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CHAOS AND FRACTAL IMPACT ON ECONOMICS

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Abstract:
Complexity is one of the most important characteristic properties of the economic behaviour. The new field of knowledge called Chaotic Dynamic Economics born precisely with the objective of understanding, structuring and explaining in an endogenous way such complexity. In this paper, and after scanning the principal concepts and techniques about the mathematics of chaos and the fractal geometry, we analyze the possibilities of application of these valuable tools to the different areas of Economics. After a set of general considerations, within a wide landscape, we tackle the study of two main economic areas: the existence of chaos in time series, and the eventual chaotic behaviour in the Capital Markets during the recent period of crisis, considering the IBEX 35 daily data in the years 2006-2013, obtaining very interesting results.

Keywords: Complexity, chaos, fractal, non-linearity, correlation dimension, fractal dimension, Lyapunov exponents, time series, capital markets, predictions.

EL IMPACTO EN LA CIENCIA ECONÓMICA DE LA TEORÍA DEL CAOS Y DE LOS FRACTALES: Teoría y aplicaciones

Resumen:
La complejidad constituye una de las características más importantes del comportamiento económico. El nuevo campo de conocimiento denominado Dinámica Caótica en Economía nació precisamente con el objetivo de comprender, estructurar y explicar de manera endógena tal complejidad. En este artículo, y tras exponer los principales conceptos y técnicas de las matemáticas del caos y de la geometría fractal, se analizan las posibilidades de aplicación de estos importantes instrumentos a las diferentes áreas de la Economía. Después de un conjunto general de consideraciones abordamos el estudio de dos áreas concretas: la existencia de caos en las series temporales, y el eventual comportamiento caótico del Mercado de Capitales durante los años de la crisis recientemente padecida, tomando los datos diarios del IBEX 35, obteniéndose resultados muy relevadores.

Palabras clave:
Complejidad, Caos, Fractales, No-linealidad, Dimensión de correlación, Dimensión fractal, Exponente de Lyapunov, Series temporales, Mercado de capitales, Predicción.

Materia: Modelos matemáticos
JEL: C02, C22.

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Introduction

Since the last half of the 20th century the concepts and ideas of chaos and fractals have had developed a very important impact on a set of scientific disciplines, such as astrophysics, general relativity and cosmology, atomic and molecular dynamics, fluid dynamics, plasma physics, etc. But at the same time they have affected many other fields as well, which in principle seem are far away from physics, such as economics or ecology, which have also turned out to be fruitful environments where fractal structures appear naturally. We are going to approach how much significant has been the impact on Economics, but before it is necessary to deal with the concepts, content and extent of chaos and fractals.[1]

Perhaps the most clear and ancient definition of chaos can be found in the words of Giordano Bruno written in his book published in 1583 in Venezia whit the title "De l´infinito universo mondi*": Now more than ever I perceive that a tiny error in the beginning causes a big difference and a serious deviation at the end; a single problem was multiplied gradually branching out into an infinite number of other, just as a root spreads in infinite branches and masses.* Also, centuries later, in 1908, appears the G. K, Chesterton´s celebrated novel "The Man Who was Thursday", and in the page 12 said: *Why do all the clerks and navvies in the railway trains look so sad and tired, so very sad and tired?.I will tell you. It is because they know that the train is going right. It is because they know that whatever place they have taken a ticket for, that place they will reach. It is because after they have passed Sloane Square they know that the next station must be Victoria, and nothing but Victoria. Oh, their wild rapture! Oh, their eyes like stars and their souls again in Eden, if the next station were unaccountably Baker Street.* These two relevant references are indeed very clear and rich in ideas about chaos, and it is an excellent starting point for our analysis that will be related, obviously, to complexity, given that it means diversity, creativeness, interactions at different levels and the possibility of emergent properties.

The paradigm of the complexity constitutes an obligatory reference to the scientific analysis on the verge of the XXI century, and this is especially true in the field of Economics, which is accustomed to navigate against the tide, and subject to a continuous dispersion, such as the one we had the occasion to emphasize in a paper published some years ago.[2] The appeal of the complexity, in the case of Economics,

[1] AGUIRRE, Jacobo; VIANA, Ricardo L. and SANJUÁN, Miguel A.F. (2009): p.381.
[2] FERNÁNDEZ DÍAZ, Andrés (2000): pp. 39-44.
There is an up-to-date edition:
FERNÁNDEZ DÍAZ, Andrés and GRAU-CARLES, Pilar (2014).

warrants its interpretation as a healthy exercise, worthy of defense before the reductionist determinism professed by our science throughout its history, succumbing all too often to the aesthetic pleasure of simplicity. Should it not be clear, and with a view to taking a stand, it should be stated, with no undue delay, that determinism is pernicious: when we are distracted we deliberately skip the complexity.

We obviously do not seek the disqualification of determinism in radical support of decisive indeterminism, nor enter into the protracted and inconclusive polemics set forth all around it. What really and powerfully attracts the attention is the fact that a natural science, such as Physics, may have spectacularly advanced, which supposes its quantum revolution, through basis on the Heisenberg's uncertainty principle and the well known interpretation of Copenhagen, due to Niels Bohr, in as much as, it reigns over a rigid determinism in the most complex sphere of the spirit, of the culture and society, or, tantamount to the same, of a social science such as Economics.

With a view to entrenching the question momentarily, and without hazard of remission to other more specific and detailed works on the matter, determinism could be understood as an abstraction and simplification, to make everyday complexity intelligible, considering, for its part, indeterminism as a consequence of our inability to explain complication, due to the fact that we do not avail of sufficient information. From this point of view, indeterminism would then be a clear consequence of complexity.

Below the broad parasol of complexity, we find ourselves in a fascinating world of concepts, terms and instruments, bustling, intertwined, and opening new horizons in almost all fields of knowledge. Dynamics, non-linearity, irregularity, order and chaos are only some of them, behaving as parts of an indivisible whole. Chaos, habitually considered a subset of complexity, constitutes, without doubt, one of the key pieces of the process, and due to this, we can speak within our scientific field of *Chaotic Dynamic Economics*.

But, can Economics be set forth and understood in terms of complexity?. Evidently, it can and should be done, especially when dealing with an empirical science situated within the scope or group of *socials*, as we have already seen, and if it is taken into account that complexity is consubstantial and ubiquitous. If we remember the three types of complexity considered by Henri Atlan in his work published in 1991, it could be easily proven that Economics exhibits or presents all of them, both quantitative as well as the essentially qualitative type. In effect, besides the probabilistic natural complexity, the most proper and direct, there is the algorithmic complexity, for instance, in the example of computer processed models of equilibrium, such as those carried out by Scarf in his well known work on matter. Likewise, one might undoubtedly speak about complexity in the appreciation of Economics, which, on the other hand, involves recognizing and admitting the existence of a subjective indeterminism in the sense already brought to hand. To all this, it is necessary to add a type of strict definition of complexity, which is habitually used in Economics in the more recent works of specialists. In them, the term *complex* is used, to refer to those cases in which dynamic long-term behaviour is more complicated than a fixed point, a cycle limit, or a torus; or tantamount to the same, when chaotic behaviour is produced.

Important lines or sub-headings of the Economic Analysis and Economic Policy fall full and can be included in the scope of the Economics of Complexity. In these scopes, subjects referent to monetary dynamics or keynesian dynamics are undertaken, problems of inflation and unemployment, the determinants of endogenous cycles, growth and distribution models, the development of exchange rates, the existence of chaos in capital markets, or the non-linearity and chaos in time series. In all of them, one tries to count on new approaches and methods that allow an analysis closer to the truth or reality, and to the intrinsic and inevitably dynamic nature of economic phenomena.

Within the specific framework of financial markets there are many studies that test for non-linear dependence on daily stock indices, or searching for evidence of chaos in the future prices of commodities. It is also very important to search for the implications of non-linear dynamics for reasons of financial risk management; the chaotic behaviour in exchange-rate series, or nonlinearities and chaotic effects in option prices. Finally, non-linear modelling with neural networks offers a very interesting and efficient approach for studying the prediction of chaotic time series, and has been utilized successfully in different branches and problems of Economics.[3]

It is necessary to remember that the major problem in time series research is the difficulty of distinguishing between deterministic chaos and a purely random process, taking into consideration that the most important characteristic of chaotic dynamical systems is their short-term predictability. Chaos is at the same time disorder and determinism. Chaos, in principle, due that is apparently disordered, make non predictable its evolution. But, on the other hand, being deterministic, and governed by systems of non-linear equations, it should be possible to predict and control once you know the mathematical relationships of the variables that influence it. As said Henri Poincaré is much better to look farther without having certainty, that don't look anything at all. Because of this we must to undertake the analysis of the main concepts, techniques and mathematics of chaos, that is, of all the weapons we need to know and to deal with an irregular and complex reality, as already we have pointed out.

Before of going on, however, it is necessary to know that complexity and chaos are intimately related to the concept of emergence. What does emergence means?. Taking into account the evolution approach, we can think that emergent evolution may be interpreted as an incessant flow of creative novelty, which implicate a special conception of the whole and the parts, farther away the simple and lineal idea of an additive process.[4] In reality there is a process of emergence when the behaviour of the

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[3] TRIPPI, Robert (1995) : pp. 467-486.
[4] FRENÁNDEZ DÍAZ, Andrés (1999_a): pp. 139-145.

overall system cannot be obtained by summing the behaviours of its constituent parts. That is, the whole is indeed more than the sum of its parts.

The habitual definition of chaos, which hallmark is the sensitive dependence on initial conditions, implies that there is no information within chaos, and it has neither form nor structure. For us, chaos may be complex and appear to be non-deterministic, but hidden within it is a wealth of information. If in an emergent phenomenon there is also some hidden information, given that, as we have seen, the whole became something more than the sum of its components, seems clear, first, the closed relation between chaos and emergence, and secondly, the help that the last one can render to the predictability of chaos. We must remember this very important consideration in the conclusions of this work.

Mathematics of Chaos and Fractal Geometry

The characteristics of irregularity and non-linearity are, among others, derived from the complexity of economic behaviour, which oblige, as stated at the beginning, the utilization of concepts and new instruments especially conceived to face challenges, which today arise within Economics, and of course, in other fields of knowledge. Amongst them, the Theory of Catastrophes, and very especially, the Mathematics of Chaos stand out.

