1}{p}}\|f\|_\infty$.

Fix $\varepsilon > 0$. For each $i = 1, \ldots, n$, we define $m_i = \inf\{x_i^*(x) : x \in \mathbb{L}\}$ and $M_i = \sup\{x_i^*(x) : x \in \mathbb{L}\}$ and find $x_{i,m}, x_{i,M} \in \mathbb{L}$ such that

$$0 \leq x_i^*(x_{i,m}) - m_i \leq \varepsilon \quad \text{and} \quad 0 \leq M_i - x_i^*(x_{i,M}) \leq \varepsilon.$$

Now, define $x_m = \bigwedge_{i=1}^n x_{i,m}$ and $x_M = \bigvee_{i=1}^n x_{i,M}$. For every $1 \leq i \leq n$ we have

$$0 \le x_i^*(x_m) - m_i \le \varepsilon \quad \text{and} \quad 0 \le M_i - x_i^*(x_M) \le \varepsilon.$$

It follows that

$$1 \ge \left(\sum_{i=1}^n |x_i^*(x_m)|^p \right)^{\frac{1}{p}} \ge \left(\sum_{i=1}^n |m_i|^p \right)^{\frac{1}{p}} - n^{\frac{1}{p}}\varepsilon,$$

and

$$1 \ge \left(\sum_{i=1}^n |x_i^*(x_M)|^p \right)^{\frac{1}{p}} \ge \left(\sum_{i=1}^n |M_i|^p \right)^{\frac{1}{p}} - n^{\frac{1}{p}}\varepsilon.$$

For every $1 \le i \le n$ we have that $y_i^* := \frac{1}{|m_i| \vee |M_i|} x_i^* \in \mathbb{L}^*$, hence

$$|f(x_i^*)| = (|m_i| \vee |M_i|)|f(y_i^*)| \le (|m_i| \vee |M_i|)\|f\|_\infty.$$

We conclude that

$$\begin{aligned} \left(\sum_{i=1}^n |f(x_i^*)|^p \right)^{\frac{1}{p}} &\le \|f\|_\infty \left(\sum_{i=1}^n (|m_i| \vee |M_i|)^p \right)^{\frac{1}{p}} \\ &\le \|f\|_\infty \left(\sum_{i=1}^n |m_i|^p + \sum_{i=1}^n |M_i|^p \right)^{\frac{1}{p}} \le 2^{\frac{1}{p}}(1 + n^{\frac{1}{p}}\varepsilon)\|f\|_\infty. \end{aligned}$$

As ε was arbitrary the claim follows.

Note that $C_{ph}(\mathbb{L}^*)$ is a Banach lattice with respect to $\|\cdot\|_p$, which contains $\{\delta_x : x \in \mathbb{L}\}$. Hence, according to the discussion before the proposition, in order to prove the second claim, it is enough to show that

$$C_{ph}(\mathbb{L}^*) = \overline{\text{lat}}^{\|\cdot\|_p}\{\delta_x : x \in \mathbb{L}\}.$$

Since the two norms are equivalent, the closure may be taken with respect to $\|\cdot\|_\infty$, and so the statement follows from Proposition 5.39. □

We conclude this section by showing that every linear lattice homomorphism on $\text{FBL}^{(p)}\langle\mathbb{L}\rangle$ attains its norm. Note that although we have just seen that the free norm of $\text{FBL}^{(p)}\langle\mathbb{L}\rangle$ is equivalent to the supremum norm, this does not guarantee that its linear lattice homomorphisms attain their norm. Also observe that it is not difficult to find very simple examples of lattice homomorphisms $x^* : \mathbb{L} \to \mathbb{R}$ which do not attain their norm: the naturals $\mathbb{N}$ with the usual order and $x^* : \mathbb{N} \to \mathbb{R}$ defined by $x^*(n) := 1 - \frac{1}{n}$; the interval $(-1, 1)$ with the usual order and x^* being the identity on $(-1, 1)$; another example is, given an AM-space X, to consider $\mathbb{L} = \{x \in X : \|x\| < 1\}$ and take as x^* the restriction to $\mathbb{L}$ of a norm-one linear lattice homomorphism defined on X.

Proposition 5.41. *Every linear lattice homomorphism on $FBL^{(p)}\langle\mathbb{L}\rangle$ attains its norm.*

Proof. According to Proposition 5.40 we may assume that $\text{FBL}^{(p)}\langle\mathbb{L}\rangle = C_{ph}(\mathbb{L}^*)$ endowed with the norm $\|\cdot\|_p$, which is equivalent to $\|\cdot\|_\infty$. Since $C_{ph}(\mathbb{L}^*)$ is a closed sublattice of $C(\mathbb{L}^*)$, by Proposition 5.21 the lattice homomorphisms on $C_{ph}(\mathbb{L}^*)$ are positive multiples of point evaluations at the members of $\mathbb{L}^*$. Hence, it is enough to show that these point evaluations attain their $\|\cdot\|_p$-norms. Let $0 \ne x^* \in \mathbb{L}^*$ and let $m = \inf\{x^*(x) : x \in \mathbb{L}\}$ and $M = \sup\{x^*(x) : x \in \mathbb{L}\}$. We will assume that $|M| \ge |m|$, which also implies that

$M > 0$ (the case $|M| \leq |m|$ is analogous). Moreover, replacing x^* with $\frac{1}{M}x^*$ if needed, we may assume that $M = 1$.

Let us first show that $\|\widehat{x^*}\| \leq 1$, where $\widehat{x^*}$ denotes the point evaluation at x^*, or equivalently, the unique extension of the lattice homomorphism x^* to $\mathrm{FBL}^{(p)}\langle\mathbb{L}\rangle$ as a linear lattice homomorphism. Indeed, if $f \in C_{ph}(\mathbb{L}^*)$ is such that $\|f\|_p \leq 1$, then $\|f\|_\infty \leq 1$, and so $\widehat{x^*}(f) = f(x^*) \leq 1$. On the other hand, for every $n \in \mathbb{N}$ there is $x_n \in \mathbb{L}$ such that $\widehat{x^*}(\delta_{x_n}) = x^*(x_n) \geq 1 - \frac{1}{n}$. By replacing x_n by $\vee_{k=1}^n x_k$ if necessary, we may assume that $(x_n)_{n=1}^\infty$ is an increasing sequence in $\mathbb{L}$. Since $\phi_p : \mathbb{L} \to \mathrm{FBL}^{(p)}\langle\mathbb{L}\rangle$ preserves lattice operations, $(\delta_{x_n}^+)_{n=1}^\infty$ defines an increasing sequence in $B_{\mathrm{FBL}^{(p)}\langle\mathbb{L}\rangle_+}$. Thus, by Proposition 5.14, $\widehat{x^*}$ attains its norm. □

Chapter 6

Stable phase retrieval in $C(K)$-spaces

In [58], Freeman, Oikhberg, Pineau, and Taylor showed that if a compact Hausdorff space K has infinitely many accumulation points, then the Banach space $C(K)$ must contain a subspace isometric to c_0 doing SPR. Motivated by this, the authors ask whether $K^{(\alpha)}$ being infinite for $\alpha > 1$ implies the existence of an isometric SPR embedding of $C[1, \omega^\alpha]$ into $C(K)$. In this chapter, we explore this question and provide an affirmative answer when α is a finite ordinal. All notions and spaces considered in this final section are always defined over the field of real numbers. This chapter is based on a joint work with García-Sánchez:

> [60] E. García-Sánchez and D. de Hevia, *Subspaces of $C(K)$-spaces doing stable phase retrieval*, in preparation.

6.1 Stable phase retrieval in Banach lattices

In some areas of physics and engineering such as speech recognition, quantum state tomography, X-ray crystallography, or electron microscopy, one often wishes to recover some vector f from an *intensity measurement* $|Tf|$, where T is a linear mapping into a function space (see, for instance, [53, 116, 143]). These sorts of problems are called *phase retrieval problems*. Observe that as T is linear, the magnitude $|Tf|$ does not change if we multiply f by a scalar of modulus 1. For this reason, we say that T does *phase retrieval* if $|Tf| = |Tg|$ implies that $f = \lambda g$ for some scalar λ with $|\lambda| = 1$. In [58], Freeman, Oikhberg, Pineau and Taylor *laid the groundwork* (and also obtained very significant results) for analyzing (stable) phase retrieval in general function spaces and, in fact, in general *Banach lattices*. These authors aim to provide a unified framework for examining very relevant problems related to phase retrieval, by drawing on the rich theory of Banach lattices. In this chapter, we will heavily rely on their work. We will begin by recalling some of the concepts and results that will be needed subsequently.

Definition 6.1 (Phase retrieval). A subspace E of a Banach lattice X is said to do *phase retrieval* if whenever $f, g \in E$ satisfy $|f| = |g|$, it follows that $f = \lambda g$ for some unimodular λ.

Note that if E contains two (non-zero) disjoint vectors f and g, then $|f+g| = |f|+|g| = |f-g|$. Since f and g are linearly independent, it follows that $f-g$ and $f+g$ are also linearly independent and, thus, E fails phase retrieval. In fact, it was pointed out in [58] that having a *disjoint pair* is the *only obstruction* to phase retrieval (see also [39, Proposition 2.1] for a proof of this fact).

Proposition 6.2 (Characterization phase retrieval). *Let E be a subspace of a Banach lattice X. Then, E fails phase retrieval if and only if there exist non-zero $f, g \in E$ such that $|f| \wedge |g| = 0$.*

In practical applications, measurements are inherently noisy. When minor measurement errors lead to significant reconstruction errors, the recovery process becomes unstable and effectively useless. This challenge has driven the study of the notion of *stable phase retrieval* (*SPR*), which aims to identify conditions and algorithms that ensure the reconstructed signal changes continuously with respect to measurements, guaranteeing robustness to noise.

Definition 6.3 (Stable phase retrieval). A subspace E of a Banach lattice X is said to do *C-stable phase retrieval* (*C-SPR*, for short), where C is a constant ≥ 1 if

$$\min_{|\lambda|=1} \|f - \lambda g\| \leq C \| |f| - |g| \|, \quad \text{for every } f, g \in E.$$

We say that E does *stable phase retrieval* (SPR) if it does C-SPR for some $C \geq 1$.

Inspired by the characterization of phase retrieval in Proposition 6.2, in [58] the authors also manage to precisely identify when a subspace of a Banach lattice does stable phase retrieval: it must not contain *almost disjoint pairs*. We say that a subspace E of a Banach lattice X contains *ε-almost disjoint pairs* if there are $f, g \in S_E$ such that $\| |f| \wedge |g| \| < \varepsilon$. If E contains ε-almost disjoint pairs for every $\varepsilon > 0$, then E is said to contain *almost disjoint pairs*. Below, we present Bilokopytov's refinement of the aforementioned characterization of SPR [28, Proposition 3.4] (see also [58, Theorem 3.4] for the original result).

Proposition 6.4 (Characterization SPR). *Let E be a subspace of a Banach lattice X and $\varepsilon > 0$. Then E contains no ε-almost disjoint pairs if and only if it does $\frac{1}{\varepsilon}$-stable phase retrieval. In particular, E does stable phase retrieval if and only if it does not contain almost disjoint pairs.*

6.2 On a question regarding SPR embeddings of $C(K)$-spaces

Freeman, Oikhberg, Pineau and Taylor devote Section 6 of [58] to studying SPR in $C(K)$-spaces. Among their most notable findings in this context is the following necessary and sufficient condition for a $C(K)$-space to contain a subspace doing SPR: given a compact Hausdorff space K, $C(K)$ has an SPR subspace if and only if K' is infinite. More precisely, they prove the following [58, Theorem 6.1] (cf. [28, Theorem 5.3]):

Theorem 6.5. *Suppose K is a compact Hausdorff space. Then, the following are equivalent:*

(i) K', the set of non-isolated points of K, is infinite.

(ii) c_0 embeds isometrically into $C(K)$ as an SPR subspace.

(iii) $C(K)$ contains an (infinite dimensional) SPR subspace.

In light of the previous result, the authors ask whether a *large* compact space K (in terms of the smallest ordinal α for which $K^{(\alpha)}$ is non-empty) guarantees the existence of spaces *bigger* than c_0 that can be embedded into $C(K)$ in an SPR way. More specifically, they ask the following:

Question 6.6. [58, Question 6.4] If $K^{(\alpha)}$ is infinite, does $C(K)$ contain an SPR copy of $C[0,\omega^\alpha]$?