The majority of authors coincide in as much as the Theory of Catastrophes and the Mathematics of Chaos can be considered as two approaches to a general theory of dynamics of discontinuities. Both have in common as a base the idea of a splitting or halving of the equilibrium at critical points, just as the fact that functional relation are, with greater frequency, of the nonlinear type. But they differ in as much as some discontinuities are set forth on a great scale: the Theory of Catastrophes, and others on a small scale: the Mathematics of Chaos. The Theory of Catastrophes is therefore a special case of the bifurcation theory accredited originally to Poincaré, which contemplates the world as essentially uniform and stable yet subject to sudden changes, the unexpected, or discontinuities on a grand scale which are produced in certain variables of state.

It is well known that the starting point of the Theory of Catastrophes can be found in the works of René Thom and Christopher Zeeman, at the end of the sixties and the beginning of the seventies. On other occasions we have, and at certain length, taken to hand this new mathematical method, in order to describe the evolution of forms in nature, by hazarding even some economic applications, and concretely, the problem of stagflation[5]. We shall not go into this any further. Instead, we shall center our attention on Chaos and its measurement, which has greater relevancy for the purposes of our analysis.

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[5] FERNÁNDEZ DÍAZ, Andrés (1999b): pp. 45-48.

It is often said that chaos is a ubiquitous phenomenon which is produced everywhere and can be observed in all fields of Science. Thus we find chaotic systems in the Hamiltonians, in the three bodies of celestial mechanics, in the physics of fluids, in lasers, in particle accelerators, in biological systems, in chemical reactions, and as we shall soon see, in no small part of the behavioural forms within the field of Economics.

Chaos can be located through the function of ***strange attractors***, by following bifurcation diagrams, or by analyzing the intricate profile of figures of fractal geometry. It should not be forgotten, in this respect, that, as Giambattista Vico said, chaos is *the raw material of natural things that, shapeless, is thirsty for form, and devours all*. We know that the essential geometrics of chaos consist of stretching and bending, as pointed out by Stephen Smale in his topologic transformations. In effect, the irregularity of movement is produced by a mechanism which is broken down into two actions. On the one hand, the spatial phase is stretched, by separating trajectories, and then it doubles back onto itself.

The exponents of Lyapunov serve to explain the first part of this process, on giving a measurement on the exponential separation of two adjacent trajectories. We can study local instability of a discrete system $x_{n+1} = f(x_n)$ in the Lyaponov sense measuring how two adjacent points separate with the iterated application of the function, that is,

$$|f^n(\mathrm{x}_0 + \epsilon) - f^n(\mathrm{x}_0)| = \epsilon^{n\lambda(x_0)} \qquad (1)$$

The limit

$$\lambda(x_0) = \lim_{n\to\infty} \lim_{\epsilon\to 0} \frac{1}{n} \log \left|\frac{f^n(x_0+\epsilon)-f^n(x_0)}{\epsilon}\right| \qquad (2)$$

or also:

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \log \left|\frac{df^n(x_0)}{dx_0}\right| \qquad (3)$$

are both (2) and (3) path expressions of the Lyapunov exponent. In a general n-dimensional system there will be *n* Lyapunov exponents, showing each of them the average rate of expansion or contraction of the phase space in each of its *n* direction under the action of the dynamical system.

These Lyapunov exponents can be used to distinguish simple dynamic attractors from complex dynamics attractors, that is, those that we could call existing traditional attractors until the Lorenz contribution in 1963, *fixed points*, *limit cycles* and *quasi-periodic torus*, from chaotic or strange attractors.

We can say that the dynamics inside fixed points, limit cycles or torus is a simple dynamics in the sense that two orbits that start arbitrary near each other, always remain adjacent, thus providing a guaranteed predictability on a long term basis. However, inside the strange or chaotic attractor the dynamical system is complex in the sense that although bounded inside the attractor is also locally instable, with recurrent but aperiodic cycles, and with sensitive dependence to initial conditions making predictions difficult beyond the very short term. Therefore “simple dynamics” and “complex dynamics”, can be detected by means of the Lyapunov exponent. More concretely, a positive Lyapunov exponent, i.e. local instability, is a necessary condition for the existence of chaos.

If we are situated in a dimension, we have only stable fixed points with the exponent λ negative. In the case of two dimensions, the attractors would be fixed points, with negative exponents, and limit cycles with $(\lambda_1, \lambda_2)=(-,0)$.

In three dimensions one would have:

$(\lambda_1, \lambda_2, \lambda_3) = (-,-,-)$ → (stable fixed point)

$(\lambda_1, \lambda_2, \lambda_3) = (0,-,-)$ → (stable limit cycles))

$(\lambda_1, \lambda_2, \lambda_3) = (-,0,0)$ → (stable torus)

$(\lambda_1, \lambda_2, \lambda_3) = (+,0,-)$ → (strange attractor)

There are many examples of strange attractors in specialized literature, beginning with Stephen Smale, who supplies through a topological transformation a base for the understanding of chaotic properties of dynamic systems. Among the stated attractors, the Lorenz attractor stands out, which takes on the form known as the wings of the butterfly, and in the one that borders on an important problem for meteorology, the one about atmospheric convection, consisting of the evolution of a layer of fluid heated from below. To this emblematic attractor, other examples should be added, such as the not less known one of Rössler, and the one by Hénon, which has an elegant structure and is quite complex. But we cannot speak of strange attractors without entering into the attractive and fascinating field of fractals.

The *chaotic attractors* are fractals. Fractals are geometric objects which have a beautiful microscopic structure that have been developed within the framework of a new form of the geometry of nature or of the complexity created by Benoît Mandelbrot, who in 1975 published his famous work called ***The Fractal Geometry of Nature***. He was educated at "Ecole Normale" and the "Normale Polytechnique" and, with his original formulation, intended to confront the unbounded formality of the Bourbaki group, thereby reestablishing the image and prestige of Henri Poincaré.

Some authors affirm very often that scientists know a fractal when they see one, but there is not universally accepted definition. It is generally acknowledge that fractals have some or all of the following properties: complicated structure at a wide range of length scales, repetition of structures at different length scales, and a fractal dimension that is not an integer.[6]

It is necessary to point out that fractality is ubiquitous in nature, as has been observed since the relation between fractality and nonlinear dynamics was established. In the context of dissipative systems, examples of fractal behaviour are numerous, noting the appearance of fractal basins in a wide variety of nonlinear oscillators such as the Duffing one or the forced damped pendulum, and multispecies competition or predator-prey models.[7]

The fractal concept involves a new idea of dimension beyond the Euclidian one, given that it dealt with the fractal or intermediate dimensions that come, in essence, into prominence if it is considered that the system does not occupy all the space that corresponds to its Euclidean dimension. In effect, the fact that the dimension may be inferior to the number of parameters or degrees of freedom, necessary to completely specify the state of the system considered, signifies that it does neither exploit all the possibilities, nor all the states theoretically possible.

The fractal should be understood as a geometric form which remains unaltered, whatever the increase in which it is observed. It could be said that, within reasonable, the fractal has the same structure on all scales, the contrary to what occurs in the phenomenon of the renormalization in which the figures notably alter when they are modified or vary.

The fundamental problem of fractals lies in knowing their dimension, which does not necessarily need to be an integer, as we have already noted. Originally, the numeric measurement of the degree of rigorousness was denominated as the Hausdorff-Besicovitch dimension; today it is called the fractal dimension. In general terms, the dimension is a measure of the occupation of space by a geometric object. Normal non-fractal objects have a Hausdorff dimension equal to its topological dimension. However, fractals have a Hausdorff dimension strictly greater than its topological dimension

One of the methods employed for carrying out the measurement of fractals is based on the concept of homotecia in euclidian geometry, which also allows the calculation of fractal dimension through basis on the concept of capacity:

$$D_0(S) = \lim_{\epsilon \to 0} \frac{\log M(\epsilon)}{\log\left(\frac{1}{n}\right)} \qquad (4)$$

where S is a subset of the n-dimensional space, and $M(\varepsilon)$ the minimum number of ε-side n-dimensional cubes necessary for covering such a subset. For small values of ε, the implicit definition shown in (4) means that:

$$M(\epsilon) \propto K \cdot \epsilon^{-D_0} \qquad (5)$$

[6] ALLIGOOD, Kathleen T.; SAUER, Tim D.; YORKE, James A. (1997): pp- 149-150.
[7] AGUIRRE, Jacobo; VIANA, Ricardo L.; SANJUÁN, Miguel A.F. (2009): p. 334.

The capacity of a point, a line or an area in the bi-dimensional space, takes the values 0, 1 and 2 respectively. That is, if we take the cubes of the side ε, the number required to cover the point would be proportional to $1/\varepsilon^0$, to cover the line to $1/\varepsilon^1$, and to cover the surface to $1/\varepsilon^2$. The dimension of fractal sets, as mentioned previously, is strictly greater that this Euclidean capacity. Thus, the Kotch curve and the Cantor set, which constitute typical examples of fractals, have fractal dimension greater that one and 0, respectively:

$$d = \frac{\log 4}{\log 3} = 1.2619 \qquad and \qquad d = \frac{\log 2}{\log 3} = 0.6309 \qquad (6)$$

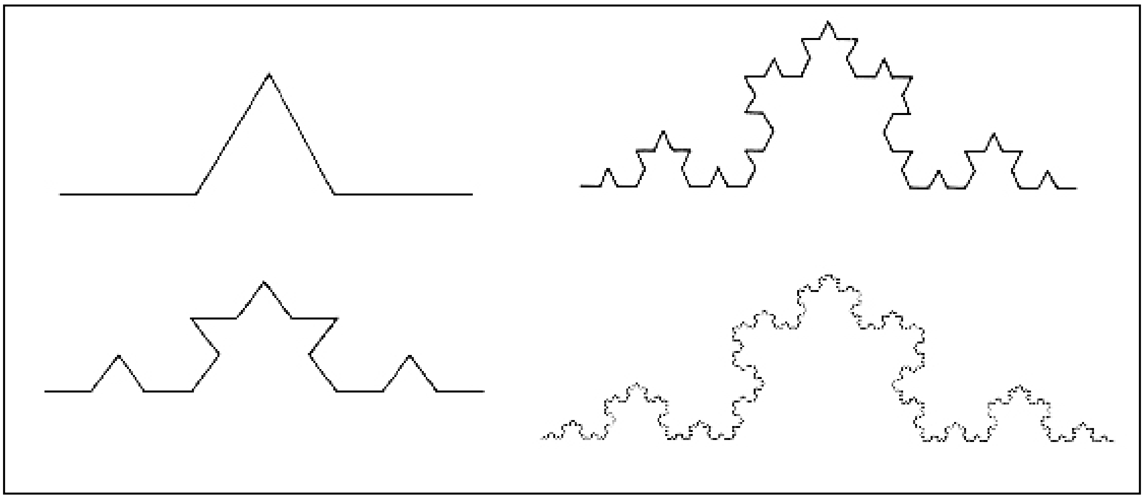

Figure 1. Kotch curve

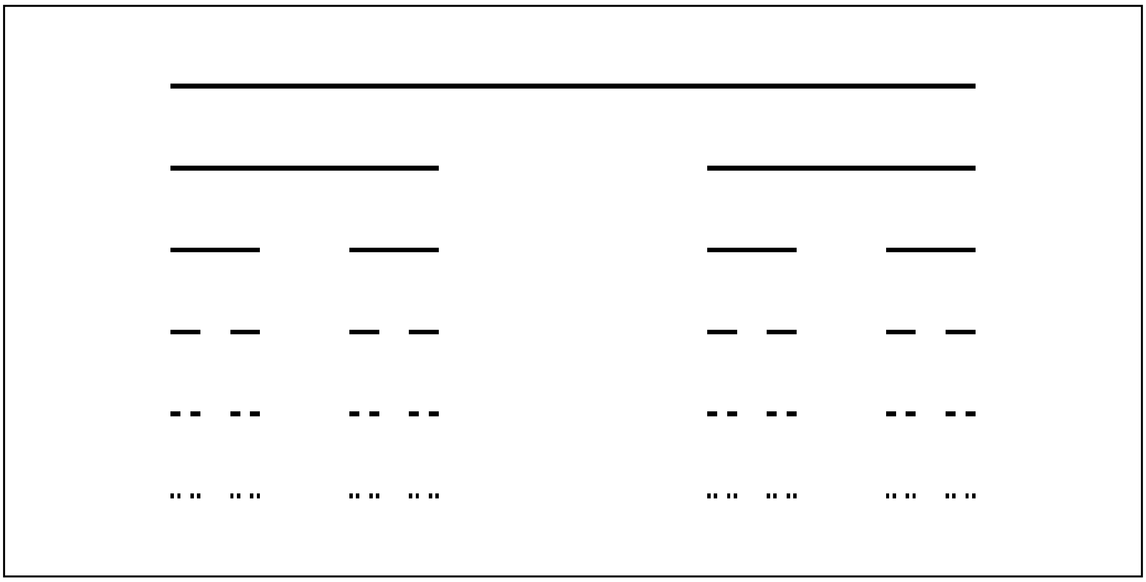

Figure 2. Cantor Set

Lyapunov exponents and fractal dimension are the two main instruments of the Mathematics of Chaos. Another key concept in the analysis of chaos and complex dynamics is bifurcation, which could be defined as a doubling

of the period of the attractors as some parameter of the system changes. With the analysis of the occurrence of this bifurcation, that is, the comparative dynamics of the system when the value of a parameter is changed, it is possible to explore the different types of attractors that the model can achieve. And the way in which the changes in the parameters move the system from one attractor to another is precisely through the bifurcation of the period.

Without forgetting the contributions of Yorke and May, and of course, the clear and decisive inspiration of Smale, the best known and illustrative analysis of comparative dynamics is the *Feigenbaum bifurcation tree*. Consider, for example, the logistic map:

$$x_{t+1} = \mu x_t(1 - x_t) \tag{7}$$

where μ is a constant situated at the interval [0,4]. When we iterate the map from an arbitrary initial condition, the attractor of this discrete dynamic system is obtained depending on the values assigned to μ:

If	$0 \leq \mu < 3$	→	a sole stable fixed point
	$\mu=3$	→	a marginally stable fixed point
	$\mu>3$	→	a fixed point becomes unstable
	$\mu=3{,}2$	→	second period cycle
	$\mu=3{,}5$	→	fourth period cycle
	$\mu=3{,}56$	→	the period has doubled to eight
	$\mu=3{,}567$	→	the period has doubled to sixteen
	$\mu=3{,}58$	→	the cascade of duplications is so rapid that the logistic map becomes chaotic

and the graph of bifurcation would be the following:

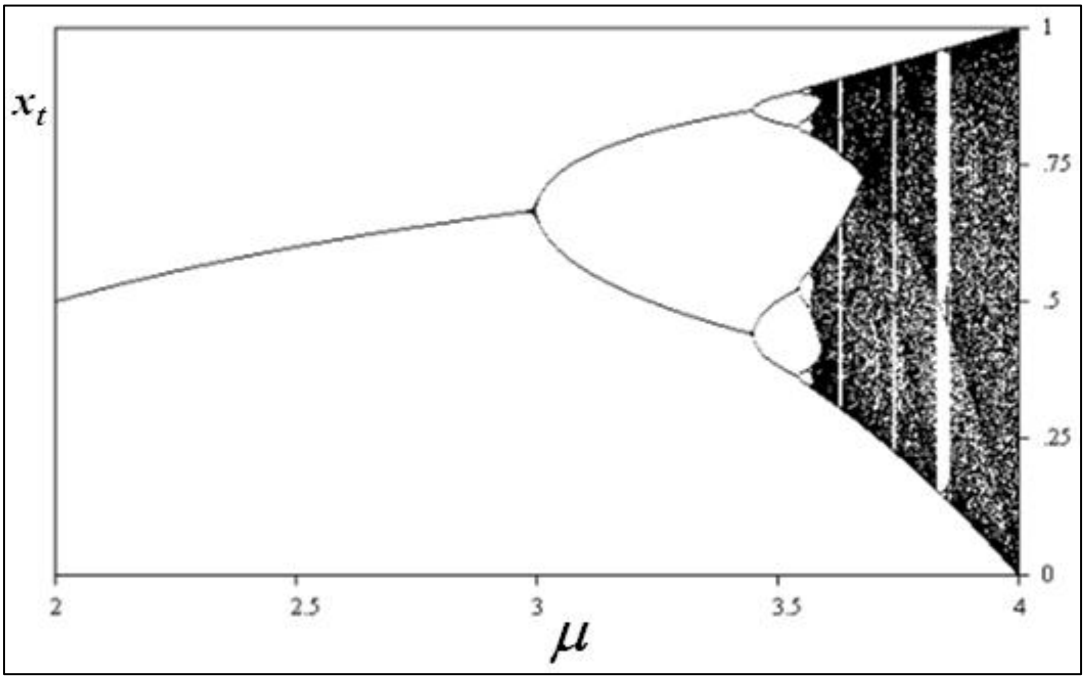

Figure 3. Feigenbaum Tree

In this way, by advancing through the duplication or the bifurcation of the period of the attractor, the Feigenbaum Tree is obtained (*figure 3*).The scale factor of the tree branches tends towards a universal Feigenbaum constant, which is equal to 4.6692, and is maintained even if we use another application. In effect, the relation

$$\delta_i = \frac{\mu_i - \mu_{i+1}}{\mu_{i+1} - \mu_{i+2}} \qquad (8)$$

converges towards this universal number, being therefore $\delta_\infty = 4.66922$.

It should be added that, in the zone of chaos, small windows or oases of order and stability can be found in the middle of disorder, illustrated in the habitual graphic representations by means of clear spots in foggy and obscure areas.

Chaos in time series

There are abundant contributions in the literature showing that it is possible to generalize the traditional theoretical economic models to show chaotic behaviour under economically plausible assumptions. However, while there is not a great difficulty to design theoretical models in regime of chaotic behaviour, there is no clear evidence that economic time series behave chaotically.

In fact, the major advances in the application of chaos theory in Economics deal with the tools to detect if the underlying true economic time series generation process is really a chaotic dynamic system. The detection of chaos in the underlying dynamics of a time series is divided into several stages.

The first step in detecting chaotic behaviour from a time series is to find evidence of nonlinear time dependence in the underlying dynamics of the system. And for that, the most widely used tests in the field of economics are the BDS test and the Hurst exponent, which are two techniques to contrast the existence of time dependence, linear or non-linear.[8]

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[8] BROCK, W.A.; DECHERT, W.D.; SCHEINKMAN, J.A. and B. LeBaron. (1996): pp. 197-235.
MANDELBROT, Benoit. (1972): pp. 259-290

To detect nonlinear dependence the Brock test is used. This test consists on filtering the time series by a general auto-regression model with a range large enough to ensure that any linear dependence has been completely removed. If, despite the linear filtering, Hurst and BDS tests continue to show evidence of time dependence, and then it must be nonlinear.

Once detected the non-linear dependence, the next step is to estimate both the Lyapunov exponent and the fractal dimension of the attractor of the underlying system generating the time series in order to test if that dynamical system presents chaotic behaviour. The main limitation of this approach is that this system is unknown. It is for this reason that a previous step (prior to the estimation of Lyapunov exponents and fractal dimension) is the recovery or reconstruction of the attractor but maintaining the qualitative properties of the underlying unknown dynamical system generating the time series.

A commonly used method of reconstruction of the attractor is the *lag method*. This method is based on the *embedding* theorem of Takens (1985), that establishes that, under certain conditions, though it will not be possible to reconstruct the orbit of the dynamical system in the original phases space, it is possible to obtain an approximation of it that result equivalent in a topological sense (equivalence in the dynamic and geometric properties), and that permit to extract all the relevant information about the unknown underlying dynamical system that generates the time series.

Once we have reconstructed the attractor from the time series, we can proceed now to estimate the fractal dimension and Lyapunov exponents to detect chaotic behaviour.

The Fractal dimension has a metric character, but using alternative measurement concepts to the traditional length, area or volume, and is usually calculated using covering formed by hyper-cubes or boxes. This method for calculating the fractal dimension is, however, little operational when working with embedding dimensions higher than two and when using time series contaminated by purely random noise, and for that reason alternative methods have been developed. Among them are the methods that use the ergodic theory to calculate a probabilistic measure of the attractor, the frequency with which the orbit visits the different parts of the attractor. Among these probabilistic dimensions, the more generally used in Economics is the correlation dimension that we explain shortly. The method consist, fundamentally, in centering an hypersphere in a point of the phase space making growth the radio r of the sphere until that all the points remain into it.