Note that for every $\alpha > 0$, $[1,\omega^\alpha]$ and $[0,\omega^\alpha]$ are homeomorphic and hence $C[0,\alpha]$ and $C[1,\alpha]$ are lattice isometric. Therefore, we will present our results using the latter space. This is because *Cantor's normal form* allows for slightly simpler notation when representing a generic point in $[1,\alpha]$. Recall that every ordinal $\alpha > 0$ can be represented uniquely as

$$\alpha = w^{\beta_1} \cdot k_1 + \ldots + \omega^{\beta_n} \cdot k_n, \tag{6.1}$$

where $n \geq 1$, $\alpha \geq \beta_1 > \ldots > \beta_n$ and $k_1, \ldots, k_n$ are non-zero natural numbers and this representation is called *Cantor's normal form.*

Our aim in this chapter is to investigate Question 6.6 *for finite ordinals* $\alpha < \omega$. As a consequence of the results that we will prove in this section, we can provide a fairly complete answer to that question for finite ordinals:

(i) If $K^{(\alpha)}$ is infinite (equivalently, $K^{(\alpha+1)} \neq \varnothing$) for some $\alpha \geq 2$ then $C(K)$ contains an isometric SPR copy of $C[1,\omega^{\alpha+1}]$ (Corollary 6.15 (*i*)).

(ii) If K' is infinite (equivalently, $K'' \neq \varnothing$), it was shown in [58, Theorem 6.1] (Theorem 6.5 for us) that c_0 embeds isometrically into $C(K)$ as an SPR subspace. If $|K''| = 1$, then $C[1,\omega^2]$ cannot be embedded into $C(K)$ doing SPR (Proposition 6.16). However, if $|K''| \geq 2$, then $C[1,\omega^2]$ is linearly isometric to an SPR subspace of $C(K)$ (Corollary 6.15 (*ii*)).

Observe that for $2 \leq \alpha < \omega$ the conclusions turn out to be stronger that what was originally conjectured in Question 6.6.

Recall that a classic result by Bessaga and Pełczyński states that $C[1,\omega^\alpha]$, with $1 \leq \alpha < \omega$, is isomorphic to c_0 [27, Theorem 1]. Therefore, every infinite-dimensional $C(K)$ space contains a subspace isomorphic to $C[1,\omega^\alpha]$ for any finite $\alpha > 0$. The *isometric situation changes drastically*: for a $C(K)$-space to isometrically contain $C[1,\omega^\alpha]$ for a *large* $\alpha > 0$, K also needs to be *large* in terms of its Cantor-Bendixson derivative.

Proposition 6.7. *Let $\alpha < \omega$ be a finite ordinal and K be a compact Hausdorff space. If $C[1,\omega^\alpha]$ embeds isometrically into $C(K)$, then $K^{(\alpha)} \neq \varnothing$.*

Proof. Let us first establish the notation. Given $\alpha \in \omega$, we will denote by $\mathbb{1}$ the constant function on $[1,\omega^\alpha]$. Given a natural number $m_1 \geq 1$, $\mathbb{1}_{m_1}$ will represent the characteristic function of the clopen set

$$(\omega^{\alpha-1} \cdot (m_1 - 1), \omega^{\alpha-1} \cdot m_1].$$

Similarly, given natural numbers $m_1, m_2 \geq 1$, $\mathbb{1}_{m_1,m_2}$ will stand for the characteristic function of the clopen set

$$\big(\omega^{\alpha-1} \cdot (m_1 - 1) + \omega^{\alpha-2} \cdot (m_2 - 1), \omega^{\alpha-1} \cdot (m_1 - 1) + \omega^{\alpha-2} \cdot m_2\big]$$

In the same way, we can define the functions $\mathbb{1}_{m_1,m_2,m_3}$, $\ldots$, $\mathbb{1}_{m_1,m_2,\ldots,m_\alpha}$. It should be noted that, given $m_1, m_2, \ldots, m_\alpha \geq 1$, $\mathbb{1}_{m_1,m_2,\ldots,m_\alpha}$ is precisely the characteristic function of the singleton

$$\big\{\omega^{\alpha-1} \cdot (m_1 - 1) + \omega^{\alpha-2} \cdot (m_2 - 1) + \cdots + \omega \cdot (m_{\alpha-1} - 1) + m_\alpha\big\}.$$

Let us start the proof. Given an isometric embedding $T : C[1,\omega^\alpha] \to C(K)$ and given $m_1, \ldots, m_\alpha \geq 1$ there must exist $t_{m_1,\ldots,m_\alpha} \in K$ such that

$$\big|\big(T\mathbb{1}+T\mathbb{1}_{m_1}+\cdots+T\mathbb{1}_{m_1,\dots,m_\alpha}\big)(t_{m_1,\dots,m_\alpha})\big|=\alpha+1. \tag{6.2}$$

Indeed, keep in mind that $\mathbb{1}+\mathbb{1}_{m_1}+\cdots+\mathbb{1}_{m_1,\dots,m_\alpha}$ has norm $\alpha+1$ in $C[1,\omega^\alpha]$ and T is norm-preserving. From the expression (6.2), given that $\|T\mathbb{1}\|=\|T\mathbb{1}_{m_1}\|=\cdots=\|T\mathbb{1}_{m_1,\dots,m_\alpha}\|=1$, we deduce the existence of some unimodular scalar $\theta_{m_1,\dots,m_\alpha}$ such that

$$\begin{aligned}T\mathbb{1}(t_{m_1,\dots,m_\alpha})&=T\mathbb{1}_{m_1}(t_{m_1,\dots,m_\alpha})=T\mathbb{1}_{m_1,m_2}(t_{m_1,\dots,m_\alpha})=\cdots\\&=T\mathbb{1}_{m_1,\dots,m_\alpha}(t_{m_1,\dots,m_\alpha})=\theta_{m_1,\dots,m_\alpha}.\end{aligned}$$

We are going to show by induction on $\alpha\in\omega$ that if for every $m_1,\dots,m_\alpha\geq 1$ we have a point $t_{m_1,\dots,m_\alpha}\in K$ satisfying (6.2), then there exists $t_0\in\overline{\{t_{m_1,\dots,m_\alpha}\,:\,m_1,\dots,m_\alpha\geq 1\}}$ which belongs to $K^{(\alpha)}$. Observe that as $|T\mathbb{1}(t_{m_1,\dots,m_\alpha})|=1$ for all natural numbers $m_1,\dots,m_\alpha$, then $|T\mathbb{1}(t_0)|=1$.

If $\alpha=0$, the previous statement is obvious: in this case, $C[1,\omega^\alpha]=\{1\}$, so $C(K)\neq\{0\}$ and $K\neq\varnothing$. Let us assume that it is true for α. Let $T:C[1,\omega^{\alpha+1}]\to C(K)$ be an isometric embedding. Following the notation established above, we know that for every $m_1,\dots,m_{\alpha+1}\geq 1$, there is $t_{m_1,\dots,m_{\alpha+1}}\in K$ and $\theta_{m_1,\dots,m_{\alpha+1}}$ of modulus 1 such that

$$\big|\big(T\mathbb{1}+T\mathbb{1}_{m_1}+\cdots+T\mathbb{1}_{m_1,\dots,m_{\alpha+1}}\big)(t_{m_1,\dots,m_{\alpha+1}})\big|=\alpha+2.$$

and

$$\begin{aligned}T\mathbb{1}(t_{m_1,\dots,m_{\alpha+1}})&=T\mathbb{1}_{m_1}(t_{m_1,\dots,m_{\alpha+1}})=T\mathbb{1}_{m_1,m_2}(t_{m_1,\dots,m_{\alpha+1}})=\cdots\\&=T\mathbb{1}_{m_1,\dots,m_{\alpha+1}}(t_{m_1,\dots,m_{\alpha+1}})=\theta_{m_1,\dots,m_{\alpha+1}}.\end{aligned}$$

In particular,

$$|(T\mathbb{1}_{m_1}+\cdots+T\mathbb{1}_{m_1,\dots,m_{\alpha+1}})(t_{m_1,\dots,m_{\alpha+1}})|=\|T\mathbb{1}_{m_1}+\cdots+T\mathbb{1}_{m_1,\dots,m_{\alpha+1}}\|=\alpha+1.$$

Now observe that for every $m_1\geq 1$, the compact $(\omega^\alpha\cdot(m_1-1),\omega^\alpha\cdot m_1]$ is homeomorphic to $[1,\omega^\alpha]$. So, if for every $m_1\geq 1$, we restrict T to $C(\omega^\alpha\cdot(m_1-1),\omega^\alpha\cdot m_1]$ (which, in this case, stands for the continuous functions on $[1,\omega^{\alpha+1}]$ which vanish outside the clopen set $(\omega^\alpha\cdot(m_1-1),\omega^\alpha\cdot m_1]$), the inductive hypothesis ensures the existence of a point $t_{m_1}\in\overline{\{t_{m_1,\dots,m_{\alpha+1}}\,:\,m_2,\dots,m_{\alpha+1}\geq 1\}}\cap K^{(\alpha)}$ with $|T\mathbb{1}_{m_1}(t_{m_1})|=1$.

Next, we are going to show that these elements $\{t_{m_1}\}_{m_1}$ are distinct. Suppose that $t_{m_1}=t_{m_1'}$ for some $m_1,m_1'\geq 1$ with $m_1\neq m_1'$. Let $\theta_{m_1},\theta_{m_1'}$ be scalars of modulus 1 such that $T\mathbb{1}_{m_1}(t_{m_1})=\theta_{m_1}$ and $T\mathbb{1}_{m_1'}(t_{m_1'})=\theta_{m_1'}$. Observe that $\theta_{m_1}^{-1}\mathbb{1}_{m_1}+\theta_{m_1'}^{-1}\mathbb{1}_{m_1'}$ has norm 1 in $C[1,\omega^{\alpha+1}]$, but $T(\theta_{m_1}^{-1}\mathbb{1}_{m_1}+\theta_{m_1'}^{-1}\mathbb{1}_{m_1'})(t_{m_1})=2$, which is a contradiction with the fact that T is norm-preserving. Hence, $(t_{m_1})_{m_1=1}^\infty$ is a sequence of distinct points in $\bigcup_{m_1=1}^\infty(\overline{\{t_{m_1,\dots,m_\alpha}\,:\,m_2,\dots,m_{\alpha+1}\geq 1\}}\cap K^{(\alpha)})\subseteq\overline{\{t_{m_1,\dots,m_\alpha}\,:\,m_1,\dots,m_{\alpha+1}\geq 1\}}\cap K^{(\alpha)}$. Thus, there exists $t_0\in\overline{\{t_{m_1,\dots,m_{\alpha+1}}\,:\,m_1,\dots,m_{\alpha+1}\geq 1\}}\cap K^{(\alpha+1)}$. Since $|T\mathbb{1}|$ is 1 at every point $t_{m_1,\dots,m_{\alpha+1}}$, $|T\mathbb{1}(t_0)|=1$ as well. In particular, $K^{(\alpha+1)}\neq\varnothing$, as we wanted to show. □

The following result provides a *converse* to the previous proposition. That is, we will now show that if $K^{(\alpha)}\neq\varnothing$, then $C[1,\omega^\alpha]$ can be isometrically embedded into $C(K)$. However, what is most interesting is that the isometric embedding $T:C[1,\omega^\alpha]\to C(K)$ we construct has several *nice properties*, among which is property $(*)$ that will appear in the next lemma. This property $(*)$ ensures that T preserves SPR subspaces, and this will simplify Question 6.6 to studying SPR embeddings between spaces of the form $C[1,\omega^\alpha]$.

Lemma 6.8. *Let K,L be compact Hausdorff spaces and $T:C(K)\to C(L)$ be an operator with the following property:*

(*) for every $t \in K$ there is $s_t \in L$ such that $Tf(s_t) = f(t)$ for every $f \in C(K)$.

Then, if E is an SPR subspace of $C(K)$, $T(E)$ is an SPR subspace of $C(L)$.

Remark 6.9. Note that if an operator $T : C(K) \to C(L)$ has the property $(*)$ defined above, then $\|Tf\| \geq \|f\|$ for every $f \in C(K)$. Therefore, if E is a subspace of $C(K)$, then it is $\|T\|$-isomorphic to $T(E)$; in particular, if $\|T\| = 1$, T is an isometric embedding.

Proof of Lemma 6.8. Let us suppose that E is a C-SPR subspace of $C(K)$ for some constant $C \geq 1$, that is, for any $f, g \in E$ we have

$$\min_{|\lambda|=1} \|f - \lambda g\| \leq C\||f| - |g|\|.$$

Given any $f, g \in C(K)$ and any $t \in K$,

$$\big(|Tf| - |Tg|\big)(s_t) = |Tf(s_t)| - |Tg(s_t)| = |f(t)| - |g(t)| = \big(|f| - |g|\big)(t),$$

and hence $\||f| - |g|\| \leq \||Tf| - |Tg|\|$. With this in mind, we obtain for any $f, g \in E$ the following:

$$\min_{|\lambda|=1} \|Tf - \lambda Tg\| \leq \|T\| \min_{|\lambda|=1} \|f - \lambda g\| \leq C\|T\|\||f| - |g|\| \leq C\|T\|\||Tf| - |Tg|\|.$$

This shows that $T(E) \subseteq C(L)$ does $C\|T\|$-SPR. □

Proposition 6.10. *Let $\alpha < \omega$ and let K be a compact Hausdorff space. If $K^{(\alpha)} \neq \varnothing$, then there exists a positive isometric embedding $T : C[1, \omega^\alpha] \to C(K)$ with the following properties:*

(1) T preserves the unit, that is, $T\mathbb{1} = \mathbb{1}$;

(2) $T\big(C_0[1, \omega^\alpha]\big) \subseteq C_0(K)$;

(3) for every $t \in [1, \omega^\alpha]$ there is $s_t \in K$ such that $Tf(s_t) = f(t)$ for every $f \in C[1, \omega^\alpha]$.

Consequently, in this situation, every SPR subspace of $C[1, \omega^\alpha]$ isometrically embeds as an SPR subspace of $C(K)$.

Proof. First, we will show by induction on $\alpha \in \omega$ that if $K^{(\alpha)} \neq \varnothing$ and V is an open subset of K such that $V \cap K^{(\alpha)} \neq \varnothing$, then there exists a positive linear isometric embedding $S : C_0[1, \omega^\alpha] \to C_0(V)$ with property $(*)$, i.e., for every $t \in [1, \omega^\alpha]$ there exists $s_t \in V$ such that $Sf(s_t) = f(t)$ for every $f \in C_0[1, \omega^\alpha]$. As usual, $C_0(K)$ stands for

$$\big\{f \in C(K) \,:\, \forall \varepsilon > 0 \;\exists\, L \subseteq K \text{ compact s.t. } |f(t)| < \varepsilon \;\forall t \notin L\big\},$$

so, in particular, $C_0[1, \omega^\alpha]$ is the space of continuous functions on $[1, \omega^\alpha]$ vanishing at ω^α. Note that if $f \in C_0(V)$, then it can be extended to K by simply defining $\widetilde{f}(t) = f(t)$ for $t \in V$ and $\widetilde{f}(t) = 0$ for $t \in K \backslash V$. Indeed, it is clear that $\widetilde{f}$ is continuous on the open subsets V and $K \backslash \overline{V}$. Now, fix any $t_0 \in \overline{V} \backslash V$ and $\varepsilon > 0$. By definition, there is a compact subset $L \subseteq V$ such that $|f(t)| < \varepsilon$ for all $t \in V \backslash L$. Thus, $K \backslash L$ is an open neighborhood of t_0 such that $|\widetilde{f}(t)| < \varepsilon$ whenever $t \in K \backslash L$.

If $\alpha = 0$, our assertion is trivial. Let us assume that it is true for $\alpha \in \omega$, and suppose that $K^{(\alpha+1)} \cap V \neq \varnothing$, for some open subset $V \subseteq K$. Fix any $t_0 \in K^{(\alpha+1)} \cap V$. Given that by definition $K^{(\alpha+1)} = (K^{(\alpha)})'$, and using the normality of K it is easy to construct a sequence of pairwise disjoint open subsets $V_n \subseteq V$ such that for every $n \geq 1$ there exists

$t_n \in V_n \cap K^{(\alpha)}$ and $t_0 \notin V_n$. Since K is a compact Hausdorff space (and thus a regular space), we can further assume that $\overline{V_n} \subseteq V$ [155, Theorem 14.3]. For every natural number n, we can use the *regularity* of K again to find an open set U_n such that $t_n \in U_n \subseteq \overline{U_n} \subseteq V_n$.

Let us consider $C_0(U_n)$. Since $[1,\omega^\alpha]$ may be identified with $(\omega^\alpha \cdot (n-1), \omega^\alpha \cdot n]$, by the inductive hypothesis, given any natural number n there exists a positive linear isometric embedding $S_n : C_0(\omega^\alpha \cdot (n-1), \omega^\alpha \cdot n] \to C_0(U_n)$ with property $(*)$. As we have mentioned before, any function in $C_0(U_n)$ may be extended by zero to $C_0(V)$. We will write $\widetilde{S_n} f := \widetilde{S_n(f)}$, so that $\widetilde{S_n}$ is an isometric embedding from $C_0(\omega^\alpha \cdot (n-1), \omega^\alpha \cdot n]$ into $C_0(V)$.

By Urysohn's lemma, for every $n \geq 1$ there is a continuous function $g_n : K \to [0,1]$ such that $g_n|_{\overline{U_n}} = 1$ and $g_n|_{K \setminus V_n} = 0$. Note that $g_n|_V \in C_0(V)$, since $\overline{V_n} \subseteq V$. For simplicity, it will be convenient for us to assume that the domain of each g_n is V.

Finally, given $f \in C_0[1, \omega^{\alpha+1}]$, we will define Sf as

$$Sf := \sum_{n=1}^{\infty} \Big(f(\omega^\alpha \cdot n) g_n + \widetilde{S_n}\big(f|_{(\omega^\alpha(n-1), \omega^\alpha n]} - f(\omega^\alpha \cdot n) \chi_{(\omega^\alpha(n-1), \omega^\alpha n]} \big) \Big) .$$

For S to be well defined, we need to show that the sequence of partial sums converges. Once this is established, it will be clear that S is a linear operator. Let $f \in C_0[1, \omega^{\alpha+1}]$, and write

$$h_n = f(\omega^\alpha \cdot n) g_n + \widetilde{S_n}\big(f|_{(\omega^\alpha(n-1), \omega^\alpha n]} - f(\omega^\alpha \cdot n) \chi_{(\omega^\alpha(n-1), \omega^\alpha n]} \big). \tag{6.3}$$

We have that $\|h_n\|_\infty \leq 3 \sup_{t \in (\omega^\alpha(n-1), \omega^\alpha n]} |f(t)|$, and the summands $(h_n)_n$ are pairwise disjoint. Therefore, if $n < m$,

$$\Big\| \sum_{k=n}^{m} h_k \Big\|_\infty = \Big\| \bigvee_{k=n}^{m} h_k \Big\|_\infty = \bigvee_{k=n}^{m} \|h_k\|_\infty \leq 3 \sup_{t \in (\omega^\alpha(n-1), \omega^\alpha m]} |f(t)| \xrightarrow{n,m\to\infty} 0,$$

so the sequence of partial sums is Cauchy and converges in $C_0(V)$, and Sf is well defined for every $f \in C_0[1, \omega^{\alpha+1}]$. Next, let us see that S is positive. Let f be a positive function in $\in C_0[1, \omega^{\alpha+1}]$. It is enough to show that for any natural number n the summand h_n defined in (6.3) is positive. If $f(\omega^\alpha \cdot n) = 0$, as $\widetilde{S_n}$ is positive on $C_0(\omega^\alpha \cdot (n-1), \omega^\alpha \cdot n]$, the term h_n is positive. Suppose now that $f(\omega^\alpha \cdot n) > 0$ and write

$$g = f|_{(\omega^\alpha(n-1), \omega^\alpha n]} - f(\omega^\alpha \cdot n) \chi_{(\omega^\alpha(n-1), \omega^\alpha n]} \in C_0(\omega^\alpha \cdot (n-1), \omega^\alpha \cdot n].$$

We decompose $g = g^+ - g^-$, and keeping in mind that f is positive, we deduce that $\|g^-\|_\infty \leq f(\omega^\alpha \cdot n)$. Since $\widetilde{S_n}$ is a norm-preserving operator which sends g^- to a function whose support is contained in U_n, then $f(\omega^\alpha \cdot n) g_n \geq \widetilde{S_n}(g^-)$. Now, observe that $h_n = \big(f(\omega^\alpha \cdot n) g_n - \widetilde{S_n}(g^-) \big) + \widetilde{S_n}(g^+) \geq 0$.

It remains to show that S is norm-preserving and has property $(*)$. We will start by checking that $\|S\| \leq 1$. Let $f \in C_0[1, \omega^{\alpha+1}]$ with $\|f\|_\infty \leq 1$. We want to see that $\|Sf\|_\infty \leq 1$. It should be observed that since S is positive we may assume that $f \geq 0$ (by changing f by its modulus $|f|$). As the summands (6.3) used to define S are pairwise disjoint, it will be enough to verify that each of them is bounded by 1. If $f(\omega^\alpha \cdot n) = 0$ the corresponding summand is bounded by 1 because $\widetilde{S_n}$ is norm-preserving, so suppose that $f(\omega^\alpha \cdot n) > 0$. Then, we have $\widetilde{S_n}(g^-) \geq 0$ because $\widetilde{S_n}$ is positive, and $g^+ \leq 1 - f(\omega^\alpha \cdot n)$, so $\|\widetilde{S_n}(g^+)\|_\infty \leq 1 - f(\omega^\alpha \cdot n)$, where

$$g = f|_{(\omega^\alpha(n-1), \omega^\alpha n]} - f(\omega^\alpha \cdot n) \chi_{(\omega^\alpha(n-1), \omega^\alpha n]}.$$

Thus,

$$0 \leq \big(f(\omega^\alpha \cdot n)g_n - \widetilde{S_n}(g^-)\big)(s) + \widetilde{S_n}(g^+)(s) \leq f(\omega^\alpha \cdot n) + 1 - f(\omega^\alpha \cdot n) = 1$$

for every $s \in U_n$, and

$$0 \leq f(\omega^\alpha \cdot n)g_n(s) + \widetilde{S_n}(g)(s) \leq f(\omega^\alpha \cdot n)g_n(s) \leq 1$$

for every $s \in V \setminus U_n$, since $\widetilde{S_n}(g)$ is supported on U_n. Therefore, any term of S is bounded by 1, and $\|S\| \leq 1$. On the other hand, fix any $t \in [1, \omega^{\alpha+1}]$. We distinguish between two situations:

- $t \neq \omega^{\alpha+1}$ and, thus, $t \in (\omega^\alpha \cdot (n-1), \omega^\alpha \cdot n]$ for some $n \geq 1$. By the inductive hypothesis, there exists $s_t \in U_n$ such that $S_n f(s_t) = f(t)$ for every $f \in C_0(\omega^\alpha \cdot (n-1), \omega^\alpha \cdot n]$; and it is also clear that $\widetilde{S_n} f(s_t) = f(t)$ for any f. Now take any $f \in C_0[1, \omega^{\alpha+1}]$ and consider the function
$$f|_{(\omega^\alpha(n-1), \omega^\alpha n]} - f(\omega^\alpha \cdot n)\chi_{(\omega^\alpha(n-1), \omega^\alpha n]}$$
which belongs to $C_0(\omega^\alpha \cdot (n-1), \omega^\alpha \cdot n]$, so it must satisfy
$$\widetilde{S_n}\left(f|_{(\omega^\alpha(n-1), \omega^\alpha n]} - f(\omega^\alpha \cdot n)\chi_{(\omega^\alpha(n-1), \omega^\alpha n]}\right)(s_t) = f(t) - f(\omega^\alpha \cdot n).$$
Since $s_t \in U_n$, $g_n(s_t) = 1$, and $Sf(s_t) = f(t)$ for every $f \in C_0[1, \omega^{\alpha+1}]$ in this case.
- $t = \omega^{\alpha+1}$. In this situation, we have to keep in mind the fact that we constructed the sequence $(V_n)_{n=1}^\infty$ in such a way that $t_0 \notin V_n$ for any $n \geq 1$. Then, $Sf(t_0) = 0 = f(\omega^{\alpha+1})$ for every $f \in C_0[1, \omega^{\alpha+1}]$.

Recall that property $(*)$ together with the fact that $\|S\| \leq 1$ implies that S is norm-preserving.

To conclude, suppose that K is a compact Hausdorff space such that $K^{(\alpha)} \neq \varnothing$. We have proved that there is a positive isometric embedding $S : C_0[1, \omega^\alpha] \to C_0(K)$ with the property that for every $t \in [1, \omega^\alpha]$ there exists $s_t \in V$ such that $Sf(s_t) = f(t)$ for every $f \in C_0[1, \omega^\alpha]$. If we define

$$Tf := f(\omega^\alpha) \cdot \mathbf{1}_K + S\big(f - f(\omega^\alpha) \cdot \mathbb{1}_{[1,\omega^\alpha]}\big), \qquad f \in C[1, \omega^\alpha],$$

it is not difficult to verify that S defines a positive linear isometric embedding of $C[1, \omega^\alpha]$ into $C(K)$ with the properties (1), (2) and (3) defined in the statement of the proposition.