We can write the correlation function between two points for a small r in this way

$$C_m(r) = r^{Dc} \tag{9}$$

where r is the sphere radio, m the embedding dimension, and D_c the correlation dimension.

Then, taking logarithm we should have

$$l_n\, C_m\,(r) = D_c\, l_n\, r$$

$$D_c = \frac{\ln Cm\ (r)}{\ln r} \qquad (10)$$

That is the correlation dimension, been demonstrated by Grassberger and Procaccia that its values are near to those of the capacity dimension without exceed them. That is

$$D_c \leq D$$

If the correlation dimensions growth with m, that is, with the embedding dimension, the process will be stochastic, and if is independent of m, the process will be deterministic.[9]

The fractal dimension, in his stead, provides a measure of the complexity of the attractor. However, the estimation of fractal dimension from a time series cannot be taken as a sufficient test for the detection of deterministic chaos. This is because firstly, it is only possible to obtain rough estimates of the true fractal dimension of an unknown dynamic system, and therefore, it is very risky to assure when this approximation is an integer or fractional. Second, because when working with economic time series, we must accept the fact that in the series there is always some random component, so that the estimate of the fractal dimension will be always biased upwards, making it difficult the detection of low-dimensional chaotic behaviour.

Therefore, the estimation of fractal dimension in the search for a non-integer dimension and not very high, must be taken as a supplement to other techniques for the detection of chaos, especially, the spectrum of Lyapunov exponents. Recall that in dissipative systems, the presence of a positive Lyapunov exponent is indicative of sensitive dependence to initial conditions, and then it is the sufficient condition to chaotic dynamics to exist.

There are several algorithms to measure the Lyapunov exponents of the underlying time series generation process. The *Wolf et al (1985) direct algorithm* may not provide a correct characterization of Lyapunov exponents of a time series with limited number of observations. Furthermore, the performance of this direct algorithm is very sensitive to the degree of noise in the data. For these reason in Economics, with time series characterized by short sample and error measured, we use indirect methods that use regressions method to estimate the underlying derivative en (4).

These regression methods to estimate indirectly Lyapunov exponents assume the existence of an unobserved dynamic model that may be chaotic

$$x_t = f(x_{t-1}, x_{t-2}, \dots, x_{t-d}) + e_t \qquad (9)$$

[9] FERNÁNDEZ DÍAZ, Andrés (1994): pp. 150-151.
GRASSBERGER, P. and PROCACCIA, I. (1983): pp. 189-208.

where t=1, 2, ..., N ; and $\{e_t\}$ a sequence of *iid* random variables. This model may be expressed in terms of a state vector $\vec{\boldsymbol{X}}_t$ an error vector $\boldsymbol{\varepsilon}_t$, and a function F: $\mathbb{R}^d \rightarrow \mathbb{R}^d$ such that

$$\vec{\boldsymbol{X}}_t = \boldsymbol{F}(\vec{\boldsymbol{X}}_{t-1}) + \boldsymbol{\varepsilon}_t \tag{10}$$

It is then possible to estimate Lyapunov exponents from the Jacobians of the map, that is, based on nonlinear regression estimates of f and F in the respective equations. There are different methods for estimating the map, but the NEGM and the NETLE methods, that take advantage of the use of multilayer feed-forward neural networks models, has been revealed as especially adequate for use with noisy data as well as with limited number of observations.[10]

We know very well the special prominence of time series in Economics, and because of this appears convenient to add some complimentary considerations on them, by highlighting those aspects which are related to the nonlinearity of chaos. It seems that this specific chapter of Economics had advanced notably, by passing from the use of classic methods, like the moving average, the adjustment of trend, the analysis of decomposition, multiple regression or auto-regressive models, and other more recent ones, such as spectral analysis, adaptive filters, the exponential smooth or Box-Jenkins analysis. But when, in dealing with some problems these instruments do not turn out to be sufficient nor adequate, generalization of the approach are attempted, based on the concept of random walk, which is what happens with martingales or processes of Itô, from a theoretical point of view, or with models such as ARCH, GARCH, EGARCH, and TAR, from an empiric point of view.

In the last years a greater deal of attention has been paid to the possibilities of martingale theory in the field of financial markets. It is also necessary to emphasize the more and more use of stochastic partial differential equations such as the following:

$$\frac{\partial W}{\partial t} = rW - rS\frac{\partial W}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} \tag{11}$$

that is, the well known Black-Sholes, in fact, a Ito's process, where W is the value of a derivative product which underlying security price is S, a random variable that follows a Wiener process, r denotes the risk-free rate of interest, and σ is the annualized standard deviation of dS. In the context of the multidimensional Black-Scholes model with

[10] McCAFFREY, D.F.; ELLNER, S.; GALLANT, A.F.R. and NYCHKA, D.W. (1992): pp. 682-695.
GENCAY, R. and DECHERT, W.D. (1992): pp. 142-157.

market imperfections are used relevant techniques, as backward stochastic differential equations and stochastic optimal control.[11]

An application to Capital Markets

Among the many applications of chaos and fractal to Economics highlights that one to the Capital Markets due to the fact that this sector of the economic activity works with a great volume of data and long time series, and because of this we think that is the most advisable to apply the theories and techniques that have been previously review.

In other works we have developed and applied the R/S analysis of the Capital Markets, and in all cases we have found fractal structure and non-periodic cycles, clear evidence that the capital markets are nonlinear systems and that the Efficient Market Hypothesis (EMH) is questionable. Let us remember that in this method, starting of a measure of correlation in function o the Hurst coefficient and of the fractal dimension that respectively would be

$$C=2^{(2H-1)} - 1 \qquad (12)$$

$$\alpha = \frac{1}{H}$$

and taking into account the variation or intervals of the Hurst coefficient, we should have three types of series:

a/ $H = 0{,}50 \longrightarrow C=2^{(2H-1)} -1=0$

b/ $0 \leq H \leq 0{,}50 \longrightarrow C=2^{(2h-1)} -1=-0{,}50$

c/ $0{,}50 \leq H \leq 1 \longrightarrow C=2^{(2h-1)} -1=1$

In the first case a random series is dealt with, in which the events are not correlated. In the second we find an anti-persistent or ergodic series, its degree of inpersistence, depending on the measurement in which H nears zero, which, as we have seen, gives rise to a negative correlation. In the third case, we have a persistent series, in which the peak or low values of the previous series repeat or maintain themselves, the major persistence being the closer the H value approaches 1. This last case would implies a total correlation or one hundred percent.

[11] A wide mathematical development of the stochastic partial differential equations may be found in:
FERNÁNDEZ DÍAZ, Andrés (2000): pp. 237-250.
Respect to martingale theory see:
FERNÁNDEZ DÍAZ, Andrés; GRAU-CARLES, Pilar (2011): pp. 267-271.

In the first case we speak of the existence of *white noise* in the habitual sense of the term, and of *pink noise* and *black noise* in the second and third cases respectively.[12]It can be interpreted that the persistent series follows a biased random walk, and that the importance of the bias depends on the distance that H is above 0,50. On the other hand, series of this type are fractals, because they can be described with brownian fractional movement, and given that, as demonstrated by Mandelbrot, the inverse of H constitutes a measurement of the fractal dimension, we can learn the dimension of the fractals in the respective series with precision.

In the concrete analysis presented in this new article, we have choose the evolution of the Madrid stock exchange during the lapse of time 2006-2013 that embraces the years of crisis that has its origin in the behaviour of financial markets.

This period is characterized by strong oscillations and high volatility in the markets that allows that techniques we have explained can throw light on the dynamical followed by the stock market in these hard times, specially starting from the collective stock exchange hysteria on Monday 21 January of 2008, due to the fear of one recession in the Unites States economy after the beginning of mortgage enormous fraud.

The analysis will be realized with the time series of IBEX 35 daily data in the course of the referred period, studying in such an analysis the time evolution of IBEX daily returns obtained as the composed returns or the first difference of logarithm of stock prices. The series that take daily data from January 2006 to December 2013 is integrated by 2019 data, and its time evolution appears in the figure nº 1.[13]

[12] FERNÁNDEZ DÍAZ, Andrés (2001): pp. 224-226.

[13] See the recent publication:
FERNÁNDEZ DÍAZ, Andrés; GRAU-CARLES, Pilar (2014): pp. 251-260.

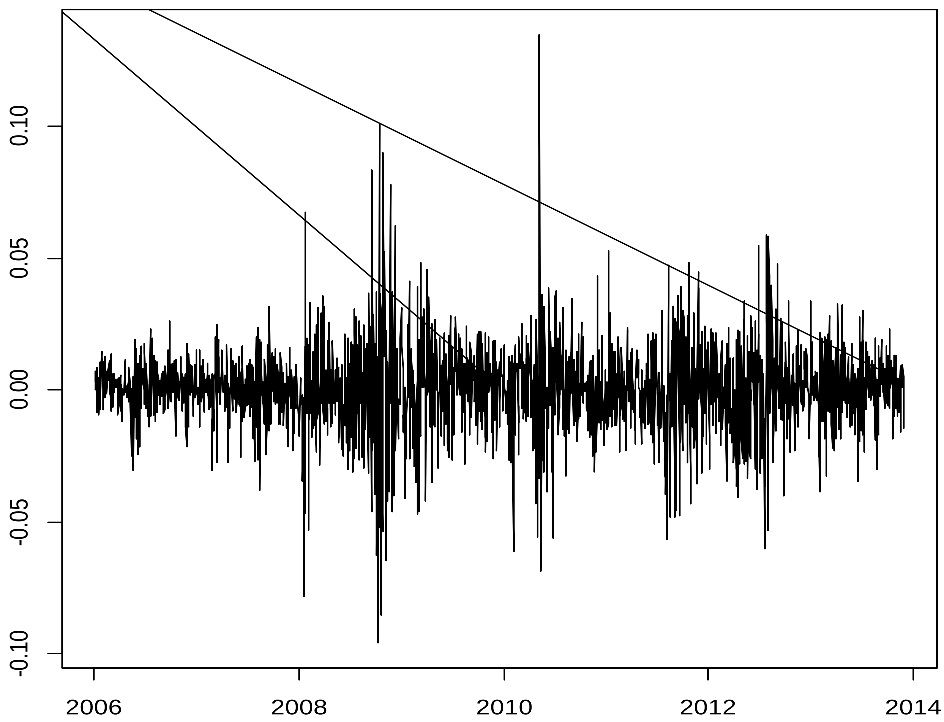

Figure 1: Time evolution of the IBEX 35 daily returns (2006-2013).