To check the last assertion of the proposition, it suffices to observe that the third property of the embedding T that we have constructed is precisely property $(*)$ of the previous lemma. □

Remark 6.11. It should be noticed that $K^{(\alpha)} \neq \varnothing$ does not guarantee the existence of a lattice isometric embedding $T : C[1, \omega^\alpha] \to C(K)$. Suppose that there exists a lattice isometric embedding $T : c \to C[0, 1]$. Let us denote $f_n = Te_n$ for every natural number n. Note that these f_n's are pairwise disjoint norm-one elements of $C[0, 1]$. Hence, for every n there is $t_n \in [0, 1]$ such that $f_n(t_n) = 1$. Passing to a subsequence, we may assume that $t_n \to t_0$ in $[0, 1]$. Note that $f_n(t_0) = 0$ for every n. Moreover, as $T\mathbb{1}(t_n) = 1$, it follows by the continuity of this function that $T\mathbb{1}(t_0) = 1$.

Let I^{t_0} be an open interval of $[0, 1]$ such that $t_0 \in I^{t_0}$ and $T\mathbb{1}(t) > \frac{1}{2}$ for every $t \in I^{t_0}$, and let N be such that $t_N \in I^{t_0}$. Now, observe that $\mathbb{1} - e_N$ and e_N are disjoint in c, so $T(\mathbb{1} - e_N)$ and Te_N are disjoint in $C[0, 1]$. Thus, $g := T(\mathbb{1} - e_N)$ is zero on $\mathrm{supp}(Te_N)$ and is equal to $T\mathbb{1}$ outside that set. Then, I^{t_0} can be represented as the union of the following two disjoint non-empty open sets

$$t_N \in A_1 := \operatorname{supp}(Te_N) \cap I^{t_0} = \left\{ t \in I^{t_0} \,:\, g(t) \leq \frac{1}{2} \right\} \text{ and } A_2 := \left\{ t \in I^{t_0} \,:\, g(t) > \frac{1}{2} \right\} \ni t_0,$$

which is a contradiction given that I^{t_0} is connected.

Using the exact same ideas from [58] to prove Theorem 6.5, it can be shown that c_0 embeds linearly isometrically into $C[1,\omega^2]$ doing SPR. Since this result will be determinant for us, we include its proof below.

Proposition 6.12. *There exists an isometric SPR embedding of c_0 into $C[1,\omega^2]$.*

Proof. For every $n \in \mathbb{N}$, we define the continuous function on $[1,\omega^2]$ given by the expression

$$x^{(n)} = \mathbb{1}_{1,n} + \frac{1}{2}\mathbb{1}_{n+1} + \frac{1}{2}\sum_{m=2}^{n} \mathbb{1}_{m,n}$$

(see Figure 6.1). It is clear that $x^{(n)}(\omega^2) = 0$, as it is supported on $[1, \omega \cdot (n+1)]$.

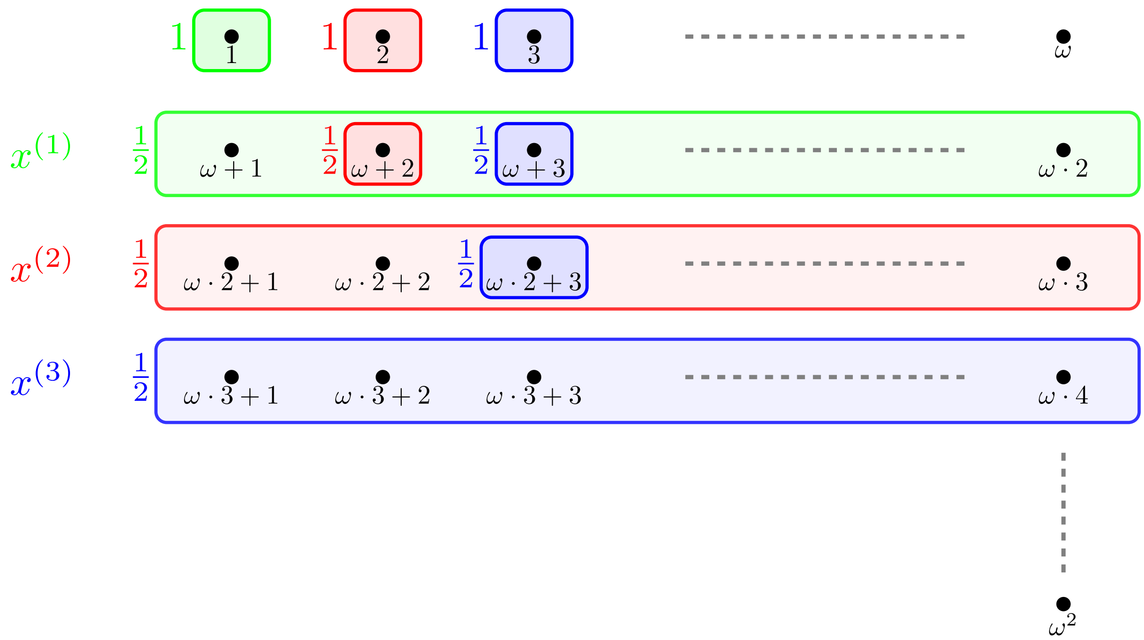


Fig. 6.1: Representation of $x^{(n)}$ in the real setting. The points in $[1,\omega^2]$ are ordered from the left to the right and from the top to the bottom.

The sequence $\left(x^{(n)}\right)_{n=1}^{\infty}$ spans an isometric copy of c_0. Indeed, given $(\alpha_n)_{n=1}^{\infty} \in c_{00}$ with $\bigvee_{n=1}^{\infty} |\alpha_n| = 1$, it follows that $x = \sum_{n=1}^{\infty} \alpha_n x^{(n)}$ takes the value α_n at the point n for every $n \in \mathbb{N}$, so $\|x\| \geq 1$. On the other hand, observe that x vanishes outside of $[1, \omega \cdot (k+1)]$ for k big enough. Moreover, the function x takes the values α_n at the point n, 0 at ω, $\frac{1}{2}\alpha_n$ at $[\omega \cdot n + 1, \omega \cdot n + n] \cup \{\omega \cdot (n+1)\}$, and $\frac{1}{2}(\alpha_n + \alpha_m)$ at $\omega \cdot n + m$ for every $m > n$ and $n \in \mathbb{N}$. In all the cases above, $|x| \leq 1$, so we can conclude that $\|x\|_\infty = 1$.

Now, let us check that $E = \overline{\text{span}}\{x^{(n)} : n \in \mathbb{N}\}$ does 3-SPR. By Proposition 6.4, it suffices to show that $\||x| \wedge |y|\|_\infty \geq \frac{1}{3}$ for every $x, y \in S_E$. Put $x = \sum_n \alpha_n x^{(n)}$ and $y = x = \sum_n \beta_n x^{(n)}$ for some $(\alpha_n)_n, (\beta_n)_n \in S_{c_0}$. Find $n, m \in \mathbb{N}$ such that $|\alpha_n| = 1 = |\beta_m|$. If $n = m$, then $|x| \wedge |y|(n) = 1$. Otherwise, assume that $n < m$. If $|\alpha_m| \geq \frac{1}{3}$, then $|x| \wedge |y|(m) = |\alpha_m| \wedge |\beta_m| \geq \frac{1}{3}$, and the same can be done when $|\beta_n| \geq \frac{1}{3}$. Finally, if $|\alpha_m|, |\beta_n| < \frac{1}{3}$, it follows that $|\alpha_n + \alpha_m|, |\beta_n + \beta_m| \geq \frac{2}{3}$, so $|x| \wedge |y|(\omega \cdot n + m) \geq \frac{1}{3}$. □

It should be noticed that we can combine the preceding result with Proposition 6.10 to deduce the implication $(i) \Rightarrow (ii)$ of Theorem 6.5. Indeed, if K' is infinite (or equivalently, K'' is non-empty), by Proposition 6.10, there exists an isometric embedding $T : C[1, \omega^2] \to C(K)$ that preserves SPR subspaces. Since Proposition 6.12 states that $C[1, \omega^2]$ contains a subspace isometric to c_0 doing SPR, we can deduce that $C(K)$ also does. We will soon show that for $\alpha > 2$ a *stronger result* holds: $C[1, \omega^\alpha]$ isometrically embeds into itself in an SPR way. This will be an immediate consequence of the following proposition.

Proposition 6.13. *Let $2 \leq \alpha < \omega$ be a finite ordinal. Then there exists an isometric SPR embedding of $C[1, \omega^\alpha]$ into $C[1, \omega^2] \oplus_\infty C[1, \omega^\alpha]$.*

Proof. Fix any $2 \leq \alpha < \omega$. Recall that by [27, Theorem 1], there exists a surjective isomorphism $R : C[1, \omega^\alpha] \to c_0$. Replacing R by $\frac{R}{\|R\|}$ and R^{-1} by $\|R\| R^{-1}$ if necessary, we may assume that $\|R\| \leq 1$. Let $C \geq 1$ be such that $\|R^{-1}\| \leq C$. Now, consider the isometric 3-SPR embedding $S : c_0 \to C[1, \omega^2]$ constructed in Proposition 6.12, and define the operator

$$\begin{aligned} T : C[1, \omega^\alpha] &\longrightarrow C[1, \omega^2] \oplus_\infty C[1, \omega^\alpha] \\ f &\longmapsto Tf = (SRf, f). \end{aligned}$$

Clearly, T is an isometric embedding, as it is contractive and $\|Tf\| \geq \|f\|$. To conclude, we will use Proposition 6.4 to show that $T(C[1, \omega^\alpha])$ does $3C$-SPR in $C[1, \omega^2] \oplus_\infty C[1, \omega^\alpha]$. Given $f, g \in S_{C[1, \omega^\alpha]}$, we have

$$\||Tf| \wedge |Tg|\|_\infty \geq \||SRf| \wedge |SRg|\|_\infty \geq \frac{1}{\|R^{-1}\|} \left\| \left| S\left(\frac{Rf}{\|Rf\|}\right) \right| \wedge \left| S\left(\frac{Rg}{\|Rg\|}\right) \right| \right\|_\infty \geq \frac{1}{3C}.$$

□

Corollary 6.14. *$C[1, \omega^2]$ embeds isometrically into $C[1, \omega^2 \cdot 2]$ in an SPR way. If $2 < \alpha < \omega$, there exists an isometric SPR embedding of $C[1, \omega^\alpha]$ into itself.*

Proof. To prove this, it is enough to observe that $C[1, \omega^2] \oplus_\infty C[1, \omega^2]$ is lattice isometric to $C[1, \omega^2 \cdot 2]$ and for $\alpha > 2$, $C[1, \omega^2] \oplus_\infty C[1, \omega^\alpha]$ is lattice isometric to $C[1, \omega^\alpha]$. The result then follows from the last proposition. □

Corollary 6.15. *Let K be a compact Hausdorff space. Then:*

(i) If there exists $2 < \alpha < \omega$ such that $K^{(\alpha)} \neq \varnothing$, then there is an isometric SPR embedding of $C[1, \omega^\alpha]$ into $C(K)$.

(ii) If $|K''| \geq 2$, then $C[1, \omega^2]$ embeds isometrically into $C(K)$ in an SPR way.

Proof. (i): Suppose that $K^{(\alpha)} \neq \varnothing$ for some $2 < \alpha < \omega$. By Proposition 6.10, there exists an isometric embedding $T : C[1, \omega^\alpha] \hookrightarrow C(K)$ which preserves the SPR subspaces of

$C[1,\omega^\alpha]$. Since, by the previous corollary, $C[1,\omega^\alpha]$ embeds into itself in an SPR way, we infer that $C[1,\omega^\alpha]$ embeds into $C(K)$ in an SPR way.

(ii): The proof of Proposition 6.10 can be adapted to show that, if $|K''| \geq 2$, there exists a positive linear isometric embedding of $C[1,\omega^2\cdot 2]$ into $C(K)$ with the property $(*)$ defined in Lemma 6.8. Indeed, it suffices to consider distinct points $s_1, s_2 \in K''$ and open neighborhoods $V_i, W_i \subseteq K$, $i=1,2$, such that W_1 and W_2 are disjoint and $s_i \in V_i \subseteq \overline{V_i} \subseteq W_i$, $i=1,2$. As $V_i \cap K'' \neq \varnothing$, by the inductive step there exist positive linear isometries $S_i : C_0[1,\omega^2] \to C_0(V_i)$ satisfying property $(*)$, $i=1,2$ (denote by $\widetilde{S}_i$ the extensions by 0 to $C(K)$), and we can find $h_i \in C(K)$ such that $0 \leq h_i \leq 1$, $h_i|_{\overline{V_i}} = 1$ and $h_i|_{K\setminus W_i}$, $i=1,2$. Therefore, the operator $T : C[1,\omega^2\cdot 2] \to C(K)$ defined as

$$Tf = f(\omega^2)h_1 + \widetilde{S_1}(f|_{[1,\omega^2]} - f(\omega^2)\chi_{[1,\omega^2]}) + f(\omega^2\cdot 2)h_2 + \widetilde{S_2}(f|_{(\omega^2,\omega^2\cdot 2]} - f(\omega^2)\chi_{(\omega^2,\omega^2\cdot 2]})$$

for every $f \in C[1,\omega^2\cdot 2]$ will satisfy the properties mentioned above. In particular, T will preserve the SPR subspaces of $C[1,\omega^2\cdot 2]$. Thus we have that $C[1,\omega^2]$ can be embedded isometrically in an SPR way into any $C(K)$-space with $|K''| \geq 2$. □

In view of the preceding corollaries, one might wonder whether it is possible to embed $C[1,\omega^2]$ into itself in an SPR way, as this is the case for $C[1,\omega^\alpha]$ when $\alpha > 2$. We will see below that this is not possible.

Proposition 6.16. *$C[1,\omega^2]$ cannot be embedded isometrically into a $C(K)$-space with $|K''| = 1$ in an SPR way.*

Proof. Let $T : C[1,\omega^2] \to C(K)$ be an isometric embedding, and consider the same notation as in Proposition 6.7. In that proposition, it was shown that this implies the existence of an element $t_0 \in K''$ such that $|T\mathbb{1}(t_0)| = 1$. We are assuming that $|K''| = 1$, so $K = \{t_0\}$.

Suppose that $T(C[1,\omega^2])$ does SPR. By Proposition 6.4, there exists a constant $C > 0$ such that for every $n, m \geq 1$, with $n \neq m$, we have

$$\||T\mathbb{1}_{1,n}| \wedge |T\mathbb{1}_{1,m}|\|_\infty \geq \frac{1}{C}.$$

Thus, for every $n \geq 1$, there exists a sequence $(s_{n,m})_{m=n+1}^\infty \subseteq K$ such that

$$|T\mathbb{1}_{1,n}| \wedge |T\mathbb{1}_{1,m}|(s_{n,m}) \geq \frac{1}{C}.$$

Fix any $n \geq 1$. We claim that the sequence $(s_{n,m})_{m=n+1}^\infty$ has infinitely many distinct elements. We will prove even more: no term of the sequence $(s_{n,m})_{m=n+1}^\infty$ can be repeated an infinite number of times. Suppose that a point $s \in K$ appears N times in the sequence, that is, there are $m_1 < m_2 < \cdots < m_N$ such that $s_{n,m_1} = s_{n,m_2} = \cdots = s_{n,m_N} = s$. Since $|T\mathbb{1}_{1,m_k}|(s_{n,m_k}) \geq \frac{1}{C}$ for $k = 1,\ldots,N$, we can find a scalar θ_k of modulus 1 such that

$$\mathbb{R} \ni \theta_k T\mathbb{1}_{1,m_k}(s_{n,m_k}) \geq \frac{1}{C}.$$

It is clear that $\sum_{k=1}^N \theta_k \mathbb{1}_{1,m_k}$ has norm 1 in $C[1,\omega^2]$, and since T is norm-preserving, the function $\sum_{k=1}^N \theta_k T\mathbb{1}_{1,m_k}$ must have norm 1. If we evaluate this function at s we have

$$1 \geq \sum_{k=1}^N \theta_k T\mathbb{1}_{1,m_k}(s_{n,m_1}) \geq \frac{N}{C},$$

so N must be less than or equal to C. Therefore, for any $n \geq 1$ we can find a point $s_n \in \overline{\{s_{n,m} : m \geq n+1\}} \setminus \{s_{n,m} : m \geq n+1\}$, which of course belongs to K'.

By the continuity of each $T\mathbb{1}_{1,n}$, we have that $|T\mathbb{1}_{1,n}(s_n)| \geq \frac{1}{C}$ for every $n \geq 1$. In addition, via the same argument that we have just presented, it is easy to check that for every $n \geq 1$, the sequence $(s_n)_{n=1}^{\infty}$ has infinitely many different points. In particular, this implies that $t_0 \in \overline{\{s_n : n \geq 1\}} \setminus \{s_n : n \geq 1\}$, as $K'' = \{t_0\}$.

Now we consider the family of functions $\mathbb{1} - \mathbb{1}_1 + \zeta_m \mathbb{1}_{1,m}$, where $m \geq 1$ and ζ_m are unimodular scalars. Observe that since these functions have norm 1 in $C[1, \omega^2]$, then $\|T\mathbb{1} - T\mathbb{1}_1 + \zeta_m T\mathbb{1}_{1,m}\|_\infty = 1$ for every $m \geq 1$. In the first paragraph of the proof we have pointed out that $T\mathbb{1}(t_0) = \theta$ for some scalar θ of modulus 1. In addition, $T\mathbb{1}_1(t_0) = 0$: indeed, it is easy to construct a sequence of distinct points $(t_n)_{n=1}^{\infty} \subseteq K'$ such that $|T\mathbb{1}_n(t_n)| = 1$ and, therefore, $T\mathbb{1}_1(t_n) = 0$ whenever $n > 1$, so that $T\mathbb{1}_1(t_0) = 0$. Fix $0 < \varepsilon < \frac{1}{2C}$ and let U^{t_0} be a neighborhood of t_0 in K such that

$$|T\mathbb{1}(t) - \theta| < \varepsilon \quad \text{and} \quad |T\mathbb{1}_1(t)| < \varepsilon \quad \text{for every } t \in U^{t_0}.$$

Since t_0 is an accumulation point of the sequence $(s_n)_{n=1}^{\infty}$, there exists $M \in \mathbb{N}$ such that $s_M \in U^{t_0}$. We also know that $|T\mathbb{1}_{1,M}(s_M)| \geq \frac{1}{C}$, so take θ_M of modulus 1 such that

$$\mathbb{R} \ni \xi_M \, T\mathbb{1}_{1,M}(s_M) \geq \frac{1}{C}.$$

Finally, we consider the function $T\big(\mathbb{1} - \mathbb{1}_1 + \theta\,\theta_M\,\mathbb{1}_{1,M}\big)$, which should have norm 1 in $C(K)$. However, if we evaluate this function at s_M we obtain the following:

$$\begin{aligned} |T\mathbb{1} - T\mathbb{1}_1 + \theta\,\theta_M\,T\mathbb{1}_{1,M}|(s_M) &= |T\mathbb{1}(s_M) - \theta + \theta + \theta\,\theta_M\,T\mathbb{1}_{1,M}(s_M) - T\mathbb{1}_1(s_M)| \\ &\geq |\theta + \theta\,\theta_M\,T\mathbb{1}_{1,M}(s_M)| - |T\mathbb{1}(s_M) - \theta| - |T\mathbb{1}_1(s_M)| \\ &= 1 + \theta_M\,T\mathbb{1}_{1,M}(s_M) - 2\varepsilon \geq 1 + \frac{1}{C} - 2\varepsilon > 1, \end{aligned}$$

so we have arrived at a contradiction. □

Since the second derivative of $[1, \omega^2]$ is the singleton $\{\omega^2\}$, the above proposition has the following immediate consequence:

Corollary 6.17. *$C[1, \omega^2]$ cannot be embedded isometrically into itself in an SPR way.*

We conclude this section by discussing the possibility of extending Theorem 6.5 to AM-spaces. In [28, Question 5.4], Bilokopytov formulates the following question:

Question 6.18. Let X be an AM-space. If X^a has infinite codimension in X, then does X contain an SPR subspace of infinite dimension?

Note that in a $C(K)$-space the order continuous part $C(K)^a$ has infinite codimension if and only if K' is infinite (recall Remark 5.31). That is, condition (i) of Theorem 6.5 can be replaced by *$C(K)^a$ has infinite codimension in $C(K)$*. This means that Question 6.18 has an affirmative answer for $C(K)$ spaces. We will now give a partial answer to this question by showing that the implication $(i) \Rightarrow (ii)$ of Theorem 6.5 does not extend to AM-spaces: we will construct an AM-space X such that X^a has infinite codimension in X but it cannot even contain subspaces isomorphic to c_0 doing SPR.

Let us consider the following AM-space:

$$X := \left\{ f \in C_0[1,\omega^2] \,:\, \frac{1}{n} f(\omega \cdot (n-1)+1) = f(\omega \cdot n) \quad \text{for all } n \geq 1 \right\}. \tag{6.4}$$

By Proposition 5.30, X^a has infinite codimension in X. Note that the norm-one lattice homomorphisms are: $\delta_{\omega \cdot n+m}$ for $n \geq 0$ and $m \geq 2$ are the coordinate functionals; $n\delta_{\omega\cdot n} \equiv \delta_{\omega\cdot(n-1)+1}$, for $n \geq 1$, are the remaining norm-one lattice homomorphisms on X.

Proposition 6.19. *c_0 does not isomorphically embed into the space X defined in (6.4) in an SPR way.*

Proof. Suppose that there exists an isomorphic embedding $T : c_0 \to X$ with the property that $T(c_0)$ does C-SPR in X. By Proposition 6.4, $T(c_0)$ does not contain C-almost disjoint pairs, so for every natural numbers $n, m \geq 1$, $n \neq m$ we have

$$\big\| |Te_n| \wedge |Te_m| \big\| \geq \frac{1}{\|T^{-1}\|C},$$

and since X is an AM-space, $\mathrm{Hom}(X,\mathbb{R})$ is 1-norming for X (Proposition 5.22), so for every $n, m \in \mathbb{N}$ there exists $x^*_{n,m} \in \mathrm{Hom}(X,\mathbb{R}) \cap S_{X^*}$ such that

$$|Te_n|(x^*_{n,m}) \wedge |Te_m|(x^*_{n,m}) \geq \frac{1}{\|T^{-1}\|C}.$$

Let us see what can be said about these lattice homomorphisms $(x^*_{n,m})_{n,m=1}^\infty$. First, note that for every $n \geq 1$, there are infinitely many distinct lattice homomorphisms in the sequence $(x^*_{n,m})_{m>n}$. In fact, by mimicking the argument of the proof of Proposition 6.16 one can check that there is no lattice homomorphism that repeats infinitely many times in this sequence (each lattice homomorphism may appear $\lfloor \|T^{-1}\|\|T\|C \rfloor$ different times at most).

Now, observe that given $n \geq 1$, the sequence $(x^*_{n,m})_{m>n}$ can only contain a finite number of lattice homomorphisms that are not coordinate functionals. Otherwise, we could find an increasing sequence of natural numbers $(m_k)_{k=1}^\infty$, with $m_1 > n$, such that $\delta_{\omega\cdot(m_k-1)+1} \in \{x^*_{n,m} \,:\, m>n\}$. Given that $\delta_{\omega\cdot(m_k-1)+1} \overset{w^*}{\to} 0$, we obtain that $0 \in \overline{\{x^*_{n,m} \,:\, m>n\}}^{w^*}$. But this is impossible, as $|Te_n|(x^*_{n,m}) \geq \frac{1}{\|T^{-1}\|C}$.

Therefore, passing to subsequences, we may assume that for every $n \geq 1$, $(x^*_{n,m})_{m>n}$ is a sequence of distinct norm-one coordinate functionals on X. Let x^*_n be an accumulation point of $(x^*_{n,m})_{m>n}$. It should be noticed that $x^*_n = \delta_{\omega\cdot k_n}$ for some $k_n \geq 1$, so $\|x^*_n\| = \frac{1}{k_n}$. It can be proven that there must be infinitely many different x^*_n's (again, we may replicate the proof of Proposition 6.16). But this is a contradiction with the fact that

$$\frac{1}{\|T^{-1}\|C} \leq |Te_n|(x^*_n) \leq \|T\|\|x^*_n\|, \text{ for all } n \geq 1.$$

□

Conclusiones

En esta tesis se da respuesta a la *principal cuestión* que nos propusimos analizar inicialmente: el **Problema del Subespacio Complementado en retículos de Banach** (**CSP**). Para ello, nos apoyamos en una construcción reciente (2023) de Plebanek y Salguero-Alarcón, que denotamos por PS_2, la cual constituye el primer (y, hasta ahora, único) ejemplo de subespacio complementado en un espacio $C(K)$ que no puede ser isomorfo a ningún espacio $C(K)$. En el Capítulo 3 probamos que PS_2 *no es isomorfo a un retículo de Banach*. Además, mostramos también que dicha construcción se puede refinar para dar asimismo una solución *negativa* al CSP en el *caso complejo*.