According to the statistical summary of data, the daily returns average for this period is very small, close to 0, while the daily volatility is around 1,7%. The smallest daily return is of -9,6%, whereas the largest is of 13,4%. The returns present a positive and small asymmetry, and a high kurtosis. These results are summarized in the Table 1. Likewise the hypothesis of normality of the data is rejected using the Jarque-Bera, and the presence of autocorrelation with the Ljung-Box test.

Table .1: Summary of statistical data

Max	13.484%
Min	-9.586%
Average	-0.005%
Standard Deviation	1.653%
Asymmetry	0.156
Kurtosis	8.823
Jarque-Bera (p-value)	0.000
Ljung-Box (20 retards (p-value)	0.000

The figures 2 and 3 show the distribution of the returns compared with the normal distribution and the graphic QQ. Both graphics exhibit that the returns distribution has wider queue that the normal distribution, so much in the positive side as much in the negative one, so that the first conclusion is that daily returns not behave as a normal distribution.

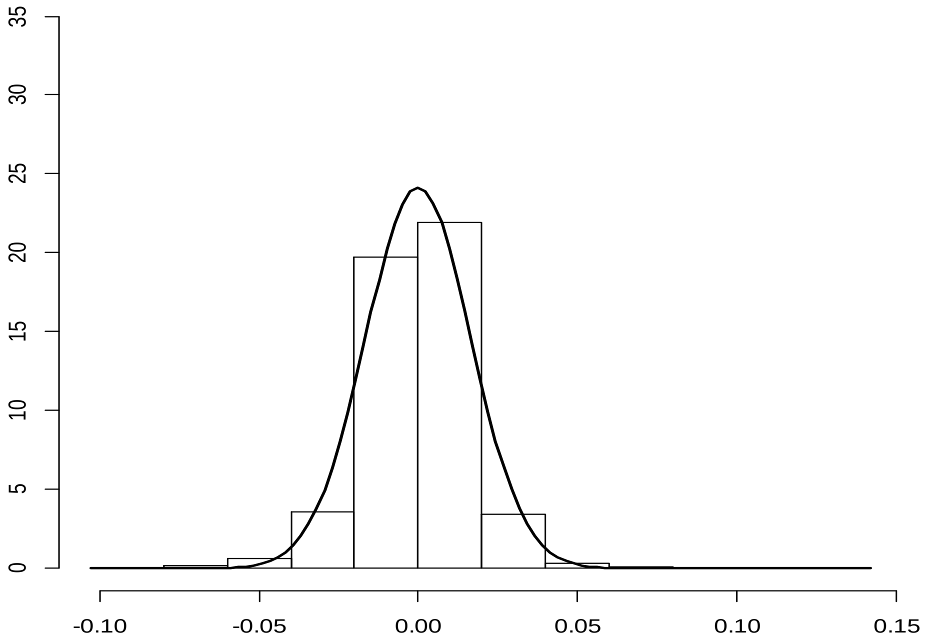

Figure .2: Frequency distribution of the returns series respect to normal Distribution.

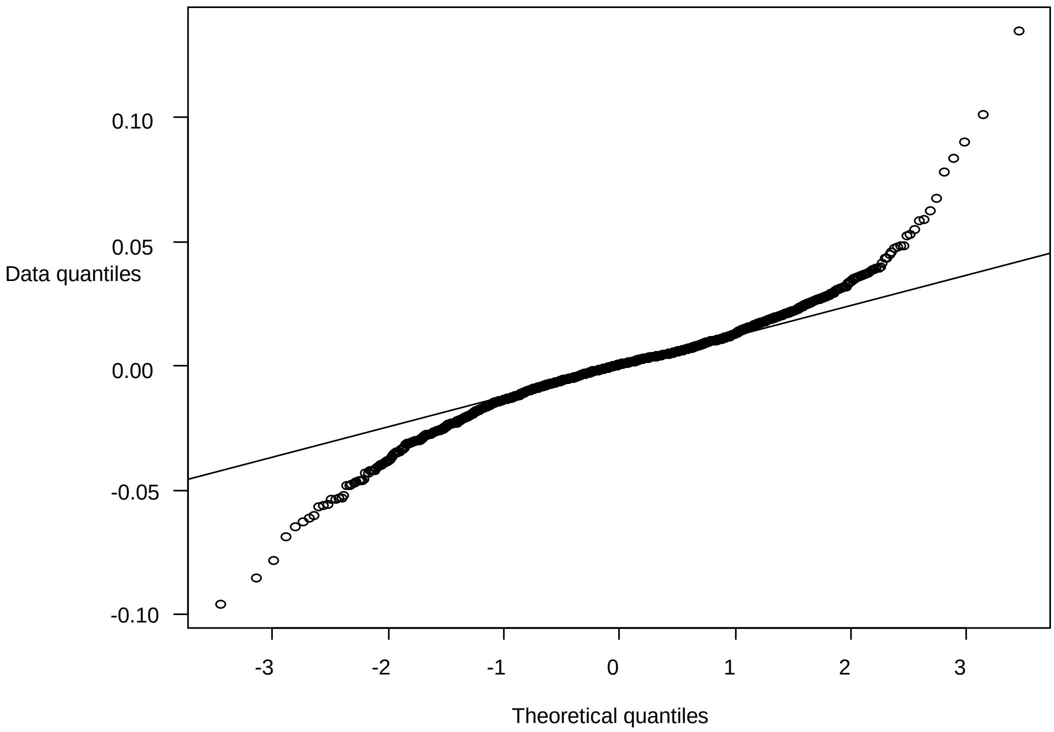

Figure .3: QQ graph of the daily returns.

If we analyze the long memory of the returns during the period using the analysis R/S or the technique DFA (Detrended Fluctuation Analysis)[14] we find that the series do not present long memory, but, on the contrary, is obtained an Hurst exponent lightly inferior to 0,5. Figure 4, shows the result of DFA, that gives an exponent H=0,47, very near to 0,5. Although the estimation of the Hurst exponent through the DFA is robust, we proceed to study its stability by calculating moving windows, for 21 days, that is the approximated number of data that there is in a month.. The results appear in the figure 5, observing that for majority of windows we obtain values of the Hurst exponent lower than 0,5, what would indicate that follows a reversion process toward the average.

[14] See: FERNÁNDEZ DÍAZ, Andrés; GRAU-CARLES, Pilar and ESCOT MANGAS, Lorenzo (2002): pp. 469-481.

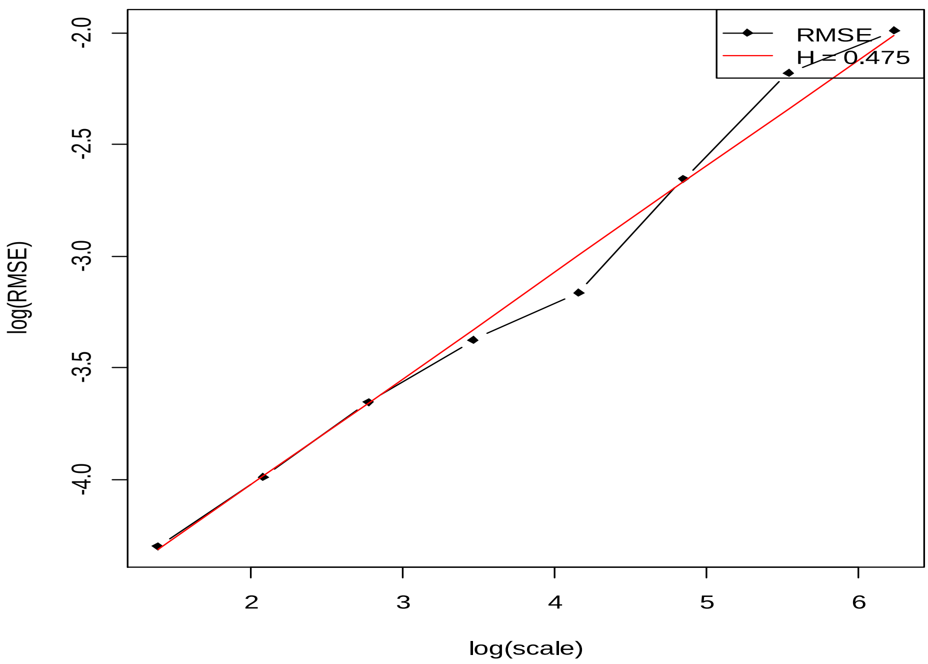

Figure 4: DFA of the returns series.

Perhaps become convenient to remember the extent and meaning of the Detrended Fluctuation Analysis method introduced by C. K. Peng et al. in a paper published in the *Physical Review* E in the year 1994.

The Detrended Fluctuation Analysis (DFA) method has been in the last decade a widely used technique for determination of fractal scaling properties and the detection of long-range correlation in noisy, non stationary time series, and has successfully been applied to diverse fields such DNA sequences, geology, solid state physics or economic time series, among others. Being more precise, we could say that was a method basically designed to research long-range correlation in non stationary series.

We must point out that advantages of Detrended Fluctuation Analysis (DFA) over conventional methods, as spectral analysis and Hurst analysis, are that it permits the detection of intrinsic self-similarity embedded in a seemingly non stationary time series, and also that avoids the spurious detection of apparent self-similarity, which may be an artifact of extrinsic trends.