Sin embargo, quedan abiertas muchas preguntas relativas al CSP y muestra de ello son los numerosos problemas que se plantean a lo largo del Capítulo 2. Entre ellos destacamos:

Pregunta 2.33. ¿Todo subespacio **separable** complementado en un retículo de Banach es isomorfo a un retículo de Banach? El espacio PS_2 no es separable. Además, está 1-complementado en un espacio $C(K)$ de una manera muy natural, por lo que, en principio, no vamos a poder dar una respuesta a este nuevo problema por medio de una construcción similar; recordemos que los subespacios separables 1-complementados de espacios $C(K)$ son isomorfos a espacios $C(K)$ [20]. Además, el CSP separable es especialmente relevante por estar conectado con dos importantes conjeturas sobre complementación en espacios de Banach clásicos (Observación 3.5).

Pregunta 2.37. ¿Todo hiperplano de un retículo de Banach es isomorfo a un retículo de Banach? Esta pregunta se discute en la Sección 2.5 y en muchas situaciones resulta muy sencillo responderla de manera afirmativa. En la Observación 2.38 mostramos cómo el *único candidato a contraejemplo* que tenemos (el espacio PS_2) no puede ser isomorfo a un hiperplano de un retículo de Banach.

Otro aspecto que queda pendiente tras este trabajo es tratar de encontrar un *contraejemplo más sencillo* al CSP, puesto que la construcción del espacio PS_2 es muy intrincada.

La demostración de la existencia y construcción del **retículo de Banach libre complejo** también era uno de los objetivos iniciales de este proyecto de tesis. Nos gustaría recalcar que la noción de *retículo de Banach complejo* (Definición 1.20) es aparentemente *mucho menos natural* que la de espacio de Banach complejo: un retículo de Banach complejo es siempre la complejificación de un retículo de Banach real con una norma muy concreta. Por esta razón, no resulta trivial que exista el retículo de Banach libre complejo generado por un espacio de Banach complejo E sin asumir propiedades adicionales en E.

La razón que nos impulsó a considerar este objeto es que ofrece un **lugar canónico** para estudiar el *Problema del Subespacio Complementado para retículos de Banach complejos*: si un espacio E está complementado en un retículo de Banach complejo, entonces está complementado en su $\mathrm{FBL}_{\mathbb{C}}[E]$ con la *mejor constante posible* (se puede imitar la prueba de la Proposición 2.11). Esto es interesante, porque existen diversos resultados positivos relativos a subespacios complementados en retículos complejos que no son ciertos en el caso real, destacando entre ellos el resultado de Kalton y Wood que afirma que todo subespacio 1-complementado en un espacio complejo con base 1-incondicional también posee una base 1-incondicional [89]. Ahora que conocemos que la respuesta al CSP tanto en el caso real como en el caso complejo es negativa, el retículo de Banach libre complejo puede jugar un papel importante para ayudarnos a encontrar respuestas afirmativas al CSP en situaciones más concretas como, por ejemplo, en el caso complejo separable y 1-complementado.

En el Capítulo 5 estudiamos los *homomorfismos reticulares que alcanzan su norma*, continuando el trabajo iniciado en [42]. En nuestro caso, lo hacemos con un enfoque más general y menos centrado en los retículos de Banach libres. En el citado artículo, los autores preguntan si todo homomorfismo reticular en un retículo de Banach *σ-Dedekind completos* alcanza su norma. Tras el Teorema 5.7 mostramos algunos ejemplos de que esto no tiene por qué ser así. Queremos destacar que la Conjetura 5.5 que plantean los autores en [42] sigue abierta (véase también nuestra Pregunta 5.33):

Conjetura. Sea E un espacio de Banach. Un funcional $x^* \in E^*$ alcanza su norma si y solo si su extensión $\widehat{x^*} \in \mathrm{FBL}[E]^*$ alcanza su norma.

Las secciones 5.3 y 5.4 del quinto capítulo se enfocan en analizar el *alcance de la norma de los homomorfismos reticulares* en **AM-espacios**. Algunos de los resultados que aparecen aportan información que puede ser muy útil para entender la estructura de esta clase de retículos de Banach. Entre ellos queremos destacar una versión del *lema de Urysohn para AM-espacios* (Corolario 5.18), la caracterización de los retículos de Banach que son AM-espacios de la Proposición 5.22 o la representación de esta clase de retículos como espacios de funciones continuas positivamente homogéneas. Aunque esto último ya fue observado por Goullet de Rugy en [65, Corollaire 1.31], se trata de un resultado poco conocido y que, al menos, había recibido muy poca atención hasta ahora. Lo probamos de nuevo en la Proposición 5.22. Otro teorema muy destacable relativo a los AM-espacios y que es determinante para demostrar que el espacio PS_2 no puede ser isomorfo a un retículo es el Corolario 3.2: los retículos de Banach que son espacios $\mathcal{L}_\infty$ son reticularmente isomorfos a AM-espacios.

Todos estos ejemplos ilustran la *importancia fundamental de los AM-espacios* en esta memoria. En el futuro esperamos seguir investigando con más detalle la estructura de esta clase de retículos de Banach porque esto aportará indudablemente luz a otras tantas preguntas que han quedado abiertas en este trabajo (Preguntas 2.32, 2.44 y 6.18).

Conclusions

In this thesis, we answer the *main question* we initially aimed to analyze: the **Complemented Subspace Problem for Banach lattices** (**CSP**). To do this, we use a recent construction (2023) due to Plebanek and Salguero-Alarcón, which we denote by PS_2, and which constitutes the first (and, until now, only) example of a complemented subspace of a $C(K)$-space that cannot be isomorphic to any $C(K)$-space. In Chapter 3, we prove that PS_2 *is not isomorphic to a Banach lattice*. Furthermore, we also show that this construction can be refined to provide a *negative* solution to the CSP in the *complex case*.

Nevertheless, many questions related to the CSP remain open, as illustrated by the numerous problems posed throughout Chapter 2. Among them, we highlight:

Question 2.33. Is every **separable** complemented subspace of a Banach lattice isomorphic to a Banach lattice? The space PS_2 is not separable. Furthermore, it is 1-complemented in a $C(K)$-space in a very natural way, so, in principle, we will not be able to answer this new problem by means of a similar construction; let us recall that separable 1-complemented subspaces of $C(K)$-spaces are isomorphic to $C(K)$-spaces [20]. Moreover, the separable CSP is particularly relevant due to its connection with two important conjectures concerning complementation in classical Banach spaces (Remark 3.5).

Question 2.37. Is every hyperplane of a Banach lattice isomorphic to a Banach lattice? This question is discussed in Section 2.5, and in many situations, it is very straightforward to answer it affirmatively. In Remark 2.38, we show how the *only candidate for a counterexample* we have (the space PS_2) cannot be isomorphic to a hyperplane of a Banach lattice.

Another aspect that remains pending after this work is to try to find a *simpler counterexample* to the CSP, since the construction of the space PS_2 is quite involved.

The proof of the existence and construction of the **complex free Banach lattice** was also one of the initial objectives of this thesis project. We would like to emphasize that the notion of a *complex Banach lattice* (Definition 1.20) is apparently *much less natural* than that of a complex Banach space: a complex Banach lattice is always the complexification of a real Banach lattice with a very specific norm. For this reason, it is not trivial that the complex free Banach lattice generated by a complex Banach space E exists without additional assumptions on the space E.

The reason that prompted us to consider this object is that it offers a **canonical place** to study the *Complemented Subspace Problem for complex Banach lattices*: if a space E is complemented in a complex Banach lattice, then it is complemented in its $\mathrm{FBL}_{\mathbb{C}}[E]$ with the *best possible constant* (the proof of Proposition 2.11 can be replicated to check this). This is interesting because there are various positive results concerning complemented subspaces in complex Banach lattices that are not true in the real case, among them standing out the result by Kalton and Wood which states that every 1-complemented subspace in a complex Banach space with a 1-unconditional basis also has a 1-unconditional basis [89]. Now that we know that the answer to the CSP in both the real and complex cases is negative, the complex free Banach lattice can play an important role in helping us find affirmative answers to the CSP in more specific situations, such as, for example, in the separable and 1-complemented complex case.

In Chapter 5, we study *norm-attaining lattice homomorphisms*, continuing the work initiated in [42]. In our case, we do so with a more general approach and less focused on free Banach lattices. In the aforementioned article, the authors ask whether every lattice homomorphism in a σ*-Dedekind complete* Banach lattice attains its norm. Following Theorem 5.7, we show some examples where this does not necessarily hold. We wish to highlight that Conjecture 5.5 posed by the authors in [42] remains open (see also our Question 5.33):

Conjecture. Let E be a Banach space. A linear functional $x^* \in E^*$ attains its norm if and only if its extension $\widehat{x^*} \in \mathrm{FBL}[E]^*$ attains its norm.

Sections 5.3 and 5.4 of the fifth chapter focus on analyzing the *norm-attainment of lattice homomorphisms* on **AM-spaces**. Some of the results presented provide information that can be very useful for understanding the structure of this class of Banach lattices. Among them, we wish to highlight a version of *Urysohn's lemma for AM-spaces* (Corollary 5.18), the characterization of Banach lattices that are AM-spaces provided in Proposition 5.22, or the representation of this class of Banach lattices as spaces of positively homogeneous continuous functions. Although the latter was already observed by Goullet de Rugy in [65, Corollaire 1.31], it is a result that is not widely known and, at least, had received very scant attention until now. We prove it again in Proposition 5.22. Another highly notable theorem concerning AM-spaces, which is crucial for proving that the space PS_2 cannot be isomorphic to a Banach lattice, is Corollary 3.2: Banach lattices that are $\mathcal{L}_\infty$-spaces are lattice isomorphic to AM-spaces.

All these examples illustrate the *fundamental importance of AM-spaces* in this dissertation. In the future, we hope to continue investigating the structure of this class of Banach lattices in more detail because this will undoubtedly shed light on many other questions that remain open in this work (Questions 2.32, 2.44, and 6.18).

References

1. Y. A. Abramovich and C. D. Aliprantis, *An invitation to operator theory*, Graduate Studies in Mathematics, vol. 50, American Mathematical Society, Providence, RI, 2002. MR 1921782
2. Y. A. Abramovich and P. Wojtaszczyk, *The uniqueness of the order in the spaces $L_p[0,1]$ and l_p*, Mat. Zametki **18** (1975), no. 3, 313–325. MR 394097
3. M. D. Acosta and M. Soleimani-Mourchehkhorti, *On the Bishop-Phelps-Bollobás property for positive functionals*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **118** (2024), no. 3, Paper No. 122, 18. MR 4756250
4. I. Aharoni and J. Lindenstrauss, *Uniform equivalence between Banach spaces*, Bull. Amer. Math. Soc. **84** (1978), no. 2, 281–283. MR 482074
5. F. Albiac and N. J. Kalton, *Topics in Banach space theory*, second ed., Graduate Texts in Mathematics, vol. 233, Springer, [Cham], 2016, With a foreword by Gilles Godefory. MR 3526021
6. C. D. Aliprantis and O. Burkinshaw, *Factoring compact and weakly compact operators through reflexive Banach lattices*, Trans. Amer. Math. Soc. **283** (1984), no. 1, 369–381. MR 735429
7. ———, *Positive operators*, Springer, Dordrecht, 2006, Reprint of the 1985 original. MR 2262133
8. D. Alspach and Y. Benyamini, *Primariness of spaces of continuous functions on ordinals*, Israel J. Math. **27** (1977), no. 1, 64–92. MR 440349
9. ———, *A geometrical property of $C(K)$ spaces*, Israel J. Math. **64** (1988), no. 2, 179–194. MR 986146
10. D. Alspach, P. Enflo, and E. Odell, *On the structure of separable $\mathcal{L}_p$ spaces* $(1 < p < \infty)$, Studia Math. **60** (1977), no. 1, 79–90. MR 500076
11. T. Andô, *Contractive projections in L_p spaces*, Pacific J. Math. **17** (1966), 391–405. MR 192340
12. R. Anisca, *Subspaces of L_p with more than one complex structure*, Proc. Amer. Math. Soc. **131** (2003), no. 9, 2819–2829. MR 1974339
13. A. Avilés, G. Martínez-Cervantes, and J. Rodríguez, *Weak*-sequential properties of Johnson-Lindenstrauss spaces*, J. Funct. Anal. **276** (2019), no. 10, 3051–3066. MR 3944288
14. A. Avilés, G. Martínez-Cervantes, J. D. Rodríguez Abellán, and A. Rueda Zoca, *Lattice embeddings in free Banach lattices over lattices*, Math. Inequal. Appl. **25** (2022), no. 2, 495–509. MR 4428584
15. A. Avilés, J. Rodríguez, and P. Tradacete, *The free Banach lattice generated by a Banach space*, J. Funct. Anal. **274** (2018), no. 10, 2955–2977. MR 3777636
16. A. Avilés and J. D. Rodríguez Abellán, *The free Banach lattice generated by a lattice*, Positivity **23** (2019), no. 3, 581–597. MR 3977585

17. A. Avilés and P. Tradacete, *Amalgamation and injectivity in Banach lattices*, Int. Math. Res. Not. IMRN (2023), no. 2, 956–997. MR 4537317

18. K. A. Baker, *Free vector lattices*, Canadian J. Math. **20** (1968), 58–66. MR 224524

19. S. Banach, *Théorie des opérations linéaires*, 1932.

20. Y. Benyamini, *Separable G spaces are isomorphic to $C(K)$ spaces*, Israel J. Math. **14** (1973), 287–293. MR 333668

21. ______, *An M-space which is not isomorphic to a $C(K)$ space*, Israel J. Math. **28** (1977), no. 1-2, 98–102. MR 458145

22. Y. Benyamini, P. Flinn, and D. R. Lewis, *A space without 1-unconditional basis which is 1-complemented in a space with a 1-unconditional basis*, Texas functional analysis seminar 1983–1984 (Austin, Tex.), Longhorn Notes, Univ. Texas Press, Austin, TX, 1984, pp. 143–149. MR 832238

23. Y. Benyamini and J. Lindenstrauss, *A predual of l_1 which is not isomorphic to a $C(K)$ space*, Israel J. Math. **13** (1972), 246–254. MR 331013

24. S. J. Bernau and H. E. Lacey, *The range of a contractive projection on an L_p-space*, Pacific J. Math. **53** (1974), 21–41. MR 358325

25. ______, *A local characterization of Banach lattices with order continuous norm*, Studia Math. **58** (1976), no. 2, 101–128. MR 625322

26. C. Bessaga and A. Pełczyński, *A generalization of results of R. C. James concerning absolute bases in Banach spaces*, Studia Math. **17** (1958), 165–174. MR 115071

27. ______, *Spaces of continuous functions. IV. On isomorphical classification of spaces of continuous functions*, Studia Math. **19** (1960), 53–62. MR 113132

28. E. Bilokopytov, *Disjointly non-singular operators and various topologies on Banach lattices*, Preprint available on arXiv (2025), 19 pp.

29. E. Bilokopytov, E. García-Sánchez, D. de Hevia, G. Martínez-Cervantes, and P. Tradacete, *Norm-attaining lattice homomorphisms and renormings of Banach lattices*, Preprint available on arXiv (2025), 29 pp.

30. E. Bilokopytov and V. G. Troitsky, *Uniformly closed sublattices of finite codimension*, Linear Multilinear Algebra **72** (2024), no. 13, 2248–2266. MR 4793051

31. E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. **67** (1961), 97–98. MR 123174

32. R. D. Bleier, *Free vector lattices*, Trans. Amer. Math. Soc. **176** (1973), 73–87. MR 311541

33. F. Bohnenblust, *A characterization of complex Hilbert spaces*, Portugal. Math. **3** (1942), 103–109. MR 8117

34. J. Bourgain, *A counterexample to a complementation problem*, Compositio Math. **43** (1981), no. 1, 133–144. MR 631431

35. ______, *Real isomorphic complex Banach spaces need not be complex isomorphic*, Proc. Amer. Math. Soc. **96** (1986), no. 2, 221–226. MR 818448

36. J. Bourgain and F. Delbaen, *A class of special $\mathcal{L}_\infty$ spaces*, Acta Math. **145** (1980), no. 3-4, 155–176. MR 590288

37. J. Bourgain, H. P. Rosenthal, and G. Schechtman, *An ordinal L^p-index for Banach spaces, with application to complemented subspaces of L^p*, Ann. of Math. (2) **114** (1981), no. 2, 193–228. MR 632839

38. A. V. Bukhvalov, A. I. Veksler, and G. Ya. Lozanovskii, *Banach lattices – some Banach aspects of their theory*, Russian Math. Surveys **34** (1979), no. 2, 159–212. MR 535711

39. M. Camúñez, E. García-Sánchez, and D. de Hevia, *A characterization of complex stable phase retrieval in Banach lattices*, Preprint available on arXiv (2025), 14 pp.

40. P. G. Casazza, N. J. Kalton, and L. Tzafriri, *Decompositions of Banach lattices into direct sums*, Trans. Amer. Math. Soc. **304** (1987), no. 2, 771–800. MR 911095

41. W. Cuellar Carrera, *A Banach space with a countable infinite number of complex structures*, J. Funct. Anal. **267** (2014), no. 5, 1462–1487. MR 3229797

42. S. Dantas, G. Martínez-Cervantes, J. D. Rodríguez Abellán, and A. Rueda Zoca, *Norm-attaining lattice homomorphisms*, Rev. Mat. Iberoam. **38** (2022), no. 3, 981–1002. MR 4413761

43. R. Deville, G. Godefroy, and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 64, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1993. MR 1211634

44. J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, Cambridge Studies in Advanced Mathematics, vol. 43, Cambridge University Press, Cambridge, 1995. MR 1342297

45. R. G. Douglas, *Contractive projections on an L_1 space*, Pacific J. Math. **15** (1965), 443–462. MR 187087

46. L. Drewnowski, A. Kamińska, and P.-K. Lin, *On multipliers and L- and M-projections in Banach lattices and Köthe function spaces*, J. Math. Anal. Appl. **206** (1997), no. 1, 83–102. MR 1429280

47. E. Dubinsky, A. Pełczyński, and H. P. Rosenthal, *On Banach spaces X for which $\Pi_2(\mathcal{L}_\infty, X) = B(\mathcal{L}_\infty, X)$*, Studia Math. **44** (1972), 617–648. MR 365097

48. N. Dunford and J. T. Schwartz, *Linear operators. Part I*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1988, General theory, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original, A Wiley-Interscience Publication. MR 1009162

49. I. S. Edelstein and P. Wojtaszczyk, *On projections and unconditional bases in direct sums of Banach spaces*, Studia Math. **56** (1976), no. 3, 263–276. MR 425585

50. P. Enflo and T. W. Starbird, *Subspaces of L^1 containing L^1*, Studia Math. **65** (1979), no. 2, 203–225. MR 557491

51. M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant, and V. Zizler, *Functional analysis and infinite-dimensional geometry*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, vol. 8, Springer-Verlag, New York, 2001. MR 1831176

52. V. Ferenczi, *Uniqueness of complex structure and real hereditarily indecomposable Banach spaces*, Adv. Math. **213** (2007), no. 1, 462–488. MR 2331251

53. J. R. Fienup, *Phase retrieval algorithms: a comparison*, Appl. Opt. **21** (1982), no. 15, 2758–2769.

54. T. Figiel, N. Ghoussoub, and W. B. Johnson, *On the structure of nonweakly compact operators on Banach lattices*, Math. Ann. **257** (1981), no. 3, 317–334. MR 637954

55. T. Figiel, W. B. Johnson, and L. Tzafriri, *On Banach lattices and spaces having local unconditional structure, with applications to Lorentz function spaces*, J. Approximation Theory **13** (1975), 395–412. MR 367624

56. P. Flinn, *On a theorem of N. J. Kalton and G. V. Wood concerning* 1*-complemented subspaces of spaces having an orthonormal basis*, Texas functional analysis seminar 1983–1984 (Austin, Tex.), Longhorn Notes, Univ. Texas Press, Austin, TX, 1984, pp. 135–144. MR 832237

57. J. Flores, F. L. Hernández, E. Spinu, P. Tradacete, and V. G. Troitsky, *Disjointly homogeneous Banach lattices: duality and complementation*, J. Funct. Anal. **266** (2014), no. 9, 5858–5885. MR 3182963

58. D. Freeman, T. Oikhberg, B. Pineau, and M. A. Taylor, *Stable phase retrieval in function spaces*, Math. Ann. **390** (2024), no. 1, 1–43. MR 4800909

59. E. García-Sánchez, D. de Hevia, and P. Tradacete, *Free objects in Banach space theory*, Cutting-edge mathematics, RSME Springer Ser., vol. 13, Springer, Cham, [2024] ©2024, pp. 100–124. MR 4823784

60. E. García-Sánchez and D. de Hevia, *Subspaces of $C(K)$-spaces doing stable phase retrieval*, in preparation.

61. N. Ghoussoub, *On spaces with local unconditional structure*, Seminar on Functional Analysis, 1979–1980 (French), École Polytech., Palaiseau, 1980, pp. Exp. No. 24, 8. MR 604405

62. J. Glück and P. Krokor, personal communication (April 2, 2025).

63. G. Godefroy and N. J. Kalton, *Lipschitz-free Banach spaces*, vol. 159, 2003, Dedicated to Professor Aleksander Pełczyński on the occasion of his 70th birthday, pp. 121–141. MR 2030906

64. Y. Gordon and D. R. Lewis, *Absolutely summing operators and local unconditional structures*, Acta Math. **133** (1974), 27–48. MR 410341

65. A. Goullet de Rugy, *La structure idéale des M-espaces*, J. Math. Pures Appl. (9) **51** (1972), 331–373. MR 385512

66. W. T. Gowers, *A solution to Banach's hyperplane problem*, Bull. London Math. Soc. **26** (1994), no. 6, 523–530. MR 1315601

67. W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. **6** (1993), no. 4, 851–874. MR 1201238

68. S. Heinrich, C. W. Henson, and L. C. Moore, Jr., *Elementary equivalence of L_1-preduals*, Banach space theory and its applications (Bucharest, 1981), Lecture Notes in Math., vol. 991, Springer, Berlin-New York, 1983, pp. 79–90. MR 714174

69. S. Heinrich, C. Ward Henson, and L. C. Moore, Jr., *Elementary equivalence of $C_\sigma(K)$ spaces for totally disconnected, compact Hausdorff K*, J. Symbolic Logic **51** (1986), no. 1, 135–146. MR 830080

70. D. de Hevia, G. Martínez-Cervantes, A. Salguero-Alarcón, and P. Tradacete, *A negative solution to the complemented subspace problem for Banach lattices*, Preprint available on arXiv (2025), 25 pp.

71. D. de Hevia and P. Tradacete, *Free complex Banach lattices*, J. Funct. Anal. **284** (2023), no. 10, Paper No. 109888, 26. MR 4552375

72. ———, *Complemented subspaces of Banach lattices*, Banach J. Math. Anal. **19** (2025), no. 4, Paper No. 60. MR 4940175

73. D. Ilišević and A. Turnšek, *On Wigner's theorem in smooth normed spaces*, Aequationes Math. **94** (2020), no. 6, 1257–1267. MR 4171843

74. R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. (2) **52** (1950), 518–527. MR 39915

75. ———, *A non-reflexive Banach space isometric with its second conjugate space*, Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 174–177. MR 44024

76. ———, *Characterizations of reflexivity*, Studia Math. **23** (1963/64), 205–216. MR 170192

77. H. Jardón-Sánchez, N. J. Laustsen, M. A. Taylor, P. Tradacete, and V. G. Troitsky, *Free Banach lattices under convexity conditions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **116** (2022), no. 1, Paper No. 15, 25. MR 4326041

78. W. B. Johnson, *On finite dimensional subspaces of Banach spaces with local unconditional structure*, Studia Math. **51** (1974), 225–240. MR 358306

79. W. B. Johnson and J. Lindenstrauss, *Some remarks on weakly compactly generated Banach spaces*, Israel J. Math. **17** (1974), 219–230. MR 417760

80. ———, *Examples of $\mathcal{L}_1$ spaces*, Ark. Mat. **18** (1980), no. 1, 101–106. MR 608329

81. W. B. Johnson, J. Lindenstrauss, and G. Schechtman, *On the relation between several notions of unconditional structure*, Israel J. Math. **37** (1980), no. 1-2, 120–129. MR 599307

82. W. B. Johnson and L. Tzafriri, *On the local structure of subspaces of Banach lattices*, Israel J. Math. **20** (1975), no. 3-4, 292–299. MR 420210

83. ______, *Some more Banach spaces which do not have local unconditional structure*, Houston J. Math. **3** (1977), no. 1, 55–60. MR 430753

84. S. Kakutani, *Concrete representation of abstract (M)-spaces. (A characterization of the space of continuous functions.)*, Ann. of Math. (2) **42** (1941), 994–1024. MR 5778