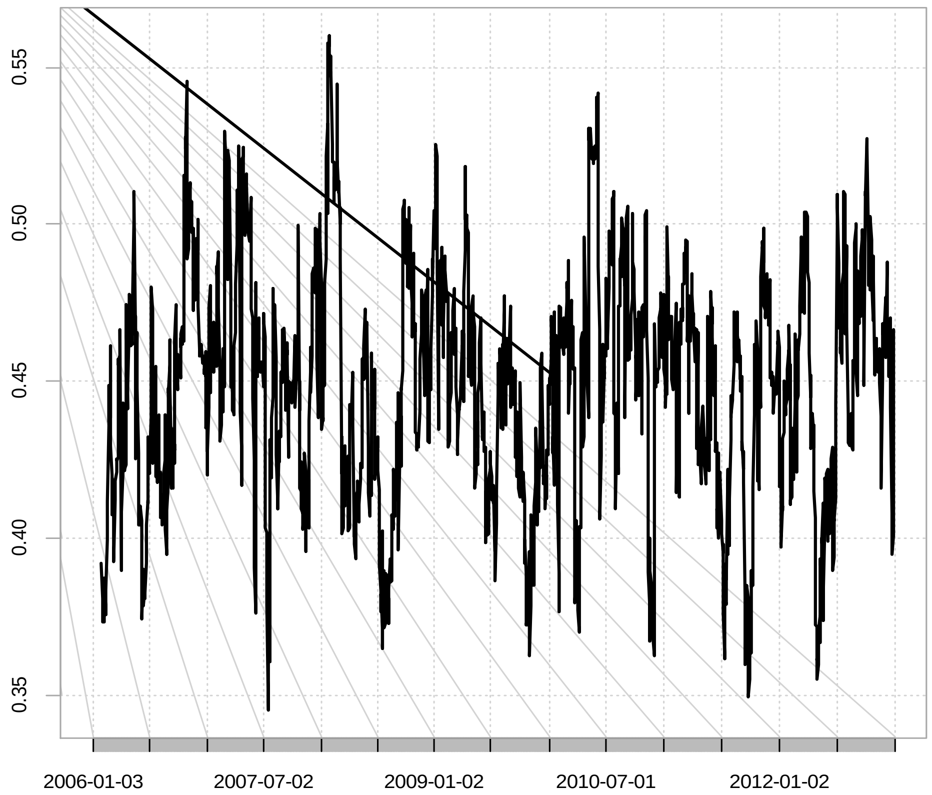

Figure 5: Evolution of the Hurst exponent with windows of a month.

In the study of the chaotic behaviour of a time series it is indispensable to realize an analysis about the dimension correlation. The method most largely used for the calculus is that one proposal by Grassberger y Proccacia (1983) that consists in constructing an integral of correlation $C(\varepsilon)$ equal to the probability of that two arbitrary points are more close that ε in the orbit of the space of states. The figure 6 shows the results of the calculus and the slope of the lineal part of graph in logarithm is the correlation dimension. The table 2, in his stead, exhibit the outcomes of the estimation of the slopes for the different embedding dimension, where we can verify that the slope is no stabilized, making it impossible to conclude on the correlation dimension of this series.

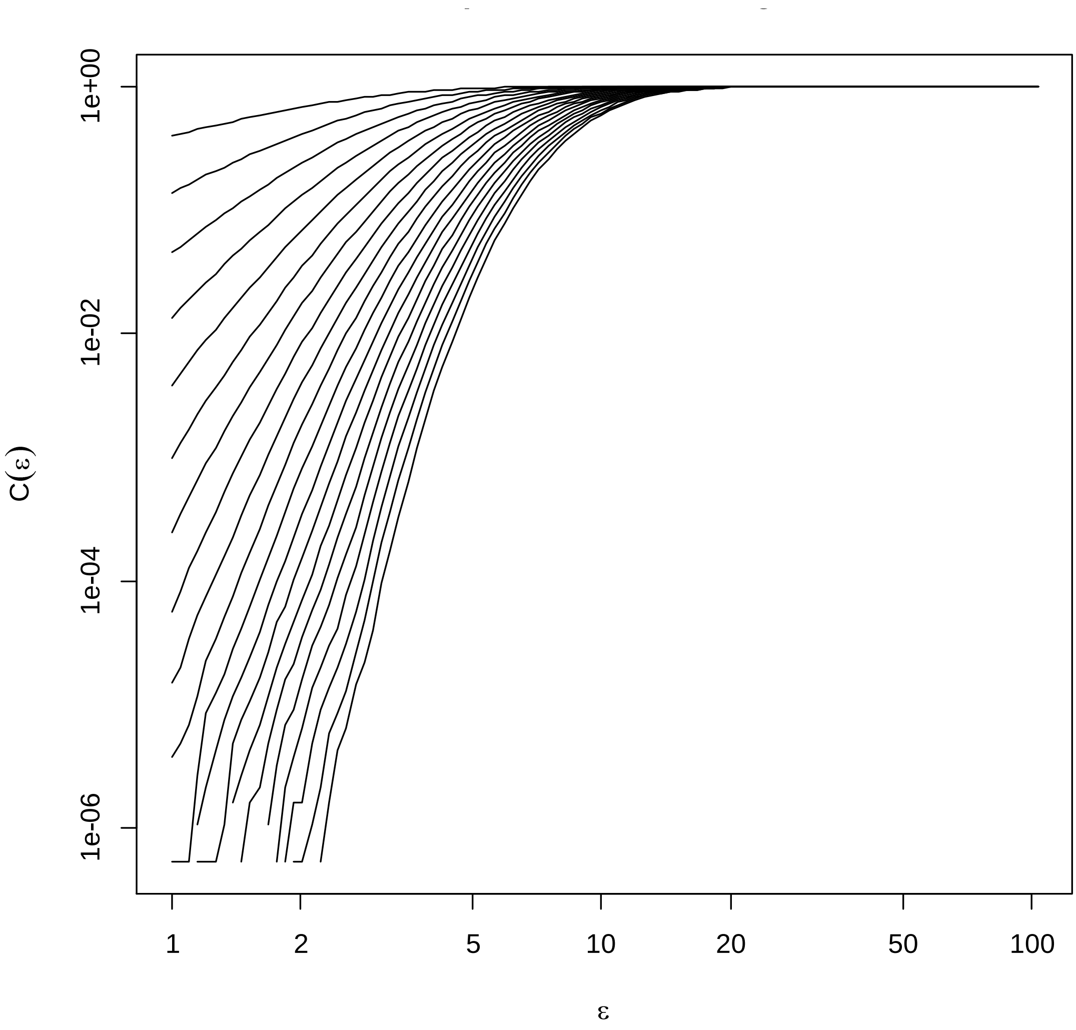

Figure 6: Integral of correlation for the IBEX 35 series.

Table 2: Slope of the integral of correlation for IBEX 35.

Embedding Dimension	10	11	12	13	14	15	16	17
Slope	3.7	4.2	4.7	5.3	5.8	6.4	7	7.6

Other characteristic of a chaotic dynamic system is, as already we have said, the existence of a positive Lyapunov exponent. The exponent is a measuring of the convergence or divergence of the near initial conditions, so that a positive exponent implies that any perturbation in the initial conditions it airs and will grow significantly. One algorithm which proved efficient for calculus of Lyapunov exponent with short series is proposed by Rosenstein, Collins and De Luca (1993) and is what we will use in the present work. The figure 7 shows the result of using such algorithm, could observe the existence of noise in the system as there are many variations in the distances.

Following the procedure of algorithm, to calculate the slope of the "middle" line, is obtained an exponent of 0,019 that is positive although very close to zero.

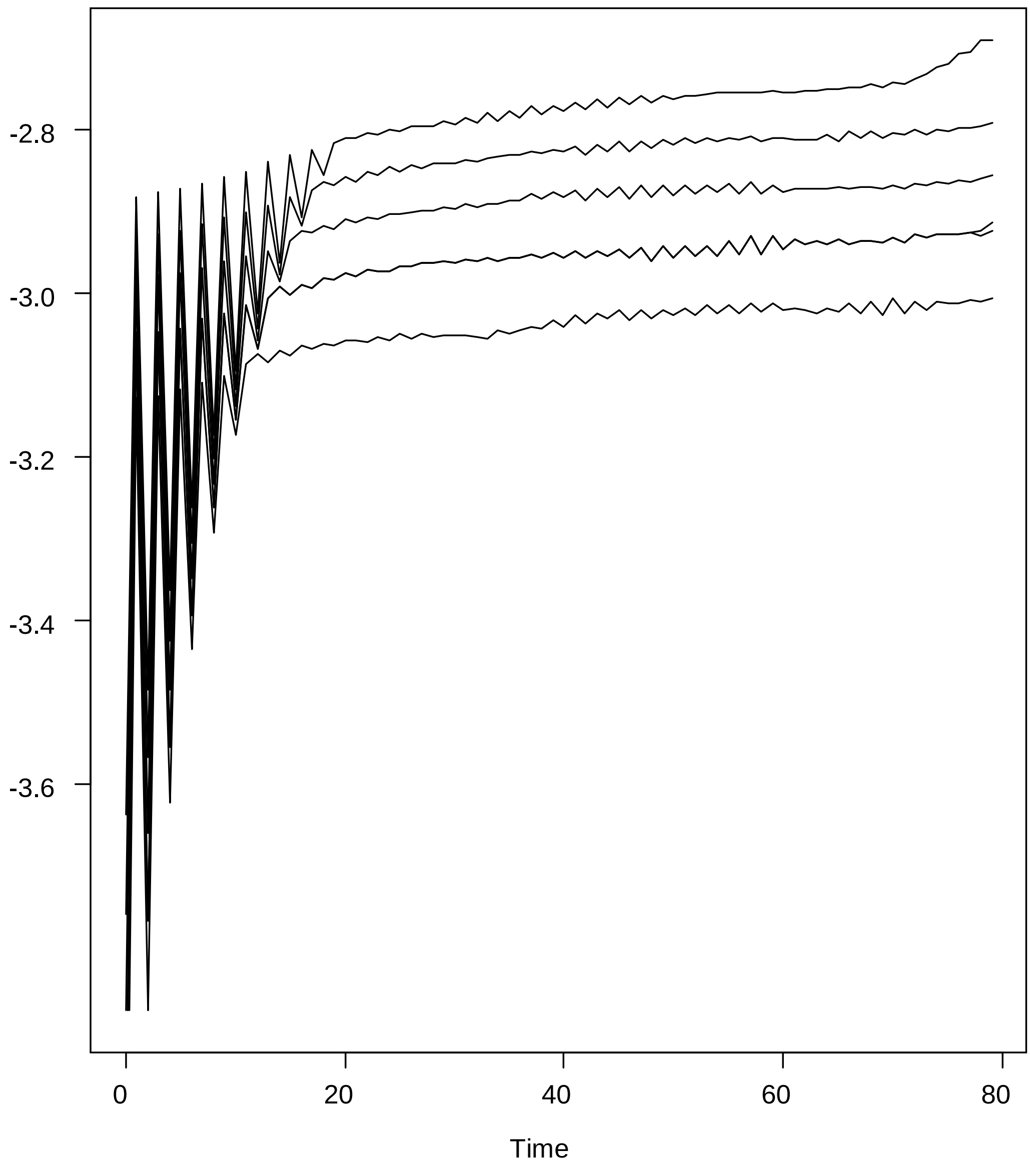

Figure 7: Result of calculus of major Lyapunov exponent according to the Rosestein algorithm.

Some time series have a cyclical behaviour difficult to detect, especially if the period is not known in advance. The periodogram and the power spectrum, calculated through a Discrete Fourier Transform (DFT), can help to find that period. Figure 8 shows the periodogram respect to the frequency, while figure 9 shows it respect to the period. In both cases is observed a very erratic behaviour that prevents the possibility of finding harmonics or clear periodic behaviour.

It is well known that time series can be considered in the frequency domain, on the one hand, or in the time domain, on the other. In this context is used the concept of

Discrete Fourier Transform (DFT) that transforms a mathematical function in other, obtaining representation in the frequency domain, being the original function other one in the time domain. As in the time domain the basic concept is the autocorrelation function, in the frequency domain corresponds to the spectrum which can be approximated well within a deterministic hypothesis, in which case one speak of harmonic analysis, well probabilistic, denominating it then spectral analysis.

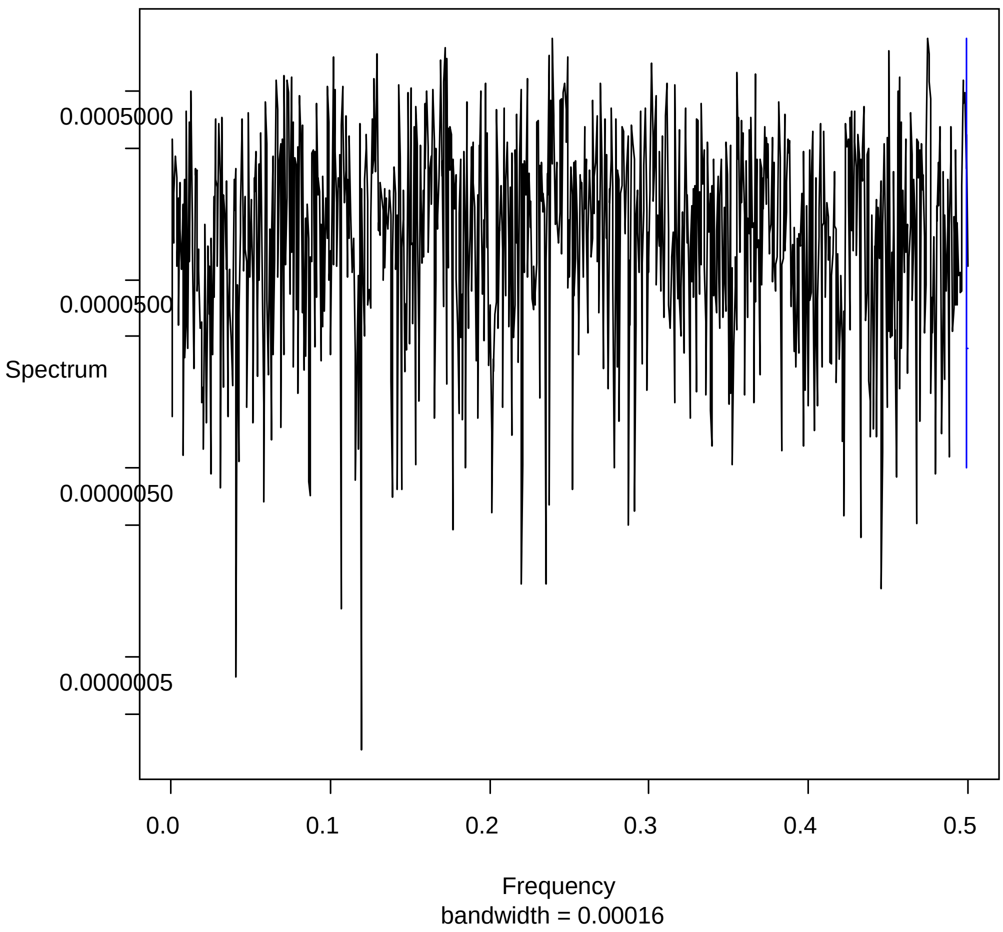

Figure 8: Periodogram of the IBEX series between 2006-2013.

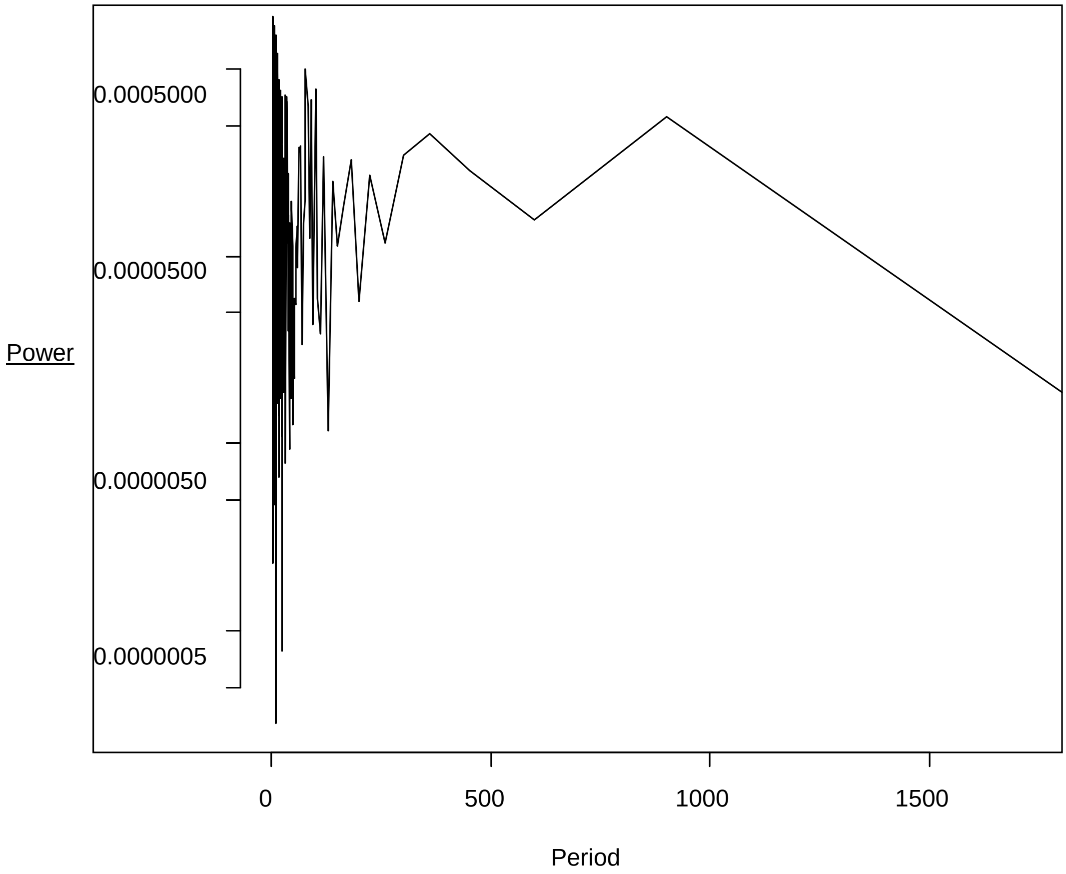

Figure 9: Periodogram of the IBEX series between 2006-2013 respect to the period.

Although the outcomes obtained until the moment do not permit to conclude about the chaotic behaviour of the IBEX series during the crisis period, we are goinng to verfy that follows a nonlinear behaviour. The recurrence diagram allow us to find the characterisdtics of the evolution in the time of phase space.

Figure 10 shows a recurrence diagram that is not homogeneous, and shows also that the returns behave in a nonlinear manner. Given that there are white areas close to dark one, it is possible affirm that coexist changes in the dynamic of the system and extreme value. The zone most clear reveal periods of less volatility, while the most darkness are the corresponding to more erratic values of the series.

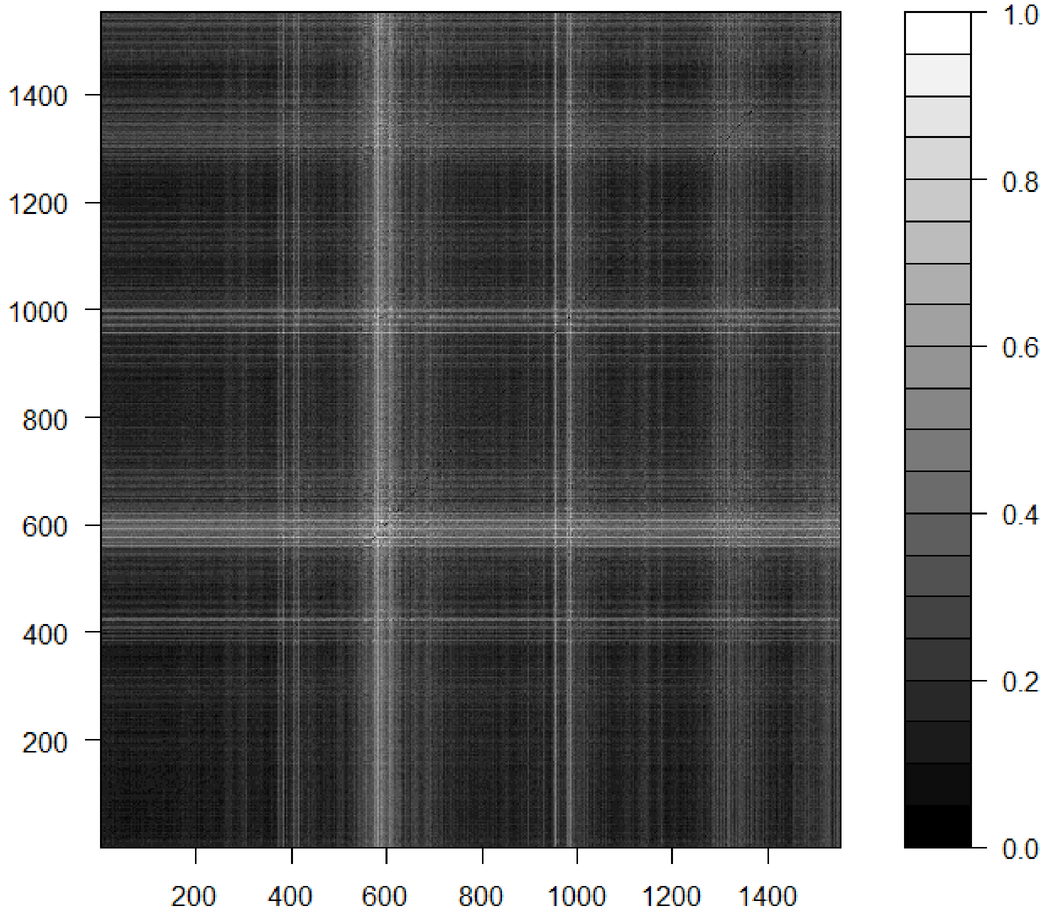

Figure 10: Recurrence Plot for the series.