85. N. J. Kalton, *An elementary example of a Banach space not isomorphic to its complex conjugate*, Canad. Math. Bull. **38** (1995), no. 2, 218–222. MR 1335101

86. ______, *The complemented subspace problem revisited*, Studia Math. **188** (2008), no. 3, 223–257. MR 2429822

87. ______, *Hermitian operators on complex Banach lattices and a problem of Garth Dales*, J. Lond. Math. Soc. (2) **86** (2012), no. 3, 641–656. MR 3000824

88. N. J. Kalton and N. T. Peck, *Twisted sums of sequence spaces and the three space problem*, Trans. Amer. Math. Soc. **255** (1979), 1–30. MR 542869

89. N. J. Kalton and G. V. Wood, *Orthonormal systems in Banach spaces and their applications*, Math. Proc. Cambridge Philos. Soc. **79** (1976), no. 3, 493–510. MR 402471

90. P. Koszmider, *Banach spaces of continuous functions with few operators*, Math. Ann. **330** (2004), no. 1, 151–183. MR 2091683

91. ______, *A survey on Banach spaces $C(K)$ with few operators*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **104** (2010), no. 2, 309–326. MR 2757243

92. P. Koszmider and N. J. Laustsen, *A Banach space induced by an almost disjoint family, admitting only few operators and decompositions*, Adv. Math. **381** (2021), Paper No. 107613, 39. MR 4206788

93. J.-L. Krivine, *Constantes de Grothendieck et fonctions de type positif sur les sphères*, Adv. in Math. **31** (1979), no. 1, 16–30. MR 521464

94. H. E. Lacey, *The isometric theory of classical Banach spaces*, Die Grundlehren der mathematischen Wissenschaften, vol. Band 208, Springer-Verlag, New York-Heidelberg, 1974. MR 493279

95. ______, *Local unconditional structure in Banach spaces*, Banach spaces of analytic functions (Proc. Pelczynski Conf., Kent State Univ., Kent, Ohio, 1976), Lecture Notes in Math., vol. Vol. 604, Springer, Berlin-New York, 1977, pp. 44–56. MR 454596

96. H. E. Lacey and P. Wojtaszczyk, *Nonatomic Banach lattices can have l_1 as a dual space*, Proc. Amer. Math. Soc. **57** (1976), no. 1, 79–84. MR 402459

97. D. R. Lewis, *Ellipsoids defined by Banach ideal norms*, Mathematika **26** (1979), no. 1, 18–29. MR 557122

98. J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. **48** (1964), 112. MR 179580

99. ______, *On complemented subspaces of m*, Israel J. Math. **5** (1967), 153–156. MR 222616

100. ______, *Decomposition of Banach spaces*, Indiana Univ. Math. J. **20** (1971), no. 10, 917–919. MR 405074

101. ______, *The geometric theory of the classical Banach spaces*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, Gauthier-Villars Éditeur, Paris, 1971, pp. 365–372. MR 425583

102. ______, *A remark on symmetric bases*, Israel J. Math. **13** (1972), 317–320. MR 331024

103. J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in L_p-spaces and their applications*, Studia Math. **29** (1968), 275–326. MR 231188

104. ______, *Contributions to the theory of the classical Banach spaces*, J. Functional Analysis **8** (1971), 225–249. MR 291772

105. J. Lindenstrauss and H. P. Rosenthal, *The $\mathcal{L}_p$ spaces*, Israel J. Math. **7** (1969), 325–349. MR 270119

106. J. Lindenstrauss and L. Tzafriri, *On the complemented subspaces problem*, Israel J. Math. **9** (1971), 263–269. MR 276734

107. ———, *Classical Banach spaces*, Lecture Notes in Mathematics, vol. Vol. 338, Springer-Verlag, Berlin-New York, 1973. MR 415253

108. ———, *Classical Banach spaces. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. Band 92, Springer-Verlag, Berlin-New York, 1977, Sequence spaces. MR 500056

109. ———, *Classical Banach spaces. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 97, Springer-Verlag, Berlin-New York, 1979, Function spaces. MR 540367

110. J. Lindenstrauss and D. E. Wulbert, *On the classification of the Banach spaces whose duals are L_1 spaces*, J. Functional Analysis **4** (1969), 332–349. MR 250033

111. H. P. Lotz, *Extensions and liftings of positive linear mappings on Banach lattices*, Trans. Amer. Math. Soc. **211** (1975), 85–100. MR 383141

112. J. L. P. Luiz and V. C. C. Miranda, *On positive norm-attaining operators between Banach lattices*, Preprint available on arXiv (2024), 15 pp.

113. W. Marciszewski and G. Plebanek, *Extension operators and twisted sums of c_0 and $C(K)$ spaces*, J. Funct. Anal. **274** (2018), no. 5, 1491–1529. MR 3778681

114. B. Maurey, *Sous-espaces complémentés de L^p, d'après P. Enflo*, Séminaire Maurey-Schwartz 1974–1975: Espaces L^p, applications radonifiantes et géométrie des espaces de Banach, École Polytech., Paris, 1975, pp. Exp. No. III, 15 pp. (erratum, p. 1). MR 397386

115. P. Meyer-Nieberg, *Banach lattices*, Universitext, Springer-Verlag, Berlin, 1991. MR 1128093

116. R. P. Millane, *Phase retrieval in crystallography and optics*, J. Opt. Soc. Am. A **7** (1990), no. 3, 394–411.

117. G. Mittelmeyer and M. Wolff, *über den Absolutbetrag auf komplexen Vektorverbänden*, Math. Z. **137** (1974), 87–92. MR 348441

118. T. Oikhberg, *Geometry of unit balls of free Banach lattices, and its applications*, J. Funct. Anal. **286** (2024), no. 8, Paper No. 110351, 27. MR 4703369

119. T. Oikhberg, M. A. Taylor, P. Tradacete, and V. G. Troitsky, *Free banach lattices*, J. Eur. Math. Soc. (JEMS) (2024), 128 pp.

120. T. Oikhberg and M. A. Tursi, *Order extreme points and solid convex hulls*, The mathematical legacy of Victor Lomonosov—operator theory, Adv. Anal. Geom., vol. 2, De Gruyter, Berlin, 2020, pp. 297–315. MR 4312044

121. B. de Pagter and A. W. Wickstead, *Free and projective Banach lattices*, Proc. Roy. Soc. Edinburgh Sect. A **145** (2015), no. 1, 105–143. MR 3304578

122. A. Pełczyński, *Projections in certain Banach spaces*, Studia Math. **19** (1960), 209–228. MR 126145

123. ———, *Sur certaines propriétés isomorphiques nouvelles des espaces de Banach de fonctions holomorphes A et H^∞*, C. R. Acad. Sci. Paris Sér. A **279** (1974), 9–12. MR 365106

124. A. Pełczyński and M. Wojciechowski, *Sobolev spaces*, Handbook of the geometry of Banach spaces, Vol. 2, North-Holland, Amsterdam, 2003, pp. 1361–1423. MR 1999199

125. ———, *Spaces of functions with bounded variation and Sobolev spaces without local unconditional structure*, J. Reine Angew. Math. **558** (2003), 109–157. MR 1979184

126. R. S. Phillips, *A characterization of Euclidean spaces*, Bull. Amer. Math. Soc. **46** (1940), 930–933. MR 3444

127. ———, *On linear transformations*, Trans. Amer. Math. Soc. **48** (1940), 516–541. MR 4094

128. G. Pisier, *Counterexamples to a conjecture of Grothendieck*, Acta Math. **151** (1983), no. 3-4, 181–208. MR 723009

129. G. Plebanek, *A construction of a Banach space $C(K)$ with few operators*, Topology Appl. **143** (2004), no. 1-3, 217–239. MR 2081013

130. G. Plebanek and A. Salguero Alarcón, *On the three space property for $C(K)$-spaces*, J. Funct. Anal. **281** (2021), no. 9, Paper No. 109193, 15. MR 4291511

131. ———, *The complemented subspace problem for $C(K)$-spaces: a counterexample*, Adv. Math. **426** (2023), Paper No. 109103, 20. MR 4592269

132. A. M. Plichko and D. Yost, *Complemented and uncomplemented subspaces of Banach spaces*, vol. 15, 2000, III Congress on Banach Spaces (Jarandilla de la Vera, 1998), pp. 335–371. MR 1823896

133. B. Randrianantoanina, *Norm-one projections in Banach spaces*, Taiwanese J. Math. **5** (2001), no. 1, 35–95, International Conference on Mathematical Analysis and its Applications (Kaohsiung, 2000). MR 1816130

134. Y. Raynaud, *The range of a contractive projection in $L_p(H)$*, Rev. Mat. Complut. **17** (2004), no. 2, 485–512. MR 2083967

135. ———, *New axiomatizable classes of Banach spaces via disjointness-preserving isometries*, Positivity **22** (2018), no. 1, 301–339. MR 3764645

136. H. P. Rosenthal, *On factors of $C([0,1])$ with non-separable dual*, Israel J. Math. **13** (1972), 361–378 (1973); correction, ibid. 21 (1975), no. 1, 93–94. MR 388063

137. ———, *On one-complemented subspaces of complex Banach spaces with a one-unconditional basis, according to Kalton and Wood*, Israel seminar on geometrical aspects of functional analysis (1983/84), Tel Aviv Univ., Tel Aviv, 1984, pp. IX, 16. MR 827285

138. ———, *The Banach spaces $C(K)$*, Handbook of the geometry of Banach spaces, Vol. 2, North-Holland, Amsterdam, 2003, pp. 1547–1602. MR 1999603

139. W. Rudin, *Functional analysis*, second ed., International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991. MR 1157815

140. C. Samuel, *Sur certains espaces $C_\sigma(S)$ et sur les sous-espaces complémentés de $C(S)$*, Bull. Sci. Math. (2) **95** (1971), 65–82. MR 285898

141. H. H. Schaefer, *Banach lattices and positive operators*, Die Grundlehren der mathematischen Wissenschaften, vol. Band 215, Springer-Verlag, New York-Heidelberg, 1974. MR 423039

142. Z. Semadeni, *Banach spaces of continuous functions. Vol. I*, Monografie Matematyczne [Mathematical Monographs], vol. Tom 55, PWN—Polish Scientific Publishers, Warsaw, 1971. MR 296671

143. Y. Shechtman, Y. C. Eldar, O. Cohen, H. N. Chapman, J. Miao, and M. Segev, *Phase retrieval with application to optical imaging: A contemporary overview*, IEEE Signal Processing Magazine **32** (2015), no. 3, 87–109.

144. I. Singer, *Bases in Banach spaces. I*, Die Grundlehren der mathematischen Wissenschaften, vol. Band 154, Springer-Verlag, New York-Berlin, 1970. MR 298399

145. A. Sobczyk, *Projection of the space (m) on its subspace (c_0)*, Bull. Amer. Math. Soc. **47** (1941), 938–947. MR 5777

146. ———, *Projections in Minkowski and Banach spaces*, Duke Math. J. **8** (1941), 78–106. MR 3443

147. A. Szankowski, *A Banach lattice without the approximation property*, Israel J. Math. **24** (1976), no. 3-4, 329–337. MR 420231

148. M. A. Taylor and V. G. Troitsky, *Bibasic sequences in Banach lattices*, J. Funct. Anal. **278** (2020), no. 10, 108448, 33. MR 4067989

149. N. Tomczak-Jaegermann, *Banach-Mazur distances and finite-dimensional operator ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 38, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989. MR 993774

150. V. G. Troitsky, *Simple constructions of* FBL(A) *and* FBL[E], Positivity **23** (2019), no. 5, 1173–1178. MR 4011244

151. A. Tselishchev, *Absence of local unconditional structure in spaces of smooth functions on the two-dimensional torus*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **491** (2020), 153–172. MR 4223668

152. L. Tzafriri, *Remarks on contractive projections in L_p-spaces*, Israel J. Math. **7** (1969), 9–15. MR 248514

153. ______, *Reflexivity in Banach lattices and their subspaces*, J. Functional Analysis **10** (1972), 1–18. MR 358303

154. D. Vuza, *Sur les espaces vectoriels réticulés complexes*, Rev. Roumaine Math. Pures Appl. **25** (1980), no. 4, 663–674. MR 577057

155. S. Willard, *General topology*, Dover Publications, Inc., Mineola, NY, 2004, Reprint of the 1970 original [Addison-Wesley, Reading, MA; MR0264581]. MR 2048350

156. P. Wojtaszczyk, *On projections and unconditional bases in direct sums of Banach spaces. II*, Studia Math. **62** (1978), no. 2, 193–201. MR 500084