Finally, the BDS statistic serve to corroborate the nonlinearity in the series studied. The table 3 exhibits the results of the statistic for the embedding dimensions 2, 3, 4, 5 and 6, and values of ϵ equal to 0,5, 1, 1,5 and 2 standard deviation of data. For all cases is obtained a p-value for testing the null hypothesis equal to 0, so the i.i.d. null hypothesis is rejected.

Table 3: BDS Statistic

	ϵ/σ			
n	0.5	1	1.5	2
2	6.36	7.1467	7.9918	8.0215
3	10.3294	10.2494	10.596	10.7371
4	14.1967	12.6289	12.2907	12.4068
5	19.2307	15.7589	14.4035	14.3591
6	25.6458	18.9488	16.3546	15.9826

Starting from all results obtained in our empirical analysis we cannot conclude that the IBEX 35 daily returns in the period of crisis be covered by a deterministic system, but so was the system would nonlinear components which are reflected in the data. However the series could also be described as a nonlinear dynamic system with noise far from the equilibrium, submissive to bifurcations or changes in the dynamical of the system. The problem with this approach is that although the dynamical of the system it could be modelling correctly, could not obtain better predictions.

Nevertheless it is necessary to emphasize that ethic, politic and institutional factors during the period analyzed can explain the irregularities and disequilibrium in the economy and in the international financial markets. The existence of corruption, the behaviour of the firms of rating with oligopoly power in the evaluation of the economic and financial policy of the different countries that work in the markets, and colluding with groups of brokers, are key elements in explaining the unusual and unconventional operation of capital markets in the years of the crisis. All that can be interpreted as evidences of that there has been something more that hazard, or what is the same, that there has been a chaotic behaviour in the evolution of the series of values, and has been seen, some ability to predict.

This conclusion, taking into account the reservations we did before, and the strong influence of external shocks considered, we believe that corresponds to and is consistent with the results obtained by applying different tests to series of IBEX 35 covering the years of the financial and economic crisis.

Conclusion

Of all that we have expounded on, within the framework of modern theories of chaos and complexity, some conclusions of interest can be extracted in relation to new approaches and techniques in the field of Economics. First of all, it is convenient to highlight, as can be easily deducted, that the importance of the application of the theory of chaos in our scientific field resides in the fact that it extends beyond the frontiers delineated for probabilistic analysis, by attempting to offer an consistent explanation of endogenous nature, to that part of behaviour which is habitually excluded on adopting a resigned attitude, refusing to interpret it as an inexplicable error, or a shock, or an unpredictable impulse. Besides is also fundamental and imperative to persevere with the intrinsically dynamic character of economic phenomena, which involves assuming the need of a feed-back or retroactive mechanism to facilitate the making of decisions, with a greater degree of rationality. Too it appears to be clear that economic phenomena are habitually nonlinear in nature, show a great sensibility to modification of the initial conditions, and often exhibit irregular behaviour.

A chaotic sequence and a random sequence look superficially the same, even though they are very different. Unlike the random sequence, the chaotic one is completely deterministic, and contains a good deal of hidden order. The use of methods to discern that order, distinguishing between chaos and randomness, can, for instance, deepen the understanding of the dynamics of financial markets and may lead to an improvement in short-term price predictability. Let us remember, among others, the long-memory, the correlation dimension, the fractal dimension, the embedding dimension, the Lyapunov exponent, the Kolmogorov entropy, the Hurst coefficient, and the BDS and Kaplan tests.

It is important to bring to light that, given the complexity of the economic universe, the great difficulty of any type of prediction of the future is evident, but this does not decrease our possibilities of intervention. It could even be said, that these possibilities increase notably from the moment that, on not being trapped by an established mechanism from the start, we are free to explore alternative future trajectories. On the other hand, we must not forget that emergence bear a great relation with complexity and chaos, and it can help to find underlying laws that we are searching behind the irregularity.

Complexity, as science of complex systems, implies a challenge to integrate a large number of disciplines, as nonlinear dynamics and theory of chaos, the statistical physics, stochastic process theory, the theory of information, the networks theory, the biology and the computational sciences. As highlighted in the Review Nature in the year 2005, there is a new line of research named “synthetic biology” that integrates different scientific areas, just as the nonlinear dynamic, the physic of complex systems, the engineering and the molecular biology. This emergent field of research is intrinsically interdisciplinary and constitutes an advance to take into account for the next years in our work about the application of chaos and complexity theories. The studies of complex networks in physics, mathematics, economics and other sciences, is also at present undergoing a very important development, which represents new possibilities in the task that we are tackling. It is clear that in the applications of chaos and complexity theories to Economics we use the methods of other disciplines (principally those of the mathematical, physical and biological sciences) wherever they seem to fit, although one must tailor their usage to one´s own discipline.

Situated concretely in the very sphere of Economics, besides the different concepts and techniques necessaries to understand the role, significance and functioning of complexity and chaos, we have seen any of the main and most recent applications in our field. Now, as summary or conclusions, we are going to emphasize some of them, pointing out its degree of efficacy, real interest and capacity of explanation.

Among the applications of mathematics of chaos and fractal geometry to Economics, that are many,[15] we must highlight those related to capital market, because the time series are longer and is easier, most profitable and useful to work with them. At present, the Efficient Market Hypothesis is not well supported by empirical evidence, and has often failed to explain the market behaviour. At the same time, markets are not well described by random walk model. Because of this we have decided dedicate an especial attention to the Capital Markets within the field of Economics. In effect, in the last part of this article we have analyzed the Madrid Stock Exchange case to detect chaotic behaviour in time series that correspond to the years of the recent (and not yet finished) crisis, that is to say, to the period existent between 2006 and 2013, using the IBEX 35 daily data.

As we see in his moment, the results of our analysis referred to the behaviour of the Madrid Stock Exchange in the years of the last crisis can be interpreted as evidences of that there has been something more that hazard, or what is the same, that there has been a chaotic-deterministic behaviour in the evolution of the series of values, and because of this, a certain capacity to predict.

Some paper finds that nonlinearities in financial assets can be product of contamination produced by shifts in the distribution of the data. Using the BDS and Kaplan tests it is shown that, some of the nonlinearities found in foreign exchange rate returns, can be the product of shifts in variance while other do not. Also, the behaviour of the volatility is studied, showing that long-range correlation modelling is able to capture long memory, but depending on the *proxy* used for the volatility, is not always able to capture all the nonlinearities of the data.

Although in this work we have not treated about this important and modern technique, it is necessary to stand out the possibilities of applying the Artificial Neural Networks (ANN) to Economics and Finance. For instance, there are multivariable model from genetic algorithms and artificial neural networks to predict the sign of the variations of stock-market indexes. Artificial Neural Networks are composed of a large number of highly interconnected processing elements working in unison to solve specific and complex problems. In more practical terms Neural Networks are nonlinear statistical data modelling or decision making tools. If we speak about the economic application, we find that Neural Networks can be used as an alternative to more traditional methods

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[15] In others works, for instance, we have considered the urban dynamics as a fractal geometry, a new way of getting a more deep and productive analysis of the structure and evolution of cities from an economic point of view. It is important to emphasize that we find in urban dynamics and in the stock market, between other fields, the existence of power laws, a relevant tool that is related to changes of phases or to critical states, where a many body system evolves from state of self-organization into a different one.

such as discriminant analysis or logistic regression. A special feature of Neural Networks that distinguishes them from traditional methods is their ability to classify data which are not linearly separable. The majority of papers that use Neural Networks for classification tasks in Economics can be found in the area of bankruptcy of economic agents, mainly banks. But probably, we find the largest share of economic applications of Neural Networks in the field of prediction of time series in the capital markets. As the efficient market hypothesis is neither correct nor generally accepted, is advisable to use nonlinear models to improve the fit and thus the prediction, taking into account that Neural Networks are flexible functional forms that allow providing effective nonlinear models. To end up, it is convenient to remember that, in the comparisons of available tests for nonlinearity and chaos we have the White´s proposed Neural Network test.

Recent works deal with the applications of the concepts of fractal and strange attractor to explain the present economic crisis, considering that in a world of nonlinearities, complexity, depending of the initial conditions, with irregularities and far from equilibrium, there is no other solution that to profit the possibilities derived of the fractal geometry. Although, as we know, this don´t convert the Black Swan in a predictable happening, is enough to work and take decisions, and become it in a Gray Swan, that is, in a new approach that avoids more of the same, and allows a better management of the crisis, leaving the conventional economic wisdom, and undertaking a set of deep structural reforms, mainly in the financial system and its international coordination.[16]

To conclude about the possibilities and usefulness of applications of chaos theory to Economics at present and in the future, it is necessary to distinguish between the pessimistic and apathetic approaches, that only pay attention to identify chaos with unpredictability, and those that tries to deep and find the way through which nonlinear dynamic and chaos can help the economy as an evolving complex system. As we say, the first group, that is, the frigid one, principally the econometricians, argues that is not possible the predictability, and because of it chaos theory don´t works, but the fact that we cannot make exact predictions of the long-term behaviour of chaotic systems does not exclude the possibility of making, more or less accurate, short-run forecasts.

In the presence of this duality, darkness or light, pessimism or optimism, let us leave the license of finishing our work with a relevant reference to the best literature, very illustrative and opportune, remembering that, between the unexplored limbo of fear and sadness perceived in the Henry James´s verses of his work “The turn of the Screw”, and the best of the worlds of Candide, the Voltaire´s famous story, we find ourselves, regarding this matter, with a present and future that belong to us and which knowledge and control depends on our capability of analysis and deluded effort.

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[16] About this issue sees: NIETO de ALBA, Ubaldo (2014): pp. 192-201.

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